Affine Term Structure Pricing with Bond Supply As Factors

by Fumio Hayashi

Slides prepared for CIGS Conference

31 May 2016
What This Paper Does: Extend Greenwood and Vayanos (2014)

  - The maturity structure \((s_t^{(1)}, \ldots, s_t^{(N)})\) is drive by a *single* factor.

- This paper:
  - The maturity structure unrestricted VAR.
  - IR of the yield curve to “local” shock to the maturity structure.

thus providing a modern formulation of Tobin (1969)’s *portfolio balance channel*. 
The Portfolio Balance Channel

Bernanke about LSAPII (his August 2010 Jackson Hole speech)

“I see the evidence as most favorable to the view that such purchases work primarily through the so-called portfolio balance channel... Specifically, the Fed’s strategy [the operation twist] relies on the presumption that different financial assets are not perfect substitutes in investors’ portfolios, so that changes in the net supply of an asset available to investors affect its yield and those of broadly similar assets.”

BOJ’s Announcement about QQE (April 4, 2013). Has part a)-d). b) is

b) An increase in JGB purchases and their maturity extension

“With a view to encouraging a further decline in interest rates across the yield curve, the Bank will purchase JGBs... In addition, ... the average remaining maturity of the Bank’s JGB purchases will be extended...."

FYI: a) base targeting, c) ETF and J-REIT, d) Inflation exit condition is 2%.
The Irrelevance Theorem?

- A casual look at recent Japanese JGB yields.
- \text{term premium} \equiv \text{yield} - \text{risk-neutral component} \text{ (average of current and future short rates), i.e.,}
  \[\text{yield} = \text{risk-neutral component} + \text{term premium}.
- Evidence in favor of the portfolio balance effect:
  - Joyce, Lasaosa, Stevens, and Tong (\textit{IJCB}, September 2011): same for the U.K.
Effect of Maturity Structure on Spread: Apr 2001 - March 2016

Spread (10 yr - 1 yr) vs BOJ holding of JGB/BOJ holding of all govt bonds
Rest of Talk

- What is ATSM?
- Greenwood-Vayanos (2014)
- the extension
- IR
ATSM Default-Free Bond Pricing (well known)

Notation:
\[ P_t^{(n)} = \text{price of } n\text{-period bonds at } t, \quad y_t^{(n)} = \text{yield on } n\text{-period bonds} = -\frac{1}{n} \log P_t^{(n)} \]

The model:
(factor dynamics) \[ f_{t+1} = c + \Phi f_t + \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \mathcal{N}(0, \Omega), \quad (1) \]
(short rate equation) \[ y_t^{(1)} = -\delta_0 - \delta'_1 f_t, \quad \text{i.e., } P_t^{(1)} = \exp \left( \delta_0 + \delta'_1 f_t \right), \quad (2) \]
(no-arb condition) \[ P_t^{(n)} = E_t \left( M_{t+1} \cdot P_{t+1}^{(n-1)} \right), \quad E_t(\cdot) \equiv E(\cdot | f_t). \]
(pricing kernel/SDF) \[ -\log(M_{t+1}) = y_t^{(1)} + \frac{1}{2} \lambda'_t \Omega \lambda_t + \lambda'_t \varepsilon_{t+1}, \quad \lambda_t \equiv \lambda_0 + \Lambda_1 f_t, \]

Can show: bond prices are exponentially affine, \[ P_t^{(n)} = \exp \left( \bar{a}_n + \bar{b}'_n f_t \right). \] The recursion:
\[
\begin{cases}
\bar{b}_1 = \delta_1, \quad \bar{b}'_n = \bar{b}'_{n-1} \Phi^Q + \bar{b}'_1 \quad (n = 2, 3, \ldots, N), \\
\bar{a}_1 = \delta_0, \quad \bar{a}_n = \bar{a}_{n-1} + \bar{b}'_{n-1} c^Q + \frac{1}{2} \bar{b}'_{n-1} \Omega \bar{b}_{n-1} + \bar{a}_1 \quad (n = 2, 3, \ldots, N),
\end{cases}
\]

with \[ \Phi^Q \equiv \Phi - \Omega \Lambda_1, \quad c^Q \equiv c - \Omega \lambda_0. \]

So \[ y_t^{(n)} = -\frac{1}{n} \bar{a}_n - \frac{1}{n} \bar{b}'_n f_t. \]
Rest of Talk

- What is ATSM?

- Greenwood-Vayanos (2014)

- the extension

- IR
Greenwood and Vayanos (2014): The Model

- Factor dynamics ((1) above): \( f_t = (f_{1t}, f_{2t})' \) in \( f_{t+1} = c + \Phi f_t + \varepsilon_{t+1} \). \( f_{2t} \) is the global shock to the maturity structure:

\[
s_{t}^{(n)} = \xi_n + \theta_n f_{2t} \quad (n = 1, 2, \ldots, N) \text{ with } \sum_{n=1}^{N} s_{t}^{(n)} = 1.
\]

- Short rate equation ((2) above): \( f_{1t} = y_{t}^{(1)} \) (i.e., \( \delta_0 = 0, \delta_1 = (1, 0)' \) in \( y_{t}^{(1)} = -\delta_0 - \delta_1' f_t \)).

- Replace the no-arb condition by an explicit model of government bond market.
  - Arbitrageuers and the gov’t (CB and Treasury).
  - The arbs’ decision problem:

\[
\max_{\{z_{t}^{(n)}\}_{n=1}^{N}} \left[ E_t(R_{t+1}) - \frac{\gamma}{2} \text{Var}_t(R_{t+1}) \right] \quad \text{subject to} \quad \sum_{t=1}^{N} z_{t}^{(n)} = 1,
\]

where

\[
R_{t+1} \equiv \sum_{n=1}^{N} \frac{P_{t+1}^{(n-1)}}{P_{t}^{(n)}} z_{t}^{(n)}.
\]

- Bond market equilibrium: \( s_{t}^{(n)} = z_{t}^{(n)}, \quad n = 2, 3, \ldots, N \).
Greenwood and Vayanos (2014): The Recursion

(ATSM recursion) \( P_t^{(n)} = \exp \left( \bar{a}_n + \bar{b}_n' f_t \right) \). The recursion:

\[
\begin{align*}
\bar{b}_1 &= \delta_1, \quad \bar{b}_n' = \bar{b}_{n-1}' \Phi^Q + \bar{b}_1' \quad (n = 2, 3, \ldots, N), \\
\bar{a}_1 &= \delta_0, \quad \bar{a}_n = \bar{a}_{n-1} + \bar{b}_{n-1}' c^Q + \frac{1}{2} \bar{b}_{n-1}' \Omega \bar{b}_{n-1} + \bar{a}_1 \quad (n = 2, 3, \ldots, N),
\end{align*}
\]

with \( \Phi^Q \equiv \Phi - \Omega \Lambda_1, \ c^Q \equiv c - \Omega \lambda_0. \)

Can show: bond prices are exponentially affine. (3) with

\[
\begin{align*}
\Phi^Q &\equiv \Phi - \Omega \Lambda_1, \ c^Q \equiv c - \Omega \lambda_0, \\
\bar{b} &\equiv \bar{b}_1 \xi_2 + \cdots + \bar{b}_{N-1} \xi_N, \quad \bar{b}_1 &\equiv \bar{b}_1 \theta_2 + \cdots + \bar{b}_{N-1} \theta_N.
\end{align*}
\]

No longer a recursion.
Sketch of Proof

Under the conjecture of exponentially affine bond prices \( P_t^{(n)} = \exp \left( \bar{a}_n + \bar{b}'_n f_t \right) \),

(a) derive arbs' FOCs under the conjecture, (b) impose \( z_t^{(n)} = s_t^{(n)} \) \( (n = 2, 3, \ldots, N) \).

Part (a):

\[
E_t \left[ \frac{P_{t+1}^{(n-1)}}{P_t^{(n)}} \right] - \frac{1}{P_t^{(1)}} = \gamma \frac{1}{2} \frac{\partial \text{Var}_t(R_{t+1})}{\partial z_t^{(n)}} \quad (n = 2, 3, \ldots, N).
\]

Part (b):

(i) Set \( z_t^{(n)} = s_t^{(n)} \) \( (n = 2, 3, \ldots, N) \).

(ii) Use \( s_t^{(n)} = \xi_n + \theta_n f_{2t} \).

by Fumio Hayashi

Affine Term Structure Pricing with Bond Supply As Factors

31 May 2016, 11 / 23
A Detour

Reproducing,

\[ E_t \left[ \frac{P_{t+1}^{(n-1)}}{P_t^{(n)}} \right] - \frac{1}{P_t^{(1)}} \approx E_t \left( \log P_{t+1}^{(n-1)} \right) - \log P_t^{(n)} + \frac{1}{2} \bar{b}_{n-1} \Omega \bar{b}_{n-1} - y_t^{(1)} \equiv r p_t^{(n)}. \]

Routine to show:

\[ y_t^{(n)} = \frac{1}{n} \sum_{i=0}^{n-1} E_t(y_{t+i}^{(1)}) + \frac{1}{n} \sum_{i=0}^{n-1} E_t(rp_{t+i}^{(n-i)}) + \frac{1}{n} A_n \quad (n = 1, 2, \ldots, N), \]

where \( A_n \equiv -\frac{1}{2} \sum_{i=0}^{n-1} \bar{b}_{n-i-1}^{(1)} \Omega \bar{b}_{n-i-1}. \)
Rest of Talk

- What is ATSM?
- Greenwood-Vayanos (2014)
- the extension
- IR
The Model: Allowing for Local Supply Shocks

\[ f_t = \left( y_t^{(1)}, s_t^{(2)}, s_t^{(3)}, \ldots, s_t^{(N)} \right)'. \]

Go back to (b-i). Key observation:

\[
\frac{1}{2} \frac{\partial \text{Var}_t(R_{t+1})}{\partial z_t^{(n)}} = \frac{\partial'}{\partial z_t^{(n)}} \Omega \left( \begin{array}{c} b_1 s_t^{(2)} + \cdots + b_{N-1} s_t^{(N)} \end{array} \right) \quad \text{(by replacing } z_t^{(i)} \text{ by } s_t^{(i)})
\]

\[
= \frac{\partial'}{\partial z_t^{(n)}} \Omega \left[ \begin{array}{ccc} b_1 & b_2 & \cdots & b_{N-1} \end{array} \right] \quad \text{(by replacing } z_t^{(i)} \text{ by } s_t^{(i)})
\]

\[
= \left( \begin{array}{c} b_1 \gamma b_2 \cdots b_{N-1} \end{array} \right) \quad \text{by } \Phi^{\Omega} = \Phi - \Omega \left[ \begin{array}{ccc} \gamma b_1 & \gamma b_2 & \cdots & \gamma b_{N-1} \end{array} \right]
\]

We obtain (3) with

\[
\begin{align*}
c^{\Omega} &\equiv c, \\
\Phi^{\Omega} &\equiv \Phi - \Omega \left[ \begin{array}{ccc} \gamma b_1 & \gamma b_2 & \cdots & \gamma b_{N-1} \end{array} \right] \quad \text{(by replacing } z_t^{(i)} \text{ by } s_t^{(i)})
\end{align*}
\]

by Fumio Hayashi
The QVE

To reproduce the recursion for $\overrightarrow{b}$:

$$
\overrightarrow{b}_1 = \delta_1, \quad \overrightarrow{b}'_n = \overrightarrow{b}'_{n-1} \Phi^Q + \overrightarrow{b}'_1 \quad (n = 2, 3, \ldots, N),
$$

$$
\Phi^Q_{(N \times N)} \equiv \Phi - \Omega \begin{bmatrix}
\gamma \overrightarrow{b}_1_{(N \times 1)} & \gamma \overrightarrow{b}_2_{(N \times 1)} & \cdots & \gamma \overrightarrow{b}_{N-1}_{(N \times 1)}
\end{bmatrix}_{(N \times (N-1))} \mathbf{S}_{((N-1) \times N)}.
$$

Write this as a quadratic vector equation (QVE)

$$
\mathbf{M}_{(N^2 \times N^2)} \overrightarrow{b}_{(N^2 \times 1)} = \mathbf{d}_{(N^2 \times 1)} - \gamma \mathbf{g}(\overrightarrow{b})_{(N^2 \times 1)},
$$

where

$$
\overrightarrow{b}_{(N^2 \times 1)} \equiv (\overrightarrow{b}_1, \ldots, \overrightarrow{b}_N) \text{ stacked}, \quad \mathbf{d}_{(N^2 \times 1)} \equiv \mathbf{1}_{(N \times 1)} \otimes \delta_1_{(N \times 1)},
$$

by Fumio Hayashi

Affine Term Structure Pricing with Bond Supply As Factors

31 May 2016, 15 / 23
The QVE (ctd.)

\[
\mathbf{M} = \begin{bmatrix}
\mathbf{I}_N & 0 & 0 & \cdots & 0 \\
\mathbf{I}_N & 0 & 0 & \cdots & 0 \\
0 & \mathbf{I}_N & \mathbf{I}_N & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \mathbf{I}_N & 0 \\
\end{bmatrix},
\]

\[
\mathbf{g}(\mathbf{b}) = \begin{bmatrix}
\mathbf{0} \\
\mathbf{S}' \\
\mathbf{B}' \\
\mathbf{\Omega} \\
\mathbf{b}_1 \\
\vdots \\
\mathbf{S}' \\
\mathbf{B}' \\
\mathbf{\Omega} \\
\mathbf{b}_{N-1} \\
\end{bmatrix}
= \begin{bmatrix}
\mathbf{P} \\
\mathbf{vec} \left( \mathbf{\overline{B}' \Omega \overline{B}} \right) \\
\end{bmatrix},
\quad
\mathbf{P} = \begin{bmatrix}
\mathbf{0} \\
\mathbf{I}_{N-1} \otimes \mathbf{S}' \\
\end{bmatrix},
\]

\[
\mathbf{\overline{B}} = \begin{bmatrix}
\mathbf{\overline{b}}_1 \\
\mathbf{\overline{b}}_2 \\
\vdots \\
\mathbf{\overline{b}}_{N-1} \\
\end{bmatrix}.
\]
Solution Methods

- Recall the QVE is: \( \mathbf{M} \mathbf{b} = \mathbf{d} - \gamma \mathbf{g}(\mathbf{b}) \).

- Case: \( \gamma = 0 \). A solution exists and it is unique.

  \[ \mathbf{b}^* \equiv \mathbf{M}^{-1} \mathbf{d}. \]

- Case: \( \gamma > 0 \). If a solution exists, there are generally more than one. Pick one that converges to \( \mathbf{b}^* \) as \( \gamma \downarrow 0 \) (as in Greenwood and Vayanos).

- This solution can be calculated by solving an appropriate differential equation (see next slide).

- Another option is the fixed-point algorithm:

  \[ \mathbf{b}^{(k+1)} = \mathbf{M}^{-1} \left[ \mathbf{d} - \gamma \mathbf{g}(\mathbf{b}^{(k)}) \right], \quad k = 0, 1, 2, \ldots \]

  No theoretical reason for the fixed point to be the particular solution we picked. In the example below, they are the same.
Solution Methods (ctd.)

Define $\mathbf{b}$ implicitly as a function of $\gamma$.

$$ f(\mathbf{b}, \gamma) = \begin{pmatrix} 0 \\ \text{(N}^2 \times 1) \end{pmatrix}, \quad f(\mathbf{b}, \gamma) \equiv \mathbf{Mb} - \mathbf{d} + \gamma \mathbf{g(b)}.$$

By the implicit function theorem, there exists an interval $U$ including 0 as an interior point and a vector-valued function of a single variable, $\mathbf{b}(\cdot): U \to \mathbb{R}^{N^2}$, such that $f(\mathbf{b}(\tilde{\gamma}), \tilde{\gamma}) = 0$ for all $\tilde{\gamma} \in U$ and its derivative $\mathbf{b}'(\cdot)$ is given by

$$ \mathbf{b}'(\tilde{\gamma}) = - \left[ \frac{\partial f(\mathbf{b}(\tilde{\gamma}), \tilde{\gamma})}{\partial \mathbf{b}'} \right]^{-1} \frac{\partial f(\mathbf{b}(\tilde{\gamma}), \tilde{\gamma})}{\partial \gamma} $$

$$ = - \left[ \mathbf{M} + \tilde{\gamma} \frac{\partial \mathbf{g(b(\tilde{\gamma}))}}{\partial \mathbf{b}'} \right]^{-1} \begin{pmatrix} \mathbf{g(b(\tilde{\gamma}))} \\ (N^2 \times 1) \end{pmatrix}. $$
Rest of Talk

- What is ATSM?
- Greenwood-Vayanos (2014)
- the extension
- IR
Specializing the Factor Dynamics of the Model

\[ \mathbf{f}_t \equiv \begin{pmatrix} y_t^{(1)}, s_t^{(2)}, s_t^{(3)}, \ldots, s_t^{(N)} \end{pmatrix} \]

The model is tightly parameterized. The parameters are: \( \gamma, N \), and the factor dynamics parameters \((c, \Phi, \Omega)\).

The factor dynamics (\( \mathbf{f}_{t+1} = \mathbf{c} + \Phi \mathbf{f}_t + \mathbf{\varepsilon}_{t+1} \)):

\[ y_t^{(1)} = c_1 + \rho y_{t-1}^{(1)} + \varepsilon_{1t}, \quad \varepsilon_{1t} \sim \mathcal{N}(0, \sigma_1^2), \]

\[ s_t^{(n)} = \begin{cases} c_n + \theta s_{t-1}^{(n+1)} + \varepsilon_{nt}, & \varepsilon_{nt} \sim \mathcal{N}(0, \sigma_n^2) \quad \text{if } n = 2, 3, \ldots, N - 1, \\ c_N + \varepsilon_{Nt}, & \varepsilon_{Nt} \sim \mathcal{N}(0, \sigma_N^2) \quad \text{if } n = N, \end{cases} \]

and \((\varepsilon_{1t}, \ldots, \varepsilon_{Nt})\) are uncorrelated.

The IR function for the VAR factor dynamics: for \( n = 2, 3, \ldots, N \),

\[ \frac{\partial \mathbf{f}'_{t+j}}{\partial \varepsilon_{nt}} \equiv \begin{pmatrix} 0, 0, \ldots, 0, \theta^j, 0, \ldots, 0 \end{pmatrix}_{(1 \times N)} \quad \text{for} \quad j = 0, 1, \ldots, n - 2, \]

\[ 0'_{(1 \times N)} \quad \text{for} \quad j = n - 1, n, \ldots. \]
Calibration

- The unit interval is a quarter. 20 years. So $N = 80$ quarters.

- Set $\gamma = 20$.

- The short rate process estimated on U.S. 3-month T-bill rate. Sample period is the Greenspan period. 1987:Q4-2007:Q4. This pins down $(c_1, \rho, \sigma_1)$.

- Pick $c_n = (1 - \theta)/N$ ($n = 2, 3, \ldots, N - 1$), $c_N = 1/N$ so that

  $$\text{steady-state value of } f_t^{(N \times 1)} = \left( \frac{c_1}{1 - \rho}, \frac{1}{N}, \ldots, \frac{1}{N} \right)'$$

  $N - 1$ elements

- $\sigma_n = \lambda/N$ ($n = 2, 3, \ldots, N$) with $\lambda = 0.01$. Results insensitive to $\lambda$.

- $\theta = 0$ or $\theta = 1$. 

by Fumio Hayashi  
Affine Term Structure Pricing with Bond Supply As Factors  
31 May 2016, 21 / 23
Average Yield Curve and Risk Premium

Yield Curve

Risk Premium

by Fumio Hayashi

Affine Term Structure Pricing with Bond Supply As Factors

31 May 2016, 22 / 23
Impulse Responses, $\theta = 1$, shock size is 1 percentage point.