Leverage Effect and Multifractality

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Multifractality $\Longleftrightarrow$ Multiple variation modes

Brownian motion

$$\Delta B(t) = B(t + \Delta t) - B(t)$$

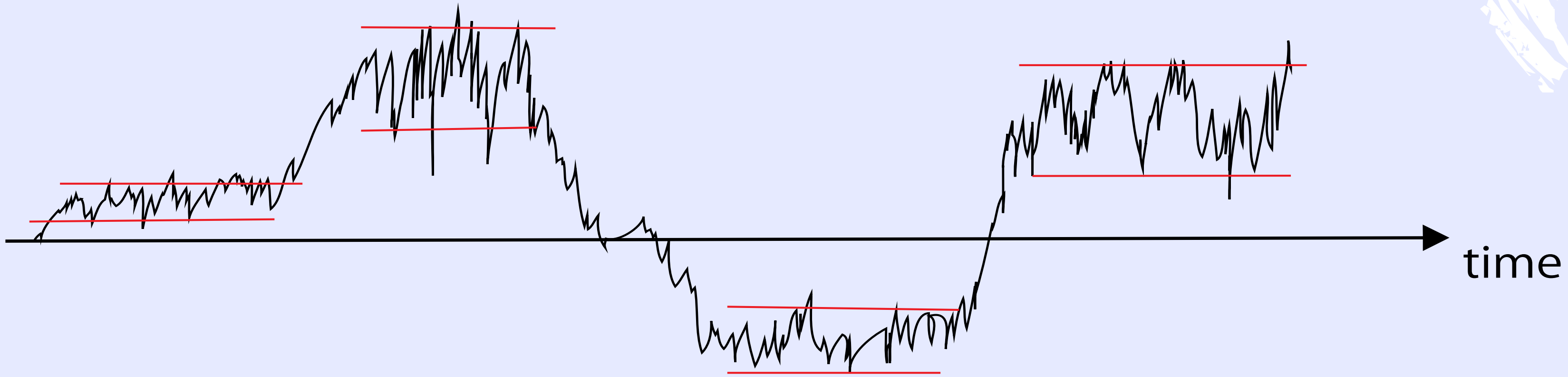
$$\sim (\Delta t)^{\frac{1}{2}} W \quad (W \sim \mathrm{N}(0,1)) \quad \text{for all } t$$

$$\Longleftrightarrow \text{ variation mode } (\Delta t)^{\frac{1}{2}}$$

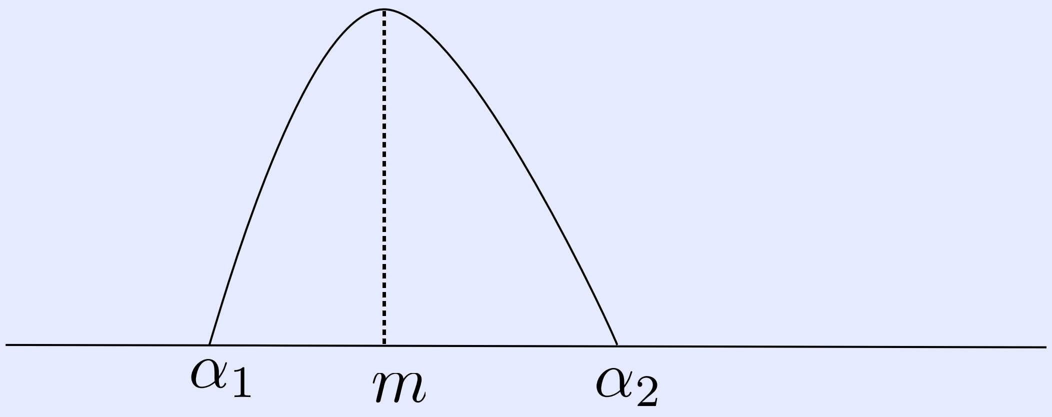
Log-return Process

$$\Delta X(t) = X(t + \Delta t) - X(t) \qquad X(t) = \log \frac{S(t)}{S(0)}$$

$$\sim (\Delta t)^{\alpha}\, W \qquad (\alpha \ : \ \text{depends on } t)$$



p.d.f. of $\alpha = \rho(\alpha)(\Delta t)^{-f(\alpha)}$ $f(\alpha)$: Multifractal spectrum

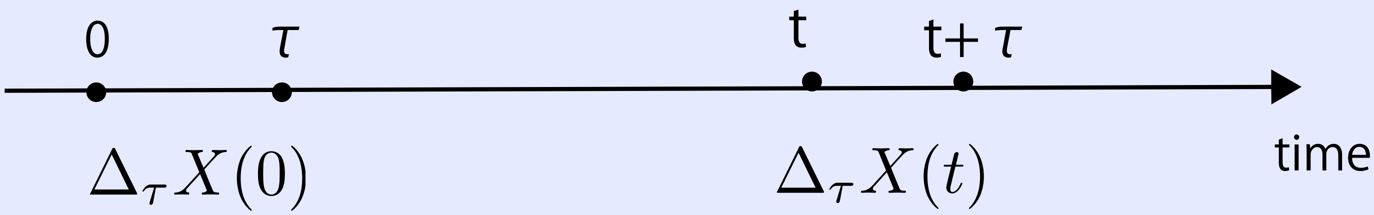


$E[|X(t)|^q] = C_q t^{\tau(q)}$ $\tau(q)$: non-linear

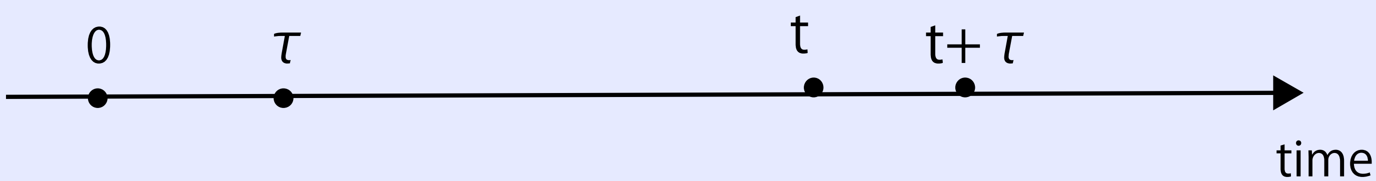
$f(\alpha) = \inf_q\{\alpha q - \tau(q)\}$: Legendre transform

Leverage Effect

$$\Delta_\tau X(t) = X(t+\tau) - X(t)$$



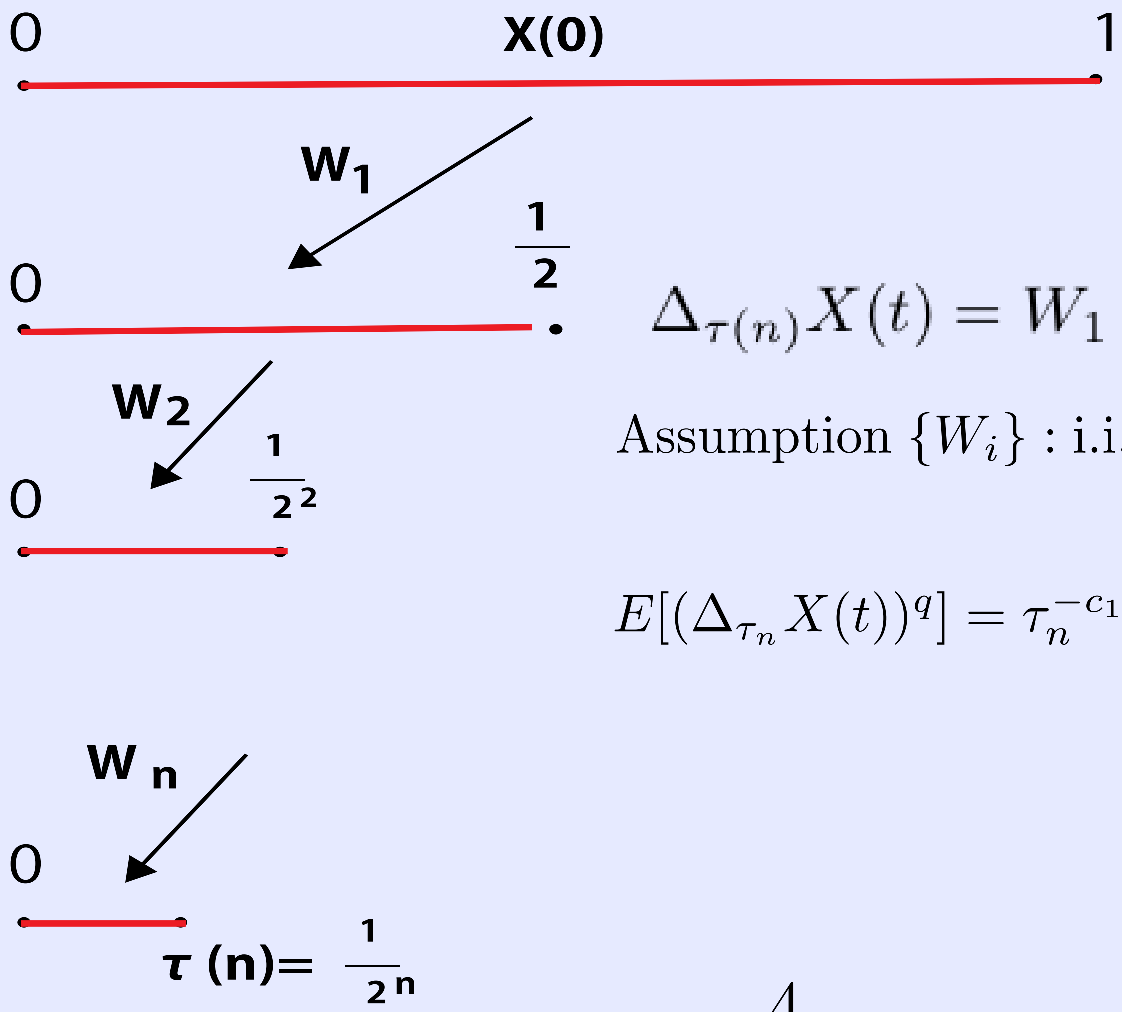
$$\mathrm{Cov}(\Delta_\tau X(0), (\Delta_\tau X(t))^2) < 0$$



$$\mathrm{Cov}((\Delta_\tau X(0))^2, \Delta_\tau X(t)) = 0$$

Mathematical Model for Multifractality

1. Multiplicative Cascade Model



$$\Delta_{\tau(n)} X(t) = W_1 \cdots W_n \cdot X(0)$$

Assumption $\{W_i\}$: i.i.d. $\log W_i \sim \mathrm{N}(\mu, \sigma^2)$

$$E[(\Delta_{\tau_n} X(t))^q] = \tau_n^{-c_1 q^2 - c_2 q}$$

2 Multifractal Random Walk

$$X(t)= \lim_{r\downarrow 0} \int_0^t \left(\frac{e^{w_r(u)}}{E[e^{w_r(\cdot)}]}\right)^{\frac{1}{2}} dB(u)$$

$\{w_r(t)\}$: log-volatility process

$w_{\lambda r}(\lambda^\alpha t) \sim w_r(t) + D_\lambda$: continuous cascade equation

(1) Bacry, Muzy C.M.P. (2003)

(2) Abry, Chainais, et al., IEET Trans on Info Th. (2009)

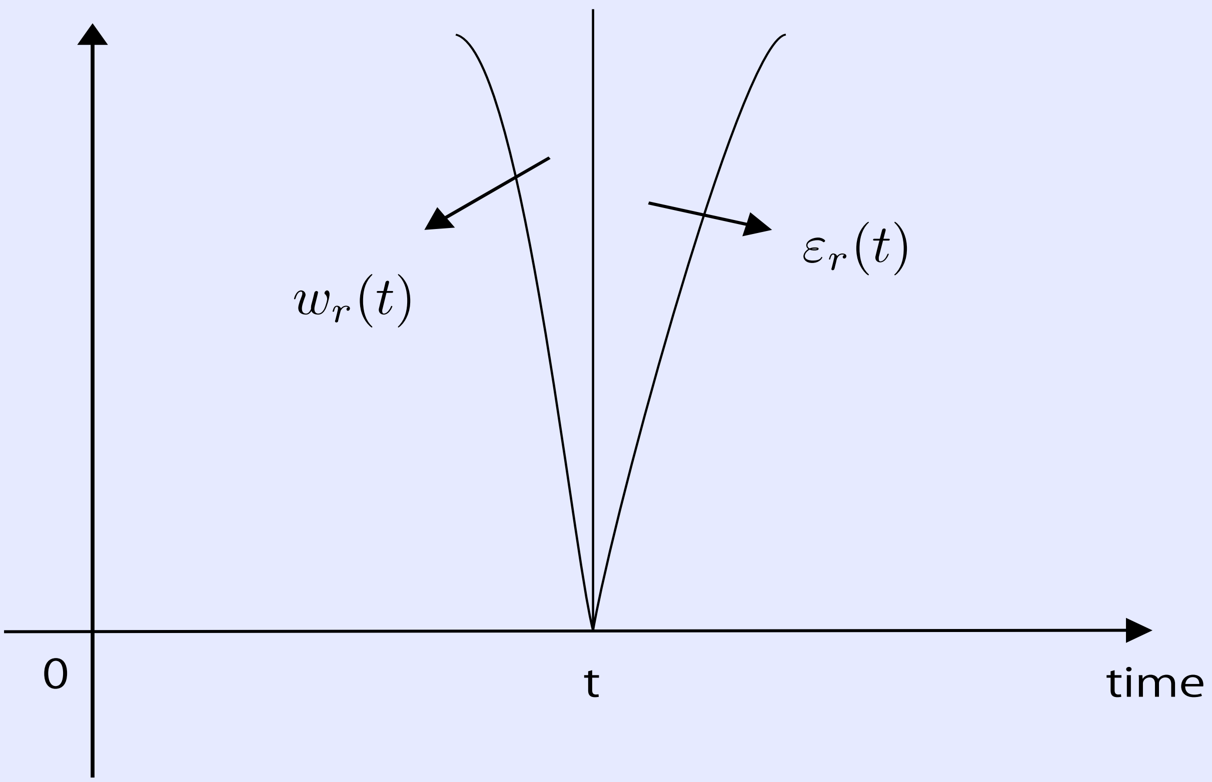
(3) Kuroda , Evol. Inst. Econo. Rev. (2016)

(4) Bacry, Duvernet, Muzy , Applied Prob.Trust (2011)

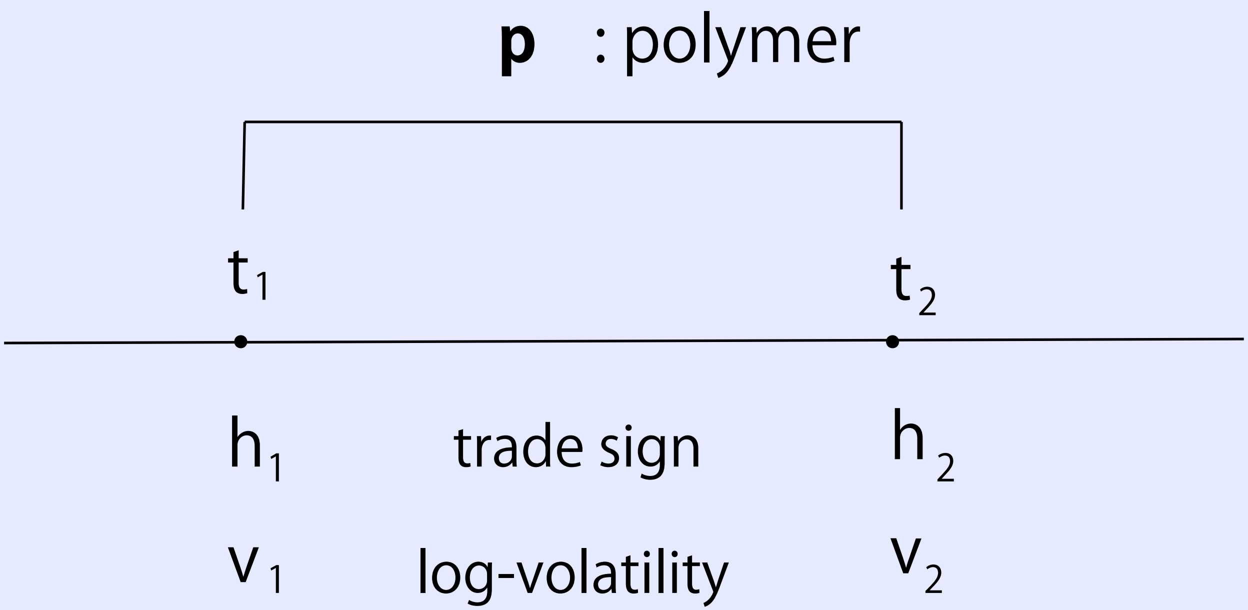
Bacry Muzy's Result on leverage Effect

$$X(t) = \lim_{r \downarrow 0} \int_0^t \varepsilon_r(u) e^{w_r(u)} du$$

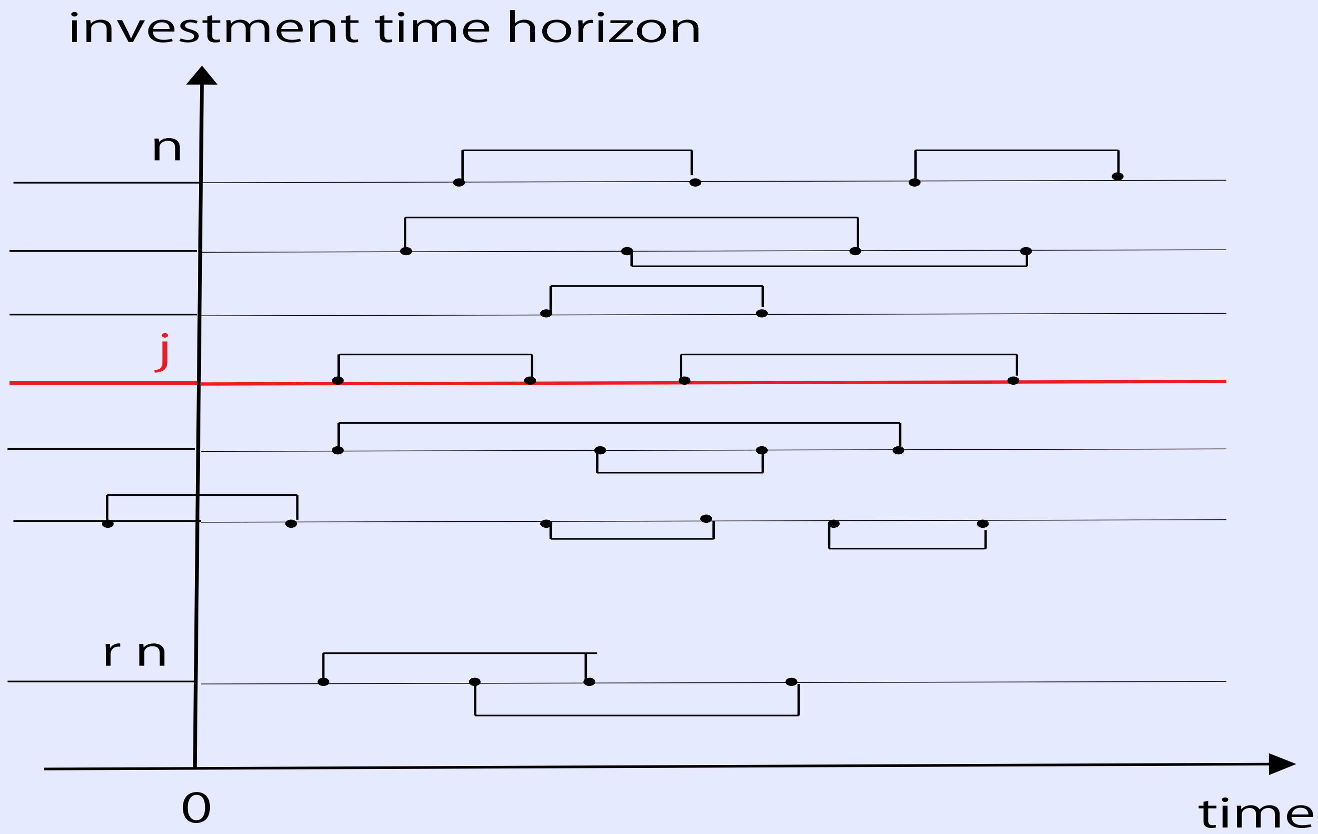
$\varepsilon(t), w_r(t)$: Correlated Gaussian Processes



Polymer Model



Interaction range : $t_2 - t_1 < T_2 n^{\gamma}$

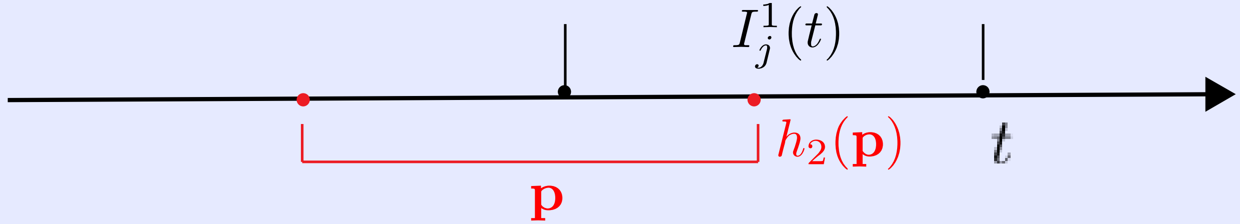


Prob. dist. of $t_2 - t_1 = \frac{c}{(t_2 - t_1)^{\beta_j}}$ for type j tradres

power-law exponent $\beta_j \downarrow$ as $j \uparrow$

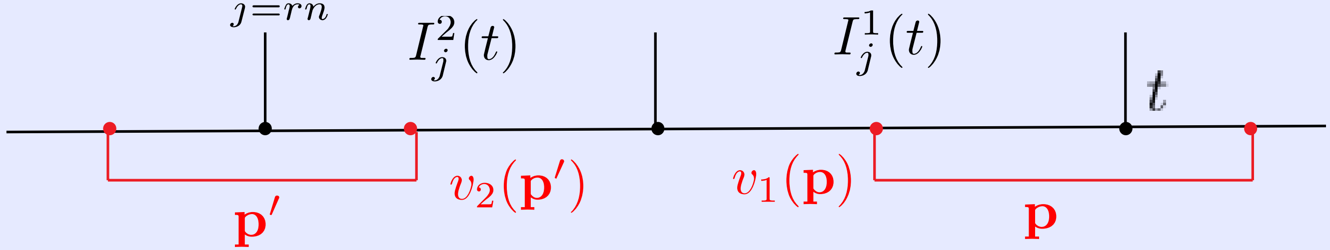
$0 < T_1 < T_2 - T_1$ $\quad I_j^1(t) = [t - T_1 j^\alpha, t)$

$$I_j^1(t) = [t - T_1 j^\alpha, t) \quad I_j^2(t) = [t - T_2 j^\alpha, t - T_1 j^\alpha)$$



$\varepsilon_t^j =$ Sum for $h_2(\mathbf{p})$ for polymers $\mathbf{p}$ such that $t_2 \in I_j^1(t)$

$$\varepsilon_t^{(n,r)} = \sum_{j=rn}^{n} \varepsilon_t^j$$



$W_t^j =$ (Sum for $v_1(\mathbf{p})$ for polymers $\mathbf{p}$ such that $t_1 \in I_j^1(t)$)

$+$(Sum for $v_2(\mathbf{p})$ for polymers $\mathbf{p}$ such that $t_2 \in I_j^2(t)$)

$W_t^{(n,r)} = \sum_{j=rn}^{n} W_t^j$

Scale Process $\hat{W}_t^{(n,r)} = \dfrac{W_{n^\alpha t}^{(n,r)}}{c(n)}$ $\hat{\varepsilon}_t^{(n,r)} = \dfrac{\varepsilon_{n^\alpha t}^{(n,r)}}{c(n)}$

Theorem 1 Under some assumptions on polymer,

$(\hat{W}_t^{(n,r)}, \hat{\varepsilon}_t^{(n,r)}) \to (w_r(t), \varepsilon_r(t)) \quad (n \to \infty)$

in a sense of finite dimensional distribution,

and $(w_r(t), \varepsilon_r(t))$ is a correlated Gaussian process with covariances $(c_r^w(t), c_r^\varepsilon(t), c_r^{\varepsilon,w}(t), c_r^{w,\varepsilon}(t))$

Remark $c_r^{w,\varepsilon}(t) = \mathrm{Cov}(w_r(t_1), \varepsilon_r(t_2)) = 0 \ (t = t_2 - t_1 > 0)$

$\Longrightarrow$ Leverage Effect

Theorem 2 Under some assumptions on polymer, the continuous cascade equation holds for $\{w_r(t)\}$

i.e. $w_{\lambda r}(\lambda^\alpha t) \sim w_r(t) + D_\lambda$ for any $0 < \lambda < 1$

$\Longrightarrow$ Multifractality for $\{X(t)\}$

$$X(t) = \lim_{r \downarrow 0} \frac{1}{E[e^{w_r(\cdot)}]} \int_0^t \varepsilon_r(u) e^{w_r(u)} du$$

Multifractality

Theorem 3 $(\alpha < \gamma)$

(1) $0 < t < T_1$

$$E[X(t)^q] = \sum_{m=0}^{\frac{1}{2}q} K_q(m) t^{-c_1 q^2 + c_2 q + c_3 m}$$

(2) $T_1 \leq t \leq T_2$

$$E[X(t)^q] = \sum_{m=0}^{\frac{1}{2}q} \sum_{0 \leq N_1 + N_2 \leq \frac{1}{2}(q^2 - q)} K_q(m, N_1, N_2) \; t^{-c_4 N_1 - c_5 N_2 + c_6 m - c_7 q}$$

Leverage Effect

Bacry-Muzy's Result on leverage Effect

$$|\mathrm{Cov}(\Delta_\tau X(0), (\Delta_\tau X(t))^2)| = O(d^{\frac{1}{2}})$$

$$(0.01 < d < 0.1)$$

$$|\mathrm{Cov}((\Delta_\tau X(0))^2, \Delta_\tau X(t))| = O(d^{\frac{3}{2}})$$

Bacry, Duvernet, Muzy , Applied Prob.Trust (2011)

Theorem 4

$$(1) \quad \mathrm{Cov}((\Delta_\tau X(0))^2, \Delta_\tau X(t)) = 0 \quad \text{if} \quad t > T_1$$

$$(2) \quad |\mathrm{Cov}(\Delta_\tau X(0), (\Delta_\tau X(t))^2)| > 0$$