Unique Equilibria in Models of Rational Asset Price Bubbles

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Abstract

Existing models of rational pure bubble models feature multiple (and often a continuum of) equilibria, which makes model predictions and policy analysis non-robust. We show that when the interest rate in the fundamental equilibrium is below the economic growth rate ($R < G$), a bubbly equilibrium with $R = G$ exists. By injecting dividends that are vanishingly small relative to aggregate income to the bubble asset, we can eliminate the fundamental steady state and resolve equilibrium indeterminacy. We show the general applicability of dividend injection through examples in overlapping generations and infinite-horizon models with or without production or financial frictions.

Keywords: bubble, dividend, equilibrium indeterminacy, growth, low interest rate.

JEL codes: D53, G12.

1 Introduction

Asset price bubbles are situations where the asset price (P) exceeds its fundamental value (V) defined by the present value of the dividend stream (D). Although it may not be possible to provide conclusive empirical evidence for or against asset price bubbles due to the difficulty associated with evaluating the fundamental value, history is abound with anecdotal evidence of bubbly episodes (Kindleberger,

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2000; Shiller, 2015). Since the seminal theoretical work by Samuelson (1958) and Tirole (1985), significant progress has been made on rational asset pricing bubbles. Especially, the 2008 financial crisis fueled a major development on the relationship between financial conditions and asset bubbles in macroeconomic models.

The literature on rational asset price bubbles has almost exclusively focused on a special case called "pure bubbles", namely assets that pay no dividend ($D = 0$), which are intrinsically useless like fiat money. Because the fundamental value of a pure bubble asset is $V = 0$, we say that an asset price bubble exists whenever $P > 0$ in equilibrium. Although pure bubble models may illustrate some important aspects of asset price bubbles, they have several severe shortcomings. First, because it is hard to find pure bubble assets in reality other than fiat money or cryptocurrency, it is unrealistic to use pure bubble assets without dividends to describe real world bubbles attached to land, housing, and stocks. Hence, formal applications of the theory to empirical or quantitative analysis are difficult. Second, asset price bubbles tend to emerge in models under tight financial conditions, which goes against stylized facts that bubbly episodes tend to be associated with loose financial conditions (Kindleberger, 2000). Finally, pure bubble models feature multiple equilibria: there often exist a continuum of bubbly equilibria as well as a fundamental equilibrium in which the price of the pure bubble asset is zero. Therefore it is not obvious which equilibrium we should select.

Although the equilibrium indeterminacy in pure bubble models has been recognized for decades, the literature has selected only one of a continuum of bubbly equilibria (a saddle path or a steady state)[1] and has advanced policy and quantitative analysis. However, the equilibrium selected is only one point within an open set and thus has measure zero. There is no scientific basis for this equilibrium selection and there is no rational reason for why heterogeneous economic agents coordinate on that equilibrium even if they are exposed to disturbances, including policy changes. In short, equilibrium indeterminacy makes model implications and predictions fragile and non-robust.

In this paper, we propose a solution to this long-standing problem in rational bubble models. We resolve the equilibrium indeterminacy by injecting a small dividend to the bubble asset. Using a general equilibrium model, when the interest rate in the fundamental equilibrium is below the economic growth rate ($R < G$),

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[1]In a typical pure bubble model such as Tirole (1985), only when the initial bubble price just equals the price that corresponds to a saddle path, the economy will converge to a steady state with positive bubbles. Otherwise, the bubble price will converge to zero, and there are a continuum of such asymptotically bubbleless paths. The bubble price on the saddle path is the largest bubble size that can be sustained.

we show that a bubbly equilibrium with $R = G$ exists. In this case we can eliminate the fundamental steady state and resolve indeterminacy by introducing an asset with vanishing dividends relative to aggregate income. Intuitively, in these circumstances, without bubbles, the interest rate would be equal to or lower than the growth rate of dividends. But then the asset price would diverge to infinity, which obviously cannot happen in equilibrium. Thus asset price bubbles become necessary for the existence of equilibrium by raising the interest rate above the dividend growth rate. Our dividend injection procedure not only shows the necessity of asset price bubbles but also eliminates all equilibria converging to the fundamental steady state. Furthermore, we show that under plausible conditions on the elasticity of substitution (between goods or production factors), there exists a locally determinate bubbly equilibrium even with slightly positive dividends. These results imply that there is a discontinuity in equilibrium determinacy between the case with $D = 0$ and the case with $D > 0$.

Our equilibrium selection by dividend injection places the existing literature of pure bubbles on firmer footing. However, there are many substantial benefits in establishing the necessity and local uniqueness of the bubbly equilibrium beyond mere theoretical satisfaction. First, it enables comparative dynamics, making model predictions robust for policy and quantitative analysis. Second, unlike pure bubble assets with no dividends, our idea of dividend injection is also a model of bubbles attached to assets yielding positive dividends. Hence, this approach would be more natural for considering realistic bubbles attached to land, housing, and stocks. Third, our approach has a potential for extensions and applications. In game theory, Carlsson and van Damme (1993) show that adding a small noise in 2×2 games with incomplete information removes equilibrium multiplicity. Morris and Shin (1998) apply this idea to select a unique equilibrium in a model of currency attacks with a continuum of traders. The selection of a unique equilibrium outcome by adding small noises has opened up enormous possibilities for applications (Morris and Shin, 2000, 2003). From this perspective, we can expect our approach to produce various applications in macroeconomic analysis with asset price bubbles.[2]

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[2]Bernanke et al. (1998) developed the quantitative macroeconomic framework called "BGG", which pioneered the large literature studying the link between asset prices and the real economy. Our approach can be embedded into their framework or other quantitative macroeconomic models so that we can conduct various policy and quantitative analysis in a way that asset prices contain a bubble. This direction is of considerable importance because policy makers want to understand leaning against the bubble policy. Our approach provides a theoretical foundation for the direction. Note that Bernanke and Gertler (1999) extended Bernanke et al. (1998) to add exogenous asset price bubbles. In contrast, in our approach, asset bubbles occur as the

The rest of the paper is organized as follows. Section 2 presents a motivating example to select a unique equilibrium in a canonical pure bubble model. Section 3 discusses the elimination of fundamental equilibria by dividend injection in a general setting and applies it to overlapping generations and infinite-horizon models with or without production. Section 4 presents an application to an entrepreneurial economy with leverage. Section 5 provides further applications to typical pure bubble models and discuss the literature. Most proofs are relegated to Appendix A.

2   Motivating example

We consider a simple overlapping generations (OLG) model with an intrinsically useless asset as in Samuelson (1958) and Gale (1973). Time is discrete and indexed by $t = 0, 1, \ldots$. Agents born at time t live for two periods and have utility function

$$\log c_t^y + \beta \log c_{t+1}^o, \tag{2.1}$$

where $\beta > 0$ and c_t^y, c_{t+1}^o denote the consumption when young and old. The initial old care only about their consumption. The time t endowments of the young and old are $(e_t^y, e_t^o) = (aG^t, bG^t)$, where $G > 0$ is the gross growth rate of the economy and $a, b > 0$. There is a unit supply of an intrinsically useless asset (pure bubble asset) with no dividend, which is initially held by the old.

Let us find all the perfect foresight equilibria of this economy. Let $P_t \geq 0$ be the price of the asset in units of time t consumption and $R_t > 0$ be the gross risk-free rate between time t and $t+1$. Since agents live only for two periods and the asset is initially held by old agents, in equilibrium it is clear that the old sell the entire asset to the young. Therefore the time t budget constraints imply

$$\text{Old:} \qquad c_t^o + P_t \cdot 0 = P_t \cdot 1 + e_t^o \iff c_t^o = bG^t + P_t,$$
$$\text{Young:} \qquad c_t^y + P_t \cdot 1 = P_t \cdot 0 + e_t^y \iff c_t^y = aG^t - P_t.$$

To support this allocation as an equilibrium, it remains to verify the Euler equation (first-order condition) of the young, which is

$$1 = \beta R_t (c_{t+1}^o / c_t^y)^{-1} = \beta R_t \frac{aG^t - P_t}{bG^{t+1} + P_{t+1}}. \tag{2.2}$$

equilibrium outcome within a general equilibrium framework.

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If the asset price equals its fundamental value, then $P_t = 0$, the equilibrium allocation is autarkic, and (2.2) implies the interest rate

$$R_t = R = \frac{bG}{\beta a}. \tag{2.3}$$

Next suppose that there is an asset price bubble, so $P_t > 0$. Then the absence of arbitrage implies $R_t = P_{t+1}/P_t$, so (2.2) implies

$$1 = \beta \frac{P_{t+1}}{P_t} \frac{aG^t - P_t}{bG^{t+1} + P_{t+1}} \iff P_{t+1} = \frac{bG^{t+1} P_t}{\beta a G^t - (1 + \beta) P_t}. \tag{2.4}$$

To solve the difference equation (2.4), let us introduce the detrended asset price $x_t := G^{-t} P_t$. Then (2.4) becomes

$$x_{t+1} = \frac{bx_t}{\beta a - (1 + \beta)x_t} \iff \frac{1}{x_{t+1}} = \frac{\beta a}{b} \frac{1}{x_t} - \frac{1 + \beta}{b}. \tag{2.5}$$

Since (2.5) is a linear difference equation in $1/x_t$, it is straightforward to solve. The general solution is

$$\frac{1}{x_t} = \begin{cases} \left(\frac{\beta a}{b}\right)^t \left(\frac{1}{x_0} - \frac{1+\beta}{\beta a - b}\right) + \frac{1+\beta}{\beta a - b} & \text{if } \beta a \neq b, \\ \frac{1}{x_0} - \frac{1+\beta}{b} t & \text{if } \beta a = b. \end{cases} \tag{2.6}$$

Since $P_t \geq 0$ and hence $x_t \geq 0$, a solution with $P_t \geq 0$ for all t is possible only if $\beta a > b$. Under this condition, noting that $x_0 = P_0$, the general solution is

$$P_t = \frac{G^t}{\left(\frac{\beta a}{b}\right)^t \left(\frac{1}{P_0} - \frac{1+\beta}{\beta a - b}\right) + \frac{1+\beta}{\beta a - b}}, \tag{2.7}$$

where $0 < P_0 \leq \frac{\beta a - b}{1 + \beta}$ is arbitrary so that $P_t \geq 0$ for all t. Using (2.7), the consumption of the young is

$$c_t^y = aG^t - P_t = G^t \frac{a\left(\frac{1}{P_0} - \frac{1+\beta}{\beta a - b}\right) + \frac{a+b}{\beta a - b}\left(\frac{b}{\beta a}\right)^t}{\frac{1}{P_0} - \frac{1+\beta}{\beta a - b} + \frac{1+\beta}{\beta a - b}\left(\frac{b}{\beta a}\right)^t} > 0,$$

so consumption is interior and we indeed have an equilibrium. We can summarize the above derivations in the following proposition.

Proposition 2.1. *The following statements are true.*

(i) If $\beta a \leq b$, the unique equilibrium is $P_t = 0$ and $(c_t^y, c_t^o) = (aG^t, bG^t)$.

(ii) If $\beta a > b$, there are a continuum of equilibria parametrized by $0 \le P_0 \le \frac{\beta a - b}{1+\beta}$, where P_t is given by (2.7) and $(c_t^y, c_t^o) = (aG^t - P_t, bG^t + P_t)$.[3]

(iii) The asymptotic behavior of the detrended variables is

$$\lim_{t \to \infty} G^{-t}(c_t^y, c_t^o, P_t) = \begin{cases} \left(\frac{a+b}{1+\beta}, \frac{\beta(a+b)}{1+\beta}, \frac{\beta a - b}{1+\beta} \right) & \text{if } \beta a > b \text{ and } P_0 = \frac{\beta a - b}{1+\beta}, \\ (a, b, 0) & \text{otherwise.} \end{cases}$$

Proposition 2.1 shows that, when $\beta a > b$, we are in an uncomfortable situation where there are a continuum of equilibria. Furthermore, each equilibrium has a different consumption allocation (the economy features real equilibrium indeterminacy) because P_t is strictly increasing in P_0 by (2.7) and the consumption of the young is $aG^t - P_t$. Thus it is not obvious which equilibrium we should select.

One possibility for equilibrium selection is to look for equilibria that are more efficient. The following proposition shows that all equilibria can be Pareto ranked.

Proposition 2.2. *Suppose $\beta a > b$ and let $E(P_0)$ be the equilibrium with initial asset price $P_0 \in \left[0, \frac{\beta a - b}{1+\beta} \right]$. If $P_0 < P_0'$, then $E(P_0')$ Pareto dominates $E(P_0)$. Furthermore, $E\left(\frac{\beta a - b}{1+\beta} \right)$ is Pareto efficient.*

By Proposition 2.2, the only equilibrium that is not Pareto dominated (and is itself efficient) is the one corresponding to $P_0 = \frac{\beta a - b}{1+\beta}$ and hence

$$(c_t^y, c_t^o, P_t) = \left(\frac{a+b}{1+\beta}G^t, \frac{\beta(a+b)}{1+\beta}G^t, \frac{\beta a - b}{1+\beta}G^t \right).$$

Hence one may argue that $E\left(\frac{\beta a - b}{1+\beta} \right)$ is the most natural equilibrium because it is the most efficient. However, this equilibrium (and any other with $P_0 > 0$) exhibits an asset price bubble. Another natural equilibrium may be $E(0)$, in which there is no asset price bubble, but it is the most inefficient equilibrium.

We now argue that a slight perturbation to the economy leads to a unique equilibrium, and only one of the equilibria of the original economy can be achieved as the perturbation becomes zero, which enables us to select a unique equilibrium.

The idea is to "inject" a small dividend to the asset. Instead of a pure bubble asset, suppose that the asset pays dividend $D_t = D_0 G_d^t$, where the initial dividend $D_0 > 0$ is sufficiently small and the dividend growth rate satisfies $G_d < G$ so that the dividend is asymptotically negligible relative to endowments. Combining the

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[3]We always use the convention $1/0 = \infty$ and $1/\infty = 0$.

Euler equation (2.2) (including the dividend) and the no-arbitrage condition

$$R_t = \frac{P_{t+1} + D_{t+1}}{P_t}, \tag{2.8}$$

the equilibrium condition becomes

$$1 = \beta \frac{P_{t+1} + D_{t+1}}{P_t} \frac{aG^t - P_t}{bG^{t+1} + P_{t+1} + D_{t+1}}$$

$$\iff P_{t+1} = \frac{bG^{t+1}P_t}{\beta aG^t - (1+\beta)P_t} - D_{t+1}. \tag{2.9}$$

Although it is not generally possible to solve the difference equation (2.9) explicitly since it is nonlinear and non-autonomous (the equation explicitly depends on time), we may study its qualitative property by applying results from the theory of dynamical systems. To this end, introduce the auxiliary variables $x_{1t} := G^{-t}P_t$ and $x_{2t} = (G_d/G)^t D_0$. Then the one-dimensional non-autonomous nonlinear difference equation (2.9) can be converted to the two-dimensional autonomous nonlinear difference equation

$$x_{1,t+1} = \frac{bx_{1t}}{\beta a - (1+\beta)x_{1t}} - \frac{G_d}{G}x_{2t}, \tag{2.10a}$$

$$x_{2,t+1} = \frac{G_d}{G}x_{2t}. \tag{2.10b}$$

Let us write (2.10) as $x_{t+1} = h(x_t)$, where $x_t = (x_{1t}, x_{2t})$ and

$$h(x_1, x_2) = \begin{bmatrix} \frac{bx_1}{\beta a - (1+\beta)x_1} - \frac{G_d}{G}x_2 \\ \frac{G_d}{G}x_2 \end{bmatrix}. \tag{2.11}$$

Solving $x = h(x)$, since $G_d < G$ we find that h has two fixed points

$$x_f^* := (0, 0) \quad \text{and} \quad x_b^* := \left(\frac{\beta a - b}{1 + \beta}, 0 \right), \tag{2.12}$$

where the subscripts f and b refer to "fundamental" and "bubbly".

The following proposition shows that, by appropriately choosing the dividend growth rate G_d, we can eliminate the fundamental steady state. Furthermore, the bubbly steady state is locally determinate in the sense that there exists a unique equilibrium path converging to the bubbly steady state.

Proposition 2.3. *Suppose $\beta a > b$ and $G_d \in \left(\frac{bG}{\beta a}, G \right)$. Then the following statements are true.*

(i) There exist no fundamental equilibria.

(ii) For any initial dividend $D_0 > 0$, there exists a unique equilibrium path $\{x_t\}_{t=0}^{\infty}$ converging to x_b^*.

The proof is technical and is deferred to the appendix. However, it is very simple to see that there cannot be any equilibrium paths converging to the fundamental steady state x_f^*. To see this, suppose an equilibrium with $x_t \to x_f^*$ exists and let $R_t > 0$ be the gross risk-free rate implied by (2.2). Then

$$
\begin{aligned}
R_t &= \frac{1}{\beta} \frac{bG^{t+1} + P_{t+1} + D_{t+1}}{aG^t - P_t} \\
&= \frac{G}{\beta} \frac{b + x_{1,t+1} + x_{2,t+1}}{a - x_{1t}} \to \frac{bG}{\beta a} < G_d
\end{aligned}
$$

as $t \to \infty$. Since the risk-free rate is asymptotically lower than the dividend growth rate, the present value of the dividend stream is infinite, which is impossible in equilibrium.

3 Equilibrium selection by dividend injection

The idea of Proposition 2.3 to eliminate a fundamental equilibrium (an equilibrium in which the bubble asset has no value) by adding asymptotically negligible but positive dividends to the bubble asset is fairly general. In this section we discuss this "dividend injection" procedure in a general setting and present applications to canonical models in the literature.

3.1 Existence of bubbly equilibrium

We start with a system of reduced-form equations

$$W' = G(R)W, \tag{3.1a}$$

$$s(R)W = B. \tag{3.1b}$$

Here W (W') is the current (next period) "aggregate wealth"; R is the gross risk-free rate; $G(R)$ is the gross growth rate of the economy given the interest rate; $s(R)$ is the "saving rate" or the demand for risk-free assets per unit of wealth, given the interest rate; and B is the aggregate supply of the risk-free asset. Thus (3.1a) and (3.1b) can be interpreted as the aggregate resource constraint and market clearing condition, respectively.

Although the system of equations (3.1) is reduced-form, it can often be derived from first principles. For instance, suppose agents have homothetic preferences and solve an optimal consumption-portfolio problem. In this setting, because consumption and the demand for particular assets are proportional to wealth due to homotheticity, we obtain equations of the form (3.1) after aggregating across agents. Although this argument is somewhat informal, we shall present specific examples in the following subsections and Section 5.

The economy may feature equilibria that are "stationary" in some sense. One example is a stationary equilibrium in which $W > 0$ is constant. Then the equilibrium interest rate is pinned down by $G(R) = 1$ using the resource constraint (3.1a). Given this interest rate, the steady state W is pinned down by $W = B/s(R)$ using the market clearing condition (3.1b). Another example is a balanced growth path in which W grows at a constant rate. For such an equilibrium, we require that the risk-free asset is in zero net supply, so $B = 0$. Then the equilibrium interest rate is pinned down by $s(R) = 0$ using the market clearing condition (3.1b). Given this interest rate, W grows at rate $G(R)$ using (3.1a).

In either case, let $G = G(R)$ be the equilibrium growth rate of the economy. In equilibrium, it could be $R \geq G$ or $R < G$. We now argue that an equilibrium with $R < G$ is implausible. To see this, introduce an asset in unit supply that pays dividend $D_t = D_0 G_d^t$ at time t, where the dividend growth rate satisfies $G_d \in (R, G)$. Then the dividend-to-wealth ratio is

$$\frac{D_t}{W_t} = \frac{D_0}{W_0}(G_d/G)^t \to 0$$

as $t \to \infty$ because $G_d < G$. Hence the asset is asymptotically irrelevant and we may conjecture that the perturbed economy will converge to the original equilibrium. However, if such an equilibrium exists, then the asset price must be infinite because the present value of time t dividend, which is approximately $D_0(G_d/R)^t$, diverges to infinity because $G_d > R$. This argument shows that an equilibrium with $R < G$ is fragile in the sense that if we introduce an asset with asymptotically negligible dividends (with growth rate between the interest rate and the economic growth rate), the original equilibrium can no longer be supported as a steady state in the perturbed economy.

But if an equilibrium with $R < G$ is implausible, are there other (plausible) equilibria? The answer is typically yes. Suppose that the growth rate of the economy satisfies $G(R) < R$ for large enough R. This is a natural condition: it trivially holds in models with exogenous growth (because $G(R) = G$ is constant);

in endogenous growth models, high interest rate typically reduces investment and hence growth. Thus if there exists a fundamental equilibrium with $R_f < G(R_f)$ and the growth rate satisfies $R > G(R)$ for large enough R, (assuming continuity) there exists $R_b \in (R_f, \infty)$ such that $R_b = G(R_b)$. To construct such an equilibrium, introduce a unit supply of pure bubble asset with price P_t, which can be thought of a limit of a vanishingly small dividend-paying asset. Then (3.1) becomes

$$W_{t+1} = R_b W_t, \tag{3.2a}$$

$$s(R_b)W_t = P_t. \tag{3.2b}$$

Using (3.2), the gross return on the bubble asset is

$$\frac{P_{t+1}}{P_t} = \frac{s(R_b)W_{t+1}}{s(R_b)W_t} = \frac{W_{t+1}}{W_t} = R_b,$$

so the no-arbitrage condition holds and we indeed have an equilibrium.

We now formalize the preceding argument.

Assumption 1. *Given the gross risk-free rate $R > 0$, the aggregate dynamics satisfies $W_{t+1} = G(R)W_t$, where $G : (0, \infty) \to (0, \infty)$ is continuous and satisfies $G(R) < R$ for large enough R.*

Assumption 2. *Given the gross risk-free rate $R > 0$ and aggregate state variable W_t, the aggregate demand for the risk-free asset is pW_t for $p \in s(R)$, where $s : \mathbb{R} \twoheadrightarrow \mathbb{R}$.*

In Assumption 2, we let the demand for the risk-free asset be a correspondence instead of a function because agents could be indifferent between two assets if they have identical returns, in which case the demand is indeterminate. Under the maintained assumptions, the following theorem shows that whenever the interest rate is lower than the economic growth rate in the fundamental equilibrium, we may construct a bubbly equilibrium in which the interest rate equals the economic growth rate.

Theorem 3.1 (Existence of bubbly equilibrium)**.** *Consider an economy satisfying Assumptions 1 and 2 and suppose the risk-free asset is in zero net supply. If there exists a fundamental equilibrium interest rate R_f with $R_f < G(R_f)$, then there exists $R_b \in (R_f, \infty)$ with $R_b = G(R_b)$. If $p \in s(R_b) \cap (0, \infty)$, then there exists a bubbly equilibrium with interest rate R_b and asset price $P_t = pW_t$.*

Proof. Immediate from the argument in the main text. □

Whenever the assumptions of Theorem 3.1 are satisfied, we may eliminate the fundamental steady state by introducing an asset with small dividends with an appropriate growth rate just as we did in Proposition 2.3. This procedure will often lead to a unique equilibrium path converging to the bubbly steady state. However, the exact implementation depends on the model specifics. In the next few subsections, we apply Theorem 3.1 and the elimination of fundamental equilibria by dividend injection to canonical models from the literature of pure bubble with overlapping generations or infinite horizon and with or without production.

3.2 OLG endowment economy (Samuelson, 1958)

We revisit the OLG model in Section 2. To apply Theorem 3.1, define the "aggregate wealth" W_t by the income of the young, so $W_t = aG^t$. Then $W_{t+1} = GW_t$, so $G(R) = G$ in (3.1a) is constant. We next solve for the "saving rate" $s(R)$ in (3.1b). Letting R be the gross risk-free rate and s be the savings of the young, the Euler equation implies

$$1 = \beta R \left(\frac{bG^{t+1} + Rs}{aG^t - s} \right)^{-1} \iff s = \frac{G^t}{1+\beta} \left(\beta a - \frac{bG}{R} \right).$$

Therefore we may define the "saving rate" in (3.1b) by

$$s(R) = \frac{s}{aG^t} = \frac{1}{1+\beta} \left(\beta - \frac{bG}{aR} \right).$$

Thus the fundamental equilibrium condition is

$$s(R) = 0 \iff R_f = \frac{bG}{\beta a},$$

which is precisely (2.3). The low interest condition is therefore

$$G > R_f = \frac{bG}{\beta a} \iff \beta a > b,$$

which is precisely the existence condition for a bubbly equilibrium in Proposition 2.1. Clearly $s(R) > 0$ for $R > R_f$, so all assumptions of Theorem 3.1 are satisfied. The bubbly equilibrium condition is $R_b = G$ and the asset price is

$$P_t = s(R_b)W_t = \frac{\beta a - b}{1+\beta} G^t,$$

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which corresponds to the bubbly steady state in (2.12).

3.3 OLG production economy (Tirole, 1985)

Tirole (1985) introduces a pure bubble asset in the Diamond (1965) overlapping generations neoclassical growth model. Since the model is well known, we only describe it briefly.[4] The number of young agents at time t is $N_t = N_0 G^t$. The young maximize utility $U(c_t^y, c_{t+1}^o)$ subject to the budget constraint $c_t^y + c_{t+1}^o/R_t = \omega_t$, where R_t is the gross risk-free rate from time t to $t+1$ and ω_t is the wage. The firm has a constant-returns-to-scale production function $F(K, N)$, hires labor N to maximize the profit $F(K, N) - \omega N$, and distributes the profit to capital owners. Letting K_t be aggregate capital, $k_t = K_t/N_t$ be capital per capita (capital-labor ratio), and $f(k) := F(k, 1)$, profit maximization implies

$$\omega_t = \omega(k_t) := f(k_t) - k_t f'(k_t).$$

The gross risk-free rate is then

$$R_{t-1} = \frac{F(K_t, N_t) - \omega_t N_t}{K_t} = f'(k_t). \tag{3.3}$$

Since ω_t and R_{t-1} are functions of k_t, the consumption of the young can be written as $c_t^y = c^y(k_t, k_{t+1})$. Hence the aggregate demand for the risk-free asset (in excess of capital) at time t is

$$S_t := N_t(\omega(k_t) - c^y(k_t, k_{t+1})) - K_{t+1} = N_t(\omega(k_t) - c^y(k_t, k_{t+1}) - Gk_{t+1}). \tag{3.4}$$

Let us show that this model fits in the framework described in Theorem 3.1. Suppose the capital-labor ratio k, wage ω, and gross risk-free rate R are constant over time. As in Section 3.2, define the "aggregate wealth" by the aggregate income of the young, so $W_t = \omega N_0 G^t$. Then $W_{t+1} = GW_t$, so Assumption 1 holds. Using (3.4), the "saving rate" is

$$s(R) := \frac{S_t}{W_t} = 1 - \frac{c^y(k, k) + Gk}{\omega(k)}, \tag{3.5}$$

where $k = (f')^{-1}(R)$. Therefore Assumption 2 holds. If the risk-free asset is in zero net supply, we obtain the fundamental equilibrium interest rate by setting $s(R) = 0$. Letting k_f be the corresponding capital per capita, we have $R_f =$

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[4]See Blanchard and Fischer (1989, §5.2) or Miao (2020, §16.3) for textbook treatment.

$f'(k_f)$. If the low interest condition $R_f < G$ holds, under standard monotonicity, concavity, and Inada conditions on f, there exists a unique $k_b < k_f$ such that $R_f = f'(k_f) < f'(k_b) = G =: R_b$. If $s(G) > 0$, there exists a bubbly equilibrium with asset price

$$P_t = s(R_b)W_t = (\omega(k_b) - c^y(k_b, k_b) - Gk_b)N_0 G^t.$$

We now show that dividend injection eliminates the fundamental equilibrium. As in Section 2, introduce an asset in unit supply with dividend $D_t = D_0 G_d^t$, where $R_f < G_d < G$. Combining the no-arbitrage condition (2.8), the definition of the risk-free rate (3.3), and the per capita demand for the asset (3.5), the equilibrium is characterized by the system of equations

$$f'(k_{t+1}) = \frac{P_{t+1} + D_{t+1}}{P_t}, \tag{3.6a}$$

$$\omega(k_t) - c^y(k_t, k_{t+1}) - Gk_{t+1} = P_t/N_t, \tag{3.6b}$$

$$D_{t+1} = G_d D_t. \tag{3.6c}$$

Here (3.6a) is the no-arbitrage condition between capital and the asset, (3.6b) is the condition that the asset absorbs the aggregate savings, and (3.6c) governs the evolution of dividends. To study asymptotically balanced growth paths, define the auxiliary variables $x_t = (x_{1t}, x_{2t}, x_{3t}) = (k_t, P_t/N_t, D_t/N_t)$. Then the system of equations (3.6) can be written as $H(x_t, x_{t+1}) = 0$, where

$$H_1(x, y) = y_2 + y_3 - \frac{x_2}{G}f'(y_1), \tag{3.7a}$$

$$H_2(x, y) = x_2 + Gy_1 + c^y(x_1, y_1) - \omega(x_1), \tag{3.7b}$$

$$H_3(x, y) = y_3 - \frac{G_d}{G}x_3. \tag{3.7c}$$

The fundamental and bubbly steady states are given by

$$x_f^* := (k_f, 0, 0) \quad \text{and} \quad x_b^* := (k_b, \omega(k_b) - c^y(k_b, k_b) - Gk_b, 0).$$

We now present a result establishing the nonexistence of fundamental equilibria after dividend injection and a sufficient condition for the existence of a locally determinate bubbly equilibrium. Before doing so, recall that with the utility

function $U(c_1, c_2)$, the elasticity of (intertemporal) substitution ε is defined by

$$\frac{1}{\varepsilon} = -\frac{\partial \log(U_1/U_2)}{\partial \log(c_1/c_2)}, \tag{3.8}$$

see for instance the discussion in Flynn et al. (2023).

Theorem 3.2. *Suppose $R_f = f'(k_f) < G$, $s(G) > 0$, and $G_d \in (R_f, G)$. Then the following statements are true.*

(i) *There exist no equilibrium paths converging to x_f^*.*

(ii) *If the elasticity of intertemporal substitution satisfies*

$$\varepsilon > 1 + \frac{G^2}{f''} \frac{\omega}{c^y(\omega - c^y)}, \tag{3.9}$$

where all functions are evaluated at k_b, then x_b^* is locally determinate. More precisely, for any (k_0, D_0) sufficiently close to $(k_b, 0)$, there exists a unique equilibrium path $\{x_t\}_{t=0}^\infty$ converging to x_b^*.

In the textbook treatment of the Tirole (1985) model, Blanchard and Fischer (1989, p. 232) state "The knife-edge case where the bubble is just such that the economy is on the saddle point path, though unlikely, is interesting". Theorem 3.2(i) reveals that the bubbly saddle point path is actually not a knife-edge case at all. It is in fact the stable "bubbleless" or fundamental equilibrium that is a knife-edge case because any small perturbations to the dividends will eliminate this equilibrium.

To our knowledge, most authors discuss the saddle path property of the bubbly steady state without showing it from first principles.[5] As we can see from Theorem 3.2(ii), the saddle path property (local determinacy) of the bubbly steady state does not come for free.[6] Note that the right-hand side of (3.9) is strictly less than 1 because $f'' < 0$. Thus the sufficient condition for local determinacy is that the elasticity of substitution is not too much below 1, which resembles the characterization of equilibrium uniqueness in general equilibrium models (Toda and Walsh, 2017).

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[5]For instance, Blanchard and Fischer (1989, p. 268, Endnote 16) state "Care must be taken in using a phase diagram to analyze the dynamics of a *difference* equation system. [. . .] Thus we must check in this case whether the system is indeed saddle point stable [. . .]. This check is left to the reader".

[6]See Calvo (1978) and Woodford (1984) for a discussion of equilibrium indeterminacy in OLG models.

As a concrete example, suppose that the production function is Cobb-Douglas with a constant depreciation rate, so $f(k) = Ak^\alpha + (1 - \delta)k$ for some $A > 0$, $\alpha \in (0, 1)$, and $\delta \in [0, 1]$. Then at the bubbly steady state $k = k_b$, we have

$$G = f'(k) = A\alpha k^{\alpha-1} + 1 - \delta,$$
$$f''(k) = A\alpha(\alpha - 1)k^{\alpha-2},$$
$$\omega(k) = f(k) - kf'(k) = A(1 - \alpha)k^\alpha.$$

Using $c^y(\omega - c^y) \leq \omega^2/4$, the lower bound in (3.9) can be bounded above as

$$1 + \frac{G^2}{f''} \frac{\omega}{c^y(\omega - c^y)} \leq 1 + \frac{4G^2}{\omega f''} = 1 - \frac{4\alpha}{(1 - \alpha)^2}\left(1 - \frac{1 - \delta}{G}\right)^{-2}.$$

Therefore we obtain a sufficient condition for local determinacy that involves only exogenous parameters G, α, δ.

3.4 Infinite-horizon endowment economy (Kocherlakota, 1992)

Kocherlakota (1992, Example 1) considers a two-agent economy (with relative risk aversion equal to 2) in which aggregate endowment grows at a constant rate, the individual endowments alternate between high and low levels every period, and agents trade a pure bubble asset subject to a shortsales constraint. This example builds on Bewley (1980), which features no growth but a general utility function. Here we follow the exposition in Hirano, Jinnai, and Toda (2022).

The two agents have constant relative risk aversion (CRRA) utility

$$\sum_{t=0}^\infty \beta^t \frac{c_t^{1-\gamma}}{1 - \gamma}.$$

The individual endowments are $(e_{1t}, e_{2t}) = (aG^t, bG^t)$ when t is odd and (bG^t, aG^t) when t is even, where $a > b > 0$ and $G > 0$ is the growth rate of the aggregate endowment. Call the agent with endowment aG^t (bG^t) "rich" ("poor"). The only asset in the economy is a pure bubble asset in unit supply, which is initially held by the poor agent.

As in Section 3.2, define the "aggregate wealth" by the income of the rich agent, so $W_t = aG^t$ and hence $W_{t+1} = GW_t$. Letting R be the gross risk-free rate and s be the savings of the rich agent (who wishes to save and hence the shortsales

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constraint does not bind), the Euler equation implies

$$1 = \beta R \left(\frac{bG^{t+1} + Rs}{aG^t - s} \right)^{-\gamma} \iff s = \frac{(\beta R)^{1/\gamma} - \frac{b}{a}G}{R + (\beta R)^{1/\gamma}} aG^t =: s(R)W_t.$$

We obtain the fundamental equilibrium interest rate by setting $s(R) = 0$. The low interest condition is therefore

$$G > R_f = \frac{1}{\beta} \left(\frac{b}{a}G \right)^\gamma \iff b < (\beta G^{1-\gamma})^{1/\gamma} a, \tag{3.10}$$

which is precisely the bubble existence condition in Hirano, Jinnai, and Toda (2022).[7] Under this condition, all assumptions of Theorem 3.1 are satisfied. The bubbly equilibrium condition is $R_b = G$ and the asset price is

$$P_t = s(R_b)W_t = \frac{a(\beta G^{1-\gamma})^{1/\gamma} - b}{1 + (\beta G^{1-\gamma})^{1/\gamma}} G^t.$$

4 Entrepreneurial economy with leverage

As a more substantive application, we consider an infinite-horizon production economy in which investment opportunities arrive stochastically and agents (entrepreneurs) can borrow subject to the leverage constraint. The model we present here is an extension of Kocherlakota (2009) with labor productivity growth and leverage.

4.1 Setup

We consider an infinite-horizon economy with a homogeneous good. Time is discrete and denoted by $t = 0, 1, \ldots$.

Agents The economy is populated by two types of agents, entrepreneurs and workers. Workers supply labor inelastically and consume the entire wage (hand-to-mouth) for simplicity. The aggregate labor supply at time t is G^t, where $G > 1$ can be interpreted as the growth rate of the population or labor productivity. There is a unit mass of entrepreneurs (agents) indexed by $i \in [0, 1]$. A typical

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[7]In addition to the low interest condition (3.10), it is necessary and sufficient for equilibrium existence that $\beta G^{1-\gamma} < 1$, which implies the Euler inequality for the poor agent and the transversality condition for individual optimality. See the discussion in Hirano, Jinnai, and Toda (2022) for details.

agent has utility function

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \log c_t, \tag{4.1}$$

where $\beta \in (0,1)$ is the discount factor and $c_t \geq 0$ is consumption.

Investment opportunities and production Each period, an investment opportunity arrives to an agent with probability $\pi \in (0,1)$, which is independent across agents and time. If an agent has an investment opportunity at time $t-1$ (is a high-productivity (H) type) and decides to install capital k, then the agent can produce goods using the production function $F(k,n)$ at time t, where n is labor input. As usual, suppose that F is homogeneous of degree 1, strictly quasi-concave, continuously differentiable, and satisfies the Inada conditions. If an agent does not have an investment opportunity (is a low-productivity (L) type), there is no production. Therefore the production function at time t can be written as $F_t(k,n) = A_{t-1}F(k,n)$, where $A_{t-1} = 1$ if the agent has an investment opportunity at time $t-1$ and $A_{t-1} = 0$ otherwise. Regardless of the investment opportunity, capital depreciates at rate $\delta \in [0,1]$.

Assets and budget constraint There is a risk-free asset in zero net supply and a pure bubble asset (land) in unit supply. Let R_t be the gross risk-free rate between time t and $t+1$, and let P_t be the land price. Suppressing the individual subscript, the budget constraint of a typical agent is

$$c_t + k_{t+1} + P_t x_t + b_t = A_{t-1}F(k_t, n_t) + (1-\delta)k_t - \omega_t n_t + P_t x_{t-1} + R_{t-1}b_{t-1}, \tag{4.2}$$

where $k_t, n_t \geq 0$ are capital and labor inputs at time t, A_{t-1} is an indicator for investment opportunity, $\omega_t \geq 0$ is the wage, and x_t, b_t are land and bond holdings.

Leverage constraint Agents are subject to the leverage constraint

$$k_{t+1} \leq \lambda(k_{t+1} + P_t x_t + b_t), \tag{4.3}$$

where $\lambda \geq 1$ is the leverage limit. Here $k_{t+1} + P_t x_t + b_t$ is total financial asset ("equity") of the agent. The leverage constraint (4.3) implies that total investment in the production technology cannot exceed some multiple of total equity, which is essentially identical to the constraint in Hirano, Jinnai, and Toda (2022). If

$\lambda = 1$ (no leverage allowed), then (4.3) reduces to

$$k_{t+1} \leq k_{t+1} + P_t x_t + b_t \iff -b_t \leq P_t x_t, \tag{4.4}$$

so borrowing $(-b_t)$ must be collateralized by land, which is exactly the constraint considered in Kocherlakota (2009).

Equilibrium The economy starts at $t = 0$ with some initial distribution of capital and land $\{(k_{i0}, x_{i,-1})\}_{i \in I}$, where $(k_{i0}, x_{i,-1}) > 0$ for all i. The definition of a rational expectations equilibrium is standard.

Definition 1 (Rational expectations equilibrium). Given the initial condition $\{(k_{i0}, x_{i,-1})\}_{i \in I}$, a *rational expectations equilibrium* consists of wages $\{\omega_t\}_{t=0}^{\infty}$, land prices $\{P_t\}_{t=0}^{\infty}$, interest rates $\{R_t\}_{t=0}^{\infty}$, and allocations $\{(c_{it}, k_{it}, n_{it}, x_{it}, b_{it})_{i \in I}\}_{t=0}^{\infty}$ such that the following conditions hold.

(i) (Individual optimization) Agents maximize the utility (4.1) subject to the budget constraint (4.2) and the leverage constraint (4.3).

(ii) (Labor market clearing) For all t, we have $\int_I n_{it}\, \mathrm{d}i = G^t$.

(iii) (Land market clearing) For all t, we have $\int_I x_{it}\, \mathrm{d}i = 1$.

(iv) (Bond market clearing) For all t, we have $\int_I b_{it}\, \mathrm{d}i = 0$.

4.2   Equilibrium analysis

We analyze the equilibrium dynamics.

Labor demand and profit For an L agent, it is clearly optimal to choose $n_t = 0$. Consider an H agent. Letting $f(k) = F(k, 1)$, we have $F(k, n) = nf(k/n)$ by homogeneity. Therefore the first-order condition for profit maximization is

$$\omega_t = f(y_t) - y_t f'(y_t), \tag{4.5}$$

where $y_t = k_t/n_t$ is the capital-labor ratio. Since $(f(y) - yf'(y))' = -yf''(y) > 0$, there exists a unique $y_t = y(\omega_t) > 0$ satisfying (4.5). The labor demand is then

$$n_t = k_t/y_t. \tag{4.6}$$

The maximized profit is

$$F(k_t, n_t) - \omega n_t = \frac{k_t}{y}[f(y) - (f(y) - yf'(y))] = f'(y)k_t.$$

Consumption and investment   Let w_t be the beginning-of-period wealth defined by the right-hand side of (4.2) maximized over n_t, so

$$w_t := (A_{t-1}f'(y_t) + 1 - \delta)k_t + P_t x_{t-1} + R_{t-1}b_{t-1}. \tag{4.7}$$

Due to log utility (4.1), the optimal consumption rule is $c_t = (1 - \beta)w_t$. Therefore the agent saves βw_t, which must equal $k_{t+1} + P_t x_t + b_t$ by accounting.

How agents allocate savings to capital (k_{t+1}), land (x_t), and bonds (b_t) depends on the interest rate and investment opportunities. In equilibrium, in order for investment to occur (so that there is demand for labor), the return on capital for H agents must exceed the risk-free rate. Furthermore, since L agents can always save at rate $1 - \delta$ by holding idle capital, the risk-free rate cannot fall below this value. Therefore we have the bound

$$f'(y_{t+1}) + 1 - \delta \geq R_t \geq 1 - \delta. \tag{4.8}$$

If the return on capital exceeds the risk-free rate, H agents will obviously choose maximal leverage. Therefore using the leverage constraint (4.3), we obtain the optimal investment rule

$$H \text{ agents:} \qquad k_{t+1} \begin{cases} = \lambda \beta w_t & \text{if } f'(y_{t+1}) + 1 - \delta > R_t, \\ \leq \lambda \beta w_t & \text{if } f'(y_{t+1}) + 1 - \delta = R_t, \end{cases} \tag{4.9a}$$

$$L \text{ agents:} \qquad k_{t+1} = \begin{cases} 0 & \text{if } R_t > 1 - \delta, \\ \text{arbitrary} & \text{if } R_t = 1 - \delta. \end{cases} \tag{4.9b}$$

Aggregation   We now aggregate the individual decision rules. Let $W_t = \int_I w_{it}\,di$ be the aggregate wealth. Let K_t^H, K_t^L be the aggregate capital holdings of H, L agents, $K_t = K_t^H + K_t^L$ the aggregate capital, and $Y_t = f'(y_t)K_t^H$ the aggregate output. Aggregating individual wealth (4.7) across all agents and using the land and bond market clearing conditions, we obtain

$$W_t = Y_t + (1 - \delta)K_t + P_t. \tag{4.10}$$

Aggregating individual savings $k_{t+1} + P_t x_t + b_t = \beta w_t$ across all agents, we obtain

$$K_{t+1}^H + K_{t+1}^L + P_t = \beta W_t. \tag{4.11}$$

Since the fraction of H agents is π, aggregating the capital investment (4.9a) across H agents, we obtain

$$K_{t+1}^H \leq \pi \lambda \beta W_t =: \phi \beta W_t, \tag{4.12}$$

with equality if the return on capital strictly exceeds the interest rate. The parameter $\phi := \pi \lambda$ can be interpreted as a measure of financial market imperfection. If $\phi \geq 1$ (so $\lambda \geq 1/\pi$), then the H type can absorb all savings. Then the economy reduces to the frictionless case with $K_{t+1}^H = \beta W_t$ and $K_{t+1}^L = P_t = 0$, which is uninteresting.

Therefore assume $\phi < 1$. In this case the H type cannot absorb all savings, so the return on capital must exceed the interest rate and (4.12) holds with equality. Combining (4.10), (4.11), (4.12), we can solve for the aggregate capital allocation $(K_{t+1}^H, K_{t+1}^L, K_{t+1})$ as a function of predetermined resources $Y_t + (1-\delta)K_t$ and the asset price P_t:

$$K_{t+1}^H = \phi \beta (Y_t + (1-\delta)K_t) + \phi \beta P_t, \tag{4.13a}$$

$$K_{t+1}^L = (1-\phi)\beta(Y_t + (1-\delta)K_t) - (1 - \beta + \phi\beta)P_t, \tag{4.13b}$$

$$K_{t+1} = \beta(Y_t + (1-\delta)K_t) - (1-\beta)P_t. \tag{4.13c}$$

We can make two observations from (4.13). First, as the leverage limit λ increases, so does $\phi = \pi \lambda$. Then K^H increases and K^L decreases, so higher leverage allows a more efficient allocation of capital. However, leverage has no direct effect on aggregate capital K because (4.13c) does not depend on ϕ: leverage affects capital only through the asset price P. Second, as the asset price P increases, K^H increases and K^L decreases, so an asset price bubble crowds out idle capital and crowds in investment.

Equilibrium   We focus on balanced growth path equilibria in which the capital-labor ratio y, wage ω, and risk-free rate R are constant. Aggregating labor demand (4.6) across H agents and using (4.12), we obtain the aggregate labor

$$G^t = N_t = \frac{K_t^H}{y} = \frac{\phi \beta}{y} W_{t-1}. \tag{4.14}$$

We next derive the aggregate wealth dynamics similar to (3.1a). At time t, by (4.11) aggregate savings is βW_t. By (4.12), fraction ϕ of it is invested by H agents as capital and the remaining fraction $1 - \phi$ is saved (either in the pure bubble asset or idle capital). Since the return on capital is $f'(y) + 1 - \delta$ and the risk-free rate is R, aggregate wealth evolves according to

$$W_{t+1} = (\phi(f'(y) + 1 - \delta) + (1 - \phi)R)\beta W_t. \tag{4.15}$$

By (4.14), aggregate wealth W_t must grow at rate G in equilibrium. Therefore the coefficient of W_t in (4.15) must equal G and we obtain the equilibrium condition

$$G(y, R) := \beta(\phi(f'(y) + 1 - \delta) + (1 - \phi)R) = G. \tag{4.16}$$

In a fundamental equilibrium ($P_t = 0$ for all t), because L agents hold idle capital, the risk-free rate must be $R = 1 - \delta$. We thus obtain the following proposition.

Proposition 4.1 (Fundamental equilibrium). *Suppose $1 \leq \lambda < 1/\pi$ and let $\phi = \pi\lambda$. Then there exists a unique fundamental balanced growth path equilibrium. The equilibrium is characterized by the following equations.*

$$\phi f'(y) + 1 - \delta = \frac{1}{\beta}G, \tag{4.17a}$$

$$\omega = f(y) - yf'(y), \tag{4.17b}$$

$$R = 1 - \delta, \tag{4.17c}$$

$$P_t = 0. \tag{4.17d}$$

The proof of Proposition 4.1 is immediate by setting $R = 1 - \delta$ in (4.16). The left-hand side of (4.17a) is the average return on capital, which pins down the capital-labor ratio y. Note that capital generates a return of $1 - \delta$ from storage, and only fraction $\phi < 1$ is used for production. The wage (4.17b) is determined by profit maximization. The fundamental interest rate (4.17c) equals the return from storage $1 - \delta$.

We next consider a balanced growth path equilibrium with an asset price bubble ($P_t > 0$ for all t). Using (4.10), we may rewrite (4.13) as

$$K_{t+1}^H = \phi\beta W_t, \tag{4.18a}$$

$$K_{t+1}^L = (1 - \phi)\beta W_t - P_t. \tag{4.18b}$$

Along a balanced growth path, the labor market clearing condition (4.14) forces

the aggregate wealth W_t to grow at rate G. By the definition of balanced growth, $K_{t+1}^H, K_{t+1}^L, P_t$ must all grow at rate G. Then the gross return on land is $P_{t+1}/P_t = G$, so the absence of arbitrage forces the equilibrium interest rate to be $R = G$. Since $G > 1 > 1 - \delta$, by (4.9b), we obtain $K_{t+1}^L = 0$. That is, L agents buy land instead of holding idle capital with low returns. Therefore (4.18b) and (4.14) imply

$$P_t = (1 - \phi)\beta W_t = \frac{1 - \phi}{\phi} y G^{t+1}. \tag{4.19}$$

We thus obtain the following proposition.

Proposition 4.2 (Bubbly equilibrium). *Suppose $1 \leq \lambda < 1/\pi$ and let $\phi = \pi\lambda$. Then there exists a unique bubbly balanced growth path equilibrium. The equilibrium is characterized by the following equations.*

$$f'(y) + 1 - \delta = \frac{1 - \beta + \phi\beta}{\phi\beta} G, \tag{4.20a}$$

$$\omega = f(y) - yf'(y), \tag{4.20b}$$

$$R = G, \tag{4.20c}$$

$$P_t = \frac{1 - \phi}{\phi} y G^{t+1}. \tag{4.20d}$$

The proof of Proposition 4.2 is immediate by setting $R = G$ in (4.16) and using (4.19). The left-hand side of (4.20a) is the equilibrium return on capital, which pins down the capital-labor ratio y. The wage (4.20b) is determined by profit maximization. The bubbly equilibrium interest rate (4.20c) equals the economic growth rate, as in Theorem 3.1. Note that in the bubbly equilibrium, the land price (4.20d) grows at the same rate as the economy.

4.3 Comparison of fundamental and bubbly equilibria

Suppose $1 \leq \lambda < 1/\pi$ so that there exist both a fundamental equilibrium E_f and a bubbly equilibrium E_b. As we have seen in (4.13), an asset price bubble crowds in investment. Therefore a natural question is whether the asset price bubble improves welfare.

To address this question, for $j \in \{f, b\}$, let y_j, ω_j, R_j be the equilibrium capital-labor ratio, wage, and risk-free rate in equilibrium E_j. Using the equilibrium condition (4.16), we obtain

$$\beta(\phi(f'(y_f) + 1 - \delta) + (1 - \phi)R_f) = G = \beta(\phi(f'(y_b) + 1 - \delta) + (1 - \phi)R_b).$$

Since $R_f = 1 - \delta < G = R_b$, it immediately follows that $f'(y_f) > f'(y_b)$ and hence $y_f < y_b$ because $f'' < 0$. Therefore the capital-labor ratio is higher in the bubbly equilibrium, and so is the wage $\omega = f(y) - yf'(y)$. Since workers are hand-to-mouth (consume their entire wage), they are clearly better off with bubbles. The welfare implication for entrepreneurs is not obvious. The asset price bubble allows low productive agents to save at a higher rate ($G > 1 - \delta$). For high productive agents, noting that they choose maximal leverage, the return on wealth is

$$R_j^H := \lambda(f'(y_j) + 1 - \delta) + (1 - \lambda)R_j.$$

Using Propositions 4.1, 4.2, and $\phi = \pi\lambda$, a straightforward calculation shows that

$$R_j^H = \frac{1}{\pi}\frac{G}{\beta} + \left(1 - \frac{1}{\pi}\right)R_j. \qquad (4.21)$$

Since $\pi < 1$ and $R_b > R_f$, the return on wealth conditional on being high productive is *lower* with bubbles due to increased borrowing costs and the reduction in the return on capital. Nevertheless, the following theorem shows that the bubbly equilibrium Pareto dominates the fundamental equilibrium.

Theorem 4.3 (Welfare). *Suppose $1 \leq \lambda < 1/\pi$. Then $y_b > y_f$, $\omega_b > \omega_f$, $R_b > R_f$, and E_b Pareto dominates E_f.*

The intuition for Theorem 4.3 is as follows. Without bubbles, L agents end up holding idle capital with low returns in equilibrium. Once bubbles emerge, they increase the rate of return on savings for L agents from $1 - \delta$ to $G > 1$, which expands the level of aggregate wealth, financing more investments of H agents, thereby increasing the capital-labor ratio y and the wage ω. This is the critical mechanism of how bubbles generate expansionary effects on y and ω. Since workers consume their wage, they are better off with bubbles. There are two reasons why entrepreneurs are also better off in the bubbly equilibrium. First, the bubble increases the land price from zero to a positive value, which increases the wealth of entrepreneurs because they own land. The rise in wealth by the appearance of bubbles has a persistent effect on future wealth of entrepreneurs because their wealth will increase multiplicatively by the factor βR, which increases their consumption and welfare in the future. Second, the bubble reduces the rate of return difference between the productive and unproductive states, enhancing consumption smoothing: the interest rate (return when unproductive) increases and the return on capital (return when productive) decreases because the capital-labor

ratio increases. Thus entrepreneurs are also better off and the asset price bubble is Pareto improving.

In our model, because agents are subject to idiosyncratic investment risk, a non-trivial wealth distribution emerges in equilibrium. Thus it is also of interest to study how asset price bubbles affect wealth inequality. To address this question, let $s_t = w_t/W_t$ be the individual wealth of a typical agent relative to the aggregate wealth. Using the optimal consumption rule $c_t = (1 - \beta)w_t$, the return on wealth (4.21), and the fact that W_t grows at rate G in equilibrium, the relative wealth dynamics becomes

$$\frac{s_t}{s_{t-1}} = \begin{cases} \frac{1}{\pi} + \left(1 - \frac{1}{\pi}\right)\frac{\beta R}{G} & \text{if } H \text{ type,} \\ \frac{\beta R}{G} & \text{if } L \text{ type,} \end{cases} \tag{4.22}$$

where R is the risk-free rate. Therefore the relative wealth satisfies Gibrat's law of proportional growth. Assuming a perpetual youth model in which agents survive with probability $\upsilon < 1$ each period (and newborn agents are endowed with the average wealth of deceased agents so that resources are recycled back), we may characterize the wealth Pareto exponent using the recent results from Beare and Toda (2022).

Proposition 4.4 (Wealth Pareto exponent). *Consider a perpetual youth model with survival probability $\upsilon \in (0, 1)$. Suppose $1 \leq \lambda < 1/\pi$ and let R be the equilibrium risk-free rate. Then the following statements are true.*

(i) *There exists a unique stationary distribution for relative wealth s_t.*

(ii) *The wealth distribution has a Pareto upper tail with exponent $\zeta > 1$, which solves*

$$\pi\left(\frac{1}{\pi} + \left(1 - \frac{1}{\pi}\right)\left(\frac{\beta R}{G}\right)\right)^\zeta + (1 - \pi)\left(\frac{\beta R}{G}\right)^\zeta = \frac{1}{\upsilon}. \tag{4.23}$$

(iii) *Letting $\zeta(R)$ be the Pareto exponent that solves (4.23), $\zeta(R)$ is strictly increasing in R if $R \leq G$. Consequently, wealth inequality is lower in the bubbly equilibrium.*

4.4 Elimination of fundamental equilibrium

In our model, we have $R_b = G > 1 - \delta = R_f$. Since the fundamental equilibrium interest rate $R_f = 1 - \delta$ is lower than the economic growth rate G, we may eliminate the fundamental equilibrium E_f by dividend injection just as we did for the Samuelson (1958) model in Proposition 2.3.

To this end, consider a perturbed economy in which land pays a constant dividend $D > 0$ every period. In this case, we need to make two changes to the derivations in Section 4.2. First, the capital-labor ratio y, wage ω, and risk-free rate R are no longer constant over time. Second, we need to include dividends to the aggregate wealth. With these changes, the equilibrium conditions can be summarized as follows:

$$W_t = (f'(y_t) + 1 - \delta)\phi\beta W_{t-1} + P_t + D, \tag{4.24a}$$

$$P_t = (1 - \phi)\beta W_t, \tag{4.24b}$$

$$\frac{\phi\beta}{y_t} W_{t-1} = G^t. \tag{4.24c}$$

Here (4.24a) comes from including dividend to (4.10), noting that the return on capital is $f'(y_t) + 1 - \delta$, and using (4.12). The condition (4.24b) is identical to (4.19). The condition (4.24c) is the labor market clearing condition, which comes from (4.14). Combining (4.24a) and (4.24b) to eliminate P_t and then using (4.24c) to eliminate W_t and W_{t-1}, the equilibrium dynamics is fully characterized by the nonlinear difference equation involving only the capital-labor ratio

$$y_{t+1} = \frac{1}{G} \frac{\phi\beta}{1 - \beta + \phi\beta} \left(y_t(f'(y_t) + 1 - \delta) + DG^{-t} \right). \tag{4.25}$$

The following proposition shows that dividend injection eliminates the (inefficient) fundamental equilibrium. Furthermore, the bubbly equilibrium is locally determinate as long as the elasticity of substitution between capital and labor is not too much below $1/2$. To state this result, recall that the elasticity of substitution ε is defined by

$$\frac{1}{\varepsilon} = -\frac{\partial \log(F_K/F_L)}{\partial \log(K/L)}. \tag{4.26}$$

Proposition 4.5 (Local determinacy of bubbly equilibrium). *Suppose $1 \leq \lambda < 1/\pi$ and $D > 0$. Then the following statements are true.*

(i) *There exist no equilibrium paths converging to the fundamental balanced growth path equilibrium E_f.*

(ii) *If the bubbly capital-labor ratio y_b satisfies*

$$y_b f''(y_b) > -2G\frac{1 - \beta + \phi\beta}{\phi\beta}, \tag{4.27}$$

for any (k_0, D) sufficiently close to $(y_b, 0)$, there exists a unique equilibrium

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path $\{y_t\}_{t=0}^{\infty}$ *converging to* y_b. *Furthermore,* (4.27) *is equivalent to*

$$\varepsilon > \frac{1}{2}\left(1 - \frac{\phi\beta}{1-\beta+\phi\beta}\frac{1-\delta}{G}\right)\frac{1}{1+y_b f'(y_b)/\omega_b}. \qquad (4.28)$$

When y_b satisfies (4.27) (or equivalently (4.28)), the bubbly steady state $(y_b, 0)$ is stable, and because the initial condition (k_0, D) is exogenous, there exists a unique equilibrium path converging to the steady state. When the reverse inequality to (4.27) holds, the steady state is unstable and the equilibrium path does not converge unless k_0 (or D) happens to take a particular value. In this case there is no scientific basis for focusing on the steady state. Proposition 4.5 tells us that focusing on the bubbly steady state is justified if the elasticity of substitution between capital and labor is not too much below $1/2$. In particular, local determinacy holds if the production function is Cobb-Douglas ($\varepsilon = 1$).

4.5   Extension with asset creation

Scheinkman (2014, p. 17) emphasizes the relationship between bubbles and asset creation. Historically, new assets have been created during bubbly episodes. For instance, during the dot-com bubble from the late 1990s to 2000, there were numerous initial public offerings (IPOs). In this subsection, we consider an extension with asset creation.

For this purpose, we consider the perpetual youth model in Section 4.3. When newborn agents enter the economy at date t, they create new assets. Suppose that the aggregate supply of assets exogenously grow according to

$$X_{t+1} = G_x X_t,$$

where G_x is the gross growth rate of assets. We also assume that each share of the asset yields a constant dividend D in every period. We can interpret this situation such that assets are stocks and new agents create new firms whose shares are sold at an IPO.

Assuming that the discount factor β already includes the survival probability, using an argument similar to the derivation of (4.25), equilibrium dynamics with

asset creation is characterized by the following difference equations:

$$y_{t+1} = \frac{1}{G}\frac{\phi\beta}{1-\beta+\phi\beta}\left(y_t(f'(y_t)+1-\delta)+\frac{DX_t}{G^t}\frac{1}{G_x}\right),$$
$$\frac{DX_{t+1}}{G^{t+1}} = \frac{G_x}{G}\frac{DX_t}{G^t}.$$

It is clear that if $G > G_x$, Proposition 4.5 remains valid.

We can also learn that in the long run, the effect of asset creation on the dynamics of y_t will be negligible, even though the growth rate of asset creation will affect the condition of bubbles, i.e., excessive asset creation (G_x becomes too high) will lead to a collapse of bubbles, which is consistent with an empirical finding by Scheinkman (2014, p. 17) that too much supply of assets leads to an implosion of bubbles.

4.6   Comparison to Hirano, Jinnai, and Toda (2022) model

We conclude this section with a comparison of the current model (which we refer to as "Model X" for brevity because growth is exogenous) to that of Hirano, Jinnai, and Toda (2022) (which we refer to as "Model N" because growth is endogenous). Although the two models share many similarities such as log utility, idiosyncratic investment risk, and leverage constraint, there are crucial differences both in terms of model assumptions and predictions.

Regarding assumptions, there are two differences. First, Model X features labor with exogenous productivity growth, whereas Model N features no labor but growth is endogenously determined by the interaction of the land and tech sectors through leverage. Second, Model X has two productivity types for entrepreneurs, whereas Model N has a continuum of types. We shall see that these differences in assumptions affect the predictions.

Regarding predictions, there are three differences. First, the bubble existence condition in Model X is unrelated to the leverage λ (as long as $\lambda < 1/\pi$ to rule out the uninteresting frictionless case[8]), whereas bubbles emerge in Model N if and only if λ exceeds some threshold. In other words, relaxing financial conditions makes bubbles inevitable in Model N, whereas financial conditions play no role in Model X. To understand this difference, recall that a sufficient condition for the emergence of bubbles is $R_f < G_d < G$, where R_f is the fundamental equilibrium interest rate, G_d is the dividend growth rate, and G is the economic growth rate.

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[8]In Model N, there is no need to consider the case $\lambda \geq 1/\pi$ because there is no point mass at the top of the productivity distribution and therefore $\pi = 0$.

In Model X, R_f is pinned down by the return from storage $1 - \delta$ and the growth rates G, G_d are exogenous. Therefore there is no room for financial conditions to affect the existence of bubbles.

Second, in Model N, wealth inequality is higher (Pareto exponent is lower) with asset price bubbles, while the opposite is true in Model X (Proposition 4.4). The intuition for this difference is the following. In an exogenous growth model, the equilibrium return on capital investment is essentially exogenous: see the first term in (4.21). Therefore the bubble only increases the interest rate and allow low productive agents to catch up, which reduces wealth inequality. In contrast, in an endogenous growth model, as the interest rate rises with an asset price bubble, capital is allocated to even more productive agents. Thus even though the low productive agents are catching up, the marginal investor becomes more productive, so wealth inequality actually goes up.

Third, in Model N, the wealth Pareto exponent depends on leverage λ, which is not the case in Model X. This difference is an artifact of assuming only two productivity types in Model X. With only two types and exogenous growth G and return from storage $1 - \delta$, the return distribution is entirely determined by exogenous parameters and leverage plays no role. In this sense, although assuming two types may seem innocuous, we need to be careful about drawing implications when the number of types is small and the number of exogenous parameters is large.

5 Further applications and related literature

In this section, we provide further applications of equilibrium selection by dividend injection and discuss the related literature.

5.1 Stochastic bubbles (Weil, 1987)

So far we have exclusively focused on perfect foresight (deterministic) equilibria but we can apply similar arguments to stochastic equilibria. As an illustrative example, consider the stochastic bubble model of Weil (1987). This model is similar to that of Tirole (1985) discussed in Section 3.3 except that agents rationally expect the bubble to pop (permanently become worthless) with some probability each period. Since this model reduces to the perfect foresight economy once the bubble asset becomes worthless, under the low interest condition we can eliminate this equilibrium by dividend injection. Consequently, stochastic bubbles due to

sunspots can be ruled out.

5.2 Public debt as private liquidity (Woodford, 1990)

Woodford (1990) develops a model with infinitely lived agents in which agents have investment opportunities every other period, private borrowing is impossible, and the only means of saving is government bonds backed by taxes. In this setting, he showed that government bonds may crowd in capital investments, instead of crowding them out. This model is a variant of Bewley (1980)'s model with growth and production. It has subsequently been applied to pure bubble models by replacing government bonds with pure bubble assets, including Kocherlakota (2009), Brunnermeier and Sannikov (2016), Hirano and Yanagawa (2017), and Kiyotaki and Moore (2019). Since the mathematical argument for equilibrium selection is similar to that in Section 3.4, we omit the details.

5.3 Models with bubble creation

Martin and Ventura (2012) develop a two-period overlapping generations model assuming that private borrowing is impossible and new (young) agents with investment opportunities can create pure bubble assets in every period. This assumption of liquidity creation by liquidity-constrained agents increases their wealth directly, generating a crowd-in effect on investments. Without the assumption, there is no crowd-in effect and there is only crowd-out effect, as in Tirole (1985). Note that the expansionary effect of pure bubble creation on the macro economy is essentially similar to that in Kocherlakota (2009), who formalizes the effect in a model in which a pure bubble asset (land) is used as collateral. Since the mathematical structure with asset creation in a two-period OLG model is similar to that of the perpetual youth model we have explored in Section 4.5, we omit the details.

5.4 Models of bubble as a safe asset

There are several economic models in which a pure bubble asset circulates in the presence of uninsurable idiosyncratic risk. Kitagawa (1994, 2001) showed in a two-period OLG economy that fiat money can circulate as a safe asset when agents are subject to idiosyncratic storage risk. Aoki et al. (2014) extend this model to an infinite-horizon setting with general shocks. Brunnermeier and Sannikov (2016) also model fiat money as self-insurance against idiosyncratic investment risk.

We briefly describe the model of Aoki et al. (2014). There are a continuum of agents with logarithmic utility (4.1). Investing k_t units of good at time t yields an output of $y_{t+1} = z_{t+1}k_t$ at time $t + 1$, where the productivity $z_t > 0$ is IID across agents and time. The agents can also trade a risk-free asset in zero net supply. Let $w_t > 0$ be the wealth of a typical agent at the beginning of time t. Given the risk-free rate R, due to logarithmic utility and IID shocks, the optimal consumption rule is $c_t = (1 - \beta)w_t$ and the optimal portfolio problem reduces to

$$\max_{\eta \geq 0} \mathrm{E}[\log(z\eta + R(1 - \eta))], \tag{5.1}$$

where $\eta \geq 0$ is the fraction of wealth invested in the technology. Using these rules and aggregating individual wealth, we obtain the system of equations

$$G(R) = \beta \, \mathrm{E}[z\eta(R) + R(1 - \eta(R))] \quad \text{and} \quad s(R) = 1 - \eta(R)$$

in (3.1), where $\eta(R) \geq 0$ is the (unique) maximizer of (5.1).

This model is a special case of Toda (2014) with log utility, one technology, and IID shocks. Because agents make the same portfolio choice due to homotheticity, a risk-free asset in zero net supply is useless for risk sharing (Toda, 2014, Proposition 10). However, as Aoki et al. (2014) show, the introduction of a pure bubble asset may improve risk sharing and hence welfare because it is in positive net supply.

The fundamental equilibrium condition $s(R) = 0$ implies $\eta(R) = 1$. Taking the first-order condition of (5.1) at $\eta = 1$, the fundamental equilibrium interest rate satisfies

$$0 = \mathrm{E}\left[\frac{z - R}{z}\right] \iff R_f = 1/\mathrm{E}[1/z].$$

The low interest condition is therefore

$$G(R_f) > R_f \iff \beta \, \mathrm{E}[z] > 1/\mathrm{E}[1/z] \iff \beta \, \mathrm{E}[z] \, \mathrm{E}[1/z] > 1, \tag{5.2}$$

which is exactly the bubble existence condition in Aoki et al. (2014).[9] Because $\eta = 0$ whenever $R > \mathrm{E}[z]$ due to concavity, it follows that $G(R) = \beta R < R$ if $R > \mathrm{E}[z]$. Thus all assumptions of Theorem 3.1 are satisfied, and there exists a bubbly equilibrium with $G(R_b) = R_b$, which is condition (39) in Aoki et al. (2014).

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[9]The productivity z in our notation corresponds to $A\theta$ in Aoki et al. (2014), where $A > 0$ is a constant and they normalize $\mathrm{E}[\theta] = 1$. Therefore (5.2) is equivalent to $\mathrm{E}[\beta/\theta] = 1$, which is their Equation (41).

5.5 Related literature

To the best of our knowledge, there are only two papers that study bubbles attached to dividend-paying assets.[10] One is Proposition 1(c) of Tirole (1985), which introduces a dividend-paying asset in an overlapping generations model. Although Tirole (1985) recognized the possibility of the nonexistence of fundamental equilibria, he did not necessarily provide a formal proof nor investigate the dynamic stability of the bubbly equilibrium. We provide a proof and a sufficient condition for the local determinacy of the bubbly equilibrium in Section 3.3.

The other paper that studies rational asset price bubbles attached to a dividend-paying asset is Hirano, Jinnai, and Toda (2022). They show that there are plausible economic models with a bubbly equilibrium but without any fundamental equilibria, implying the necessity of bubbles. These models feature multiple sectors with different output elasticities. In particular, they present an endogenous growth model with entrepreneurs and show the emergence of asset price bubbles with lax leverage. We present a similar model but with exogenous growth that builds on the model of Kocherlakota (2009) in Section 4. Relative to these papers, our contribution is that we use (asymptotically negligible) dividends as an equilibrium selection mechanism and a theoretical foundation of pure bubble models.

Our idea of introducing small positive dividends to select an equilibrium is conceptually analogous to the equilibrium refinement literature in game theory (see van Damme (1987) for a review) and monetary economics. For instance, in both the perfect equilibrium of Selten (1975) and the proper equilibrium of Myerson (1978), players assign infinitesimally small but positive probabilities to all pure strategies that are not best responses. To rule out hyperinflationary equilibria in OLG models with money, Brock and Scheinkman (1980) discuss the possibility of introducing small mandatory savings (social security) or small utility from holding money.

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[10]As discussed in the introduction, the existing literature on rational asset price bubbles has almost exclusively focused on pure bubbles. See Galí (2021), Plantin (2021), Kocherlakota (2022), and Guerron-Quintana et al. (2022) for the recent development of pure bubble models. Allen et al. (2022) studied a risk-shifting model in which there is asymmetric information between lenders and borrowers. This asymmetric information encourages borrowers to gamble on risky assets, allowing asset prices to exceed fundamentals. Although their approach is different from ours because we focus on rational bubbles and abstract from asymmetric information, the two different approach should be viewed as complementary, not as a substitute.

6 Conclusion

If a model does not make determinate predictions, its value as a tool for economic analysis is severely limited. Although the equilibrium indeterminacy in pure bubble models has been recognized for decades, the literature has focused on only one of a continuum of bubbly equilibria corresponding to a saddle path or a steady state, with no scientific basis.[11] We have challenged this long-standing unsolved problem in the literature and proposed a solution. Our proposed idea of dividend injection not only places the existing models of pure bubbles on firmer footing but also makes applications to policy and quantitative analysis possible in a robust way. Furthermore, our idea escapes the criticism that it is unrealistic to use pure bubble assets without dividends to describe real world bubbles attached to land, housing, and stocks.

A Proofs

Proof of Proposition 2.2. The initial old's consumption $c_0^o = b + P_0$ is strictly increasing in P_0. Let us show that the equilibrium utility of generation t is strictly increasing in P_0, which implies that $E(P_0')$ Pareto dominates $E(P_0)$ if $P_0 < P_0'$. Using $x_t = G^{-t} P_t$ and (2.5), the equilibrium utility of generation t is

$$\log(aG^t - P_t) + \beta \log(bG^{t+1} + P_{t+1})$$
$$= t \log G + \log(a - x_t) + \beta(t+1)\log G + \beta \log\left(b + \frac{bx_t}{\beta a - (1+\beta)x_t}\right)$$
$$= (1+\beta)\log(a - x_t) - \beta \log(\beta a - (1+\beta)x_t) + \text{constant},$$

where "constant" is a term independent of x_t. Let

$$f(x) = (1+\beta)\log(a - x) - \beta \log(\beta a - (1+\beta)x).$$

Then $0 < x < \frac{\beta}{1+\beta}a$, we have

$$f'(x) = -\frac{1+\beta}{a-x} + \frac{\beta(1+\beta)}{\beta a - (1+\beta)x} = \frac{(1+\beta)x}{(a-x)(\beta a - (1+\beta)x)} > 0.$$

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[11] Awaya et al. (2022) obtain a unique bubbly equilibrium but their model is based on information frictions and is not a macroeconomic model with investment and production, which is quite different from standard macroeconomic models with pure bubbles and rational expectations.

Since x_t is strictly increasing in $x_0 = P_0$ by (2.6), it follows that the equilibrium utility of generation t is strictly increasing in P_0.

In the bubbly equilibrium $E\left(\frac{\beta a - b}{1+\beta}\right)$, we have $R = G$. Its Pareto efficiency follows from detrending the economy and applying Proposition 5.6 of Balasko and Shell (1980). $\qquad\square$

Proof of Proposition 2.3. The proof of (i) is technical and similar to that of Proposition 2.2 of Hirano, Jinnai, and Toda (2022), so we omit.

We prove (ii). Using the definition of h in (2.11) and x_b^* in (2.12), we obtain the Jacobian

$$Dh(x_b^*) = \begin{bmatrix} \frac{\beta a}{b} & -\frac{G_d}{G} \\ 0 & \frac{G_d}{G} \end{bmatrix}.$$

Since by assumption $\beta a > b$ and $G_d < G$, the eigenvalues of $Dh(x_b^*)$ are $\beta a / b > 1$ and $G_d / G \in (0,1)$. Therefore x_b^* is a saddle point. The local stable manifold theorem (see Irwin (1980, Theorems 6.5 and 6.9) and Guckenheimer and Holmes (1983, Theorem 1.4.2)) implies that for any sufficiently small $x_{20} > 0$ (hence sufficiently small $D_0 > 0$), there exists a unique orbit $\{x_t\}_{t=0}^\infty$ converging to x_b^*. Since the difference equation (2.10) is autonomous (does not explicitly depend on time), the same argument applies if we start the economy from any $t = T$. Therefore for sufficiently small $x_{2T} > 0$ (hence sufficiently small $D_0(G_d/G)^T$), there exists a unique orbit $\{x_t\}_{t=T}^\infty$ converging to x_b^*. Since $G_d < G$, it follows that for large enough $T > 0$, there exists a unique orbit $\{x_t\}_{t=T}^\infty$ converging to x_b^*. But clearly (2.10) can be solved backwards because (2.10b) is linear and (2.10a) can be solved for x_t as

$$x_{1t} = \frac{\beta a}{b}\left(\frac{1}{x_{1,t+1} + \frac{G_d}{G}x_{2t}} + \frac{1+\beta}{b}\right)^{-1}.$$

Therefore by backward induction, there exists a unique orbit $\{x_t\}_{t=0}^\infty$ converging to x_b^* and the equilibrium is locally determinate. $\qquad\square$

Proof of Theorem 3.2. The nonexistence of an equilibrium path converging to x_f^* follows from the same argument as in the proof of Proposition 2.3. To prove the local determinacy of x_b^*, we apply the implicit function theorem and the local stable manifold theorem. We divide the rest of the proof in several steps.

Step 1. Calculation of the Jacobian.

Using the definition of H in (3.7) and $G = f'(k_b)$, we obtain

$$D_x H(x_b^*) = \begin{bmatrix} 0 & -1 & 0 \\ c_1^y - \omega' & 1 & 0 \\ 0 & 0 & -G_d/G \end{bmatrix}, \qquad D_y H(x_b^*) = \begin{bmatrix} -\frac{k_b}{G} f''(k_b) & 1 & 1 \\ G + c_2^y & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Suppose for the moment that

$$G + c_2^y = G + \partial c^y / \partial k' > 0. \tag{A.1}$$

Since $D_y H(x_b^*)$ is block upper triangular, we have $\det(D_y H(x_b^*)) = -G - c_2^y < 0$ by (A.1). Therefore applying the implicit function theorem, we can solve $H(x, y) = 0$ locally as $y = h(x)$. To simplify notation, let $p = \omega' - c_1^y$, $q = G_d/G$, $r = -\frac{k_b}{G} f''(k_b)$, and $s = G + c_2^y$. Then $s > 0$ by (A.1) and

$$Dh(x_b^*) = -[D_y H(x_b^*)]^{-1} D_x H(x_b^*) = \begin{bmatrix} p/s & -1/s & 0 \\ -pr/s & 1 + r/s & -q \\ 0 & 0 & q \end{bmatrix}.^{12}$$

Step 2. If (A.1) holds, then $p > 0$, $q \in (0, 1)$, and $s > 0$.

Since $R_f < G_d < G$, we have $q \in (0, 1)$. Since $f'' < 0$, we have $r > 0$. Let us show $p > 0$. Noting that $\omega(k) = f(k) - kf'(k)$, we have $\omega'(k) = -kf''(k) > 0$. Therefore it suffices to show $c_1^y < 0$. Substituting the budget constraint into the utility function, the utility maximization reduces to

$$\max_{c^y} U(c^y, f'(k')(\omega(k) - c^y)).$$

The first-order condition is

$$U_1 - f'(k')U_2 = 0 \iff M(c^y, f'(k')(\omega(k) - c^y)) - f'(k') = 0, \tag{A.2}$$

where $M := U_1/U_2$ is the marginal rate of substitution. Applying the implicit function theorem, we obtain

$$\frac{\partial c^y}{\partial k} = -\frac{M_1 f'(k')\omega'(k)}{M_1 - f'(k')M_2}, \tag{A.3a}$$

$$\frac{\partial c^y}{\partial k'} = -\frac{(M_2(\omega(k) - c^y) - 1)f''(k')}{M_1 - f'(k')M_2}. \tag{A.3b}$$

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[12] We have calculated this matrix using MATHEMATICA to avoid human errors.

The quasi-concavity of U implies $M_1 < 0$ and $M_2 > 0$. Since $f' > 0$ and $\omega' > 0$, we obtain $c_1^y = \partial c^y/\partial k < 0$.

Step 3. If (A.1) holds, then $Dh(x_b^*)$ has one eigenvalue in $(1, \infty)$ and two eigenvalues in $(0, 1)$ and the bubbly equilibrium is locally determinate.

Since $Dh(x_b^*)$ is block upper triangular, one eigenvalue is $q \in (0, 1)$. The other two eigenvalues are those of the first 2×2 block. Its characteristic function is

$$\Phi(z) = \det \begin{bmatrix} z - p/s & 1/s \\ pr/s & z - 1 - r/s \end{bmatrix}$$
$$= (z - p/s)(z - 1 - r/s) - pr/s^2.$$

Then $\Phi(0) = p/s > 0$ and $\Phi(1) = -r/s < 0$, so one eigenvalue is in $(0, 1)$ and the other is in $(1, \infty)$.

Since $x_0 = (x_{10}, x_{20}, x_{30}) = (k_0, P_0/N_0, D_0/N_0)$, the number of predetermined variables (x_{10} and x_{30}, so two) equals the number of eigenvalues of $Dh(x_b^*)$ less than 1 in absolute value. Therefore by the local stable manifold theorem, x_b^* is locally determinate.

Step 4. (A.1) and (3.9) are equivalent.

To complete the proof of Theorem 3.2, it remains to show the equivalence of (3.9) and (A.1). Let $\sigma = 1/\varepsilon$ and $M = U_1/U_2$. Setting $t = c_1/c_2$, it follows from the definition of the elasticity of intertemporal substitution (3.8) that

$$\sigma = -t \frac{\partial \log M(tc_2, c_2)}{\partial t} = -tc_2 \frac{M_1}{M} = -\frac{c^y}{f'} M_1, \qquad \text{(A.4)}$$

where we have used (A.2). Similarly,

$$\sigma = -t \frac{\partial \log M(c_1, c_1/t)}{\partial t} = \frac{c_1}{t} \frac{M_2}{M} = (\omega - c^y) M_2. \qquad \text{(A.5)}$$

Therefore using (A.4) and (A.5) to eliminate M_1, M_2 from (A.3b) and the bubbly steady state condition $G = f'(k_b)$, we obtain

$$\text{(A.1)} \iff G - \frac{(\sigma - 1)f''}{-\sigma \frac{f'}{c^y} - f' \frac{\sigma}{\omega - c^y}} > 0$$
$$\iff \sigma \frac{G^2}{f''} \left(\frac{1}{c^y} + \frac{1}{\omega - c^y} \right) + \sigma - 1 < 0$$
$$\iff \varepsilon = \frac{1}{\sigma} > 1 + \frac{G^2}{f''} \frac{\omega}{c^y(\omega - c^y)} \iff \text{(3.9)}. \qquad \square$$

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Proof of Theorem 4.3. We have already shown all claims in the main text except that the entrepreneurs are better off in the bubbly equilibrium. Since agents have log utility, the value function of the agents (which is a measure of welfare) is a weighted sum of log initial wealth and expected log return on wealth. (See, for instance, the proof of Proposition B.2 of Hirano, Jinnai, and Toda (2022).) Hence to show that entrepreneurs are better off in the bubbly equilibrium, it suffices to show that both the log initial wealth and the expected log return on wealth increase.

If there are no bubbles, land is worthless, whereas with bubbles, land has positive value. Therefore the initial wealth of agents is higher in the bubbly equilibrium because agents are endowed with nonnegative land.

Let R_j^H be the return on wealth for an H agent in the equilibrium E_j defined by (4.21). Letting $x = R_j$ be the equilibrium risk-free rate, the expected log return on wealth can be written as

$$g(x) := \pi \log \left(\frac{G}{\pi\beta} + \left(1 - \frac{1}{\pi} \right) x \right) + (1 - \pi) \log x.$$

Differentiating this and rearranging terms, we obtain

$$g'(x) = \frac{\pi - 1}{\frac{G}{\pi\beta} + (1 - 1/\pi)x} + (1 - \pi)\frac{1}{x} = \frac{1 - \pi}{\pi} \frac{G/\beta - x}{x \left(\frac{G}{\pi\beta} + (1 - 1/\pi)x \right)} > 0$$

because $x \le G < G/\beta$. Therefore the entrepreneur welfare is increasing with the risk-free rate, and since $R_b > R_f$, entrepreneurs are strictly better of in the bubbly equilibrium. □

Proof of Proposition 4.4. We only provide a sketch since the proof is similar to that in Hirano, Jinnai, and Toda (2022). According to the Pareto exponent formula for Markov multiplicative processes with ɪɪᴅ shocks (Beare and Toda, 2022, Equation (2)), the Pareto exponent is determined by $v \, \mathrm{E}[G_s^\zeta] = 1$, where G_s is the random gross growth rate of the relative size s_t. The formula (4.23) immediately follows by setting G_s to the right-hand side of (4.22).

Let us show that $\zeta(R)$ is above 1 and is increasing in R. To this end, let $r = \beta R / G$ and

$$\Psi(z, r) := \pi \left(\frac{1}{\pi} + \left(1 - \frac{1}{\pi} \right) r \right)^z + (1 - \pi)r^z - \frac{1}{v}.$$

Since $R \leq G$, we have $r = \beta R/G \leq \beta < 1$. Therefore

$$\frac{1}{\pi} + \left(1 - \frac{1}{\pi}\right) r > \frac{1}{\pi} + \left(1 - \frac{1}{\pi}\right) = 1 > r \tag{A.6}$$

and $\Psi(1, r) = 1 - 1/v < 0 < \infty = \Psi(\infty, r)$. The intermediate value theorem implies that there exists $z = \zeta \in (1, \infty)$ such that $\Psi(\zeta, r) = 0$. Since Ψ is a weighted sum of exponential functions of z, it is strictly convex in z. Therefore ζ is unique and $(\partial \Psi / \partial z)(\zeta, r) > 0$ by the Proof of Proposition 5 of Beare and Toda (2022). Furthermore,

$$\frac{\partial \Psi}{\partial r}(\zeta, r) = \zeta(\pi - 1)\left(\left(\frac{1}{\pi} + \left(1 - \frac{1}{\pi}\right) r\right)^{\zeta-1} - r^{\zeta-1}\right) < 0$$

by (A.6) and $\zeta > 1$. Therefore the implicit function theorem implies $\partial \zeta / \partial r = -(\partial \Psi / \partial r)/(\partial \Psi / \partial z) > 0$, so ζ is strictly increasing in R. $\square$

Proof of Proposition 4.5. The nonexistence of an equilibrium converging to the fundamental balanced growth path follows by the same argument as in Proposition 2.3.

To prove the local determinacy of the bubbly steady state, define the auxiliary variables

$$\xi_t = (\xi_{1t}, \xi_{2t}) = \left(y_t, \frac{\phi\beta}{1 - \beta + \phi\beta} DG^{-t-1}\right).$$

Then (4.25) can be rewritten as $\xi_{t+1} = h(\xi_t)$, where $h : \mathbb{R}_{++}^2 \to \mathbb{R}^2$ is defined by

$$h_1(\xi_1, \xi_2) = \frac{1}{G} \frac{\phi\beta}{1 - \beta + \phi\beta} \xi_1(f'(\xi_1) + 1 - \delta) + \xi_2,$$
$$h_2(\xi_1, \xi_2) = \frac{1}{G}\xi_2.$$

Let $\xi^* = (y_b, 0)$ be the bubbly steady state, where y satisfies (4.20a). Using the steady state condition (4.20a), a straightforward calculation yields the Jacobian

$$Dh(\xi^*) = \begin{bmatrix} \frac{1}{G} \frac{\phi\beta}{1-\beta+\phi\beta}(f'(y_b) + 1 - \delta + y_b f''(y_b)) & 1 \\ 0 & 1/G \end{bmatrix}$$
$$= \begin{bmatrix} 1 + \frac{1}{G} \frac{\phi\beta}{1-\beta+\phi\beta} y_b f''(y_b) & 1 \\ 0 & 1/G \end{bmatrix}.$$

Therefore the eigenvalues of $Dh(\xi^*)$ are

$$\lambda_1 = 1 + \frac{1}{G} \frac{\phi\beta}{1 - \beta + \phi\beta} y_b f''(y_b) \tag{A.7}$$

and $\lambda_2 = 1/G \in (0, 1)$. Since the initial condition (k_0, D) is exogenous, the steady state is locally determinate if $|\lambda_1| < 1$. Since $f'' < 0$, we have $\lambda_1 < 1$ by (A.7). Therefore it suffices to check $\lambda_1 > -1$, which is equivalent to (4.27).

Finally, let us show the equivalence of (4.27) and (4.28). Setting $y = K/L$ and using $F_K = f'(y)$ and $F_L = f(y) - yf'(y)$ in (4.26), a straightforward calculation yields

$$\frac{1}{\varepsilon} = -y \frac{f f''}{f'(f - yf')}.$$

Therefore

$$yf''(y) = -\frac{1}{\varepsilon} \frac{f'\omega}{yf' + \omega} = -\frac{1}{\varepsilon} \frac{f'}{1 + yf'/\omega},$$

where we have used $\omega = f - yf'$. Substituting this expression into (4.27) and using the steady state condition (4.20a), we obtain (4.28). $\qquad\square$

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