CIGS The Canon Institute for Global Studies

CIGS Working Paper Series No. 23-015E

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2023.8

General Incorporated Foundation
The Canon Institute for Global Studies
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Leverage, Endogenous Unbalanced Growth, and Asset Price Bubbles*

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August 17, 2023

Abstract

We present a new theoretical framework to think about asset price bubbles in dividend-paying assets. We study a general equilibrium macro-finance model with a positive feedback loop between capital investment and land price, whose magnitude is affected by financial leverage. As leverage is relaxed beyond a critical value, a phase transition occurs from balanced growth of a stationary nature where land prices reflect fundamentals (present value of rents) to unbalanced growth of a nonstationary nature where land prices grow faster than rents, generating a land price bubble. Unbalanced growth dynamics and bubbles are associated with financial deregulation and technological progress.

Keywords: land bubble, leverage, nonstationarity, phase transition, unbalanced growth.

JEL codes: D52, D53, E44, G12.

*This paper is a significant revision of an earlier manuscript circulated under the title "Necessity of Rational Asset Price Bubbles in Two-Sector Growth Economies" (Hirano, Jinnai, and Toda, 2022). We thank Fumio Hayashi, Keiichi Kishi, Nobuhiro Kiyotaki, Jianjun Miao, José Scheinkman, Joseph Stiglitz, and seminar participants at various institutions and conferences for comments and encouragement.

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1 Introduction

An asset price bubble is a situation in which the asset price exceeds its fundamental value defined by the present value of dividends. History has repeatedly witnessed bubbly dynamics of asset prices. For instance, Kindleberger (2000, Appendix B) documents 38 bubbly episodes in the 1618–1998 period. In addition, bubbly fluctuations of asset prices have often been associated with financial crises, with significant economic and social costs (Jordà et al., 2015). Therefore, there is a substantial interest among policymakers, academics, and the general public in understanding why and how asset price bubbles emerge in the first place.

Since the seminal paper by Lucas (1978), the macro-finance theory has typically assumed outright that the asset price equals its fundamental value. The dominant view of the literature seems to be that bubbles are either not possible in rational equilibrium models or even if they are, a situation in which asset price bubbles occur is a special circumstance and hence fragile.[1] In fact, there is a fundamental difficulty in generating bubbles attached to dividend-paying assets in rational equilibrium models including the Lucas tree model. Importantly, according to the Bubble Characterization Lemma in the independent work by some of the authors (Hirano and Toda, 2023a), a bubble exists if and only if future dividend yields are summable.[2] With positive dividends, this is true only if the price-dividend ratio grows, implying that the essence of asset price bubbles is nonstationarity. Although economists have long been trained and accustomed to studying stationary models, to understand asset price bubbles in dividend-paying assets, we need to depart from them.

The primary purpose of this paper is to take the first step towards building a macro-finance theory to think about asset price bubbles in dividend-paying assets. We show that the key to understand asset price bubbles in dividend-paying assets is a world of nonstationarity characterized by *unbalanced growth*, unlike a world of stationarity characterized by balanced growth. Even a small deviation from the world of stationarity will lead to vastly different insights on asset pricing. Furthermore, we show the tight connection between financial leverage and unbalanced growth, that is, whether the economy exhibits balanced or unbalanced growth is endogenously determined and crucially depends on the level of financial leverage.

[1] This view is summarized well by the abstract of Santos and Woodford (1997): "Our main results are concerned with nonexistence of asset pricing bubbles in those economies. These results imply that the conditions under which bubbles are possible are relatively fragile."

[2] See Appendix B for a precise definition of asset price bubbles and the statement of the Bubble Characterization Lemma.

We consider a simple incomplete-market dynamic general equilibrium model with a continuum of infinitely-lived heterogeneous agents. There is a representative firm with a standard neoclassical production function, where capital and land (in fixed aggregate supply) are used as factors of production. Because land yields positive rents, it may be interpreted as a variant of the Lucas (1978) tree with endogenous dividends. Agents can save by investing in a portfolio of capital and land. Each period, agents are hit by productivity shocks and decide how much capital investment they make using leverage and how much to save through holding land. In this model, capital investment and land price reinforce each other, with endogenous changes in land rents, generating a positive feedback loop: when the land price goes up, aggregate wealth increases, leading to large investments, which in turn increase land rents, future wealth, and the demand for land. The current land price is determined reflecting future changes in land rents and prices, which in turn affects the current aggregate wealth. Importantly, leverage affects the magnitude of this interaction. There are two possibilities for the long run behavior of the economy. One possibility is that the economy converges to the steady state. Another possibility is that the financial leverage of agents is sufficiently high so that the economy grows endogenously. We find that whether land prices reflect fundamentals or contain a bubble crucially depends on which growth regime the economy falls into.

We prove the Land Bubble Characterization Theorem, which establishes the tight link between leverage, the growth behavior of the economy, and asset pricing implications. When leverage is below a critical value, the interaction between land prices and capital investment is not strong enough to sustain growth and the economy converges to the steady state of zero growth (because land is a fixed factor) in the long run. In this case, aggregate capital, land price, and land rent all grow at the same rate in the long run, therefore exhibiting *balanced growth*, and land prices reflect fundamentals. However, when leverage exceeds the critical value, the positive feedback loop between capital investment and land price becomes so strong that the macro-economy suddenly loses its balanced growth property and the economy takes off to endogenous growth. While land prices grow at the same rate as the economy driven by the demand for land as a store of value, land rents grow at a slower rate driven by the demand for land as a production factor, generating a gap between the growth rates of land prices and rents. This *unbalanced growth* causes the price-rent ratio to rise and leads to a land price bubble. In the land bubble economy, the determination of the land price is purely demand-driven, i.e., the price continues to rise due to sustained demand growth arising from economic growth. In contrast, when the land price reflects fundamentals, it equals the present value of land rents and hence its determination is supply-driven. The demand-driven positive feedback loop is a distinctive feature of the land bubble economy. In this way, our Theorem implies that as the leverage is relaxed beyond the critical value, the qualitative growth behavior and the asset pricing implications of the economy abruptly change, which we refer to as a *phase transition*. With the phase transition, the macro-economy shifts from a stationary world of balanced growth and fundamental value to a nonstationary world of unbalanced growth and bubble.

Moreover, by considering a standard constant elasticity of substitution (CES) production function where capital and land are used as inputs, we provide a complete characterization of the fundamental region and the land bubble region in terms of the underlying parameters of the economy such as financial leverage, overall productivity of the economy, and the elasticity of substitution in the production function. We show that for all values of the elasticity, the land bubble region always emerges if leverage gets sufficiently high. The leverage threshold decreases as the overall productivity of the economy rises (in the sense of first-order stochastic dominance), indicating the positive connection between technological innovations and asset price bubbles. This result is consistent with the narrative "asset price bubbles tend to appear in periods of excitement about innovations" highlighted by Scheinkman (2014, p. 22).

Our model incorporating the possibility of unbalanced growth dynamics provides a new perspective on constructing macro-finance theory. So long as the model allows for *only* balanced growth of a stationary nature in the long run, by model construction, asset price bubbles attached to dividend-paying assets are impossible because land prices grow at the same rate as land rents in the long run. However, once the model features some mechanism that allows for unbalanced growth of a nonstationary nature, asset price bubbles emerge. To demonstrate the usefulness of our theory, we present a numerical example showing how changes in leverage or productivities lead to the emergence and collapse of land price bubbles and provide a discussion in light of Japan's experience in the 1980s. In addition, we interpret a special case of our model as a two-sector economy with capital- and land-intensive sectors and show how the interactions between the two sectors generate unbalanced growth dynamics with uneven growth rates, generating land overvaluation.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 characterizes the equilibrium system and the asymptotic behavior of the model. Section 4 shows that relaxing the leverage constraint leads to endogenous unbalanced growth and land price bubbles. Section 5 discusses the related literature. Proofs are deferred to Appendix A. Appendix B formally defines asset price bubbles and discusses the Bubble Characterization Lemma of Hirano and Toda (2023a).

2 Model

Agents The economy is populated by a continuum of agents with mass 1 indexed by $i \in I = [0,1]$.[3] A typical agent has the utility function

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \log c_t, \tag{2.1}$$

where $\beta \in (0,1)$ is the discount factor and $c_t \geq 0$ is consumption.

Productivity The economy features no aggregate uncertainty but agents are subject to idiosyncratic risk. Every period, each agent independently draws investment productivity $z_t \geq 0$ from a cumulative distribution function (cdf) Φ satisfying the following assumption.

Assumption 1. $\Phi : [0,\infty) \to [0,1]$ *is differentiable and satisfies* $\phi = \Phi' > 0$ *and* $\int_0^\infty z \, \mathrm{d}\Phi(z) < \infty$.

The IID assumption is inessential but simplifies the analysis.[4] The differentiability of Φ implies that the productivity distribution has full support and no point mass except possibly at $z = 0$. This assumption is inessential but simplifies the analysis by avoiding cases. The condition $\int_0^\infty z \, \mathrm{d}\Phi(z) < \infty$ implies that the mean productivity is finite, which is necessary for guaranteeing that aggregate capital is finite.

Production Production uses capital and land as inputs, whose quantities are denoted by K and X. There is a representative firm with neoclassical production function $f(K, X)$. Markets for production factors are competitive and inputs

[3] It is well known that using the Lebesgue unit interval as the agent space leads to a measurability issue. We refer the reader to Sun and Zhang (2009) for a resolution based on Fubini extension. Another simple way to get around the measurability issue is to suppose that there are countably many agents and define market clearing as $\lim_{I\to\infty} \frac{1}{I}\sum_{i=1}^{I} x_{it} = X_t$, where x_{it} is agent i's demand at time t and X_t is the per capita supply.

[4] For a related model in a Markovian setting, see Hirano and Toda (2023a, §5).

are paid their marginal products. After production, capital depreciates at rate $\delta \in (0,1)$. To simplify notation, following Coleman (1991), we introduce the function

$$F(K,X) = f(K,X) + (1-\delta)K. \tag{2.2}$$

For our analysis, only the function F (not f) matters, as it constitutes aggregate wealth and hence plays an important role for the aggregate wealth dynamics.

We impose the following assumption on F.

Assumption 2. *$F : \mathbb{R}^2_{++} \to \mathbb{R}_+$ is homogeneous of degree 1, concave, continuously differentiable with positive partial derivatives, and satisfies*

$$\lim_{K\to\infty} \frac{F(K,1)}{K} = \lim_{K\to\infty} F_K(K,1) =: m > 0, \tag{2.3a}$$

$$\lim_{K\to 0} \frac{F(K,1)}{K} = \lim_{K\to 0} F_K(K,1) = \infty. \tag{2.3b}$$

Note that the assumption $m > 0$ in (2.3a) is natural because F includes the term $(1-\delta)K$ in (2.2), and (2.3b) is the standard Inada condition. A typical example satisfying Assumption 2 is the constant elasticity of substitution (CES) production function

$$f(K,X) = A\left(\alpha K^{1-\rho} + (1-\alpha)X^{1-\rho}\right)^{\frac{1}{1-\rho}}, \tag{2.4}$$

where $A, \alpha > 0$ are parameters and $\rho > 0$ is the reciprocal of the elasticity of substitution between capital and land.[5]

Land and REIT The aggregate supply of land is exogenous and normalized to 1. In the background, there are perfectly competitive financial intermediaries who securitize land into real estate investment trusts (REITs), which agents can invest in arbitrary amounts. Because the economy features no aggregate uncertainty, REITs are simply risk-free assets; we thus often refer to REITs as bonds.

The gross risk-free rate between time t and $t+1$, denoted by R_t, is determined as follows. Let K_t, P_t be the aggregate capital and land price (excluding current rent) at time t. Because the aggregate supply of land is 1, the aggregate land rent at time $t+1$ is $F_X(K_{t+1},1)$. Therefore the gross risk-free rate (return on land) is

$$R_t := \frac{P_{t+1} + F_X(K_{t+1},1)}{P_t}. \tag{2.5}$$

[5] The case $\rho = 1$ reduces to the Cobb-Douglas form $AK^\alpha X^{1-\alpha}$ by taking the limit $\rho \to 1$.

Budget constraint At time t, a typical agent starts with wealth w_t carried over from the previous period. The time t budget constraint is

$$c_t + i_t + b_t = w_t, \tag{2.6}$$

where $c_t \geq 0$ is consumption, $i_t \geq 0$ is investment, and $b_t \in \mathbb{R}$ is bond holdings. Note that $b_t > 0$ corresponds to saving and $b_t < 0$ corresponds to borrowing, where in the latter case $-b_t > 0$ is the amount borrowed. An agent with productivity z_t who invests i_t generates capital $k_{t+1} = z_t i_t$ at time $t + 1$.[6] Therefore the time $t + 1$ wealth is defined by

$$w_{t+1} := \underbrace{F_K(K_{t+1}, 1) z_t i_t}_{\text{income from capital}} + \underbrace{R_t b_t}_{\text{income from REIT}}. \tag{2.7}$$

Leverage constraint Agents are subject to the collateral or leverage constraint

$$0 \leq \underbrace{i_t}_{\text{investment}} \leq \lambda \underbrace{(i_t + b_t)}_{\text{equity}}, \tag{2.8}$$

where $\lambda \geq 1$ is the exogenous leverage limit. Here $i_t + b_t = w_t - c_t$ is the net worth ("equity") of the agent after consumption. The leverage constraint (2.8) implies that capital investment cannot exceed some multiple of total equity, which is standard in the literature as well as commonly used in practice.[7] Note that since $i_t \geq 0$ and $\lambda \geq 1 > 0$, (2.8) implies that equity must be nonnegative: $i_t + b_t \geq i_t/\lambda \geq 0$. Furthermore, solving (2.8) for b_t and noting that $1 - 1/\lambda \geq 0$, we obtain

$$\underbrace{-b_t}_{\text{borrowings}} \leq \underbrace{(1 - 1/\lambda)}_{\text{collateral ratio}} i_t, \tag{2.9}$$

so the leverage constraint (2.8) is equivalent to the borrowing constraint (2.9).

---

[6] Thus agents know their productivities before investment. This timing convention follows that of Kocherlakota (2009) and provides a trade motive among agents while keeping the model analytically tractable.

[7] According to standard accounting practices for constructing the balance sheet, equity equals asset (i_t) minus liability ($-b_t$), which is $i_t + b_t$. The leverage is defined as the ratio between asset and equity. The leverage constraint (2.8) is identical to Equation (4) of Moll (2014) except the difference of discrete versus continuous time. See his Footnote 18 for a discussion of microfoundations of this type of constraint.

3 Equilibrium

The economy starts at $t = 0$ with some initial allocation of capital and land (k_0, x_{-1}) across agents. Loosely speaking, a competitive equilibrium is defined by individual optimization and market clearing. We first characterize the individual behavior and then formally define and analyze the rational expectations equilibrium.

3.1 Individual behavior

Because the economy features no aggregate risk, the sequence of aggregate capital and land price $\{(K_t, P_t)\}_{t=0}^{\infty}$ is deterministic. Individual agents take these aggregate variables as given and maximize utility (2.1) subject to the budget constraints (2.6), (2.7) and the leverage constraint (2.8).

Due to log utility, as is well known, the optimal consumption rule is $c_t = (1-\beta)w_t$. How agents allocate savings $w_t - c_t = \beta w_t$ to capital investment or REIT depends on their productivity. If an agent with productivity z_t invests, the gross return on investment is $F_K(K_{t+1}, 1)z_t$. Therefore an agent invests if and only if

$$F_K(K_{t+1}, 1)z_t > R_t \iff z_t > \bar{z}_t := \frac{R_t}{F_K(K_{t+1}, 1)}, \tag{3.1}$$

where $\bar{z}_t$ is the productivity threshold for investment and R_t is the gross risk-free rate in (2.5).[8] Whenever (3.1) holds, the agent chooses maximal leverage to invest (so the leverage constraint (2.8) binds). Therefore the optimal asset allocation is

$$(i_t, b_t) = \begin{cases} (0, \beta w_t) & \text{if } z_t \le \bar{z}_t, \\ (\lambda \beta w_t, (1-\lambda)\beta w_t) & \text{if } z_t > \bar{z}_t. \end{cases} \tag{3.2}$$

3.2 Rational expectations equilibrium

We now derive equilibrium conditions. Because land is securitized into REIT, which is a risk-free asset, the market capitalization of bonds equals the land price P_t. Therefore aggregating the time t wealth ((2.7) with time shifted by 1) across agents, and using the definition of the risk-free rate (2.5), aggregate wealth be-

[8]If $F_K(K_{t+1}, 1)z_t = R_t$, the agent is indifferent between capital investment and REIT and hence the portfolio is indeterminate. We need not worry about such cases because Φ is atomless by Assumption 1 and hence the measure of indifferent agents is zero.

comes

$$
\begin{aligned}
W_t &:= F_K(K_t,1)K_t + R_{t-1}P_{t-1} \\
&= F_K(K_t,1)K_t + P_t + F_X(K_t,1) \\
&= F(K_t,1) + P_t,
\end{aligned} \tag{3.3}
$$

where the last equality uses the homogeneity of F. As noted before, we can see that F (not f) constitutes aggregate wealth. Multiplying z_t to investment in (3.2), aggregating individual capital $k_{t+1} = z_t i_t$ across agents, and noting that productivities are independent across agents, we obtain

$$
K_{t+1} = \beta\lambda W_t \int_{\bar{z}_t}^{\infty} z \, \mathrm{d}\Phi(z). \tag{3.4}
$$

Aggregating bond holdings b_t in (3.2) across agents and noting that the market capitalization of bonds equals the land price P_t, we obtain

$$
\begin{aligned}
P_t &= \beta W_t \int_0^{\bar{z}_t} \mathrm{d}\Phi(z) + \beta(1-\lambda)W_t \int_{\bar{z}_t}^{\infty} \mathrm{d}\Phi(z) \\
&= \beta W_t(\lambda\Phi(\bar{z}_t) + 1 - \lambda),
\end{aligned} \tag{3.5}
$$

where we need $\bar{z}_t > \Phi^{-1}(1-1/\lambda)$ so that $P_t > 0$. Therefore we may define a rational expectations equilibrium as follows.

Definition 1. Given the initial aggregate capital K_0, a *rational expectations equilibrium* consists of sequences of aggregate capital $\{K_{t+1}\}_{t=0}^{\infty}$, aggregate wealth $\{W_t\}_{t=0}^{\infty}$, land price $\{P_t\}_{t=0}^{\infty}$, gross risk-free rate $\{R_t\}_{t=0}^{\infty}$, and productivity threshold $\{\bar{z}_t\}_{t=0}^{\infty}$ such that, (i) R_t satisfies (2.5), (ii) $\bar{z}_t > \Phi^{-1}(1-1/\lambda)$ satisfies (3.1), (iii) W_t satisfies (3.3), (iv) K_{t+1} satisfies (3.4), and (v) P_t satisfies (3.5).

According to Definition 1, the equilibrium is characterized by a system of five nonlinear difference equations in five unknowns. The following proposition shows that we can reduce the equilibrium to a two-dimensional dynamics.

Proposition 1. *Suppose Assumptions 1, 2 hold and define the functions*

$$
W(K,\bar{z}) := \frac{1}{1-\beta(\lambda\Phi(\bar{z})+1-\lambda)} F(K,1), \tag{3.6a}
$$

$$
P(K,\bar{z}) := \frac{\beta(\lambda\Phi(\bar{z})+1-\lambda)}{1-\beta(\lambda\Phi(\bar{z})+1-\lambda)} F(K,1), \tag{3.6b}
$$

where we restrict the domain to $(K, \bar{z}) \in (0, \infty) \times (\Phi^{-1}(1 - 1/\lambda), \infty)$ *so that* $W, P > 0$*. Given the initial aggregate capital* K_0*, the equilibrium is characterized by the two-dimensional dynamics*

$$K_{t+1} = \beta\lambda W(K_t, \bar{z}_t) \int_{\bar{z}_t}^{\infty} z \, \mathrm{d}\Phi(z), \tag{3.7a}$$

$$\bar{z}_t = \frac{P(K_{t+1}, \bar{z}_{t+1}) + F_X(K_{t+1}, 1)}{F_K(K_{t+1}, 1)P(K_t, \bar{z}_t)}. \tag{3.7b}$$

Interestingly, this model produces the financial accelerator: the real economy and the land price reinforce each other. To see this formally, an increase in the land price P_t raises the current aggregate wealth W_t by (3.3). But an increase in W_t raises the next period's aggregate capital K_{t+1} and wealth W_{t+1} through investment and production: see (3.4). Finally, this increased wealth feeds back into the land price through the demand for savings: see (3.5). As we shall see below, whether this positive feedback loop can sustain economic growth and high asset valuation depends on how high the leverage λ is.

3.3 Asymptotic behavior

In this section we study the asymptotic behavior of the model qualitatively. Intuitively, there are two possibilities for the long run behavior of the model. One possibility is that the economy converges to a steady state. Another possibility is that the economy endogenously grows. The main reason for focusing on the asymptotic behavior is that whether or not economic growth takes off is crucial for asset pricing implications. To determine whether the economy will grow or not, we first present a heuristic but intuitive argument, followed by formal propositions.

Suppose that a rational expectations equilibrium exists, and as $t \to \infty$, conjecture that there exist constants $k, w, p > 0$ and growth rate $G > 1$ such that

$$K_t \sim kG^t, \quad W_t \sim wG^t, \quad P_t \sim pG^t. \tag{3.8}$$

Suppose for the moment that the production function takes the CES form (2.4).

Then the land rent is given by

$$
\begin{aligned}
r_t := F_X(K_t,1) &= A(1-\alpha)\left(\alpha K_t^{1-\rho}+1-\alpha\right)^{\frac{\rho}{1-\rho}} \\
&\sim \begin{cases} A(1-\alpha)\alpha^{\frac{\rho}{1-\rho}}k^\rho G^{\rho t} & \text{if } \rho<1, \\ A(1-\alpha)k^\alpha G^{\alpha t} & \text{if } \rho=1, \\ A(1-\alpha)^{\frac{1}{1-\rho}} & \text{if } \rho>1. \end{cases}
\end{aligned} \tag{3.9}
$$

Regardless of the value of ρ, we have $r_t/G^t \to 0$ as $t\to\infty$, so (2.5) implies $R_t \sim G$. Substituting (3.8) into (3.7), assuming $\bar{z}_t \to \bar{z}$, and using (2.3a), we obtain

$$
\begin{aligned}
kG &= \frac{\beta\lambda mk\int_{\bar{z}}^\infty z\,\mathrm{d}\Phi(z)}{1-\beta(\lambda\Phi(\bar{z})+1-\lambda)}, \\
\bar{z} &= \frac{G}{m}.
\end{aligned}
$$

Canceling k and eliminating $\bar{z}$, we obtain the long-run growth rate condition

$$
G/m = \frac{\beta\lambda\int_{G/m}^\infty z\,\mathrm{d}\Phi(z)}{1-\beta(\lambda\Phi(G/m)+1-\lambda)}. \tag{3.10}
$$

In order for the economy to grow as conjectured, we need $G>1$. We obtain the leverage threshold for determining growth ($G>1$) or no growth ($G=1$) by setting $G=1$ in (3.10) and solving for λ:

$$
\bar{\lambda} := \frac{1-\beta}{\beta}\frac{1}{\int_{1/m}^\infty (mz-1)\,\mathrm{d}\Phi(z)}. \tag{3.11}
$$

To ensure that the numerator of (3.6b) is positive (so $\Phi(\bar{z})>1-1/\lambda$) at $(\lambda,\bar{z}) = (\bar{\lambda},1/m)$, we introduce the following assumption.

Assumption 3. *The model parameters satisfy*

$$
\beta m\frac{\int_{1/m}^\infty z\,\mathrm{d}\Phi(z)}{1-\Phi(1/m)} > 1. \tag{3.12}
$$

Assumption 3 has a natural interpretation. Let Z be a random variable with cdf Φ. Then (3.12) is equivalent to

$$
\beta\,\mathrm{E}[mZ \mid mZ \geq 1] > 1.
$$

Because mz is the gross return on capital for an agent with productivity z when aggregate capital is infinite (hence the marginal product of capital takes the asymptotic value m in (2.3a)) and the propensity to save is β, the condition (3.12) roughly says that the wealth of productive agents can grow and the take-off of economic growth is possible.

The following proposition characterizes the asymptotic behavior of the model.

Proposition 2. *Suppose Assumptions 1–3 hold and define the leverage threshold $\bar{\lambda}$ by (3.11). Then the following statements are true.*

(i) If $\lambda < \bar{\lambda}$, the dynamics (3.7) has a steady state $(K, \bar{z})$.

(ii) If $\lambda > \bar{\lambda}$, there exists a unique $G > 1$ solving (3.10).

Proposition 2 has two implications. First, when leverage λ is below the threshold $\bar{\lambda}$ in (3.11), a steady state of the aggregate dynamics (3.7) exists. Thus, if the initial aggregate capital K_0 equals this steady state value, then a rational expectations equilibrium with constant aggregate variables (stationary equilibrium) exists. Second, when leverage λ exceeds the threshold $\bar{\lambda}$, a unique growth rate $G > 1$ consistent with the heuristic argument above exists. Of course, this does not necessarily justify the heuristic argument, which we turn to next.

The elasticity of substitution between capital and land plays a crucial role in determining the asymptotic behavior of the model. However, we need to distinguish the elasticity of the production function f (which is $1/\rho$ in the CES case (2.4)) and the elasticity of substitution between K and X in the function F in (2.2), denoted by σ below in (3.13). As it turns out, it is this σ that plays a key role for generating land price bubbles. In this sense, we identify which elasticity matters.

When we consider the elasticity of substitution between K and X in the function F, it is defined by the percentage change in relative factor inputs with respect to the percentage change in relative factor prices

$$\sigma(K, X) = -\frac{\partial \log(K/X)}{\partial \log(F_K/F_X)}. \tag{3.13}$$

The following lemma shows that any production function satisfying Assumption 2 has elasticity of substitution above 1 at high capital levels.

Lemma 3.1. *If F satisfies (2.3a), then*

$$\liminf_{K \to \infty} \sigma(K, 1) \geq 1.$$

In particular, if f takes the CES form (2.4), then

$$\lim_{K\to\infty} \sigma(K,1) = \begin{cases} 1/\rho & \text{if } \rho < 1, \\ 1/\alpha & \text{if } \rho = 1, \\ \infty & \text{if } \rho > 1. \end{cases} \tag{3.14}$$

Note that the right-hand side of (3.14) is always above 1 regardless of model parameters. Motivated by Lemma 3.1, we introduce the following assumption.

Assumption 4. *The elasticity of substitution between capital and land defined by (3.13) exceeds 1 at high capital levels:*

$$\liminf_{K\to\infty} \sigma(K,1) \geq \sigma > 1.$$

According to Lemma 3.1, σ always exceeds 1 at high capital levels, so Assumption 4 is relatively weak. In particular, it is satisfied for the CES production function (2.4). Although the elasticity of f $(1/\rho)$ does not matter, Epple, Gordon, and Sieg (2010) empirically find that it also exceeds 1.

The importance of the *intertemporal* elasticity of substitution in macro-finance models is well known (Bansal and Yaron, 2004; Pohl, Schmedders, and Wilms, 2018; Iachan, Nevov, and Simsek, 2021). The analogy here is only superficial because (i) the relevant elasticity of substitution in our model is between capital and land, not between consumption in different periods, and (ii) macro-finance models typically assume outright that the asset price equals its fundamental value.

The following proposition establishes the existence of an equilibrium with endogenous growth, which justifies the heuristic argument above.

Proposition 3. *Suppose Assumptions 1–4 hold and let $\bar{\lambda}$ be as in (3.11). Suppose $\lambda > \bar{\lambda}$ and let $G > 1$ be as in Proposition 2. Then for any sufficiently large $K_0 > 0$, there exists a unique equilibrium satisfying the order of magnitude (3.8).*

4 Leverage, unbalanced growth, and asset prices

In this section, we study the asset pricing implications of the model.

4.1 Unbalanced growth and land bubble

We say that the model dynamics exhibits *balanced growth* if aggregate capital, land prices, and rents all grow at the same rate (potentially equal to zero) in the long run. Otherwise, we say that the model dynamics exhibits *unbalanced growth*. As we shall see below, whether the economy exhibits balanced or unbalanced growth is crucial for asset pricing implications.

Define the date-0 price of consumption delivered at time t (the price of a zero-coupon bond with face value 1 and maturity t) by $q_t = 1/\prod_{s=0}^{t-1} R_s$, with the normalization $q_0 = 1$. The *fundamental value* of land at time t is defined by the present value of rents

$$V_t := \frac{1}{q_t} \sum_{s=1}^{\infty} q_{t+s} r_{t+s}, \tag{4.1}$$

where $r_t := F_X(K_t, 1)$ is the land rent at time t. We say that land is *overvalued* or exhibits a *bubble* if $P_t > V_t$.

If we focus on the CES production function (2.4), then the order of magnitude of rents (3.9) satisfy

$$r_t \sim G^{t/\sigma},$$

where $\sigma > 1$ is the elasticity of substitution at high capital levels in (3.14). As discussed after (3.9), the interest rate R_t converges to $G > G^{1/\sigma}$. Therefore we also have the order of magnitude of the fundamental value

$$V_t \sim G^{t/\sigma}.$$

In contrast, we know from Proposition 3 that $P_t \sim G^t$. Therefore, in the long run, land prices grow faster than rents and hence we will have a land bubble ($P_t > V_t$). Moreover, the fact that the economy grows faster than rents implies that the land bubble economy exhibits unbalanced growth and the price-rent ratio will continue to rise. The following theorem, which is the main result of this paper, formalizes this argument.

Theorem 1 (Land Bubble Characterization)**.** *Suppose Assumptions 1–4 hold and define the leverage threshold $\bar{\lambda}$ by (3.11). Then the following statements are true.*

(i) If $\lambda < \bar{\lambda}$, in any equilibrium converging to the steady state, we have $P_t = V_t$ for all t. The economy exhibits balanced growth and the price-rent ratio converges.

(ii) If $\lambda > \bar{\lambda}$, in the equilibrium in Proposition 3, we have $P_t > V_t$ for all t. The economy exhibits unbalanced growth and the price-rent ratio diverges to ∞.

A distinctive feature of the bubble economy is that the land price is purely demand-driven. Despite the fact that the interest rate asymptotically converges, the land price continues to increase without bound, deviating from the growth rate of land rents. This is purely driven by sustained demand growth for land arising from economic growth. On the other hand, when the land price reflects fundamentals, the price movement is mainly supply-driven because the price is determined as the present discount value of land rents, which is a supply factor generated from land. The demand-driven financial accelerator is a key feature of the land bubble economy.

Theorem 1 also provides important information on detecting land price bubbles. During the bubble, the price-rent ratio shows an upward trend, while the ratio becomes stable if the land price reflects its fundamental value. Hence, a sustained increase in the ratio could be used as an early warning signal for land price bubbles.[9]

4.2 Phase transition and comparative statics

Theorem 1 tells us that there exists a critical value of the financial leverage $\bar{\lambda}$ in (3.11) above which a phase transition to unbalanced growth dynamics occurs, leading to land price bubbles. This implies that as the economy develops financially, it will lead to land price bubbles. Intuitively, when the financial condition is sufficiently relaxed, aggregate capital starts to grow rapidly and land prices are pulled by growing aggregate capital, rising at a faster rate than land rents, therefore exhibiting bubbles. Interestingly, when leverage is low enough, in the steady state equilibrium, even if leverage changes, there is no impact on the long-run economic growth rate: the economy exhibits *exogenous* growth. However, once leverage gets higher beyond the critical value, the behavior abruptly changes to *endogenous* growth.[10]

[9] Of course, in reality, if policymakers decide that the observed price-rent ratio appears to be too high, they tend to introduce a leverage regulation. If it is tightened sufficiently so that the land bubble is no longer sustainable, it will surely collapse and the economy will return to the fundamental regime. With loosening and tightening of leverage due to policy changes (in a way contrary to private agents' expectations), the economy might switch back and forth between two regimes, with upward and downward movements in the price-rent ratio. In reality, this process might repeat itself.

[10] In the growth literature, when the production function takes the CES form (2.4) with elasticity of substitution $1/\rho > 1$, it is known that the average and marginal productivity of capital converge to a positive constant as $K \to \infty$, which can generate endogenous growth. The possibility of endogenous growth with this property with a general production function was already recognized by Solow (1956, p. 72). See also Barro and Sala-i-Martin (2004, pp. 68–69). Jones and Manuelli (1990) study an endogenous growth model along this line, in which their condition G is similar to our Assumption 3. In our paper, m captures the asymptotic slope and is positive regardless of the value of ρ. Our paper connects this property of generating endogenous growth to financial leverage and shows that the level of financial leverage determines whether

Using the explicit expression for the leverage threshold (3.11) for generating bubbles, we obtain the following comparative statics. We provide a complete characterization of the fundamental region and the land bubble region in the elasticity-leverage plane.

Proposition 4 (Comparative statics). *Let $\bar{\lambda}(\beta, m, \Phi)$ be the leverage threshold in (3.11). Then $\bar{\lambda}$ is decreasing in β, m, Φ, where for the productivity distribution we use first-order stochastic dominance: $\Phi_1 \preceq \Phi_2$ if $\Phi_1(z) \geq \Phi_2(z)$ for all z. Furthermore, if the production function takes the CES form (2.4), then $\bar{\lambda}$ is constant for $\rho \geq 1$ and decreasing in the elasticity of substitution $1/\rho$ for $\rho < 1$.*

Proposition 4 implies that more patience, higher marginal product of capital, and higher productivity all decrease the leverage threshold for generating bubbles and hence make bubbles more likely to emerge. Intuitively, these changes strengthen the positive feedback loop between capital investment and land prices, allowing the take-off of economic growth.

As an example, set $A = 1$ in the CES production function (2.4) and

$$\Phi(z) = 1 - \mathrm{e}^{-z/\zeta}, \tag{4.2}$$

so productivity is exponentially distributed with mean ζ. By the proof of Proposition 4, we have

$$m = \begin{cases} \alpha^{\frac{1}{1-\rho}} + 1 - \delta & \text{if } \rho < 1, \\ 1 - \delta & \text{if } \rho \geq 1. \end{cases}$$

Using integration by parts, the denominator in (3.11) can be calculated as

$$\begin{aligned} \int_{1/m}^{\infty} (mz - 1) \, \mathrm{d}\Phi(z) &= \int_{1/m}^{\infty} (1 - mz) \, \mathrm{d}(1 - \Phi(z)) \\ &= [(1 - mz)(1 - \Phi(z))]_{1/m}^{\infty} + \int_{1/m}^{\infty} m(1 - \Phi(z)) \, \mathrm{d}z \\ &= m \int_{1/m}^{\infty} \mathrm{e}^{-z/\zeta} \, \mathrm{d}z = m\zeta \mathrm{e}^{-\frac{1}{m\zeta}}. \end{aligned}$$

Therefore the leverage threshold (3.11) reduces to

$$\bar{\lambda} = \frac{1 - \beta}{\beta m \zeta} \mathrm{e}^{\frac{1}{m\zeta}}. \tag{4.3}$$

Figure 1 divides the parameter space into regions where a land bubble emerges or

the economy exhibits exogenous and balanced growth or endogenous and unbalanced growth.

not, where the horizontal axis is the elasticity of substitution $1/\rho$ and the vertical axis is leverage λ. Depending on whether leverage λ exceeds the critical value $\bar{\lambda}$, a land bubble emerges or the land price reflects its fundamental value. When $\rho \geq 1$, because $m = 1 - \delta$, the leverage threshold (4.3) is independent of ρ. When $\rho < 1$, consistent with Proposition 4, the boundary of the bubbly and fundamental regime is downward-sloping in $1/\rho$.

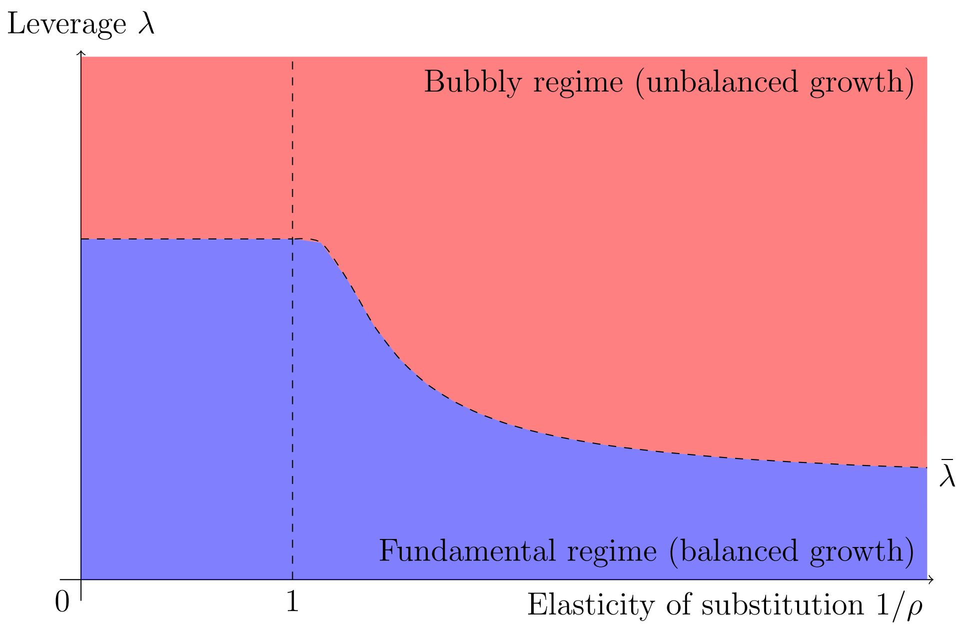

Figure 1: Phase transition of equilibrium land price regimes.

Note: parameter values are $\beta = 0.95$, $\alpha = 0.5$, $\delta = 0.08$, and $\zeta = 1/\log(100/10)$ so that $\Pr(z > 1) = 0.1$.

Proposition 4 has two important implications. First, as we can see in Figure 1, for all values of the elasticity parameter ρ, including the Cobb-Douglas production function with $\rho = 1$, the land bubble region will *always* emerge when the financial leverage gets sufficiently high. Second, the fact that the leverage threshold decreases as the overall productivity of the economy increases (in the sense of first order stochastic dominance) implies that technological innovations and asset price bubbles are closely linked. This result is consistent with the stylized fact that "asset price bubbles tend to appear in periods of excitement about innovations" highlighted by Scheinkman (2014, p. 22). In addition, Scheinkman (2014) also points out that bubbles may have positive effects on innovative investments and economic growth by facilitating finance. Even in our model, bubbles ease financing, allowing the take-off of economic growth, which in turn sustains growing bubbles.

4.3 Two-sector economy

Consider the case with $\rho = 0$ in the CES production function (2.4). Then

$$f(K, X) = A\alpha K + A(1 - \alpha)X.$$

This case can be interpreted as a two-sector economy in which the capital-intensive sector uses an AK technology and land produces a constant rent $D := A(1 - \alpha)$ every period such as agriculture or extraction of natural resources. Although the generative mechanism of land bubbles is identical, the notion of unbalanced growth is more intuitive. As long as the economy stays in the fundamental regime, the two sectors keep a balance and grow at the same rate in the long run. But once the financial leverage gets sufficiently high, the two sectors suddenly lose their balance and begin to grow at uneven rates. Production in the capital-intensive sector expands rapidly, which in turn has positive spillover effects on the land sector, increasing the demand for land and pushing land prices up faster than dividends, generating land price bubbles.

Moreover, the meaning of the demand-driven financial accelerator is also clearer. When $\rho = 0$, land rents are constant. In addition, in the land bubble economy, the interest rate asymptotically converges, as does the fundamental value of land. Despite these facts, the land price will continue to rise in the bubbly regime (though the land price may go up and down due to unexpected changes in leverage or productivities). This is purely driven by sustained demand growth arising from economic growth. This property of the demand-driven financial accelerator is in sharp contrast with the macro-finance literature we discuss in Section 5. For instance, in the model of Kiyotaki and Moore (1997), land is used as a factor of production. Since the interest rate is exogenous in their model, fluctuations in land prices reflect those of dividends, which are a supply factor.

4.4 Numerical example

We present a numerical example that shows how changes in leverage or productivities lead to the emergence and collapse of land price bubbles. As our focus is to present a theoretical framework that can be used as a stepping stone for a variety of applications, we consider a minimal illustrative example.

Consider the CES production function (2.4) and the exponential productivity distribution (4.2). A straightforward calculation yields

$$\int_{\bar{z}}^{\infty} z \, \mathrm{d}\Phi(z) = (\bar{z} + \zeta)\mathrm{e}^{-\bar{z}/\zeta}.$$

Therefore (3.7a) simplifies to

$$K_{t+1} = \frac{\beta\lambda(\bar{z}_t + \zeta)\mathrm{e}^{-\bar{z}_t/\zeta}}{1 - \beta + \beta\lambda\mathrm{e}^{-\bar{z}_t/\zeta}} F(K_t, 1). \tag{4.4}$$

We numerically solve the model using the algorithm discussed in Appendix C.

To illustrate the dynamics of land price bubbles and their collapse, suppose that financial condition (leverage) gets loose or the productivity of the economy becomes high so that the economy enters the bubbly regime. Suppose agents believe that these "good" conditions will persist forever and they actually do for a while. Once expectations change this way, macroeconomic variables such as aggregate capital, consumption, and land prices will all continue to rise with higher growth rates, while containing a land price bubble and showing a rise in the price-rent ratio. Suppose, however, that at some point, the situation unexpectedly changes and financial regulation severely tightens leverage or the productivity of the economy sufficiently declines so that the bubble is no longer sustainable. Then the bubble will surely collapse, and all macroeconomic variables will decline with a fall in the price-rent ratio. In this way, our model can describe the short-term onset of a bubble and its collapse.

Figure 2 shows two such numerical examples. The baseline parameter values are those used in Figure 1 and we set $\lambda = 1$ and $\rho = 1$ so that agents are initially self-financing and the production function is Cobb-Douglas. The economy is initially in the fundamental steady state up to time $t = 0$, where we normalize the land price and rent to 1 for visibility. At $t = 0$, either the leverage increases to $\lambda = 2$ (left panel) or the average productivity increases to $\zeta = 1/\log(100/15)$ (right panel). In both cases, the economy takes off to the bubbly regime and land prices grow faster than land rents, as we can see from the steeper slope of the former. At $t = 10$, the parameters unexpectedly reverse to the baseline values and the economy reenters the fundamental regime. Then the bubble collapses and both land prices and rents decline.[11] In the course of these dynamics with the

[11]Note that during the leverage-driven bubble, the land price declines at $t = 0$. This is because output is predetermined through previous investments, and relaxing leverage makes agents substitute from bond to capital investment. The opposite happens at the tightening stage and leads to land speculation. In contrast, during the productivity-driven bubble, this substitution effect is absent.

onset of land price bubbles and their collapse, the price-rent ratio rises and falls substantially.

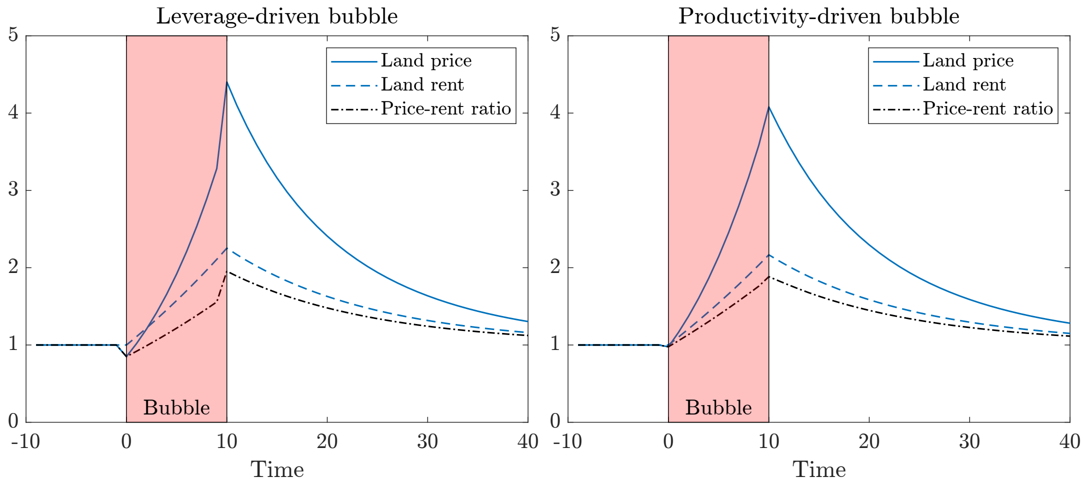

Figure 2: Emergence and collapse of land bubbles driven by high leverage or productivity.

In fact, this explanation is similar to what happened during Japan's so-called "bubble period" in the 1980s. Various financial deregulation such as capital and interest rate liberalization and the introduction of derivatives such as convertible bonds and warrants in the early 1980s set the stage. Following the Plaza Accord on September 22, 1985 (that aimed to let the U.S. dollar depreciate relative to the Japanese yen and Deutsche Mark to curb the U.S. trade deficits), the Japanese yen appreciated from 240 JPY/USD to 150 within a year, causing a severe contraction in the manufacturing sector referred to as "*endaka fukyō*" (yen-appreciation recession). The Bank of Japan cut the official discount rate from 5.0% to 2.5% to stimulate the Japanese economy, which led to an expansion in the financial sector. At that time, Japan was dubbed "number one" (Vogel, 1979) and Tokyo was believed to become a global financial hub.[12] The "Japan money" flowed into the real estate sector and substantially increased the land price. According to the calculations in Noguchi (1990), as of 1987, the price-rent ratio of the Marunouchi business district in Tokyo was 20 times that of the inner city of London. A popular urban legend at that time was that the land price of the Imperial Palace (1.15 square kilometers) was equivalent to the entire state of California (423,970 square

[12]The 1985 report by the National Land Agency titled "*Shuto Kaizō Keikaku*" (Capital Remodeling Plan) predicted that the demand for office space in Tokyo would increase from 3,700ha in 1985 to 8,700ha (equivalent of 250 skyscrapers) by year 2000 to house corporate headquarters and international financial services. For a discussion of the Japanese economy and economic policies in the 1980s, see Noguchi (1994) and Ishii (2011).

kilometers).[13] The easy money also made consumers extravagant: people flocked to expensive restaurants, discos, and ski resorts, drank expensive French wines like Romanée-Conti and Château Latour, and bid up 1,000 yen bills along streets to secure taxi rides.[14] Following the official discount rate hike from 2.5% to 6.0% in May 1989–August 1990 and the introduction of the Real Estate Loan Total Quantity Restriction by the Ministry of Finance on March 27, 1990, the land "bubble" collapsed. Land price indices in all six major cities started to decline in 1991, but the process was gradual as in Figure 2, unlike stock prices that sharply fell at the beginning of 1990. In the popular press, the 1990s were dubbed "*hyōgaki*" (ice age, referring to the cooled down job market for new college graduates) and "*ushinawareta jyūnen*" (lost decade; literally "lost ten years"). Economic stagnation continues to date and this phrase has been revised several times to "lost twenty years" and "lost thirty years".

5 Related literature and discussion

The macro-finance literature stresses the importance of the financial accelerator, including Greenwald and Stiglitz (1993), Kiyotaki and Moore (1997), Bernanke, Gertler, and Gilchrist (1999), He and Krishnamurthy (2013), and Brunnermeier and Sannikov (2014), among others. In this literature, asset prices (land or stocks yielding positive dividends) reflect fundamentals. In contrast, we identify economic conditions under which land prices exceed the fundamental value, containing a bubble.

Regarding macro-finance models with land where land plays the dual role of factor for production and means of saving, our paper is related to McCallum

[13]We were unable to trace the source of this quote. The earliest in print we found is on p. 103 of the book "*Chikyū Jidai no Shin Shiten*" (A New Perspective in a Global Era) by business consultant Kenichi Ohmae published in December 1988, who writes "the land price of the Imperial Palace equals that of the entire California". According to an interview article in the October 23, 2000 issue of *Nikkei Business*, Taro Kaneko, former Ministry of Finance bureaucrat and then the president of Marusan Security, wrote a letter in 1988 addressed to the employees stating "The total land value in Japan is estimated to be 4 times of U.S. The land area is 1/25, so the unit price is really 100 times. The land price of the Imperial Palace is about the same as California. Even if the Japanese economy is booming, we cannot expect such an abnormal disparity to be sustainable. Moreover, the Japanese population will decline. Therefore, you should refrain from purchasing housing for the time being". The first time such a quote appeared in English seems to be the November 19, 1989 article of *Chicago Tribune*.

[14]The 2007 Japanese science fiction comedy film "Bubble Fiction: Boom or Bust" vividly illustrates this situation. The extravagance in the society was even noticeable to elementary school children like the authors, who watched the cartoon "*Obocchamakun*" whose main character was a spoiled wealthy boy.

(1987), Mountford (2004), and Stiglitz (2015). A critical difference from these papers is that we derive land price implications under unbalanced growth, while these papers focus on balanced growth. Hansen and Prescott (2002) study a model with land and unbalanced growth but abstract from asset pricing.

Regarding unbalanced growth, our paper is related to Baumol (1967), Matsuyama (1992), and Buera and Kaboski (2012). There are two main differences from our paper. First, we use homothetic preferences and whether the economy exhibits balanced or unbalanced growth depends on the financial leverage, while these papers use non-homothetic preferences to generate unbalanced growth and do not consider financial frictions. Second, these papers abstract from asset pricing, including Acemoglu and Guerrieri (2008) who consider unbalanced growth under homothetic preferences.

In terms of deriving asset pricing implications under unbalanced growth, the present paper is related to Hirano and Toda (2023b), with three main differences. First, the present paper employs an infinite-horizon model with heterogeneous agents, while Hirano and Toda (2023b) consider a two-period overlapping generations model. Second, our model features endogenous unbalanced growth, while growth is exogenous in Hirano and Toda (2023b). This difference is substantial because it highlights the importance of leverage in generating bubbles. Third, land is the primary means of saving in Hirano and Toda (2023b), while in this paper agents have access to multiple savings vehicles.

Our paper is related to the large literature of rational asset price bubbles. A well-known difficulty in this literature is that in order to support rational asset price bubbles, the present value of the aggregate endowment must be infinite; see Kocherlakota (1992, Proposition 4) and Santos and Woodford (1997, Theorem 3.3) for this type of results. Infinite present value could be considered "exotic" (Werner, 2014). Apart from stylized overlapping generations (OLG) models such as Samuelson (1958) and Tirole (1985) in which individual optimality and infinite present value of the aggregate endowment may be consistent due to finite lives, it is necessary to consider models with financial frictions. With sufficient financial constraints, individual optimality and infinite present value of the aggregate endowment may be consistent because financial constraints can prevent agents from capitalizing the infinite present value of endowments.[15] The first models of this

[15]In some models with financial frictions such as Geanakoplos (2010), Fostel and Geanakoplos (2012), and Simsek (2013), the fundamental theorem of asset pricing fails because financial constraints bind and first-order conditions hold with inequalities for all agents. In such models, the asset price may exceed the valuation of any agent even in two period models. In this paper we only consider models in which the fundamental theorem of asset pricing holds.

kind are Bewley (1980) and Scheinkman and Weiss (1986). Since the 2008–2009 financial crisis, many papers on financial conditions and asset price bubbles have been written including Kocherlakota (2009, 2013), Farhi and Tirole (2012), Hirano and Yanagawa (2017), and Miao and Wang (2018), among others. See Barlevy (2018) and Guerron-Quintana et al. (2023) for surveys of this literature.

However, existing rational bubble models with financial frictions have three severe shortcomings. First, perhaps due to the necessity of nonstationarity for attaching bubbles to dividend-paying assets discussed in the introduction and the fact that economists have been trained to study stationary models, the literature has almost exclusively focused on a special case called "pure bubbles", which are assets that pay no dividends and hence are intrinsically worthless like fiat money.[16] While pure bubble models are very useful to analyze fiat money or cryptocurrency, it is difficult to apply the theory for empirical or quantitative analysis of realistic bubbles attached to dividend-paying assets like stocks, land, or housing.[17] Second, pure bubble models suffer from equilibrium indeterminacy: there exists an equilibrium in which the bubble asset has no value, and there also exist a continuum of bubbly equilibria, making model predictions non-robust.[18] Equilibrium indeterminacy also implies that the theory cannot explain why and how the bubble starts. Third, by its very nature the existence of rational bubbles rests on financial frictions, and thus bubbles are more likely to arise when financial conditions get tighter, which contradicts stylized facts that bubbly episodes tend to be associated with loose financial conditions (Kindleberger, 2000). Our model circumvents all these shortcomings of pure bubble models. Namely, in our model (i) the bubble is attached to a dividend-paying asset (land), (ii) the equilibrium is determinate, and (iii) asset price bubbles emerge as the leverage constraint is relaxed. Moreover, the critically important point we have discovered is that to understand asset price bubbles in dividend-paying assets, we need to think about a nonstationary world instead of a stationary world. To our knowledge, our paper is the first in macro-finance that stresses the importance of nonstationarity

[16] The only exceptions we are aware of other than our own works are the example in Wilson (1981, §7) and Tirole (1985).

[17] On this point, we thank Nobuhiro Kiyotaki for continuously encouraging us to construct a macro-finance model with asset price bubbles in dividend-paying assets.

[18] See Simsek (2021) for similar criticisms. Although the equilibrium indeterminacy in pure bubble models has been recognized for decades, the literature has selected only one of a continuum of bubbly equilibria (a saddle path or a steady state) and has advanced policy and quantitative analysis. However, the equilibrium selected is only one point within an open set and thus has measure zero. There is neither scientific basis for this equilibrium selection nor rational reason for why heterogeneous economic agents coordinate on that equilibrium even if they are exposed to disturbances, including policy changes.

in generating asset price bubbles in dividend-paying assets and shows a positive connection between loose financial conditions (financial accelerator) and asset price bubbles. Our model can be used as a stepping stone for a variety of applied analyses.

6 Conclusion

Since the Lucas (1978) asset pricing model, significant progress has been made in the modern macro-finance theory of asset prices. Although the term "asset price bubbles" are commonly discussed in the popular press and among policymakers, the conventional wisdom in the literature suggests that asset price bubbles are either not possible or even if they are, a situation in which bubbles occur is a special circumstance and hence fragile. Indeed, there are fundamental difficulties in generating asset overvaluation in dividend-paying assets, including the Lucas tree model, as discussed in the introduction. In this paper, we have challenged this long-standing conventional view and have presented a new macro-finance framework to think about asset price bubbles.

The new finding we have uncovered is that whether asset prices reflect fundamentals or contain a bubble critically depends on whether the economy exhibits balanced growth of a stationary nature or unbalanced growth of a nonstationary nature. Asset pricing implications in a world of stationarity and a world of nonstationarity are markedly different. Using a dynamic general equilibrium macro-finance model, we have established the Land Bubble Characterization Theorem, showing that whether the economy exhibits balanced or unbalanced growth is endogenously determined and crucially depends on the level of the financial leverage. There exists a critical value of leverage above which a phase transition to unbalanced growth dynamics occurs, leading to land bubbles, while below which the economy keeps balanced growth in the long run and land prices reflect the fundamental value.

This novel result of the tight link between unbalanced growth and asset price bubbles has an important insight on the methodology of model building. That is, so long as we construct a macro-finance model with land (or dividend-paying assets like stocks or housing) that allows for *only* balanced growth in the long run, by model construction, bubbles are impossible because land prices will grow at the same rate as land rents in the long run. However, once we incorporate the possibility of the phase transition to unbalanced growth dynamics, the land bubble region will *always* emerge if the financial leverage or the overall productivity of the economy get sufficiently high.[19] Unbalanced growth is precisely about a world of nonstationarity.

In this paper, we have tried to take the first step toward building a macro-finance theory to think about asset price bubbles in dividend-paying assets like Lucas trees. Hence, we abstract from many realistic elements such as aggregate risk, wage rigidity, price stickiness, and defaults involved with borrowing, etc. Obviously, introducing aggregate uncertainty will make our model more realistic and more appropriate to capture recurrent fluctuations in asset prices, including bubbles. We plan to work on this direction in subsequent research. Introducing wage rigidity and/or price stickiness and/or defaults along the lines of Barlevy (2018) and Allen, Barlevy, and Gale (2022) would be an important direction to describe realistic costs associated with a fall in asset prices including bubbles and to think about appropriate government policy.

Economists have long been trained and accustomed to studying models with stationarity. However, as our paper has shown, the key to understand asset price bubbles in dividend-paying assets is a world of nonstationarity. We hope our framework will open a new door to asset pricing in macro-finance models and will lead to a variety of applied analyses.

A Proofs

Proof of Proposition 1. Using (3.3) and (3.5), we may solve for W_t, P_t, which yields $W_t = W(K_t, \bar{z}_t)$ and $P_t = P(K_t, \bar{z}_t)$ in (3.6a) and (3.6b). Substituting these equations into (3.4), we obtain (3.7a). Finally, (3.7b) follows from the definition of $\bar{z}_t$ in (3.1) and R_t in (2.5). □

We prove Proposition 2 by establishing a series of lemmas. Below, Assumptions 1–3 are in force. If a steady state $(K, \bar{z})$ exists, then (3.6a)–(3.7b) imply

$$K = \frac{\beta\lambda F(K,1)}{1-\beta(\lambda\Phi(\bar{z})+1-\lambda)} \int_{\bar{z}}^{\infty} z \,\mathrm{d}\Phi(z), \tag{A.1a}$$

$$\bar{z} = \frac{1}{F_K(K,1)} \left(1 + \frac{1-\beta(\lambda\Phi(\bar{z})+1-\lambda)}{\beta(\lambda\Phi(\bar{z})+1-\lambda)} \frac{F(K,1)-KF_K(K,1)}{F(K,1)}\right), \tag{A.1b}$$

where we have used $F(K,1) = KF_K(K,1) + F_X(K,1)$.

[19] To the best of our understanding, in the macro-finance papers discussed in the literature review, the models are constructed so that only balanced growth arises in the long run. Without unbalanced growth, there cannot be a phase transition to the bubble economy, even if there is a feedback loop between asset prices and aggregate investments.

Lemma A.1. *If $\lambda < \bar{\lambda}$, for any $\bar{z} > \Phi^{-1}(1 - 1/\lambda)$, there exists a unique $K > 0$ solving* (A.1a).

Proof. We can rewrite (A.1a) as

$$\frac{1 - \beta(\lambda\Phi(\bar{z}) + 1 - \lambda)}{\beta\lambda \int_{\bar{z}}^{\infty} z \,\mathrm{d}\Phi(z)} = \frac{F(K, 1)}{K}. \tag{A.2}$$

Due to the concavity of F, the right-hand side of (A.2) is strictly decreasing and tends to m as $K \to \infty$ by (2.3a). Therefore (A.2) has a unique solution if and only if $\Phi(\bar{z}) > 1 - 1/\lambda$ and

$$\begin{aligned} &\frac{1 - \beta(\lambda\Phi(\bar{z}) + 1 - \lambda)}{\beta\lambda \int_{\bar{z}}^{\infty} z \,\mathrm{d}\Phi(z)} > m \\ \iff &g(\bar{z}) := 1 - \beta(\lambda\Phi(\bar{z}) + 1 - \lambda) - \beta\lambda m \int_{\bar{z}}^{\infty} z \,\mathrm{d}\Phi(z) > 0. \end{aligned}$$

By Assumption 1, we have $g'(\bar{z}) = \beta\lambda\phi(\bar{z})(mz - 1)$, so $g'(\bar{z}) \gtrless 0$ if $\bar{z} \gtrless 1/m$. Therefore g achieves a minimum at $\bar{z} = 1/m$, and

$$\begin{aligned} &0 < g(1/m) = 1 - \beta(\lambda\Phi(1/m) + 1 - \lambda) - \beta\lambda m \int_{1/m}^{\infty} z \,\mathrm{d}\Phi(z) \\ \iff &1 - \beta > \beta\lambda \int_{1/m}^{\infty} (mz - 1) \,\mathrm{d}\Phi(z) \iff \lambda < \bar{\lambda}, \end{aligned}$$

where the last inequality uses (3.11). Therefore if $\lambda < \bar{\lambda}$, then $g(\bar{z}) \geq g(1/m) > 0$, and there exists a unique $K > 0$ satisfying (A.1a). □

Lemma A.2. *If $\lambda < \bar{\lambda}$, the dynamics* (3.7) *has a steady state.*

Proof. For any $\bar{z} > \Phi^{-1}(1-1/\lambda)$, let $K(\bar{z})$ be the $K > 0$ in Lemma A.1. As $\bar{z} \uparrow \infty$, the left-hand side of (A.2) tends to ∞, so $K(\bar{z}) \to 0$ by the Inada condition. Then the left-hand side of (A.1b) tends to ∞, while the right-hand side remains finite (because $F_K(K, 1) \to m > 0$ and $\Phi(\bar{z}) \to 1$). As $\bar{z} \downarrow \Phi^{-1}(1 - 1/\lambda)$, we have $\lambda\Phi(\bar{z}) + 1 - \lambda \downarrow 0$, so the right-hand side of (A.1b) tends to ∞, while the left-hand side remains finite. Therefore by the intermediate value theorem, there exits $\bar{z}$ solving (A.1b) for $K = K(\bar{z})$, and hence there exists a steady state. □

Lemma A.3. *There exists a unique $G = G(\lambda)$ satisfying* (3.10). *Furthermore, $G(\lambda) > 1$ if and only if $\lambda > \bar{\lambda}$.*

Proof. Letting $v = G/m$, we can rewrite (3.10) as

$$\Psi(v, \lambda) := v(1 - \beta(\lambda\Phi(v) + 1 - \lambda)) - \beta\lambda \int_v^\infty z \,\mathrm{d}\Phi(z) = 0. \tag{A.3}$$

Clearly Ψ is continuously differentiable in v, λ. Let $\phi = \Phi'$. Then

$$\begin{aligned}\Psi(0, \lambda) &= -\beta\lambda \int_v^\infty z \,\mathrm{d}\Phi(z) < 0,\\ \Psi(\infty, \lambda) &= \infty \times (1 - \beta) = \infty.\end{aligned}$$

Therefore by the intermediate value theorem, there exists $v > 0$ such that $\Psi(v, \lambda) = 0$. Since Φ is a cdf and hence $\Phi(v) \leq 1$, we obtain

$$\begin{aligned}\frac{\partial\Psi}{\partial v} &= 1 - \beta(\lambda\Phi(v) + 1 - \lambda) - v\beta\lambda\phi(v) + \beta\lambda v\phi(v)\\ &= 1 - \beta(\lambda\Phi(v) + 1 - \lambda) \geq 1 - \beta > 0,\end{aligned}$$

so the value of v is unique. When $\Psi(v, \lambda) = 0$, (A.3) implies that

$$\begin{aligned}\frac{\partial\Psi}{\partial \lambda} &= v\beta(1 - \Phi(v)) - \beta \int_v^\infty z \,\mathrm{d}\Phi(z)\\ &= v\beta(1 - \Phi(v)) - \frac{1}{\lambda} v(1 - \beta(\lambda\Phi(v) + 1 - \lambda))\\ &= -\frac{v}{\lambda}(1 - \beta) < 0.\end{aligned}$$

By the implicit function theorem, $v'(\lambda) = -(\partial\Psi/\partial\lambda)/(\partial\Psi/\partial v) > 0$. Therefore $G'(\lambda) > 0$. Since $G(\lambda)$ is strictly increasing, we have $G(\lambda) > 1$ if and only if $\lambda > \bar{\lambda}$, where $G(\bar{\lambda}) = 1$. Setting $G = 1$ in (3.10), $\lambda = \bar{\lambda}$ solves

$$\begin{aligned} 1 &= \frac{m\beta\lambda \int_{1/m}^\infty z \,\mathrm{d}\Phi(z)}{1 - \beta(\lambda\Phi(1/m) + 1 - \lambda)}\\ \iff 1 - \beta(\lambda\Phi(1/m) + 1 - \lambda) &= m\beta\lambda \int_{1/m}^\infty z \,\mathrm{d}\Phi(z)\\ \iff \beta\lambda \int_{1/m}^\infty (mz - 1) \,\mathrm{d}\Phi(z) &= 1 - \beta,\end{aligned}$$

which gives (3.11). ∎

Proof of Proposition 2. Immediate from Lemmas A.1–A.3. ∎

Proof of Lemma 3.1. Let $X = 1$ and $\rho(K, X) = 1/\sigma(K, X)$. Using (3.13) and applying l'Hôpital's rule, we obtain

$$\limsup_{K\to\infty} \rho(K,1) = \limsup_{K\to\infty} \frac{\log(F_X/F_K)}{\log K} = 1 + \limsup_{K\to\infty} \frac{\log \frac{F_X}{KF_K}}{\log K}. \tag{A.4}$$

Therefore to prove the claim, it suffices to show $F_X \leq KF_K$ for large enough K. Since $X = 1$ and F is homogeneous of degree 1, we have $F = KF_K + F_X$, so

$$\frac{1}{K}(KF_K - F_X) = \frac{1}{K}(2KF_K - F) = 2F_K - \frac{F}{K} \to 2m - m = m > 0,$$

implying $F_X < KF_K$ for large enough H.

If f takes the CES form (2.4), a straightforward calculation yields

$$\frac{F_K}{F_X} = \begin{cases} \frac{\alpha}{1-\alpha}K^{-\rho} + \frac{1-\delta}{A(1-\alpha)(\alpha K^{1-\rho}+1-\alpha)^{\frac{\rho}{1-\rho}}} & \text{if } \rho \neq 1, \\ \frac{\alpha}{1-\alpha}K^{-1} + \frac{1-\delta}{A(1-\alpha)}K^{-\alpha} & \text{if } \rho = 1. \end{cases}$$

Taking the logarithm, dividing by $\log K$, and letting $K \to \infty$, we obtain (3.14). □

We need the following lemma to prove Proposition 3.

Lemma A.4. *Let $\bar{K} > 0$ and suppose that $\sigma(K,1) \geq \sigma$ for $K \geq \bar{K}$. Let $\rho = 1/\sigma$. If $K \geq \bar{K}$, then*

$$\frac{F_X}{F_K}(K,1) \leq \frac{F_X}{F_K}(\bar{K},1)(K/\bar{K})^{\rho}. \tag{A.5}$$

Proof. Setting $K = \mathrm{e}^k$ and $X = 1$ in (3.13), we obtain

$$\rho(\mathrm{e}^k, 1) = \frac{\mathrm{d}}{\mathrm{d}k} \log \frac{F_X}{F_K}(\mathrm{e}^k, 1).$$

Integrating both sides from $k = \log \bar{K}$ to $k = \log K$ and applying the intermediate value theorem for integrals, there exists $\bar{k} \in (\log \bar{K}, \log K)$ such that

$$\begin{aligned} \rho(\mathrm{e}^{\bar{k}}, 1)\log(K/\bar{K}) &= \int_{\log \bar{K}}^{\log K} \rho(\mathrm{e}^k, 1)\,\mathrm{d}k \\ &= \log \frac{F_X}{F_K}(K,1) - \log \frac{F_X}{F_K}(\bar{K},1). \end{aligned} \tag{A.6}$$

Taking the exponential of both sides of (A.6) and letting $M := (F_X/F_K)(\bar{K},1)$, we obtain

$$\frac{F_X}{F_K}(K,1) = M(K/\bar{K})^{\rho(\mathrm{e}^{\bar{k}},1)}.$$

Since $K \geq \bar{K}$ and $\rho(\mathrm{e}^{\bar{k}}, 1) \leq \rho := 1/\sigma$, it follows that

$$\frac{F_X}{F_K}(K, 1) \leq M(K/\bar{K})^\rho,$$

which is (A.5). □

Proof of Proposition 3. We establish the claim by applying the local stable manifold theorem (see Irwin (1980, Theorems 6.5 and 6.9) and Guckenheimer and Holmes (1983, Theorem 1.4.2)), which is essentially linearization and evaluating the magnitude of eigenvalues. Since a complete proof is technical and tedious (see the appendix of Hirano and Toda (2023a) for a rigorous argument in a related model), we only provide a sketch.

Let $G = G(\lambda) > 1$. Then the steady state productivity threshold is $\bar{z} := G/m$. Around the steady state, combining (3.6a), (3.7a), and (2.3a), the capital dynamics is approximately

$$\begin{aligned} K_{t+1} &= \frac{\beta\lambda m \int_{\bar{z}}^\infty z \,\mathrm{d}\Phi(z)}{1 - \beta(\lambda\Phi(\bar{z}) + 1 - \lambda)} K_t = GK_t \\ \iff \xi_{1,t+1} &= \frac{1}{G}\xi_{1t}, \end{aligned}$$

where we define $\xi_{1t} = 1/K_t$. Linearizing (3.6b) with respect to (K, z) around $(K, z) = (\infty, \bar{z})$, we have

$$P(K, z) \approx \frac{b}{1-b} mK + \frac{\beta\lambda\phi(\bar{z})}{(1-b)^2} mK(z - \bar{z}), \tag{A.7}$$

where $b := \beta(\lambda\Phi(\bar{z}) + 1 - \lambda) \in (0, \beta)$. By Lemma A.4, we have $F_X(K_t, 1) \sim G^{\rho t} \ll G^t \sim K_t$, where $\rho = 1/\sigma < 1$ by Assumption 4. Substituting (A.7) into (3.7b) and using $K_{t+1} = GK_t$, we obtain

$$z = \frac{G}{m} \frac{\frac{b}{1-b} m + \frac{\beta\lambda\phi(\bar{z})}{(1-b)^2} m(z' - \bar{z})}{\frac{b}{1-b} m + \frac{\beta\lambda\phi(\bar{z})}{(1-b)^2} m(z - \bar{z})},$$

where $(z, z') = (\bar{z}_t, \bar{z}_{t+1})$. Solving for z', we obtain

$$z' - \bar{z} = \frac{1}{\bar{z}} \left(z + \frac{b(1-b)}{\beta\lambda\phi(\bar{z})} \right) (z - \bar{z}).$$

Therefore if we define $\xi_{2t} = \bar{z}_t - \bar{z}$ and $\xi_t = (\xi_{1t}, \xi_{2t})$, the dynamics of ξ_t near the steady state 0 is approximately $\xi_{t+1} = A\xi_t$ for the matrix

$$A = \begin{bmatrix} 1/G & 0 \\ 0 & 1 + \frac{b(1-b)}{\beta\lambda\bar{z}\phi(\bar{z})}. \end{bmatrix}$$

Clearly, the eigenvalues of A are $\lambda_1 = 1/G \in (0,1)$ and $\lambda_2 = 1 + \frac{b(1-b)}{\beta\lambda\bar{z}\phi(\bar{z})} > 1$ because $b \in (0,1)$. Since $\xi_{10} = 1/K_0$ is exogenous but $\xi_{20} = \bar{z}_0 - \bar{z}$ is endogenous, for any sufficiently large K_0 (hence ξ_{10} sufficiently close to 0), by the local stable manifold theorem, there exists a unique equilibrium path converging to the steady state 0. □

Proof of Theorem 1. Suppose $\lambda < \bar{\lambda}$ and consider any equilibrium converging to the steady state. Then by definition r_t/P_t converges to a finite positive number, so $\sum_{t=1}^{\infty} r_t/P_t = \infty$. By Lemma B.1, we have $P_t = V_t$.

Next, suppose $\lambda > \bar{\lambda}$ and consider the equilibrium in Proposition 3. By Assumption 4, we have $\rho := 1/\sigma < 1$. Take any $\bar{\rho} \in (\rho, 1)$. Then we can take $\bar{K} > 0$ such that $\sigma(K,1) \geq 1/\bar{\rho}$ for all $K \geq \bar{K}$. By Proposition 3, we have $K_t \sim G^t \to \infty$. Therefore for large enough t, we have $K_t > \bar{K}$. By Lemma A.4, we have

$$r_t = F_X(K_t,1) \leq F_K(K_t,1)\frac{F_X(\bar{K},1)}{F_K(\bar{K},1)}(K_t/\bar{K})^{\bar{\rho}} \sim m\frac{F_X(\bar{K},1)}{F_K(\bar{K},1)}(G^t/\bar{K})^{\bar{\rho}}.$$

Therefore for large enough t, we have the order of magnitude

$$\frac{r_t}{P_t} \sim G^{(\bar{\rho}-1)t},$$

which is summable because $\bar{\rho} < 1$. By Lemma B.1, we have $P_t > V_t$ for all t. □

Proof of Proposition 4. That $\bar{\lambda}$ is decreasing in β and m are obvious. The monotonicity with respect to Φ follows from the definition of first-order stochastic dominance and noting that $z \mapsto \max\{mz - 1, 0\}$ is increasing in z.

If the production function takes the CES form (2.4), as $K \to \infty$, we have

$$\begin{aligned} F_K(K,X) &= A\left(\alpha K^{1-\rho} + (1-\alpha)X^{1-\rho}\right)^{\frac{\rho}{1-\rho}} \alpha K^{-\rho} + 1 - \delta \\ &= A\alpha\left(\alpha + (1-\alpha)(X/K)^{1-\rho}\right)^{\frac{\rho}{1-\rho}} + 1 - \delta \\ &\to m := \begin{cases} A\alpha^{\frac{1}{1-\rho}} + 1 - \delta & \text{if } \rho < 1, \\ 1 - \delta & \text{if } \rho \geq 1. \end{cases} \end{aligned}$$

Clearly m is decreasing in ρ because $\alpha \in (0,1)$. Therefore $\bar{\lambda}$ is decreasing in

$\sigma = 1/\rho$. □

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Online Appendix

B Definition and characterization of bubbles

This appendix defines asset price bubbles and provides an exact characterization. The discussion is based on Hirano and Toda (2023a, §2).

We consider an infinite-horizon, deterministic economy with a homogeneous good and time indexed by $t = 0, 1, \dots$. Consider an asset with infinite maturity that pays dividend $D_t \geq 0$ and trades at ex-dividend price P_t, both in units of the time-t good. In the background, we assume the presence of rational, perfectly competitive investors. Free disposal of the asset implies $P_t \geq 0$. Let $q_t > 0$ be the Arrow-Debreu price, i.e., the date-0 price of the consumption good delivered at time t, with the normalization $q_0 = 1$. The absence of arbitrage implies

$$q_t P_t = q_{t+1}(P_{t+1} + D_{t+1}). \tag{B.1}$$

Iterating the no-arbitrage condition (B.1) forward and using $q_0 = 1$, we obtain

$$P_0 = \sum_{t=1}^{T} q_t D_t + q_T P_T. \tag{B.2}$$

Noting that $P_t \geq 0$, $D_t \geq 0$, and $q_t > 0$, the infinite sum of the present value of dividends

$$V_0 := \sum_{t=1}^{\infty} q_t D_t \tag{B.3}$$

exists, which is called the *fundamental value* of the asset. Letting $T \to \infty$ in (B.2), we obtain

$$P_0 = \sum_{t=1}^{\infty} q_t D_t + \lim_{T\to\infty} q_T P_T = V_0 + \lim_{T\to\infty} q_T P_T. \tag{B.4}$$

We say that the *transversality condition* for asset pricing holds if

$$\lim_{T\to\infty} q_T P_T = 0. \tag{B.5}$$

When the transversality condition (B.5) holds, the identity (B.4) implies that $P_0 = V_0$ and the asset price equals its fundamental value. If $\lim_{T\to\infty} q_T P_T > 0$, then $P_0 > V_0$, and we say that the asset contains a *bubble*.

Note that in deterministic economies, for all t we have

$$P_t = \underbrace{\frac{1}{q_t}\sum_{s=1}^{\infty} q_{t+s}D_{t+s}}_{=:V_t} + \frac{1}{q_t}\lim_{T\to\infty} q_T P_T.$$

Therefore either $P_t = V_t$ for all t or $P_t > V_t$ for all t, so the economy is permanently in either the bubbly or the fundamental regime. Thus, a bubble is a permanent overvaluation of an asset, which is a feature of rational expectations.

In general, checking the transversality condition (B.5) directly could be difficult because it involves q_T. The following lemma provides an equivalent characterization.

Lemma B.1 (Bubble Characterization)**.** *If $P_t > 0$ for all t, the asset price exhibits a bubble if and only if $\sum_{t=1}^{\infty} D_t/P_t < \infty$.*

Proof. Because the economy is deterministic, the interest rate is defined by the asset return, so $R_t = (P_{t+1} + D_{t+1})/P_t$. Using the definition of q_t, we obtain

$$\begin{aligned} q_T P_T &= P_T \prod_{t=0}^{T-1} \frac{1}{R_t} = P_T \prod_{t=0}^{T-1} \frac{P_t}{P_{t+1} + D_{t+1}} \\ &= P_0 \prod_{t=1}^{T} \frac{P_t}{P_t + D_t} = P_0 \left(\prod_{t=1}^{T} \left(1 + \frac{D_t}{P_t}\right)\right)^{-1}. \end{aligned}$$

Expanding terms and using $1 + x \le \mathrm{e}^x$, we obtain

$$P_0 \exp\left(-\sum_{t=1}^{T} \frac{D_t}{P_t}\right) \le q_T P_T \le P_0 \left(1 + \sum_{t=1}^{T} \frac{D_t}{P_t}\right)^{-1}.$$

Letting $T \to \infty$, we have $\lim_{T\to\infty} q_T P_T > 0$ if and only if $\sum_{t=1}^{\infty} D_t/P_t < \infty$. □

C Solution algorithm

This appendix explains how we numerically solve the model in Section 4.4.

C.1 Steady state

We first find the steady state. If $\lambda > \bar{\lambda}$, letting $\bar{z} = G/m$ in (3.10), we obtain

$$\bar{z} = \frac{\beta\lambda(\bar{z} + \zeta)\mathrm{e}^{-\bar{z}/\zeta}}{1 - \beta + \beta\lambda\mathrm{e}^{-\bar{z}/\zeta}}.$$

We numerically solve this equation for $\bar{z}$ and obtain $G = m\bar{z}$.

If $\lambda < \bar{\lambda}$, we proceed in two steps. First, we take arbitrary $\bar{z} > 0$, set $\bar{z}_t = \bar{z}$ and $K_{t+1} = K_t = K$ in (4.4), and solve for K. Let

$$C = \frac{\beta\lambda(\bar{z} + \zeta)\mathrm{e}^{-\bar{z}/\zeta}}{1 - \beta + \beta\lambda\mathrm{e}^{-\bar{z}/\zeta}}.$$

If $C(1 - \delta) < 1$, then the steady state K for the CES production function (2.4) is

$$K(\bar{z}) := \begin{cases} \left(\frac{\left(\frac{1-C(1-\delta)}{CA}\right)^{1-\rho} - \alpha}{1-\alpha}\right)^{\frac{1}{\rho-1}} & \text{if } \rho \neq 1, \\ \left(\frac{1-C(1-\delta)}{CA}\right)^{\frac{1}{\alpha-1}} & \text{if } \rho = 1. \end{cases}$$

If $C(1 - \delta) \geq 1$, just set $K(\bar{z}) = \infty$. Finally, set $K_{t+1} = K_t = K(\bar{z})$ and $\bar{z}_{t+1} = \bar{z}_t = \bar{z}$ in (3.7b) and solve for $\bar{z}$.

C.2 Transition dynamics

To solve for the transition dynamics, we take some large T and start with a guess $\{\bar{z}_t\}_{t=0}^T$, where we set $\bar{z}_T$ to the steady state value computed above. Given initial aggregate capital K_0, we then generate the sequence of aggregate capital $\{K_t\}_{t=0}^T$ using (4.4). Using (3.7b), we update $\{\bar{z}_t\}_{t=0}^T$ by

$$\bar{z}_t^{\text{new}} = \frac{P(K_{t+1}, \bar{z}_{t+1}) + F_X(K_{t+1}, 1)}{F_K(K_{t+1}, 1)P(K_t, \bar{z}_t)}.$$

Finally, we find $\{\bar{z}_t\}_{t=0}^T$ by minimizing the equilibrium error

$$\sum_{t=0}^T (\bar{z}_t^{\text{new}} - \bar{z}_t)^2.$$

In practice, to reduce the dimensionality, we parametetrize $\{\bar{z}_t\}_{t=0}^T$ by a small number of values $\{\bar{z}_{t_j}\}_{j=1}^J$ (say $J = 10$) and use spline interpolation to calculate $\{\bar{z}_t\}_{t=0}^T$.