The Canon Institute for Global Studies

CIGS Working Paper Series No. 23-001E

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2023.1

General Incorporated Foundation
The Canon Institute for Global Studies
一般財団法人　キヤノングローバル戦略研究所
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Debt-Ridden Borrowers and Persistent Stagnation

Keiichiro Kobayashi and Daichi Shirai *

January 2023 (Substantial revision. First version: March 2012)

Abstract

Persistent stagnation often follows a financial crisis. We construct a model in which a debt buildup in the corporate sector can persistently depress the economy, even when there are no structural changes. We consider endogenous borrowing constraints on short-term and long-term debt. A firm is referred to as *debt-ridden* when its long-term debt is so large that it can never decrease even though the firm pays all income in each period to the lender. A debt-ridden firm continues inefficient production permanently, and the emergence of a substantial number of debt-ridden firms causes a persistent recession. Further, if the initial debt exceeds a certain threshold, the firm intentionally chooses to increase borrowing and, thus, becomes debt-ridden. We numerically show successive productivity shocks or a large wealth shock can generate debt-ridden firms. The relief of debt-ridden borrowers from excessive debt may be effective for economic recovery.

JEL Classification Numbers: E30, G01, G30

Keywords: borrowing constraint, debt overhang, secular stagnation, nonlinear solution methods.

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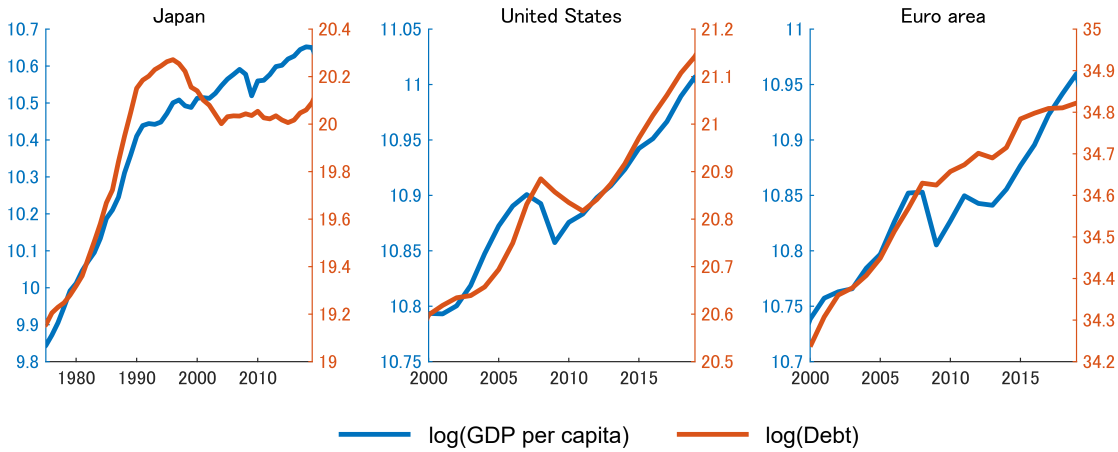
Previous versions of this paper were circulated under the title "Debt-Ridden Borrowers and Productivity Slowdown" and "Debt-Ridden Borrowers and Economic Slowdown." The model is substantially revised from those in the previous versions. We thank Tomoyuki Nakajima for their helpful comments and fruitful discussions on this project and other related topics. We also thank Kosuke Aoki, Hiroki Arato, R. Anton Braun, Yasuo Hirose, Selahattin Imrohoroglu, Masaru Inaba, Nobuhiro Kiyotaki, Huiyu Li, Daisuke Miyakawa, Tsutomu Miyagawa, Kengo Nutahara, Ryoji Ohdoi, Vincenzo Quadrini, Yosuke Takeda, Hajime Tomura, Ichihiro Uchida, Yuichiro Waki and the seminar participants at the CIGS Conference on Macroeconomic Theory and Policy 2012, the 2012 JEA autumn meeting, Aichi University, the 2012 GRIPS macro conference, the Summer Workshop on Economic Theory (SWET 2015) at the Otaru University of Commerce, the 11th Dynare conference, the 2015 JEA autumn meeting, 2016 AMES (Kyoto), 2016 ESEM (Geneva), the 18th Macro Conference (Osaka), 2017 CEF (New York), and 2017 EEA (Lisbon) for their insightful comments and valuable discussions. Kobayashi gratefully acknowledges the financial support from the JSPS Grant-in-Aid for Scientific Research (C) (No. 23330061) for the project "Analysis of dynamic models with systemic financial crises." Shirai gratefully acknowledges the financial support from JSPS KAKENHI Grant Number 19K13690. All remaining errors are ours.

1 Introduction



Sources: BIS total credit statistics; OECD quarterly national accounts; World bank, *World Development Indicators*;

Figure 1: Non-financial corporate debt and GDP per capita

The decade after a financial crisis tends to be associated with low economic growth (Cerra and Saxena, 2008; Reinhart and Rogoff, 2009; Reinhart and Reinhart, 2010). Cerra and Saxena (2008) show that the economy tends to stagnate for an extended period in the aftermath of banking and/or currency crises. Reportedly, financial constraints were tightened both during and after the Global Financial Crisis (GFC). See , for example, Altavilla, Darracq Paries and Nicoletti (2015). However, which factors caused the tightening of these financial constraints and whether this tightening can cause a persistent slowdown in economic growth remain unclear.

The data from the developed economies shows that buildups of debt are associated with recessions. Figure 1 shows the buildup of corporate debt and the GDP per capita in Japan, the United States, and the Euro area.

In the 1990s in Japan, the slowdown of the GDP per capita is associated with the plateau state of corporate debt. The slowdown of GDP in the United States in the aftermath of the GFC in 2008–2009 seems to be associated with higher growth of debt. The GDP in the Euro area did not return to the pre-crisis trend after the GFC, which is also associated with a high level of debt.

Recent empirical studies also show that sizeable corporate debt negatively affects GDP growth (e.g., Cecchetti, Mohanty and Zampolli, 2011; Mian, Sufi and Verner, 2017). Giroud and Mueller (2017) find that the establishments of more highly leveraged firms experienced more significant employment losses during and after the GFC in the United States. Duval, Hong and Timmer (2017) also show that highly leveraged firms experienced significant and persistent drops in total factor productivity (TFP) growth in the aftermath of the GFC.

This study proposes a theoretical model in which the buildup of debt induces an endogenous tightening of the borrowing constraints and prolongs stagnation persistently in a stochastic economy where the TFP shocks or wealth shocks hit the firms. Our theory demonstrates that inefficiency due to

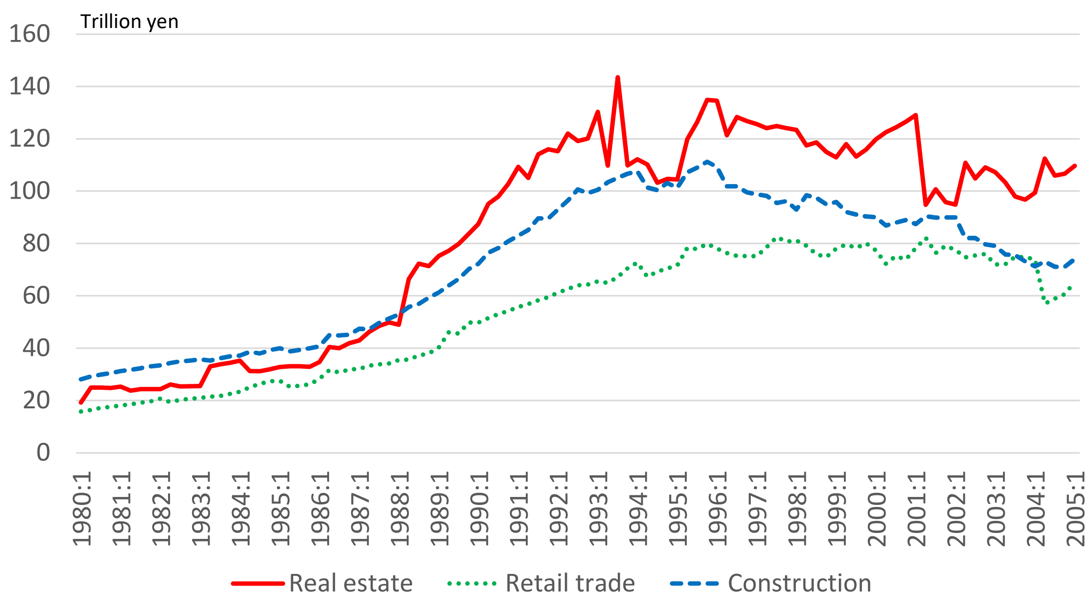


Figure 2: Debt by industries in Japan

Note: The data are seasonally adjusted by X12-ARIMA.

Source: Ministry of Finance, Policy Research Institute, *Financial Statements Statistics of Corporations by Industry*

the buildup of debt can continue persistently, which is consistent with the debt supercycle hypothesis (Rogoff, 2016; Lo and Rogoff, 2015). Our model also shows a theoretical possibility that the borrowers may intentionally choose to increase debt and stay debt-ridden for an extended period when the initial debt exceeds a certain threshold.

Our theory may be able to explain the observations of corporate borrowers in the 1990s in Japan that increased their borrowings to exceed repayable amounts and stayed debt-ridden for a decade. The cases of Sogo and Daiei, two giants of Japanese general merchandise stores, are typical examples. Although the real estate bubble collapsed at the beginning of the 1990s, Sogo and Daiei doubled their borrowings and bought more land in the mid-1990s when they were already known as overly indebted. Sogo finally went bankrupt in 2000 with total debt of JPY 1,870 billion. Daiei was de facto nationalized in 2004 with a debt of JPY 1,630 billion. Our theory may be useful to understand the cases of debt-ridden "industries" in Japan that increased their debt after the economy went down at the peak of February 1991. Figure 2 shows that the debt in the sectors of real estate, retail trade, and construction increased in 1992–93 and stayed high for almost a decade.

Our theoretical contribution to the literature is to show that a buildup of debt can persistently tighten borrowing constraints and cause the aggregate inefficiency that can continue indefinitely. Thus, our theory provides a rationale for heterodox policy recommendations for the government interventions that facilitate partial debt forgiveness in the private sector (see Geanakoplos, 2014). In standard models of financial friction such as Carlstrom and Fuerst (1997) and Bernanke, Gertler and Gilchrist (1999), the buildup of debt generates inefficiency only for a few periods. Jermann and Quadrini (2012) and Albuquerque and Hopenhayn (2004) show in their models of long-term debt that inefficiency due to the buildup of debt can continue for finite periods. Our result that inefficiency can continue indefinitely thus contrasts sharply with the findings in the literature and suggests new causality from a financial crisis to persistent stagnation in the post-crisis period.

Our model of financial contracts has endogenous borrowing constraints that arise because of borrowers' lack of commitment, and lenders can choose whether to liquidate defaulting firms or forgive them. The market is incomplete, and debt and equity are the only available financial instruments. Firms cannot relax the borrowing constraints by raising funds from external investors because of market frictions that prevent them from issuing new equity quickly. There is a distinction between inter-period and intra-period loans in this economy. The borrowing constraint binds more tightly as the initial amount of inter-period debt is larger. As the borrowing constraint tightens, firms cannot raise sufficient intra-period debt for working capital, which leads to inefficient production. When the amount of inter-period debt exceeds the maximum repayable amount, firms fall into a *debt-ridden* state in which they can repay no more than the interest payments even though they pay all of their income in each period. As a result, the amount of debt does not decrease. Therefore, debt-ridden firms continue inefficient production permanently. Moreover, when the debt exceeds a certain threshold, a firm may choose to increase borrowing and intentionally become debt-ridden because the gain from additional borrowing can exceed the inefficiency of the additional tightening of the borrowing constraint. This result implies that an overly indebted firm may rationally choose to become and then stay debt-ridden. Although our model is a simple modification of that of Jermann and Quadrini (2012), there is a significant difference in that the debt-ridden state arises naturally in our model. By contrast, it does not exist in Jermann and Quadrini (2012). This distinction is due to a difference in settings that a portion of output can serve as the collateral for borrowing in our model, whereas it cannot in Jermann and Quadrini's model.

In our model, persistent inefficiency is not caused by permanent changes in structural parameters, whereas in existing models, persistent recessions are usually caused by persistent changes in parameters. See, for example, Christiano, Eichenbaum and Trabandt (2015) and Bianchi, Kung and Morales (2019) for the GFC, Cole and Ohanian (2004) for the Great Depression, and Kaihatsu and Kurozumi (2014a) for the lost decade of Japan. Several authors have argued that persistent shocks that cause persistent recessions are exogenous changes in the structural parameters, such as the risk shock in Christiano, Motto and Rostagno (2014) and the financial shock in Jermann and Quadrini (2012). In this study, we consider temporary shocks on the TFP, whereas there is no change in the parameters. In our model, the TFP evolves with the Markov process, and the debt builds up if the low productivity continues for an extended period.

Our model implies a policy recommendation distinct from most existing models in which exogenous shocks on the structural parameters cause persistent recessions. The policymaker can only mitigate these shocks by conducting accommodative monetary and fiscal policies or designing ex-ante financial regulations. In our model, debt restructuring or debt forgiveness for overly indebted borrowers restores aggregate efficiency and enhances economic performance. Restoring economic efficiency does not necessitate the physical liquidation of debt-ridden firms but rather their relief from excessive debt. This argument is in line with the policy recommendations of partial debt forgiveness by Geanakoplos (2014).

1.1 Intuition on the borrowing constraint

To illustrate how persistent inefficiency can arise in our model, let us consider a simple model of a firm that produces output $f(q_t)$ from input q_t in period t, where $f'(q_t) > 0$ and $f''(q_t) < 0$. The first-best solution that maximizes the social surplus, $f(q_t) - q_t$, is attained by q^*, where q^* solves $f'(q_t) = 1$. In period t, this firm initially holds debt b_{t-1}. The firm repays b_{t-1} and then borrows new debt $\frac{b_t}{R_t}$, where R_t is the loan rate. Thus, the firm's dividend is

$$\pi_t = f(q_t) - q_t - b_{t-1} + \frac{b_t}{R_t}.$$

Given b_{t-1}, the firm chooses b_t such that $\pi \geq 0$, that is, the dividend π_t cannot be negative. The firm chooses q_t and b_t to maximize the present discounted value of future dividends, that is, $\mathbb{E}_t\left[\sum_{j=0}^{\infty} \beta^{t+j}\pi_{t+j}\right]$, where $\mathbb{E}_t$ is the expectation as of period t and β is the time discount factor, subject to the non-negativity constraint, $\pi_t \geq 0$, and the borrowing constraint on the working capital q_t:

$$q_t \leq \phi f(q_t) + \max\left\{\xi S_t - \frac{b_t}{R_t},\ 0\right\},$$

where ϕ and ξS_t are exogenously given to satisfy $0 \leq \phi < 1$ and $\xi S_t > 0$. This borrowing constraint is formally introduced in the model of Section 2. As we argue in Section 2, it is a version of the earnings-based borrowing constraint (Drechsel, 2022; Lian and Ma, 2021) , which is widely observed in the U.S. and other developed economies. The appearance of the max operator is a unique feature of this borrowing constraint, which is derived in the Appendix A as a no-default condition under the distinction between the short-term debt (q_t) and the long-term debt (b_t).

As we see numerically in Section 5, our model demonstrates that when the inter-period debt $\frac{b_t}{R_t}$ is small, the borrower reduces the debt by repaying as much as possible to the bank. By contrast, when the debt is larger than a threshold, the firm increases the debt by borrowing additional amounts so that it intentionally moves to the debt-ridden state. The intuition of this remarkable prediction can be given as follows from the above borrowing constraint. First, let us consider the case where the current debt b_{t-1} is so small that $\frac{b_{t+j}}{R_{t+j}} < \xi S_{t+j}$ for all future periods $t+j$, with $j = 0, 1, 2, \cdots$. In this case, the borrowing constraints in the future periods are $q_{t+j} \leq \phi f(q_{t+j}) + \xi S - \frac{b_{t+j}}{R_{t+j}}$. Therefore the borrower can relax the borrowing constraints in the future by repaying as much as possible in the current period and reducing b_t. The relaxed borrowing constraints give rise to increases in the borrower's payoffs in the future. Thus, it is optimal for the borrower to repay as much as possible if the current debt is small. Conversely, let us consider the case where the current debt b_{t-1} is so large that $\frac{b_{t+j}}{R_{t+j}} > \xi S_{t+j}$ for all future period $t+j$. In this case, the borrowing constraints in the future periods are $q_{t+j} \leq \phi f(q_{t+j})$. Therefore the borrower cannot relax the borrowing constraints in the future by reducing b_t by some marginal amount in the current period. In this case, repaying the current debt just decreases the borrower's payoff today and never increases the borrower's efficiency in the future. Thus, it is not optimal for the borrower to repay debt in the current period. On the contrary, it can be optimal to borrow more in the current period. This is because the new borrowing increases the borrower's utility today while it does not worsen the future efficiency as it does not change the borrowing constraint in the future from $q_{t+j} \leq \phi f(q_{t+j})$.

1.2 Related literature

Our theory is related to the literature on debt overhang, such as Myers (1977), Krugman (1988), and Lamont (1995). Debt overhang is an inefficiency typically due to the coordination failure between incumbent and new lenders. By contrast, inefficiency is generated in our model even though incumbent lenders provide new money. Debt overhang typically causes inefficiency in the short run. However, in our study, inefficiency can continue permanently. Jungherr and Schott (2022) analyze the persistent inefficiency due to debt overhang in the economy where long-term debt exists. In their framework, the borrowing firm's default decision becomes inefficient because it considers only the cost for the buyers of newly issued bonds and neglects the cost for the existing bondholders. This externality causes high debt levels to reduce only gradually during recessions. The externality is due to the multiplicity of lenders in Jungherr and Schott (2022), while our model generates persistent stagnation despite no coordination failure among lenders. Our model is also closely related to that of Kobayashi, Nakajima and Takahashi (2022), who analyze a version of the debt overhang effect, in which an excessively large debt makes the lender lose the commitment, which in turn discourages the borrower from investing.

Our study is also closely related to the work of Caballero, Hoshi and Kashyap (2008). They define "zombie lending" as the provision of a de facto subsidy from banks to unproductive firms and argue that congestion by zombie firms hinders the entry of more productive firms and lowers aggregate productivity. In this study, we make the complementary point to their argument that even an intrinsically productive firm can become inefficient when it is debt-ridden. This offers a notably different policy implication. Caballero et al. (2008) imply that the physical liquidation of zombie firms is desirable, whereas our theory implies that zombie firms can restore high productivity if they are relieved of their excessive debt. Fukuda and Nakamura (2011) report that the majority of firms identified as zombies by Caballero et al. (2008) recovered substantially in the first half of the 2000s. This observation seems consistent with our model.

In the macroeconomic literature, endogenous borrowing constraints are introduced by the seminal works of Kiyotaki and Moore (1997), Carlstrom and Fuerst (1997), and Bernanke et al. (1999), which spawned the large body of literature on dynamic stochastic general equilibrium (DSGE) models with financial frictions. The borrowing constraints in an economy in which intra-period and inter-period loans exist are analyzed by Albuquerque and Hopenhayn (2004), Cooley, Marimon and Quadrini (2004), and Jermann and Quadrini (2006, 2007, 2012). The modeling method in this study is closest to that of Jermann and Quadrini (2012). The endogenous borrowing constraint in our model can be classified as the earning-based constraint. Drechsel (2022) raises a concept of the earning-based borrowing constraint that determines the borrowing limit as a multiple of the current earnings, which is a proxy of the future earnings of the borrower, that the lender can seize in the future. Compared to this, Drechsel (2022) argue that there is another type of borrowing constraint, which is the collateral constraint that gives the debt limit as a portion of the collateral asset, such as land. Lian and Ma (2021) show that the earnings-based borrowing constraint is widely observed in the United States and that 80% of corporate debt is subject to the earnings-based constraint.

Furthermore, our model is similar to that of Guerrón-Quintana and Jinnai (2014) in that a temporary shock affects economic performance persistently, although there is a significant difference in the policy implications. In our model, the emergence of debt-ridden borrowers due to negative productivity shocks causes a persistent recession. Thus, debt restructuring (i.e., wealth redistribution from lenders to borrowers) restores aggregate efficiency. By contrast, debt restructuring has no effect in the model of Guerron-Quintana and Jinnai because, in their model, the financial crisis is caused by a shock to the parameters of financial technology. Khan, Senga and Thomas (2017) also quantitatively analyzes a persistent recession in the aftermath of a financial crisis. Their heterogeneous firm model implies that a credit shock leads to persistent stagnation because it tightens the borrowing constraints that give rise to the persistence of low productivity due to less entry of new firms and more exits of incumbents. Ottonello and Winberry (2020) build a heterogenous agent New Keynesian model that extends the Khan et al. (2017)'s framework. They show that the aggregate effect of monetary policy depends on the distribution of default risk. Another study closely related to ours is Ikeda and Kurozumi (2014). They build a medium-scale DSGE model with financial friction à la Jermann and Quadrini (2012) and endogenous productivity growth à la Comin and Gertler (2006). Their study differs from ours in that Ikeda and Kurozumi (2014) also posit that a financial crisis is an exogenous shock on financial parameters.

These studies argue that endogenous productivity slowdowns in the endogenous growth models generate persistent recessions after financial crises. Other similar studies, such as Ates and Saffie (2021), argue that sudden stops reduce firm entries but raise the average productivity of entrants. Further, Queraltó (2020) argues that financial frictions reduce firm entries and innovations in an expanding variety model. Our model is different from these endogenous growth models in that we show that stagnation can be persistent in the aftermath of a financial crisis, even in a model without endogenous productivity growth.

Our study is also closely related to the literature on the secular stagnation hypothesis. This hypothesis shows that a persistent stagnation can be caused by, for example, a sunspot shock under nominal rigidities (Benigno and Fornaro, 2017), a change in debt limit (Eggertsson, Mehrotra and Robbins, 2019), and wealth inequality under rich people having lower propensities to consume (Mian, Straub and Sufi, 2021). By contrast, our hypothesis in this paper is that debt buildup due to a financial crisis can cause persistent inefficiency and lower output for years.

Our model is solved by a fully nonlinear solution method. As our model has a kink at the switch of the borrowing constraint, it cannot be solved by standard linearization. Following Hirose and Sunakawa (2019), we adopt a Smolyak algorithm developed by Judd, Maliar, Maliar and Valero (2014) and an index function approach to deal with occasionally binding constraints. This approach is applied mainly for the analysis of the effective lower bound (ELB) of the monetary policy (e.g., Hirose and

Sunakawa, 2023). Our study is one of few research that use it in the analysis of theme other than the ELB.

The remainder of this paper is organized as follows. In the next section, we present the model. In Section 3, we analyze the debt dynamics. In Section 4, we adjust the model for quantitative simulation. Simulation results are provided in Section 5. Section 6 presents our concluding remarks.

2 The model

Time is discrete and continues from zero to infinity: $t = 0, 1, 2, \cdots, \infty$. We consider a closed economy in which the final good is produced competitively from varieties of intermediate goods. The intermediate goods firms are monopolistic competitors, producing their respective varieties of intermediate goods from capital and labor inputs. The main players are intermediate goods firms, and they face borrowing constraint that limits financing intra-period borrowing for working capital and inter-period debt. Increased debt tightens the borrowing constraint for working capital. The household is a lender and supplies labor, capital, intra-period loan for working capital, inter-period loan at market prices, and buys consumer goods from the firm.

2.1 Final goods firm

The final good is produced competitively from intermediate goods $y_{i,t}$, where $i \in [0,\ 1]$, by the following production function:

$$Y_t = \left(\int_0^1 y_{i,t}^{\eta} di \right)^{\frac{1}{\eta}},$$

where $0 < \eta < 1$. Since the final good producer maximizes $Y_t - \int_0^1 p_{i,t} y_{i,t} di$, where $p_{i,t}$ is the real price of intermediate good i, perfect competition in the final goods market implies that

$$p_{i,t} = p(y_{i,t}) = A_{i,t} y_{i,t}^{\eta-1},$$

where $A_{i,t} \equiv a_{i,t}^{\eta} Y_t^{1-\eta}$, and $a_{i,t}$ is the productivity of firm i that is an exogenous shock to the firm.

2.2 Intermediate goods firm

A representative household owns a mass of intermediate goods firms. Firm i produces variety i monopolistically and can borrow funds from the household. Given the productivity $a_{i,t}$, firm i produces intermediate good i from capital $k_{i,t}$ and labor $l_{i,t}$ by the following production function:

$$y_{i,t} = a_{i,t} k_{i,t}^{\alpha} l_{i,t}^{1-\alpha}.$$

Each firm i employs labor $l_{i,t}$ and capital $k_{i,t}$, and produces intermediate goods $y_{i,t}$. The productivity shock $a_{i,t}$ evolves stochastically. The transition of productivity is a Markov process, which is determined exogenously and is taken as given by the household and firms. Hereafter, we omit the subscript i for simplicity. The firm's gross revenue in period t is given by

$$F_t(A_t, k_t, l_t) = p_t y_t = A_t k_t^{\alpha\eta} l_t^{(1-\alpha)\eta}.$$

Firms use equity and debt, where debt is not state-contingent. We focus on the case with initial debt stock b_{-1} at $t = 0$, where $\frac{b_{-1}}{R_{-1}}$ is the amount of inter-period debt at the end of the previous period, and R_t is the gross rate of corporate loans. This study assumes that firms hold inter-period debt because it offers tax advantages.[1] Thus, R_t is determined by

$$R_t = 1 + (1 - \tau) r_t,$$

where τ represents the tax benefit. The debt $\frac{b_{t-1}}{R_{t-1}}$ at the end of period $t - 1$ grows at the gross rate R_{t-1} to become b_{t-1} at the beginning of period t. Specifically, the firm owes $(1 + r_{t-1})\frac{b_{t-1}}{R_{t-1}}$ to the lender. Hence, the firm has to pay this amount to the lender, whereas it obtains a transfer from the government as a tax advantage, amounting to $\tau r_{t-1}\frac{b_{t-1}}{R_{t-1}}$. Thus, the net payment by the firm is $(1 + r_{t-1})\frac{b_{t-1}}{R_{t-1}} - \tau r_{t-1}\frac{b_{t-1}}{R_{t-1}} = b_{t-1}$.

The cost of the capital and labor inputs for the firm is given by q_t:

$$q_t \geq r_t^K k_t + w_t l_t.$$

The firm needs to borrow working capital, q_t, from the bank as an intra-period loan and pay the household in advance of production, as in Albuquerque and Hopenhayn (2004), Cooley, Marimon and Quadrini (2004), and Jermann and Quadrini (2006, 2007, 2012). We define $f_t(q_t)$ by

$$f_t(q) = \max_{k,l} F_t(A_t, k, l),$$
$$\text{subject to} \quad r_t^K k + w_t l \leq q.$$

Thus, the solution implies

$$f_t(q) = A_t \left(\frac{\alpha}{r_t^K}\right)^{\alpha\eta} \left(\frac{1-\alpha}{w_t}\right)^{(1-\alpha)\eta} q^{\eta}.$$

The budget constraint for the firm is given by

$$\pi_t \leq f_t(q_t) - q_t - b_{t-1} + \frac{b_t}{R_t},$$

where π_t is the payment to the firm's owner as a dividend. The intra-period loan $q_t = w_t l_t + r_t^K k_t$ is subject to the following borrowing constraint (derived in Appendix A):

$$q_t \leq \phi f_t(q_t) + \max\left\{\xi S_t - \frac{b_t}{R_t},\ 0\right\}, \tag{1}$$

where $0 \leq \phi < 1,\ \ 0 \leq \xi \leq 1$, and S_t is the present discounted value of future earnings of the firm, which is the value that the lender can obtain by taking control of the firm. S_t is taken as given by the lender and borrower. The equilibrium condition that S_t must satisfy is later given by (9), and ξ is the probability that the lender can successfully take control of the firm when it defaults on the debt.

[1] This assumption is a shortcut to formulating the motivation for holding debt. As is well known, with asymmetric information and costly state verification, the optimal contract takes the form of debt (e.g., Townsend, 1979; Gale and Hellwig, 1985).

The nature of the earnings-based borrowing constraint: The above borrowing constraint is a variant of the earning-based borrowing constraint, which is widely observed in the U.S. and other developed economies (see Drechsel, 2022; Lian and Ma, 2021). The long-term debt b_t is covered by a part of the future earnings ξS_t, while the intra-period debt q_t is covered by the earnings in the current period $\phi f(q_t)$ and the net earnings of the future periods $\max\left\{\xi S_t - \frac{b_t}{R_t},\ 0\right\}$. Hence, the borrowing constraint in our model is a natural variant of an earnings-based borrowing constraint. Moreover, this borrowing constraint is a natural extension of that in Jermann and Quadrini (2012). If we assume the constraint be $q_t \leq \max\left\{\xi S_t - \frac{b_t}{R_t},\ 0\right\}$, this constraint is the same as that in Jermann and Quadrini (2012), [2] as it says the working capital loan, q_t, should be covered by the collateral, ξS_t subtracted by the future liability, $\frac{b}{R}$. We obtain the borrowing constraint (1) by adding an assumption that the bank can take a part of output, $\phi f_t(q_t)$, as collateral. By setting $\phi = 0$, we can make our borrowing constraint replicate the same one as Jermann and Quadrini (2012).

Throughout this analysis, we assume that

$$\phi < \eta,$$

which means that production becomes inefficient when the borrowing constraint is

$$q_t \leq \phi f_t(q_t). \tag{2}$$

This constraint corresponds to the case where $\frac{b_t}{R_t} > \xi S_t$. If $\frac{b}{R}$ is smaller than ξS, the borrowing constraint becomes

$$q_t \leq \phi f_t(q_t) + \xi S_t - \frac{b_t}{R_t}. \tag{3}$$

This borrowing constraint is qualitatively similar to those in Jermann and Quadrini (2012) and Kiyotaki and Moore (1997).

We define $q_{z,t}$ as the solution to $q_{z,t} = \phi f_t(q_t)$. Given the assumption that $\phi < \eta$, it is shown that $q_{z,t} < q^*$, where q^* is the first-best value that solves $f'(q) = 1$, and production is inefficient. The firm's owner has no liquid assets and cannot pay any positive amount to the firm, as in Albuquerque and Hopenhayn (2004). Therefore, the dividend must be non-negative:

$$\pi_t \geq 0. \tag{4}$$

The firm cannot avoid the non-negativity constraint (4) by soliciting equity investment from outside investors because of market frictions. The details of the market frictions are not specified in this analysis, and we assume that the firm cannot issue new equities timely, even if the new money can generate a positive surplus by relaxing the borrowing constraint and even if outside investors are willing to buy new equities. This assumption can be justified by market frictions such as a lack of commitment and asymmetric information.

[2] More precisely, there is a difference that the borrowing constraint in Jermann and Quadrini (2012) is a collateral constraint because S_t in their model is the value of the collateralized capital stock and not the present value of the future earnings.

Value of the firm: We can distinguish two states for the firm: the *normal state* and the *debt-ridden state*. In the normal state, the firm continues to repay debt and obtains positive amounts of payoffs with positive probabilities. The firm in the normal state has the option to borrow an additional amount in the current period to fall into the debt-ridden state. In a debt-ridden state, the firm's debt is so large that the firm's profit in each period is not enough to cover the interest payment of the debt. *The debt-ridden state is the state where the firm continues to pay all profits to the bank every period indefinitely, the amount of debt never decreases, and the payoff to the firm stays at zero forever.* The firm in the normal state has a binary choice of whether to stay in the normal state or fall into the debt-ridden state, whereas the firm in the debt-ridden state has no choice other than staying debt-ridden.[3] The firm may choose to become debt-ridden because there are cases where the gain from tax advantage of having a large debt is strictly larger than the marginal cost of tightening the borrowing constraint. Thus, given that the firm was in the normal state in period $t-1$, the value of the firm V_t is given by:

$$V_t = \max \left\{ V_t^N,\ V_t^Z \right\},$$

where V_t^N is the value of the firm in the normal state, and V_t^Z is the value of the firm that is initially in the normal state to move into the debt-ridden state by borrowing additional funds in period t. V_t^N evolves by the following dynamic programming equation:

$$\begin{aligned} V_t^N &= \max \pi_t + \mathbb{E}_t \left[m_{t+1} V_{t+1} \right], \qquad (5) \\ \text{subject to} \quad \pi_t &= f_t(q_t) - q_t - b_{t-1} + \frac{b_t}{R_t}, \\ q_t &\le \phi f_t(q_t) + \max \left\{ \xi S_t - \frac{b_t}{R_t},\ 0 \right\}, \\ \pi_t &\ge 0, \end{aligned}$$

where $\mathbb{E}_t$ is the expectation operator as of period t, m_{t+1} is the stochastic discount factor given as an outcome of the household's problem and is defined later by (10). V_t^Z, the value of the firm that borrows additionally to become debt-ridden, is determined by the following dynamic programming equation:

$$V_t^Z = \max \ (1-\phi) f_t(q_{z,t}) + \mathbb{E}_t [m_{t+1} b_{z,t+1}] - b_{t-1}, \qquad (6)$$

$$\text{subject to} \ \ b_{z,t} = (1+\tau)(1-\phi) f_t(q_{z,t}) + \mathbb{E}_t \left[m_{t+1} b_{z,t+1} \right], \qquad (7)$$

$$q_{z,t} = \phi f_t(q_{z,t}), \qquad (8)$$

where $b_{z,t}$ is the expected value that the bank can obtain from the debt-ridden firm. The borrowing constraint is (8) permanently because the face value of debt, which we explain shortly, is so immense that $\xi S_t < \frac{b}{R}$ for all future periods. Thus, the production is given by $f_t(q_{z,t})$ every period, and

[3] Specifically, the debt-ridden firm has the option to exit. The value of exiting is zero, while the value of staying debt-ridden is also zero because all earnings are taken by the bank forever. We assume that the firm owner obtains nonpecuniary utility from continuing the operation so that the firm chooses to stay debt-ridden rather than to exit, even though both options give zero as the pecuniary payoffs.

equation (7) refers to when the bank takes all earnings of the firm and the tax advantage for all future periods.[4] V_t^Z is the total value that the firm obtains by intentionally falling into the debt-ridden state by borrowing $\mathbb{E}_t[m_{t+1}b_{z,t+1}]$ additionally. The firm can obtain the cash flow, $(1-\phi)f_t(q_{z,t}) + \mathbb{E}_t[m_{t+1}b_{z,t+1}] - b_{t-1}$, at the most in the current period, while the firm obtains zero in all future periods once it falls into the debt-ridden state. The amount of the working capital $q_{z,t}$ for the debt-ridden firm is given by the borrowing constraint (8) and, thus, labor input $l_{z,t}$ and capital input $k_{z,t}$ are inefficient:

$$\begin{aligned} k_{z,t} &= \left[\phi A_t \left(\frac{r_t^K}{\alpha}\right)^{(1-\alpha)\eta-1} \left(\frac{1-\alpha}{w_t}\right)^{(1-\alpha)\eta}\right]^{\frac{1}{1-\eta}}, \\ l_{z,t} &= \frac{(1-\alpha)r_t^K k_{z,t}}{\alpha w_t}. \end{aligned}$$

Here we have a caveat on the face value of the inter-period debt, b_t, in the debt-ridden state. As explained above, the face value b_t does not appear in determining allocations and payoffs. The firm decides whether or not to move to the debt-ridden state from the normal state by comparing the payoff in the normal state and that in the debt-ridden state that is not dependent on the face value of debt b_t. See Appendix B for the evolution of the face value.

The seizure value S_t is the firm's value when the bank seizes the firm. Since S_t is the value of a brand-new firm without existing debt, the condition that determines S_t is

$$S_t = \max_b \mathbb{E}_t \left[m_{t+1}V_{t+1}(b)\right] + \frac{b}{R_t}. \tag{9}$$

Difference between inter- and intra-period debt: We have the following difference between inter-period debt b_{t-1} and intra-period debt q_t. The firm has the chance to default on its inter-period debt b_{t-1} at the beginning of period t, and it will do so only if the continuation value of the firm is negative, $V_t < 0$. However, this outcome never occurs because the firm's dividend is non-negative ($\pi_t \geq 0$), as is the continuation value ($V_t \geq 0$). Thus, the firm never defaults on its inter-period debt b_{t-1}. The firm has the chance to default on intra-period debt q_t at the end of period t, which we analyze in the Appendix A, in which the borrowing constraint (1) for q_t is given as the no-default condition. Thus, the firm does not default on q_t in equilibrium.

Timing of events: The events in a given period t occur in the following way. The firm and bank enter period t with outstanding debt of b_{t-1}. At the beginning of the period, the firm has the chance to default on b_{t-1}, and it will do so if the continuation value is negative (which never happens). Subsequently, the firm borrows intra-period debt q_t, employs labor and capital by paying q_t, and produces output $f_t(q_t)$. The firm repays b_{t-1} and borrows new inter-period debt $\frac{b_t}{R_t}$ by paying $b_{t-1} - \frac{b_t}{R_t}$. Finally, it repays intra-period debt q_t to the bank. At this point, the firm has the chance to default on q_t. After repaying q_t, the firm pays out the remaining amount, $\pi_t = f_t(q_t) - q_t - b_{t-1} + \frac{b_t}{R_t}$, to the firm owner as a dividend.

[4] Note that the maximum amount of tax benefit is $\tau(1-\phi)f_t(q_{z,t})$ when all the earnings $(1-\phi)f_t(q_{z,t})$ is used as an interest payment of the debt.

2.3 Household

A representative household solves the following problem:

$$\max_{C_t, L_t, D_t, K_t} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t U(C_t, L_t) \right],$$

subject to the budget constraint

$$C_t + K_t + \frac{D_t}{1 + r_t} + T_t \leq w_t L_t + (r_t^K + 1 - \delta) K_{t-1} + D_{t-1} + \int_0^1 \pi_{i,t} di,$$

where β is the subjective discount factor, C_t is consumption, L_t is total labor supply, K_t is capital stock, δ is the depreciation rate of capital, D_t is inter-period lending to the firms, and T_t is a lump-sum tax. The period utility is

$$U(C, L) = \left[\ln C_t - \gamma_L \frac{L_t^{1+\nu}}{1 + \nu} \right],$$

where $\nu > 0$ is the elasticity of labor supply, and γ_L is the coefficient of labor disutility relative to contemporaneous consumption utility.

Let m_t be $\frac{\lambda_{t+1}}{\lambda_t}$, where λ_t is the Lagrange multiplier associated with the budget constraint for the representative household. The FOC with respect to C_t implies

$$m_{t+1} = \frac{\beta^{t+1} \partial U(C_{t+1}, L_{t+1}) / \partial C_{t+1}}{\beta^t \partial U(C_t, L_t) / \partial C_t} \tag{10}$$

The FOC with respect to K_t and D_t implies

$$\frac{1}{1 + r_t} = \mathbb{E}_t \left[\frac{1}{r_{t+1}^K + 1 - \delta} \right] = \mathbb{E}_t \left[m_{t+1} \right].$$

Thus, m_t is the stochastic discount factor.

2.4 Competitive equilibrium

The market-clearing conditions are

$$\begin{aligned}
&C_t + K_t - (1 - \delta) K_{t-1} = Y_t, \\
&\int_0^1 l_{i,t} di = L_t, \\
&\int_0^1 k_{i,t} di = K_{t-1}, \\
&\int_0^1 \frac{b_{i,t}}{R_t} di = \frac{D_t}{1 + r_t}.
\end{aligned}$$

A competitive equilibrium consists of sequences of prices $\{r_t, r_t^K, w_t, m_t\}$, a household's decisions $\{C_t, L_t, K_t, D_t\}$, firms' decisions $\{\pi_t, l_t, k_t, b_t\}$, such that *(i)* the representative household and firms solve their respective optimization problems, taking prices as given, and *(ii)* the market-clearing conditions and equilibrium conditions are all satisfied.

Kobayashi and Shirai (2022) provides proof of the existence of the competitive equilibrium with certain restrictions on parameters for a partial equilibrium version of our model, in which the prices are given exogenously.

The observed TFP is defined as follows:

$$TFP_t = \frac{Y_t}{K_t^{\alpha} L_t^{1-\alpha}}.$$

We will see in Section 5 that the observed TFP declines when negative productivity shocks or a wealth shock hits the economy. In the simulation of Section 5, the observed TFP remains low even after the negative productivity shock is gone, as long as the borrowing constraint is tightened. It is because the shocks hit only a subgroup of intermediate firms and cause the misallocation of labor and capital between the firms hit by the shocks and the other firms. As it is well known in the literature (e.g., Chari, Kehoe and McGrattan, 2007), the misallocation of inputs lowers the observed TFP.

3 Debt dynamics

In this section, we characterize the debt dynamics of our model. In our model, state variables are debt (b_{t-1}), the capital stock (K_{t-1}), interest rate (r_{t-1}), and productivity (a_t).[5] These state variables govern the policy functions. This section presents policy functions and value functions as functions of debt, given that other state variables are constant. The severity of the borrowing constraint varies with the size of the debt, which affects the shape of policy functions. Details of analytical results are described in Kobayashi and Shirai (2022).

Figure 3 shows the policy functions $b_t = b(b_{t-1})$ and $q_t = q(b_{t-1})$ and the value function $V_t = V(b_{t-1})$ in the case that $a_t = 1$. To characterize the debt dynamics, we divide four regions for debt level: debt-ridden, large debt, medium-sized debt, and small debt. The horizontal axis in Figure 3 shows these four regions.

Debt-ridden where $\xi S < \frac{b}{R}$, $\pi = 0$, and $V = 0$: When the debt b_{t-1} satisfies $b_{t-1} \in [B_Z, +\infty)$, the firm intentionally borrows additional money and falls into the debt-ridden state. The value of B_Z is determined by solving $V_t^N = V_t^Z$. Resultant of the new borrowing, the firm obtains a positive payoff when it moves to the debt-ridden state, where the payoff of becoming debt-ridden is $V_t^Z > 0$. The firm's payoff is zero forever once it falls into the debt-ridden state, that is, $V_{t+j} = 0$ for $j \geq 1$, where the firm falls into debt-ridden in period t. In Section 5.2, we will numerically confirm that when negative productivity shocks hit the economy over a long period, firms intentionally move to the debt-ridden state. The inefficiency due to the tighter borrowing constraint continues indefinitely once the firm enters the debt-ridden state.

Large debt where $\xi S < \frac{b}{R}$, $\pi = 0$, and $V > 0$: There exists B_L such that for $b_{t-1} \in [B_L, B_Z]$ it is the case that $\xi S_t < \frac{b_t}{R_t}$ and $\pi_t = 0$ in equilibrium. In this region, we call b_{t-1} a large debt,

[5]Since the social surplus (19) in the Online Appendix is determined by lagged interest rate, r_{t-1} works as a state variable.

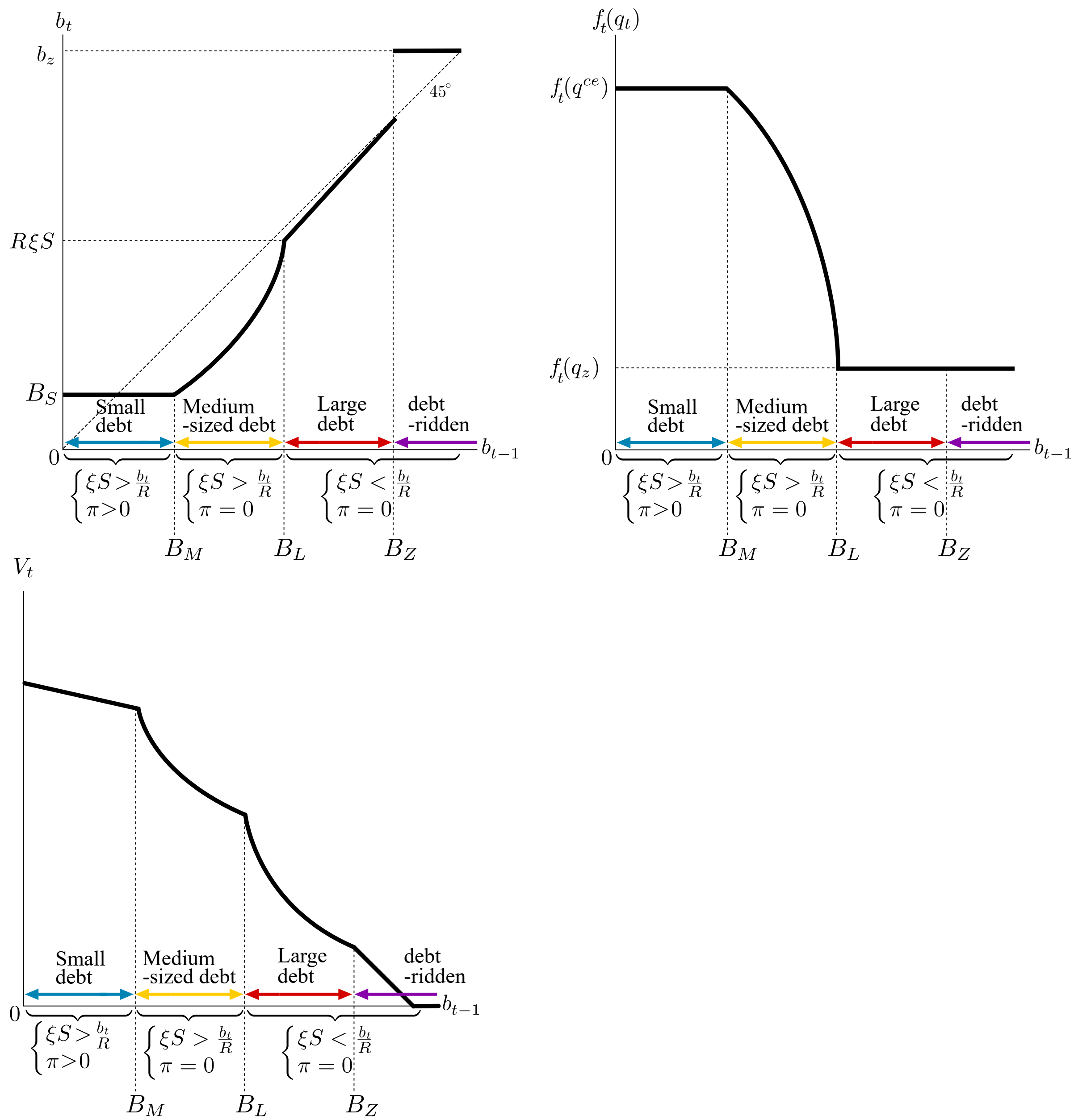


Figure 3: Policy functions and value function

and the firm is severely inefficient because the borrowing constraint is tight ($q_t \leq \phi f_t(q_t)$) and the non-negativity constraint is binding. In this region, the borrowing firm pays all earnings to the bank, and debt decreases, whereas the decrease in debt is slow because the output $f_t(q_{z,t})$ and the earnings $(1-\phi)f_t(q_{z,t})$ are both small.

Lemma 1. *When b_{t-1} is large, that is, $b_{t-1} \in [B_L,\ B_Z]$, the policy function for debt is $b_t = R_t[b_{t-1} - (1-\phi)f(q_z)]$.*

Proof. As $\pi_t = 0$ and $q_t \leq \phi f_t(q_t)$, the budget constraint can be written as $b_t = R_t[b_{t-1} - (1-\phi)f(q_z)]$. □

This lemma implies that the speed of the decrease in debt is slow when debt is large. Because $(1-\phi)f(q_z)$ is considerably small, and R_t is larger than one, the policy function of b_t in Figure 3 is close to the 45-degree line for $b_{t-1} \in [B_L, B_Z]$. The figure indicates that the economy can suffer from extreme persistence of inefficiency if b_{t-1} falls into the region where $B_L < b_{t-1} \leq B_Z$. This mechanism is a key ingredient for persistent stagnation that appears when the economy is deeply indebted in a time of financial crisis.

Medium-sized debt where $\xi S > \frac{b}{R}$, $\pi = 0$, and $V > 0$: There exists B_M such that for $b_{t-1} \in [B_M, B_L]$ it is the case that $\xi S_t \geq \frac{b_t}{R_t}$ and $\pi_t = 0$ in equilibrium. Although the non-negativity constraint is binding, the borrowing constraint is not so tightly binding as for the large debt, and the production inefficiency is not as severe as for the large debt. Thus, the policy function $b_t = b(b_{t-1})$ in Figure 2 shows that debt decreases rapidly in the region where $b_{t-1} \in [B_M, B_L]$.

Small debt where $\xi S > \frac{b}{R}$, $\pi > 0$, and $V > 0$: For $b_{t-1} \in [0,\ B_M]$, it is the case that $\xi S_t \geq \frac{b_t}{R_t}$ and $\pi_t > 0$ in equilibrium. In this case, the non-negativity constraint is nonbinding, and the firm attains the constrained-efficient production q^{ce}, and there exists B_S such that the firm optimally chooses $b_t = B_S$ for all $b_{t-1} \in [0,\ B_M]$. In this constrained-efficient equilibrium, B_S is determined such that the marginal gain from the tax advantage of additional borrowing equals the marginal cost from tightening the borrowing constraint. The following conditions must be satisfied on the equilibrium:

$$\xi S > B_S, \tag{11}$$

$$B_S > 0. \tag{12}$$

The first condition requires that the borrowing constraint (1) must be (3) rather than (2) in the constrained-efficient equilibrium. The second condition requires that firms are not net lenders to other firms or households in the constrained-efficient equilibrium.

4 Settings for numerical simulation

Following Khan and Thomas (2013) we divide intermediate goods firms into two groups: *risky firms* with measure ζ and *safe firms* with measure $1-\zeta$. The productivity level of safe firms is time-invariant

Parameters	Values	Description	Source or Target
α	0.3	Cobb–Douglas production function	
β	0.99	Subjective discount factor	
δ	0.025	Depreciation rate	
η	0.7	Intermediate goods elasticity, $1/(1-\eta)$	
ν	1	Labor supply elasticity	Ikeda and Kurozumi (2019)
γ_L	5.039	Labor disutility weight	Steady state labor supply $\overline{L} = 1/3$
ϕ	0.3577	Collateral ratio of revenue	average debt/GDP ratio, NIPA
ξ	0.065	Collateral ratio of foreclosure value	average debt/GDP ratio, NIPA
ζ	0.13	Debt-ridden firms ratio	Banerjee and Hofmann (2018)
τ	0.35	Corporate tax rate	Jermann and Quadrini (2012)

Table 1: Calibrated parameters

at all times, $a_{n,t} = 1$, where the subscript n denotes variables associated with safe firms. By contrast, the productivity level of risky firms, $a_{d,t}$, is time-variant and follows a two-state Markov chain process, where we put the subscript d for risky firms. The market-clearing conditions are modified as follows:

$$\zeta l_{d,t} + (1-\zeta) l_{n,t} = L_t,$$
$$\zeta k_{d,t} + (1-\zeta) k_{n,t} = K_{t-1},$$
$$\zeta \frac{b_{d,t}}{R_t} + (1-\zeta) \frac{b_{n,t}}{R_t} = \frac{D_t}{1+r_t}.$$

When risky firms' productivity is low due to the realization of low state, their debt will increase as the amount repaid is smaller than the interest. When risky firms' debt $b_{d,t}$ increases to $\xi S_t < b_{d,t}/R_t$, the borrowing constraint becomes (2) and working capital financing becomes severely constrained. If the level of debt is $b_{d,t} < B_{Z,t}$ and the low state does not continue, then the debt will decrease to the optimal level over time, and the borrowing constraint will return to (3). If the low state is prolonged, debt may exceed the threshold, that is, $b_{d,t} \geq B_{Z,t}$. In this case, risky firms borrow new money $\mathbb{E}_t[m_{t+1} b_{z,t+1}]$ to increase the debt to gain the tax advantage, and risky firms become debt-ridden firms. The level of debt does not decrease because the level of debt is too large, and paying all earnings to the bank cannot cover the interest payment. As a result, the borrowing constraint remains (2), and risky firms will remain permanently inefficient due to the severe constraint. ζ is the percentage of firms that can be trapped in a debt-ridden state.

By contrast, the debt level of safe firms is always small and never exceeds the threshold ($B_{L,t}$). Hence, the borrowing constraint for safe firms is always (3).

4.1 Calibration

We set the capital share in the Cobb–Douglas production function at $\alpha = 0.3$, the subjective discount factor at $\beta = 0.99$, and the depreciation rate at $\delta = 0.025$, as the economy is modeled at quarterly frequencies. The parameter for the elasticity of substitution is set at $\eta = 0.7$, which is a standard value as most studies set at the value between [0.6 0.9]. The elasticity of the labor supply is set at $\nu = 1$, following the literature. The coefficient of labor disutility relative to contemporaneous consumption utility $\gamma_L = 5.039$ is chosen to make a steady-state labor supply 1/3. These are the standard settings used in prior studies.

The collateral ratio of foreclosure value ξ and of revenue ϕ are calibrated for the U.S. economy. We set $\xi = 0.065$ as the working capital borrowing is usually nearly 7% of the corporate value in the reality (see, e.g., Galindo, 2021).[6] Subsequently, ϕ is chosen to have a steady-state ratio of total debt over value-added equal to 1.648. This value is the average ratio over the period 1984:I-2017:IV for liability of the non-financial corporate business from the Board of Governors of the Federal Reserve System, *Financial Accounts of the United States* and the Bureau of Economic Analysis, *NIPA Tables*. The required value is $\phi = 0.3577$.

Following Jermann and Quadrini (2012), the mean tax rate is set to $\tau = 0.35$. The borrowing constraint is always binding because the firm borrows inter-period debt to exploit the tax advantage.

The ratio of risky firms is set to $\zeta = 0.13$, which is estimated as the average zombie firm ratio of 14 advanced countries in 2016 by Banerjee and Hofmann (2018). We use the zombie firms as a proxy of the debt-ridden firms in our model, and as we see later in the simulation, risky firms become debt-ridden firms when they are hit by low productivity shocks. Thus, we can approximate the ratio of risky firms to that of zombie firms.[7]

The economy evolves through changes in the productivity of risky firms. The productivity shocks, $a_{d,t}$, follow the two-state Markov chain with realizations $\{a_{d,high}, a_{d,low}\}$ and transition matrix:

$$\begin{bmatrix} p_{high} & 1 - p_{high} \\ 1 - p_{low} & p_{low} \end{bmatrix}.$$

The realization $a_{d,high}$ corresponds to the productivity level during normal times in business cycles, and $a_{d,low}$ corresponds to the productivity level being low (financial crises). p_{high} is the probability of continuing normal time, $Pr[a_{t+1} = a_{d,high}|a_t = a_{d,high}]$, while $1 - p_{low}$ is the probability of escape from crisis conditions, $Pr[a_{t+1} = a_{d,high}|a_t = a_{d,low}]$. As previous empirical studies have shown large negative productivity shocks are one of the main causes of financial crises. Therefore, We set $p_{high} = 0.9941$ and $p_{low} = 0.9219$ so that the average duration of the financial crisis in our quarterly model is 12.8 quarters, and the economy spends 7 percent of the time in the crisis state. These numbers are the facts about financial crises summarized by Reinhart and Rogoff (2009) as well as Khan and

[6] In addition, ϕ and ξ are chosen to satisfy the equilibrium conditions, that is, (11) and (12), and fit the data. The value ξ that can satisfy these conditions is limited to a narrow range of [0.0540, 0.0763] and is set to 0.065.

[7] There may be a slight difference between zombie firms and debt-ridden firms. The zombie firms include the firms that are intrinsically unproductive but kept afloat by banks, whereas the debt-ridden firms are intrinsically productive, but the debt burden makes them inefficient. However, we could argue that most of the zombie firms, in reality, are debt-ridden firms because, in Japan, a substantial proportion of the firms that were classified as "zombie firms" in the 1990s recovered to become non-zombie firms in 2000 (see Fukuda and Nakamura, 2011). This observation implies that most zombie firms are intrinsically productive and debt-ridden.

Thomas (2013). Note that unconditional probability distributions evolve according to

$$[Pr(a_{t+1} = a_{d,high}),\quad Pr(a_{t+1} = a_{d,low})] = [Pr(a_t = a_{d,high}),\quad Pr(a_t = a_{d,low})] \begin{bmatrix} p_{high} & 1 - p_{high} \\ 1 - p_{low} & p_{low} \end{bmatrix}.$$

Concerning the realizations, $a_{d,high}$ and $a_{d,low}$, we calibrate our model to capture the aggregate inefficiency of the financial crises in the U.S. economy. We set $a_{d,high} = 1$ and $a_{d,low} = 0$ using evidence on banking crises from Reinhart and Rogoff (2014). They show that the average peak-to-trough decline for the U.S. real per capita GDP across nine major financial crises is about 9 percent, which is replicated when $a_{d,t}$ changes from $a_{d,high} = 1$ to $a_{d,low} = 0$. In Online Appendix E, we perform the simulation using an alternative setting for $a_{d,low}$ that is greater than zero and show that the main results are robust.

5 Simulation

To implement numerical simulation, we need to solve the model using a non-linear global solution method to handle two occasionally binding constraints: the borrowing constraint and the non-negative constraint for dividends.[8] These occasionally binding constraints generate policy functions with kinks and non-linearity. Since a standard numerical approximation method cannot solve the non-linear policy function, we apply a Smolyak-based projection method proposed by Judd et al. (2014) and the index function approach to account for non-linearity in policy functions. Our simulation code is the modified version of Shirai (2021) who extends Hirose and Sunakawa (2019) to apply two occasionally binding constraints. The full set of equilibrium conditions is available in Online Appendix C, and the details of the method are described in Online Appendix D.

5.1 Temporary shocks can induce a persistent recession

This subsection compares the main model with a frictionless real business cycle (RBC) model to show that the financial friction makes the inefficiency due to a negative productivity shock more persistent in our model than in the RBC model. Firstly, we construct a version of the standard RBC model with no financial frictions. The settings for final goods firm and representative household is the same as in Section 2.1 and 2.3. The difference between the main model and the RBC model is that the intermediate goods firm does not face the borrowing constraint nor the non-negative constraint for the dividend in the RBC model. It is also assumed that there is no tax advantage for interest payments. Thus, there is no distinction between debt and equity in the RBC model. Hence, we omit debt in the RBC model. The value of the intermediate goods firm is determined by the following dynamic programming equation:

$$\begin{aligned} & V_t^F = \max \pi_t^F + \mathbb{E}_t \left[m_{t+1}^F V_{t+1}^F \right], \\ \text{subject to} \quad & \pi_t^F = f_t(q_t^F) - q_t^F, \end{aligned}$$

[8] Accurately speaking, the borrowing constraint is always binding in the sense that it holds with equality. We abuse the usage of the term "binding" in a way that the borrowing constraint is called binding if it is (2) with equality, and non-binding if it is (3) with equality.

where superscript F denotes variables associated with the RBC model.

Figure 4 shows the responses to a negative productivity shock in the main model and the RBC model. The vertical axis is an index of deviation from the steady-state equilibrium and represents 1 when the value of the corresponding variable is equal to that in the steady-state equilibrium.

Initially, the economy was in the steady-state equilibrium with $a_{d,t} = 1$. In period 10, the low state is realized, and the productivity level of firms of the ratio ζ (i.e., risky firms) falls to zero. Since the firms' productivity falls to zero, they can not repay the debt, and their debt increases each period in the main model.[9] In period 110, the high state is realized, and the productivity level recovers to $a_{d,t} = a_{d,high} = 1$. However, in the main model, the production of risky firms remains inefficient for some periods because the borrowing constraint is (3) due to a large increase in debt during the low state. In addition, they repay as much debt as possible by setting the dividend to zero. The decreases in the total output, the observed TFP, and the social welfare are more persistent, reflecting that inefficiencies are more prolonged than in the RBC model due to the large debt and the tighter borrowing constraint. The decrease in the observed TFP is caused by misallocations of labor and capital between the risky firms and the safe firms.

Thus, the debt accumulation during the low state results in the tighter borrowing constraint (2) and a long-lasting inefficient production. The greater the amount of debt accumulated, the more inefficient production continues.

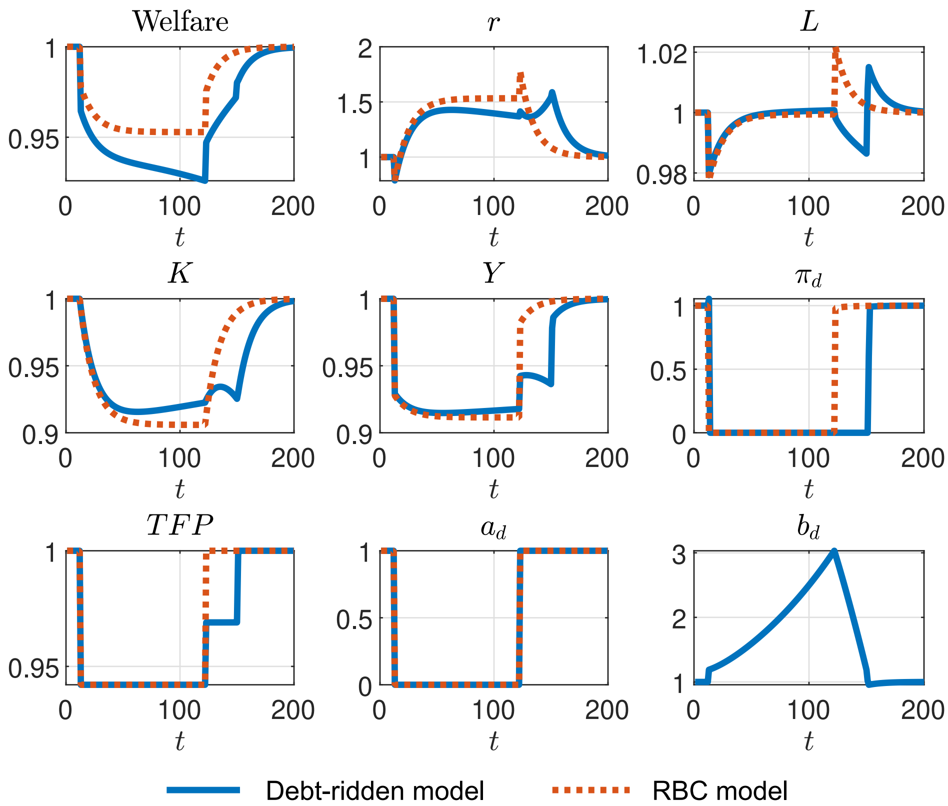
5.2 Prolonged shocks leads to the debt-ridden state

In this subsection, we show that firms become debt-ridden endogenously when the low state persists for a considerably long period. In our model, the low productivity state is modeled as the financial crisis, and the transition probabilities are calibrated to the duration of the financial crisis. Figure 5 shows the simulation results when the low state is prolonged. From the 10th period to the 259th period, the productivity a_d is low, and the firm's output remains at zero. The debt continues to increase at the rate of interest because the debt cannot be repaid at all. As the level of debt increases, the value of V^Z comes closer to the value of V^N. Eventually, V^Z exceeds V^N in the 226th period and the risky firms increase their borrowing in the 226th period and become debt-ridden firms from then on. Even if productivity returns to the high state in the 260th period, macro variables do not return to their pre-crisis levels because the production of debt-ridden firms remains inefficient permanently. Thus, the total output, the observed TFP, and the social welfare are all staying lower than the pre-crisis levels. Note that in Figure 5, we show $b_{z,t}$ defined in (7) as the debt (b_d) after the firm becomes debt-ridden.[10]

The figure suggests that this situation is one explanation for the failure of GDP to return to its pre-crisis trend in many countries since the GFC. Recent empirical studies have shown evidence of a downturn due to corporate debt (e.g., Cecchetti, Mohanty and Zampolli, 2011; Mian, Sufi and Verner, 2017), and our model provides one mechanism that explains the recession due to the buildup of corporate debt.

[9] Online Appendix E shows that the debt accumulation also occurs even if $a_{d,low} > 0$.

[10] The value of $b_{z,t}$ is not the face value of debt but the present discounted value of the payoff for the lender in the debt-ridden state. See the arguments about b_z in Section 2.2



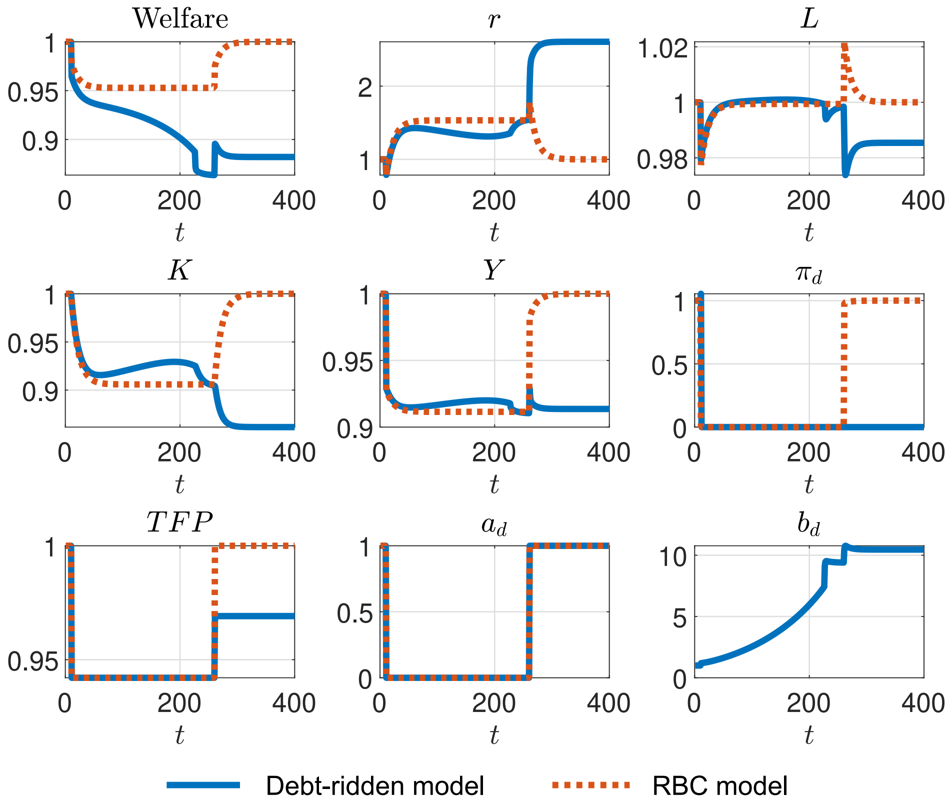
Notes: The vertical axis shows the index as 1 = the steady state value.
π_d: Risky firm's dividend, f_d: Risky firm's revenue (production),
a_d: Risky firm's productivity, b_d: Risky firm's debt

Figure 4: RBC model vs. Debt-ridden model

The figure shows that firms become debt-ridden only after they experience more than 200 periods (50 years) of low productivity. This requirement may seem unrealistic because the required period of low productivity is too long. The requirement of an unrealistically long duration of low productivity is an artifact due to our simplifying assumption that the productivity shock is the only exogenous shock to the economy. As in many DSGE models, introducing various shocks may shorten the period required to make the risky firms debt-ridden. For example, a buildup of debt due to a collapse of an asset-price bubble would induce $V^N < V^Z$ and make the firms choose debt-ridden immediately, and a significantly large decrease in capital stock due to a capital quality shock would also have the same result.

5.3 Persistent effect of a wealth shock

In Section 5.2, we have shown that a negative productivity shock can cause intermediate goods firms to accumulate debt, and the accumulation of debt has a persistent negative impact. We also showed that a prolonged period of low productivity could lead firms to a debt-ridden state. In this section,



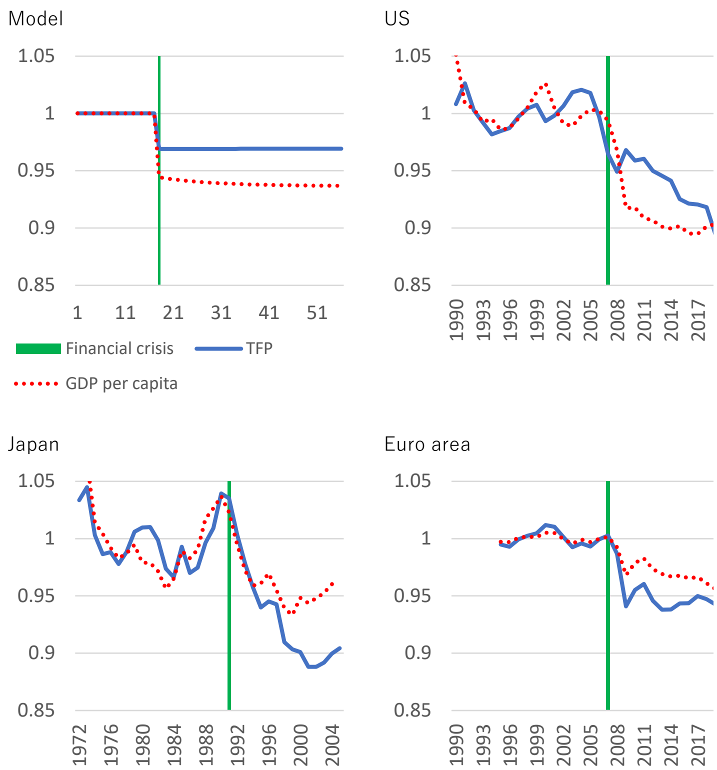
Notes: The vertical axis shows the index as 1 = the steady state value.
π_d: Risky firm's dividend, f_d: Risky firm's revenue (production),
a_d: Risky firm's productivity, b_d: Risky firm's debt

Figure 5: Long-term depression and debt-ridden firms

we show that if a wealth shock in the financial crisis causes many firms to become debt-ridden, our model can explain that the economy can suddenly fall into a persistent stagnation after the crisis, as we observe in the aftermath of the GFC.

Existing literature has examined the exogenous redistribution of assets and liabilities due to wealth shocks as a driving force of the financial crisis (Carlstrom and Fuerst, 1997; Bernanke et al., 1999; Christiano, Motto and Rostagno, 2010; Christiano, Trabandt and Walentin, 2011; Kaihatsu and Kurozumi, 2014b). A typical example of a wealth shock is a collapse of the asset-price bubble, which reduces the net worth of the borrowers. Following these literature, we examine the response of the economy to a wealth shock that redistributes wealth from intermediate goods firms to households (lenders), whereby risky firms take on debt over a threshold B_Z. In this exercise, we introduce wealth shocks instead of productivity shocks and assume that the level of productivity is constant over time. It is common knowledge that wealth shocks follow the two-state Markov chain. In the low state, the lender owes an additional large debt. The transition probabilities are assumed to be the same as in section 4.1. The wealth shock causes risky firms to become overly indebted. Figure 6 shows the results. This figure also shows the actual detrended GDP and the observed TFP for the United States, Japan, and the

Euro area. The negative shocks in these series correspond to the bursting of the land price in 1991 in the case of Japan and that of the housing price in 2007 in the cases of the United States and the Euro area. The simulation result in Figure 6 shows that GDP and the observed TFP are permanently stagnant when risky firms become debt-ridden due to a negative wealth shock. Our model shows that the financial crisis, which is modeled as a negative wealth shock, can cause prolonged stagnation in the aftermath of the crisis.



Notes: The actual data frequency is annual. These series are detrended by a linear time detrend. The sample period is from 1990 to 2007 for the U.S., from 1972 to 1990 for Japan, and from 1995 to 2007 for the Euro area. In Japan, TFP is classified as the "market economy" sectors, which excludes education, medical services, government activities, and imputed housing rent.

Sources: US: Fernald (2012); OECD, *Quarterly National Accounts*. Japan: Cabinet Office, Government of Japan, *Annual Report on National Accounts*; The Research Institute of Economy, Trade and Industry, *JIP 2014 database*. Euro area: World bank, *World Development Indicators*; European Commission, *AMECO database*.

Figure 6: Wealth shock

6 Conclusion

Persistent stagnation in the aftermath of financial crises is often observed and requires a convincing theoretical explanation and sensible policy recommendations. Thus, this study analyzed the debt dynamics under the borrowing constraint, in which short-term borrowing and long-term borrowing are distinguished. This distinction introduces endogenous changes in the nature of the borrowing constraint associated with changes in the amounts of long-term and short-term debt. It has been shown that when the long-term debt increases, the borrowing constraint for the short-term debt becomes tighter, and inefficiency is made persistent. In particular, it was shown that when the long-term debt exceeds the upper limit, the long-term debt can never decrease, and the inefficiency continues indefinitely. This is what we call the debt-ridden state. A unique feature of our model is that when the long-term debt exceeds a threshold, the borrower intentionally chooses to borrow new money to increase the long-term debt and become debt-ridden. We show in our numerical simulation that a succession of many negative productivity shocks or a large negative wealth shock can make the borrowers choose to become debt-ridden. Our results also imply that temporary shocks can induce the borrowers intentionally accumulate debt, whereby the whole economy falls into persistent stagnation. These results can be regarded as an explanation for the persistent stagnation often observed in the aftermath of financial crises.

Policy implications of our theory are straightforward: debt restructuring or debt forgiveness for overly indebted borrowers after a financial crisis may be effective to escape from persistent stagnation. This is because, in our explanation, there are no technological or structural changes that cause persistent stagnation after the financial crises. However, the accumulation of debt alone can plunge the economy into a deep and prolonged recession. Our policy recommendation, that is, debt restructuring, is complementary to policy implications of existing literature that emphasize monetary and fiscal stabilization.

Our theory focuses on the debt buildup in the corporate sector, and massive debt restructuring in the corporate sector by government intervention may increase the government debt instead. The integrated analysis of private and public debt should be the agenda for future research. The key implication of this study is that debt restructuring deserves to be considered more seriously as a policy measure for recovery from persistent recessions in the aftermath of financial crises.

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A Derivation of the borrowing constraint

Here, we describe the events that follow a counterfactual default on q_t and derive the borrowing constraint (1) as the no-default condition. Our argument is similar to that of Jermann and Quadrini (2012).

As described in the text, the firm owes an inter-period debt $\frac{b_t}{R_t}$ and intra-period debt q_t at the end of period t, where b_t is to be repaid in period $t+1$ and q_t is to be repaid in period t. At the end of period t, the firm has the chance to default on q_t.

Now, we consider what would happen if the firm defaults on q_t. Once the firm defaults, the bank unilaterally seizes a part of the firm's revenue, $\phi f_t(q_t)$, where $0 \leq \phi < 1$.[11] The amount of seizures, $\phi f_t(q_t)$ may be interpreted ex-ante as collateral that the bank can legitimately seize when the firm defaults. After the seizure, the firm and bank renegotiate the conditions for the firm to continue to operate. Following Jermann and Quadrini (2012), we assume that the firm has all the bargaining power in the renegotiation. The bank has acquired the right to seize the firm at this stage. Here, the seizure of the firm means that the bank takes control of the firm. Recall that S_t defined by (9) is the present discounted value of a firm's future earnings. When the bank chooses seizure, it successfully operates the firm by itself and recovers value S_t with probability ξ, whereas the firm is destroyed with probability $1-\xi$. When the firm is destroyed, the bank obtains nothing. Thus, the expected value that the bank can obtain by liquidation is ξS_t. By contrast, if the bank decides to allow the firm to continue to operate, it can recover its inter-period debt in the next period, the present value of which is $\frac{b_t}{R_t}$. The renegotiation agreement depends on whether ξS_t is larger or smaller than $\frac{b_t}{R_t}$.

A.1 Normal state

First, we consider the case where the firm is in a normal state; that is, the firm is not debt-ridden. In this case, $\frac{b_t}{R_t}$ is smaller than $\mathbb{E}_t[m_{t+1}b_{z,t+1}]$, defined by (7), and thus it is feasible to repay $\frac{b_t}{R_t}$ fully.

- **Case where $\xi S_t > \frac{b_t}{R_t}$:** The firm has to make a payment that leaves the bank indifferent between liquidation and allowing the firm to continue to operate. Thus, the firm has to make payment $\xi S_t - \frac{b_t}{R_t}$ and promise to pay $(1+r_t)\frac{b_t}{R_t}$ at the beginning of the next period. Therefore, the ex-post default value for the firm is

$$(1-\phi)f_t(q_t) - b_{t-1} + \frac{b_t}{R_t} - \left\{\xi S_t - \frac{b_t}{R_t}\right\} + \mathbb{E}_t\left[m_{t+1}V_{t+1}^N\right].$$

- **Case where $\xi S_t \leq \frac{b_t}{R_t}$:** In this case, the optimal choice for the bank is to wait until the next period, when $(1+r_t)\frac{b_t}{R_t}$ is due. In period t, the bank receives no further payments. Thus, the

[11] Because the firm has paid $b_{t-1} - \frac{b_t}{R_t}$, the remaining value of the resources it possesses is $f_t(q_t) - b_{t-1} + \frac{b_t}{R_t}$ after defaulting on q_t. Thus, if the bank were to seize $\phi f(q)$ from the remaining output only, then the seizure should have been feasible only if

$$\phi f_t(q_t) \leq f_t(q_t) - b_{t-1} + \frac{b_t}{R_t}. \tag{13}$$

However, we assume for simplicity of the analysis that the bank can take $\phi f_t(q)$ from the firm owner's pocket and not just from the remaining output of the firm. Thus, here we assume that the bank seizure is not constrained by (13).

ex-post default value for the firm is

$$(1-\phi)f(q_t) - b_{t-1} + \frac{b_t}{R_t} + \mathbb{E}_t\left[m_{t+1}V_{t+1}^N\right].$$

Therefore, the default value is expressed as

$$(1-\phi)f_t(q_t) - b_{t-1} + \frac{b_t}{R_t} - \max\left\{\xi S_t - \frac{b_t}{R_t},\ 0\right\} + \mathbb{E}_t\left[m_{t+1}V_{t+1}^N\right].$$

Enforcement requires that the value of not defaulting is no smaller than the value of default; that is,

$$f_t(q_t) - b_{t-1} + \frac{b_t}{R_t} - q_t \geq (1-\phi)f_t(q_t) - b_{t-1} + \frac{b_t}{R_t} - \max\left\{\xi S_t - \frac{b_t}{R_t},\ 0\right\},$$

which can be rearranged as (1).

A.2 Debt-ridden state

In the debt-ridden state, the value that the bank can expect to obtain by waiting until the next period is $\mathbb{E}_t[m_{t+1}b_{z,t+1}]$, which is larger than ξS_t. Therefore, the optimal choice for the bank in response to the firm's default on q_t is to allow the firm to continue and wait until the next period. Noting that the value for the firm in the debt-ridden state is zero, the ex-post default value for the firm is

$$(1-\phi)f_t(q_t) - b_{t-1} + \frac{b_t}{R_t} + \mathbb{E}_t[m_{t+1} \times 0].$$

Enforcement implies that

$$f_t(q_t) - b_{t-1} + \frac{b_t}{R_t} - q_t + \mathbb{E}_t[m_{t+1} \times 0] \geq (1-\phi)f_t(q_t) - b_{t-1} + \frac{b_t}{R_t} + \mathbb{E}_t[m_{t+1} \times 0],$$

which can be rearranged to $q_t \leq \phi f_t(q_t)$.

B Evolution of Face Value of Debt

When the borrower moves to the debt-ridden state, the face value b_t and the loan rate $\hat{R}_t$ can be assumed as follows such that the present value $b_t/\hat{R}_t$ equals the borrowed amount, $\mathbb{E}_t[m_{t+1}b_{z,t+1}]$. Suppose that the firm moves from the normal state to the debt-ridden state in period t by borrowing $\mathbb{E}_t\left[m_{t+1}b_{z,t+1}\right]$. In this case, the bank makes the loan of the face value b_t at the loan rate $\hat{R}_t$, which are determined by

$$\frac{b_t}{\hat{R}_t} = \mathbb{E}_t\left[m_{t+1}b_{z,t+1}\right].$$

The face value b_t is chosen arbitrarily large such that $r_t b_t > \max_{a_{t+1}}(1+\tau)(1-\phi)f_{t+1}(q_{z,t+1})$, where r_t is the market rate for safe loans. The face value $\{b_{t+j}\}_{j=1}^{\infty}$ of the debt-ridden firm evolves by

$$b_{t+j-1} = (1+\tau)(1-\phi)f_{t+j}(q_{z,t+j}) + \frac{b_{t+j}}{1+r_{t+j}}.$$

Note that $\{b_{t+j}\}_{j=1}^{\infty}$ is increasing and is not repaid in full.[12]

[12] The transversality condition for the face value b_t is not satisfied. However, it is not a problem because the allocation of resources in the debt-ridden state is not dependent on the face value of debt b_t, which is just a number on the balance sheet.

For Online Publication: Appendices

C Equilibrium Conditions

This Appendix lists the complete set of equilibrium conditions for the model.

C.1 Household optimality conditions

The optimality conditions for the household problem described in subsection 2.3 are

$$
\begin{aligned}
w_t &= \gamma_L C_t L_t^{\nu}, \\
\frac{1}{C_t} &= \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} (1 - \delta + r_{t+1}^K) \right], && (14) \\
\frac{1}{(1+r_t)} &= \mathbb{E}_t [m_{t+1}], && (15)
\end{aligned}
$$

where $m_{t+1} \equiv \beta \frac{C_t}{C_{t+1}}$.

C.2 Intermediate goods firms optimality conditions

The conditions for the intermediate goods firm problem described in subsection 2.2 are

$$
\begin{aligned}
& V_{i,t} = \max \left\{ V_{i,t}^N, V_{i,t}^Z \right\}, \\
& V_{i,t}^N = \max \ \pi_{i,t} + \mathbb{E}_t \left[m_{t+1} V_{i,t+1} \right], && (16) \\
& \pi_{i,t} = f_{i,t}(q_{i,t}) - q_{i,t} - b_{i,t-1} + \frac{b_{i,t}}{R_t}, \\
& q_{i,t} \le \phi f_{i,t}(q_{i,t}) + \max \left\{ \xi S_t - \frac{b_{i,t}}{R_t}, \ 0 \right\}, \\
& \pi_{i,t} \ge 0, \\
& \begin{cases} \text{if } \xi S_t - \dfrac{b_{i,t}}{R_t} > 0, & \dfrac{1}{R_t} \left(1 - \dfrac{\mu_{i,t}}{\lambda_{i,t}} \right) = \mathbb{E}_t \left[m_{t+1} \dfrac{\lambda_{i,t+1}}{\lambda_{i,t}} \right], \\ \text{if } \xi S_t - \dfrac{b_{i,t}}{R_t} \le 0, & \dfrac{1}{R_t} = \mathbb{E}_t \left[m_{t+1} \dfrac{\lambda_{i,t+1}}{\lambda_{i,t}} \right], \end{cases} && (17) \\
& 1 + \lambda_{\pi_{i,t}} - \lambda_{i,t} = 0, \\
& r_t^K = \alpha \eta \frac{1 + \phi \frac{\mu_{i,t}}{\lambda_{i,t}}}{1 + \frac{\mu_{i,t}}{\lambda_{i,t}}} \frac{f_{i,t}(q_{i,t})}{k_{i,t}}, \\
& w_t = (1-\alpha) \eta \frac{1 + \phi \frac{\mu_{i,t}}{\lambda_{i,t}}}{1 + \frac{\mu_{i,t}}{\lambda_{i,t}}} \frac{f_{i,t}(q_{i,t})}{l_{i,t}}, \\
& \begin{cases} \pi_{i,t} > 0, & \lambda_{\pi,t} = 0, \\ \pi_{i,t} = 0, & \lambda_{\pi,t} > 0, \end{cases} \\
& y_{i,t} = a_{i,t} k_{i,t}^{\alpha} l_{i,t}^{1-\alpha}, \\
& R_t = 1 + (1-\tau) r_t,
\end{aligned}
$$

$$S_t = \mathbb{E}_t\left[m_{t+1}V_{n,t+1}\right] + \frac{b_{n,t}}{R_t},$$

where $i \in \{n, d\}$.

$$\begin{aligned}
V_t^Z &= \max f_{z,t}(q_{z,t}) - q_{z,t} - b_{t-1} + \mathbb{E}_t\left[m_{t+1}b_{z,t+1}\right], \\
b_{z,t} &= (1+\tau)(1-\phi)f_{z,t}(q_{z,t}) + \mathbb{E}_t\left[m_{t+1}b_{z,t+1}\right], \\
q_{z,t} &= \phi f_{z,t}(q_{z,t}), \\
k_{z,t} &= \left[\phi A_t\left(\frac{r_t^K}{\alpha}\right)^{(1-\alpha)\eta-1}\left(\frac{1-\alpha}{w_t}\right)^{(1-\alpha)\eta}\right]^{\frac{1}{1-\eta}}, \\
l_{z,t} &= \frac{(1-\alpha)r_t^K k_{z,t}}{\alpha w_t}.
\end{aligned} \tag{18}$$

C.3 Final goods firm optimality conditions

The conditions for the final goods firm problem are described in subsection 2.1 are

$$\begin{aligned}
Y_t &= \left[\zeta y_{d,t}^\eta + (1-\zeta)y_{n,t}^\eta\right]^{\frac{1}{\eta}}, \\
A_t &= a_t^\eta Y_t^{1-\eta}.
\end{aligned}$$

C.4 Exogenous Processes

The productivity shock for risky firms $a_{d,t}$ follows the two-state Markov process.

C.5 Market clearing conditions

$$\begin{aligned}
&C_t + K_t - (1-\delta)K_{t-1} = Y_t, \\
&\zeta l_{d,t} + (1-\zeta)l_{n,t} = L_t, \\
&\zeta k_{d,t} + (1-\zeta)k_{n,t} = K_{t-1}, \\
&\zeta\frac{b_{d,t}}{R_t} + (1-\zeta)\frac{b_{n,t}}{R_t} = \frac{D_t}{1+r_t}.
\end{aligned}$$

D Solving the model

To solve our DSGE model, we closely follow Shirai (2021). Shirai (2021) analyzes the effectiveness of fiscal policy when the debt level is high and extends the algorithm provided by Hirose and Sunakawa (2019) to handle two occasionally binding constraints (OBC). This algorithm combines several methods developed in recent years to solve the model nonlinearly, taking into account the zero lower bound of the nominal interest rate. For more detail on the solution method, see Appendix C on Shirai (2021), and Hirose and Sunakawa (2019).

In our model, there are two OBC: the borrowing constraint (1) and the non-negative dividend constraint (4). Regarding the situation where the two constraints bind or do not bind, four different cases need to be considered. However, the definition of B_L implies that if the borrowing constraint is (2), the non-negativity or the limited liability shall always bind. Hence, we need to consider only three cases and not the situation where the borrowing constraint is (2) and the limited liability constraint does not bind. The three cases are below:

	borrowing constraint	non-negativity constraint
nn	$q_t = \phi f(q_t) + \xi S_t - \dfrac{b_t}{R_t}$	Not bind
nb	$q_t = \phi f(q_t) + \xi S_t - \dfrac{b_t}{R_t}$	bind
bb	$q_t = \phi f(q_t)$	bind

The labels nn, nb, and bb correspond to the state of each constraint.

It is well known that the OBC generates policy function with kinks. The standard approximation method, such as using the Chebyshev polynomials function, has difficulty dealing with kinks. This study adapts an index function approach to deal with kinks. The index function combines the three policy functions, $\psi_{x,nn}$, $\psi_{x,nb}$, and $\psi_{x,bb}$, to generate one new policy function ψ_x for each endogenous variable,

$$\psi_x = \mathbb{1}_{nn}\psi_{x,nn} + \mathbb{1}_{nb}\psi_{x,nb} + (1 - \mathbb{1}_{nn} - \mathbb{1}_{nb})\,\psi_{x,bb},$$

where x represents each endogenous variable, and $\mathbb{1}$ is an index function and defined by:

$$\begin{aligned}
\mathbb{1}_{nn} &= 1 \qquad \text{if } \xi S_t - b_t > 0 \text{ and } \pi_t > 0,\\
&= 0 \qquad \text{otherswise},\\
\mathbb{1}_{nb} &= 1 \qquad \text{if } \xi S_t - b_t > 0 \text{ and } \pi_t \leq 0,\\
&= 0 \qquad \text{otherswise},\\
1 - \mathbb{1}_{nn} - \mathbb{1}_{nb} &= 1 \qquad \text{if } \xi S_t - b_t \leq 0 \text{ and } \pi_t \leq 0,\\
&= 0 \qquad \text{otherswise}.
\end{aligned}$$

Three policy functions, $\psi_{x,nn}$, $\psi_{x,nb}$, and $\psi_{x,bb}$ are smooth functions without kinks, assuming that the OBC does not switch even if the state variables change. For example, $\psi_{x,nn}$ is assumed that the borrowing constraint is always (3) and the limited liability constraint never binds even if the debt level is so high and $\xi S_t - \frac{b_t}{R_t} < 0$. Subsequently, $\psi_{x,nn}$ is a smooth function and has no kinks. Even though each policy function is smooth, kinks can emerge at the switchings of the policy function.

D.1 The total value of the firm

The fixed point iteration is a method to find the solution by computing iteration until the fixed point is found. This method is similar to the value function iteration. As explained above, to deal with occasionally binding constraints, we solve three models assuming that the status of each constraint (binding or non-binding) is invariant. One of these three is a model in which the non-negativity constraint for the dividend is always binding. In this model, the dividend is always zero, and the firm value V_t converges to zero even though it should be positive. In our theory, as long as the level of debt does not exceed B_Z, the firm's equity value is strictly greater than zero ($V_t > 0$) because the debt level eventually returns to its optimal level. The numerical result that $V_t = 0$ is inconsistent with the theory. To avoid this inconsistency, we define the total value of the firm W_t as the sum of the values of the firm owner and the lender:

$$W_t = \frac{(1+r_{-1})}{R_{-1}} b_{-1} + V_t.$$

The total value of the firm is the sum of debt and the present discounted values of dividends. Note that the bank receives $\frac{1+r_{-1}}{R_{-1}} b_{-1}$ from the firm, whereas the net payment for the firm is b_{-1} because the government provides it with a tax advantage $\frac{\tau r_{-1}}{R_{-1}} b_{-1}$. As with V_t, the firm faces the binary choice of whether to stay in the normal state or fall into the debt-ridden state, whereas the firm in the debt-ridden state has no choice other than staying debt-ridden. Thus, given that the firm was in the normal state in period $t-1$, the total value of the firm W_t is given by:

$$W_t = \max \left\{ W_t^N ,\ W_t^Z \right\},$$

where W_t^N is the total value of the firm in the normal state, and W_t^Z is the total value of the firm that is initially in the normal state to move into the debt-ridden state by borrowing additional funds in period t. W_t^N evolves by the following dynamic programming equation:

$$\begin{aligned} W_t^N &= \max_{q,b} \frac{r_{-1}\tau}{R_{t-1}} b_{t-1} + f_t(q_t) - q_t + \mathbb{E}\left[m_{+1} W_{t+1} \right], \qquad (19) \\ \text{subject to} \quad & f_t(q_t) - q_t - b_{t-1} + \frac{b_t}{R_t} \geq 0, \\ & q_t \leq \phi f_t(q_t) + \max \left\{ \xi S_t - \frac{b_t}{R_t},\ 0 \right\}. \end{aligned}$$

W_t^Z, the total value of the firm that borrows additionally to become debt-ridden, is determined by the following dynamic programming equation:

$$\begin{aligned} W_t^Z &= \frac{r_{-1}\tau}{R_{-1}} b_{-1} + f(q_{z,t}) - q_{z,t} + \mathbb{E}[m_{t+1} b_{z,t+1}] \\ \text{subject to} \quad & b_{z,t} = (1+\tau)(1-\phi) f(q_{z,t}) + \mathbb{E}\left[m_{+1} b_{z,t+1} \right], \\ & q_{z,t} \leq \phi f_t(q_{z,t}). \end{aligned}$$

Thus, these dynamic programming problems are equivalent to (5) and (6). We solve the numerical optimization problem using W_t instead of V_t.

D.2 Fixed-point iteration

In this subsection, we solve our DSGE model using the fixed-point iteration with the index function approach. We approximate expectation terms in Euler equations and the value functions and solve them by the fixed-point iteration. To approximate functions, we use a Smolyak-based projection method proposed by Judd et al. (2014) and construct the Smolyak polynomials using extrema of second-order Chebyshev polynomials and unidimensional second-order Chebyshev polynomials.

Following Gust, Herbst, López-Salido and Smith (2017) and Hirose and Sunakawa (2019), we define the expectation functions for expectation terms of the right-hand side in Euler equations (Equation (14), (15), and (17)) and the value function (Equation (16), (18), and (19)) as follows:

$$
\begin{aligned}
e_{C,jj}(h) &\equiv \frac{1}{\beta}\mathbb{E}\left[\frac{C'}{1-\delta+r^{K\prime}}\right], && jj = nn, nb, bb,\\
e_{r,jj}(h) &\equiv \frac{\mathbb{E}\left[C'\right]}{\beta C} - 1, && jj = nn, nb, bb,\\
e_{\mu_n,jj}(h) &\equiv \mathbb{E}\left[m'\frac{\lambda'}{\lambda}\right] R, && jj = nn, nb, bb,\\
e_{V,jj}(h) &\equiv \mathbb{E}\left[m'V'\right], && jj = nn, nb, bb,\\
e_{W,jj}(h) &\equiv \mathbb{E}\left[m'W'\right], && jj = nn, nb, bb,\\
e_{b_z,jj}(h) &\equiv \mathbb{E}\left[m'b_z'\right], && jj = nn, nb, bb,
\end{aligned}
$$

where $h = [K, b_n, b_d, r_{-1}]$ and jj is an index for regimes. In this Appendix, to clarify the notation, we shall use letters without time-subscript to denote current period values and a prime to denote the next period's value.

Step 1 Set an upper bound and a lower bound for each state variable.

Step 2 Set realizations $\{a_{d,high}, a_{d,low}\}$ and transition matrix:

$$
\begin{bmatrix} p_{high} & 1-p_{high} \\ 1-p_{low} & p_{low} \end{bmatrix}.
$$

of the technology shocks, a_d, which follow a 2-state Markov chain.

Step 3 Make an initial guess for expectation functions and value functions:

$$
\begin{aligned}
e^{(0)}_{C,j,jj,g} &= \frac{\overline{C}}{\beta(1-\delta+\overline{r})}, && \text{for } j = 1, 2, \cdots, J, \quad jj = nn, nb, bb, \quad g = high, low,\\
e^{(0)}_{r,j,jj,g} &= \overline{r}, && \text{for } j = 1, 2, \cdots, J, \quad jj = nn, nb, bb, \quad g = high, low,\\
e^{(0)}_{\mu_n,j,jj,g} &= \beta\overline{R} && \text{for } j = 1, 2, \cdots, J, \quad jj = nn, nb, bb, \quad g = high, low,\\
e^{(0)}_{V,j,jj,g} &= \beta\overline{V}, && \text{for } j = 1, 2, \cdots, J, \quad jj = nn, nb, bb, \quad g = high, low,\\
e^{(0)}_{W,j,jj,g} &= \beta\overline{W}, && \text{for } j = 1, 2, \cdots, J, \quad jj = nn, nb, bb, \quad g = high, low,\\
e^{(0)}_{b_z,j,jj,g} &= \beta\overline{b}_z, && \text{for } j = 1, 2, \cdots, J, \quad jj = nn, nb, bb, \quad g = high, low,
\end{aligned}
$$

where j is an index for state variables, J is the total number of grid points and equal to 9 in our setting, and overbars indicate the steady-state value of the corresponding variable.

Step 4 Compute the coefficients for Smolyak polynomials $\boldsymbol{\theta}$:

$$\begin{aligned}
\boldsymbol{\theta}_{C,jj,g} &= \mathfrak{T}(\mathcal{H})^{-1}e^{(i-1)}_{C,jj,g},\\
\boldsymbol{\theta}_{r,jj,g} &= \mathfrak{T}(\mathcal{H})^{-1}e^{(i-1)}_{r,jj,g},\\
\boldsymbol{\theta}_{\mu_n,jj,g} &= \mathfrak{T}(\mathcal{H})^{-1}e^{(i-1)}_{\mu_n,jj,g},\\
\boldsymbol{\theta}_{V,jj,g} &= \mathfrak{T}(\mathcal{H})^{-1}e^{(i-1)}_{V,jj,g},\\
\boldsymbol{\theta}_{W,jj,g} &= \mathfrak{T}(\mathcal{H})^{-1}e^{(i-1)}_{W,jj,g},\\
\boldsymbol{\theta}_{b_z,jj,g} &= \mathfrak{T}(\mathcal{H})^{-1}e^{(i-1)}_{b_z,jj,g},
\end{aligned}$$

where $\boldsymbol{\theta}_{*,jj,g} = [\theta_{*,jj,g,0},\ \theta_{*,jj,g,1},\ \cdots\ ,\theta_{*,jj,g,J}]'$, $e^{(i-1)}_{*,jj,g} = [e^{(i-1)}_{*,1,jj,g},\ \cdots\ e^{(i-1)}_{*,J,jj,g}]'$, $* = \{C,\ r,\ \mu_n,\ V,\ W,\ b_z\}$, $\mathcal{H}$ is a Smolyak grid point, and $\mathfrak{T}(\mathcal{H})$ is a Smolyak basis function. For more detail on the Smolyak grid points and the Smolyak basis function, see Judd et al. (2014) and Appendix C.1 of Shirai (2021).

Step 5 Choose a grid: $h_j = [K_j, b_{d,j}, b_{n,j}, r_j]$. Exogenous variables are set using the grid: $K_{t-1} = K_j,\ b_{d,t-1} = b_{d,j},\ b_{n,t-1} = b_{n,j},\ r_{t-1} = r_j$.

Step 6 Taking as given the productivity ($a_{d,g}$) and the expectation functions previously obtained,

$$\begin{aligned}
&C_{j,jj,g} = e^{(i-1)}_{C,j,jj,g}(h_j),\\
&r_{j,jj,g} = e^{(i-1)}_{r,j,jj,g}(h_j),\\
&\frac{\mu_{n,j,jj,g}}{\lambda_{n,j,jj,g}} = 1 - e^{(i-1)}_{\mu_n,j,jj,g}(h_j),\\
&\mathbb{E}[m'_{j,jj,g}V'_{n,j,jj,g}] = e^{(i-1)}_{V,jj,g}(h_j),\\
&\mathbb{E}[m'_{j,jj,g}W^{N'}_{j,jj,g}] = e^{(i-1)}_{W,jj,g}(h_j),\\
&\mathbb{E}[m'_{j,jj,g}b'_{z,j,jj,g}] = e^{(i-1)}_{b_z,j,jj,g}(h_j),
\end{aligned}$$

Step 7 Solve the dynamics equations for each regime $jj = nn,\ nb,\ bb, \quad g = high, low$

$$\begin{cases}
\text{If regime is in } nn, & \dfrac{\mu_{d,j,jj,g}}{\lambda_{d,j,jj,g}} = \dfrac{\mu_{n,j,jj,g}}{\lambda_{n,j,jj,g}}\\
\text{If regime is in } nb, & \dfrac{\mu_{d,j,jj,g}}{\lambda_{d,j,jj,g}} \text{ is given by solving the nonlinear equation (20).}\\
\text{If regime is in } bb, & \begin{cases} \dfrac{1+\phi\frac{\mu_{d,j,jj,g}}{\lambda_{d,j,jj,g}}}{1+\frac{\mu_{d,j,jj,g}}{\lambda_{d,j,jj,g}}} = \dfrac{\phi}{\eta},\\ \dfrac{\mu_{d,jj,g}}{\lambda_{d,jj,g}} = \dfrac{(\eta-\phi)}{\phi(1-\eta)} \end{cases}
\end{cases}$$

Calculate each equation sequentially at time t.

$$\Omega_{j,jj,g} \equiv \frac{f(q_{d,j,jj,g})}{f(q_{n,j,jj,g})} = \left(\frac{\frac{1+\phi\frac{\mu_{d,j,jj,g}}{\lambda_{d,j,jj,g}}}{1+\frac{\mu_{d,j,jj,g}}{\lambda_{d,j,jj,g}}}}{\frac{1+\phi\frac{\mu_{n,j,jj,g}}{\lambda_{n,j,jj,g}}}{1+\frac{\mu_{n,j,jj,g}}{\lambda_{n,j,jj,g}}}} \left(\frac{a_{d,j}}{a_{n,j}}\right)^{\eta} \right)^{\frac{1}{1-\eta}}$$

$$\Psi_{y,j,jj,g} \equiv \left[\zeta\Omega^{\eta}_{j,jj,g}\left(\frac{a_{d,j}}{a_{n,j}}\right)^{\eta} + 1 - \zeta\right]^{\frac{1}{\eta}},$$

$$\Psi_{k,j,jj,g} \equiv \zeta\Omega_{j,jj,g} + 1 - \zeta,$$

$$k_{n,j,jj,g} = \frac{K_j}{\Psi_{k,j,j}},$$

$$l_{n,j,jj,g} = \left[\frac{(1-\alpha)\eta \frac{1+\phi\frac{\mu_{n,j,jj,g}}{\lambda_{n,j,jj,g}}}{1+\frac{\mu_{n,j,jj,g}}{\lambda_{n,j,jj,g}}}\Psi_{y,j,jj,g}^{1-\eta} a k_{n,j,jj,g}^{\alpha}}{\Psi_{k,j,jj,g}^{\nu}\gamma_L C_{j,jj,g}}\right]^{\frac{1}{\alpha+\nu}},$$

$$L_{j,jj,g} = \Psi_{k,j,jj,g} l_{n,j,jj,g},$$

$$w_{j,jj,g} = \gamma_L C_{j,jj,g} L_{j,jj,g}^{\nu},$$

$$y_{n,j,jj,g} = a k_{n,j,jj,g}^{\alpha} l_{n,j,jj,g}^{1-\alpha},$$

$$Y_{j,jj,g} = \Psi_{y,j,jj,g} y_{n,j,jj,g},$$

$$A_{j,jj,g} = a^{\eta} Y_{j,jj,g}^{1-\eta},$$

$$A_{d,j,jj,g} = a_{d,g}^{\eta} Y_{j,jj,g}^{1-\eta},$$

$$r_{j,jj,g}^{K} = \frac{\alpha\eta\left(1+\phi\frac{\mu_{n,j,jj,g}}{\lambda_{n,j,jj,g}}\right)}{1+\frac{\mu_{n,j,jj,g}}{\lambda_{n,j,jj,g}}}\frac{A_{j,jj,g} k_{n,j,jj,g}^{\alpha\eta} l_{n,j,jj,g}^{(1-\alpha)\eta}}{k_{n,j,jj,g}},$$

$$K'_{j,jj,g} = Y_{j,jj,g} - C_{j,jj,g} + (1-\delta)K_j,$$

$$q_{n,j,jj,g} = w_{j,jj,g} l_{n,j,jj,g} + r_{j,jj,g}^{K} k_{n,j,jj,g},$$

$$f(q_{n,j,jj,g}) = A_{j,jj,g} k_{n,j,jj,g}^{\alpha\eta} l_{n,j,jj,g}^{(1-\alpha)\eta},$$

$$R_{j,jj,g} = 1 + r_{j,jj,g}(1-\tau),$$

$$k_{d,j,jj,g} = \left[\frac{1+\phi\frac{\mu_{d,j,jj,g}}{\lambda_{d,j,jj,g}}}{1+\frac{\mu_{d,j,jj,g}}{\lambda_{d,j,jj,g}}}\eta A_{d,j,jj,g}\left(\frac{r_{j,jj,g}^{K}}{\alpha}\right)^{(1-\alpha)\eta-1}\left(\frac{1-\alpha}{w_{j,jj,g}}\right)^{(1-\alpha)\eta}\right]^{\frac{1}{1-\eta}},$$

$$l_{d,j,jj,g} = \frac{(1-\alpha) r_{j,jj,g}^{K} k_{d,j,jj,g}}{\alpha w_{j,jj,g}},$$

$$k_{z,j,jj,g} = \left[\phi A_{d,j,jj,g}\left(\frac{r_{j,jj,g}^{K}}{\alpha}\right)^{(1-\alpha)\eta-1}\left(\frac{1-\alpha}{w_{j,jj,g}}\right)^{(1-\alpha)\eta}\right]^{\frac{1}{1-\eta}},$$

$$l_{z,j,jj,g} = \frac{(1-\alpha) r_{j,jj,g}^{K} k_{z,j,jj,g}}{\alpha w_{j,jj,g}},$$

$$q_{d,j,jj,g} = w_{j,jj,g} l_{d,j,jj,g} + r_{j,jj,g}^{K} k_{d,j,jj,g},$$

$$f(q_{d,j,jj,g}) = A_{d,j,jj,g} k_{d,j,jj,g}^{\alpha\eta} l_{d,j,jj,g}^{(1-\alpha)\eta},$$

$$f(q_{z,j,jj,g}) = A_{d,j,jj,g} k_{z,j,jj,g}^{\alpha\eta} l_{z,j,jj,g}^{(1-\alpha)\eta},$$

$$b'_{n,j,jj,g} = \frac{R_{j,jj,g}}{1-\xi}\left[\phi f(q_{n,j,jj,g}) - q_{n,j,jj,g} + \xi e_{V,jj,g}^{(i-1)}(h_j)\right],$$

$$S_{j,jj,g} = e_{V,jj,g}^{(i-1)}(h_j) + \frac{b'_{n,j,jj,g}}{R_{j,jj,g}},$$

$$\pi_{n,j,jj,g} = f(q_{n,j,jj,g}) - q_{n,j,jj,g} - b_{n,j} + \frac{b'_{n,jj,g}}{R_{j,jj,g}},$$

$$V_{n,j,jj,g} = \pi_{n,j,jj,g} + \mathbb{E}[m'_{j,jj,g}V'_{n,j,jj,g}],$$
$$\begin{cases} \text{If } jj = nn, & b'_{d,j,jj,g} = R_{j,jj,g}\left[\phi f(q_{d,j,jj,g}) - q_{d,j,jj,g} + \xi S_{j,jj,g}\right], \\ & \pi_{d,j,jj,g} = f(q_{d,j,jj,g}) - q_{d,j,jj,g} - b_{d,j} + \frac{b'_{d,j,jj,g}}{R_{j,jj,g}}, \\ \text{If } jj = nb, & b'_{d,j,jj,g} = R_{j,jj,g}\left[\phi f(q_{d,j,jj,g}) - q_{d,j,jj,g} + \xi S_{j,jj,g}\right], \\ & \pi_{d,j,jj,g} = 0, \\ \text{If } jj = bb, & b'_{d,j,jj,g} = R_{j,jj,g}\left[b_{d,j} - f(q_{d,j,jj,g}) + q_{d,j,jj,g}\right], \\ & \pi_{d,j,jj,g} = 0, \end{cases}$$
$$R_j = 1 + r_j(1-\tau), \qquad \Longleftrightarrow R_{t-1} = 1 + r_{t-1}(1-\tau),$$
$$W^N_{j,jj,g} = \frac{r_j\tau}{R_j}b_{d,j} + f(q_{d,j,jj,g}) - q_{d,j,jj,g} + \mathbb{E}[m'_{j,jj,g}W^{N'}_{j,jj,g}],$$
$$b_{z,j,jj,g} = (1+\tau)(1-\phi)f(q_{z,j,jj,g}) + \mathbb{E}[m'_{j,jj,g}b'_{z,j,jj,g}],$$

If regime is in nb, solve for $\mu_{d,j,jj,g}/\lambda_{d,j,jj,g}$ with the equation below numerically: [13]

$$0 = f(q_{d,j,jj,g}) - q_{d,j,jj,g} - b_{d,j} + \frac{b'_{d,j,jj,g}}{R_{j,jj,g}}. \tag{20}$$

Interpolate between grids using Smolyak polynomials:

$$\begin{aligned} \hat{e}_C(h'_{j,jj,g};\boldsymbol{\theta}_{C,jj,g}) &= \mathfrak{T}(\varphi(h'_{j,jj,g}))\boldsymbol{\theta}_{C,jj,g}, \\ \hat{e}_r(h'_{j,jj,g};\boldsymbol{\theta}_{r,jj,g}) &= \mathfrak{T}(\varphi(h'_{j,jj,g}))\boldsymbol{\theta}_{r,jj,g}, \\ \hat{e}_{\mu_n}(h'_{j,jj,g};\boldsymbol{\theta}_{\mu_n,jj,g}) &= \mathfrak{T}(\varphi(h'_{j,jj,g}))\boldsymbol{\theta}_{\mu,jj,g}, \\ \hat{e}_V(h'_{j,jj,g};\boldsymbol{\theta}_{V,jj,g}) &= \mathfrak{T}(\varphi(h'_{j,jj,g}))\boldsymbol{\theta}_{V,jj,g}, \\ \hat{e}_W(h'_{j,jj,g};\boldsymbol{\theta}_{W,jj,g}) &= \mathfrak{T}(\varphi(h'_{j,jj,g}))\boldsymbol{\theta}_{W,jj,g}, \\ \hat{e}_{b_z}(h'_{j,jj,g};\boldsymbol{\theta}_{b_z,jj,g}) &= \mathfrak{T}(\varphi(h'_{j,jj,g}))\boldsymbol{\theta}_{b_z,j,jj,g}, \end{aligned}$$

where $h'_{j,jj,g} = [K'_{j,jj,g},\ b'_{n,j,jj,g},\ b'_{d,j,jj,g},\ r_{j,jj,g}]$. The domain of Chebyshev polynomials is the interval $[-1,1]$, and in order to approximate a function by the Chebyshev polynomials, it is necessary to transform the interval $h_j \in [h^{\min},\ h^{\max}]$ into the interval of $x_j \in [-1,1]$, $h^{\min}$ and $h^{\max}$ are each state variable's maximum and minimum values chosen to encompass a wide interval. For each state variable in h, we use $\varphi : [h^{\min},\ h^{\max}] \to [-1,\ 1]$ for $\{K, b_n, b_d, r\}$,

$$x_j = \varphi(h_j) = \frac{2(h_j - h^{\min}) - (h^{\max} - h^{\min})}{h^{\max} - h^{\min}}.$$

In calculating $t+1$, $a_{d,g'}$ is given. Firstly, we assume that $\xi S'_{j,jj,g'} - \frac{b''_{d,j,jj,g'}}{R'_{j,jj,g'}} > 0$, $\pi'_{d,j,jj,g'} > 0$,

[13] For example, fsolve is a numerical solver in Matlab.

the regime is in nn, and the expectation term is given by the followings:

$$\begin{aligned}
&C'_{j,jj,g'} = \hat{e}_C(h'_{j,jj,g'}; \boldsymbol{\theta}_{C,nn,g'}),\\
&r'_{j,jj,g'} = \hat{e}_r(h'_{j,jj,g'}; \boldsymbol{\theta}_{r,nn,g'}),\\
&\frac{\mu'_{n,j,jj,g'}}{\lambda'_{n,j,jj,g'}} = 1 - \hat{e}_{\mu_n}(h'_{j,jj,g'}; \boldsymbol{\theta}_{\mu,nn,g'}),\\
&\mathbb{E}[m''_{j,jj,g'} V''_{n,j,jj,g'}] = \hat{e}_V(h'_{j,jj,g'}; \boldsymbol{\theta}_{V,nn,g'}),\\
&\mathbb{E}[m''_{j,jj,g'} W^{N''}_{j,jj,g'}] = \hat{e}_W(h'_{j,jj,g'}; \boldsymbol{\theta}_{W,nn,g'}),\\
&\mathbb{E}[m''_{j,jj,g'} b''_{z,j,jj,g'}] = \hat{e}_{b_z}(h'_{j,jj,g'}; \boldsymbol{\theta}_{b_z,nn,g'}).
\end{aligned}$$

Calculate Step 7 for time $t+1$. Check $\pi'_{d,j,jj,g'} > 0$, and if it is satisfied, go to Step 8. If it is not satisfied, next we assume that $\xi S'_{j,jj,g'} - \frac{b''_{d,j,jj,g'}}{R'_{j,jj,g'}} > 0$, $\pi'_{d,j,jj,g'} < 0$, the regime is in nb, and calculate as the following:

$$\begin{aligned}
&C'_{j,jj,g'} = \hat{e}_C(h'_{j,jj,g'}; \boldsymbol{\theta}_{C,nb,g'}),\\
&r'_{j,jj,g'} = \hat{e}_r(h'_{j,jj,g'}; \boldsymbol{\theta}_{r,nb,g'}),\\
&\frac{\mu'_{n,j,jj,g'}}{\lambda'_{n,j,jj,g'}} = 1 - \hat{e}_{\mu_n}(h'_{j,jj,g'}; \boldsymbol{\theta}_{\mu,nb,g'}),\\
&\mathbb{E}[m''_{j,jj,g'} V''_{n,j,jj,g'}] = \hat{e}_V(h'_{j,jj,g'}; \boldsymbol{\theta}_{V,nb,g'}),\\
&\mathbb{E}[m''_{j,jj,g'} W^{N''}_{j,jj,g'}] = \hat{e}_W(h'_{j,jj,g'}; \boldsymbol{\theta}_{W,nb,g'}),\\
&\mathbb{E}[m''_{j,jj,g'} b''_{z,j,jj,g'}] = \hat{e}_{b_z}(h'_{j,jj,g'}; \boldsymbol{\theta}_{b_z,nb,g'}).\\
&\frac{\mu'_{d,j,jj,g'}}{\lambda'_{d,j,jj,g'}} \text{ is given by solving the nonlinear equation (20).}
\end{aligned}$$

Calculate Step 7 for time $t+1$ and go to Step 8.

Step 8 Calculate expectation terms:

$$\begin{aligned}
\mathbb{E}\left[\frac{C'_{j,jj,g}}{\beta(1-\delta+r^{K'}_{j,jj,g})}\right] &= \sum_{g'=\{high,\ low\}} prob(a_{d,g'}|a_{d,g}) \frac{C'_{j,jj,g'}}{\beta(1-\delta+r'_{j,jj,g'})},\\
\mathbb{E}\left[\frac{C'_{j,jj,g}}{\beta C_{j,jj,g}}\right] - 1 &= \sum_{g'=\{high,\ low\}} prob(a_{d,g'}|a_{d,g}) \left(\frac{C'_{j,jj,g'}}{\beta C_{j,jj}} - 1\right),\\
\mathbb{E}\left[\frac{1}{C'_{j,jj,g}}\right] \beta C_{j,jj,g} R_{j,jj,g} &= \sum_{g'=\{high,\ low\}} prob(a_{d,g'}|a_{d,g}) \frac{1}{C'_{j,jj,g'}} \beta C_{j,jj,g} R_{j,jj,g},\\
\mathbb{E}\left[m'_{j,jj,g} V'_{n,j,jj,g}\right] &= \sum_{g'=\{high,\ low\}} prob(a_{d,g'}|a_{d,g}) \frac{\beta C_{j,jj,g}}{C'_{j,jj,g'}} V'_{j,jj,g'},\\
\mathbb{E}\left[m'_{j,jj,g} W'_{j,jj,g}\right] &= \sum_{g'=\{high,\ low\}} prob(a_{d,g'}|a_{d,g}) \frac{\beta C_{j,jj,g}}{C'_{j,jj,g'}} W'_{j,jj,g'},\\
\mathbb{E}\left[m'_{j,jj,g} b'_{z,j,jj,g}\right] &= \sum_{g'=\{high,\ low\}} prob(a_{d,g'}|a_{d,g}) \frac{\beta C_{j,jj,g}}{C'_{j,jj,g'}} b'_{z,j,jj},
\end{aligned}$$

Step 9 Next, substitute in the policy functions:

$$\begin{aligned}
\psi^{(i)}_{*,j,jj,g} &= *_{j,jj,g},\\
e^{(i)}_{C,j,jj,g} &= \mathbb{E}\left[\frac{C'_{j,jj,g}}{\beta(1-\delta+r^{K'}_{j,jj,g})}\right],\\
e^{(i)}_{r,j,jj,g} &= \mathbb{E}\left[\frac{C'_{j,jj,g}}{\beta C_{j,jj,g}}\right]-1\\
e^{(i)}_{\mu_n,j,jj,g} &= \mathbb{E}\left[\frac{1}{C'_{j,jj,g}}\right]\beta C_{j,jj,g}R_{j,jj,g}\\
e^{(i)}_{V,j,jj,g} &= \mathbb{E}\left[m'_{j,jj,g}V'_{n,j,jj,g}\right],\\
e^{(i)}_{W,j,jj,g} &= \mathbb{E}\left[m'_{j,jj,g}W'_{d,j,jj,g}\right],\\
e^{(i)}_{b_z,j,jj,g} &= \mathbb{E}\left[m'_{j,jj,g}b'_{z,j,jj,g}\right],
\end{aligned}$$

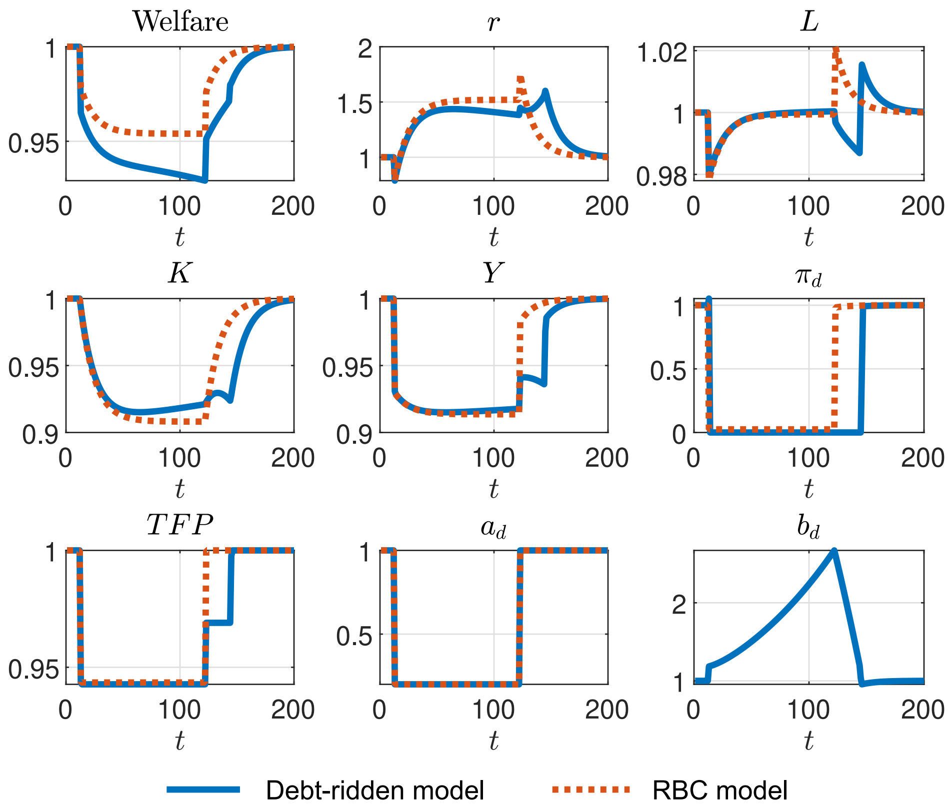
where $\psi^{(i)}_{*,jj}(h;\boldsymbol{\theta})$ is a policy function, $*_{j,jj,g} = \Big\{C_{j,jj,g},\ V_{n,j,jj,g},\ K'_{j,jj,g},\ r_{j,jj,g},\ w_{j,jj,g},\ \pi_{n,j,jj,g},\ \pi_{d,j,jj,g},$ $b'_{n,j,jj,g}, b'_{d,j,jj,g}, y_{n,j,jj,g}, y_{d,j,jj,g},\ \mu_{n,j,jj,g}, \mu_{d,j,jj,g}\Big\}$.

Step 10 If $||\psi^{(i)} - \psi^{(i-1)}|| > 10^{-6}$, update the policy functions and expectation functions by $\psi(i) = \delta_\psi \psi^{(i-1)} + (1-\delta_\psi)\psi^{(i)}$ and $e^{(i)} = \delta_\psi e^{(i-1)} + (1-\delta_\psi)e^{(i)}$, respectively, where δ_ψ is set to 0.8, and go back to Step 5. If $||\psi^{(i)} - \psi^{(i-1)}|| \leq 10^{-6}$, end.

E Robustness analysis

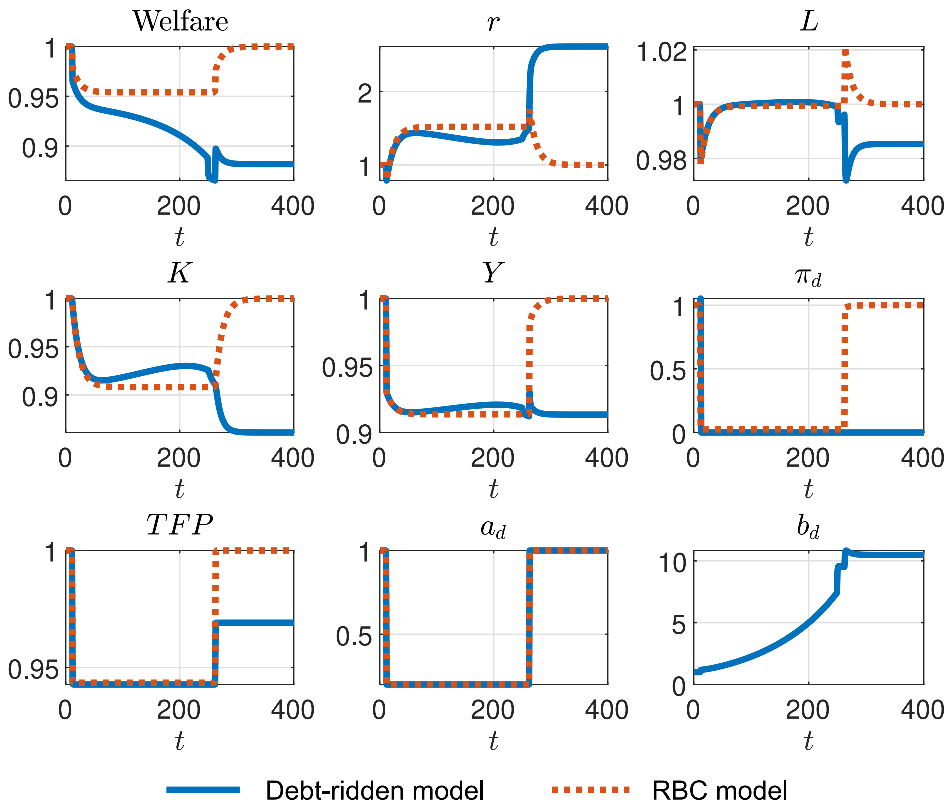
In the main text of this paper, we calibrate $a_{d,low} = 0$ to fit the average peak-to-trough decline for the U.S. real per capita GDP across nine major financial crises. However, this setting might seem extreme. In this appendix, we show that the main results are robust even if we set that $a_{d,low} = 0.2$.

Figure 7 and Figure 8 look almost identical to Figure 4 and Figure 5, respectively. Figure 7 also shows persistence due to the accumulation of debt. Figure 8 also shows that risky firms fall into the debt-ridden state. However, if $a_{d,low} > 0.2903$, risky firms do not fall into the debt-ridden state because $W^N_t \leq W^Z_t$ do not happen.



Notes: The vertical axis shows the index as 1 = the steady state value.
π_d: Risky firm's dividend, f_d: Risky firm's revenue (production),
a_d: Risky firm's productivity, b_d: Risky firm's debt

Figure 7: RBC model vs. Debt-ridden model: $a_{d,low} = 0.2$



Notes: The vertical axis shows the index as 1 = the steady state value.
π_d: Risky firm's dividend, f_d: Risky firm's revenue (production),
a_d: Risky firm's productivity, b_d: Risky firm's debt

Figure 8: Long-term depression and debt-ridden firms: $a_{d,low} = 0.2$