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Land Speculation and Wobbly Dynamics with Endogenous Phase Transitions[#]

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Abstract

This paper examines the global macro-dynamics of a dynamic model with capital and land with rational expectations. Through the interactions between capital accumulation and land prices, the economy experiences phase transitions, endogenously moving from back and forth from situations with unique and multiple momentary equilibria. Consequently, there can be a plethora of rational expectation equilibria trajectories, without any smooth convergence properties, neither converging to a steady state or even to a limit cycle—what we call "wobbly" macro-dynamics. The price of land and other key macro variables (wages, interest rates, output, consumption, wealth, capital stock) endogenously fluctuate within a well-identified range with repeated boom-bust cycles. The key disturbance to the economy is endogenous; even with rational expectations, there can be real estate booms, with resource allocation deteriorating as land prices increase, crowding out productive investments; but such unsustainable land price booms inevitably are followed by a crash. We analyze the set of parameter values for which wobbly fluctuations occur, show that with some parameter values, the only r.e. trajectories involve such wobbly dynamics, demonstrate how changes in parameters affect global macro-dynamics, and show how policy interventions can affect stability and social welfare.

Key words: Interactions between land prices and capital, Critical point, Endogenous phase transitions, Endogenous crash

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1: Introduction

Land speculation is a big concern in many countries. There are, in particular, worries that it contributes to macroeconomic instability. This paper provides a theoretical model to understand economic fluctuations associated with real estate booms and busts from a new perspective.

To examine the effect of land speculation on macroeconomic fluctuations, we construct a standard two-period overlapping generations model, where there is a competitive economy with capital, land, and labor. We show that under not implausible conditions, multiplicity of momentary equilibria can arise. We explore the implications of this for global macro-dynamics, showing that there can be a plethora of rational expectation equilibria trajectories, without any smooth convergence properties, neither converging to a steady state or even to a limit cycle—what we call "wobbly" macro-dynamics. The price of land and other key macro variables (wages, interest rates, output, consumption, wealth, capital stock) endogenously fluctuate within a well-identified range with repeated boom-bust cycles.

Key to understanding wobbly fluctuations is that ever-increasing or ever-decreasing land prices are unsustainable, but whether land prices increase or decrease, and at what rate, depends on the interest rate, and that depends on the capital stock. Because of the multiplicity of momentary equilibrium, an economy with seemingly exploding land prices can suddenly find itself with land prices crashing. Importantly, for our analysis, these fluctuations, marked by booms and busts in land prices, occur as part of equilibrium outcomes, in a way totally consistent with rational expectations. By contrast, in standard macro models, neither seemingly explosive paths nor persistent wobbly can be part of an equilibrium trajectory.

There are complex interactions between capital accumulation and land prices. Land crowds out productive capital, and that means that higher land prices lead to less capital accumulation. That, in turn, leads to higher interest rates, necessitating still faster increases in land prices. In the standard models, this potential "explosion" of prices leads to the saddle point property: a unique initial price converging to the steady state. But in wobbly dynamics, there is an alternative: the economy eventually "switches" to a low return momentary equilibrium, inducing an endogenous crash in land prices. Being able to make such a switch requires, however, that there be multiple equilibria; and whether there are multiple equilibria itself depends on the price of land. As the price of land changes, the economy goes through endogenous phase transitions, moving from a state with a unique equilibrium to one with multiple equilibria back to one where there is a unique equilibrium.

In our wobbly economy, the key "disturbance" to the economy is endogenous, i.e., endogenous changes in expectations and land prices. When individuals have bullish expectations and expect the returns to land to be high, land prices rise. If they remain bullish for an extended period of time, land prices will continue to increase. But, once prices pass a critical level, individuals' expectations can suddenly change into bearish, with returns expected to be low; the land price boom then breaks, and prices start to decline. If the price of land exceeds another (higher) critical threshold, they *must* become bearish if the trajectory is to be consistent with rational expectations. We emphasize that this occurs in a rational expectations framework with common knowledge.

A symmetric analysis applies to the crash: decreases in land prices which look like implosions inconsistent with rational expectations can still be part of a rational expectations trajectory. When land prices fall far enough, there is again an endogenous phase transition, and when prices fall below that level, they can, at any moment, start to increase as expectations suddenly turn bullish; and when land prices fall still further, below another critical level, expectations on a r.e. trajectory *must* turn bullish (for otherwise, there would be a price implosion that would be inconsistent with r.e.)

While the assumption of rational expectations does not constrain the economy to a single trajectory, it imposes strong constraints, which limit, for instance, both the maximum and minimum values of the price of land. Once the land price increases beyond a certain threshold, there must be an endogenous collapse. That, while in general in wobbly dynamics, there are many feasible trajectories going forward, all agents know that when land prices exceed a certain level, land prices will collapse, and therefore trajectories which might put land prices above that threshold are avoided. Similarly, there *may* be a lower bound (greater than zero) to land prices. These bounds on land prices in turn generate bounds on other relevant macroeconomic variables.

We also show the sense in which the presence of land may exacerbate instability. It can induce a large change in global dynamics, creating opportunities for the economy to fluctuate even when in the absence of land, there is a unique momentary equilibrium, a unique steady-state, and a unique dynamic trajectory starting from any initial capital stock. This may be true even when land is totally unproductive.

In the case of the fixed coefficients technology upon which much of the analysis here is focused, there can be episodic involuntary unemployment. In the model with land, a steady state equilibrium may not exist, even when an economy with exactly the same parameters without land has a steady state with full employment. In such a case, the only possible r.e. trajectory entails wobbles. More broadly, the presence of land can have adverse effects on the *range* within which output and employment fluctuate.

Contrast with standard rational expectations models

The analysis of this paper reaches conclusions markedly different from those concerning the dynamics of rational expectations economies in models with an infinitely lived representative agent. We've already noted one important difference: the multiplicity of paths.

Another is that in our model, economic agents only need to look one period forward, to ensure that next period's price is within the range for which there is a feasible rational expectations path. This is in sharp contrast with the standard models in which economic agents need to formulate expectations infinitely far into the future and figure out the (typically unique) equilibrium path. The difficulty of doing so, especially in the absence of a complete set of futures markets, outside the representative agent model, has become a standard critique of prevailing dynamic models assuming rational expectations.[1]

[1] In the representative agent model, there is no trade—and in that sense, no real economy. Individuals can ascertain the dynamic path satisfying the transversality conditions by introspection. But when there are many individuals, this doesn't work. Without markets to coordinate, there is no assurance that the economy will be on a sustainable trajectory. Indeed, with two assets, the market can be on a trajectory that satisfies the capital

Contrast with standard overlapping generations models

The results of our paper also contrast with much of the standard life-cycle literature. Earlier studies of life cycle models with land have shown that in general there cannot exist steady states that are dynamically inefficient (see the discussion of the related literature below), thereby resolving the concerns raised by Diamond (1965) on oversaving. For instance, in the case of zero labor growth and technological change, dynamic inefficiency requires a negative return to capital. But a negative return to capital would imply an infinite value to land, so long as it yielded any positive return. But we go beyond steady state analysis: in our wobbly economy with land, there still can be dynamic inefficiency. Indeed, many of the wobbly trajectories that we identify exhibit periods of such inefficiency. At the same time we emphasize that wobbly dynamics can occur even if the economy is dynamically efficient. Dynamic inefficiency is not a necessary condition for wobbly dynamics to arise.

A particularly analytically interesting case is that in which land is nothing more than an alternative store of value, i.e. yields no dividends. When land nonetheless has a positive price, it is often referred to as a bubble, with its value today depending simply on the ability to sell the asset onto the next generation. We show that under some parameter conditions, there is no steady-state with positive values of land, that is, *the only possible rational expectations equilibrium is a wobbly economy,* and that there are indeed such rational expectations wobbly dynamics. Land prices endogenously fluctuate, rising and falling exponentially, without converging or diverging, neither exploding nor asymptotically converging to zero. These dynamics are substantially different from those in the standard rational expectations bubble literature.

1-1: Related literature

This paper is related to a vast literature on macroeconomics with sunspots or indeterminacy.[2] There are, however, substantial differences between our analysis and these models, both in assumptions and in the resulting dynamics. Most importantly, we revert back to the simplest overlapping generations model, where there is a competitive economy with productive capital, land, and labor, without the frictions and increasing returns that marked this more recent literature.

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arbitrage equation for any finite period going forward, and still not satisfy the transversality conditions. Having established that this was so led Frank Hahn to claim that this was the "golden nail in the coffin of capitalism." See Hahn (1966) and Shell and Stiglitz (1967).

[2] Cass and Shell (1983) is one of the earliest papers in the literature. Farmer (2016, 2020) provides an extensive survey on some of the more recent literature, including classic contributions by Reichlin (1986), Woodford (1986), Muller and Woodford (1988), and Benhabib and Farmer (1994, 1999) that focused on local indeterminacy, i.e., a steady state is locally unique but there are multiple dynamic paths converging to the steady state. Calvo (1978), Woodford (1984), and Muller and Woodford (1988) showed that the presence of land does not rule out local indeterminacy.

Several studies emphasize the role of increasing returns to scale in generating local indeterminacy (Benhabib and Farmer (1994, 1999)). Matsuyama (1991) examined global dynamics, establishing indeterminacy *provided* the (exogenously assumed) increasing returns is sufficiently large.

Farmer (2016, 2020) also identifies a set of second-generation models, which establish the existence of a multiplicity of steady states generated by introducing some frictions, such as search frictions or frictions in nominal wages or prices or a zero lower bound on the interest rates (see also Kocherlakota 2011 and 2020).

There is also a large literature (Calvo (1978), Scheinkman (1980), Woodford (1984), Tirole (1985), McCallum (1987), Muller and Woodford (1988), Rhee (1991), and Mountford (2004)), showing the dynamic efficiency of competitive economies with rational expectations with land.[3] By contrast, as noted in the Introduction, our paper demonstrates that dynamic inefficiency can still arise in a model with land.

There is a much more limited literature trying to reconcile asset booms that crash with a modicum of rationality. Abreu and Brunnermeier (2003) do so in a model with dispersed opinions about the timing of collapse. In our model, dispersion in beliefs plays no role: all agents have the same, rational expectations. Most importantly, Abreu and Brunnermeier (2003) is a partial equilibrium model without investment and production, while our paper has a dynamic general equilibrium framework with investment and production, in which there is a strong interaction between land prices and capital; and in our model (unlike theirs) there is a critical price at which the land price boom *must* break.

Our paper is perhaps most closely related to Matsuyama (2013) who constructs an OLG model with credit frictions. There are two main differences from Matsuyama (2013). In Matsuyama's paper, the momentary equilibrium is always unique, while in our paper the economy endogenously moves between a state with a unique momentary equilibrium and a state with multiplicity of momentary equilibrium. Moreover, our paper mainly focuses on the interactions between asset prices and global macro-dynamics—it is high land prices which endogenously lead eventually to a real estate crash *along a rational expectations path*-- while Matsuyama's model abstracts from asset prices.[4] In Matsuyama's model, an endogenous deterioration in credit allocation results in the endogenous collapse in output.

There are several studies that have explored the relationship between asset prices and macroeconomic fluctuations, including Bernanke, Gertler, and Gilchrist (1999), Kiyotaki and Moore (1997), and Brunnermeier and Sanikov (2014). However, there are substantial differences between these papers and ours. Firstly, these papers focus on sustainable asset price increases, constructing standard models with a saddle point, in which it is simply hypothesized that somehow the economy finds the saddle point trajectory. Policy interventions can result in a higher *level* of land prices, but in the long run, land prices are stable (or increase at the rate of land augmenting technological progress.) In our paper, episodically land prices increase at a rate which is unsustainable, later, endogenously

[3] There is also vast literature on sunspot in monetary economies in an overlapping generations model starting from Samuelson (1958) (see Gale 1973; Azariadis 1981; Grandmont 1985; Azariadis and Guesnerie 1986). In that literature, the existence of fiat money plays a key role, analogous to land in our model. There are, however, substantial differences between that literature and our paper. Firstly, other than Grandmont's paper, the literature focuses on endowment economies, and in his model there is no capital and the dynamical system is one-dimensional. By contrast, our model has capital and land, and the deep interactions between capital and land prices play a crucial role in generating fluctuations. Secondly, these papers showed the existence of deterministic cycles with various periods. By contrast, our main analysis focuses on wobbly fluctuations instead of deterministic cycles, though we also show the possibility of deterministic cycles.

[4] Still another paper that is somewhat related to ours is Allen, Barlevy, and Gale (2021). They focus on the misallocation of resources during asset booms within an overlapping generations endowment economy. By introducing costly default exogenously (a fall in output when an asset boom ends, associated with an exogenous change in assets' dividends), they show that borrowers undertake excessively risky investments during asset booms.

crashing.[5] Secondly, in their papers, the key shock to the economy is exogenous, i.e., exogenous changes in productivity, while in ours, it is endogenous, i.e., endogenous changes in expectations. Thirdly, in their papers, asset price increase leads to improved resource allocation, because the increased value of collateral provides more scope for entrepreneurship.[6] By contrast, in our paper, resource allocation deteriorates as land prices increase, in the sense that increasing land prices crowd out capital accumulation.

The present paper is an extension of Hirano and Stiglitz (2021a), which studies an OLG model without land, and where depending on the level of capital, there will either be a unique or multiple momentary equilibrium. Here, we introduce another store of value, productive land. As we will see, this markedly changes global dynamics.

2. The Model with Land

2.1: The basic model

We construct a standard two-period overlapping generations model where there is a competitive economy with productive capital, land, and labor. The two-period overlapping generations model is the simplest model with heterogeneous agents—there are just the young and the old. It illustrates speculative behaviour of heterogeneous agents trading assets with each other—individuals buying an asset largely on the basis of beliefs (here assumed to be rational) of what they can sell it for. Here there is a simple basis of this heterogeneity: the young buy land from the old, their parents' generation, in anticipation of exiting the market by selling that land to the next generation, that of their children.[7]

In each period young agents are born and live for two periods. Each young person is endowed with one unit of labor when young, and supplies it inelastically, receiving wage income, w_t. Each young person also has e units of consumption goods as an endowment (e.g., which can be thought of as "other fixed income," such as "dividends" from the ownership of trees),[8] and saves a fraction s_t of the total income $W_t \equiv w_t + e$. The savings of each young person at date t finances land holdings and capital investment at date t. Capital investment at date t becomes capital stock at date $t+1$, which determines the rate of return on capital and wages that period. What is not saved determines first period

[5] Guido et al. (2019) proposed an empirical method of the rational expectations framework to allow for temporarily unstable paths by introducing multiplicative sunspot shocks. Although their paper explores US inflation dynamics through the lens of a simple New Keynesian model, and focuses on empirical analysis, our paper is conceptually related to their paper, in the point that both focus on temporarily explosive paths.

[6] In their papers, entrepreneurs need collateral to get finance; there is a scarcity of this collateral; increasing the effective supply allows more entrepreneurship. To the contrary, instead of using fixed assets like land as collateral, much of small business lending is secured by accounts receivable, which increase as production and sales increase (see Lian and Ma (2021)). Also, these models typically don't have a fully articulated theory of collateral/borrowing constraints, which should be endogenous, based explicitly on models of imperfect information and/or noise, as in Stiglitz and Weiss (1981).

[7] There are of course other ways of modeling asset trading in a model with heterogeneous agents. In Hirano and Yanagawa (2017), asset trading occurs between agents with different productivities, and in Scheinkman and Xiong (2001) or Guzman and Stiglitz (2021), it occurs between agents with heterogeneous beliefs. Random shocks affecting different individuals differently—with some possibly facing credit rationing—can also give rise to asset trading. See the discussion in section 7.

[8] We provide discussions about the role of e in Hirano and Stiglitz (2021a).

consumption of $c_{1t} \geq 0$, and the return on savings plus asset sales to the next generation determine second period consumption $c_{2t} \geq 0$.

The production function of this economy is $Y_t = F(K_t, L_t, T_t)$, where Y_t is output at time t, K_t is aggregate capital stock at date t, and L_t and T_t are labor force and land at date t. The aggregate supply of land is fixed $T_t = T$. Output per capita can be written as $y_t = f(\frac{K_t}{L_t}, \frac{T_t}{L_t})$. For simplicity, we also assume the labor force is fixed $L_t = L$. This means that $\frac{T_t}{L_t}$ is fixed, and, without loss of generality, we write $y_t = f(k_t)$ where $k_t \equiv \frac{K_t}{L_t}$ and $f_k(k_t)$ is the rental rate of capital, $w_t = f_L(k_t)$ is the wage rate, and $f_T(k_t)$ is the rental rate of land. We normalize L and T at unity, and take produced output (which can be used either for consumption or investment) as our numeraire.

Total returns to owning land have to equal the return to capital, the rental rate minus depreciation, which equals the interest rate.[9] That is, the capital arbitrage equation is

$$(1) \quad \frac{f_T(k_{t+1})}{P_t} + \frac{P_{t+1}}{P_t} = f_k(k_{t+1}) + 1 - \delta = 1 + r_{t+1},$$

where P_t is the market price of land at date t, δ is the rate of depreciation of capital, and $1 + r_{t+1}$ is the interest rate between period t and $t + 1$.

The competitive equilibrium is defined as a set of prices $\{1 + r_t, P_t, w_t\}_{t=0}^{\infty}$ and quantities, $\{c_{1t}, c_{2t}, k_t, y_t\}_{t=0}^{\infty}$, given initial k_0, such that (i) each young agent chooses consumption, land holdings, and capital investment to maximize expected utility under the budget constraints, and (ii) the competitive market clearing condition for goods, land, capital and labor are all satisfied.

The savings/capital accumulation equation is written as

$$(2) \quad k_{t+1} + P_t = s_t(w_t + e).$$

Other things being equal, higher land prices reduce capital investment. *This is the obvious sense in which land holdings crowd out real capital accumulation.*

Equilibrium paths consistent with rational expectations have to satisfy (1) and (2) for all dates. Given P_t and k_t, if s_t were fixed, we could easily solve (1) and (2) for P_{t+1} and k_{t+1}.

But suppose that the saving rate is a function of the interest rate, which in turn depends on k_{t+1}. Then equation (2) can be rewritten as specifying k_{t+1} as a function (correspondence) of k_t and P_t.[10]

$$(3) \quad \Omega(k_{t+1}, P_t) \equiv \frac{k_{t+1}}{s_t(k_{t+1})} + \frac{P_t}{s_t(k_{t+1})} = w(k_t) + e \equiv W(k_t, e).$$

The case of $P_t = 0$

As we have fully explored in Hirano and Stiglitz (2021a), in the model without land (in which case we don't have the term $\frac{P_t}{s_t(k_{t+1})}$), under quite general and plausible conditions regarding the utility and the production functions, Ω is not monotonic in k_{t+1}. The solid line in Figure

[9] We ignore risk, so the actual return on any rational expectations path has to be the same.
[10] Similar results as those presented here hold if the savings rate is also a function of the individual's income.

1-1 illustrates this. Define $\overline{\Omega}$ as the local maximum for Ω and $\underline{\Omega}$ as the local minimum, and correspondingly, $W(\overline{k}, e) = \overline{\Omega}$ and $W(\underline{k}, e) = \underline{\Omega}$. Then for any value of $\underline{k} < k_t < \overline{k}$, there exists three solutions to (3), i.e. three momentary equilibria.

To determine when Ω is not monotonic in k_{t+1}, we differentiate $\Omega(k_{t+1})$ with respect to k_{t+1}.

(4) $\Omega'(k_{t+1}) = \frac{1}{s_t}\left(1 - \frac{dlog(s_t)}{dlog(1+r_{t+1})}\frac{dlog(1+r_{t+1})}{dlog(k_{t+1})} - \frac{P_t}{k_{t+1}}\frac{dlog(s_t)}{dlog(1+r_{t+1})}\frac{dlog(1+r_{t+1})}{dlog(k_{t+1})}\right)$

where $\frac{dlog(s_t)}{dlog(1+r_{t+1})}$ is the interest rate elasticity of savings. $\frac{dlog(1+r_{t+1})}{dlog(k_{t+1})}$ is the elasticity of the interest rate with respect to the capital stock. These elasticities depend on the intertemporal elasticity of substitution in consumption (IES) and the elasticity substitution between capital and labor (ES), respectively. When $P_t = 0$, all we require for multiplicity of momentary equilibria is that $\frac{dlog(s_t)}{dlog(1+r_{t+1})}\frac{dlog(1+r_{t+1})}{dlog(k_{t+1})} > 1$ for some values of k_{t+1} and < 1 for others.

In Hirano and Stiglitz (2021a), we investigated in some detail the special case of CES production and utility functions of the form:

$$Y_t = A\left(\alpha(\frac{K_t}{\omega_1})^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(\frac{L_t}{\omega_2})^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \text{ and } u_t = \left((a_1)^{\frac{1}{\theta}}(c_{1t})^{\frac{\theta-1}{\theta}} + (a_2)^{\frac{1}{\theta}}(c_{2t})^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}},$$

where σ is the elasticity of substitution. A is a productivity parameter, and $\frac{1}{\omega_1}$ and $\frac{1}{\omega_2}$ are parameters reflecting capital and labor productivity, respectively. $\alpha \in (0,1)$ reflects capital intensity in production. θ is the IES. a_1 and a_2 are weights on consumption in the individual's first and second periods (i.e., when young and old, respectively); the relative value is isomorphic to a discount factor. When $\frac{a_1}{a_2} > 1$, individuals put more weight on consumption when young rather than consumption when old, which is equivalent to discounting future consumption. (In Appendix A, we provide a formal analysis.) In the remainder of the paper, we assume $\frac{a_1}{a_2} > 1$.

Under these functional forms, we showed that if $\mathrm{IES} < 1$, the saving rate is a decreasing function of the interest rates, and that if both the ES and IES are sufficiently small (sufficiently less than unity), Ω is non-monotonic in k and there will exist a multiplicity of momentary equilibria. If individuals believe that the interest rates will be low, they save a lot, generating a high level of capital, thereby leading to low interest rates. Conversely, if they believe that the interest rates will be high, they save less and less savings finances less investments, thereby leading to a low level of capital and high interest rates. When the elasticity of substitution is low, a low level of capital accumulation leads to a low share of labor in national income, sustaining the low level of accumulation and the high interest rate.

The presence of multiplicity of momentary equilibria can generate a plethora of trajectories consistent with rational expectations. Figure 2-1 illustrates a typical wobbly trajectory when $P_t = 0$. From Figure 1-1, we derive the correspondence between k_t and k_{t+1}. Within a certain range of k_t, for each k_t, there are three values of k_{t+1} and depending on people's beliefs, the economy wobbles without converging. But in the figure, it is clear that, over the long run, there are bounds on k: $\underline{k} \le k_t \le \overline{k}$, implying bounds on the relevant other variables.

The General Case

In this paper we ask, how does the presence of land change wobbly dynamics? Multiplicity of momentary equilibria can still occur under the same general conditions. We can see in equation (3) that an increase in P_t shifts the function Ω up but (at least for small P_t) there still exists a multiplicity of momentary equilibria for some values of k_t. The dotted line in Figure 1-1 illustrates the situation. The reasoning is changed only slightly: given P_t, if individuals believe that the interest rates will be low, they save a lot, financing a high level of capital accumulation *beyond their land purchases*, thereby leading to low interest rates. Conversely, if they believe that the interest rates will be high, they save less and less savings finances less capital accumulation *beyond their land holdings*, leading to high interest rates.

As before, we can translate these results into a relationship between k_t and k_{t+1}, and see how the correspondence between k_t and k_{t+1} is affected by an increase in P_t. Going back to Figure 1-1, the increase in P_t doesn't shift Ω up uniformly: the amount by which it shifts up is proportional to $\frac{1}{s_t}$. If the savings rate increases with k_{t+1}, it means that an increase in P_t increases $\Omega(k_{t+1})$ more for low values of k_{t+1} than for high values. That means that as the price of land increases, we might go from a situation where corresponding to a particular k_t there were three values of k_{t+1}, now there is a single value of k_{t+1} (that is, initially $\underline{\Omega} \le W(k_t, e) \le \overline{\Omega}$ but as P_t increases, $\underline{\Omega} > W(k_t, e)$; or alternatively, we might go from a situation where there was a single value of k_{t+1}, to one in which there is now multiple values of k_{t+1}, i.e. initially, $\overline{\Omega} < W(k_t, e)$, with the increase in P_t, $\overline{\Omega} > W(k_t, e)$: There is an endogenous transition between regimes in which there is a unique momentary equilibrium and multiple momentary equilibria. Within a certain range of P_t, there are still multiple values of k_{t+1}, given k_t but once land prices reach critical values, there is a single value of k_{t+1}.

This in turn means that if the land price gets too high, for a low value of $W(k_t, e)$ there is a unique equilibrium, entailing low k_{t+1}, high returns, and thus (as we will later show) explosive price dynamics. Thus, before P_t reaches such levels, the economy must switch to the high savings-low return equilibrium, which in turn leads to the crash of real estate prices. Similarly, on the downside: if P_t falls below a certain level, there is a unique momentary equilibrium entailing, for a particular high value of $W(k_t, e)$, low interest rates; we will later show that this entails P_t imploding. In the analysis below, we will explicitly derive those critical values.

Moreover, because Ω is shifted up more for low values of k_{t+1} than for high values, at least for some production and utility functions, including the Leontief utility function upon which we focus in this paper, the range of k_t for which there are multiple equilibria, is increased as P_t rises.[11] This implies an increased range of variability in economic activity.

[11] That is, both $\overline{\Omega}$ and $\underline{\Omega}$ are increased, with the former increasing more than the latter. The increase in the corresponding critical values of k depends on the value of $-(1/kf'')$, the value of which changes with k depending on k'''. However, consider the case where the utility function is of the Leontief form and the production function is of a general CES. $\Omega = (k_{t+1} + P_t)(1 + \frac{a_1}{a_2}(1 + r_{t+1})$. When P_t increases, Ω increases by

Land price dynamics

So far, we have focused on the dynamics of k_t. From (1) we can see the dynamics of P_t which is interlinked with that of k_t:

(1') $\frac{P_{t+1}}{P_t} = 1 + r_{t+1}(k_{t+1}) - \frac{D(k_{t+1})}{P_t} >$ or < 1 as $P_t >$ or $< \frac{D(k_{t+1})}{r_{t+1}(k_{t+1})}$

where $D(k_{t+1}) \equiv f_T(k_{t+1})$: land prices go up or down depending on where P_t is greater or less than $\frac{D(k_{t+1})}{r_{t+1}(k_{t+1})}$.

(1') and (2) define (together with the standard boundary value conditions, $0 \leq \{k_{t+1}, P_t\} < \infty$) the set of r.e. dynamic trajectories. The basic insight of wobbly-dynamics is that because of the multiplicity of momentary equilibria, k_t can suddenly increase dramatically, causing the interest rate to fall, leading prices of land to start declining: while previously, it may have looked as if the economy was on a trajectory with an explosive real estate boom, but with the fall in r, the real estate boom collapses.[12] Of course, we have to check *simultaneously* movements in k and P, showing that they are consistent with wobbly dynamics. The following analysis does this.

Figure 2-2 illustrates wobbly dynamics with endogenous fluctuations in land prices. The curve giving k_{t+1} as a function of k_t constantly moves up and down as the price of land fluctuates. Associated with these fluctuations in land prices, capital and output also fluctuate.

At the same time, we can trace out the dynamics of P, from equation (1'), noting that if the economy selects a high return (a low k) momentary equilibrium, the interest rate r will be high, so the curve giving P_{t+1} as a function of P_t will be steep—land prices will look like they are exploding. As land prices increase, land speculation crowds out capital accumulation, so the rate of interest increases even more. Moreover, wages decrease, lowering capital accumulation further. The explosion in land prices accelerates. But, of course, along a rational expectations equilibrium this can't continue forever, and the market knows this. Thus, at some time, the economy must select a low return equilibrium, leading land prices to collapse, and return to a sustainable level. This has to happen before land prices rise so high that there is a unique momentary equilibrium—providing the upper bound on land prices within wobbly dynamics.

This provides the heuristics of the two-dimensional dynamics. In Hirano and Stiglitz (2021a), where we analyzed dynamics with only capital, we showed the essence of the analysis can be illustrated with the Leontief production and utility functions, in which all possible trajectories can easily be traced out. The Leontief production and utility functions correspond to the limiting case of constant elasticity production and consumption functions,

$1 + \frac{a_1}{a_2}(1 + r_{t+1})$. And $1 + r_{t+1}$ is larger the smaller k_{t+1}. Hence the range of k_t where there are multiple momentary equilibria increases.

[12] The dynamics may seem counterintuitive, since we often think of real estate booms as associated with low interest rates (as in the early years of this century). But we focus here on rational expectations trajectories, where (for given land prices), a lower interest rate means a lower overall return to holding land, and if there are positive land rents, this may necessitate a *fall* in land prices. Our sequel Hirano and Stiglitz (2021b) introduces credit frictions into the present model where capital and land are used as collateral, and shows land price booms associated with low interest rates.

of $\sigma \to 0$ and $\theta \to 0$, respectively. There, we showed that the results of the limiting Leontief model held more generally. The same is true here. Accordingly, in this paper we focus on that case.

2.2: A Parametric model: An analytically tractable Case

The utility function is

(5) $u_t = \min(\frac{c_{1t}}{a_1}, \frac{c_{2t}}{a_2})$.

The optimal consumption between the working period and the retirement period satisfies $\frac{c_{1t}}{a_1} = \frac{c_{2t}}{a_2}$.

The production function is simplified to be

(6) $Y_t = A \min\left[\frac{K_t}{\omega_1}, \frac{L_t}{\omega_2}\right] + DT_t$,

where the return to land is fixed and does not depend on the amount of labor or capital. In a later section we discuss the case where D endogenously changes depending on the amount of capital or labor. $k_t = \frac{\omega_1}{\omega_2} \equiv k^f$ is the per capita capital level that just generates full employment and full capital utilization. If $k_t < k^f$, where there is capital shortage, involuntary unemployment occurs, while $k_t > k^f$, where there is capital surplus, full employment is achieved but there is idle capital.[13]

Capital is assumed to depreciate at the fixed rate δ, and the net return to investment when capital is scarce, $\frac{A}{\omega_1} - \delta$, is assumed positive, while the net return when capital is abundant is just $-\delta$.[14]

The Leontief utility function is the extreme of intertemporal consumption smoothing. Under the Leontief utility function given by (5), the saving rate is given by $s_t = \frac{1}{1+\frac{a_1}{a_2}(1+r_{t+1})}$ and it is a decreasing function of the interest rate. Then the savings/capital accumulation equation is written as

(7) $k_{t+1} + P_t = \frac{w_t + e}{1+\frac{a_1}{a_2}(1+r_{t+1})}$.

The capital arbitrage equation becomes

[13] Another interpretation would be that there are different sectors. In one (real estate), land is the primary input (and we simplify by assuming it is the only input. In a later section, we relax this assumption). In the other, capital and labor are the main inputs for production. With this interpretation, our analysis shows how fluctuations in the real estate sector affect production and employment in the other sector.
There is still another interpretation in which D are the returns on equity, itself the income of firms after paying for the costs of labor and capital. Firms are assumed to pay a fixed dividend D on existing shares. Because of the presence of asymmetric information, firms cannot raise new equity (see Greenwald, Stiglitz, and Weiss 1984; Myers and Majluf 1984). More generally, some firms exit from the market in each period and new firms enter the market in such a way that aggregate dividends remain constant. With these interpretations, P_t could be interpreted as stock prices.

[14] The later discussion will make clear that if that is not the case, the price of land has to diminish to zero.

(8) $\frac{D}{P_t} + \frac{P_{t+1}}{P_t} = 1 + r_{t+1} = \begin{cases} \frac{A}{\omega_1} + 1 - \delta \quad if\ k_{t+1} < \frac{\omega_1}{\omega_2} \\ 1 - \delta \quad if\ k_{t+1} > \frac{\omega_1}{\omega_2} \end{cases}$

The key property of the price dynamics which simplifies the analysis is that the price dynamics simply depend on whether $k_{t+1} > or < \frac{\omega_1}{\omega_2}$, i.e. on whether there is unused capital or labor. At $k_{t+1} = k^f$, i.e., the borderline case, the distribution of income is indeterminate (though there may be a unique distribution consistent with the prior period's rational expectations), and with an indeterminate distribution of income, k_{t+1} is indeterminate.

(7) and (8) define the dynamic system for which we will provide a complete global analysis. The model entails five key production and technology parameters, $\frac{A}{\omega_1}, \frac{A}{\omega_2}$, D, e, δ, and one key taste parameter $\frac{a_1}{a_2}$. We solve for six endogenous variables at each date t, $\{y$, k, employment, w, r, and $P\}$. In the following sections, we will demonstrate the remarkable richness of dynamics that can be generated by such a simple model.

2.3: The Implications of the savings-investment equation

Key to preventing prices of land from exploding along a trajectory with a real estate boom (or imploding after the crash) is the existence of multiple momentary equilibria, which allows the economy to switch from a high return equilibrium to a low or vice versa. Earlier, we noted that whether there were multiple momentary equilibria depends on the value of P. We now investigate the ranges of values in our specific model for which there are multiple momentary equilibria.

The function Ω and W are written as

(9) $\Omega(k_{t+1}, P_t) \equiv (k_{t+1} + P_t)\left(1 + \frac{a_1}{a_2}(1 + r_{t+1})\right) = w_t + e \equiv W(w_t, e).$

where r_{t+1} depends on whether $k_{t+1} > or < \frac{\omega_1}{\omega_2}$. Figure 1-2 (which redraws Figure 1-1 for the specific preferences and technology assumed here) illustrates.
$\Omega(k_{t+1}, P_t)$ increases linearly with k_{t+1}, with slope $1 + \frac{a_1}{a_2}\left(1 + \frac{A}{\omega_1} - \delta\right)$, until k^f is reached, then jumps down, and then increases again linearly but now at a lower slope, $1 + \frac{a_1}{a_2}(1 - \delta)$. Moreover, $W(k_t, e) = e$ or $\frac{A}{\omega_2} + e$ depending on whether $k_t < or > \frac{\omega_1}{\omega_2}$. As we can see, the relationship doesn't change much compared to the general case. That is, given k_t and P_t, there can be multiple values of k_{t+1}, consistent with rational expectations.

Review of model without land

We first focus on the case where in the absence of land, there are multiple momentary equilibria both when there is a capital shortage and a capital surplus, which implies that in the absence of land, there exists wobbly dynamics. Setting $P_t = 0$, note that $\Omega(0,0) = 0$. Define

$\Omega_1 \equiv \frac{\omega_1}{\omega_2}\left[1 + \frac{a_1}{a_2}\left(1 + \frac{A}{\omega_1} - \delta\right)\right],$

the value of Ω at just full employment in the capital shortage regime. Define

$$\Omega_2 \equiv \frac{\omega_1}{\omega_2}\left[1 + \frac{a_1}{a_2}(1 - \delta)\right] < \Omega_1 ,$$

the value of Ω at just full employment in the capital surplus regime.

Then a necessary and sufficient condition for there to be multiple equilibria in both the capital shortage and capital surplus regimes is that

(A2a) $\Omega_1 \geq \frac{A}{\omega_2} + e,$

i.e., when there is a capital surplus, so workers appropriate all of national income, the economy can switch to the capital shortage regime; and

(A2b) $\Omega_2 \leq e.$

i.e., when there is a capital shortage, so capital holders receive all of national income, the economy can switch to the capital surplus regime.

If $\frac{a_1}{a_2} > 1$, then $\Omega_1 - \frac{A}{\omega_2} > \Omega_2$ and so there exists values of e for which (A2a) and (A2b) can both be satisfied. When (A2) holds, without land, there exists no stable steady state in the sense that at that steady state k^*, if individuals believed next period there were to be a capital surplus, there exists another r.e. momentary equilibrium in which that is the case, and similarly if they believed that there were to be a capital scarcity. In any steady state, there exists multiple momentary equilibria, so that the economy can move out of the steady state into wobbly dynamics.

We will show below that when there is wobbly dynamics without land, there is always wobbly dynamics with land if D is sufficiently small; but that even when there does not exist multiple momentary equilibria without land, there may exist multiple equilibria with land, so that wobbly dynamics could occur. Thus, the range of parameter values in which wobbly dynamics occurs is increased.

Critical values of land prices

As we have noted, an increase in P_t shifts the function Ω up: Thus, whether there can still be multiplicity of momentary equilibria depends on the level of P_t. By continuity, if (A2) are strictly satisfied, for small P_t there still exists multiple momentary equilibria. We now investigate more precisely the conditions under which multiple equilibria occur. To do this, we derive several critical values of land prices. We first consider only trajectories where there is strictly a capital shortage or surplus. Even if we restrict ourselves to this case, we will show that there can be multiple dynamic paths all consistent with rational expectations in which land prices can endogenously fluctuate without converging.

Wobbly dynamics requires only that when prices are seemingly exploding and becoming high, with the economy in a high return regime, it can switch into a low return regime; and when prices are imploding, and becoming too low, with the economy in a low return regime, it can switch into a high return regime. In Figure 1-2, the dotted line shows how an increase in P_t shifts Ω up. We define the point B where the "low return" line of Ω hits the full employment line $\frac{\omega_1}{\omega_2}$. As P_t increases, B moves up, and eventually, there does not exist a low return (high k) equilibrium when $W_t = e$. There either exists no equilibrium, or only the high return (low k) equilibrium. The maximum value of P before the high k equilibrium disappears is called P_2. It is the solution to

(10a) $\left(\frac{\omega_1}{\omega_2}+P_2\right)\left(1+\frac{a_1}{a_2}(1-\delta)\right)=e$; or $P_2 \equiv \frac{e-\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)}$.

Clearly, for there to exist wobbly dynamics, $P_2>0$, i.e.,

(11a) $e>\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}\equiv e_2$.

Otherwise, once in the capital shortage (high return) equilibrium, the economy could never switch out, and the price of land would increase without bound.[15]

Similarly, define the value of P_t, P_3, where land holdings just crowd out capital accumulation enough that at the high wage there is a capital shortage. P_3 satisfies

(10b) $\left(\frac{\omega_1}{\omega_2}+P_3\right)\left(1+\frac{a_1}{a_2}\left(\frac{A}{\omega_1}+1-\delta\right)\right)=\frac{A}{\omega_2}+e$; or $P_3=\frac{e-\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}-\left(\frac{a_1}{a_2}-1\right)\frac{A}{\omega_2}}{1+\frac{a_1}{a_2}\left(\frac{A}{\omega_1}+1-\delta\right)}$.

$P_2>P_3$ if

(11b) $e>\frac{\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}}{\frac{a_1}{a_2}}=\frac{e_2}{\frac{a_1}{a_2}}$.

So long as $\frac{a_1}{a_2}>1$, i.e., so long as individuals put more weight on consumption when young than when old (which is equivalent to discounting future consumption), (11b) is satisfied if (11a) holds. Note that if $P_3<0$, if the economy is in the low return equilibrium, it can always switch into the high return equilibrium. Moreover,

(11c) $P_3 < or > 0$ as $e < or > \left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}+\left(\frac{a_1}{a_2}-1\right)\frac{A}{\omega_2}=e_2+\left(\frac{a_1}{a_2}-1\right)\frac{A}{\omega_2}\equiv e_3$.

Assume that the economy is in a high return equilibrium, with land prices increasing. Once land prices get near P_2 from below, the only rational expectations equilibrium trajectory is one entailing switching to a low return regime—otherwise, the price the following period will exceed P_2, and it will not be possible to switch to a low return regime, so that prices would have to explode. Once it switches (and the switch can occur well before reaching P_2), land prices start to fall. But they can't fall too far, for we know if they fall below P_3, the unique momentary equilibrium is the low return equilibrium and prices would fall forever, eventually becoming zero or negative (if the capital arbitrage equation is to be satisfied). Hence, so long as the economy switches back to the high return regime before P reaches P_3, prices won't implode.

Tightening the bounds[16]

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[15] Later, we will establish that so long as $P_t>D/((A/\omega_1)-\delta)$ (which will obviously be the case if $D=0$), $P_{t+1}>P_t$.

[16] Later in this paper, when we bring in the possibility of a momentary equilibrium with just full employment,

We can put somewhat tighter bounds on fluctuations by observing that as P increases, land holdings crowd out capital accumulation. There is a critical value of P, P_1, such for any P_t higher than P_1, when wages are low, there is (at most) a single momentary equilibrium entailing capital surplus. P_1 is given by the solution to

(10c) $P_1\left(1+\frac{a_1}{a_2}(\frac{A}{\omega_1}+1-\delta)\right)=e$; or $P_1 \equiv \frac{e}{\left(1+\frac{a_1}{a_2}\left(\frac{A}{\omega_1}+1-\delta\right)\right)}$.

P_1 is the value of P where $\Omega(0,P)=e$, i.e. Ω intersects the vertical axis at e (where wages are zero) and is labelled A in Figure 1-2. Depending on parameter values, $P_1 > P_2$ or $P_2 > P_1$.[17] Also, $P_1 > P_3$ if

(11d) $\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}+\left(\frac{a_1}{a_2}-1\right)\frac{A}{\omega_2}>0.$

This condition does not depend on e. It is clear that if $\frac{a_1}{a_2}>1$, (11d) is always satisfied.

Thus, all that is required for switching to be possible (looking just as the savings-investment equation) is that $P_3 < P_2$, sufficient conditions for which are that (11a) and $\frac{a_1}{a_2}>1$ both be satisfied.

A full dynamic analysis, however, has to look *simultaneously* at the capital arbitrage and the savings-investment equations and has to consider the possibility of a "just full employment" equilibrium. In the general case, that turns out to be conceptually straightforward, but notationally complex, so we first focus in the next section on a special case of interest in its own right, where $D=0$. But in either case, an individual with rational expectations, simply knowing the structure of the economy as we have laid it out, can ascertain whether, given k_0 a particular value of P can be consistent with a rational expectations equilibrium going forward simply by seeing whether the price of land lies within certain bounds. There is a wide range of values of P consistent with rational expectations: there is a fundamental indeterminacy. But this indeterminacy allows us to analyse the price dynamics largely separately from the dynamics of k.

3. Unproductive land (land bubbles)

In this section, we consider the case where $D=0$ so land is just a store of value. Land may have value today simply because it can be sold tomorrow—i.e. it has value tomorrow. Such situations have come to be called "bubbles". From (1), when $D=0$, land prices grow at the rate of the return to capital. Since in both the capital shortage and capital abundance regimes, the return to capital is fixed, this means that in the former, land prices increase exponentially, and in the latter they fall exponentially. This greatly simplifies the analysis.

In the following sections, focusing on trajectories with $P_t>0$, we will show that under some parameter values there is no steady-state, so *the only rational expectations*

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we will see that we will have to loosen the bounds.

[17] $P_1 > or < P_2$ as $e < or > \left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}\left(1+\frac{a_1}{a_2}\left(\frac{A}{\omega_1}+1-\delta\right)\right)/\frac{a_1}{a_2}\frac{A}{\omega_1}$.

equilibrium is a wobbly economy.[18] Later, we will analyse how the set of parameter values for which a steady state and a wobbly rational expectations exists changes as D increases.

3-1: The existence and non-existence of a steady state

We first derive conditions under which a steady-state with positive values of land bubbles exists. If a steady state exists, the net return on capital must be zero—if it is positive, land prices must be ever increasing; if negative, ever decreasing. But this means that the steady state must entail just full employment. The savings rate is then just $\frac{1}{1+\frac{a_1}{a_2}}$. Moreover, the exhaustion of product equation gives us

(12) $\omega_1(r^* + \delta) + \omega_2 w^* = A,$

which determines $w^* = \frac{A}{\omega_2} - \frac{\omega_1}{\omega_2}\delta$ when $r^* = 0$.[19] In steady state P^* is constant, and at $r^* = 0$, can take on any value. Thus, if savings are just sufficient to sustain full employment,[20]

(13) $P^* = \frac{\frac{A}{\omega_2} - \frac{\omega_1}{\omega_2}\delta + e}{1+\frac{a_1}{a_2}} - \frac{\omega_1}{\omega_2}.$

The existence of a steady state with land having a positive value requires $P^* > 0$, i.e. $\frac{\frac{A}{\omega_2} - \frac{\omega_1}{\omega_2}\delta + e}{1+\frac{a_1}{a_2}} > \frac{\omega_1}{\omega_2}$.

Proposition 1. There exists a steady state with positive land prices in a bubble economy if and only if

(14) $e > \frac{\omega_1}{\omega_2}\left[1 + \delta + \frac{a_1}{a_2} - \frac{A}{\omega_1}\right] = e_2 + \left(\frac{a_1}{a_2}\delta + \delta - \frac{A}{\omega_1}\right)\frac{\omega_1}{\omega_2} \equiv \hat{e},$

where, it will be recalled, $e_2 \equiv \left(1 + \frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}$. (14) is always satisfied if $\frac{A}{\omega_1} > 1 + \delta + \frac{a_1}{a_2}$, i.e. if the productivity of capital is high enough. If $\frac{A}{\omega_1} < 1 + \delta + \frac{a_1}{a_2}$, for $P^* > 0$, e has to be sufficiently large.

The intuition is that to sustain just full-employment k and positive land bubbles, the productivity of the economy, $\frac{A}{\omega_1}$ or/and e, has to be large enough so that the economy can generate enough savings, recognizing that some savings is being diverted to holding land. Proposition 1 implies that if $e \leq \hat{e}$, no steady state with land bubbles exists. If there exists a steady state with a land bubble, the wage is set at w^*, with$0 \leq w^* \leq \frac{A}{\omega_2}$.

[18] In addition, there is another set of trajectories, that where $P_t = 0$ for all t—i.e. the rational expectations trajectories of a landless economy.

[19] Observe that $w^* > 0$ under our hypothesis that $\frac{A}{\omega_1} > \delta$.

[20] This is derived directly from the steady-state investment equals savings equation: $k^f + P^* = s^*(w^* + e)$, where $s^* = \frac{1}{1+\frac{a_1}{a_2}}$ and $P^* \geq 0$.

Stability of the full employment steady state

It is clear from Figure 1-2[21] that that implies that at that wage there are three momentary equilibria. That in turn implies that the steady state full employment equilibrium is not stable in the sense defined earlier. Of course, we have to check that such a deviation is consistent with a r.e. trajectory going off infinitely far into the future. We now show that that is in general the case, by exploring in greater detail wobbly dynamics.

Two limiting cases

Before doing that, however, we need to describe the two other possible steady states that can arise, one entailing $k_t = 0$ for all t; the other $P_t = 0$ for all t. The latter are the steady states analysed in Hirano and Stiglitz (2021a) for a model with no land. In the case of Leontief preferences and technologies, there are three possible steady states when $P_t = 0$ for all t:

a) Capital shortage, with $k^* = \frac{e}{1+\frac{a_1}{a_2}\left(1+\frac{A}{\omega_1}-\delta\right)} < \frac{\omega_1}{\omega_2}$

b) Capital surplus, with $k^* = \frac{\frac{A}{\omega_2}+e}{1+\frac{a_1}{a_2}(1-\delta)} > \frac{\omega_1}{\omega_2}$

c) Full employment of both labor and capital, with $k^* = \frac{\omega_1}{\omega_2}$ and $0 \leq w^* \leq \frac{A}{\omega_2}$.

We define $e_0 \equiv \frac{\omega_1}{\omega_2}\left[1+\frac{a_1}{a_2}\left(1+\frac{A}{\omega_1}-\delta\right)\right] = e_2 + \frac{a_1}{a_2}\frac{A}{\omega_2}$ and $e_{00} \equiv \frac{\omega_1}{\omega_2}\left[1+\frac{a_1}{a_2}(1-\delta)\right] - \frac{A}{\omega_2}$. Then

i) if $e_{00} < e < e_0$, steady states with a capital shortage, a capital surplus, and with just full employment of both labor and capital can exist,

ii) if $e > e_0$, only a capital surplus steady state can exist, and

iii) if $e < e_{00}$, only a capital shortage steady state can exist.

The no capital (effectively a pure endowment economy) steady state entails $r = 0$ and $P = P^*$ with P sufficiently high that it absorbs all of savings. From (9) we have

$\Omega(0, P^*) \equiv \left(1+\frac{a_1}{a_2}\right)P^*$,

And $w^* = 0$, so

$P^* = \frac{e}{1+\frac{a_1}{a_2}}$.

Provided $\delta < \frac{A}{\omega_1}$, however, this steady state is not really a competitive equiibrium, since any firm could pay a return of 0, and invest in capital with a positive net return. When $\delta > \frac{A}{\omega_1}$, capital accumulation is dynamically inefficient. A social security system in which each

[21] The critical feature of the figure is that $P_2 > P_3$, i.e. the price at which the capital surplus equilibrium disappears when $W_t = e$ is higher than the price at which the capital shortage equilibrium disappears when $W_t = \frac{A}{\omega_2} + e$.

generation agrees to transfer some of its endowment when young to the old would be a Pareto improvement.

3.2: Wobbly dynamics in the case with $D = 0$

The conditions for wobbly dynamics can now be easily ascertained and compared to those for the existence of a steady state. The price dynamics are now given by

(8') $P_{t+1} - P_t = (1 + r_{t+1} - 1)P_t = r_{t+1}P_t$.

implying that in the high return regime, prices rise at the rate $\frac{A}{\omega_1} + 1 - \delta$; i.e.

(8a') $P_{t+1} = \left(\frac{A}{\omega_1} + 1 - \delta\right) P_t$

And in the low return regime, prices fall at the rate $1 - \delta$; i.e.

(8b') $P_{t+1} = (1 - \delta)P_t$.

Price dynamics with $D = 0$ are illustrated in Figure 3-1, showing the price initially rising exponentially, then falling.

When $D = 0$, the lowest possible land price satisfying both the capital arbitrage and the saving-investment equations along a r.e. trajectory is $P_{min} \equiv max\{0, P_3\}$.

Consider first the case where $P_3 < 0$. Assume for the moment that $P_1 > P_2$. Consider a trajectory which begins with P_t small ($< P_2$). There are multiple momentary equilibria. Assume it chooses the high return equilibrium. The price of land starts to rise. So long as the economy switches back to a low return equilibrium before $P_t = P_2$, it can be on a rational expectations trajectory. If it switches, then the price of land falls exponentially. It can then switch back, at any time, to the high return equilibrium.

As P_t increases, Ω shifts up, to the point where eventually point B in Figure 1-2 rises above the $W_t = e$ line. Just prior to hitting that point, the only equilibrium is that with low returns, for if the economy were to remain in the high return regime, the price the following period would exceed P_2, and would have to increase thereafter. Then, once the economy switches to the low return equilibrium, land prices start falling.

Within the bounds of land prices where there are multiple momentary equilibria, the economy with land can go from one momentary equilibrium to another, with prices of land rising and falling by the arbitrage equation, but not exploding; endogenously fluctuating within a certain range, neither converging nor diverging.

With rising and falling land prices and capital accumulation, aggregate wealth defined as $(k_t + P_t)$ is also rising and falling. Aggregate consumption also shows large swings, i.e., with the savings rate low in the high return regime but wages low, and conversely in the low return regime. (We explore this in more details in Appendix B for the more general case where $D \geq 0$.)[22] Associated with these fluctuations in land prices, other key macro variables (wages, interest rates, output, employment, capital stock) all fluctuate without converging.[23]

The case of $P_3 < 0$ but $P_1 < P_2$, is similar except the moment the price exceeds P_1, the economy switches into the low return regime.

The dynamics for the case of $P_3 > 0$ is the same, with the analysis only slightly more complex. Assume $P_1 > P_2$ and that initially P_t lies between P_3 and P_2. Assume the economy chooses the high return regime. Then P_t increases exponentially. So long as it switches back to the low return equilibrium before P_t exceeds P_2, it can be on a rational expectations trajectory. The case where $P_1 < P_2$ can be handled similarly, except now, when P exceeds P_1, the only equilibrium is the low return equilibrium, and the economy immediately switches to falling prices.

Critical Points and Endogenous phase transitions

Note that there may be a unique momentary equilibrium *consistent with a r.e. trajectory,* even if at those land price, there is still multiple momentary equilibria according to the savings-investment equation. For if the economy doesn't select the "right" equilibrium, the capital arbitrage equations result in land prices moving to values where there is a unique momentary equilibrium, such that going forward prices either implode or explode. We now derive those critical points, which help refine the boundaries of the economy's fluctuations.

If land prices get near P_2 from below, there exists a land price level

$$\hat{P} \equiv min\left(\frac{P_2}{\frac{A}{\omega_1}+1-\delta}, P_1\right)$$

such that if $\hat{P} < P_t$, the low return equilibrium is the unique momentary equilibrium. The reason is obvious: if land prices were to continue to rise (in the high return equilibrium), price the next period would exceed P_2, which would imply that they have to increase forever, inconsistent with a rational expectations trajectory. Hence, after P_t exceeds $\hat{P}$, land prices start to fall, that is, a rational expectations trajectory *must* endogenously become bearish, leading to an endogenous crash. The "bearish" momentary equilibrium with

[22] In standard macroeconomics, there is a unique momentary equilibrium and the behavior of macroeconomy is accordingly deterministic. This view corresponds to the deterministic behavior in classical mechanics in physics. On the other hand, the view of macroeconomy in this paper has a loose similarity with the view of the world in quantum mechanics. There is, however, no theory identifying which of the multiple momentary equilibria the economy chooses at any point of time, and therefore no well-defined set of relative frequencies associated with different states.

[23] The dynamics of real capital can also easily be described. As the price of land rises, capital continues to be crowded out and the curve giving the value of k_{t+1} for any value of k_t accordingly continues to shift down. If the price of land gets too high, then for small values of k_t, the only possible momentary equilibrium entails a low level of k_{t+1}. This means the economy enters into the explosive region. Hence just prior to this date, land prices must fall. A similar analysis holds when the land price gets very low.

collapsing land prices is the *unique* momentary equilibrium.[24] [25] From another angle, as land prices are rising explosively, the crowding out effect gets strong, so the resource allocation deteriorates over time. Once the deterioration exceeds a certain threshold, the endogenous crash in land prices occurs.

The same logic applies as prices fall. Consider the case where $P_3 > 0$. If the economy remains in a low return regime, eventually $P_t < P_{min} \equiv \max\{P_3, 0\}$, after which P would continue to fall, inconsistent with a rational expectations trajectory. If $P_3 \leq 0$, land prices can fall exponential towards zero.

In the low return regime, the land price at date $t+1$ can be written as $P_{t+1} = (1-\delta)P_t$. Hence, P_{t+1} must satisfy $P_{t+1} = (1-\delta)P_t \geq P_{min}$ or $P_t \geq \frac{P_{min}}{1-\delta} \equiv \underline{P}$. This means that when $P_t < \underline{P}$, the only equilibrium is the high return equilibrium—expectations *must* be bearish along a r.e. trajectory. At $\underline{P}$ there is an endogenous phase transition from multiple equilibria to a unique equilibrium. When $P_3 \leq 0$, $\underline{P} = 0$. If $P_3 > 0$, $\underline{P} > 0$, and there is a strictly positive lower bound to land prices, and an even higher bound to land prices at which there can be multiple equilibria. As we will see later, $\underline{P} > 0$ in the case of $D > 0$.

If the initial land price is in the price range of $P_{min} \leq P_t \leq \underline{P}$, the bullish momentary equilibrium with increasing land prices is the unique momentary equilibrium. Once land prices enter into the range of $\underline{P} < P_t < \hat{P}$, then there is a multiplicity of equilibria, with bullish and bearish expectations both being possible. In other words, the economy endogenously enters into a fragile state in which land prices *can* fall at any time. But, this doesn't necessarily mean land prices must fall in this price range. So long as individuals have bullish expectations, land prices can still continue to rise. But, once land prices reach $\hat{P}$, the "bearish" momentary equilibrium with collapsing land prices is the unique momentary equilibrium and the price of land *must* collapse. This means even during the upward movement, the economy goes through endogenous phase transitions twice from a state with a unique momentary equilibrium with bullish expectations to a state with multiplicity of equilibrium where bullish and bearish expectations are both possible, and then to a state with a unique momentary equilibrium with bearish expectations.[26]

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[24] Note that the generation born at date X when $\hat{P} < P_t$ buy land at high prices and sell it at low prices at date $X+1$, which may look inconsistent with utility maximization but as long as all returns from asset holding fall simultaneously (total returns to land equal the return to capital), the analysis is perfectly consistent with individual rationality. The capital arbitrage equation is satisfied. To save for the retirement, there is no choice other than capital investment or buying land. In these bad states, land is decreasing in value, but capital also yields a negative return. Whether such trajectories exhibit collective rationality is a question to which we turn later in the paper.

[25] Land prices will continue to rise so long as the economy is in the capital scarcity regime; when $\hat{P} = P_1$, eventually $P_1 < P_t$. Immediately after land prices exceed P_1, they must fall. This is because for $P_t > P_1$, the low return to land is the only equilibrium rate of return. At that moment there is an endogenous phase transition, from a situation where there are multiple momentary equilibria to one where there is a unique momentary equilibrium, associated with a capital surplus and declining land prices.

[26] There is a rich set of possible phase transitions. For instance, if P_t falls below $\underline{P}$, the economy goes from multiple equilibria to a unique equilibrium: the next equilibrium has to be a capital shortage equilibrium. But at $t+1$ there may be a unique equilibrium (if $\left(\frac{A}{\omega_1} + 1 - \delta\right)(1-\delta)$ is sufficiently less than 1, for then the upward movement still could leave the economy below $\underline{P}$). It might be several periods before the economy

Necessary and sufficient conditions for wobbles

For wobbly dynamics to exist, there is only one more set of conditions that have to be satisfied. When it switches from the high return regime to the low return regime, price falls to 1 – δ of its previous value. When it falls from $min\{P_1, P_2\}$, it cannot fall below P_3, and similarly, when the price increases from near P_3, it cannot exceed P_2.

This means that a *necessary* condition for wobbly dynamics (limiting our analysis now to momentary equilibria with either a capital shortage or surplus) is

$P_2 \geq \left\{P_3\left(\frac{A}{\omega_1} + 1 - \delta\right), 0\right\}$ and $min\{P_1, P_2\} \geq \frac{P_3}{1-\delta}$,

that is,

Proposition 2a. A necessary condition for wobbly dynamics with D = 0 is that

(15a) $P_3 \leq \min\left\{\frac{P_2}{\frac{A}{\omega_1}+1-\delta}, P_1(1-\delta), P_2(1-\delta)\right\}$.

If the economy switches from the high return regime to the low return regime as it approaches $\hat{P}$, arriving at a value of P that is greater than P_3, so that it can once again rise; and if, when it switches from the low return regime at $\underline{P}$ to the high return regime, arriving at a value of P that is sufficiently low that it can once again switch to a low return regime—then clearly wobbly dynamics can be sustained. Thus, Proposition 2b provides sufficient conditions for wobbly dynamics:

Proposition 2b. Sufficient conditions for wobbly dynamics with $D = 0$ to exist are

(15b) $P_3 \leq min\left(\frac{P_2(1-\delta)}{\frac{A}{\omega_1}+1-\delta}, P_1(1-\delta)\right)$,

that is,

(a) $P_2 > 0$ and $P_3 \leq 0$.
(b) $P_2 > 0, P_3 > 0, \frac{P_2}{\frac{A}{\omega_1}+1-\delta} < P_1$, and $\frac{P_2}{\frac{A}{\omega_1}+1-\delta}(1-\delta) \geq P_3$.
(c) $P_2 > 0, P_3 > 0, P_1 < \frac{P_2}{\frac{A}{\omega_1}+1-\delta}$, and $P_1(1-\delta) \geq P_3$.

Because P_i is a function of e, for each of these cases, we can define critical values of e, say, as a function of the other parameters, for which wobbly dynamics exists.

We can define e_{12} as the values of e for which $P_1 = \frac{P_2}{\frac{A}{\omega_1}+1-\delta}$; e_{23} as the values of e for which $\frac{P_2}{\frac{A}{\omega_1}+1-\delta}(1-\delta) = P_3$; and e_{13} as the values of e for which $P_1(1-\delta) = P_3$. e_2 and e_3, defined earlier are just the values of e at which $P_2 = 0$ and $P_3 = 0$, respectively:

exceeds $\underline{P}$ during which there is a unique equilibrium. Similarly for the phase transitions associated with downward movements in prices.

$e_3 = \left(1 + \frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2} + \left(\frac{a_1}{a_2} - 1\right)\frac{A}{\omega_2} = e_2 + \left(\frac{a_1}{a_2} - 1\right)\frac{A}{\omega_2} > e_2$ if $\frac{a_1}{a_2} > 1$ as we have assumed.

We can thus establish

Proposition 2c. Sufficient conditions for wobbly dynamics with $D = 0$, corresponding to the three cases identified in Proposition 2b, are that

(a) $e_2 < e \leq e_3$

(b1) $\delta < \frac{a_2}{a_1}, e_3 < e < e_{23}\ and\ \widehat{\left(\frac{A}{\omega_1}\right)} < \frac{A}{\omega_1},$

(b2) $\delta < \frac{a_2}{a_1}, e_3 < e < min\{e_{23}, e_{12}\}\ and\ \delta < \frac{A}{\omega_1} < \widehat{\left(\frac{A}{\omega_1}\right)}$

(b3) $\frac{a_2}{a_1} \leq \delta < \frac{A}{\omega_1}, e_3 < e < min\{e_{23}, e_{12}\}$

(c1) $\frac{a_2}{a_1} \leq \delta < \frac{A}{\omega_1}, min\{e_3, e_{12}\} < e < e_{13}$

(c2) $\delta < \min\{\frac{A}{\omega_1}, \frac{a_2}{a_1}\} < \widehat{\left(\frac{A}{\omega_1}\right)}, min\{e_3, e_{12}\} < e < e_{13}$

where $\widehat{\left(\frac{A}{\omega_1}\right)} \equiv \frac{\delta\left(1+\frac{a_1}{a_2}(1-\delta)\right)}{1-\frac{a_1}{a_2}\delta}$.

Appendix C shows that the set of values satisfying these restrictions, in each of the cases, is non-empty. While there is a rich set of parameters for which wobbly dynamics exists, wobbly dynamics cannot occur if given $\frac{A}{\omega_1}$, e is too small or too large.

Here, we focus on (a) and (b1). Figure 4-1 and 4-2 shows the regions for the special case of $\frac{\omega_1}{\omega_2} = 1$. The figures take all of the parameters except e and $\frac{A}{\omega_1}$ as given.

In case (a), where $P_3 \leq 0$, $P_2 > 0$ is both the necessary and sufficient condition for wobbly dynamics. This means that in case (a) $e_2 < e \leq e_3$ is the necessary and sufficient condition for wobbly dynamics. We illustrate the wobbly and non-wobbly region, respectively, in Figure 4-1. Wobbly dynamics occurs when $P_3 < 0$, i.e. e lies below e_3, which is a positively sloped straight line; and $P_2 > 0$, i.e. above e_2, which is a horizontal line. Thus, the relevant parameter space is divided into two regions, between e_3 and e_2, in which there are wobbles, and below e_2 and/or and above e_3 in which wobbles cannot exist.[27]

For the conditions of case (b1) to be satisfied, e has to be above $e_3 = \left(1 + \frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2} + \left(\frac{a_1}{a_2} - 1\right)\frac{A}{\omega_2}$, and below $e_{23} = \left(1 + \frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}\left\{1 + \left(\frac{A}{\omega_1} + 1 - \delta\right)\left(\frac{a_1}{a_2} - 1\right)\right\}$, which is a straight line with slope $\left(1 + \frac{a_1}{a_2}(1-\delta)\right)\left(\frac{a_1}{a_2} - 1\right)$ greater than that

[27] Note too that at $A = 0$, $e_3 = e_2$. We limit our attention to the case where the net productivity of capital is positive, i.e. $\frac{A}{\omega_1} > \delta$.

of e_3 and with an intercept at $A = 0$ greater than that for e_3.[28] The other two conditions for (b1), that $\delta < \frac{a_2}{a_1}$ and $\widehat{\left(\frac{A}{\omega_1}\right)} < \frac{A}{\omega_1}$ can be easily satisfied, e.g. for small enough δ or large enough A, as depicted in Figure 4.2. Thus, the wobbly region satisfying sufficient conditions is below e_{23}, above e_3, and to the right of $\widehat{\left(\frac{A}{\omega_1}\right)}$. These are sufficient conditions; wobbly dynamics *may* occur even for some values of the parameters above e_{23} because necessary conditions (15a) is a looser condition than sufficient conditions (15b), so the parameter space of e satisfying (15a) is wider.

Wobbly dynamics and steady states

The parameter space can now be divided into four regions, depending on whether there exists both a r.e. steady state and wobbly dynamics, neither, or only one or the other.

Consider again case (a), for instance. Recall that a necessary and sufficient condition for a r.e. steady state is that $e > \hat{e}$. From (14), $\hat{e}$ is a negatively sloped line with the slope -1 (in the case where $\frac{\omega_1}{\omega_2} = 1$), depicted in Figure 4.1. Hence, between e_3 and e_2, and below the line $\hat{e}$, the only rational expectations equilibrium is a wobbly economy (focusing on $P_t > 0$). Above the lines $\hat{e}$ and e_2, but below e_3 both a steady state and wobbly dynamics exist. Above $\hat{e}$ and e_3, there exists a steady state, but no wobbly dynamics; and below $\hat{e}$ and above e_3 there exists no r.e. trajectory with $P_t > 0$, i.e., the only r.e. trajectories are those where land has a zero price.

A similar analysis holds for case (b1) except now, there always exists a just-full employment steady state equilibrium.[29]

Wobbly dynamics with and without land

We can also compare the set of parameters for which wobbly dynamics exists with and without land. Hirano and Stiglitz (2021a) establish that without land, wobbly dynamics exists if and only if

(16) $e_0 - \frac{A}{\omega_2}\frac{a_1}{a_2} = \left(1 + \frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2} < e < \left(1 + \frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2} + \left(\frac{a_1}{a_2} - 1\right)\frac{A}{\omega_2} = e_0 - \frac{A}{\omega_2}$

(16) is nothing but Condition (a) in Proposition 2c. Thus, it is clear that even when without land no wobbly dynamics can arise, with land, wobbly dynamics can occur, i.e., cases (b) and (c).

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[28] Assuming as we do throughout that $\frac{a_1}{a_2} > 1$. The intercept of e_3 is $\left(1 + \frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}$, the intercept of e_{23} is $\left(1 + \frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}\left\{1 + (1-\delta)\left(\frac{a_1}{a_2} - 1\right)\right\}$.

[29] When we solve $\hat{e} = e_3$ for $\frac{A}{\omega_1}$, we have $\frac{A}{\omega_1} = \frac{\delta\left(1+\frac{a_1}{a_2}\right)}{\frac{a_1}{a_2}}$. Since $\frac{\delta\left(1+\frac{a_1}{a_2}\right)}{\frac{a_1}{a_2}} < \widehat{\left(\frac{A}{\omega_1}\right)}$ if $\frac{a_1}{a_2} > 1$, there is always a just full employment steady state. Also, in case (b1), we cannot divide the parameter space as neatly as we can as in case (a) because $e < e_{23}$ is a sufficient but not necessary condition for wobbly dynamics, except in the case where $\delta = 0$, where it is also necessary. In the more general case, there is a line $\widehat{e_{23}}$ above e_{23}. Above that line, wobbly dynamics does not exist, but between that line and e_{23} wobbly dynamics may or may not exist.

A particularly interesting case is that where the presence of land speculation makes a large difference for global dynamics. To see this, we examine the case where (b1) holds, but without land, there is a unique momentary equilibrium (i.e. (16) is not satisfied) and a unique steady-state where full-employment with a capital surplus, which is the case if

(17) $e > e_0 \equiv e_3 + \frac{A}{\omega_2}$.

Intuitively, large enough e finances more capital, so without land, there is only a capital surplus momentary equilibrium. To see that even when (17) is satisfied, there are a wide set of parameters for which wobbly dynamics exists with land, take again the special case $\frac{\omega_1}{\omega_2} = 1$, and focus on case (b1). Then, from (16) and (17), above the line e_0 without land there is a unique momentary equilibrium and a unique steady state.

Moreover, $e_{23}\left(\frac{A}{\omega_1} = \widehat{\left(\frac{A}{\omega_1}\right)}\right) > e_0\,(\frac{A}{\omega_1} = \widehat{\left(\frac{A}{\omega_1}\right)})$ if and only if $\left[\delta\left(1 + \frac{a_1}{a_2}(1-\delta)\right) + (1-\delta)(1 - \frac{a_1}{a_2}\delta)\right]\left(\frac{a_1}{a_2} - 1\right) > \frac{a_1}{a_2}\delta$, which can be satisfied for δ small enough if (recall that in (b1), $1 > \frac{a_1}{a_2}\delta$.) And e_0 is a linearly increasing function of $\frac{A}{\omega_1}$ with the slope of $\frac{a_1}{a_2}$. Hence, (at least) for sufficiently small δ, the parameter space satisfying $e_0 < e < e_{23}$ and $\frac{A}{\omega_1} > \widehat{\left(\frac{A}{\omega_1}\right)}$ is non-empty. Thus, even when in the absence of land, the only long run r.e. trajectory entails a steady state with full employment, the presence of land bubble speculation can change global dynamics, generating the possibility of endogenous fluctuations between periods with high wages and output with full employment and periods of low wages and output and involuntary unemployment.

We have thus established with land yielding no dividends:

Proposition 2d. (i) If there exists wobbly dynamics without land (for a given set of parameters describing the economy), then there exists wobbly dynamics with land, but the converse is not true, so the set of parameters for which wobbly dynamics exists with land is greater than the set without land. (ii) There are parameters for which, with land, there is no steady state but there exists rational expectations wobbly trajectories. (iii) There are parameters for which without land, there is a unique momentary equilibrium and a unique steady-state where full-employment is achieved but with land, wobbly dynamics exists. (iv) There are parameters for which, with land, there exists both wobbly dynamics and a steady state. (v) There are parameters for which the only r.e. equilibrium entails $P_t = 0$. (vi) There are parameters for which, with land, there exists a steady state with a positive land price but no wobbly dynamics.

Initial conditions

Given an initial value of k, k_0, there is a wide range of initial values of P_0 consistent with a r.e. trajectory. If $k_0 < k^f$, i.e. initially there is capital scarcity, then so long as $P_0 < P_2$, the economy can switch into the capital abundance regime, and prices can start to fall. Similarly, if $k_0 > k^f$, i.e. initially there is capital abundance, then so long as $P_0 > P_3$ the economy can switch into the capital scarcity regime, and prices can start to rise.

These results should be contrasted with those for the standard model with an infinitely lived individual, in which there is a unique value of P_0 for any k consistent with rational expectations. Here, there is a wide range of P_0 (but still bounded) consistent with rational expectations.

3.3. Trajectories with (occasional) full employment

The trajectories considered so far oscillate between capital shortage and full employment. But we have also noted the possibility of a just full employment momentary equilibrium. This increases the set of possible r.e. trajectories, because rather than "choosing" the high k (low r) or low k (high r) equilibria, the economy may move to the full employment equilibrium, with the (rationally expected) return that just generates full employment. Of course, once there, the economy will, in general, have again multiple momentary equilibria, either moving then to a capital scarcity equilibrium, a capital shortage equilibrium, or staying at full employment.

Consider, for instance, (a) in Proposition 2b. Once P_t exceeded $\hat{P}$, we argued—ignoring the possibility just discussed-- that the economy had to switch to the capital surplus regime. But it could have switched to the just full employment regime. We now analyze that situation.

We first show under some conditions, the upper bound to land prices is still P_2. That is, *if land prices get near P_2 from below, land prices will surely fall, even if there are still two momentary equilibria for prices above P_2.*

Suppose that the economy at date $t-1$ expected the high return regime at t, i.e., $W_t = e$, and at date t, $P_t \geq \hat{P}$, and the economy switches into the just full employment equilibrium. If $P_t = \hat{P}$, then r_{t+1} satisifies

$$\left(\frac{\omega_1}{\omega_2} + \hat{P}\right)\left(1 + \frac{a_1}{a_2}(1 + r_{t+1})\right) = e.$$

Let us, for instance, consider the case of $\hat{P} = \frac{P_2}{\frac{A}{\omega_1}+1-\delta}$ (we can similarly analyse the case of $\hat{P} = P_1$.) If $r_{t+1} < 0$, then $P_{t+1} < P_t$, so land prices fall even if the economy switches to the just full employment equilibrium. From the above equation and the definition of $\hat{P}$, we can ascertain that $r_{t+1} < 0$ entails

$$e\left[\frac{\left(\frac{A}{\omega_1}-\delta\right)\left(1+\frac{a_1}{a_2}(1-\delta)\right)-\frac{a_1}{a_2}\delta}{\left(1+\frac{a_1}{a_2}\right)\left(1+\frac{a_1}{a_2}(1-\delta)\right)}\right] < \frac{\omega_1}{\omega_2}\left(\frac{A}{\omega_1} - \delta\right).$$

There are two cases to consider.

(i) If

$$\frac{A}{\omega_1} \leq \delta + \frac{\frac{a_1}{a_2}\delta}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)},$$

r_{t+1} would always be negative.

(ii) On the other hand, if $\frac{A}{\omega_1} > \delta + \frac{\frac{a_1}{a_2}\delta}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)}$,

we have that $r_{t+1} < 0$ if

$$e < \frac{\frac{\omega_1}{\omega_2}\left(\frac{A}{\omega_1}-\delta\right)\left(1+\frac{a_1}{a_2}\right)\left(1+\frac{a_1}{a_2}(1-\delta)\right)}{\left[\left(\frac{A}{\omega_1}-\delta\right)\left(1+\frac{a_1}{a_2}(1-\delta)\right)-\frac{a_1}{a_2}\delta\right]} \equiv e_m,$$

Therefore, in case (a) of Proposition 2b discussed above, if $e_2 < e < min\{e_m, e_3\}$, if $P_t \geq \hat{P}$ land prices must fall, irrespective of whether the economy chooses the capital surplus regime or the just full employment regime, though they fall more if the economy moves to the capital surplus regime. Note that because as $\frac{A}{\omega_1} \to \delta + \frac{\frac{a_1}{a_2}\delta}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)}$, $e_m \to \text{infinity}$, for $\frac{A}{\omega_1}$ near enough to $\frac{\frac{a_1}{a_2}\delta}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)}$, it is *always* true in case (a) that when P exceeds $\hat{P}$, there has to be a price collapse.

By the same reasoning as above, even if $e_3 > e > \max\{e_m, e_2\}$ there is an upper bound to land prices greater than $\hat{P}$. We call that upper bound $\widehat{P_2}$.

We now calculate $\widehat{P_2}$. Returning to Figure 1-2, note that in the intermediate regime (just full employment), w_t will normally be strictly positive, so the value of P_t at which there can be multiple equilibrium is higher than the case for $w_t = 0$. But $\widehat{P_2}$ can only be achieved if the previous period r is positive, which means from the exhaustion of product equation

$$\omega_1(r_{t+1} + \delta) + \omega_2 w_{t+1} = A,$$

we can define a bound on wages:

$$w_{t+1} \leq \frac{A}{\omega_2} - \delta\,\frac{\omega_1}{\omega_2}.$$

And that means that if

$$\left(\frac{\omega_1}{\omega_2} + P_{t+1}\right)\left(1 + \frac{a_1}{a_2}(1-\delta)\right) > \frac{A}{\omega_2} - \delta\frac{\omega_1}{\omega_2} + e,$$

the only equilibrium is the capital scarcity equilibrium, so price continues to rise forever. Thus, the upper bound to land prices is given by

$$\widehat{P_2} = \frac{\frac{A}{\omega_2}-\delta\frac{\omega_1}{\omega_2}+e}{1+\frac{a_1}{a_2}(1-\delta)} - \frac{\omega_1}{\omega_2}$$

We can similarly define a new lower bound to prices, $\widehat{P_3}$.

The phase transitions are, in some sense, more interesting in this case. Under some trajectories, we move from three momentary equilibria to two, and then to one. In others, we follow the pattern discussed earlier, from three equilibria to a unique equilibrium. We can, as before, define maximum and minimum values of P, $\hat{\hat{P}}$ and $\underline{\underline{P}}$ above which and below which there is a unique r.e. momentary equilibrium.

While we can still identify an upper and lower bound to land prices and the other relevant variables in the economy, including the intermediate value enriches the set of feasible wobbly trajectories and the possible patterns of endogenous phase transitions. And because the range of prices has been increased and the possible magnitudes of the changes in prices decreased (because the interest rate in the upswing may be lower and in the downswing large), the range of parameters for which wobbly dynamics exists can be greater than analyzed earlier in this paper.

Land price boom-bust cycles

Figure 5 illustrates endogenous land price boom-bust cycles for the case where $P_3 > 0$. Within the bounds $\underline{P} \leq P_t \leq \hat{P}$, a boom can always crash, and a downturn can always be reversed. Once prices go above $\hat{P}$, there has to be a crash, and once prices go below $\underline{P}$, there has to be a boom.

3.4. Discussion

The dynamics in our wobbly bubble economy is substantially different from that in the standard rational bubble literature (see, for instance, Tirole 1985; Farhi and Tirole 2012; Hirano and Yanagawa 2017). That literature showed that dynamics of bubbles can exhibit three patterns.

The first is that bubbles become too large and explode. Because such dynamic paths cannot be sustained, backward induction argument rules out such explosive paths from being part of a r.e. trajectory. A second pattern is that if an initial bubble price is lower than the price corresponding to the saddle-point path, the bubble price continues to decrease monotonically, converging to zero.

By contrast, in our wobbly bubble economy, *seemingly* explosive paths can occur fully consistent with rational expectations, but before P reaches a critical level, land prices start to decline; and after falling on a path *seemingly* converging to zero, land prices can start rising again before hitting a critical threshold.

The third pattern is the saddle point trajectory: if the initial bubble price is set just right, the economy converges to a steady state with a positive price of land. Along this saddle path, prices can increase over time but the extent of the increase decreases over time---otherwise there would not be convergence to a steady-state. This means that over the long run bubbles must grow at exactly the same rate as the economy.

By contrast, in our bubble economy, there is a wide range of initial prices consistent with r.e. trajectories. The economy wobbles without converging to a steady-state. While the average rate of price increase must equal that of the economy, there are periods in which land prices grow faster, others in which it grows more slowly. During land booms, when the return to land is high (there is capital scarcity), land prices grow faster than the economy. Indeed the economy's "real" growth rate (the increase in GDP, which excludes from "income" individuals' capital gains) is negative; the increase in land values crowds out real capital accumulation. In our wobbly economy, paths that look like they are unsustainable—and would be unsustainable if they continued forever—can still occur temporarily, and indeed can persist for a long time. Such a land bubble cannot occur in standard models with rational expectations because such a land bubble would have to be explosive.

Moreover, in the standard rational bubble model such as Tirole (1985), asset bubbles are so effective in crowding out unproductive capital that there cannot be over-savings, thus restoring dynamic efficiency to the OLG model. By contrast, in our fuller analysis of a dynamic OLG model with land, there can be periods in which there is over-savings (the net return to capital is negative). Moreover, associated with exponentially rising land prices, productive capital is crowded out, reducing output and employment. Furthermore, for those parameter values where wobbly dynamics can occur, even if a steady-state exists, that steady state is not stable, in the particular sense defined above.

3-5: $\boldsymbol{n + l}$ Period deterministic bubble cycles

The existence of a wobbly bubble economy also implies the existence of deterministic bubble cycles, and because of the simple exponential growth and decline of prices, they are easy to calculate. Given an initial land price bubble $P_0 < P_2$, a land bubble with l periods of expansion and n periods of contractions will have $P_n = \left(\frac{A}{\omega_1} + 1 - \delta\right)^l (1-\delta)^n P_0$. Given the set of parameters describing the economy, any integer values of l and $n \geq 1$ for which $P_n = P_0$ holds i.e. for which $\left(\frac{A}{\omega_1} + 1 - \delta\right)^l (1-\delta)^n = 1$, generates a deterministic cycle. Taking logs, we require:

$$l/n = -\ln(1-\delta)/\ln\left(\frac{A}{\omega_1} + 1 - \delta\right).$$

In general, the RHS of this equation is not an integer, but for long enough cycles, there are integer values of n and l such that the ratio is arbitrarily close to the RHS. Alternatively, by changing the two key parameters ($\frac{A}{\omega_1}$, δ) the RHS takes on an integer value. This leads to the following Proposition providing a sufficient condition for deterministic cycles.[30] .

Proposition 3 (Existence of $\boldsymbol{n + l}$ period bubble cycles): For the parameter values for which $P_3 \leq 0 \; and \; P_2 > 0$, there always exist an approximate deterministic bubble cycle, and for any given value of δ, there exists values of $\frac{A}{\omega_1}$ for which there exists fixed period deterministic bubble cycles.

When $P_3 \leq 0$, the lower bound of the cycle can be arbitrarily close to zero, so that it can take an arbitrarily large number of periods before reaching the upper bound (i.e. before reaching $\hat{P}$.

3.6. Comparative statics: *How the nature of fluctuations depends on key parameters*

We can more fully characterize the dynamics by considering what happens as key parameters change; in particular, we establish conditions under which either the boom or the bust may be of long duration, with large asymmetries in expansion and contraction. The land price boom and bust is governed by (8), and in particular the slope of the relevant lines

[30] It should be clear that if $P_3 > 0$, for some values of the parameters, there also exists deterministic cycles.

in Figure 3-1, $\left(\frac{A}{\omega_1} + 1 - \delta\right)$ in the boom, $1 - \delta$ in the bust. Depending on the relative magnitudes, the pace of decline can be greater or less than the pace of increase.

Consider the downward movement for the case where $P_3 > 0$.[31] Let us denote date t as the date where land prices start to fall. Then land prices at date $t + \hat{n}$ can be written as

$P_{t+\hat{n}} = (1-\delta)^{\hat{n}} P_t$. We can then calculate the number of periods for any given P_t before P falls to P_3:

$$\hat{n} = \frac{\ln(P_3/P_t)}{\ln(1-\delta)} = \frac{\ln\left[\frac{e-\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}-\left(\frac{a_1}{a_2}-1\right)\frac{A}{\omega_2}}{1+\frac{a_1}{a_2}(\frac{A}{\omega_1}+1-\delta)}/P_t\right]}{\ln(1-\delta)} = \frac{\ln\left[\frac{e-\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}-\left(\frac{a_1}{a_2}-1\right)\frac{A}{\omega_2}}{1+\frac{a_1}{a_2}(\frac{A}{\omega_1}+1-\delta)}\right]-\ln[P_t]}{\ln(1-\delta)}$$

from which it follows that an increase in δ decreases $\hat{n}$. An increase in δ increases $P_3$and increases the pace with which prices fall. As δ goes to zero, $\hat{n}$ goes to infinity.

More generally, a change in δ changes both the pace of decline and the possible range of decline, i.e. both the highest level to which prices can go before declining, and the lowest price they can go before increasing. For instance, if $\hat{P} = P_1$, then the range of decline, reflected in $\frac{P_3}{P_1}$ is decreased, so that the longest length of a bust is decreased, both because the speed of decline is increased and the ratio of the peak boom to the trough depression price is decreased.[32] Changes in other parameters do not affect the pace of decline, but do affect P_3 and the range of decline.

The upward movement can be analysed similarly. Given an initial price of P_t, we can ask how many periods will it take before the upper bound $\hat{P}$ is reached?[33] Prices increase at the rate of $\frac{A}{\omega_1} + 1 - \delta$, and the number of periods before the upper bound is reached $\hat{m}$ is given by

$$\hat{m} = \frac{\ln(\hat{P})-\ln(P_t)}{\ln\left(\frac{A}{\omega_1}+1-\delta\right)}.$$

Focusing on the case where $P_2 \leq P_1$, this can be rewritten as

$$\hat{m} = \frac{\ln\left[\frac{e-\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)}\right]-\ln(P_t)}{\ln(\frac{A}{\omega_1}+1-\delta)} - 1.$$

An increase in δ lowers the interest rate, slows the rate of increase in land prices, and therefore leads to an increase in the length of the potential land boom. Similarly, for a decrease in $\frac{A}{\omega_1}$. Changes in other parameters do not affect the pace of decline, but do affect

[31] Obviously, we also restrict ourselves to parameters for which wobbly dynamics exists.
[32] P_3 is the lowest price that land can go in a r.e. trajectory, but as we noted earlier, when the price falls below $\underline{P}$, prices must start increasing. Conversely, while P_2 is the highest price along a r.e. trajectory, prices start to decrease once they exceed $\hat{P}$.
[33] We recall that if we include the possibility of just full employment momentary equilibria, the upper bound of prices may exceed $\hat{P}$.

$\hat{P}$ the maximum possible range of price of increase, and therefore the maximal length of decline.

There are thus large asymmetries in the effects of changes in parameters, e.g. an increase in δ leads to shorter but steeper busts but slower and potentially longer lasting booms. When returns in the high return regime are low (a low interest rate environment during an expansion)[34], the boom is long, while if $1-\delta$ is low the bust is short and steep. These asymmetries in dynamics can be seen in Figure 3-1.

4. The General Case of $D \geq 0$.

The previous section laid out a full analysis of the rational expectations trajectories in the case of a pure bubble economy, identifying parameter values for which there exist wobbly dynamics as well as a steady state. Here, we consider the more general case. Not surprisingly, the analysis of the previous section when $D = 0$ is, in most respects, the limiting case of that of this section.[35] Since the analysis is so similar, we limit our attention to the key modifications, entailing (a) the capital arbitrage equation; (b) comparative statics; and (c) the steady state.

4.1: The implications of the capital arbitrage equations

We first investigate the restrictions on the set of rational expectations paths imposed by the capital arbitrage equations. (8) can be rewritten as

(8") $P_{t+1} - P_t = (1 + r_{t+1} - 1)P_t - D = r_{t+1}P_t - D$.

When $k_{t+1} < \frac{\omega_1}{\omega_2}$, i.e. there is capital scarcity, the above equation can be written as

(8a) $P_{t+1} - P_t = \left(\frac{A}{\omega_1} - \delta\right) P_t - D$;

or

(8a') $P_{t+1} = \left(\frac{A}{\omega_1} + 1 - \delta\right) P_t - D$

When $k_{t+1} > \frac{\omega_1}{\omega_2}$, (8) can be rewritten as

(8b) $P_{t+1} - P_t = -\delta P_t - D$;

or

(8b') $P_{t+1} = (1 - \delta)P_t - D$

[34] This is suggestive of the real estate bubble of the first decade of this century. In the concluding remarks, we comment on the applicability of our model to such a situation.

[35] There is one important exception: in steady state, the value of land D/r goes to infinity when r goes to zero if $D > 0$. And the price of land can take on any value in the limiting case of $D = 0, r = 0$. This shows up in some of the mathematics below.

(8a') and (8b') are depicted in Figure 3-2, the former with a slope > 1, the other < 1. Hence, in the low return equilibrium, land prices are always declining; in the high return equilibrium, increasing, provided $P_t > \frac{D}{\frac{A}{\omega_1}-\delta}$.

From (8a), during the boom, land prices increase slightly less than exponentially, but as the boom continues, the rate of increase in land prices increases, converging (as D/P_t gets small), to $\left(\frac{A}{\omega_1} + 1 - \delta\right)$. Conversely, from (8b), during the bust, land prices fall faster than exponentially, with the rate of decrease increasing over time.

Constraints on P_t

We can now define several constraints on the feasible r.e. trajectories:

(i) If $P_t < \frac{D}{\frac{A}{\omega_1}-\delta} \equiv q_h$, we have $P_{t+1} < P_t$ in the high return state. q_h is the point where P_t in the high return state intersects the 45 degree line in Figure 3-2. $q_h > 0$ since $\frac{A}{\omega_1} - \delta > 0$.[36] From this it follows that any P_t less than q_h cannot be part of an equilibrium trajectory, because once such a price is attained, the arbitrage equation implies that the price of land goes down and eventually becomes negative.

(ii) If $P_t < \frac{D}{1-\delta} \equiv q_l$, there is no capital shortage equilibrium because if that were the case, total returns to land would be greater than the low return to capital. q_l is the point where P_t intersects the horizontal axis in the low return state.[37] $q_l > 0$, provided only that $\delta < 1$. If $\delta = 1$, obviously no one would want to invest in capital if they expected there to be a capital surplus.

(iii) If $P_t < \frac{D}{\frac{A}{\omega_1}+1-\delta} < \min\{q_h, q_l\}$, there is no capital surplus equilibrium because total returns to land would then be greater than the return to capital. $\frac{D}{\frac{A}{\omega_1}+1-\delta}$ is the point where P_t intersects the horizontal axis in the high return state.

There are thus two cases: $\frac{A}{\omega_1} > 1$, so $q_h < q_l$ and $\delta < \frac{A}{\omega_1} < 1$, where the reverse inequality holds.[38] In the former case, land prices may fall below q_l, after which there is a unique equilibrium next period entailing a capital shortage, and land prices start to rise. In the other case, obviously, land prices can never get to q_l. In either case, q_h is the lowest possible land price (looking just at the capital arbitrage equation).

4.2: Wobbly dynamics in the case with $D \geq 0$

A necessary condition for wobbly dynamics

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[36] Many of our results do not depend on this being true. In a life cycle model, individuals will save and invest in capital even if the net return is zero, but only if there is no alternative store of value, such as money.

[37] It should be obvious that a negative land price cannot be a part of an equilibrium trajectory. No one would want to sell land at a negative price.

[38] $\frac{A}{\omega_1}$ is the output capital ratio normally assumed to be less than one.

We can put the restrictions provided by the capital arbitrage equation and those from the savings-investment equation derived earlier concerning the values of P for which there can be multiple equilibria together.

Wobbly dynamics involving just capital shortage and surplus regimes exists provided only that $\max\{q_h, q_l, P_3\}$ is sufficiently less than P_2, so that when P_t is near P_2 from below and the economy switches to the low return regime, P_{t+1} is not less than $\max\{q_h, q_l, P_3\}$, and when the economy is near $\max\{q_h, q_l, P_3\}$ and switches to the high return regime, P_{t+1} is not greater than P_2.

It follows that a necessary condition for wobbly dynamics (*limited to momentary equilibria with either a capital shortage or surplus*) is[39]

(18) $P_2 \geq \max\{q_h, q_l, P_3\}\left(\frac{A}{\omega_1} + 1 - \delta\right) - D \; and \; P_2 \geq \frac{\max\{q_h, q_l, P_3\} + D}{1-\delta}$.

Phase Transitions with D > 0

In the previous section, we explained the critical role of phase transitions—when the number of momentary equilibria consistent with rational expectations trajectories changes endogenously. The analysis is little changed from the case of $D = 0$, simply somewhat more complex. Now the critical price above which there is a unique equilibrium (ignoring for the moment the possibility of the just full employment equilibrium) is

$$\hat{P} = \frac{P_2 + D}{\frac{A}{\omega_1} + 1 - \delta} \; if \; P_2 \leq P_1 \; or \; \hat{P} = min\left(\frac{P_2 + D}{\frac{A}{\omega_1} + 1 - \delta}, P_1\right) \; if \; P_2 > P_1$$

Above $\hat{P}$, there is a phase transition from three equilibria to a unique equilibrium.

The lower bound to prices is also modified. Now if the economy were in the low return regime, if prices fall below P_{min}, they would have to fall forever:

$$P_{min}(D) \equiv max\{q_h, P_3\}.$$

Also, $\underline{P}$ is the price below which land prices must start rising.[40] Now

$$\underline{P} \equiv \frac{P_{min}(D) + D}{1-\delta}.$$

Because when $D > 0$, $\underline{P} > 0$, even when $P_3 \leq 0$, in contrast to the case where $D = 0$. Hence the range within which prices must fluctuate is more restricted.

A sufficient condition for wobbly dynamics

A sufficient condition for an endogenous oscillations between the high return regime associated with asset price booms and the low return regime associated with asset price busts and vice versa is given by (if D is not too large)

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[39] As before, once we introduce the possibility of a full employment momentary equilibria, (18) is replaced with a weaker condition.

[40] This follows from observing that now the price difference equation is given by $P_{t+1} = (1-\delta)P_t - D$, so that we require $P_{t+1} = (1-\delta)P_t - D \geq P_{min}$.

(19a) $\widehat{P(D)} \equiv min\left(\frac{P_2+D}{\frac{A}{\omega_1}+1-\delta}, P_1\right) > \frac{P_{\min}(D)+D}{1-\delta} \equiv \underline{P}.$

And

(19b) $\underline{P}\left(\frac{A}{\omega_1}+1-\delta\right) - D < P_2.$

(19a) ensures that even if the price of land falls from $\hat{P}$, the economy won't implode, i.e., the economy can switch back to the high return regime. (19b) ensures that even if the economy has a boom beginning at $\underline{P}$, it won't explode. (19b) is always satisfied if (19a) is satisfied.

For instance, if we consider the parameter space where (16) holds, in which case $P_1 > 0$, $P_2 > 0$ and $P_3 < 0$, (19a) can be written as

(19c) $\hat{P} \equiv min\left(\frac{P_2+D}{\frac{A}{\omega_1}+1-\delta}, P_1\right) > \frac{\max(q_h, q_l)+D}{1-\delta},$

where the left hand side is strictly positive. Then, it is clear that if $D \to 0$, the right hand side goes to zero, so this condition is satisfied. More generally, as $D \to 0$, the conditions (19a) and (19b) converge to the sufficient condition (15b) of Proposition 2b, demonstrating the existence of wobbly dynamics. This means, in turn, that *when there is wobbly dynamics without land, there is always wobbly dynamics with land if D is sufficiently small*.

However, it should be clear that as D increases, the necessary condition for wobbly dynamics (18) won't be satisfied. Hence wobbly dynamics cannot occur for D large enough.

While D has no effect on $P_1, P_2, or\ P_3$, an increase in D increases the lower bound of P and increases the pace of decrease in P, which is why wobbly dynamics becomes infeasible. But these effects go to zero as D goes to zero, so that wobbly dynamics for D near zero are the same as wobbly dynamics at $D = 0$, and the fact that there is a range of values of parameters for which wobbly dynamics exists at $D = 0$ implies that there is also a range of values for which wobbly dynamics exists at $D > 0$. Figure 4 therefore illustrates wobbly regions and non-wobbly regions for D sufficiently small. Similarly, Proposition 2 showed that even when there is no wobbly dynamics without land, i.e., (16) does not hold, wobbly dynamics with land could occur. This result continues to hold for $D > 0$, at least for D not too large.[41]

While so far in our discussion of $D > 0$, we have restricted ourselves to the case where total returns to owning land take either a high or a low value, even if we include the borderline case, where capital accumulation is such as to lead to $k_{t+1} = \frac{\omega_1}{\omega_2}$ (and the expected, and in a r.e. trajectory, realized returns on capital are such as to generate precisely that level of capital accumulation), the results would not change except as we noted in section 3, the upper bound on P could be higher and the lower bound on P could be lower. It is still the case that if the price of land gets sufficiently high, the economy will hit the explosive region and hence a reversal must occur. As before, allowing an intermediate value to returns simply enriches the set of feasible rational expectations trajectories.

[41] All critical values of e will be affected by D but as $D \to 0$, they are identical to those values when $D = 0$.

We can summarize these results in the following Proposition.

Proposition 4 (Existence and characterization of the wobbly economy with endogenous fluctuations in land prices): [42] [43] [44]

(i) In a rational expectations wobbly trajectory, land prices must oscillate within bounds.[45]

(ii) A sufficient condition for endogenous oscillation is (19a) and (19b).[46] For D small enough, the set of parameters $\{\frac{A}{\omega_1}, \frac{A}{\omega_2}, D, e, \delta, \frac{a_1}{a_2}\}$ for which wobbly dynamics exists is approximately the same set as for the case for $D = 0$, and is non-empty.

(iii) There exists critical values of P, $\underline{P}$ and $\hat{P}$, such if the price falls below $\underline{P}$ or exceeds $\hat{P}$, there is an endogenous phase transition from a state with three momentary equilibrium to a state with two momentary equilibrium, and there are other critical values ($\underline{\underline{P}}$ and $\hat{\hat{P}}$) such that if the price falls below or above those values there is a phase transition from two momentary equilibria to a unique momentary equilibrium.

4.3. Steady States

If $D > 0$, there may be a steady state with a positive interest rate—unlike the case of $D = 0$ where the only steady state entailed a zero interest rate. Such an equilibrium entails

(20) $P^* = \frac{D}{\frac{A}{\omega_1} - \delta} \equiv q_h$ and $k^* = \frac{e}{1 + \frac{a_1}{a_2}\left(\frac{A}{\omega_1} + 1 - \delta\right)} - \frac{D}{\frac{A}{\omega_1} - \delta}$,

where k^* has to satisfy

(21) $0 \leq k^* \leq \frac{\omega_1}{\omega_2}$.

(20) and (21) give bounds on e as a function of D under which there exists a capital shortage steady state equilibrium[47]:

[42] Our result is reminiscent of catastrophe theory (see Varian (1979) for an application in economics). In catastrophe theory, if a parameter value exceeds a threshold, then the economy must jump to the low equilibrium because the high equilibrium suddenly disappears. There are differences. In catastrophe theory, it is an exogenous change in a parameter value that leads to an abrupt change. By contrast, in our model, the economy endogenously approaches the critical point. The crash in land prices occurs endogenously and the key disturbance to the economy is endogenous.

[43] In Appendix D, we demonstrate that our analysis goes through even if we introduce technological progress.

[44] One of the criticism of an economic model with multiple equilibria is that anything can happen. That is not true in our model. Even if the land economy wobbles, the price of land fluctuates within a well-identified range. Moreover, the number of momentary equilibria changes in a precise way, according to endogenous state variables.

[45] With the possible exception of the initial period, when $P_1 < P_2$, with $k_t < k^f$ and $P_2 > P_0 > P_1$ in which case there will be a unique momentary equilibria, entailing a capital surplus.

[46] (18) provides a necessary condition for oscillations involving only capital shortage or capital surplus regimes.

[47] $k^* = 0$ when the steady-state $P^* = P_1$.

(22) $e_L(D) \equiv \frac{D}{\frac{A}{\omega_1}-\delta}\left[1+\frac{a_1}{a_2}\left(\frac{A}{\omega_1}+1-\delta\right)\right] = \frac{\frac{D}{\frac{\omega_1}{\omega_2}}}{\frac{A}{\omega_1}-\delta}(e_2 + \frac{a_1}{a_2}\frac{A}{\omega_2}) \leq e \leq \left(\frac{\omega_1}{\omega_2}+\frac{D}{\frac{A}{\omega_1}-\delta}\right)\Big[1 + \frac{a_1}{a_2}\left(\frac{A}{\omega_1}+1-\delta\right)\Big] \equiv e_H(D) \equiv \left(1+\frac{D/\frac{\omega_1}{\omega_2}}{\frac{A}{\omega_1}-\delta}\right)(e_3 + \frac{A}{\omega_2}) = \left(1+\frac{D/\frac{\omega_1}{\omega_2}}{\frac{A}{\omega_1}-\delta}\right)e_0 = \left(1+\frac{\frac{D}{\frac{\omega_1}{\omega_2}}}{\frac{A}{\omega_1}-\delta}\right)(e_2 + \frac{a_1}{a_2}\frac{A}{\omega_2})$,

where $e_H > e_L \geq 0$ so, for any set of parameters $\{\frac{a_1}{a_2}, \frac{A}{\omega_1}, \frac{\omega_1}{\omega_2}, D, \text{and } \delta\}$ there exists a range of values of e satisfying (22). Note $e_H(D) > e_0 > e_3 > e_2$ (so long as $\frac{a_1}{a_2} > 1$, as we have assumed throughout) and as $D \to 0$, $e_H(D) \to e_0 > 0$ and $e_L(D) \to 0$. Thus a sufficient condition for a capital shortage steady state is that D be small enough.

Even when the capital shortage steady-state exists, it is not stable in a sense to be explained now. Consider first the case where $q_h < P_3$. We have already shown that if land price P_t is ever less than q_h, there is a price implosion: there does not exist a r.e. trajectory from such an initial condition. Similarly for a small upward price perturbation. With a large upward perturbation, above P_3, there may be multiple r.e. momentary equilibria going forward, but all trajectories will have to remain within the bounds defined earlier. The economy won't return to the steady state.

In the case of $P_3 < q_h < P_2$, then again, there is no r.e. trajectory beginning from P near but slightly below P^*, but there are (possibly multiple) trajectories beginning from P above P^* (but below P_2), but again, not (in general) returning to the steady state.

In addition, there may exist "just" full employment steady states with

(23) $k^{**} = \frac{\omega_1}{\omega_2} = \frac{w^{**} + e}{1+\frac{a_1}{a_2}(1+r^{**})} - P^{**}$,

with

(24) $1-\delta \leq 1 + r^{**} \leq \frac{A}{\omega_1} + 1 - \delta$,

where

(25a) $P^{**} = \frac{D}{r^{**}}$, and (25b) $\omega_1(r^{**}+\delta) + \omega_2 w^{**} = A$

Substituting (25a) and (25b) into RHS of (23), steady states that generate just full employment have to satisfy

(26) $\frac{\omega_1}{\omega_2} = \frac{\frac{A}{\omega_2} - \frac{\omega_1}{\omega_2}(r^{**}+\delta)+e}{1+\frac{a_1}{a_2}(1+r^{**})} - \frac{D}{r^{**}}$, with $r^{**} > 0$.

The question is, does there exist a value of r^{**} satisfying (24) for which (26) holds.

Rewrite (26) as

(26’) $\frac{A}{\omega_2} + e = \left[1 + \frac{a_1}{a_2}(1 + r^{**})\right]\left(\frac{\omega_1}{\omega_2} + \frac{D}{r^{**}}\right) + \frac{\omega_1}{\omega_2} r^{**} + \frac{\omega_1}{\omega_2}\delta.$

For any value of parameters $\{\frac{a_1}{a_2}, \frac{\omega_1}{\omega_2}, D, and\ \delta\}$ and $r^{**} > 0$, we can find values of e and A such that (26’) is satisfied. Indeed direct calculations in Appendix E show that $r^{**} > 0$ exists if and only if D is sufficiently small and if

(14’) $e > \frac{\omega_1}{\omega_2}\left[1 + \delta + \frac{a_1}{a_2} - \frac{A}{\omega_1}\right] + D\frac{a_1}{a_2} = e_2 + \left(\frac{a_1}{a_2}\delta + \delta - \frac{A}{\omega_1}\right)\frac{\omega_1}{\omega_2} + D\frac{a_1}{a_2} \equiv \hat{\hat{e}},$

with obviously $\hat{\hat{e}}(D = 0) = \hat{e}$. Since $\hat{\hat{e}}(D)$ is an increasing function of D, the range of values of e for which there exists a just-full employment steady state equilibrium diminishes as D increases.

Multiple steady states

Comparing (14’) and (22), it is clear that there are parameter values for which no steady state exists, two steady states exists, or only a capital shortage steady state. Take the special case where $\frac{\omega_1}{\omega_2} = 1$, and D arbitrarily small. Then a capital shortage steady state requires $e \leq 1 + \frac{a_1}{a_2}\left(\frac{A}{\omega_1} + 1 - \delta\right)$. The boundary, $e = 1 + \frac{a_1}{a_2}\left(\frac{A}{\omega_1} + 1 - \delta\right)$ is a positively sloped line as a function of $\frac{A}{\omega_1}$, taking $\frac{a_1}{a_2}$ and δ as fixed, with intercept $1 + \frac{a_1}{a_2}(1 - \delta)$, with e taking the value of $1 + \frac{a_1}{a_2}$ at $\frac{A}{\omega_1} = \delta$. On the other hand, a just full employment steady state requires e $\geq 1 + \delta + \frac{a_1}{a_2} - \frac{A}{\omega_1}$. The boundary $e = 1 + \delta + \frac{a_1}{a_2} - \frac{A}{\omega_1}$ is a negatively sloped line as a function of $\frac{A}{\omega_1}$, with intercept $1 + \delta + \frac{a_1}{a_2} > 1 + \frac{a_1}{a_2} - \frac{a_1}{a_2}\delta$, and takes on the value $1 + \frac{a_1}{a_2}$ at $\frac{A}{\omega_1} = \delta$. Thus, there are three regions, depicted in Figure 4-3: That where there is only a full employment state steady, only a capital shortage steady state, and two steady states. [48]

Enriching the set of rational expectations trajectories

The preceding section showed that, for $D = 0$, depending on parameter values, the only r.e. trajectory could be wobbly, or the only r.e. equilibrium could be a steady state, or there could be both a steady state *and* wobbly dynamics. The analysis for D small proceeds in much the same way as before, except now the condition for the existence of a steady state is given by (14’) and (26). By superimposing these conditions on the earlier derived conditions for wobbly dynamics, we can identify parameter values which are consistent with each form of r.e. trajectory.

Focusing, for instance, on the case of $P_3 < 0$ (Figure 4-1), now, for small D, the condition for the existence of a capital shortage steady state is just $e \leq \left(e_3 + \frac{A}{\omega_2}\right)$. It is thus apparent that in the entire wobbly region, there exists a capital shortage steady state, but for $e_3 < e \leq \left(e_3 + \frac{A}{\omega_2}\right)$, there exists either only a capital shortage steady state or both a capital

[48] As in our earlier analysis in general each of the steady states is not stable. When $P_3 > 0$, the economy may or may not be able to wobble off P^{**} by a change in expectations, depending on parameter values. But, when there is a perturbation, the economy won’t return to the steady state.

shortage steady state and a just full employment steady state. In this region, there does not exist wobbly dynamics. For the region above $e_3 + \frac{A}{\omega_2}$ there only exists a full employment steady state.

On the other hand, in the case identified earlier as (b1), where $P_3 > 0$ and depicted in Figure 4-2, the locus of values of e for which there exists a capital shortage steady state, given by $e \leq e_3 + \frac{A}{\omega_2}$ has a lower slope than e_{23} which implies that the wobbly region is divided into two parts, where there exists a capital shortage steady state (as well as a full employment steady state) and that in which there does not.[49]

4.4: Comparative Statics[50]

How the nature of fluctuations depends on key parameters

Earlier we analysed how the dynamics was affected by changes in key parameters. Those results are unchanged, at least for small D. Here, we ask: how does an increase in D affect dynamics.

As we have already noted, as D increases, the pace of increase of prices (in the capital shortage regime) is decreased, and by itself that makes booms longer lasting. But an increase in D, while have no effect on P_2 does increase P_{min} if $P_{min} = q_h$ and also does increase $\hat{P}$ if $\hat{P} = \frac{P_2+D}{\frac{A}{\omega_1}+1-\delta}$ but the former effect is larger than the latter effect. Hence, there is some ambiguity in whether a larger D can increase the length of a boom. On the other hand, a larger D increases the pace of decline, and thus the (maximum) duration of a bust is shortened.

If $\frac{A}{\omega_1} \to \delta$, while keeping q_h unchanged, by lowering D, the slope of the land price dynamics under the high return regime becomes close to unity in Figure 3-2. This implies that land prices increase very slowly. As $D \to 0$, $q_h = q_l \to 0$. This means that if $P_3 < 0$ for sufficiently small D, the time it takes for land prices to fall to P_{min} will be arbitrarily large.

Moreover, D can affect the very nature of global dynamics. For instance, the conditions under which there exists a capital shortage steady state (14) or a just full employment steady state (25) or wobbly dynamics all depend on D. A change in D may result in wobbly dynamics existing when previously such trajectories did not, or not existing, when previously they did.

5. Discussion

5-1: The presence of land does not eliminate dynamic inefficiency

Earlier, we noted that focusing on steady states, there cannot be dynamic inefficiency in an overlapping generations model with land. But this steady-state analysis is misleading. Dynamic inefficiency can arise in the transitional dynamics of the wobbly economy. That is,

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[49] In addition, by looking at Figure 4, we can see that there may or may not exist areas within the wobbly region where there exists a just full employment steady state.

[50] In Appendix F, we present deterministic cycles in the case with $D > 0$.

in the capital surplus region where $k_t > \frac{\omega_1}{\omega_2}$, the return to capital is $1 - \delta < 1$ and this is obviously dynamically inefficient: the economy could reduce its investment the first period, having more consumption without reducing output in the next period. In our model, the economy wobbles between a state with involuntary unemployment and one with dynamic inefficiency, suggesting that there might be scope for government intervention.

While in our model, wobbly dynamics is associated with episodic dynamic inefficiency, Section 5-3 shows that wobbly dynamics does not require episodic inefficiency: there are fully efficient r.e. trajectories that wobble.

5-2: Difficulty of differentiating wobbly dynamics and unsustainable speculative bubbles

We have shown that the land price increases crowd out real investment, so that associated with large fluctuations in land prices, there is a large swing in the land price/GDP ratio: $\frac{P_t}{Y_t}$ continues to rise over time until there is a reversal and once land prices start falling, the ratio starts decreasing.[51] Aggregate wealth defined as $k_t + P_t$ is also rising together with the increase in the price of land, even though productive capital is crowded out, so the wealth-output ratio ($(K_t + P_t)/Y_t$) is going up and down in tandem with the rise and fall in land prices.[52]

Several economies have experienced fluctuations along these lines: For instance, countries such as the U.S., Japan, Spain, and the U.K. have experienced large swings in land and housing prices, with the $\frac{P_t}{Y_t}$ ratio increasing substantially during the booms and then decreasing.

Hence this ratio might seem to be an early warning signal that the trajectory the economy is on is not sustainable.[53] However, such dynamics do not necessarily mean that the economy is not on a rational expectations trajectory.[54] Land prices in our model move with the underlying fundamentals of the economy, in particular satisfying the capital arbitrage equation. *Seemingly* unsustainable dynamics can occur even in the case of $D = 0$. Hence it

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[51] If we interpret land purchase as being intermediated through banks, this means credit/GDP ratio increases so long as land prices continue to rise and then the ratio starts decreasing once land prices start falling. See our sequel Hirano and Stiglitz (2021b) that extends the current model to include the case where land and real capital are used as collateral in borrowing.

[52] Note that in this polar model, if we define wealth as $s_t W_t$, it is constant during the boom (equal to $s_t e$), and again constant ($= s_t(w_t + e)$) during the bust, with a discrete increase at the time of switching from boom to bust with the associated overinvestment, because both wages and the savings rate increase.

Note too that during the boom, while national income, as conventionally defined, is decreasing as land speculation crowds out real investment, Haig-Simons income, including capital gains, is constant; while during the bust, national income as conventionally defined, is constant, Haig-Simons income is increasing, since capital losses (from the decreasing land prices) are becoming smaller

[53] These movements in the ratio are unlikely to occur in local dynamics with a unique dynamic path, unless we assume continuous exogenous shocks. On the local saddle path trajectory, asset prices can increase over time but the extent of the increase has to decrease over time; otherwise, the economy will not converge to a steady-state. In other words, land prices cannot grow faster the economy's growth rate in those models because if that were the case, land prices would explode and such explosive paths are ruled out. This suggests that at least on the local saddle path near the equilibrium, the ratio moves monotonically.

[54] Although our analysis shows that such volatility is not *necessarily* inconsistent with rational expectations, in at least some of these episodes there are other indicia of extensive deviations from rational expectations.

is hard to tell whether the actual price movements and macro-dynamics which seemingly look unsustainable are driven by changes in the fundamental values or driven by pure speculative bubbles.

Some members of the Federal Reserve defended their failure to intervene in the housing bubble as it arose before 2008, claiming that one cannot tell whether there is a bubble until after it breaks. In our model, policymakers do not need to know in advance whether there is a bubble or not: Putting aside the obvious point that all government intervention in practice occurs in the context of uncertainty, government intervention can still be justified (see section 6), consistent with the policy perspective put forward by Borio and Lowe (2002). Land holdings crowd out capital accumulation, reducing output and wages, and in our polar Leontief model, increasing involuntary unemployment in the capital shortage regime.[55]

5-3: Land price boom and labor and capital reallocation into the real estate sector

So far, we have explored wobbly dynamics with constant productivity of land, i.e., constant D. But it is easy to show that wobbly dynamics can occur even if D depends on the amount of capital or labor allocated to the sector generating the rents.

Reallocation of capital in a capital surplus regime

Assume when there is idle capital, it is reallocated to the land sector, and that increases.[56] Assume moreover that the marginal productivity of capital in the land sector is sufficiently low that when there is a capital shortage, no capital is allocated there, so that D is fixed.

When the productivity of land is increased in the capital surplus region, the rate of decline in land price may either or decrease: it may increase, because D is higher, but it may decrease, because now the net marginal return to capital is higher. ((Now, it is $D_k - \delta$), while before it was $-\delta$.)

If $r_{t+1} > 0$, i.e. if $D_k - \delta > 0$ (which will be the case if δ is small enough), the capital arbitrage equation defines a price, $\frac{D(k_{t+1})}{r_{t+1}(k_{t+1})}$, above which, even in the capital surplus regime, prices would continue to rise. This is an increasing function of k_{t+1}. For a switch from rising to falling prices, we must be sure that P_t lies below that critical value; define the minimum value that $\frac{D(k_{t+1})}{r_{t+1}(k_{t+1})}$ can attain by d^*.[57] Then an economy can make a successful switch so long as it switches before P_t exceeds $\min\{d^*, P_2\}$. This provides a possibly lower upper bound to land prices.[58] But if D_k is small enough, $d^* > P_2$, so the upper bound will remain unaltered.

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[55] In this sense, our paper is consistent with evidence of misallocation during asset booms in Borio, Kharroubi, Upper, and Zampolli (2015) and Charles, Hurst, and Notowidigdo (2018).

While Bernanke and Gertler (1999, 2001) provide some theoretical justification for not intervening, that position has been widely criticized. As we noted, all policy is conducted under uncertainty; there was clearly a high probability (based on past data) that there was a bubble, and the claim, based on the presumption that markets are efficient, that the chairman of the Fed made that it would be better to clean up the afterwards than to intervene in the market was obviously wrong, but even seemed so at the time. See Stiglitz (2010).

[56] The aggregate production function is given by $Y_t = A \min\left[\frac{K_t}{\omega_1}, \frac{L_t}{\omega_2}\right] + D(K_t, T_t)$.

[57] That is associated with there being no surplus capital to be allocated, i.e. $D(k_f)/r(k_f)$.

[58] This is a sufficient condition for a switch, not a necessary condition.

On the other hand, now, prices can fall to max $\{P_3, q_l, q_h)$ before switching back to the capital scarcity regime, where now $q_l = \frac{D(k_{t+1})}{1+ r_{t+1}(k_{t+1})}$, the value of P_t where the price line (giving P_{t+1} as a function of P_t) crosses the horizontal axis,. If D_k is small enough, it is clear that wobbles still exist. Indeed, since it is possible that q_l is now lower than it was in our earlier analysis (because r_{t+1} is higher), it is possible that the range of wobbles has increased.

A reallocation of labor in a capital shortage regime

Now we examine the effect of labor reallocation between the capital intensive (manufacturing) sector and the real estate sector when there is a shortage of capital, and how it affects wobbly dynamics.

Assume $D(L_R)$ takes on a linear form, such that $0 < D' < \frac{A}{\omega_2}$, ensuring that if there is sufficient capital, we allocate all labor to the manufacturing sector, but when there is a shortage of capital, the residual goes to the real estate sector. L_R is the amount of labor employed in the real estate sector. That means that the wage in the capital shortage regime is higher than in the case examined in earlier sections (greater than zero), and the return to capital is accordingly lower. That in turn means that *the price increases more slowly*, both because D is higher and r is lower. And it also means that the maximum land price has increased. Accordingly, the set of parameters for which wobbles (entailing capital shortages) exist is greater, the bounds on the wobbles are greater, the parameter sets for which wobbles exist is increased,[59] and the (maximum) length of the expansion is longer.

Also, with this reallocation of labor, output in the real estate sector also changes together with rising and falling land prices. Because D has increased as a result of the reallocation of labor, P increases more slowly, which means that there is less crowding out. Accordingly, so long as the land price boom doesn't extend longer, aggregate output is higher.[60] But as we just noticed, the land price boom may be more prolonged, so there may be more crowding out in these periods.

Still more complex r.e. trajectories with capital shortage

Assume now $D(L_R)$ takes on a *piece wise* linear form with diminishing returns, with $w = w_1$ for $L_R > L_1$ and $w = w_2 > w_1$ when $L_R < L_1$, or equivalently when $k > k_1$. That in turn means that depending on the value of k, r takes on 3 values. And that, in turn, means $\Omega(k_{t+1}, P_t)$ takes on a sawtooth pattern: There can now exist not just 3 momentary equilibria, but five, as illustrated in Figure 6.

There can now also exist multiple steady states—excessive focus on real estate can lead to lower levels of GDP. Steady state entails

$$s\big(r(k)\big)(w(k) + e) = \frac{D}{r} + k.$$

The LHS is unambiguously increasing is k, but because as k increases, both D and r decrease, the RHS may be increasing or decreasing in k, and there can be multiple

[59] The parameter set for which a capital shortage steady state has, in addition, shifted.
[60] That is, higher than it would be with fixed D. If D_L is small, the increase in output in the real estate sector doesn't offset the crowding out effect in manufacturing.

intersections. In our Leontief model, s and w both jump up at k_1 and k^f, but are constant within those intervals. w is indeterminate at k_1. D decreases linearly with k, with a slope that decreases (i.e. is more negative, with the line being steeper) beyond k_1. r jumps down at k_1(because wages jump up) and so the RHS jumps up. Thus there can exist, in addition to the just full employment steady state identified earlier, three others, one below k_1, one at k_1, and one above k_1.

The range of oscillations has increased, since as P increases, the wage in the capital shortage regime is either w_1 or $w_2 > 0$, depending on the value of k. Now the upper bound on P depends on the value of k. If $k > k_1$, then P can increase to a higher value than if $k < k_1$ because wages are higher. Thus, not only is the pace of increase in prices slowed, but the range over which prices can increase is increased. On both accounts, booms can last longer.

As earlier, *without appropriately chosen government policies, more efficient use of labor and capital may end up magnifying wobbly fluctuations*.

The discussion in this section has focused on what (in the absence of the ability of the real estate sector to absorb capital and labor) would have been capital shortage (labor surplus) and capital surplus (labor shortage) regimes. We have identified three distinct effects—on the rate of interest (r), on D, and on wages—in each of the regimes, and how these affect the bounds of variability in P (and therefore of other variables), the pace of increase or decrease in P (and therefore the length of booms and busts), the set of parameters for which wobbly dynamics exists, and other aspects of global dynamics, including the set of steady states. But as we have noted earlier, once the just full employment equilibria are taken into account, the bounds of variability in P are increased as are the set of parameters for which wobbly dynamics exist.

The rich set of wobbly dynamics exhibited in these models show that wobbly dynamics is fully consistent not only with rational expectations, but also dynamic efficiency (because even in the "bust," the marginal return to capital is strictly positive).

6. Role of government policies

In this section, we consider the impact of a variety of government policies on economic outcomes, stability and welfare.

6-1: Effect of land tax on the wobbly economy and land price fluctuations

Effect on steady state

Since land holding crowds out capital accumulation, one might expect that a land tax, with proceeds distributed to workers, increases capital accumulation in the steady state. This is correct. Consider a tax on D, the proceeds of which are paid to young individuals. Such a tax is equivalent to a reduction in D by τD and an increase in e of the same amount. From (20) it is apparent that the steady state land value decreases, and the steady state value of k increases, both because total income of the young is higher and land values are lower. The land tax both increases savings and reduces "crowding out". Correspondingly, output and employment increase, but because the economy is still in the capital shortage regime, wages are unchanged. [61]

The range of values of e (given all the other parameters) for which the steady state exists shifts downwards—with the lower and upper bounds of e changing by the same amount.

Implications of the instability of the steady state

But steady-state analysis may not be of much relevance because the steady state, as we showed earlier, is unstable, and once the economy wobbles off the steady-states, it won't return. Moreover, as we have noted, there may not exist steady states. Accordingly, we now focus on the impact of such taxes on wobbly dynamics.

Effect on magnitude of fluctuations

It is often suggested that a tax on the returns to land would reduce land speculation and contribute to economic stability. But, at least in a rational expectations model, a land tax *may* end up increasing the magnitude of the price fluctuations.

Straightforward calculations show P_1 and P_3 are increased by the same amount, while P_2 is increased by a somewhat larger amount, so that if P_3 is the lower bound, depending on whether P_1 > or < P_2, the range of price fluctuations is unchanged, simply shifted up, or increased.

On the other hand, if q_h is the lower bound, the lowering of D lowers the lower bound, and the range of price fluctuations is unambiguously increased, whether P_1 is greater or less than P_2. Of course, for a large enough tax, the binding constraint either at the bottom or top may change.[62]

6-2: Capital Gains Taxes

In steady state, of course, there are no capital gains. Capital gains and losses occur during the capital shortage and capital surplus regimes, respectively. Assume again that the tax revenue is transferred to the young. This implies that P_1 and P_2 are increased. If losses are not tax deductible, then P_3 is unchanged, so that again the range of price fluctuations is increased.[63]

Moreover, from the capital arbitrage equation, the pace of price increase has to be increased to offset the effects of the tax. Contrary to what was intended, the capital gains tax leads to faster increases in land prices. It is ambiguous whether booms are shorter or longer, since while prices increase at a faster rate, the upper bound on prices has also increased, but busts can last longer.

We emphasize, however, that these results are very dependent on our underlying assumption that the economy is always on a rational expectations trajectory.

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[61] Of course, in the just-full employment equilibrium, what adjusts is w^{**} (falling) and r^{**} (increasing). Since s is accordingly smaller, total income of workers is higher (taking into account the distribution of the revenue of the dividend tax) and steady state utility is accordingly higher.

[62] This discussion ignores the effects on the expanded possible magnitude of fluctuations if we include the possibility of just-full employment momentary equilibria. The analyses in that case are similar.

[63] If the government makes up for a portion of capital losses, financed by wage taxes, then P_3 is lowered, and the range of price fluctuations is increased even more.

6-3: Effect of capital subsidy/tax on the wobbly economy

We now consider the effect of capital subsidy/tax. Let us denote τ^c and τ_t^w as the capital subsidy rate and the wage tax rate at date t. Total wage taxes equal capital subsidies, that is, $\tau_t^w w_t = \tau^c(f_k(k_t) + 1 - \delta)k_t$.[64] (A negative value of τ^c and τ_t^w means a capital tax with its proceeds transferred to young generations.)

The savings/capital accumulation equation is written as

$$k_{t+1} + P_t = \frac{w_t + e - \tau_t^w w_t}{1 + \frac{a_1}{a_2}(1 + r_{t+1})}.$$

The capital arbitrage equation is

$$1 + r_{t+1} = (1 + \tau^c)(f_k(k_{t+1}) + 1 - \delta) = \frac{D}{P_t} + \frac{P_{t+1}}{P_t}.$$

First, assume the economy is in the capital surplus region, and consider a tax perturbation at only time t. Because of the capital subsidy, the saving rate at date t decreases, lowering k at date $t+1$. Also, after tax income decreases at $t+1$ (because of the wage tax), reducing k at date $t+2$. Hence this policy leads to lower capital, reducing the magnitude of the oversaving problem. Subsequent periods are unaffected, so long as the economy still remains in the same regime. But from the capital arbitrage equation, we see that the price of land must fall more slowly, so any feasible timing of the switch from capital surplus to capital shortage regime is still feasible. Note, however, that this perturbation is not a Pareto improvement: the t[th] generation is better off, the next worse off. Moreover, while the tax is just a transfer of income from one generation to the next, the reduction in k—which because the economy is in a capital surplus region has no effect on output—means that aggregate consumption has increased. And since initially, consumption prior to the tax perturbation was identical every period (since wages and the rate of interest were the same at t and $t+1$), with any egalitarian social welfare function, a small enough perturbation is welfare increasing.

Next, let us consider a capital tax in the high return regime. Assume the government now imposes a tax on the return to capital, using the proceeds to provide transfers to the young (or provide unemployment benefits). The capital arbitrage equation becomes (recalling that $\tau^c < 0$ represents a tax)

$$P_{t+1} = (1 + \tau^c)\left(\frac{A}{\omega_1} + 1 - \delta\right)P_t - D.$$

For any P_t, the more negative τ^c, the smaller P_{t+1}. This means that with a capital tax, land prices increase more slowly, again ensuring that any feasible timing of the switch from the capital shortage regime to the capital surplus regime is still feasible.[65]

Moreover, because the return to capital is lowered, the savings rate is increased, so k_{t+1} is increased, and because workers' incomes are increased at $t+1$ (because of the wage

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[64] It will be clear from the analysis below that it makes no difference whether the subsidy (tax) is imposed on net or gross returns to capital.

[65] The percentage change is written as

$$\frac{P_{t+1} - P_t}{P_t} = (1 + \tau^c)\left(\frac{A}{\omega_1} + 1 - \delta\right) - 1 - \frac{D}{P_t}.$$

Hence the introduction of the capital tax results in smaller upward price movements.

subsidy), aggregate savings is increased at $t+1$, so k_{t+2} is increased. Thus, with a capital tax, the crowding-out effect is reduced and more resources flow to productive capital rather than land speculation. Again, the change is not a Pareto improvement, since consumption of the t^{th} generation has decreased, that of the next has increased. But because the economy is in a capital shortage regime, the return on capital is high, so that aggregate consumption increases, and since initially utility (consumption) in the relevant periods is the same, the increase in the aggregate consumption means an increase in social welfare with any equalitarian social welfare function.

In summary, this k dependent policy leads to more employment in the high return regime and reduces the magnitude of dynamic inefficiency in the low return regime. Note that the policy just described retains the multiplicity of momentary equilibria, and hence the economy can still wobble.

6-4: Government policy that just attains full employment

Our analysis implies that without appropriately chosen government policy, the price of land and other key macro variables endogenously fluctuate without converging between the dynamically inefficient region and the region associated with involuntary unemployment. Here we examine whether it is possible for the government to induce the economy to attain just full employment as the unique equilibrium, while achieving dynamic efficiency and eliminating wobbles.

Earlier, we provided conditions under which there existed a steady state with just full employment, but argued that those steady states are not stable. Now, assume the government announces that it fully commits to assuring an after tax return of r^{**}(the full employment steady state interest rate), financing any shortfall of returns out of a tax on wages. If individuals believe it, then the land price dynamics follows according to $P_{t+1} = (1+r^{**})P_t - D$. The only possible rational expectations equilibrium is $P_t = P_{t+1} = P^{**}$. That in turn implies that the steady state with full employment k^f can be achieved as the unique r.e. equilibrium.[66] Moreover, since $r^{**} \geq 0$ ($r^{**} = 0$ when $D = 0$), that steady state is also dynamically efficient.

Proposition 5: With credible commitment to a capital income tax and a wage tax, the government can achieve just full employment k^f as the unique rational expectations equilibrium, and the equilibrium is dynamically efficient.

In a sequel (Hirano and Stiglitz (2021c)) we establish a more general result on the ability of the government to use tax policy to steer the economy, ensuring a unique r.e. dynamic trajectory. We show, in particular, its ability to implement the trajectory which maximizes intertemporal social welfare.

[66] The analysis of section 3 showed that there might not be a just full employment steady state with positive land prices, simply because savings at a zero interest rate wouldn't suffice. But we can increase savings by imposing a lump sum tax on the elderly, used to finance a lump sum payment to the young. (In the case of $D = 0$, there is never any problem of oversaving, because we can increase the value of land to absorb excess savings.) In section 4, again we showed that there might not be a just full employment steady state. With D positive, we require a positive interest rate. Taxing interest and rents, and redistributing proceeds to workers, increases savings.

7. Concluding Remarks

(a) *Robustness*

The two-period overlapping generations model has a very simple structure, reflecting the speculative behaviour of economic agents who enter the market and buy assets, and then exit from the market by selling those assets. To be sure, a model such as that developed here oversimplifies, in particular in assuming that there are only two periods. It would be possible to construct more realistic life-cycle models in which individuals work for N periods, followed by M periods of retirement. Nonetheless, the key qualitative dynamic patterns can be demonstrated in an analytically tractable two-period life cycle model.

Moreover, as Woodford (1986) showed, the mathematical structure of the overlapping generations model is formally analogous to that of infinitely-lived agents models with borrowing constraints in which some agents are finance-constrained, while others are not. The behaviour of economic agents that expect (never expect) to be financially constrained is much like that of finite (infinite) lived agents as described in the current paper.[67] In this interpretation of our model, the "one period" in the overlapping generations model does not have to be the biological working life span and could be relatively short.

Furthermore, Mueller and Woodford (1988) considered a mixed model with both finite (two period lives) and infinite lived agents. They proved that the dynamic properties of the system are more akin to that of the overlapping generations model than to that of the model with *just* infinite lived agents.[68] The Mueller and Woodford (1988)'s result is suggestive that our result that there is a plethora of rational expectations trajectories in global dynamics would hold even were there to exist some infinitely lived agents.[69] In a sequel (Hirano and Stiglitz (2021c)), we explore a mixed model with both infinitely lived entrepreneurs (who have investment opportunities and who face credit frictions where capital and land can be used as collateral) and life-cycle savers with two period lives. Even in this setting, we show that there is a plethora of equilibrium trajectories consistent with rational expectations. The economy experiences endogenous and unsustainable booms followed by endogenous crash repeatedly.

While we employ rational expectations, such an assumption may be somewhat extreme. Instead, we can consider threshold effects: assume, for instance, that so long as the outcome is within 95% of expectations, we don't change our course (beliefs), but, if the outcome is outside the range, we change our course discretely. We might refer to this as

[67] Woodford (1988) also showed that the mathematical structure for the existence of sunspot equilibria in an overlapping generations monetary exchange economy as shown in Azariadis (1981) is identical to that for the existence of sunspot equilibria in an infinitely lived agents monetary model in which the cash-in-advance constraint is always binding.

[68] Showing in particular that local indeterminacy near the steady state continues to be possible even when infinite lived agents own a large fraction of total wealth, so long as their consumption is not too great a part of total consumption. Mueller and Woodford (1988) also proved that dimension of indeterminacy increases as the number of goods increases. Geanakoplos and Polemarchakis (1991) derived a similar result. These results suggest that with more complexity, the dimension of multiplicity of equilibria may increase. In a sequel (Hirano and Stiglitz 2021b), we extend the current idea to include credit frictions. Consistent with these earlier results, it is shown that it is even easier to have multiplicity of momentary equilibria and that the number of momentary equilibria increases.

[69] There are other models combining finite lived agents with those with infinite lives where the steady state results, in certain regimes, are driven by the infinitely lived agents. See, e.g. Stiglitz (2018).

“consistent” expectations. It is easy to establish that there can be an even larger set of trajectories consistent with “consistent expectations” and that wobbly dynamics exist for a wider range of parameters.

(b) Extensions

Our model can be extended into several directions, such as introducing credit frictions, e.g. where land and capital are used as collateral for borrowing (Hirano and Stiglitz 2021b, 2021c); or introducing heterogeneous agents in each generation to examine how fluctuations in asset prices affect inequality; or introducing differential returns across assets (e.g. associated with differences in risk); or assuming imperfectly flexible (real) wages. In the latter case, for instance, we might generate unemployment even in neoclassical models with significant substitutability between capital and labor.

Some of these extensions allow us to address key policy issues, such as the effect of financial sector deregulation on cyclical fluctuations. Preliminary results, for instance, indicate that the crowding out effect of increases in land prices that follow deregulation (with the resulting increase in credit availability) may outweigh the effects of the expansion of entrepreneurship from the increased value of collateral that has been emphasized in some earlier literature (Bernanke and Gertler 1989; Kiyotaki and Moore 1997; Brunnermeier and Sanikov 2014). Embedding our model with land in a New Keynesian framework should enable us to examine the two-way feedback between asset price fluctuations that look explosive and inflation rates; and then derive monetary and financial policies that can lead to both greater goods and asset price stability.

(c) From pure theory towards a more realistic model

This paper should be viewed largely as an exercise in pure theory—understanding more fully one of the standard workhorse models in economics. We have extended the standard OLG model to include both capital and land to understand the effect of land speculation on global macro-dynamics, showing that multiplicity of momentary equilibria are pervasive and exploring in particular the implications of this multiplicity for global macro-dynamics. While when we began this research program, we were not sanguine about the analytic tractability of even such a simple model, we have succeeded in providing an analytic framework that has allowed us to get precise results for the complex set of non-linear correspondences generated by the simple overlapping generating model—far more complex that seems to have been realized by the longstanding literature. While broadly confirming widespread views that land crowds out productive capital and can give rise to economic volatility, we have demonstrated the remarkable richness of dynamics that can be generated by such a simple model, in particular, the wobbly dynamics which neither converge nor diverge and may not even have regular periodicities, but are marked by endogenous phase transitions, with the economy passing from situations where there is a unique equilibrium into those where there is a multiplicity of momentary equilibria and back again. In doing so, we have also shown that many of the standard results on OLG models with land have to be modified: there *can* exist over saving even in an economy with land. The economy repeatedly—but not always—engages in overinvesting.

We have also seen that the presence of land may change the dynamics of the economy: it may create opportunities for wobbles when in the absence of land there were not. When $D = 0$ (a pure bubble economy), there is an (unstable) equilibrium in which land doesn’t matter; but the existence of even unproductive land opens up a wide range of new dynamic trajectories, including the possibility of wobbly trajectories with a positive price of land, even when there does not exist *any* steady state rational expectations trajectory in which land has a positive price.

We have shown for those values of the parameters in which there are wobbly dynamics both with and without land, the presence of land means that the value of the capital stock in the "low k" equilibrium and in the "high k" equilibrium is lower than in the absence of land. In each case, land has crowded out capital investment. In the low k equilibrium, this is of particular concern, because the crowding out of productive capital leads to a higher level of involuntary unemployment. On the other hand, in the high k equilibrium, it reduces an inefficiency associated with overinvesting in capital, when capital is the only store of value.

Since the theory provides no prediction on the average value of k, but only on the values of k between which the economy bounces, we cannot say anything about the average value of k, nor can we even say whether the long run benefits of reducing overinvestment exceeds the costs associated with the increased unemployment in the low k states. But as we have noted above, the presence of land means that the range of values of k has been shifted down, and we have identified a set of government interventions that are welfare increasing within an equalitarian social welfare function.

We have shown too that the model has different policy implications for much of the conventional wisdom about tax policies in OLG models: while there is a sense in which land and capital gains taxation (with proceeds rebated to workers) does reduce crowding out of productive investment by land speculation, within a model strictly adhering to rational expectations, such taxation may increase rather than reduce volatility.

The results of our analysis differs even more starkly with those of r.e. with an infinitely lived representative agent, where corresponding to any initial capital stock, there is typically a unique land price consistent with rational expectations, and a unique trajectory from there to a steady state—with no volatility or wobbles.

The wobbly fluctuations in our model have many features that are associated with those observed in the economy: there are episodically unsustainable asset price increases, the key disturbance to the economy is endogenous, and as real estate prices rise, land speculation crowds out real investment. Our model provides a theoretical foundation to allow for temporarily explosive paths in asset prices.

Our fluctuations exhibit another similarity to real world fluctuations—the important role played by expectations. At the center of our analysis is that there can be multiple equilibria—if there are expectations of high returns to capital, then, there is an equilibrium consistent with those expectations; if expectations are more pessimistic, then there is another momentary equilibrium consistent with those expectations. This is in contrast with the standard model for which, in effect, transversality conditions dictate what expectations *must* be if the economy is to be moving along a r.e. trajectory.

But there are some properties of our model that are at odds with observed cyclical fluctuations. Land booms are often associated with low (nominal) rates of interest; in our model, with high rates of interest—though declines in real estate prices are often associated with recessions, also marked by low (nominal) interest rates[70].

Preliminary work on some of the extensions described earlier suggest that not only do the models with those extensions generate fluctuations, but that those fluctuations have a greater verisimilitude to observed fluctuations. For instance, our extended model where land and capital are used as collateral can give rise to land booms associated with low interest rates (Hirano and Stiglitz 2021b, 2021c). This extension including credit frictions enables us to analyse cyclical movement in the disparity between borrowing rates and the average returns to capital. Booms such as that preceding the 2008 crash may be associated with low borrowing rates, but high returns to capital.[71] So too, lags in adjustments of (real) wages can give rise to fluctuations in the capital stock even when (or especially when) the interest elasticity of savings is positive.[72]

Thus, the model of this paper should be viewed as the beginning of a rich research agenda. It can be thought of as a prototype of how to analyse global dynamics in an economy with multiple assets in the presence of multiplicity of momentary equilibria.

In particular, we have highlighted a *new mechanism* by which economic fluctuations can arise: endogenous changes in asset prices (land) change, in a sense, the structure of the economy, moving the economy among regimes with unique or multiple momentary equilibria.

The contrast with the standard neoclassical model with an infinitely lived representative agent, with fully flexible wages and prices and no market distortions where fluctuations are generated by exogenous shocks is stark: we show the dramatic change in results when individuals are assumed more realistically to be finite lived. In particular, fluctuations can be generated internally, even along rational expectations trajectories retaining all the other assumptions—fully flexible wages and prices and no market distortions. This paper has shown, moreover, that government intervention can be welfare enhancing, both reducing the extent to which land speculation crowds out productive investment and increasing the stability of the economy. The results obtained here, in the context of the simplest OLG model with land, and preliminary results obtained in some of the extensions described above, suggest that it may be desirable to re-examine the robustness of some of the standard policy precepts.

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[70] And more recently by low real interest rates.

[71] The disparities between deposit rates, lending rates, the returns to capital, and the shadow price of capital play an important role in the theories of cyclical fluctuations developed by Greenwald and Stiglitz. See, e.g. Greenwald and Stiglitz (1993) and Stiglitz and Greenwald (2003).

[72] When wages are low, the return to capital is high, and with a large positive interest elasticity, this can give rise to high levels of investment, increasing the demand for labor, and eventually leading to high wages. See Akerlof and Stiglitz (1967).

Appendix A: Discounting and the value of $\frac{a_1}{a_2}$

Consider the following utility function:

$$u_t = \left((a_1)^{\frac{1}{\theta}}(c_{1t})^{\frac{\theta-1}{\theta}} + (a_2)^{\frac{1}{\theta}}(c_{2t})^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}$$

Straightforward transformations allow us to rewrite the utility function as

$$\widehat{u_t} = (c_{1t})^{\frac{\theta-1}{\theta}} + \left(\frac{a_2}{a_1}\right)^{\frac{1}{\theta}} (c_{2t})^{\frac{\theta-1}{\theta}} \equiv v_{1t}(c_{1t}) + \mu v_{2t}(c_{2t})$$

Where μ is the individual's discounting of second period's utility = $\left(\frac{a_2}{a_1}\right)^{\frac{1}{\theta}}$. Thus, μ < 1 if and only if $\frac{a_1}{a_2} > 1$.

As $\theta \to 0$, we have the Leontief utility function:

$$u_t = min\left\{\frac{c_{1t}}{a_1}, \frac{c_{2t}}{a_2}\right\}$$

f $\frac{a_1}{a_2} = 1$, $c_{1t} = c_{2t}$. When people put equal weight on today's and tomorrow's consumption, $c_{1t} = c_{2t}$. f $\frac{a_1}{a_2} > 1$, $c_{1t} = \frac{a_1}{a_2} c_{2t}$. When people put more weight on today's consumption rather than tomorrow's consumption, we have $c_{1t} > c_{2t}$. If $\frac{a_1}{a_2} < 1$, $c_{1t} = \frac{a_1}{a_2} c_{2t}$. When people put more weight on tomorrow's consumption rather than today's consumption, we have $c_{1t} < c_{2t}$.

Appendix B: Land price dynamics and aggregate wealth and aggregate consumption

We will show that associated with endogenous fluctuations in land prices, there are also large fluctuations in aggregate income, wealth and consumption.

Aggregate output (given our normalizations) is given by

$$y_t = \frac{A}{\omega_1}\min\left[k_t, \frac{\omega_1}{\omega_2}\right] + D$$

So the fluctuations that we have observed in k_t are mirrored (on the downside) by fluctuations in y_t. y_t is unaffected by fluctuations in k_t in excess of $\frac{\omega_1}{\omega_2}$.

Using (7), aggregate wealth, $k_t + P_t$, in the high return regime can be written as

$$k_t + P_t = \frac{e}{1+\frac{a_1}{a_2}(\frac{A}{\omega_1}+1-\delta)} + P_t - P_{t-1} = \frac{e}{1+\frac{a_1}{a_2}(\frac{A}{\omega_1}+1-\delta)} + \left(\frac{A}{\omega_1} - \delta\right) P_{t-1} - D.$$

As prices increase, wealth increases, though we also know that k_t decreases, and hence so does y_t. The "capital gains" effect overwhelms the capital accumulation effect. Stiglitz (2015) has argued that that has been the case in the US and some other advanced countries, and (in a more general model) helps reconcile Piketty's (2018) observation of increasing wealth with stagnating wages.

Using (7), aggregate wealth, $k_t + P_t$, in the low return regime can be written as

$$k_t + P_t = \frac{e+\frac{A}{\omega_2}}{1+\frac{a_1}{a_2}(1-\delta)} + P_t - P_{t-1} = \frac{e+\frac{A}{\omega_2}}{1+\frac{a_1}{a_2}(1-\delta)} + (-\delta)P_{t-1} - D$$

Now, wealth is decreasing even as k_t increases.

Aggregate consumption at any date t can be written as

$$C_t \equiv C_{1t} + C_{2t} = y_t + e - (1-\delta)k_t - k_{t+1},$$

and C_{2t} are aggregate consumption of young and old generations at date t, respectively.

Aggregate consumption is just the sum of the consumption levels of the young and old:

$$C_t = \frac{W(w_t,e)\frac{a_1}{a_2}(1+r_{t+1})}{1+\frac{a_1}{a_2}(1+r_{t+1})} + (1+r_t)s_{t-1}\, W(w_{t-1},e) = \frac{W(w_t,e)}{1+\frac{a_1}{a_2}(1+r_{t+1})} + \frac{W(w_{t-1},e)(1+r_t)}{1+\frac{a_1}{a_2}(1+r_t)}$$

This can take on just eight values, depending on whether the economy is in a capital shortage or surplus regime at dates $t-1$ (determining wage and therefore retirement capital and consumption of the elderly at t), t (determining wages of the young at t and the return on capital of the old at t) and $t+1$ (determining savings rate of the young at t). We can thus easily calculate the range of values that aggregate consumption can take. W takes on two values, W_H and W_L, s takes on two values, s_H and s_L and r takes on two values, r_H and r_Lr$_L$.

The highest value of aggregate consumption occurs in one of the following two states: (a) When the economy is in capital surplus at $t-1$, so wages are high, and in a capital shortage regime at t, so returns to capital are high, ensuring a high life time income for $t-1$ and therefore a high consumption of the elderly at time t. But the capital shortage at t means wages at t are low; but even so, if there is capital shortage at $t+1$, the savings rate will be low. (b) When the economy is in capital surplus at $t-1$, so wages are high, but it is again in a capital surplus regime at time t, so that savings at $t-1$ are high; but wages at t are accordingly high. Whether (a) or (b) generates a higher level of aggregate consumption depends on the level of savings (i.e. the value of $\frac{a_1}{a_2}$), the difference in W (i.e. the value of $\frac{A}{\omega_2}$), and the difference in returns (i.e. the value of $\frac{A}{\omega_1}$). We can similarly calculate the lowest value of aggregate consumption, and analyze how changes in the parameters affect the disparity between the highest and lowest values.

Appendix C: Proof of the existence of a non-empty set of parameter values in Proposition 2

First, we consider (a). Since $e_2 < e_3$, it is obvious that there exists a value of e for which (a) can be satisfied.

Next, we consider (b) and (c). In (b) and (c), $P_3 > 0$, which is equivalent to

(C1) $e > e_3$.

We first consider (b1). We are going to prove the existence of parameter values that satisfy $\frac{P_3}{1-\delta} \leq \frac{P_2}{\frac{A}{\omega_1}+1-\delta} \leq P_1$.

$\frac{P_2}{\frac{A}{\omega_1}+1-\delta} \leq P_1$ can be written as

(C2) $e\left[\frac{a_1}{a_2}\frac{A}{\omega_1}-\left(\frac{A}{\omega_1}-\delta\right)\left(1+\frac{a_1}{a_2}(1-\delta)\right)\right] \leq \left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}\left(1+\frac{a_1}{a_2}\left(\frac{A}{\omega_1}+1-\delta\right)\right).$

Hence if

(C3) $\frac{a_1}{a_2}\frac{A}{\omega_1} \leq \left(\frac{A}{\omega_1}-\delta\right)\left(1+\frac{a_1}{a_2}(1-\delta)\right),$

(C2) is obviously satisfied. (C3) can be rewritten as

(C3) $\delta\left(1+\frac{a_1}{a_2}(1-\delta)\right) \leq \frac{A}{\omega_1}\left(1-\frac{a_1}{a_2}\delta\right)$

A necessary condition for this is that $\frac{a_1}{a_2}\delta < 1$. If $\frac{a_1}{a_2}\delta < 1$, we can rearrange (C3).

(C3') $\frac{A}{\omega_1} \geq \delta\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\left(1-\frac{a_1}{a_2}\delta\right).$

$\frac{P_3}{1-\delta} \leq \frac{P_2}{\frac{A}{\omega_1}+1-\delta}$ can be written as

(C4) $e \leq \left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}\left\{1+\left(\frac{A}{\omega_1}+1-\delta\right)\left(\frac{a_1}{a_2}-1\right)\right\} \equiv e_{23}.$

Therefore, there exists a non-empty set of parameters if (C1) and (C2) and (C4) can hold simultaneously. That is, sufficient conditions are that

(C5) $e_3 < e < e_{23}$; $\frac{a_1}{a_2}\delta < 1$; and $\frac{A}{\omega_1} \geq \delta\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\left(1-\frac{a_1}{a_2}\delta\right).$

Since $e_{23} > e_3$, there exists a non-empty set of parameter values for which (C5) can be satisfied. Figure 4-2 illustrates such parameter space in the $\left\{e, \frac{A}{\omega_1}\right\}$ plane.

Next, we consider (b2). If

(C6) $\frac{a_1}{a_2}\frac{A}{\omega_1} > \left(\frac{A}{\omega_1}-\delta\right)\left(1+\frac{a_1}{a_2}(1-\delta)\right),$

(C2) can be rewritten as

(C7) $e \leq \frac{\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}\left(1+\frac{a_1}{a_2}\left(\frac{A}{\omega_1}+1-\delta\right)\right)}{\left[\frac{a_1}{a_2}\frac{A}{\omega_1}-\left(\frac{A}{\omega_1}-\delta\right)\left(1+\frac{a_1}{a_2}(1-\delta)\right)\right]} \equiv e_{12}.$

(C6) can be written as $\delta\left(1+\frac{a_1}{a_2}(1-\delta)\right) > \frac{A}{\omega_1}\left(1-\frac{a_1}{a_2}\delta\right)$. This condition is always satisfied if $\frac{a_1}{a_2}\delta > 1$.

,If $\frac{a_1}{a_2}\delta < 1$, we can rearrange (C6).

(C6’) $\frac{A}{\omega_1} < \delta\left(1 + \frac{a_1}{a_2}(1-\delta)\right) / \left(1 - \frac{a_1}{a_2}\delta\right)$

Then, by rearranging (C1), (C4), (C7), and (C6’), we can derive the conditions for $\left\{e, \frac{A}{\omega_1}\right\}$ that ensure that the inequalities $\frac{P_3}{1-\delta} \leq \frac{P_2}{\frac{A}{\omega_1}+1-\delta} \leq P_1$ are satisfied:

(C8) $e_3 < e \leq \min\{e_{23}, e_{12}\}$ and

if $\frac{a_1}{a_2}\delta < 1$, in addition: $\frac{A}{\omega_1} < \delta\left(1 + \frac{a_1}{a_2}(1-\delta)\right) / \left(1 - \frac{a_1}{a_2}\delta\right)$,

For $\frac{a_1}{a_2}(>1)$ near 1, $e_3 < e_{12}$. We also know $e_{23} > e_3$. Hence, by continuity, (at least) for $\frac{a_1}{a_2}(>1)$ near 1, there exists a non-empty set of parameter values in the $\left\{e, \frac{A}{\omega_1}\right\}$ plane for which (C9) can be satisfied.

Finally, we consider (c). We are going to prove there exists a non-empty set of parameter values that satisfy $\frac{P_3}{1-\delta} \leq P_1 \leq \frac{P_2}{\frac{A}{\omega_1}+1-\delta}$.

$\frac{P_3}{1-\delta} \leq P_1$ can be written as

(C9) $e \leq \frac{1}{\delta}\left[\left(1 + \frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2} + \left(\frac{a_1}{a_2} - 1\right)\frac{A}{\omega_2}\right] \equiv e_{13}$.

$P_1 \leq \frac{P_2}{\frac{A}{\omega_1}+1-\delta}$ is the reverse inequality analysed above, which holds iff

(C10) $\frac{a_1}{a_2}\frac{A}{\omega_1} \geq \left(\frac{A}{\omega_1} - \delta\right)\left(1 + \frac{a_1}{a_2}(1-\delta)\right)$,

which can be written as $\delta\left(1 + \frac{a_1}{a_2}(1-\delta)\right) \geq \frac{A}{\omega_1}\left(1 - \frac{a_1}{a_2}\delta\right)$, which is automatically satisfied if $\frac{a_1}{a_2}\delta \geq 1$.

If $\frac{a_1}{a_2}\delta < 1$, (C10) can be rewritten

(C10’) $\frac{A}{\omega_1} \leq \delta\left(1 + \frac{a_1}{a_2}(1-\delta)\right) / \left(1 - \frac{a_1}{a_2}\delta\right)$.

And solving $P_1 \leq \frac{P_2}{\frac{A}{\omega_1}+1-\delta}$ under (C10’) yields

(C11) $e \geq \frac{\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}\left(1+\frac{a_1}{a_2}\left(\frac{A}{\omega_1}+1-\delta\right)\right)}{\left[\frac{a_1}{a_2}\frac{A}{\omega_1} - \left(\frac{A}{\omega_1}-\delta\right)\left(1+\frac{a_1}{a_2}(1-\delta)\right)\right]} \equiv e_{12}$

(c) is non-empty if there is a set of parameters satisfying (C1), (C9), (C10’) and (C11).

By rearranging these conditions, we can derive the condition on $\left\{e, \frac{A}{\omega_1}\right\}$ that ensure that the inequalities $\frac{P_3}{1-\delta} \leq P_1 \leq \frac{P_2}{\frac{A}{\omega_1}+1-\delta}$ are satisfied. That is,

(C12) (i) $If\ \frac{a_1}{a_2}\delta \geq 1\ max\{e_3, e_{12}\} < e \leq e_{13}$.

(ii) If $\frac{a_1}{a_2}\delta < 1$, $max\{e_3, e_{12}\} < e \leq e_{13}$ and $\frac{A}{\omega_1} < \delta\left(1 + \frac{a_1}{a_2}(1-\delta)\right) / \left(1 - \frac{a_1}{a_2}\delta\right)$

$e_3 > e_{12}$ is equivalent to $\left(1 + \frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2} + \left(\frac{a_1}{a_2} - 1\right)\frac{A}{\omega_2} > \frac{\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}\left(1+\frac{a_1}{a_2}\left(\frac{A}{\omega_1}+1-\delta\right)\right)}{\frac{a_1}{a_2}\frac{A}{\omega_1} - \left(\frac{A}{\omega_1}-\delta\right)\left(1+\frac{a_1}{a_2}(1-\delta)\right)}$,

which can be written as $\left(\frac{a_1}{a_2} - 1\right)\frac{A}{\omega_1} > \frac{\left(1+\frac{a_1}{a_2}(1-\delta)\right)^2\left(1+\frac{A}{\omega_1}-\delta\right)}{\frac{a_1}{a_2}\frac{A}{\omega_1} - \left(\frac{A}{\omega_1}-\delta\right)\left(1+\frac{a_1}{a_2}(1-\delta)\right)}$, which can further be written as $\left(\frac{a_1}{a_2} - 1\right) > \frac{\left(1+\frac{a_1}{a_2}(1-\delta)\right)^2\left(\frac{1-\delta}{\frac{A}{\omega_1}}+1\right)}{\frac{A}{\omega_1}\left(-1+\frac{a_1}{a_2}\delta\right)+\delta\left(1+\frac{a_1}{a_2}(1-\delta)\right)}$. This inequality is satisfied for large enough $\delta(< 1)$ and large enough $\frac{a_1}{a_2}$. And it is clear that if $\delta < 1$, $e_3 < e_{13}$. Hence there exists a non-empty set of parameter values in the $\left\{e, \frac{A}{\omega_1}\right\}$ plane for which (C12) can be satisfied, at least for large enough $\delta(< 1)$ and large enough $\frac{a_1}{a_2}$.

Appendix D: Introducing growth

In this appendix, we show that our earlier analysis goes through if introduce labor and land augmenting technological progress..

The production function of the economy is now given by

(D1) $Y_t = A \min\left[\frac{K_t}{\omega_1}, \frac{H_t L_t}{\omega_2}\right] + H_t D T_t$,

where H_t is labor and land augmenting technological progress. H_t grows at the rate of h. That is,

(D2) $H_{t+1} = (1 + h)H_t$

We also assume that endowment grows at the same rate. $e_t = H_t e$. If the growth rate of endowment is different from h, the endowment "sector" will grow or shrink over time relative to other production sectors.

We define $k_t \equiv \frac{K_t}{H_t L_t}$ as capital per efficiency unit of labor/land and $\phi_t \equiv \frac{P_t}{H_t L_t}$ as land price per efficiency unit of labor/land, respectively.

Then the savings-investment function is written as

(D3) $\Omega(k_{t+1}, \phi_t) \equiv (k_{t+1}(1+h) + \phi_t)\left(1 + \frac{a_1}{a_2}(1 + r_{t+1})\right) = \frac{W_t}{H_t L_t}$

with

$$1 + r_{t+1} = \begin{cases} \frac{A}{\omega_1} + 1 - \delta & if\ k_{t+1} < \frac{\omega_1}{\omega_2} \\ 1 - \delta & if\ k_{t+1} > \frac{\omega_1}{\omega_2} \end{cases}$$

and

$$\frac{W_t}{H_t L_t} = \frac{w_t + H_t e}{H_t L_t} = \begin{cases} \frac{A}{\omega_2} + e & if\ k_{t+1} < \frac{\omega_1}{\omega_2} \\ e & if\ k_{t+1} > \frac{\omega_1}{\omega_2} \end{cases}$$

The capital arbitrage equation is

$$\frac{H_t D}{P_t} + \frac{P_{t+1}}{P_t} = 1 + r_{t+1}$$

which can be written as

(D4) $\phi_{t+1} = \left(\frac{1 + r_{t+1}}{1+h}\right)\phi_t - D.$

The dynamics of this economy is characterized by (D3) and (D4), which is identical to our earlier analysis, except that there is an additional parameter, h, which affects both the savings-investment function and the capital arbitrage equation. From (D3), the function Ω shifts up with an increase in the growth rate h and from (D4), the pace of the land price increase slows down, while the price decline becomes faster. It is clear that wobbly dynamics still exist.

Appendix E: Analysis of steady-states with just full-employment

From equation (25) we can derive the following quadratic equation regarding r^{**}.

$$v(r^{**}) \equiv (1 + \frac{a_1}{a_2})\frac{\omega_1}{\omega_2}(r^{**})^2 + \left\{\left(1 + \frac{a_1}{a_2}\right)\frac{\omega_1}{\omega_2} + D\frac{a_1}{a_2} - \left[\frac{\omega_1}{\omega_2}\left(\frac{A}{\omega_1} - \delta\right) + e\right]\right\} r^{**} + D(1 + \frac{a_1}{a_2}) = 0$$

with $v''(r^{**}) > 0$ and $v(r^{**} = 0) = D\left(1 + \frac{a_1}{a_2}\right) > 0$.

Solving for r^{**} yields

$$r^{**} = \frac{-b \pm \sqrt{b^2 - 4\left(1 + \frac{a_1}{a_2}\right)^2 \frac{\omega_1}{\omega_2} D}}{2\left(1 + \frac{a_1}{a_2}\right)\frac{\omega_1}{\omega_2}}.$$

where $b = \left(1 + \frac{a_1}{a_2}\right)\frac{\omega_1}{\omega_2} + D\frac{a_1}{a_2} - \left[\frac{\omega_1}{\omega_2}\left(\frac{A}{\omega_1} - \delta\right) + e\right].$

A solution with positive r^{**} requires

(F.1) $b < 0$, i.e. $e > \left(1 + \frac{a_1}{a_2}\right)\frac{\omega_1}{\omega_2} + D\frac{a_1}{a_2} - \frac{\omega_1}{\omega_2}\left(\frac{A}{\omega_1} - \delta\right)$.

Therefore, if $D > 0$ is sufficiently small, and condition (F.1) holds, there are two real solutions generating $r^{**} > 0$. In particular (using a Taylor series expansion).[73]

$$r^{**} \approx \frac{\frac{\omega_1}{\omega_2}\left(\frac{A}{\omega_1} - \delta\right) + e}{\left(1 + \frac{a_1}{a_2}\right)\frac{\omega_1}{\omega_2}} - 1 \text{ and } r^{**} \approx \left(1 + \frac{a_1}{a_2}\right)D$$

It is clear that for small enough e, $r^{**} \leq \frac{A}{\omega_1} - \delta$, i.e. the solution generates feasible values of r^{**}.

Appendix F: Deterministic cycles

In the text, we showed for the case of $D = 0$ that the existence of the wobbly dynamics implies the existence of deterministic cycles of multiple periodicities. Here we present period-2 and period-3 cycles for $D > 0$.

Proposition (Existence of Period-2 Cycles and Period-3 Cycles):

● **(6-1)** Suppose that individuals' beliefs alternate, i.e., they have bullish and bearish expectations in odd and even periods, respectively. Denoting date s as an even period, then land prices at date $s + 2$ can be written as $P_{s+2} = \left(\frac{A}{\omega_1} + 1 - \delta\right)(1 - \delta)P_s - \left(\frac{A}{\omega_1} + 1 - \delta\right)D - D$. If $\left(\frac{A}{\omega_1} + 1 - \delta\right)(1 - \delta) - 1 > 0$, $P_s = \frac{\left(\frac{A}{\omega_1} + 1 - \delta + 1\right)D}{\left(\frac{A}{\omega_1} + 1 - \delta\right)(1 - \delta) - 1} > 0$ and $P_{s+1} = \frac{(2 - \delta)D}{\left(\frac{A}{\omega_1} + 1 - \delta\right)(1 - \delta) - 1} > 0$, then we have $P_s = P_{s+2}$, and $P_{s+1} = P_{s+3}$. P_s satisfies $\frac{D}{1 - \delta} < P_{s+1} < P_s \leq P_2$, with the last inequality being satisfied if D is small enough. We have thus characterized the two period cycle.

● **(6-2)** Suppose that individuals have bearish expectations for two consecutive periods starting from period j followed by bullish expectations for one period. Then land prices at period $j + 3$ can be written as $P_{j+3} = \left(\frac{A}{\omega_1} + 1 - \delta\right)(1 - \delta)^2 P_j - \left(\frac{A}{\omega_1} + 1 - \delta\right)(1 - \delta)D - \left(\frac{A}{\omega_1} + 1 - \delta\right)D - D$. If $\left(\frac{A}{\omega_1} + 1 - \delta\right)(1 - \delta)^2 - 1 > 0$ and $P_j = \frac{\left(\left(\frac{A}{\omega_1} + 1 - \delta\right)(2 - \delta) + 1\right)D}{\left(\frac{A}{\omega_1} + 1 - \delta\right)(1 - \delta)^2 - 1}$, then we have $P_j = P_{j+3}$, where $\frac{D}{1 - \delta} < P_j$. That is, land prices at date $j + 3$ come back to the same value of date j. Period-3 cycles occur.[74] [75]

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[73] Obviously, for the former solution to be meaningful, $1 + r^{**} < \left(\frac{A}{\omega_1} - \delta\right)$, which in turn requires that e not be too large. When $b^2 - 4\left(1 + \frac{a_1}{a_2}\right)^2 \frac{\omega_1}{\omega_2} D = 0$, there is a unique positive solution, $r^{**} = \left(D \big/ \frac{\omega_1}{\omega_2}\right)^{1/2}$.

[74] We could generalize the argument by considering period-3 cycles in which individuals have bullish expectations for one period starting from P_0 followed by bearish expectations for two consecutive periods. (As in the two period cycles, we require D to be not too large.)

[75] Grandmont (1985) developed an overlapping generations model with fiat money which also gave rise to deterministic cycles. In Grandmont (1985), given the current real money balance, there are two values of real money balances in the next period consistent with rational expectations. The presence of two values generates

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the unimodal relationship between the current real money balance and the one in the next period. By applying the theory of unimodal maps, Grandmont (1985) proved the existence of deterministic cycles with various periods. There are several differences between his paper and ours. Firstly, in his paper, there is no investment and capital and the dynamical system is one dimensional. By contrast, our model has capital and productive land, so the dynamical system has two dimensions. There are deep interactions between land prices and capital date t and those values date $t+1$, affecting how phase transitions occur endogenously. Secondly, given k_t and P_t, there are at least *three* values of k_{t+1} consistent with r.e.. With three values, the relationship between k_t and k_{t+1} is not unimodal but shows more nonlinearity, which leads to the wobbly dynamics we have shown including seemingly explosive paths. Thirdly, in Grandmont (1985), applying the theory of unimodal maps the existence of a steady-state plays a crucial role. By contrast, in our model, under some conditions, no steady-state exists and the only possible rational expectations equilibrium is a wobbly economy.

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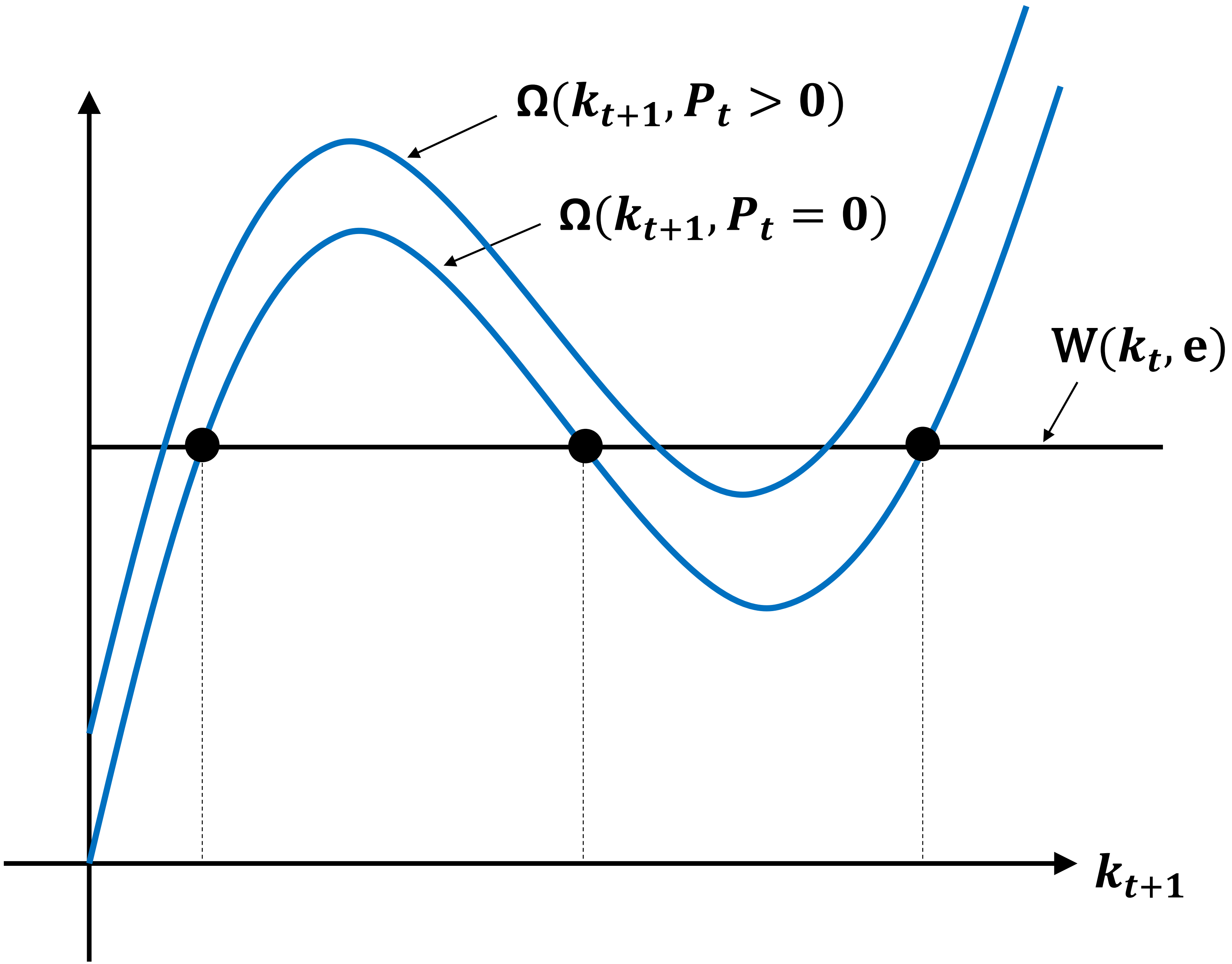

Figure 1-1: The existence of multiplicity of equilibria in a general case

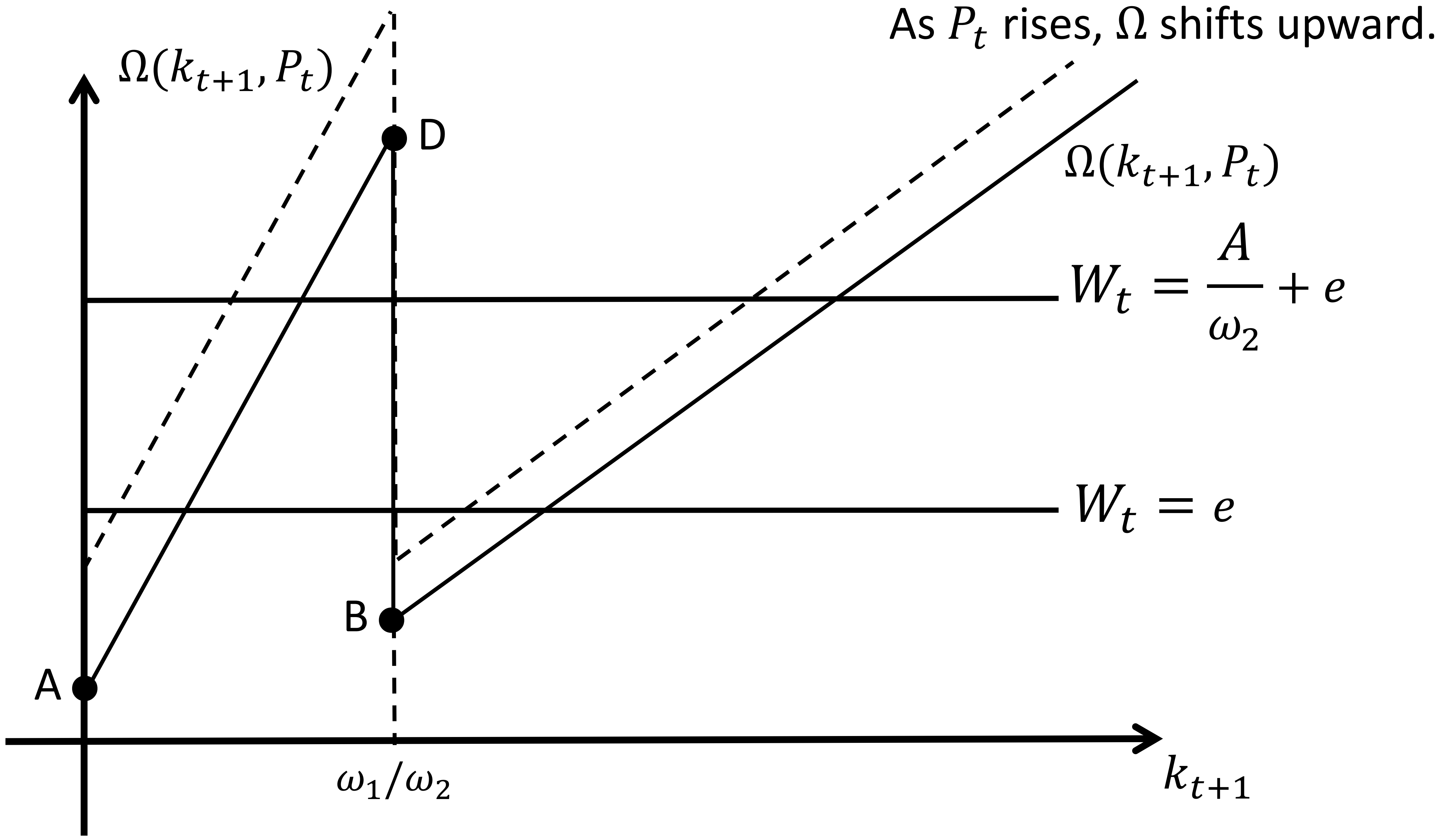

Figure 1-2: The existence of multiplicity of equilibria in the Leontief case

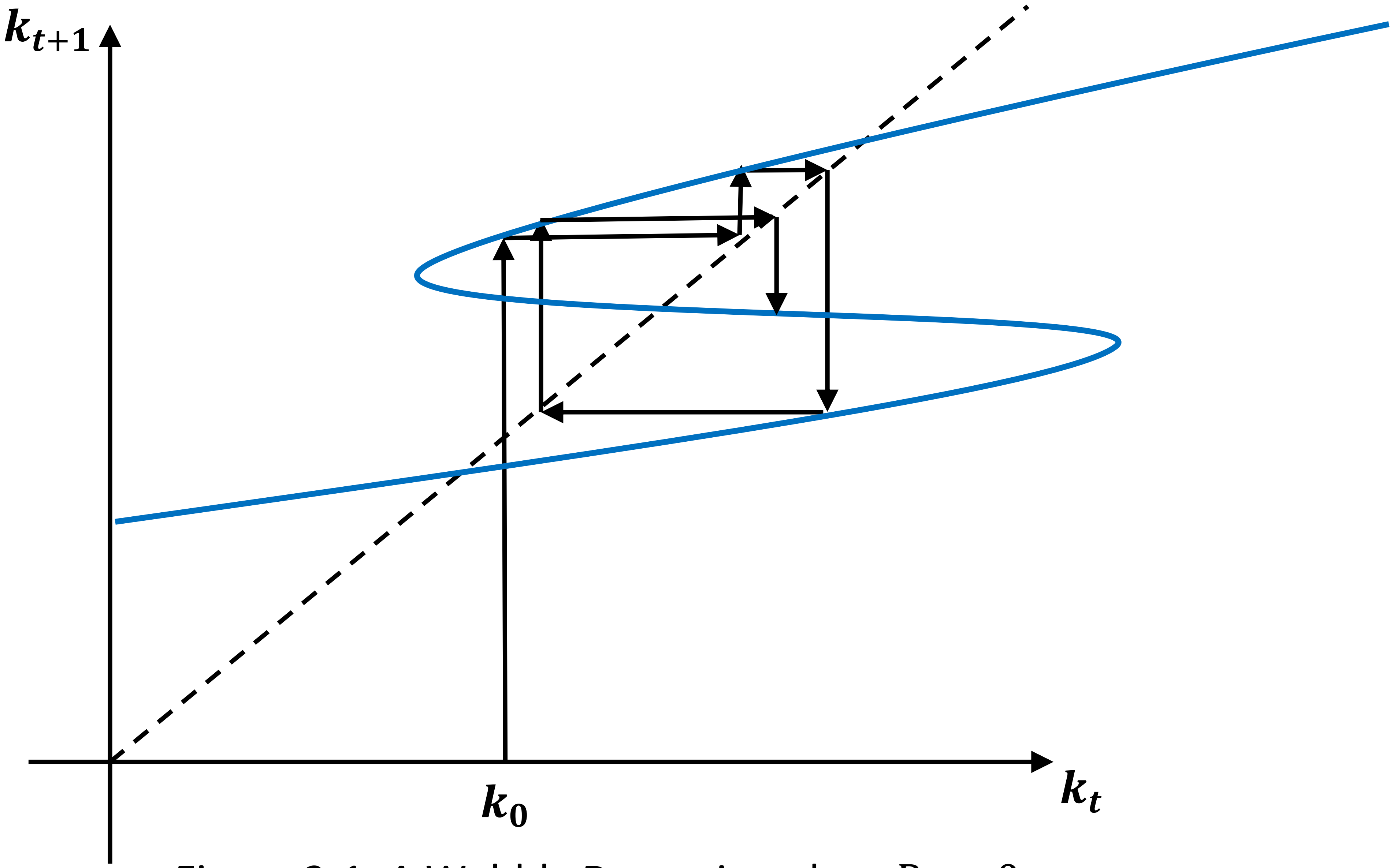

Figure 2-1: A Wobbly Dynamics when $P_t = 0$

In the case with land, the curve moves up and down with changes in P_t

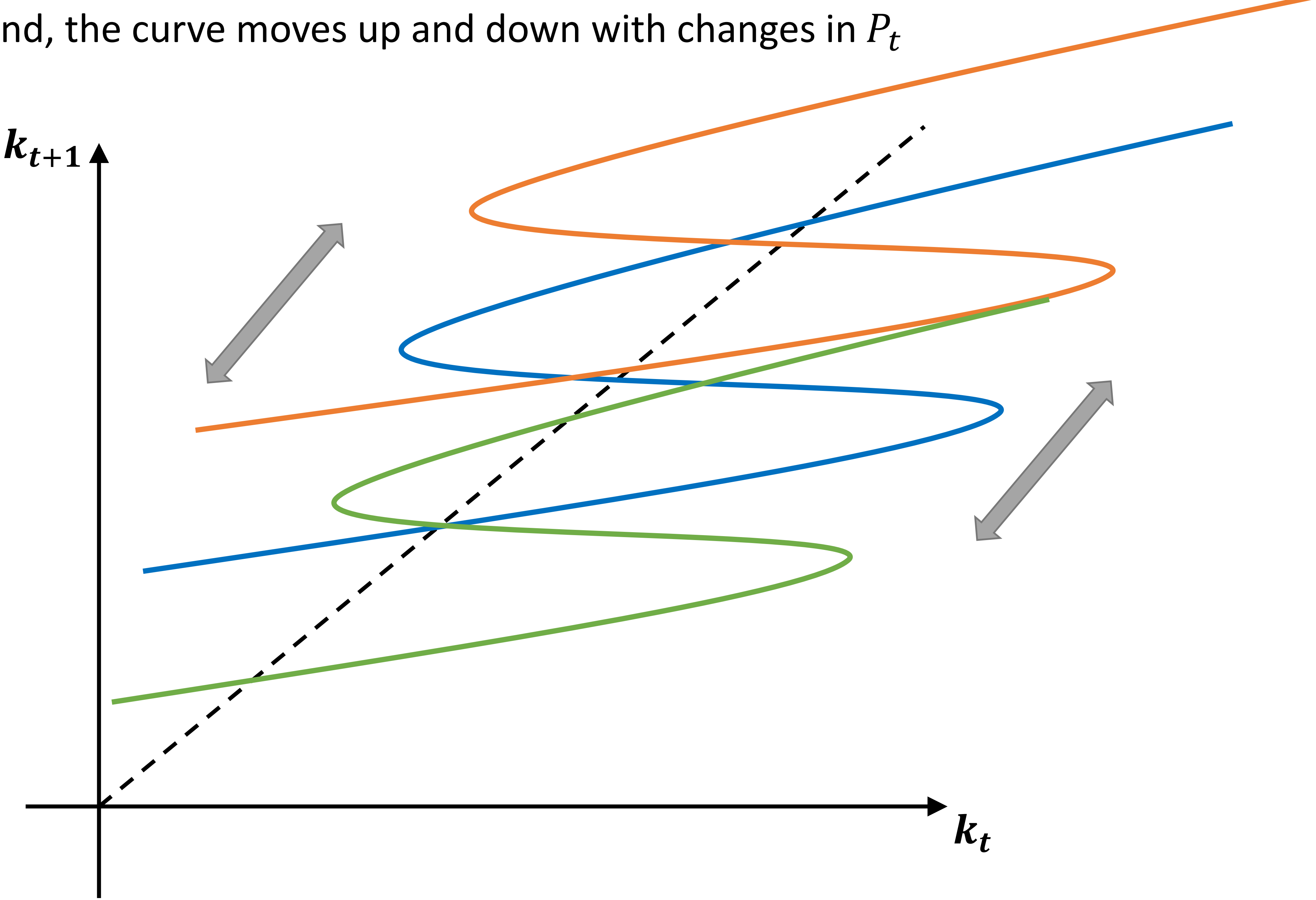

Figure 2-2: Dynamics of real capital when P_t constantly changes

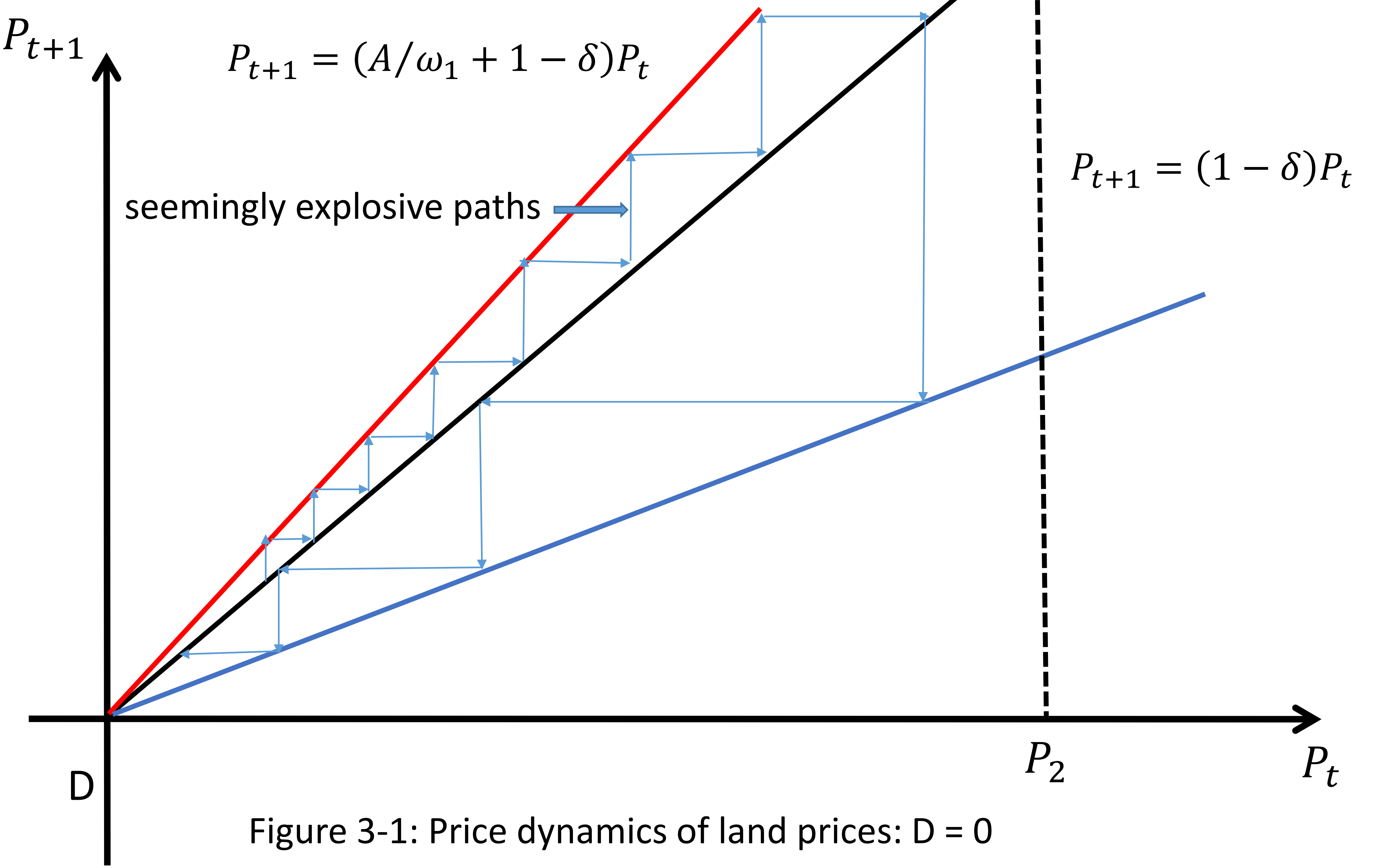

Figure 3-1: Price dynamics of land prices: D = 0

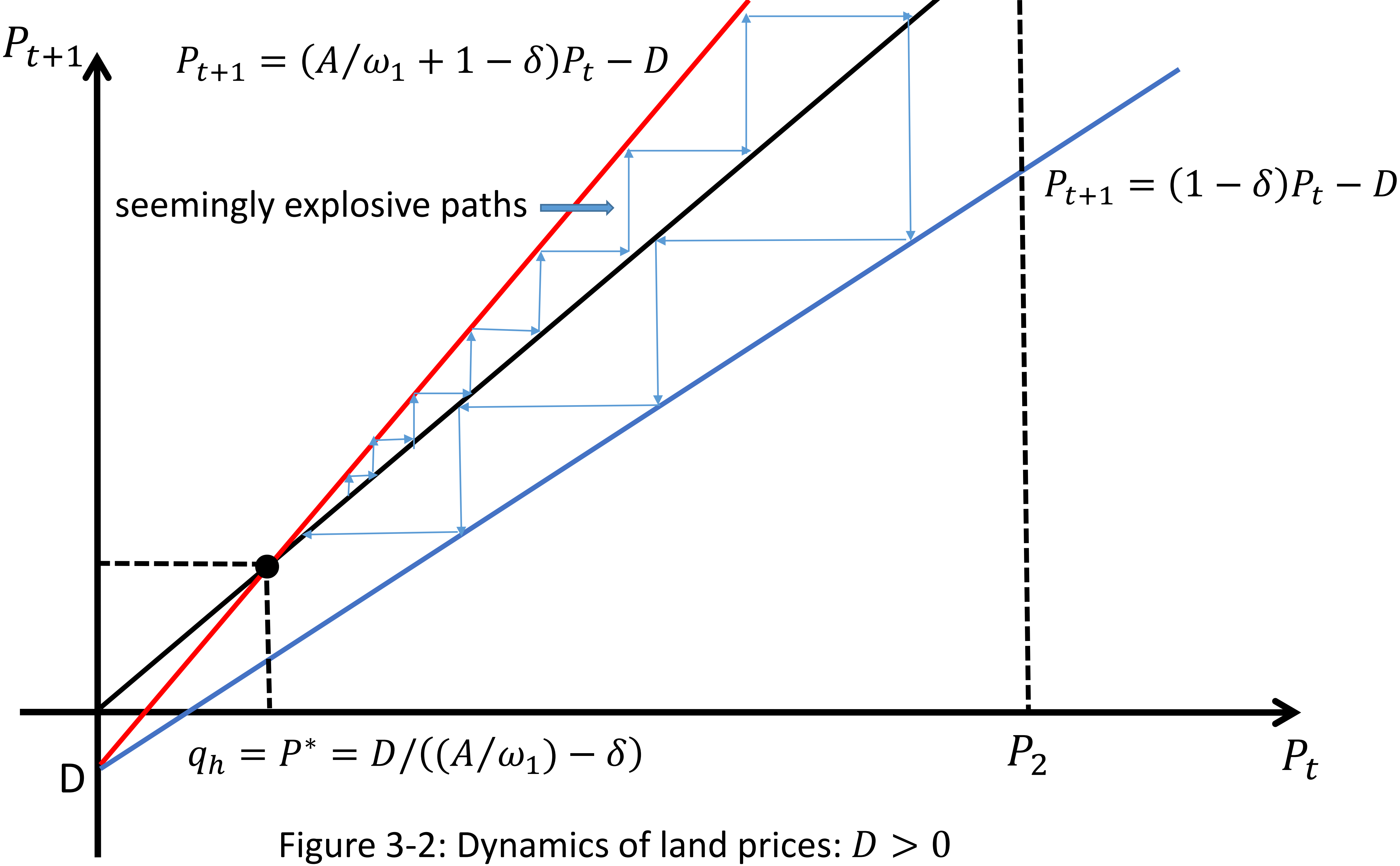

Figure 3-2: Dynamics of land prices: $D > 0$

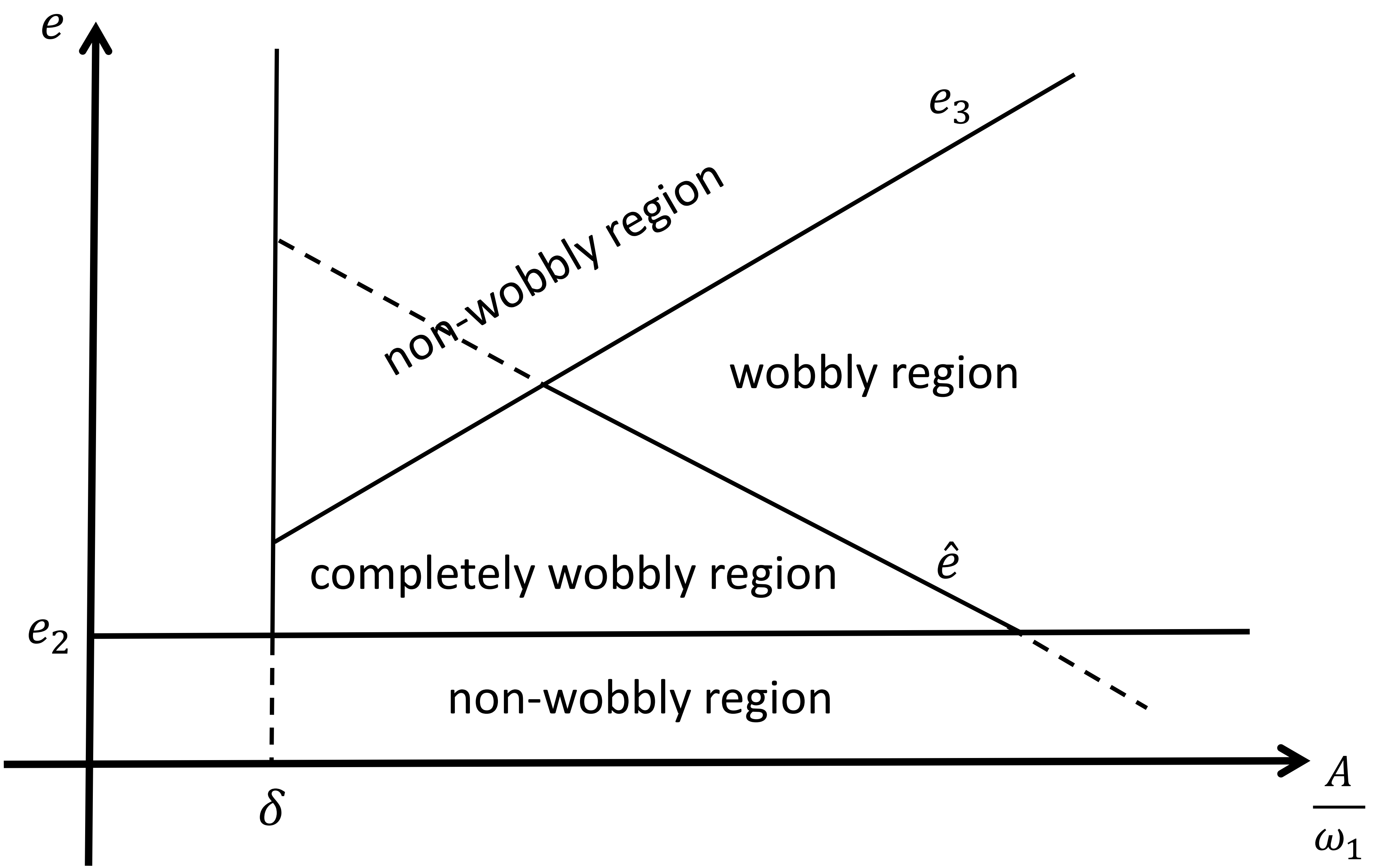

Figure 4-1: $e_2 < e < e_3$ and $\frac{A}{\omega_1} > \delta$

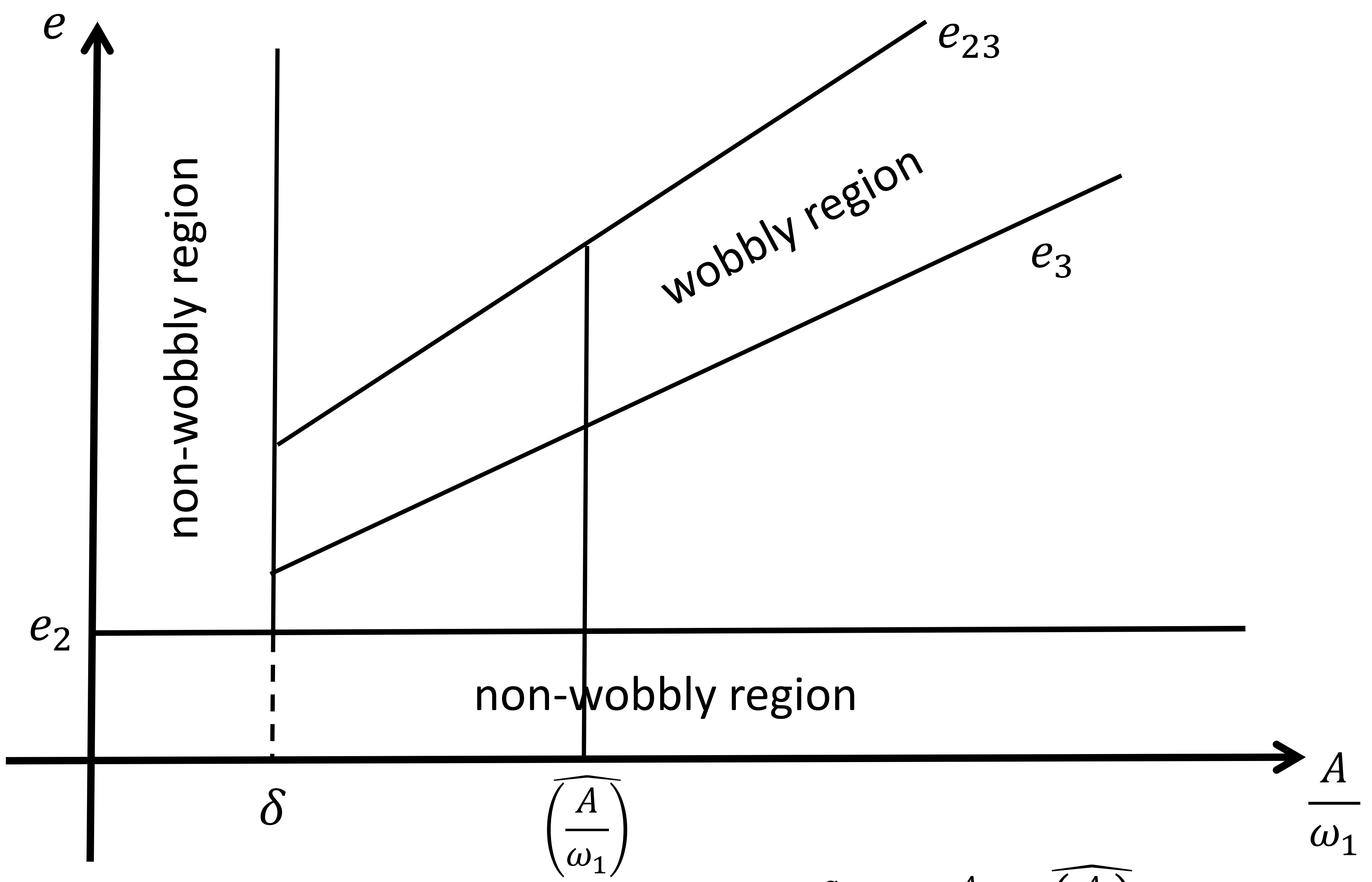

Figure 4-2: $e_3 < e < e_{23}$ and $\delta < \frac{a_2}{a_1}$ and $\frac{A}{\omega_1} > \widehat{\left(\frac{A}{\omega_1}\right)}$

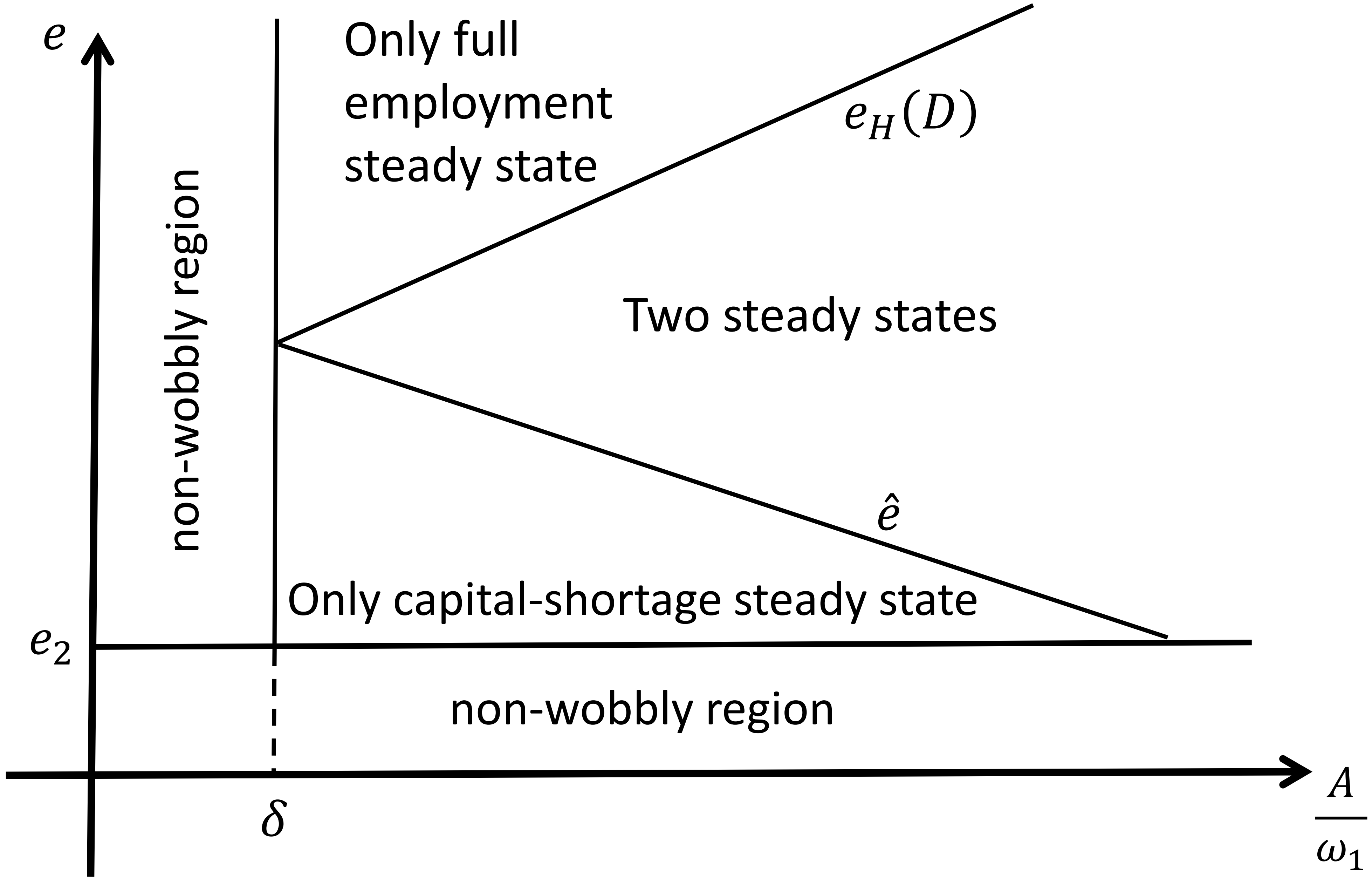

Figure 4-3: Steady-state characterization

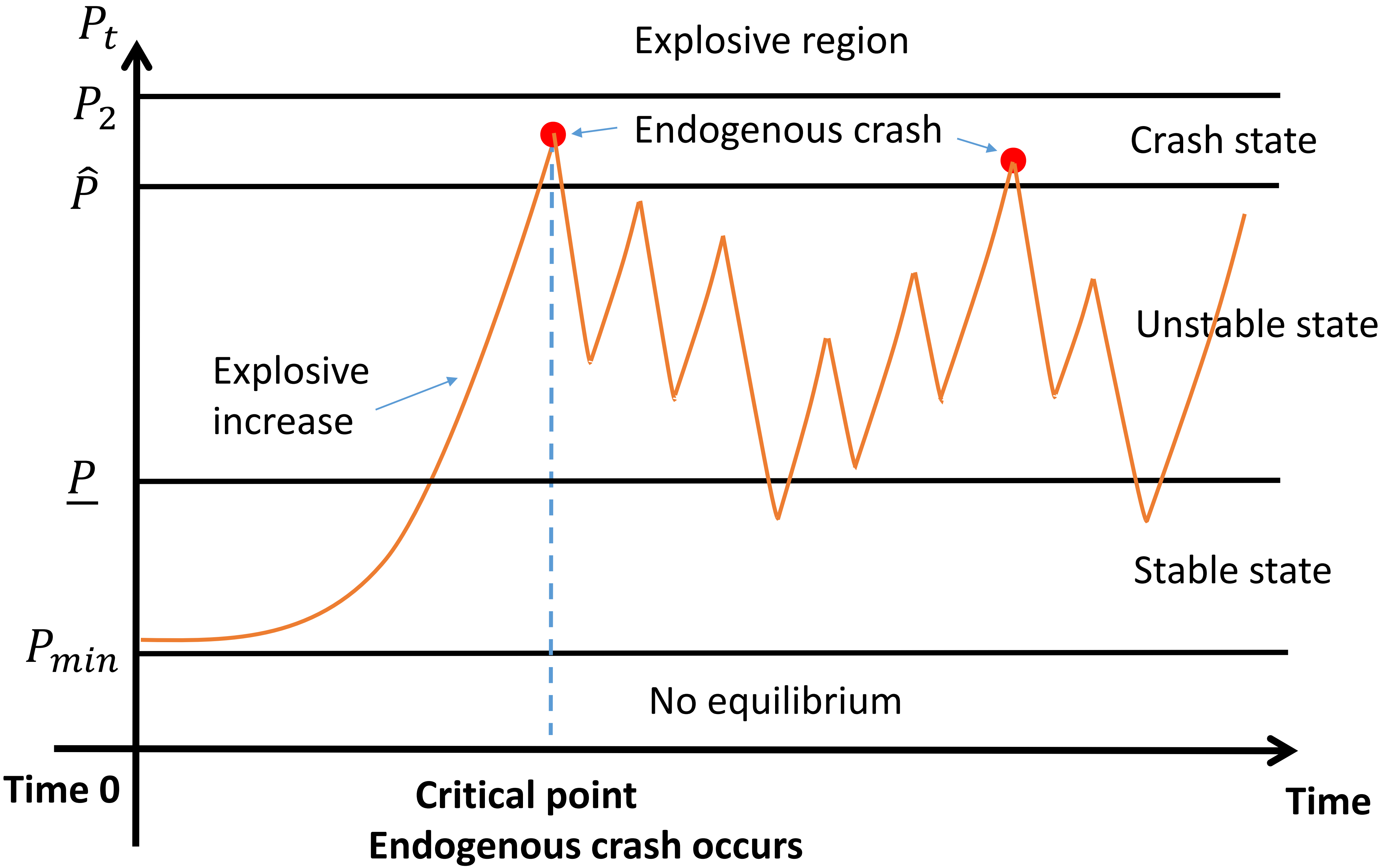

Figure 5: Endogenous Phase Transitions and Endogenous Crash

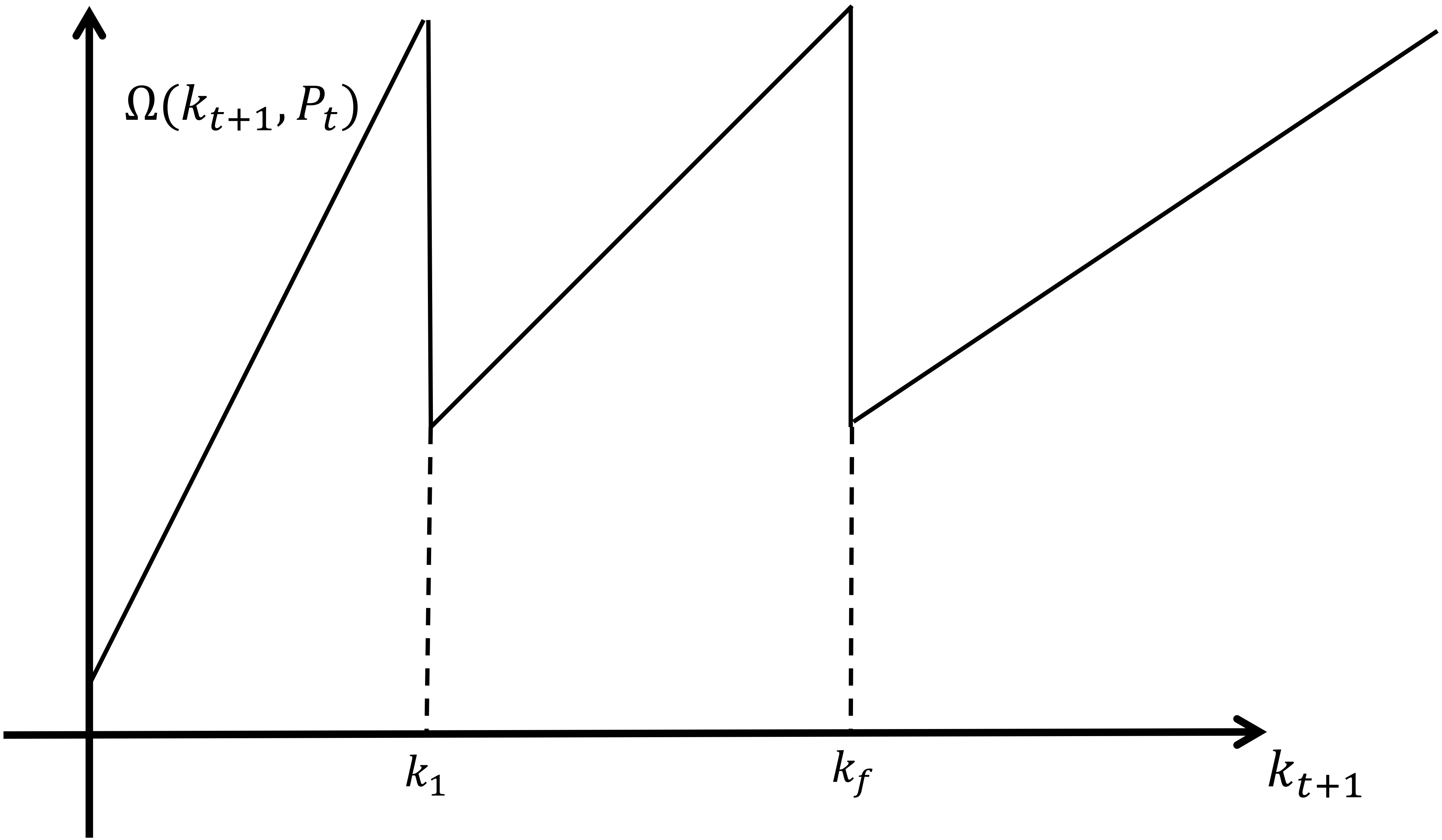

Figure 6: Sawtooth pattern with five momentary equilibria