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Growth Promotion Policies when Taxes Cannot be Raised*

Katsunori Minami† Ryo Horii‡

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Abstract

This paper examines the feasibility and effectiveness of growth promotion policies, such as R&D subsidies, when political constraints prevent the government from raising revenue. With sufficiently high life-cycle saving motives and high R&D productivity, our variety-expansion model shows that the growth rate exceeds the interest rate, and the government can finance its expenses while perpetually rolling over the debt. The economy is nevertheless dynamically efficient, because the social rate of return of R&D exceeds the growth rate. Given this situation, debt-financed R&D subsidies always accelerate technological progress in the short run. However, long-term growth can be improved only when R&D productivity exceeds another threshold. Our estimates suggest that this condition holds for most advanced countries, including the U.S., but not for low-growth countries such as Japan, Italy, and Greece. We then build a second-generation endogenous growth model that nests the baseline case, and show that low population growth is one reason why those countries do not meet the condition. We also extend the theory to include publicly funded basic research, unsustainable policies, asset income taxation, and a country premium.

Keywords: Endogenous growth, R&D, Public debt, Perpetual debt rollover, R minus G, Research subsidy, Basic research

JEL Classification Codes: O38, O41, E62

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1 Introduction

Once one starts to think about [growth], it is hard to think about anything else.

— Robert Lucas Jr. (1988)

As Lucas (1988) mentioned, economic growth is one of the most important objectives in economics. How can we achieve faster economic growth? Modern theories of endogenous growth provide a surprisingly simple answer. In the long run, economic growth is determined by technological change. Technological change is realized via R&D. Therefore, by promoting R&D, e.g., through government subsidies for R&D, economic growth can be accelerated. Nevertheless, many economies face slower economic growth than they desire. Even when they know that R&D promotion policies enhance growth, there are often insufficient funds to implement those policies. Governments typically have a limited ability to raise taxes, which constrains their ability to promote R&D and, therefore, economic growth.

Recent studies on the political economy suggest that it is difficult for the government to raise tax rates for political reasons. For example, Bassetto and Miller (2025) considered an upper bound on tax revenue due to political constraints on austerity. Jiang, Sargent, Wang and Yang (2022) assumed a similar constraint based on the political considerations of Keynes (1923).[1] However, existing studies of long-term growth typically disregard such constraints by implicitly assuming that the government can levy lump-sum taxes to implement policies (e.g. Romer, 1990; Grossman and Helpman, 1991a, Ch. 3). Some studies include distortions caused by factor-income taxation but still assume that the government can set any tax rate.[2]

In contrast, this paper considers an extreme situation in which political constraints do not allow the government to increase its tax revenue.[3] If the government wants to support R&D, the expenses must be financed entirely by public debt. The government also cannot raise taxes in the future to repay the debt. Therefore, the debt must be rolled over perpetually. In other words, we consider the environment in which growth-promotion policies must be financed by a Ponzi scheme. While common sense suggests that such policies would not be sustainable, O'Connell and Zeldes (1988) showed that the

[1] Through an overlapping generations political economy model, Song, Storesletten and Zilibotti (2012) demonstrated that intergenerational conflict can prevent the government from raising taxes if the political power of the old is stronger than that of the young. See also Alesina and Passalacqua (2016) and Yared (2019) for the literature review.

[2] For example, Ferraro and Peretto (2020) examined the case where the government can use a distortionary labor tax in the context of fiscal consolidation.

[3] We relax this assumption in Section 7.3, where the government is allowed to levy an asset income tax.

government can run a "rational" Ponzi scheme if the rate of economic growth is higher than the interest rate on governmental debt, i.e., $g > r$.

As we show in Section 2, the inequality $g > r$ holds widely across time and countries. Given this finding, this paper asks whether the growth rate can be increased when the government finances R&D subsidies entirely by issuing debt. The result is not obvious because such policies affect both g and r. The direct effect of R&D subsidies is to induce private firms to engage in more R&D, which will speed up technological progress. However, the government's debt increases the equilibrium interest rate in financial markets and may crowd out private R&D investments. More precisely, a higher interest rate implies that the present value of future profits realized by R&D is lower, thereby discouraging R&D.

In Section 3, we present a baseline model that combines a standard variety-expansion R&D model with a continuous-time overlapping-generations model in which individuals' labor productivity deteriorates as they age. This effect creates a life-cycle saving motive. We then show in Section 4 that $g > r$ holds in equilibrium if individuals have enough life-cycle saving motives and if the R&D productivity is high enough. However, this result does not always mean that the growth rate can be enhanced by debt-financed R&D promotion policies. For these policies to increase the long-term growth rate, the R&D productivity must be even greater. In other words, the $g - r$ gap needs to be not only positive but also greater than a positive threshold value. Our estimates of the R&D productivity show that most advanced countries, including the United States, satisfy this condition. However, the estimates from Japan, Italy, and Greece fall short of the threshold. In those countries, our theory suggests that R&D subsidies accelerate technological progress only temporarily, although output can continue to be higher than without the subsidies for several hundred years.

In Section 5, we examine whether the R&D promoted by these policies is beneficial to the economy. This question matters because the government in our model runs a rational Ponzi scheme of the kind studied by O'Connell and Zeldes (1988). What makes such a scheme feasible is not the length of individuals' planning horizons but the continuous arrival of new cohorts, which provides the economy with an infinite number of agents. In the rational bubble literature, Diamond (1965) and Tirole (1985) showed that an unproductive asset such as government debt can improve welfare when $g > r$, because that condition implies that the economy is dynamically inefficient.[4] In contrast, even when $g > r$ holds, we show that the economy in our model is dynamically efficient, but still, the government

[4]In later studies, Olivier (2000) and Hirano and Toda (2025) introduced bubbles for productive assets.

can provide research subsidies financed by a Ponzi scheme that can improve the welfare of future generations. Specifically, we show that the social rate of return of R&D, which includes the benefits from intertemporal knowledge spillovers, is always at least as high as the growth rate.

In Section 6, we examine how these results are affected when we explicitly introduce population growth or decline. We build a second-generation endogenous growth model with a changing population size, which nests the baseline model as a special case. When the population grows faster, the debt per person is diluted more quickly. Rolling over the debt then becomes easier, and the crowding-out effect of a given amount of debt weakens, so debt-financed growth-promotion policies are more likely to be effective. The opposite holds in countries where the population is shrinking. To assess the implications of this model for individual countries, we calibrate it using each country's population history since 1820. We find that Japan, Italy, and Greece cannot even roll over their debt permanently, given their current low population growth.[5]

Section 7 presents four extensions of the baseline model. According to the baseline model, Japan, Italy, and Greece cannot enhance growth through debt-financed subsidies because their research productivities are too low. In Section 7.1, we extend the model to allow funds from debt rollovers to be used on expanding basic research, aiming to improve the country's research productivity. We calibrate the extended model using the data on basic research expenditure in each country, as well as the estimate of the effectiveness of basic research on private R&D from Jaffe (1989). Then, among these three problematic countries, we find that Japan and Greece can improve long-term growth by directing funds to basic research, whereas Italy might not. We also derive the conditions that determine whether the growth-maximizing policy should direct the funds to research subsidies, basic research, or the mix of these.

Section 7.2 relaxes the setting by allowing the government to implement unsustainable policies. When the government sets a subsidy rate above the sustainable level, it will eventually face a situation in which the debt-GDP ratio is too high to justify. We assume that, at that time, there is some kind of intervention, subsidies are terminated, and the government is forced to enter a period of fiscal austerity. We show that such an unsustainable policy can nevertheless deliver higher consumption than a sustainable policy of modest scale for a long period, possibly several hundred years. The analysis shows that voters may support such a policy depending on their age and wealth holdings.

[5]We also find that Portugal can roll over its debt, but research subsidies do not raise its long-term growth rate.

Motivated by the fact that statutory rates on capital income have historically been revised more often than those on labor income, Section 7.3 relaxes the assumption that the government cannot raise any taxes, allowing it to adjust an asset income tax. It turns out that, even in this case, debt-financed research subsidies are necessary to maximize long-term growth. In Section 7.4, we confirm that the main qualitative results carry over to a small open economy in which government debt raises the country premium. The condition $g > r$ still requires that private R&D productivity is sufficiently high, and raising long-term growth by a marginal debt-financed subsidy requires an even higher productivity.

Related Literature

This paper is at the crossroads of two important strands of literature: the public finance literature and the endogenous growth one. In the public finance literature, a body of studies examined the sustainability of government debt in an environment where $g > r$ holds in equilibrium (e.g. Blanchard, 2019; Mehrotra and Sergeyev, 2021; Reis, 2021; Ball and Mankiw, 2023; Barro, 2023; Kocherlakota, 2023). While these studies are not particularly focused on how the funds from issuing debts are used, our model shows that whether debts are helpful depends on the use of the funds, even when $g > r$ holds. In our study, debts themselves are growth-reducing because they increase the equilibrium interest rates and always reduce the market value of innovation. Regarding the point that the growth rate is endogenous, our study is close to Saint-Paul (1992) and King and Ferguson (1993), who developed AK-type endogenous growth models in which $g > r$ holds in equilibrium. However, in those AK-type models, the long-term growth is determined solely by capital accumulation, while we consider an R&D-based variety-expansion growth model where the incentives for innovation matter. By explicitly modeling the R&D process, we are able to examine the effect of debt-financed R&D promotion policies on technological innovations, which unarguably serve as an important source of economic growth.

In the endogenous growth literature, starting with Romer (1990), a long line of works has explored the effects of R&D subsidies on economic growth. Among recent studies, Wan and Zhang (2021), Iwaisako (2023), Chu, Furukawa and Ji (2016), Minniti, Parello and Segerstrom (2013), and Grossmann, Steger and Trimborn (2013) examined the effects under various specifications of economic growth.[6] While these studies assume balanced budgets,

[6] Wan and Zhang (2021) examined the growth effect of R&D subsidies financed by a distortionary labor tax in a variety-expansion model and applied the result to the U.S. economy. Using a variety-expansion model in the context of heterogeneous industries, Iwaisako (2023) derived the optimal R&D subsidies. Chu et al. (2016) and Minniti et al. (2013) considered the effect of R&D subsidies in the Schumpeterian growth model. The effectiveness of R&D subsidies in semi-endogenous growth models is examined in Grossmann et al. (2013) and Cozzi (2023). See also the references in Chu and Wang (2022).

we focus on the fact that the introduction of government subsidies is not necessarily paired with increases in taxes.[7]

This paper is also related to Angeletos, Lian and Wolf (2024) in that both show that future tax increases are not necessary after debt-financed policies are implemented. While the abovementioned authors consider a short-term stimulus policy in a new Keynesian setting, we consider a long-term growth promotion policy in an R&D-based endogenous growth model.

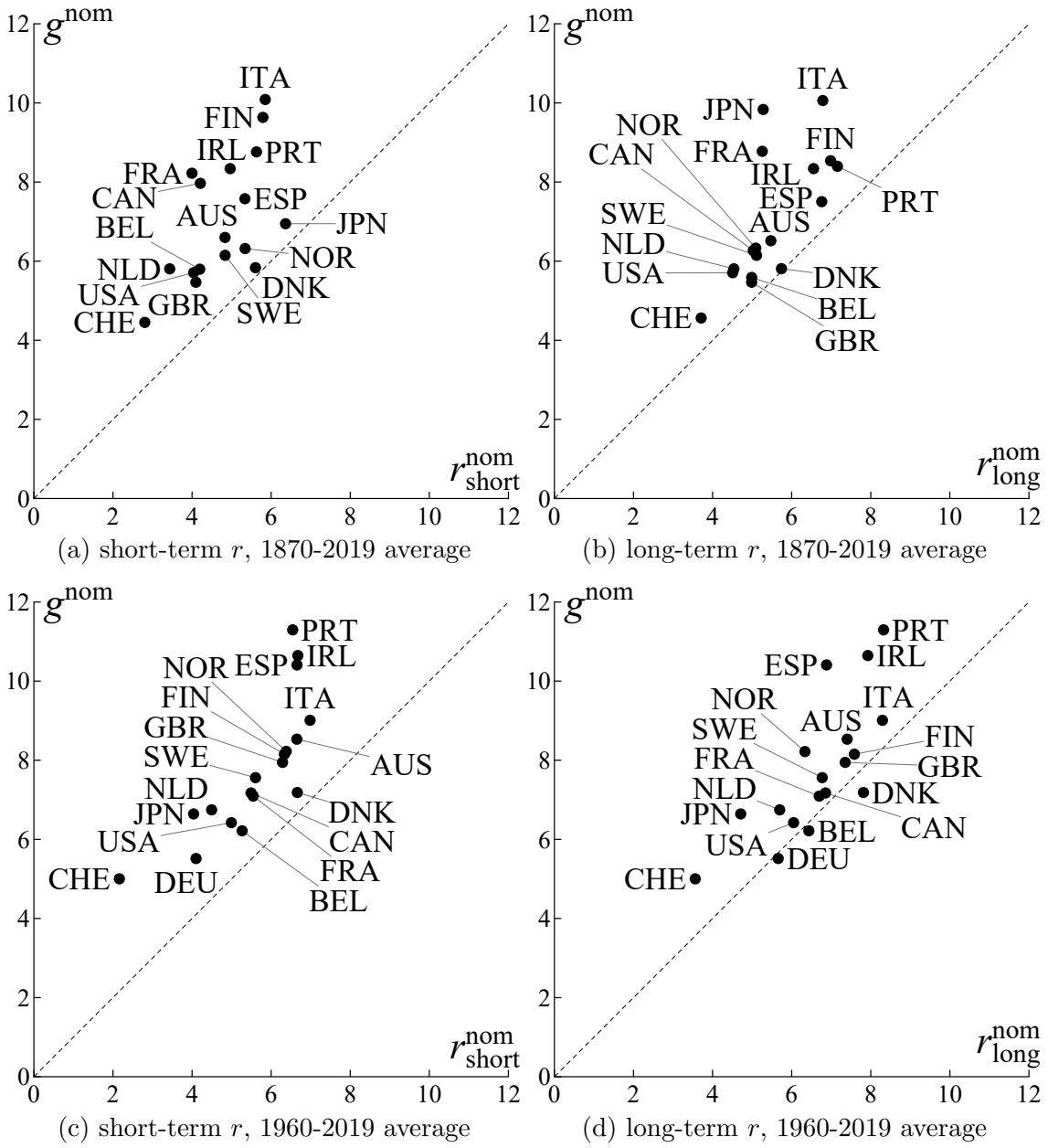
2 The $g > r$ Condition: Evidence

This section briefly presents an empirical basis for the $g > r$ condition and discusses the feasibility of financing long-term policies by exploiting the surplus generated by perpetually rolled-over debt under this condition. Figure 1 shows the cross-country scatter plot of the nominal interest rate and the nominal economic growth rate, which are averaged for the 1870-2019 period (panels a,b) and 1960-2019 period (panels c,d).[8] We use the Jordà-Schularick-Taylor Macrohistory Database (Jordà, Schularick and Taylor, 2017; Jordà, Knoll, Kuvshinov, Schularick and Taylor, 2019), which documents the macroeconomic and financial data for 18 advanced countries from 1870. Panels (a) and (c) show the relationship between the short-term interest rate and the nominal GDP growth rate.[9] For all 18 countries in the database, the nominal growth rate exceeds the short-term nominal interest rate, both for the full 1870-2019 sample and for the 1960-2019 subsample. Panels

[7] Peretto (2011) and Ferraro and Peretto (2020) also considered debt-financed fiscal policies in R&D-based growth models. Since their models have an infinitely lived representative household, the interest rate exceeds the growth rate on the balanced growth path. Hence, the debt must eventually be repaid or stabilized by raising taxes or cutting spending in the future. Our analysis complements these studies by considering the case in which taxes cannot be raised even in the future, so that the debt must be rolled over perpetually, which is possible only when $g > r$.

[8] For each country, the growth rate and the interest rate are averaged over the years in which both are available. Because of the hyperinflation in the early 1920s, the average growth rate of Germany lies above the vertical range of panels (a) and (b), and Germany is not shown in these panels. When the short-term interest rate is used, the average nominal growth rate of Germany is 20.6%, and the average interest rate is 4.0%. When the long-term interest rate is used, they are 13.4% and 5.1%, respectively. The two averages of the growth rate differ because the sets of years in which both are available are different.

[9] The short-term interest rate series in the database is a composite whose underlying instrument varies across countries and periods. It is the central-bank discount rate in earlier periods (for example, the Bank of Spain rate until 1924 and the National Bank of Belgium rate until 1964), the money market or interbank rate in the post-war decades for France, Germany, Spain, and Japan, the three-month Treasury bill rate in Canada and Ireland, and a commercial short-term lending rate in the United Kingdom and the United States. The database also reports the yield on short-term government bills for 16 of the 18 countries, all except Canada and Ireland. When we use this yield instead of the short-term interest rate, the average nominal growth rate exceeds it in all 16 countries, both for the full 1870-2019 sample and for the 1960-2019 subsample.



(a) short-term r, 1870-2019 average

(b) long-term r, 1870-2019 average

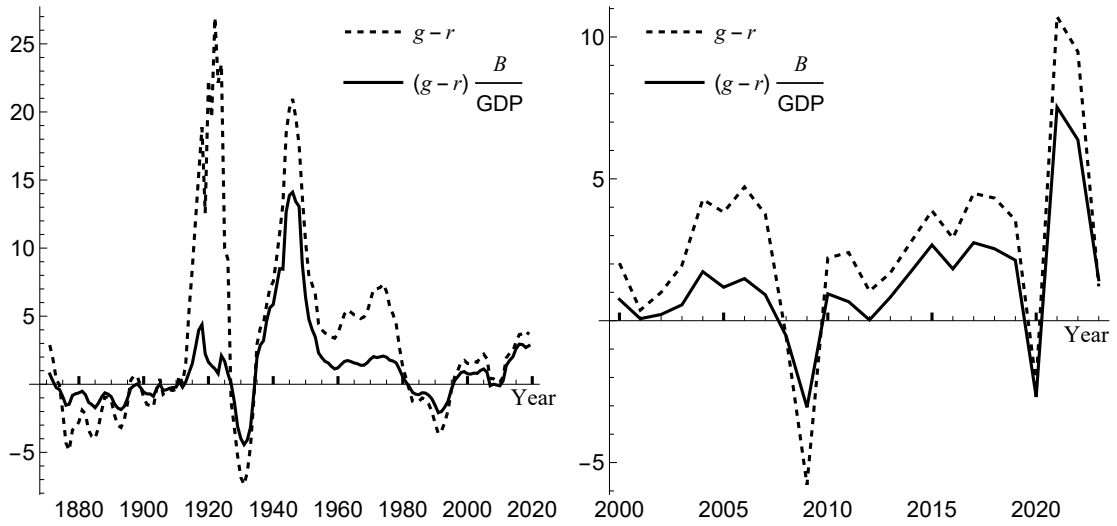
(c) short-term r, 1960-2019 average

(d) long-term r, 1960-2019 average

Figure 1: The scatter plot of the nominal interest and growth rates (%)

(b) and (d) show the results when the long-term nominal interest rate is used.[10] Even though the long-term interest rate is typically higher than the short-term rate, the nominal GDP growth rate is still higher than the long-term interest rate across all 18 countries when averaged over the entire period. When we focus only on the post-1960 period, the

[10] The post-World War II series is mostly from the IMF International Financial Statistics "Government Bonds, percent per annum" yield, extended by the OECD "Long-term interest rates". For the pre-World War II period, when domestic long-term government debt is thinly recorded, the series for several countries (Australia, Canada, France, Japan, the Netherlands, Portugal, and Sweden) is taken to be the UK consol yield plus a country-specific spread following Clemens and Williamson (2004). The maturity is ten years, with two exceptions. The United Kingdom uses the twenty-year par yield on British Government Securities from 2009 onwards, and Belgium uses the yield on a perpetual-maturity debt until 1912.



(a) 18 advanced countries' average (b) 37 OECD countries' average

Figure 2: Time series of $g - r$ gap and the ratio of the surplus from debts to the GDP (%)

difference between g and r tends to be narrower, but $g > r$ still holds for 15 countries, with Germany, Belgium, and Denmark failing to meet the condition by narrow margins of 0.14, 0.22, and 0.63 percentage points, respectively. In practice, governments finance the debt through bonds of various maturities, and they can choose maturities to meet their objectives. The fact that $g > r$ holds for the short-term interest rate suggests that, on the margin, a government can finance additional debt at $r < g$ by issuing short-term bonds if reducing interest payments is its priority.

Our finding is consistent with observations in existing studies. Mehrotra and Sergeyev (2021) reported that in the 1946–2006 period, the median value of $g - r$ was 1.0% for the United States and 0.8% for the average of 17 advanced countries. Blanchard (2019) noted that $g > r$ has been the historical norm rather than the exception in the U.S. since 1950. Mauro and Zhou (2021) analyzed data on average effective borrowing costs for 55 countries over a span of up to 200 years and reported that $g > r$ prevails for both advanced and emerging economies.

The broken line in Figure 2 illustrates the time series of the cross-country average $g - r$ gap across 18 advanced countries in the Macrohistory data (panel a), and across 37 OECD countries (panel b).[11] Although $g - r$ was negative before World War I, during the Great Depression (and briefly during the Great Recession), and in the 1980s, it was positive the

[11] In both panels, we use short-term interest rates for r and GDP growth at the aggregate level for g. In panel (a), GDP and the short-term interest rate are taken from the Jordà-Schularick-Taylor Macrohistory Database (Jordà et al., 2017, 2019), covering the 18 advanced economies listed in Figure 1. The cross-country average for each year uses observations available in that year, and the plotted line is a five-year centered moving average of the cross-country average. In panel (b), GDP and the short-term interest rate are obtained from the OECD Data Explorer.

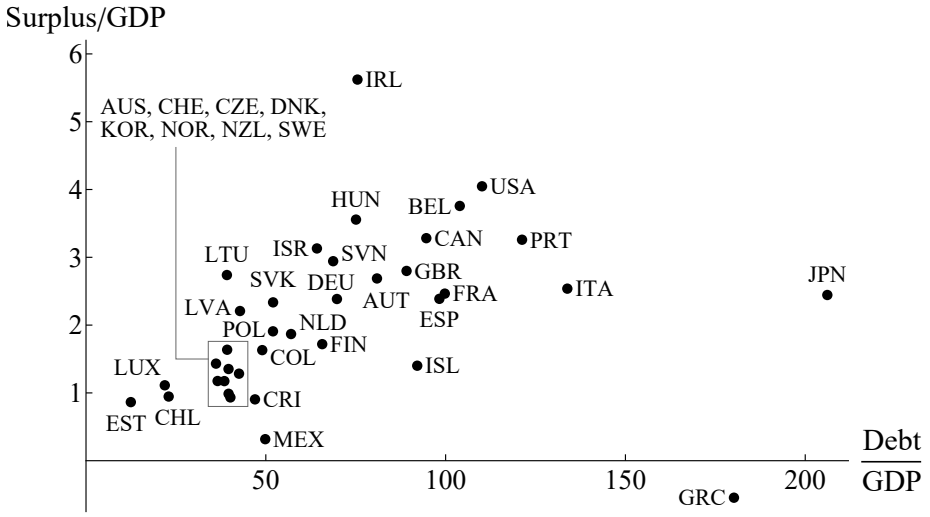


Figure 3: Debt/GDP ratio and surplus from government debt (%, 2010-2023 average)

majority of the time. When $g - r$ is positive, the debt-GDP ratio, B/GDP, shrinks by $(g - r)(B/\text{GDP})$ if the primary balance (the government revenue minus the government expenditure excluding interest payments on its debts) is zero. In other words, if $g - r$ is positive, the government can spend $(g - r)B$ over its revenue without increasing the debt-GDP ratio. The solid line in Figure 2 shows the ratio of this surplus to GDP (i.e., $(g - r)(B/\text{GDP})$).[12] The figure shows that the surplus is significant. On average, the OECD countries during 2000-2023 enjoyed 1.34% of GDP as the surplus from debts.[13]

Since the 2000s, the surplus/GDP ratio has been on the rise partly because of the increasing debt-GDP ratios among countries. The relationship between surplus and debt can be seen more precisely in Figure 3, which presents a scatter plot of B/GDP and $(g - r)(B/\text{GDP})$ from OECD countries.[14] For countries where the debt/GDP ratio is not too high (i.e., below 150%), there is a positive relationship between this ratio and the surplus/GDP ratio. This finding suggests that a moderate level of government debt can be helpful in financing long-term policies, given the government's constraint on raising taxes. Notably, however, the surplus is not necessarily large (in fact, it is negative for Greece) when the debt-GDP ratio is too high. When government debt is too high, the economy cannot maintain a positive $g - r$ gap, and the surplus vanishes. This pattern is consistent

[12]The data on the debt-GDP ratio in panel (a) are from the Jordà-Schularick-Taylor Macrohistory Database (Jordà et al., 2017, 2019). Those in panel (b) are from the IMF World Economic Outlook (General Government Gross Debt, percent of GDP).

[13]Excluding the Great Recession period (2008-2009) and the COVID-19 trough (2020), the average was 1.83% of GDP.

[14]The data sources are the same as in Figure 2 (panel b). Turkey is omitted because the OECD Data Explorer does not provide its short-term interest rate series, plausibly because the country has experienced periods of extreme inflation.

with our theory. While the condition $g > r$ enables perpetual debt financing to a certain extent, too much exploitation of the condition eventually eliminates the equilibrium with $g > r$.

3 Baseline Model

3.1 Individuals

We consider a continuous-time overlapping generations model based on the perpetual youth framework of Blanchard (1985), extended to incorporate exogenous population growth following Buiter (1988). Let N_t be the population at time t. New individuals are born at the constant rate $\beta > 0$, so that βN_t individuals are born per unit of time, and each individual faces a constant Poisson death rate $\mu > 0$. The size of the cohort born at time s and still alive at time t ($\geq s$) is $N_{s,t} = \beta N_s e^{-\mu(t-s)}$. Integrating this over all surviving cohorts gives $N_t = \int_{-\infty}^{t} \beta N_s e^{-\mu(t-s)} ds$, and differentiating it with respect to t yields $\dot{N}_t = (\beta - \mu) N_t$. The population thus grows at the constant rate $\beta - \mu$. Since β is constant, $N_s = N_t e^{-(\beta-\mu)(t-s)}$, and the cohort size can be written as

$$N_{s,t} = \beta N_t e^{-\beta(t-s)}. \tag{1}$$

Each individual supplies inelastically $e^{-\delta(t-s)}$ units of effective labor, where $t - s$ is their age.[15] The effective labor supply decreases with age at the rate of $\delta > 0$ because of the deterioration in productivity and the declining ability to work longer hours as individuals age.[16] Because of this aging effect, individuals have an incentive to make life-cycle savings. In this sense, the parameter δ represents the strength of individuals' saving motives. Then, from (1), the aggregate labor supply is

$$L_t = \int_{-\infty}^{t} N_{s,t} e^{-\delta(t-s)} ds = \frac{\beta}{\beta + \delta} N_t. \tag{2}$$

The expected utility of a generation s individual, accounting for their mortality, is given by

$$U_s = \int_{s}^{\infty} (\ln c_{s,t}) e^{-(\rho+\mu)(t-s)} dt, \tag{3}$$

[15] More precisely, $t - s$ is the period after each individual starts their economic activity. If the starting age is 20, the actual age of an individual is $20 + (t - s)$.

[16] Empirically, age-earnings profiles are inverted-U shaped rather than monotonically declining (Mincer, 1974; Murphy and Welch, 1990). We adopt the exponential specification because it keeps the aggregation across generations tractable. In this sense, δ is best viewed as a reduced-form proxy for the strength of the life-cycle saving motive rather than as a literal description of the age profile of productivity.

where $c_{s,t}$ is the amount of consumption by a generation-s individual at time t and where $\rho > 0$ is the discount rate. Observe that individuals further discount the future by their survival probability, $e^{-\mu(t-s)}$. We assume that the discount rate is not too high. In particular, we assume that the discounting is smaller than the sum of the aging effect and population growth ($\rho < \delta + \beta - \mu$) so that individuals have enough incentives to save for their later age (see Rachel and Summers, 2019).

Let $k_{s,t}$ be the real asset holding of a generation-s individual and r_t be the real interest rate on bonds. Following Blanchard (1985), we assume that there is a perfect market for annuities. Thus, the rate of return on annuities is $r_t + \mu$ for all generations, independent of when they are born. From $r_t + \mu > r_t$, individuals hold all their assets in the form of annuities. We also normalize the price of the final goods to one. Then, the budget constraint is given by

$$\dot{k}_{s,t} = (r_t + \mu)\, k_{s,t} + e^{-\delta(t-s)} w_t - c_{s,t}, \tag{4}$$

where w_t is the real wage per unit of effective labor. The newborn generation has zero financial assets, which means that $k_{t,t} = 0$. To summarize, each individual maximizes the expected utility (3) subject to the budget constraint (4), initial condition $k_{t,t} = 0$, and the usual non-Ponzi game condition. In this standard setting, the Euler equation for individuals is $\dot{c}_{s,t} = (r_t - \rho)c_{s,t}$.

Let us define aggregate consumption, C_t, and aggregate asset holding, K_t, as

$$C_t = \int_{-\infty}^{t} c_{s,t} N_{s,t} ds, \quad K_t = \int_{-\infty}^{t} k_{s,t} N_{s,t} ds. \tag{5}$$

In the following, we derive the dynamics for aggregate consumption in terms of C_t and K_t. Using the Leibniz integral rule, we can differentiate C_t in (5) as follows:

$$\dot{C}_t = \int_{-\infty}^{t} \dot{c}_{s,t} N_{s,t} ds + \int_{-\infty}^{t} c_{s,t} \dot{N}_{s,t} ds + N_{t,t} c_{t,t}. \tag{6}$$

On the right-hand side (RHS), the first term represents the sum of the changes in consumption by existing individuals. Using the Euler equation of individuals, $\dot{c}_{s,t} = (r_t - \rho)c_{s,t}$, this term can be written as $(r_t - \rho)C_t$. In the second term, the change in the cohort size $\dot{N}_{s,t}$ can be written $-\mu N_{s,t}$ from (1). Therefore, this term becomes $-\mu C_t$. This value represents the decline in aggregate consumption due to the death of a fraction μ of individuals per unit time. The last term is the sum of the consumption by newly born individuals. In

Appendix A.1, we show that the consumption of a newborn is

$$c_{t,t} = \frac{\beta+\delta}{\beta}\big(\bar{c}_t - (\rho+\mu)\overline{k}_t\big), \tag{7}$$

where $\bar{c}_t = C_t/N_t$ is average consumption and $\overline{k}_t = K_t/N_t$ is the average asset holding. Equation (7) can be interpreted as follows. Recall that newborns do not have financial assets ($k_{t,t} = 0$). Given the log utility, the consumption propensity out of assets is $\rho+\mu$. Therefore, the lack of financial assets induces a newborn to consume $(\rho+\mu)\overline{k}_t$ less than the average consumption, $\bar{c}_t$. However, newborns have a greater ability to work than average individuals do. A newborn's working ability (labor supply) is 1, whereas that of an average individual is $L_t/N_t = \beta/(\beta+\delta)$. Therefore, the RHS of (7) is multiplied by $(\beta+\delta)/\beta$. By substituting (7) into (6) and using $N_{t,t} = \beta N_t$, we obtain the Euler equation for aggregate consumption.

$$\dot{C}_t = (r_t - \rho + \beta - \mu + \delta)C_t - (\rho+\mu)(\delta+\beta)K_t. \tag{8}$$

The aggregate Euler equation differs from that for individuals in three respects. First, it depends on the population growth rate, $\beta-\mu$, because the size of the economy grows with population, and so does the aggregate consumption. The remaining two effects concern generational turnover. When older individuals are replaced by younger ones, does it increase or decrease the aggregate consumption? The answer depends on whether younger individuals consume more (or less) than older individuals. There are two opposing effects. Since the labor productivity of individuals declines at rate δ with age, older individuals, who are replaced by younger ones, tend to have lower income. However, those older individuals hold a portion of the aggregate asset K_t, while the new generation is born with zero assets. In sum, generational turnover replaces existing agents with new ones who have higher incomes but less financial wealth. The income effect positively affects $\dot{C}_t$ by δC_t, while the wealth effect reduces $\dot{C}_t$ by $(\rho+\mu)(\delta+\beta)K_t$.

In the baseline model developed in Sections 3–5, we focus on the case of a stationary population by setting $\beta = \mu$. The population N_t and the aggregate labor supply L_t are then constant over time, and we denote them simply by N and L. The case of a growing or shrinking population ($\beta \neq \mu$) is examined in Section 6, where we study a second-generation R&D-based growth model.

3.2 Supply Side

The supply side of this economy is purposefully close to that of the standard variety-expansion model of Grossman and Helpman (1991a). It consists of three sectors, namely,

the final goods sector, the intermediate goods sector, and the R&D sector. In the final goods sector, a representative firm competitively produces final goods X_t from a continuum of varieties of intermediate goods $x_t(i)$, where i is the index of intermediate goods. Each variety is produced in a different industry. The production function is given by

$$X_t = \left[\int_0^{n_t} x_t(i)^\alpha di \right]^{1/\alpha}, \tag{9}$$

where n_t is the number of intermediate goods available at time t and where $\alpha \in (1/2, 1)$ is a production parameter.[17] Let $p_t(i)$ be the price of intermediate good i. The representative final goods firm maximizes its profit, which is given by

$$X_t - \int_0^{n_t} p_t(i) x_t(i) di. \tag{10}$$

The first-order condition for profit maximization implies that the demand function of intermediate goods is

$$x_t(i) = p_t(i)^{-\frac{1}{1-\alpha}} X_t. \tag{11}$$

In the intermediate goods sector, there are n_t intermediate goods firms, each of which produces its own variety of goods, $x_t(i)$. The production of one unit of $x_t(i)$ requires one unit of labor; therefore, its profit is given by $\pi_t(i) = (p_t(i) - w_t)x_t(i)$. Given that $x_t(i)$ is determined by the demand function (11), profit-maximizing pricing implies

$$p_t(i) = \frac{w_t}{\alpha}, \quad x_t(i) = \left(\frac{\alpha}{w_t} \right)^{\frac{1}{1-\alpha}} X_t, \tag{12}$$

$$\pi_t(i) = (1-\alpha) \left(\frac{\alpha}{w_t} \right)^{\frac{\alpha}{1-\alpha}} X_t. \tag{13}$$

The above result shows that all intermediate goods firms produce the same amount of output. Therefore, the output of each intermediate goods firm can be written as $x_t(i) = L_t^P / n_t$ for all i, where L_t^P is the total amount of labor for the production of goods. By substituting this output into the final goods production function (9), we obtain

$$X_t = n_t^{\frac{1-\alpha}{\alpha}} L_t^P. \tag{14}$$

The R&D sector has a representative R&D firm, which competitively creates new goods

[17] As we show in (12), α represents the inverse of the markup rate. $\alpha \in (1/2, 1)$ means the rate is between 1 and 2.

using labor for R&D, L_t^R. The number of goods increases as follows:[18]

$$\frac{\dot{n}_t}{n_t} = A\frac{L_t^R}{L}, \tag{15}$$

where $A > 0$ is a parameter that measures the productivity of R&D. When 1% of the labor force is used for R&D, the number of intermediate goods increases A% per year.[19] While we treat A as an exogenous parameter in the baseline model, we endogenize it in Section 7.1.

Equation (15) implies that the creation of a new intermediate good requires L/An_t units of labor. We assume that the government subsidizes a fraction $\theta \in [0, 1)$ of the R&D cost. Then, the private cost of developing a new intermediate good is $(1-\theta)w_tL/An_t$. Let v_t be the value of an intermediate goods firm. Then, the free entry condition for R&D is

$$v_t \leq (1-\theta)\frac{w_tL}{An_t} \text{ with equality if } \dot{n}_t > 0. \tag{16}$$

3.3 Government

As explained above, the government subsidizes a fraction θ of the cost of R&D. Since the presubsidy aggregate cost of R&D is $w_tL_t^R$, the amount of government expenditure is $\theta w_tL_t^R$. We assume that the government cannot collect taxes and that all expenditures are financed by government debt. Then, the amount of government debt, B_t, evolves according to

$$\dot{B}_t = r_tB_t + \theta w_tL_t^R. \tag{17}$$

If the government holds net assets ($B_t < 0$), it may provide the subsidy without raising taxes even when $g < r$, because the returns from these assets can pay for the subsidy.[20] In practice, however, only a limited number of countries, such as oil-producing countries, hold such net assets. Therefore, in the main text, we focus on the case where the government at $t = 0$ does not hold net assets, $B_0 \geq 0$. Given this initial condition, Equation (17) implies that B_t does not become negative.

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[18]While this baseline model sticks to the standard formulation, it is possible to generalize the R&D equation (15) by adding a curvature parameter $\zeta \in (0, 1]$ and assuming $\dot{n}_t/n_t = A(L_t^R/L)^\zeta$. The case of $\zeta < 1$ applies, for example, when too much R&D results in duplication of ideas (see Horii and Iwaisako, 2007).

[19] We can also write equation (15) as $\dot{n}_t = an_tL_t^R$, where $a = A/L$. Although the latter convention is more standard, we use A rather than a because its value is easier to interpret from the data. Also, we can interpret this formulation as a special limiting case of the second-generation endogenous growth model, where population growth is introduced. See Section 6.5 for the relationship.

[20]The analysis in Appendices A.5–A.8 includes such cases.

Since there is no tax revenue, the repayment and interest of each bond must be refinanced entirely by issuing new bonds. Therefore, the government is running a Ponzi scheme.[21] The lenders expect that their bonds will be repaid and invest in them only when the economy is on an equilibrium path that continues into the infinite future. If there is no such equilibrium path, the repayment plan will fail in finite time. When lenders expect this, they will not buy bonds even before failure, which means that under this Ponzi scheme, the repayment will also fail at that earlier time. By backward induction, the government can sell bonds only when lenders expect repayment to continue forever, which is equivalent to the economy having an equilibrium path that continues into the infinite future. As examined by O'Connell and Zeldes (1988), this scheme requires an infinite flow of new generations.[22] With the overlapping-generations structure, explained in Section 3.1, we investigate the possibility that the government uses the funds from this scheme to promote economic growth.

4 Equilibrium

4.1 The $g - r$ Gap

A necessary condition to run a rational Ponzi scheme is that the growth rate is higher than the interest rate. Given the economy described in the previous section, here, we examine the gap between the growth rate and the interest rate in equilibrium. First, we derive the real interest rate.[23] The consumers hold all their assets in the form of annuities, and the annuity company invests the assets in government bonds and the shares of intermediate goods firms. Therefore, the equilibrium in the asset market is

$$K_t = B_t + n_t v_t. \tag{18}$$

Since risks are fully diversified, the expected return on holding the shares of an intermediate goods firm should be equal to the interest rate on bonds. This no-arbitrage condition can

[21] Note that this situation is in contrast to that of individuals, who maximize their lifetime utility subject to the no-Ponzi-game condition. The difference in the ability to borrow between the government and individuals reflects reality. Without collateral, individuals usually cannot borrow large amounts of money for various reasons (e.g., the risk of running away). We can rewrite the model with a borrowing constraint for individuals, which yields the same result as the present setting in the steady state.

[22] In the analysis of O'Connell and Zeldes (1988), a rational Ponzi game is feasible only when the economy is dynamically inefficient. In our model, the government can run the game even though the economy is dynamically efficient, as we show in Section 5.

[23] Note that, in the model, g and r are real variables. However, when we calculate $g - r$ from the data, as shown in Figures 2 and 3, we use the nominal growth rate and the nominal interest rate because the result is the same.

be written as

$$r_t = \frac{\pi_t + \dot{v}_t}{v_t}, \tag{19}$$

where π_t is given by (13).[24] In the main text, we focus on the case where the amount of R&D is positive, leaving the discussion of the case of $\dot{n}_t = 0$ for the Appendix A.5. Then, v_t is given by (16) with equality. Both π_t and v_t depend on the wage level, which we now derive. Since the final goods sector is competitive, the maximized profit of the final goods firm in Equation (10) should be zero. We can eliminate $x_t(i)$ and $p_t(i)$ from the amount of profit (10) using (11) and (12). Then, the zero profit condition determines the market wage as

$$w_t = \alpha n_t^{\frac{1-\alpha}{\alpha}}. \tag{20}$$

Eliminating π_t and v_t from (19) yields the real interest rate in equilibrium.[25]

$$r_t = \frac{A}{\alpha}\left(\frac{1-\alpha}{1-\theta}\frac{L_t^P}{L} - (2\alpha - 1)\frac{L_t^R}{L}\right). \tag{21}$$

Next, we consider the growth rate. The GDP of this economy is defined as the sum of consumption expenditures C_t, private investment expenditures for R&D, $(1-\theta)w_t L_t^R$, and government expenditures, $\theta w_t L_t^R$. Note that the final output is used only for consumption; therefore, the equilibrium in the goods market means that $C_t = X_t$. Then, using Equations (14) and (20), the GDP can be written as

$$\text{GDP}_t = n_t^{\frac{1-\alpha}{\alpha}}(L_t^P + \alpha L_t^R). \tag{22}$$

Note that the labor market equilibrium requires the sum of the two types of labor inputs to coincide with the labor supply L:

$$L_t^P + L_t^R = L. \tag{23}$$

From $\alpha < 1$, the GDP is greater when more labor is used for production, given the value of n_t. The reason is that there is a positive markup in the intermediate goods sector, whereas there is no markup in the R&D sector.

Let us also define full-production GDP, Y_t, as the level of GDP that would obtain if all

[24] Equation (19) implicitly assumes that the value of the intermediate firm is differentiable with respect to time. This assumption holds as long as the underlying parameters are constant. In Section 7.2, we extend the analysis to the case where the subsidy rate has to change when the government debt reaches a critical value, at which point v_t jumps and therefore Equation (19) cannot be defined.

[25] The derivation is given in Appendix A.2.

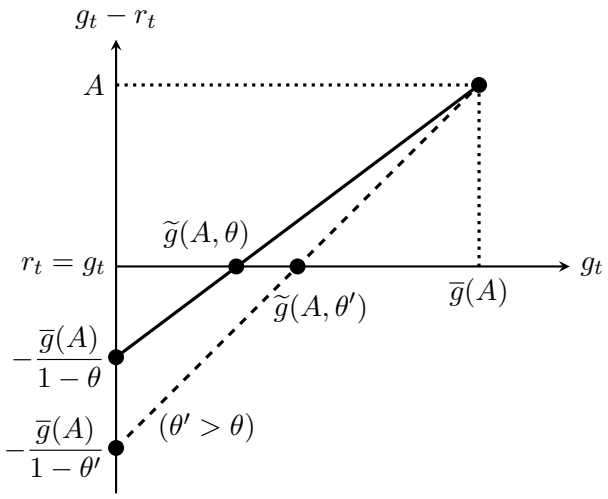


Figure 4: Relationship between the $g_t - r_t$ gap and the growth rate.

labor were used for production, given the number of intermediate goods developed by that time, n_t. With $L_t^P = L$ and $L_t^R = 0$, Equation (22) reduces to

$$Y_t = n_t^{\frac{1-\alpha}{\alpha}} L. \tag{24}$$

Since Y_t is proportional to $n_t^{\frac{1-\alpha}{\alpha}}$, it serves as an index of the technology available at time t, and it gives the upper bound for GDP_t under that technology. Note that this bound is attained only when all labor is allocated to production, which means that there is no economic growth. We use Y_t for normalization in the dynamic analysis below because it depends only on the predetermined variable n_t. From (15), the growth rate of full-production GDP in (24) is given by

$$g_t \equiv \frac{\dot{Y}_t}{Y_t} = \overline{g}(A)\frac{L_t^R}{L}, \text{ where} \tag{25}$$

$$\overline{g}(A) \equiv \frac{1-\alpha}{\alpha} A \tag{26}$$

is the maximum growth rate of full-production GDP when all labor is used for R&D. Note that on any balanced growth path (BGP), g_t is constant. Then, Equation (25) means that L_t^R is constant; the same holds for $L_t^P = L - L_t^R$. In this case, from (22) and (24), the growth rate of GDP_t coincides with $\dot{Y}_t/Y_t$. Therefore, g_t also represents the growth rate of GDP in the long run. Below, we simply call g_t the growth rate unless otherwise noted.

Now, we are ready to derive the gap between the growth rate and the interest rate. By eliminating L_t^P and L_t^R from (21) via (25) and (23), we can represent the $g - r$ gap in

terms of g_t:

$$g_t - r_t = s(\theta)(g_t - \widetilde{g}(A,\theta)), \tag{27}$$
$$\text{where } s(\theta) = \frac{1-\alpha\theta}{(1-\alpha)(1-\theta)} > 1, \quad \widetilde{g}(A,\theta) = \frac{1-\alpha}{1-\alpha\theta}\overline{g}(A) \in (0, \overline{g}(A)].$$

As shown in Figure 4, given the parameters, the $g_t - r_t$ gap is positively and linearly related to g_t, with a slope of $s(\theta) > 1$. In particular, condition $g_t > r_t$ holds when g_t is greater than the threshold at $\widetilde{g}(A,\theta)$. Therefore, when the growth rate is endogenized, we find that the possibility of running the government's Ponzi scheme depends on whether the growth rate is higher than the $\widetilde{g}(A,\theta)$ threshold.

Note that the threshold depends on the country's research productivity A and the subsidy rate θ. When the rate of R&D subsidies, θ, is increased, the thick line in Figure 4 rotates counterclockwise, which lowers $g_t - r_t$ for a given growth rate (as shown by the dashed line). Accordingly, the threshold $\widetilde{g}(A,\theta)$ in (27) increases with θ. While research subsidies may raise the growth rate, the threshold rises as well, so g_t must be even higher to keep $g_t > r_t$. The remaining question is how g_t is determined. To answer this, we need to consider the equilibrium dynamics, including the demand side of the economy.

4.2 Equilibrium Dynamics

We now derive the dynamics of the growth rate using the result from the previous subsection and the demand side of the economy, as explained in Section 3.1. In this model, final goods are used only for consumption. Therefore, the equilibrium of the goods market implies $X_t = C_t$. Combining the aggregate Euler equation (8) with the market-clearing conditions and (27), we obtain[26]

$$\dot{g}_t = (\overline{g}(A) - g_t)\big(s(\theta)(g_t - \widetilde{g}(A,\theta)) - \delta + \rho\big) + \overline{g}(A)(\rho+\mu)(\delta+\mu)\left(D_t + \frac{\alpha(1-\theta)}{A}\right), \tag{28}$$

where $D_t \equiv B_t/Y_t$ denotes the ratio of government debt to full-production GDP. We call it the debt-GDP ratio for brevity. Equation (28) shows that the dynamics of g_t depend on the debt-GDP ratio because the asset holdings of individuals affect the aggregate Euler equation as discussed at the end of Section 3.1.

Next, we turn to the dynamics of D_t. From (17), (20), (25) and (27), the time derivative of D_t is given by

$$\dot{D}_t = -s(\theta)(g_t - \widetilde{g}(A,\theta))D_t + \frac{\theta\alpha}{\overline{g}(A)}g_t. \tag{29}$$

[26]Appendix A.2 provides the formal derivation of (28) and (29) and the interpretation of each term.

The first term $-s(\theta)(g_t-\widetilde{g}(A,\theta))D_t$ equals $(r_t-g_t)D_t$, the rate at which the debt-GDP ratio would change if the primary balance were zero. The second term is the ratio of government spending on subsidies to full-production GDP. This term accelerates the increase in D_t.

Equations (28) and (29) constitute a dynamic system in the two variables g_t and D_t. Since Y_t depends only on the predetermined variable n_t, D_t is a state variable, and its initial value $D_0 = B_0/Y_0$ serves as the initial condition of the equilibrium dynamics.[27] Below, we construct a phase diagram for this system. Setting $\dot{g}_t = 0$ in (28) and $\dot{D}_t = 0$ in (29) gives the two loci

$$\dot{g}_t = 0: \quad D_t = \frac{\overline{g}(A) - g_t}{\overline{g}(A)(\rho+\mu)(\delta+\mu)}\Big(\delta - \rho - s(\theta)\big(g_t - \widetilde{g}(A,\theta)\big)\Big) - \frac{\alpha(1-\theta)}{A}, \tag{30}$$

$$\dot{D}_t = 0: \quad D_t = \frac{\alpha\theta}{\overline{g}(A)s(\theta)}\frac{g_t}{g_t - \widetilde{g}(A,\theta)}. \tag{31}$$

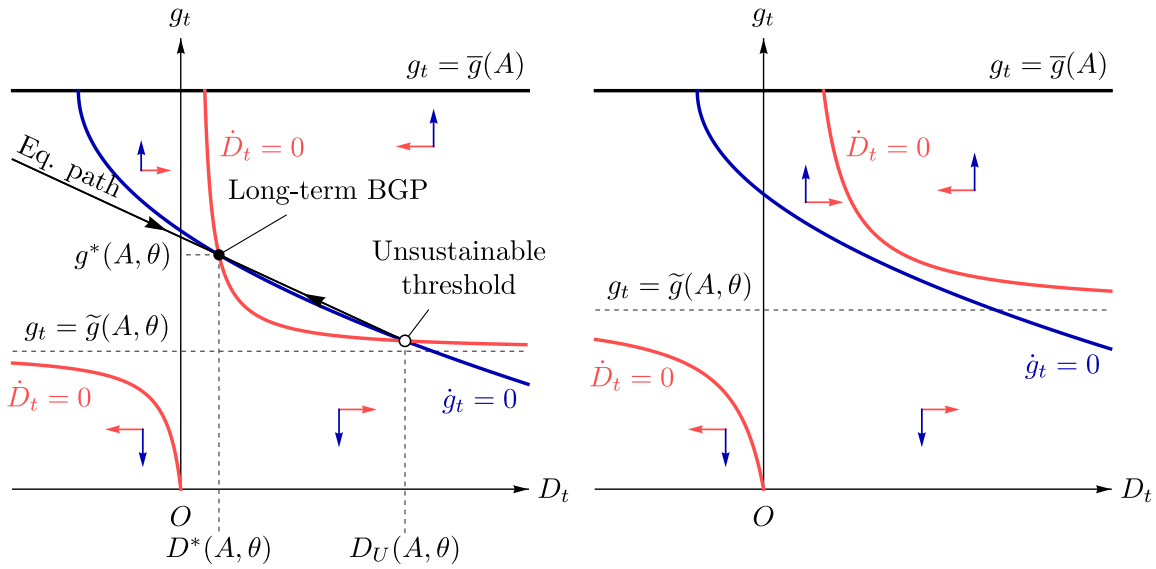
A pair of g_t and D_t represents a steady state when they satisfy (30) and (31) at the same time. Appendix A.4 explains how steady states of this economy can be calculated. Note that the equilibrium dynamics represented by (28) and (29) assume that the free entry condition for R&D, (16), holds with equality, which is true whenever $g_t > 0$. When no firm enters the market, which means $g_t = 0$, the dynamics of the economy are represented by another set of equations and steady states are defined accordingly. We examine such a case in Appendix A.5, while focusing on the case of positive R&D in the main text.

Another presumption behind the dynamics is that the government can sell bonds to households. As discussed in Section 3.3, this is possible only when households think the economy is on an equilibrium path that continues forever. This is true if and only if the dynamic system of Equations (28) and (29) (and the corresponding equations for the case of $g_t = 0$) with the initial debt-GDP ratio D_0 has a feasible solution that continues for all $t \geq 0$. Ignoring the unrealistic situation where the government holds net assets that are far larger than its GDP,[28] the following proposition shows that the existence of such an equilibrium path depends on whether the economy has steady states and whether the initial debt-GDP ratio is below a threshold.

Proposition 1 (Existence, Nonexistence, and Uniqueness of Equilibrium) *Suppose*

[27]The denominator of $D_t \equiv B_t/Y_t$ is full-production GDP rather than actual GDP. While calling D_t the debt-GDP ratio for brevity is a slight abuse of wording, this analytical choice makes the phase-diagram analysis both intuitive and tractable. Since GDP_t depends on the jump variables L_t^P and L_t^R, it is not possible to use B_t/GDP_t as an initial condition. Although it becomes more mathematically demanding, it is possible to describe the dynamics by g_t and B_t/GDP_t instead of g_t and D_t, as we explain in Appendix A.3.

[28]The lower bound in Proposition 1 is violated only when the government holds net assets exceeding $\min\{\alpha/(\delta-\rho), \alpha/(\rho+\mu)\}$ times full-production GDP, which is more than 30 times GDP under the parameter values in Table 1. Even the largest sovereign wealth funds amount to less than ten times GDP.



(a) When the two loci intersect, the economy has two steady states. From any $D_0 < D_U(A,\theta)$, the economy converges to the long-term BGP.

(b) When the $\dot{g}_t = 0$ locus and the $\dot{D}_t = 0$ locus do not intersect, there is no equilibrium path.

Figure 5: Possible Configurations of Phase Diagram in D_t and g_t.

that $D_0 > -\min\{\alpha/(\delta-\rho),\, \alpha/(\rho+\mu)\}$.

(a) Except for knife-edge cases (i.e., cases that disappear under an infinitesimal perturbation to the parameters), the economy has either two or zero steady states.

(b) Suppose that the economy has two steady states, and let $D^*(A,\theta) < D_U(A,\theta)$ *be their debt-GDP ratios.*

(i) If $D_0 < D_U(A,\theta)$*, there exists a unique equilibrium path, and it converges to the steady state with* $D^*(A,\theta)$*.*

(ii) If $D_0 > D_U(A,\theta)$*, there is no equilibrium path.*

(c) If the economy has no steady state, there is no equilibrium path.

Proof: See Appendices A.6 and A.7.

Throughout the paper, we disregard knife-edge cases that would occur only by chance. In addition to the cases mentioned in part (a) of the proposition, we do not consider the case of $D_0 = D_U(A,\theta)$ because D_0 is predetermined and unlikely to be exactly on the threshold.[29] We call the steady state with $D^*(A,\theta)$ the long-term BGP, and $D_U(A,\theta)$ the unsustainable threshold, because there is no equilibrium path when D_0 exceeds it. Figure 5 illustrates the two cases of Proposition 1 in the phase diagram. The $\dot{g}_t = 0$ locus, given by (30), is a parabola open to the right, and the $\dot{D}_t = 0$ locus, given by (31), is a hyperbola. Depending on the parameters, A and θ in particular, they do or do not intersect. Panel (a)

[29] If $D_0 = D_U(A,\theta)$ happens, the dynamic system implies that $D_t = D_U(A,\theta)$ holds for all t. However, this is an unstable situation in that if there is even an infinitesimal perturbation to a parameter or D_t, either the economy loses equilibrium, or it converges to the long-term BGP.

shows the case when the two loci intersect twice in the region $D_t \geq 0$, and the economy has two steady states. There is a unique equilibrium line (saddle path) that connects the two steady states. Given $D_0 < D_U(A, \theta)$, the initial value of g_0 is determined so that (D_0, g_0) is on the equilibrium line, and then the economy converges toward the long-term BGP. On the long-term BGP, the growth rate $g^*(A, \theta)$ is higher than the interest rate $r^*(A, \theta)$, because g_t is higher than the critical value $\widetilde{g}(A, \theta)$, as defined in (27). Therefore, the government can obtain a surplus from debt, with which it covers the expense of the R&D subsidy. Note that, even when the economy has the long-term BGP, it cannot converge to it if D_0 is larger than the threshold $D_U(A, \theta)$. In this case, there is no equilibrium path, which means that the government cannot sell bonds to lenders; i.e., it goes bankrupt. Therefore, even though a modest amount of debt could give a surplus to the government, it loses everything once the debt goes above the threshold. Panel (b) exhibits the case where the two loci do not intersect. In this case, there is no equilibrium path regardless of the initial value of D_t. As we show below, it happens when the government tries to implement subsidies that are so aggressive that they cannot be maintained even with $D_0 = 0$. The government cannot sell the bonds, and it will be forced to cancel such a policy.

4.3 Debt Sustainability and the Effect of Research Subsidies

Now, we are ready to answer one of the paper's main questions: When can the government roll over its debt indefinitely and use the proceeds to subsidize research? The phase diagrams presented in the previous subsection suggest that the answer depends both on the research productivity of the economy, A, and the rate at which the government subsidizes R&D, θ.

In this subsection, we first provide a division of the parameter space that answers the questions graphically, and then present formal propositions that state how such a division can be obtained. Figure 6 shows the result, which we numerically calculated using the same set of parameters as we use in Section 4.4 (parameter values are given in Table 1). The shaded area, which is below the $\overline{\theta}(A)$ curve, gives the parameter region in which the economy has a long-term BGP with $g > r$, as in Panel (a) of Figure 5. More specifically, suppose the research productivity is higher than a threshold

$$\overline{A} \equiv \frac{(\rho + \mu)(\delta + \mu)}{\delta - \rho}. \tag{32}$$

Then, the government can set a positive subsidy rate $\theta \in (0, \overline{\theta}(A))$, providing firms with research subsidies while raising no taxes, and the economy still has a long-term BGP

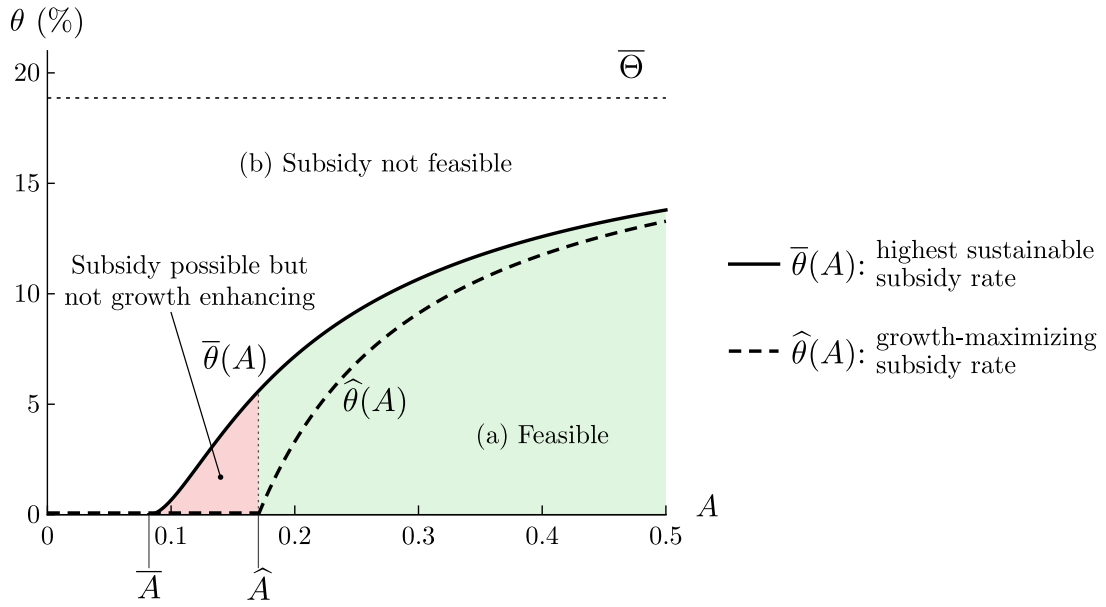


Figure 6: Sustainable and growth-maximizing subsidy rates. The economy has a long-term BGP with $g > r$ only if the pair of the research productivity (A) and the subsidy rate (θ) falls within the shaded area. Given A, the rate of subsidy that maximizes the long-term growth is given by the dashed curve.

with $g > r$. This is exactly the equilibrium in which the government rolls over its debt indefinitely and uses the proceeds to subsidize research. However, a subsidy rate above the $\overline{\theta}(A)$ curve cannot be implemented. If the government tried to set such a rate, the economy would have no long-term BGP, as in Panel (b) of Figure 5, and there would be no equilibrium path (Proposition 1). As explained in Section 3.3, lenders would then not lend to the government, and the government would be unable to obtain the funds for the subsidy. There is no $\overline{\theta}(A)$ curve when $A < \overline{A}$, which means that the economy has no long-term BGP with $g > r$ regardless of the subsidy rate. In that case, the government cannot finance the research subsidy by debt rollover.

The figure also shows the answer to the other main question: If such a policy is possible, will it increase the growth rate? The dashed curve, $\widehat{\theta}(A)$, shows the subsidy rate that maximizes the long-term growth rate, given A. Note that $\widehat{\theta}(A)$ is positive only when the research productivity exceeds a second threshold

$$\widehat{A} \equiv 2\overline{A} + \frac{\delta - \rho}{2}, \tag{33}$$

which is higher than $\overline{A}$. Only when $A > \widehat{A}$ does setting a positive research subsidy rate lead to a higher growth rate. In particular, when $A \in (\overline{A}, \widehat{A})$, providing research subsidies financed by debt rollover is feasible, but it never increases the long-term growth. In fact, setting $\theta = 0$ is the best long-term growth policy.

The following two propositions state these properties formally and give the thresholds and the two curves explicitly.

Proposition 2 (Sustainability of Debt-Financed Subsidy) *The economy has a long-term BGP with $g^*(A,\theta) > r^*(A,\theta)$ if and only if $A > \overline{A}$ and $\theta < \overline{\theta}(A)$ hold. The upper bound is*

$$\overline{\theta}(A) = \left(1 + \frac{27(1-\alpha)(\delta-\rho)A\,\overline{A}}{\Omega(A)}\right)^{-1}, \tag{34}$$

where $\Omega(A) \equiv 2\big(A^2+(\delta-\rho)(\delta-\rho-A+3\overline{A})\big)^{3/2} - 2(A+\delta-\rho)^3 + 9(\delta-\rho)(A+\delta-\rho)(A-\overline{A})$. *The upper bound is strictly increasing in A and converges to a finite $\overline{\Theta}$ as $A \to \infty$.*
Proof: See Appendix A.8.

Proposition 3 (Growth-Maximizing Subsidy) *Debt-financed research subsidies can raise the long-run growth rate if and only if $A > \widehat{A}$. When $A > \widehat{A}$, the long-run growth rate is maximized at the subsidy rate*

$$\widehat{\theta}(A) = \left(1 + 4\frac{1-\alpha}{\delta-\rho}\frac{\overline{A}}{1-\widehat{A}/A}\right)^{-1} \in (0, \overline{\theta}(A)), \tag{35}$$

which is strictly increasing in A, and converges to $\overline{\Theta}$ as $A \to \infty$. The maximized long-run growth rate is

$$\widehat{g}(A) = \overline{g}(A)\left(1 - \frac{4\overline{A}\alpha}{\delta-\rho+4\overline{A}}\right) \in \big(g^*(A,0), \overline{g}(A)\big). \tag{36}$$

Proof: See Appendix A.9.

These two propositions establish that the intuition we obtained from Figure 6 holds generally. They show that the possibility of subsidizing research by debt rollover and its effectiveness are determined by the pair of research productivity A and subsidy rate θ. While we discuss the way to estimate A in the next subsection, we can provide a useful corollary of Proposition 3 that does not require knowledge of A.

Corollary 1 (The $g-r$ Gap and the Effect of R&D Subsidy) *Suppose that the economy has a long-term BGP with $g^*(A,\theta) > r^*(A,\theta)$. Then, the gap $g^*(A,\theta) - r^*(A,\theta)$ is strictly decreasing in θ. The long-term growth can be increased by raising θ if and only if $g^*(A,\theta) - r^*(A,\theta) > (\delta-\rho)/2$.*
Proof: See Appendix A.9.

In this form, we can gauge the possibility of raising the long-term growth rate through a debt-financed research subsidy by looking at the economy's $g-r$ gap. If the gap is

Table 1: Parameters for numerical simulations

Parameter	Description	Value	Source
ρ	Discount rate	0.01	Standard
μ	Mortality rate	1/(80.3-20)	OECD (2023b)
δ	Labor deterioration rate	1/(64-20)	OECD (2023a)
α	Inverse of the markup rate	1/1.2	Ball and Mankiw (2023)
A	R&D productivity ($A > \widehat{A}$)	0.399[a]	Estimated from U.S. data
	R&D productivity ($A < \widehat{A}$)	0.120[a]	Estimated from Japanese data

[a] In the numerical simulations, we use the higher-precision values of A obtained from the estimation.

sufficiently large, the corollary suggests there is room to enhance growth by increasing θ. However, as θ gets higher, the $g - r$ gap will shrink because of the crowding-out effect of the government debt. The corollary states that it is best to stop raising the subsidy rate when the long-term $g - r$ gap reaches $(\delta - \rho)/2$, provided that the long-term growth rate is the objective. In the following subsection, we examine how these propositions and the corollary apply to data from various countries.

4.4 Numerical Simulations

In the following, we examine the effect of research subsidies via numerical simulations. Table 1 summarizes the parameters used in the simulations below. We set a standard value for the discount rate $\rho = 0.01$. The mortality rate is $\mu = 1/(80.3 - 20)$, where 80.3 is the average life expectancy in the OECD in 2021 (OECD, 2023b), and 20 is the age from which we assume that the agents start economic activities. The deterioration rate of labor productivity δ is set to $1/(64 - 20)$, where 64 is the average normal retirement age in the OECD as of 2022 (OECD, 2023a).[30] We set $\alpha = 1/1.2$ so that the markup rate of intermediate goods is $1/\alpha = 1.2$, following Ball and Mankiw (2023).[31] Given these parameters, we find that the threshold values of A are $\overline{A} = 0.0821$ and $\widehat{A} = 0.171$, and that the limit of the sustainable subsidy rate is $\overline{\Theta} = 18.9\%$.

To account for the difference in research productivity across countries, we separately estimate the value of A for each of the 24 countries that were included in the OECD by 1973. We focus on those developed countries because they can be seen as already on the BGP in the year 2000. Given that all investment in our model takes the form of R&D and

[30] Using similar data from the OECD, Rachel and Summers (2019) developed a probabilistic retirement model, whereas we consider the continuous depreciation of the working ability to simplify the analysis. The evolution of the average productivity is the same in these two specifications.

[31] In our model, α is a parameter in the production function, determining the elasticity of substitution across intermediate goods. We assume that the production function is the same across countries, and use a common value for α.

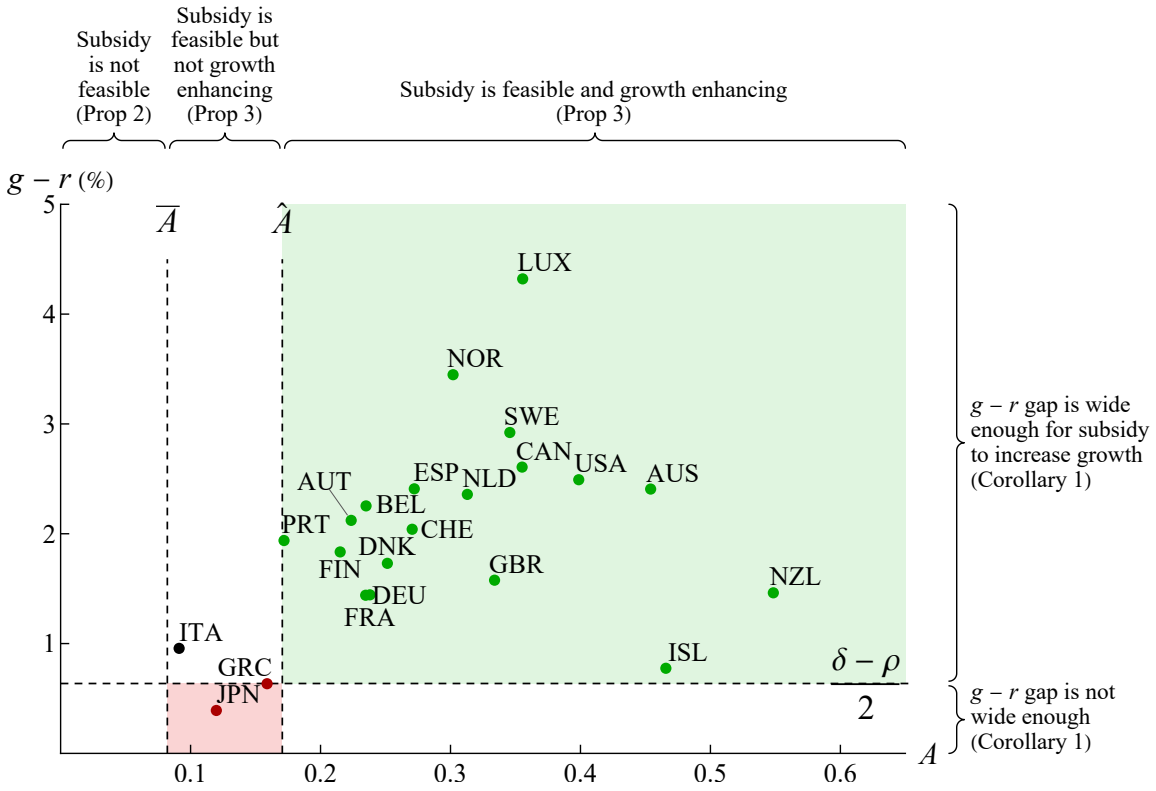


Figure 7: Estimates of research productivity A and the actual $g - r$ gaps.

that R&D uses only labor, the value of A in country i can be estimated as[32]

$$A_i = \alpha \left(1 + \frac{\alpha}{1-\alpha} \frac{1}{(wL^R/\text{GDP})_i} \right) g_i, \tag{37}$$

where we use the average real GDP growth rate and the average investment-GDP ratio of country i for the 2000–2023 period to specify g_i and $(wL^R/\text{GDP})_i$.[33] Figure 7 plots the estimates of A along with the average value of the $g - r$ gap for the same 2000–2023 period.[34] We find that all the estimates of A_i exceed the threshold value of $\overline{A} = 0.0821$. Then, Proposition 2 predicts that those economies have a long-term BGP with $g > r$. Indeed, Figure 7 confirms that the average values of $g - r$ in the 2000–2023 period are

[32]The derivation of this estimation equation is given in Appendix A.10.

[33] Both the real GDP growth rate and the investment-GDP ratio (the share of gross capital formation in GDP) are obtained from the Penn World Table version 11.0 (Feenstra, Inklaar and Timmer, 2015) and averaged over the 2000–2023 period. The $g - r$ gap is from the same data used in Figures 2 and 3. Alternatively, we could use the percentage of R&D expenditure in GDP (3.59% in the U.S. in 2022) to estimate $(wL^R/\text{GDP})_i$. This alternative method leads to much larger estimates of A_i, with which all 24 countries satisfy both $A_i > \overline{A}$ and $A_i > \widehat{A}$ by wide margins. We instead use all investment, which gives smaller estimates of A_i, to examine whether the conditions hold under a stricter criterion. The results for individual countries in this paper are obtained under this stricter criterion.

[34]Ireland ($A = 0.747$, $g - r = 6.11\%$) is omitted from the plot for readability. Turkey ($A = 0.858$) is not shown because its $g - r$ gap is not available due to a lack of short-term interest rate data.

positive for the countries shown.[35] Except for Japan, Italy, and Greece, all other countries exceed the threshold at $\widehat{A} = 0.171$. These countries also have $g - r$ gaps that are greater than $(\delta - \rho)/2$, the threshold in Corollary 1, which can be checked without estimating A.[36] Proposition 3 and Corollary 1 state that it is possible to increase growth via debt-financed research subsidies in those countries. We take the United States ($A_{\text{USA}} = 0.399$) as a representative of these countries and examine the effect of the policy in detail below. We take Japan ($A_{\text{JPN}} = 0.120$) as a representative of the countries with $A_i < \widehat{A}$.

Figure 8 numerically plots the long-term growth rate, interest rate, and debt-GDP ratio against the rate of research subsidies θ. Column (a) shows the baseline case of $A_{\text{USA}} = 0.399 > \widehat{A}$. The upper panel shows that the long-term growth rate, $g^*(A_{\text{USA}}, \theta)$, increases with θ until the subsidy rate reaches $\widehat{\theta}(A_{\text{USA}})$. It is possible to increase θ slightly more than that, but a rate above $\overline{\theta}(A_{\text{USA}})$ cannot be implemented, confirming the predictions of Propositions 2 and 3 and Proposition 1. The gap between $g^*(A_{\text{USA}}, \theta)$ and $r^*(A_{\text{USA}}, \theta)$ is wider than $(\delta - \rho)/2$ when $\theta = 0$, gradually shrinks as θ increases, and coincides with $(\delta - \rho)/2$ at the growth-maximizing subsidy rate $\widehat{\theta}(A_{\text{USA}})$, confirming Corollary 1.

For the case of countries with low research productivity, as represented by Japan (see Column (b)), the growth rate is always decreasing with the research subsidy. This is consistent with Proposition 3, which implies that the long-term growth-maximizing policy is $\theta = \widehat{\theta}(A_{\text{JPN}}) = 0$. In light of Corollary 1, we can understand this result by noting that $g^*(A_{\text{JPN}}, \theta) - r^*(A_{\text{JPN}}, \theta)$ is already lower than $(\delta - \rho)/2$ when $\theta = 0$. The $g - r$ gap is too low to justify the debt-financed subsidy, and the small gap comes from the fact that the research productivity is low in the first place.[37]

In the lower-left panel, $D^*(A_{\text{USA}}, \theta)$ shows the response of the long-term debt-GDP

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[35] The gaps in the data tend to be larger than those in the calibrated model. One reason is that our model considers only the life-cycle saving motive under a perfect annuity market. In reality, few retirees annuitize a substantial part of their wealth (Benartzi, Previtero and Thaler, 2011), and individuals save against the risk of a long life with high medical expenses (De Nardi, French and Jones, 2010). Moreover, the elderly do not use up their wealth before they die, partly because of bequest motives (Kopczuk and Lupton, 2007; Lockwood, 2018). These saving motives, which our model abstracts from, raise the demand for assets and lower the interest rate relative to the growth rate. Our theory shows that $g > r$ can hold and debt-financed policies can promote growth even without these additional saving motives, which would make the condition $g > r$ even easier to satisfy.

[36] Note that the values for $g - r$ in Figure 7 are not the values predicted by the theory, but the actual numbers from the historical data. They are measured independently of the calibration of A. Therefore, even though Proposition 3 and Corollary 1 are theoretically equivalent if the actual economy is exactly represented by the model, they could give different predictions when applied to the calibrated values of A and the data on $g - r$. Still, our results show that the two criteria give remarkably similar predictions, with the only exception of Italy, where the $g - r$ gap is larger than $(\delta - \rho)/2$ despite $A < \widehat{A}$. This exception can be explained by the saving motives discussed in footnote 35, which make the actual gap larger than the gap in the calibrated model.

[37] In Section 7.1, we examine the possibility of enhancing the research productivity in the private sector by funding public basic research.

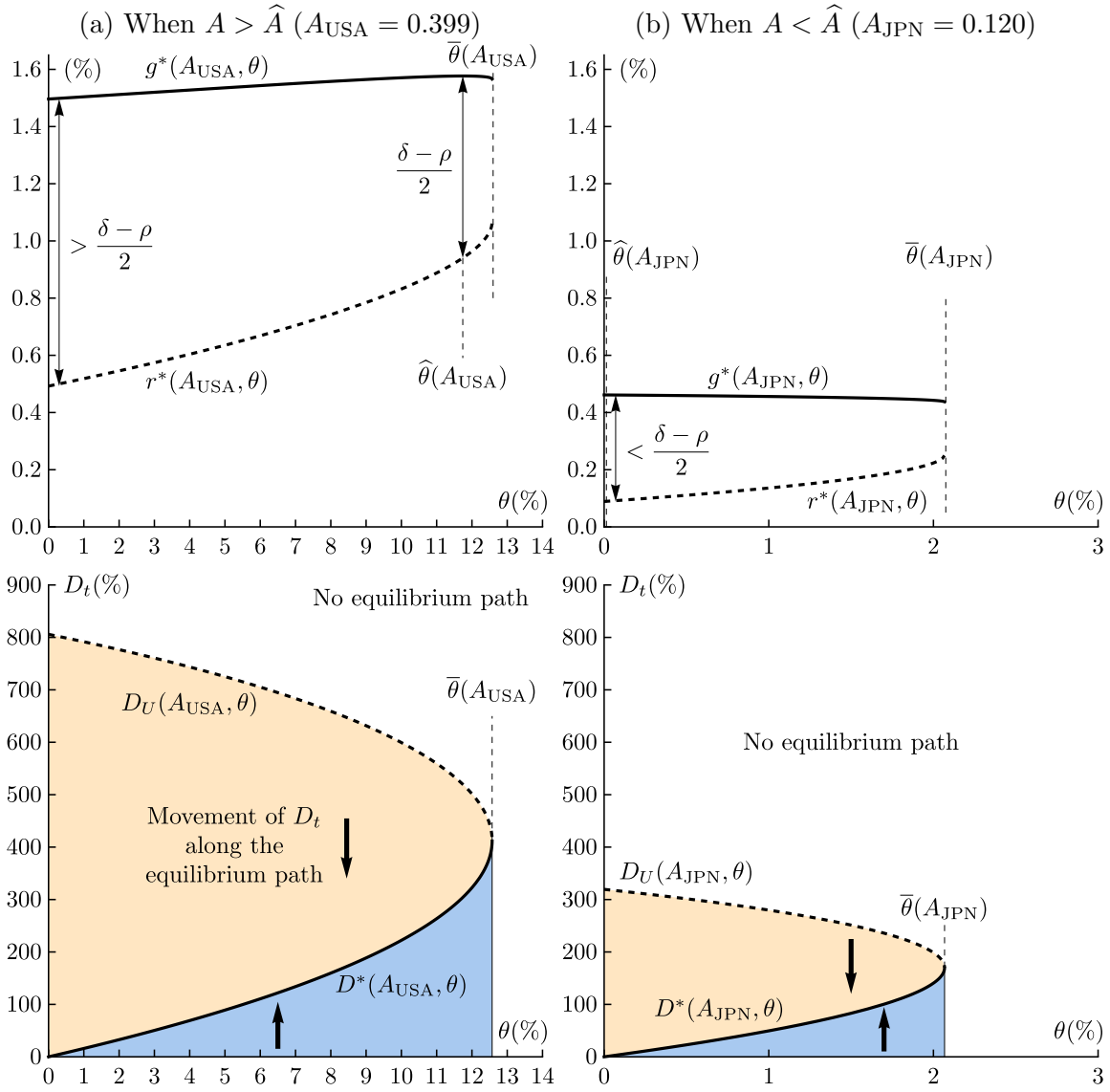


Figure 8: The long-term effect of the research subsidy rate θ on the growth rate $g^*(A, \theta)$, the interest rate $r^*(A, \theta)$, and the debt-GDP ratio $D^*(A, \theta)$ on the long-term BGP, and on the unsustainable threshold $D_U(A, \theta)$.

ratio as the subsidy rate changes.[38] The long-term debt-GDP ratio gradually increases with θ, but when the subsidy rate comes close to the upper bound, $\overline{\theta}(A_{\text{USA}})$, the slope becomes steeper, reaching infinity at $\theta = \overline{\theta}(A_{\text{USA}})$. This means that when the economy is on the verge of fiscal collapse (i.e., losing its equilibrium path), even a slight increase in government expenditure can lead to a large change in the debt-GDP ratio. In that case, the benefit of marginally increasing the subsidy to R&D will be dwarfed by the substantial increase in the interest rate, caused by the significant increase in the long-term level of

[38]Strictly speaking, the values we report in those panels are the ratio of the government debt to the full-production GDP. Conventionally, the debt-GDP ratio is defined as the ratio of the government debt to the actual GDP. Appendix A.3 shows that the conventionally defined debt-GDP ratio is higher than the values in these panels. However, the two indices yield similar values, with the difference at most 3.4%.

government debt. Therefore, setting the highest possible subsidy rate $\theta = \overline{\theta}(A_{\text{USA}})$ is not the best policy to increase growth. This is why $\widehat{\theta}(A_{\text{USA}})$ is lower than $\overline{\theta}(A_{\text{USA}})$.

The $D_U(A, \theta)$ curves in the lower panels show the unsustainable threshold. The arrows in the panels represent the direction of movement of D_t on the equilibrium path. The economy has no equilibrium path if D_t exceeds $D_U(A, \theta)$, and a subsidy rate above $\overline{\theta}(A)$ cannot be implemented (outside the shaded region in each panel). Observe that $D_U(A, \theta)$ is decreasing in θ, whereas $D^*(A, \theta)$ is increasing, and the two meet at $\theta = \overline{\theta}(A)$. A higher subsidy raises the long-term debt-GDP ratio and lowers the limit that the economy can sustain. When the two coincide, the long-term BGP disappears, and no subsidy rate above $\overline{\theta}(A)$ can be sustained. Therefore, even when a higher θ increases the growth rate, it entails two kinds of costs, namely, it increases the long-term debt-GDP ratio, $D^*(A, \theta)$, and it reduces the unsustainable threshold, $D_U(A, \theta)$.[39]

The lower-right panel shows the same for low-A countries, as exemplified by Japan. While the shape is qualitatively similar to that of the United States, it shows that the unsustainable threshold is much lower (319% for Japan and 806% for the United States when $\theta = 0$). Correspondingly, the highest sustainable subsidy rate is much lower in Japan (2.07%) than in the United States (12.57%). The United States can maintain a higher θ because its higher research productivity enables the country to sustain a larger amount of debt with a wider $g - r$ gap, which helps the government enjoy a larger surplus from the debt, $(g - r)B_t$.

4.5 Transitional Effects

Thus far, we have shown that debt-financed R&D subsidies enhance long-term growth in economies with high A (which means high $g^*(A, 0)$), but they are detrimental to long-term growth when A is low (i.e., $g^*(A, 0)$ is low). From this finding, it can be tempting to conclude that the policy will be beneficial for economies only when the growth rate before the policy is already sufficiently high. However, this conclusion may not hold for two reasons. First, the growth rate may change during the transition from the old BGP to the new long-term BGP. Second, the transition may take a very long time. If the growth rate is higher than its long-term value for a long period, the eventual difference in the GDP level can be significant. Indeed, the following proposition shows that the subsidy raises the

[39]The debt-GDP ratio could be (temporarily) affected by various kinds of shocks, although we do not explicitly consider these shocks. If the current debt-GDP ratio is close to the unsustainable threshold, $D_U(A, \theta)$, these shocks may push D_t above $D_U(A, \theta)$, which would make the government go bankrupt. Therefore, the decline in $D_U(A, \theta)$ can be viewed as a cost, as it undermines the sustainability of government debt.

growth rate of full-production GDP in the short run, regardless of A.

Proposition 4 (Short-Run Effect of the Subsidy) *From any point on an equilibrium path with positive R&D, an unanticipated permanent increase (decrease) in θ raises (reduces) g_t at that time, as long as the economy still has an equilibrium path under the new rate.*

Proof: See Appendix A.11.

Figure 9 shows the transitional response of the economy when the subsidy rate θ is permanently raised from zero to the neighborhood of the highest sustainable rate, $\overline{\theta}(A)$. At time $t = 0$, the subsidy rate is raised from 0 to 11.74% for the economy with $A_{\text{USA}} = 0.399$ (column a) and to 2% for the economy with $A_{\text{JPN}} = 0.120$ (column b).[40] The other parameters are the same as those in the previous subsection. The transitional paths of g_t and D_t are obtained in closed form, because the equilibrium path is a straight line in the (D_t, g_t) plane (see Appendix A.12). As shown in the upper panels, the growth rate of full-production GDP, g_t, increases immediately after the R&D subsidy is introduced, even when $A < \widehat{A}$, confirming Proposition 4.

The panels in the second row show the path of r_t.[41] As time passes, the growth rate gradually decreases, while the interest rate increases, shrinking the gap between g_t and r_t. It takes hundreds of years for the $g_t - r_t$ gap to converge to the long-term value. The reason for the slow convergence can be observed from the panels in the third row. The panels show that it takes a long time for the debt-GDP ratio to increase. Since g_t moves along the equilibrium path as D_t slowly changes, and r_t is determined by g_t, their moves are also sluggish.

The bottom panels show the percentage deviation of paths of full-production GDP Y_t, aggregate consumption C_t, and actual GDP, given by (22), from their respective paths without research subsidies. The formal definitions and derivations of these quantities are explained in Appendix A.12. The response of the deviation of Y_t is monotonically increasing when $A > \widehat{A}$ and is hump-shaped when $A < \widehat{A}$. Since the growth rate of Y_t is g_t, this pattern directly reflects the results shown in the upper panels. Immediately after the research subsidy is introduced, the aggregate consumption jumps down to some extent

[40] For the high-A country (calibrated to the U.S.), the subsidy rate is chosen to maximize the long-term growth rate. For the low-A country, the policy that maximizes long-term growth is $\theta = 0$, so we choose a round number that is slightly below the highest sustainable rate, $\overline{\theta}(A) = 2.068\%$.

[41] At the time of the introduction, r_t either jumps up or down. As shown in Equation (21), there are two forces that affect the return of R&D investment r_t immediately. First, a higher θ directly reduces the private cost, which increases the rate of return. However, when the subsidy increases the labor allocated to R&D, production shrinks, as well as profit. The panels show that the first effect dominates for $A = 0.399$, whereas the second effect is slightly larger when $A = 0.120$.

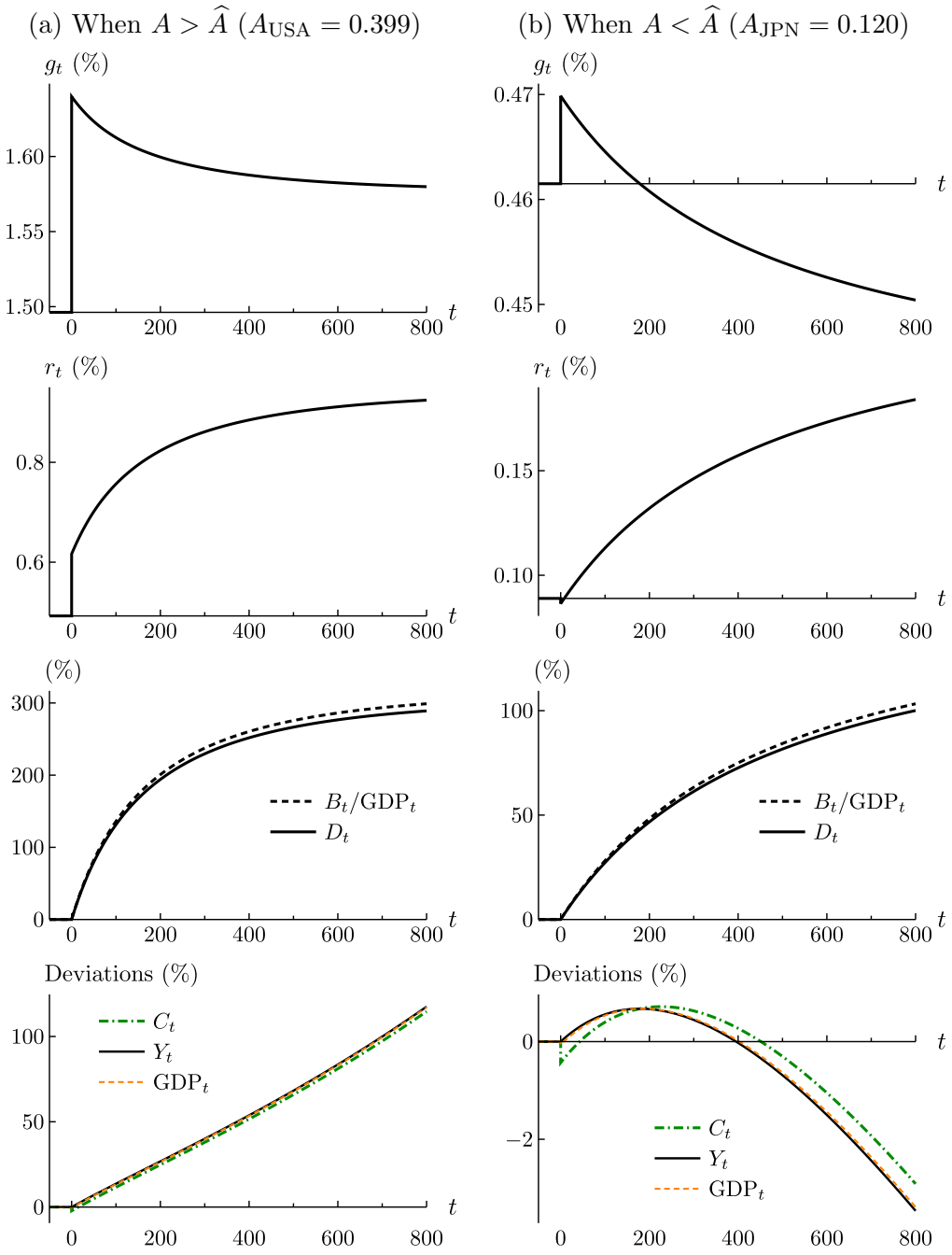


Figure 9: Transitional dynamics after research subsidies are introduced. In each panel, the horizontal axis is drawn at the level of the balanced growth path before the policy change.

because the subsidy induces more resources to be devoted to R&D than to the production of consumption goods. After about 15 years in column (a) and about 40 years in column (b), aggregate consumption overtakes the case of no subsidy, owing to increased productivity (i.e., a higher level of full-production GDP Y_t). The response of the deviation in GDP is between those of Y_t and C_t.[42]

[42]The GDP is the sum of the values of consumption and R&D. In our model, consumption goods are

Even when $A < \widehat{A}$, column (b) suggests that debt-financed R&D subsidies have sizable positive effects on the levels of GDP and consumption for several centuries. The growth rate is higher than the no-policy case for almost two centuries, and the elevated production technology keeps output higher for about four centuries. Of course, the negative effect eventually dominates the initial positive response, but it happens only centuries in the future. Given that the actual policy time horizon is far shorter, such long-term effects could be ignored in policymaking. In theory, the desirability of such a policy depends on how we discount the welfare of future generations relative to current generations.

5 Social Rate of Return of R&D and Dynamic Efficiency

In the previous sections, we have examined the conditions under which the government can roll over its debt perpetually and the conditions under which debt-financed research subsidies raise the long-term growth rate. Suppose that these conditions are met. Is such a policy desirable? Since we explicitly consider overlapping generations of consumers, there is no single welfare measure on which all generations agree. As shown in Column (b) of Figure 9, a policy can be beneficial for the current generation but detrimental to distant future generations.

We therefore consider a more modest question. Given that research subsidies increase R&D investments, we ask whether the increased investments are wasteful. In the existing literature, this question is treated as the question of dynamic efficiency. Previous studies have shown that the economy is dynamically inefficient if the rate of return on investment is lower than the rate of economic growth (e.g., Tirole, 1985; Abel, Mankiw, Summers and Zeckhauser, 1989). In this case, the aggregate consumption for all periods can be increased by reducing investment, so stimulating investment is actually welfare-reducing. For example, consider a standard neoclassical growth model in discrete time, and suppose that the economy is on a BGP where investment grows at the rate g and the social rate of return on investment is $r^s < g$. If investment is reduced by one percent in every period from some date onward, consumption at that date increases by the amount of the reduction. In each subsequent period, the investment saved in that period is larger than the output lost from the earlier reduction, and consumption increases as well.[43] Reducing investment

produced with a positive markup, whereas R&D is subject to free entry without markups. Therefore, when more resources are devoted to R&D, the GDP slightly decreases because the total value of markups decreases.

[43]The output lost in each period, including the undepreciated capital, is $1 + r^s$ times the reduction of investment in the previous period, whereas the investment saved in the same period is $1 + g$ times as large. Hence consumption increases by $(g - r^s)/(1 + g)$ percent of that period's investment.

is therefore Pareto improving when $r^s < g$. This possibility is a particular concern in the literature on rolling over debts because most papers focus on the case of $r < g$.

In this section, we show that when the research subsidy raises the long-term growth rate, the R&D investments promoted by the subsidy are not wasted, in the sense that they allow the economy to enjoy higher consumption at some future date. This claim is closely related to dynamic efficiency. Formally, we prove that on any BGP of our model, if today's R&D is reduced to increase today's consumption, for example by cutting the research subsidy, then aggregate consumption must fall at some future date. At that date, the consumption of at least one generation necessarily falls. Conversely, if the policy succeeds in increasing R&D, which requires the condition of Proposition 3 in the long run, consumption can be raised at some point in the future. This property corresponds to dynamic efficiency in the neoclassical growth model, in which a reduction in today's investment necessarily reduces consumption at some point in the future.

In our model, R&D investments have positive externalities on future R&D. In Equation (15), the output of R&D, $\dot{n}_t$, depends positively on today's stock of knowledge capital n_t, i.e., the cumulative number of ideas produced in the past. Similarly, today's R&D creates new knowledge, which will be used in the future. Therefore, the social rate of return on investment r^s, including this externality, is higher than the interest rate r. Thus, we cannot say that $r^s < g$ holds even when the interest rate on the government bond is lower than the growth rate. In fact, in the following, we show that the social rate of return of R&D investments in our model is higher than the rate of economic growth on any BGP. Following Jones and Williams (1998), we define the social rate of return as

Definition 1 *On the BGP, the social rate of return of R&D (r^s) is defined as follows:*

$$r^s \equiv \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\frac{dX_{t+\Delta t}}{-dX_t} - 1 \right), \tag{38}$$

where $dX_t < 0$ and $dX_{t+\Delta t} > 0$ are the changes in X_t and $X_{t+\Delta t}$, respectively, from the respective BGP values when the following operations are conducted in the model, where time is discretized with an infinitesimally small Δt.

(a) At time t, L_t^R marginally increases by a small value $dL_t^R > 0$.

(b) At time $t + \Delta t$, $L_{t+\Delta t}^R$ decreases by $dL_{t+\Delta t}^R < 0$. The magnitude of $dL_{t+\Delta t}^R$ is determined so that all variables at and after time $t + 2\Delta t$ are on the original BGP.

Note that in this definition, the final goods production X_t at time t decreases by $dX_t < 0$ because the economy allocates more resources to R&D. This result is considered an increase in the R&D investment measured in terms of final goods. At time $t+\Delta t$, the return on this

R&D investment is harvested by increasing final goods production by $dX_{t+\Delta t} > 0$ so that the reduction in R&D investment at $t + \Delta t$ exactly offsets the effect of the initial increase in R&D. The social rate of return of R&D is measured by how large the magnitude of the harvest ($dX_{t+\Delta t}$) is relative to the initial R&D cost ($-dX_t$). Since this value is the social rate of return for the duration of Δt, the rate of return for unit time is given by dividing it by Δt, as in (38). We determine the value of r^s as follows.

Proposition 5 (Social Rate of Return) *In any BGP of this model, the social rate of return of R&D, defined by Definition 1, is $r^s = \overline{g}(A)$; therefore, $r^s \geq g$ holds.*
Proof: See Appendix B.1.

Proposition 5 holds regardless of the value of θ, as long as the BGP exists. Note that $\overline{g}(A)$ is given by (26), which represents the maximum rate of economic growth when all labor is used for R&D. Therefore, the social rate of return $r^s = \overline{g}(A)$ is always higher than the rate of economic growth g. This result implies that the economy satisfies a necessary condition for dynamic efficiency ($r^s \geq g$), even though $r < g$. The following proposition establishes the essence of dynamic efficiency in a more direct way.

Proposition 6 (Dynamic Efficiency) *For any BGP in this model, it is not possible to increase the aggregate consumption for a short period while keeping the aggregate consumption outside this interval unchanged.*
Proof: See Appendix B.2.

Given the state of technology (i.e., n_t), the only way to increase aggregate consumption is by reducing R&D and allocating more resources to production. If the amount of R&D investments is reduced today, then the proof of the proposition shows that it is inevitable to reduce aggregate consumption sometime in the future. This property holds in any BGP, including the BGP with positive R&D subsidies. Therefore, a reduction in the research subsidy today (which reduces R&D investments) will lead to a reduction in consumption in the future. In other words, the R&D investments induced by the subsidy are never a waste of resources.

These results can be contrasted with the rational Ponzi games studied by O'Connell and Zeldes (1988). In their analysis, what enables the government to roll over its debt is not the length of individuals' planning horizons but the entry of an infinite number of new agents. The same structure is present in our model. The government is the only agent that lasts forever, while households are replaced by newborn cohorts in the overlapping-generations structure, and this is what makes perpetual debt rollover possible. In their

analysis, however, the rollover is feasible only when the economy is dynamically inefficient. The same condition characterizes the emergence of rational bubbles in Diamond (1965) and Tirole (1985).

Why is the rollover feasible in our model even though the economy is dynamically efficient? The reason is that the interest rate and the social rate of return of R&D are two different rates. The interest rate is kept below the growth rate by the life-cycle saving motives of individuals, and this is all that the rollover requires. Dynamic efficiency instead concerns the social rate of return, which exceeds the private return because of the intertemporal knowledge spillovers and remains above the growth rate by Proposition 5.[44] Hence the rolled-over debt itself does not improve welfare, unlike the bubbles in the studies above, and whether the debt benefits the economy depends on how the proceeds are used.

6 Effects of Population Growth

In Sections 3–5, we examined the case of the constant population by assuming that the birth rate β is the same as the mortality rate μ. This section studies how the results change when we relax this simplification. Population growth (or decline) matters for two reasons. First, a growing population expands the size of the economy and changes the rate at which a given stock of debt is diluted by GDP growth. Second, in the overlapping-generations environment, population growth changes the share of young households and the age composition of effective labor. This affects aggregate saving and the equilibrium $g - r$ gap. To examine these additional effects in a tractable manner, we present a simple second-generation endogenous growth model, which includes both horizontal expansion and vertical innovation. We then show that the baseline model is a special case of this new framework.

6.1 A Simple Second-Generation Endogenous Growth Model: Setup

The household side is identical to that in Section 3.1, except that we no longer impose $\beta = \mu$. As we have shown in that section, the rate of population growth becomes $g_L = \beta - \mu$. We assume that β and μ do not change over time, which means that g_L is also constant. As in the baseline model, the final goods are produced from the intermediate goods of different industries. The model differs from the baseline model in two respects. First,

[44] Recent studies (e.g., Blanchard, 2019; Ball and Mankiw, 2023; Barro, 2023) obtain dynamic efficiency despite $r < g$ in neoclassical settings, where the wedge between the marginal product of capital and the interest rate arises from risk premiums or markups on capital production. The wedge in our model comes instead from the knowledge spillovers of R&D.

the quality of the intermediate good of each industry is improved by R&D, which we call vertical innovation. Second, new industries are created on the basis of new ideas, which occur to individuals, and we call this process horizontal innovation.

We assume that genuinely new ideas that lead to the creation of new industries occur to individuals at a small rate of $f > 0$ per unit of effective labor and per unit of time. Since the aggregate supply of effective labor is L_t, the rate of creation of new industries is fL_t. At the same time, each industry faces a risk of becoming obsolete. Such obsolescence occurs at the rate of $\varepsilon > 0$, and we assume that $g_L + \varepsilon > 0$. Hence, the number of industries evolves according to

$$\dot{n}_t = fL_t - \varepsilon n_t. \tag{39}$$

Given that the rate of population growth is constant and that the economy has a sufficiently long history, the effective labor per industry becomes[45]

$$\frac{L_t}{n_t} = \frac{g_L + \varepsilon}{f}. \tag{40}$$

In the following, we consider the economy after this ratio has converged.

Let $x_t(j)$ and $q_t(j)$ denote the quantity and the quality of the good of industry $j \in [0, n_t]$. A representative firm competitively produces the final goods according to

$$X_t = \left[\int_0^{n_t} \big(q_t(j) x_t(j) \big)^{\alpha} dj \right]^{1/\alpha}, \tag{41}$$

where $\alpha \in (1/2, 1)$ as in the baseline model. We assume that a new industry starts with the average quality of the existing industries. Let $p_t(j)$ denote the price of the good of industry j. Then the demand for this good is $x_t(j) = q_t(j)^{\alpha/(1-\alpha)} p_t(j)^{-1/(1-\alpha)} X_t$, and its price elasticity is $1/(1-\alpha)$. Equation (41) differs from the production function (9) of the baseline model only in that each good has a quality. In the baseline model, the quality of a good cannot be improved, and R&D is conducted to create new industries.

In each industry, the good is produced by a single firm, the leader, which owns the core technology of the industry.[46] The leader hires workers, and one unit of effective labor produces one unit of the good. In addition, workers sometimes obtain ideas for new businesses in the course of their work. Since workers do not have the resources to

[45] From $\dot{L}_t/L_t = g_L$ and (39), we obtain a differential equation for n_t/L_t. Solving this differential equation yields $n_t/L_t = f/(g_L+\varepsilon) + \big(n_0/L_0 - f/(g_L+\varepsilon)\big) e^{-(g_L+\varepsilon)t}$. When t is sufficiently large, it becomes $f/(g_L+\varepsilon)$.

[46] When former leaders also exist, we follow Assumption 1 of Akcigit and Kerr (2018) on pricing. In a two-stage game, each firm pays a fee that is arbitrarily close to zero in the first stage, and the firms that paid the fee announce their prices in the second stage. Since only the leader can recover the fee, only the leader enters the second stage and sets the monopoly price.

commercialize such ideas by themselves, they sign an employment contract with the firm, under which they hand over the ideas to the firm and, in return, their wage is higher by the expected value of the ideas. For simplicity, we assume that commercializing an idea requires no additional cost.[47] Let v_t^{new} denote the value of a firm when a new business is spun off and becomes the leader of a new industry. The firm whose workers find a new idea earns this amount by selling the shares in the new firm. When the leader of industry j hires $\ell_t^P(j)$ units of effective labor, it produces $x_t(j) = \ell_t^P(j)$, and its expected profit is $p_t(j)x_t(j) + fv_t^{\text{new}}\ell_t^P(j) - w_t\ell_t^P(j)$, where w_t is the wage per unit of effective labor. Maximizing this profit with respect to $\ell_t^P(j)$ gives the price $p_t(j) = (w_t - fv_t^{\text{new}})/\alpha$, and the maximized expected profit is $\pi_t(j) = (1-\alpha)p_t(j)x_t(j)$. Since individuals receive w_t per unit of effective labor, the analysis in Section 3.1, including the aggregate Euler equation (8), holds without change.

Outsiders conduct R&D to take over the position of the leader. When an outsider uses $\ell_t^R(j)$ units of effective labor for R&D in industry j, its research yields a core technology that the leader cannot follow with a Poisson arrival rate of $a\ell_t^R(j)$. In that case, the outsider becomes the new leader of the industry, and the previous leader loses its profit. The parameter a measures how likely the research is to yield such a core technology. It depends not only on the nature of technology but also on institutions and the degree of competition. In an environment where it is very difficult to take over the position of the leader, a is low. In a market where leaders are frequently replaced because of institutions or the environment, a is high. As in the baseline model, the government subsidizes a fraction $\theta \in [0, 1)$ of the wage paid for R&D and finances the subsidy by issuing debt, so that its budget constraint is (17). For each unit of effective labor that an outsider hires for R&D in industry j, it obtains the value of the leader, $v_t(j)$, with probability a per unit time, and new ideas worth v_t^{new} with probability f. The cost of hiring the worker net of the subsidy is $(1-\theta)w_t$. Since outsiders can freely enter R&D, the following free-entry condition holds in equilibrium:

$$av_t(j) + fv_t^{\text{new}} \leq (1-\theta)w_t, \quad \text{with equality if } \ell_t^R(j) > 0. \tag{42}$$

Regardless of whether it yields a core technology, the research of outsiders raises the technology level of the industry, that is, the quality of its good, and its results are also used in other industries. This is because whether the leader can follow a new technology is not

[47] The results do not change if commercializing an idea requires additional costs. As long as these costs are lower than the expected cost of replacing the leader of an existing industry, an idea is commercialized as soon as it is obtained. In that case, the increase in the wage is smaller by the cost of commercialization.

known until the technology is disclosed. Outsiders therefore disclose all their technologies, and in most cases, the leader can easily adopt them without being replaced. As a result, the quality rises according to

$$\frac{\dot{q}_t(j)}{q_t(j)} = \omega_{\text{in}} \ell_t^R(j) + \omega_{\text{out}} \frac{L_t^R}{n_t}, \tag{43}$$

where $L_t^R \equiv \int_0^{n_t} \ell_t^R(j) dj$ is the aggregate effective labor used for R&D. The first term represents spillovers within the industry, and ω_{in} measures their strength. The second term represents spillovers from other industries, which arise because the research in each industry spills out to other industries. The parameter ω_{out} measures the strength of these spillovers.[48]

Lemma 1 (Symmetric Equilibrium) *Suppose that* $\omega_{\text{in}} \in \big(0, a(1-\alpha)/\alpha\big)$.

(a) In any equilibrium, leaders do not conduct research.

(b) If, in addition, $q_0(j)$ *is the same for all industries, then in any equilibrium,* $\ell_t^R(j)$ *and* $q_t(j)$ *are the same for all industries at every* t.

Proof: See Appendix C.1.

The first result holds because what a leader gains from its own research is smaller than what an outsider gains from the same research.[49] The second result holds because the value of the leader is lower in an industry where outsiders conduct more research. In the following, we assume that $\omega_{\text{in}} \in \big(0, a(1-\alpha)/\alpha\big)$.[50] By Lemma 1, $q_t(j)$, $\ell_t^R(j)$, and $x_t(j)$ are the same for all industries, and we omit the index j in the following. In particular, $v_t(j) = v_t^{\text{new}}$ for all j, and writing v_t for this common value, the free-entry condition (42) becomes

$$v_t \leq \frac{(1-\theta)w_t}{a+f}, \quad \text{with equality if } L_t^R > 0. \tag{44}$$

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[48] The spillovers increase with the aggregate R&D, L_t^R, but decrease with the number of industries, n_t. As emphasized by Peretto and Smulders (2002), when there are more industries, the technologies of different industries are farther apart, and it becomes more difficult for an industry to make use of the results of research in other industries.

[49] As in standard models such as Aghion and Howitt (1992) and Grossman and Helpman (1991a), leaders do not conduct research in our model. Some models, such as Peretto (1998), assume that leaders conduct research, but leaders are not replaced in those models. Our setting is complementary to them.

[50] If $\omega_{\text{in}} > a(1-\alpha)/\alpha$, the value of the leader is higher in an industry where outsiders conduct more research. Then, even if research is divided among several industries, a small perturbation that creates a difference in research makes the quality and the value of the leader higher in the industry with more research, and all outsiders move to that industry. Research is therefore concentrated in a single industry, and the economy effectively consists of a single industry. Since this is unrealistic, we exclude this case.

6.2 Equilibrium Dynamics

Let L_t^P denote the aggregate effective labor used for production. By Lemma 1, each leader produces $x_t = L_t^P/n_t$, and outsiders use $\ell_t^R = L_t^R/n_t$ units of effective labor for R&D in each industry. The labor market clears when $L_t^P + L_t^R = L_t$, the goods market clears when $C_t = X_t$, and the asset market clears when $K_t = B_t + n_t v_t$. Substituting $x_t = L_t^P/n_t$ into (41) gives

$$X_t = n_t^{\frac{1-\alpha}{\alpha}} q_t L_t^P. \tag{45}$$

The demand for each good then implies $p_t = n_t^{(1-\alpha)/\alpha} q_t$. Combining this price with $p_t = (w_t - f v_t)/\alpha$ gives

$$w_t - f v_t = \alpha n_t^{\frac{1-\alpha}{\alpha}} q_t, \tag{46}$$

and the profit of each leader is $\pi_t = (1-\alpha) p_t x_t = (1-\alpha) X_t/n_t$. In this section, we define full-production output, Y_t, as the output of the final goods when all effective labor is used for production:[51]

$$Y_t = n_t^{\frac{1-\alpha}{\alpha}} q_t L_t. \tag{47}$$

We define two combined parameters. The first is the relative propensity for vertical innovation, $\kappa \equiv a/f$. When κ is larger, replacing leaders through vertical R&D is easier relative to horizontal innovation, which requires new ideas that arise only in proportion to population. Hence, κ affects the direction of innovation. The second is the step size, defined as $\lambda \equiv (\omega_{\text{in}} + \omega_{\text{out}})/a$. Suppose that each industry uses one unit of effective labor for R&D. Then the replacement of leaders occurs with probability of a, which means the average duration between leader replacements is $1/a$. During this time, the quality increases by $\lambda = (\omega_{\text{in}} + \omega_{\text{out}})/a$ both due to within- and across-industry spillovers.[52]

By Lemma 1, the rate of quality growth in (43) is the same in all industries and equals $\lambda a L_t^R/n_t$. Substituting (40) into this rate, we obtain the rate of vertical innovation,

$$g_{q,t} \equiv \frac{\dot{q}_t}{q_t} = \overline{g_q}(\kappa, g_L) \frac{L_t^R}{L_t}, \qquad \overline{g_q}(\kappa, g_L) \equiv \lambda\kappa(g_L + \varepsilon), \tag{48}$$

where $\overline{g_q}(\kappa, g_L)$ is the rate of vertical innovation when all effective labor is used for R&D. By (40), the rate of horizontal innovation, $\dot{n}_t/n_t$, equals the population growth rate g_L.

[51] In this section, we call Y_t the full-production output rather than the full-production GDP because GDP now includes the value of new ideas, as discussed in Appendix C.9.

[52] While we use an example of $\ell_t^R = 1$, the step size does not depend on ℓ_t^R. Precisely, the quality increases by $e^\lambda - 1$, which is approximately equal to λ when λ is small. In the quality ladder model of Grossman and Helpman (1991a), the step size is the ratio of the qualities of successive leaders, which corresponds to e^λ in our model.

From (47), the growth rate of full-production output is then

$$g_t \equiv \frac{\dot{Y}_t}{Y_t} = \frac{g_L}{\alpha} + g_{q,t}. \tag{49}$$

Horizontal innovation contributes $(1-\alpha)g_L/\alpha$ to growth, and vertical innovation contributes $g_{q,t}$. Their sum is the growth rate per capita, and adding the population growth rate g_L gives g_t.

As in the baseline model, the expected return on holding the shares of a leader must equal the interest rate. A leader loses its position when an outsider obtains a new core technology, which occurs at the rate $a\ell_t^R$, or when its industry becomes obsolete, which occurs at the rate ε. The no-arbitrage condition is therefore

$$r_t = \frac{\pi_t + \dot{v}_t}{v_t} - a\ell_t^R - \varepsilon. \tag{50}$$

When vertical R&D is conducted, the free-entry condition (44) holds with equality. Using it together with (46) and (48) to eliminate π_t and v_t from (50), we can represent the $g-r$ gap in terms of $g_{q,t}$:[53]

$$\begin{aligned} &g_t - r_t = \frac{1-\alpha}{\alpha\lambda}\, s(\theta^e)\big(g_{q,t} - \widetilde{g}_q(\kappa, g_L, \theta)\big), \\ &\text{where } \widetilde{g}_q(\kappa, g_L, \theta) = \frac{1-\alpha}{1-\alpha\theta^e}\,\overline{g_q}(\kappa, g_L) - \frac{\alpha\lambda(1-\theta^e)}{1-\alpha\theta^e}\,(g_L+\varepsilon). \end{aligned} \tag{51}$$

The function $s(\cdot)$ is the same as in (27), but it is evaluated at

$$\theta^e \equiv \frac{\theta w_t}{w_t - f v_t} = \frac{(a+f)\theta}{a+\theta f}, \tag{52}$$

which we call the effective subsidy rate on vertical R&D. Of the wage w_t that an outsider pays for a unit of effective labor in R&D, the outsider recovers fv_t as the expected value of new ideas. The cost of vertical R&D is therefore $(w_t - fv_t)L_t^R$, whereas the government subsidizes a fraction θ of the whole wage $w_t L_t^R$. The effective subsidy rate is the ratio of the subsidy to this cost. The second equality in (52) follows from the free-entry condition (44) with equality. As in the baseline model, the $g-r$ gap increases linearly with $g_{q,t}$, and $g_t > r_t$ holds if and only if $g_{q,t} > \widetilde{g}_q(\kappa, g_L, \theta)$.

When vertical R&D is conducted, the government budget constraint (17), the aggregate Euler equation (8), and the market-clearing conditions give the dynamics of this system in

[53]The derivation is given in Appendix C.2.

$D_t = B_t/Y_t$ and $g_{q,t}$,[54]

$$\dot{g}_{q,t} = \big(\overline{g_q}(\kappa, g_L) - g_{q,t}\big)\Big(\frac{1-\alpha}{\alpha\lambda}\, s(\theta^e)\big(g_{q,t} - \widetilde{g_q}(\kappa, g_L, \theta)\big) - (\delta - \rho + g_L)\Big) + \overline{g_q}(\kappa, g_L)(\rho + \mu)(\delta + \beta)\Big(D_t + \frac{\alpha(1-\theta^e)}{\kappa(g_L + \varepsilon)}\Big). \tag{53}$$

$$\dot{D}_t = -\frac{1-\alpha}{\alpha\lambda}\, s(\theta^e)\big(g_{q,t} - \widetilde{g_q}(\kappa, g_L, \theta)\big) D_t + \frac{\alpha\theta^e}{\overline{g_q}(\kappa, g_L)}\, g_{q,t}. \tag{54}$$

In (53), $D_t + \alpha(1-\theta^e)/\big(\kappa(g_L + \varepsilon)\big)$ is the ratio of aggregate assets to full-production output, K_t/Y_t. In (54), the first term is the negative of the surplus from government debt, $-(g_t - r_t)D_t$, and the second term is government spending on subsidies, $\theta w_t L_t^R/Y_t$. Both are measured relative to full-production output. These equations can be interpreted in the same way as (28) and (29) in the baseline model. As in the baseline model, Proposition 1 holds for $D_0 \geq 0$ with g_t replaced by $g_{q,t}$ (Appendices C.4 and C.5). The economy therefore has an equilibrium path if and only if it has two steady states and the initial ratio of debt to full-production output is below the unsustainable threshold, and the path converges to the long-term BGP. In the following, we examine the long-term BGP.

6.3 Population Growth and the Effect of Research Subsidies

We now examine how population growth changes the answers to the two main questions of this paper: Can the government roll over its debt indefinitely and use the proceeds to subsidize research, and does such a policy raise the long-run growth rate? In the baseline model, the answers depend on the research productivity A, as shown in Figure 6. Figure 10 shows the corresponding results in the second-generation model, where the horizontal axis is the population growth rate g_L. We find that population growth affects the relationship between g and r through different mechanisms depending on the propensity for vertical innovation, $\kappa = a/f$. As the population rises faster, each existing industry receives more new workers and more R&D, raising vertical innovation. This is the main mechanism when κ is larger. However, when κ is small, vertical innovation is dormant. Still, higher population growth creates more new ideas from individuals, enhancing horizontal innovation.

The left column shows the result for the case when $\kappa > \alpha/(1-\alpha)$. We use the value of κ from the U.S. as an example for this case (see Section 6.4 for calibration of κ). When the subsidy rate is lower than $\overline{\theta}(\kappa, g_L)$ shown in the upper panel, the government can finance research subsidies by rolling over its debt. In this case, the economy has a long-term BGP

[54] In this section, $D_t = B_t/Y_t$ denotes the ratio of government debt to the full-production output. When we consider the limit of $f \to 0$, as we do in Section 6.5, the full-production output coincides with full-production GDP, and therefore the definition of D_t becomes the same as in the baseline model.

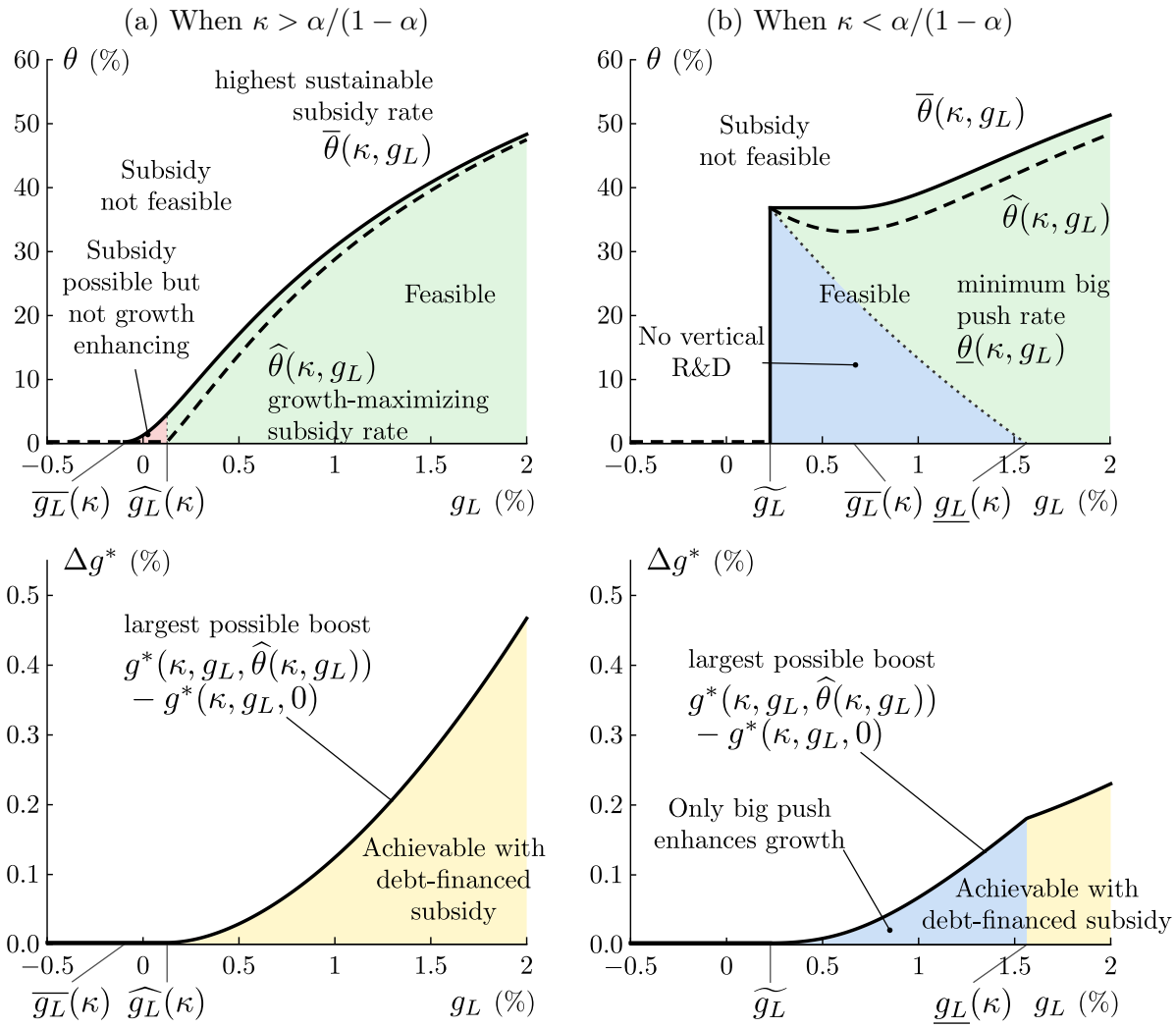


Figure 10: Sustainable and growth-maximizing subsidy rates against population growth. The parameters are those in Table 1 with $\varepsilon = 0.01$ and $\lambda = (1-\alpha)/\alpha$. Column (a) uses $\kappa_{\text{USA}} = 8.60$, and column (b) uses $\kappa_{\text{ITA}} = 2.79$.

with $g > r$, and the equilibrium path is the same as in Panel (a) of Figure 5, where g is replaced by g_q (Appendix C.5). As in the baseline model, if the government tried to set θ above $\overline{\theta}(\kappa, g_L)$, the economy would have no long-term BGP, and there would be no equilibrium path. Lenders would then not lend to the government, and such a subsidy rate cannot be implemented. The upper bound $\overline{\theta}(\kappa, g_L)$ is positive if and only if g_L is higher than the threshold $\overline{g_L}(\kappa)$, which is defined as the larger solution to the quadratic equation in g_L,[55]

$$(1+\kappa)(g_L+\varepsilon)(\delta-\rho+g_L) = (\rho+\mu)(\delta+\mu+g_L). \tag{55}$$

In the absence of subsidy ($\theta = 0$), the growth rate of aggregate output, which includes the effect of population growth, becomes higher than the interest rate, if and only if $g_L > \overline{g_L}(\kappa)$.

[55] See the proof of Proposition 7 in Appendix C.7.

The threshold $\overline{g_L}(\kappa)$ is decreasing in κ, because a higher propensity for vertical innovation helps raise the growth rate, and therefore, an economy can achieve $g > r$ with slower population growth. However, similarly to the case of the baseline model, the policy does not necessarily raise growth. When $\overline{g_L}(\kappa) < g_L < \widehat{g_L}(\kappa)$, any positive subsidy lowers the long-run growth rate. The policy raises the long-run growth rate only when g_L exceeds the second threshold $\widehat{g_L}(\kappa)$. These results are the counterparts of Propositions 2 and 3 in the baseline model, except that the answers depend on κ and g_L, rather than A. The lower panel shows that the largest possible increase in the long-run growth rate, $\Delta g^* \equiv g^*\big(\kappa, g_L, \widehat{\theta}(\kappa, g_L)\big) - g^*(\kappa, g_L, 0)$, rises with g_L.

The right column shows the result for the case when the propensity for vertical innovation is relatively low, $\kappa < \alpha/(1-\alpha)$. We use the value of κ from Italy as an example for this case. In this case, the government can finance research subsidies by rolling over its debt only when g_L is higher than the threshold $\widetilde{g_L}$, which is defined as the larger solution to the quadratic equation in g_L,

$$(g_L + \varepsilon)(\delta - \rho + g_L) = (1-\alpha)(\rho + \mu)(\delta + \mu + g_L). \tag{56}$$

Unlike $\overline{g_L}(\kappa)$, the threshold $\widetilde{g_L}$ does not depend on κ. This is because the propensity for vertical innovation is so low in these economies that they do not conduct any vertical R&D at the threshold level of g_L. In other words, the debt rollover becomes sustainable only when horizontal innovation alone can sustain $g > r$.

This result also has implications for the effectiveness of the subsidies for vertical R&D. Even when $g > r$ holds thanks to horizontal innovation, the subsidies for vertical R&D have a positive effect on growth only when they are big enough. In other words, growth responds only to a big push. Subsidies will not give those economies enough incentives to do vertical R&D when the subsidy rate is smaller than $\underline{\theta}(\kappa, g_L)$. This constraint is not binding when g_L is sufficiently large ($g_L > \underline{g_L}(\kappa)$) because in that case fast population growth gives enough incentives for vertical innovation even with smaller κ.

The following propositions formally establish the results shown in Figure 10.

Proposition 7 (Sustainability with Changing Population) *The economy has a long-term BGP with $g^* > r^*$ if and only if the following two conditions are met.*

(i) When $\kappa > \alpha/(1-\alpha)$, g_L must be higher than $\overline{g_L}(\kappa)$, which is strictly decreasing in κ. When $\kappa \leq \alpha/(1-\alpha)$, g_L must be higher than $\widetilde{g_L}$.

(ii) When $g_L > \overline{g_L}(\kappa)$, θ must be lower than $\overline{\theta}(\kappa, g_L)$, which is strictly increasing in g_L. When $\widetilde{g_L} < g_L \leq \overline{g_L}(\kappa)$, which can occur only if $\kappa < \alpha/(1-\alpha)$, θ must be lower than

$\alpha - (1-\alpha)\kappa$, *which does not depend on* g_L.

The defining equation for $\overline{\theta}(\kappa, g_L)$ *is given in Appendix C.7.*

Proof: See Appendix C.7.

Proposition 8 (Long-Run Growth Effects with Changing Population)

(i) Debt-financed research subsidies can raise the long-run growth rate if and only if

$$g_L > \begin{cases} \widehat{g_L}(\kappa) & \text{if } \kappa > \alpha/(1-\alpha), \\ \widetilde{g_L} & \text{if } \kappa \leq \alpha/(1-\alpha). \end{cases}$$

Here, $\widehat{g_L}(\kappa)$ *is strictly decreasing in* κ*, and* $\widetilde{g_L}$ *is the same threshold as in Proposition 7.*

(ii) When this condition is met, the growth-maximizing subsidy rate $\widehat{\theta}(\kappa, g_L)$ *lies in the interval* $\big(0, \overline{\theta}(\kappa, g_L)\big)$.

(iii) When $\widetilde{g_L} < g_L < \underline{g_L}(\kappa)$*, which can occur only if* $\kappa < \alpha/(1-\alpha)$*, the long-run growth rate does not increase if* $\theta \in \big(0, \underline{\theta}(\kappa, g_L)\big)$.

The defining equation for $\widehat{g_L}(\kappa)$ *is given in Appendix C.8, and those for* $\underline{g_L}(\kappa)$ *and* $\underline{\theta}(\kappa, g_L)$ *are given in Appendix C.6.*

Proof: See Appendix C.8.

Propositions 7 and 8 divide the (g_L, κ) plane into four regions, as shown in Figure 11. When g_L is low, the government cannot sustain the subsidies by rolling over its debt. For higher g_L, the subsidies are sustainable, and they raise the long-run growth rate if g_L is high enough. When $\kappa > \alpha/(1-\alpha)$, there is an intermediate range of g_L in which the subsidies are sustainable but do not raise growth. When $\kappa < \alpha/(1-\alpha)$, there is instead a range in which only a big push enhances growth.

6.4 Cross-Country Evidence

To see in which of the four regions in Figure 11 each country lies, we determine the values of κ and g_L for each of the 24 countries used in Section 4.4. For ρ, μ, δ, and α, we use the same values as in the baseline model (Table 1). We set $\varepsilon = 0.01$, which means that an industry becomes obsolete after 100 years on average.[56] We set $\lambda = (1-\alpha)/\alpha$, at which

[56]This is in line with studies of product life cycles and technology diffusion. For 25 new product markets in the United States, Agarwal and Gort (1996) find that the stages from the first commercial introduction to maturity last about 50 years in total on average. Grübler (1996) reports that the time required for a diffusion process to unfold from 10 to 90 percent of its extent is between 40 and 60 years on average, and Grübler and Nakićenović (1991) find that it is about 90 years for the substitution of primary energy sources in the United States. If we instead set $\varepsilon = 1/150$, the classification of the 24 countries in Figure 11 does not

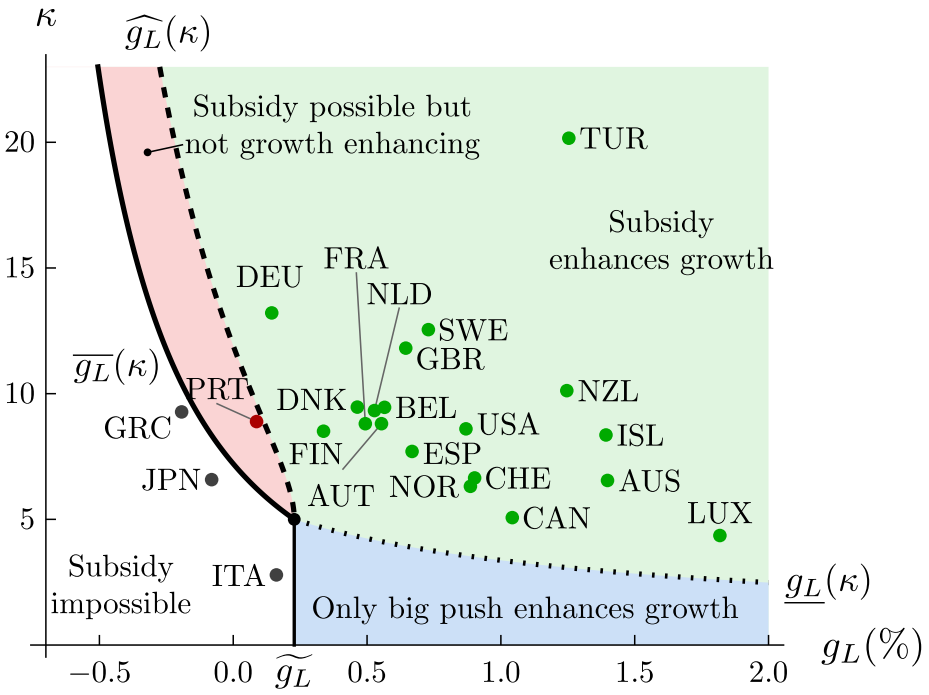


Figure 11: Growth-promoting conditions and calibrated country values in the (g_L, κ) plane.

the step size approximately equals the markup, as in the standard quality ladder model.[57]

The population of each country affects the results in two ways. First, the past population affects the current strength of vertical innovation. Since ideas for new industries occur to individuals, the number of people in the past determines the stock of these ideas, which determines the number of industries today. Therefore, if an economy had a large population relative to today, that economy now has relatively many industries when compared to the current population, which means that the number of workers per industry is small. This weakens the speed of vertical innovation because it depends on the number of researchers, as well as the demand for the goods the industry produces, both of which will be small in a country with low population growth. For this reason, when we estimate κ from the average GDP growth rate and the average investment-GDP ratio of each country over 2000–2023, we take into account its population since 1820.[58] Second, the future population growth affects whether the government can sustain the subsidies by rolling over its debt. We assume that the average population growth rate of each country over 2000–2023

change. If we set $\varepsilon = 1/50$, Japan and Italy lie in the region where only a big push enhances growth, and the other 22 countries lie in the region where the subsidies raise growth. When industries become obsolete faster, the number of workers per industry in (40) depends less on the population growth rate. Therefore, countries with slow population growth become less likely to fail the conditions in Propositions 7 and 8.

[57] If we instead set $\lambda = \ln(1/\alpha)$, at which the step size e^{λ} exactly equals the markup $1/\alpha$, the estimates of κ rise by about 10%, and only Portugal changes its classification. With $\lambda = (1-\alpha)/\alpha$, the assumption $\omega_{\text{in}} < a(1-\alpha)/\alpha$ in Lemma 1 holds whenever $\omega_{\text{out}} > 0$.

[58] We use the population since 1820 or the earliest available year. Appendix C.9 describes the details of the estimation.

continues in the future, and we use it as g_L.

The dots in Figure 11 represent the pairs of g_L and κ for the 24 countries.[59] If their current population growth rates continue, Japan, Italy, and Greece lie in the region where the government cannot sustain the subsidies by rolling over its debt.[60] In the baseline model, these three countries can roll over their debt, although the subsidies do not raise their long-run growth rates. Once population growth is taken into account, the result for them becomes more pessimistic. Portugal lies in the region where the subsidies are sustainable but do not raise growth. The other 20 countries lie in the region where the subsidies raise growth. No country lies in the region where only a big push enhances growth.

The second-generation model also provides a more structural interpretation of the cross-country differences in the effects of the subsidies. The estimates of κ for Japan and Greece are not particularly low compared with those of the other countries. Hence, these two countries fail to sustain the subsidies not because of their low propensity for vertical innovation, but because of their low population growth. When the population grows slowly, or even shrinks, the number of workers becomes small relative to the number of industries created from the ideas of past generations. When each industry has fewer workers, the R&D within the individual industries becomes less intensive, which reduces overall growth. If their population grew as fast as that of the United States, they would lie in the region where the subsidies raise growth. In contrast, the estimate of κ for Italy is the lowest among the 24 countries, and it is below $\alpha/(1-\alpha)$. As shown in the right column of Figure 10, which uses the value of κ from Italy, a slightly higher population growth rate would make the subsidies sustainable in Italy. However, even with the population growth rate of the United States, the subsidies for vertical R&D would raise growth in Italy only if the subsidy rate is high enough.

6.5 Nesting the Baseline Model

The model presented in this section is more general than the baseline model in that it includes population growth and two directions of innovation. In this subsection, we show that the baseline model, which focuses on variety-expansion innovation, as well as another widely used setting of quality-ladder innovation, can be interpreted as special cases of the current model.

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[59] Ireland ($g_L = 1.36\%$ and $\kappa = 44.6$) lies above the vertical range of the figure.

[60] This result concerns the long run, after the number of workers per industry has adjusted to the steady-state value given the current population growth rate.

Note that, using Equation (26) from the baseline model, the growth rate of the current model can be written as

$$g_t = \overline{g}(A_t)\frac{L_t^R}{L_t} + \frac{g_L}{\alpha}, \tag{57}$$

if we define $A_t \equiv \big(\alpha\lambda/(1-\alpha)\big)aL_t/n_t$. The first term is the contribution of vertical innovation, and it has the same form as the growth rate (25) of the baseline model. Here, A_t represents the productivity of vertical R&D, which is determined by a and λ, as well as the effective labor per industry, L_t/n_t. A higher a raises A_t because leaders are replaced more frequently, and a larger λ raises it because each replacement brings a larger improvement in quality. A larger L_t/n_t raises A_t because, for a given share of R&D labor, more research is conducted in each industry. In the long run, L_t/n_t converges to $(g_L+\varepsilon)/f$ by (40), and A_t becomes

$$A^* \equiv \frac{\alpha\lambda}{1-\alpha}\,\kappa(g_L+\varepsilon). \tag{58}$$

Hence, A^* is lower when f is higher, because new ideas create more industries and R&D is spread over them. It is higher when ε is higher, because more industries become obsolete, and when g_L is higher, because many new workers enter existing industries.

We first show that this model nests the standard quality-ladder model where the population and the number of industries are constant. In the standard quality-ladder model, the labor supply is constant, $L_t = L$ for all t, and the number of industries is one, $n_t = 1$ for all t. The former requires $g_L = 0$, that is, $\beta = \mu$. Given $g_L = 0$, the latter requires $\varepsilon/f = L$ by (40). Moreover, no industry is created or becomes obsolete in the standard quality-ladder model. Therefore, we take the limit of $f \to 0$ and $\varepsilon \to 0$ while keeping $\varepsilon/f = L$. We also set $\lambda = (1-\alpha)/\alpha$ so that the step size approximately equals the markup, as in Grossman and Helpman (1991a). In this limit, the wage and the cost of R&D no longer include the value of new businesses. Hence, the leader sets the usual monopoly price, and the effective subsidy rate θ^e equals θ. Since industries no longer become obsolete, the leader loses its position only when an outsider succeeds in R&D. Leaders are replaced more frequently when more labor is used for R&D, and each replacement raises the quality by the step size λ. The economy keeps growing through vertical innovation alone, under constant population. Therefore, the main parts of the model coincide with those of the standard quality-ladder model.[61] Appendix C.10 shows the coincidence formally.

The baseline model and the limit of the current model are based on two different prototypes of first-generation endogenous growth models: variety expansion and quality ladders. Nevertheless, the growth properties of the two settings are almost identical. Note

[61] The markup differs because of the form of the final goods production function.

that, without population growth, $g_t = g_{q,t}$ holds. Then, the dynamic equations for the second-generation model, (54) and (53), become exactly the same as those for the baseline model, (28) and (29), once we replace A by $A^* = aL$. Therefore, the macroeconomic implications are identical between the two settings. Indeed, Propositions 2 and 3 in the baseline model are a special case of Propositions 7 and 8 (see Appendix C.10). In addition, by substituting $A = A^* = aL$, the R&D equation for the baseline setting (15) can be written as $\dot{n}_t/n_t = aL_t^R$, which is standard in the variety-expansion model.[62]

7 Extensions

In this section, we consider four extensions of the baseline model. In the first subsection, we consider the setting in which research productivity A can be enhanced by publicly funded basic research, which is also financed by perpetually rolled-over debt. In the second subsection, we explore the possibility that the government may adopt a policy that cannot be sustained in the long run. The third subsection examines the possibility that the government can levy taxes on capital income, while it cannot resort to a labor tax or other non-distortional taxation. In the final subsection, we confirm that the main properties of our analysis continue to hold in a small-country open-economy model, once we explicitly introduce country premiums.

7.1 Basic Research

The baseline model has shown that the growth-promoting effects of debt-financed R&D subsidies critically depend on the research productivity of private firms, A. Countries with low R&D productivity typically struggle with low growth, but our baseline results suggest that subsidizing R&D will not solve the problem. What alternative policies should we consider? Is the problem of low R&D productivity unsolvable?

Recent studies suggest that the government can influence this parameter by funding basic research, which is typically conducted in government agencies and universities. For example, Akcigit, Hanley and Serrano-Velarde (2020) considered spillovers across different types of research activities and concluded that the policies geared toward basic public research improve welfare. Gersbach, Sorger and Amon (2018) reported that basic research extends the knowledge base for applied research.[63] Given these studies, this section con-

[62] This explains why we assumed (15) in the baseline model.

[63] They considered a variety-expansion model where private applied R&D and publicly funded basic research are complementary in creating new goods. In a related study, Gersbach, Schneider and Schneller (2013) introduced publicly funded basic research into a quality-ladder endogenous growth model, where

siders a simple extension of our model, where the government can improve the research productivity of private firms by investing more resources into publicly funded basic research. We consider two sets of problems: (i) whether debt-financed basic research can raise the long-term growth in countries with low R&D productivity, and (ii) how the funds obtained by rolling over debts should be allocated between funding public research and subsidizing private R&D.

In this extension, the government chooses the number of basic researchers, L^B, in addition to the private R&D subsidy rate $\theta \geq 0$.[64] When the government increases L^B from L_0^B, the productivity of private R&D responds with a constant elasticity of $\phi > 0$. This means[65]

$$A = A_0 \left(\frac{L^B}{L_0^B} \right)^{\phi}, \quad L^B \geq L_0^B, \tag{59}$$

where A_0 is private R&D productivity before the policy change.[66] Basic researchers are employed by the government at the market wage w_t. The total labor supply, L^{Total}, is constant, and we write $L \equiv L^{\text{Total}} - L_0^B$ for the labor available to the private sector before the policy change, so that the labor available to private production and private R&D is $L^{\text{Total}} - L^B$. The wage payments for the initial level of basic research employment, $w_t L_0^B$, are financed by an existing labor income tax at the rate of L_0^B / L^{Total}.[67] We maintain our assumption that the tax rates cannot be raised. Therefore, the additional expenditure associated with increasing L^B, $w_t(L^B - L_0^B)$, must be financed by issuing government bonds. The other parts of the model are the same as those in Section 3.

We define L^{Total} and L, as well as the existing tax, in this way so that the extended model exactly coincides with the baseline model when we set $L^B = L_0^B$. In other words,

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public funding increases the probability of success in quality-upgrading innovation. Leon-Ledesma and Shibayama (2023) extended Romer (1990) by introducing the basic research sector subsidized by the government. Parui and Prettner (2025) combined a variety-expansion model with human capital and analyzed the tradeoff between spending for basic research and healthcare.

[64] We omit the time subscript in L^B because it is a policy variable.

[65] For the same political reason, we assume that the government cannot reduce basic research employment below its initial level L_0^B.

[66] In our analysis, basic research conducted in other countries and the knowledge that spills over from it are taken as given and included in A_0. The elasticity ϕ captures only the effect of a country's basic research on the productivity of its own private R&D. In reality, basic research in one country also raises A_0 in other countries. If this externality is taken into account, the amount of basic research that maximizes growth in the world as a whole would be even larger than our analysis suggests. Since basic research and subsidies for private R&D are financed from the same limited funds that a sustainable level of government debt can provide, the growth-maximizing combination of the two policies would also change. In particular, it may involve smaller subsidies for private R&D. Gersbach, Schetter and Schmassmann (2023) show that basic research is underprovided in an equilibrium without international coordination, and a similar consideration would apply to our model.

[67] Since the tax lowers the after-tax wage of every individual by the same factor, it does not alter the behavior of individuals, as shown in Appendix D.

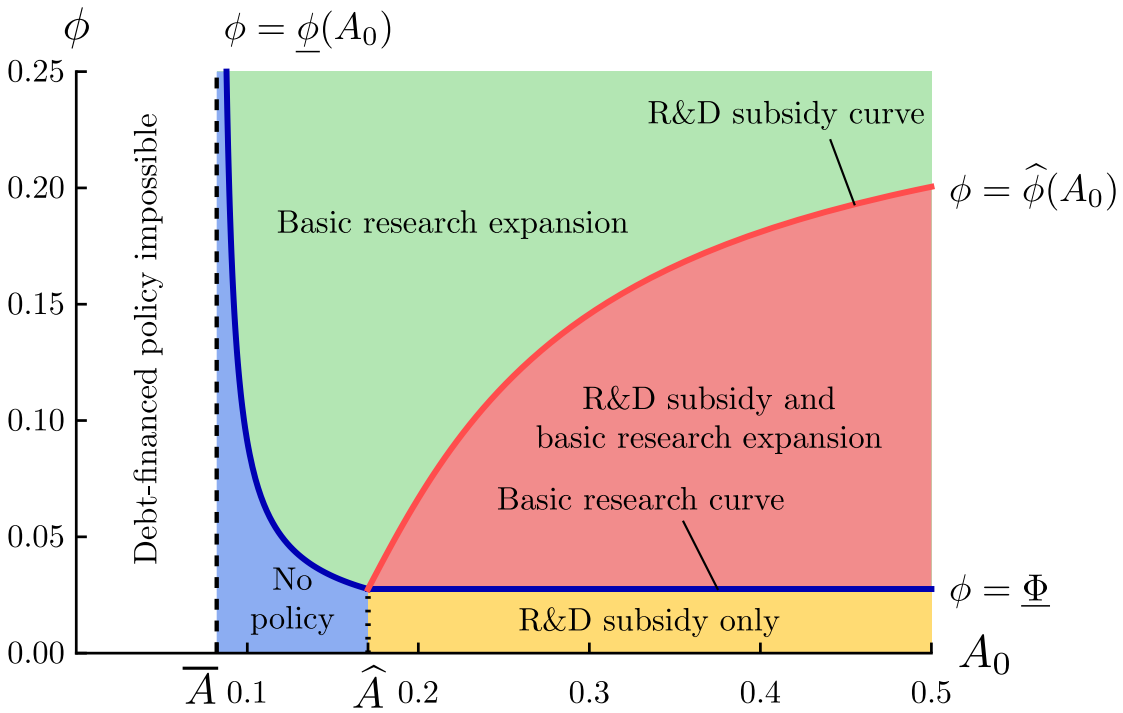


Figure 12: Growth-maximizing policy in the (A_0, ϕ) plane under the U.S. calibration.

the result from the baseline model can be seen as the special case where we impose the restriction $L^B = L_0^B$ in this extended model. In Appendix D, we derive the resulting dynamic system. Similarly to the baseline model, for a given policy pair (θ, L^B), the economy is described by a two-dimensional system in the debt-GDP ratio D_t and the growth rate g_t. As in the baseline model, Proposition 1 holds for $D_0 \geq 0$ (Appendix D.3). The economy therefore has an equilibrium path if and only if it has two steady states and the initial debt-GDP ratio is below the unsustainable threshold, and the path converges to the long-term BGP. We focus on the long-run growth rate on the long-term BGP.

When compared to the baseline model, the government now has another policy instrument. This addition affects the desirable policy through various channels. First, an increase in L^B raises private R&D productivity, A, through (59). The strength of this productivity effect depends on the effectiveness of basic research ϕ. Then, the resulting level of A affects the effectiveness of research subsidies, as we have confirmed in the baseline model. Second, both additional basic research and research subsidies are financed by government debt; therefore, their relative effectiveness matters when allocating funds across policies. Third, both policies increase the debt-GDP ratio and crowd out private R&D. Finally, allocating more labor to basic research reduces the amount of labor available for private production and R&D.

With all these considerations, Figure 12 graphically categorizes the growth-maximizing

policy in the (A_0, ϕ) plane. In the figure, we numerically calculate the location of boundary curves using the parameters from the U.S. calibration, described below. Observe that there are two curves that divide the (A_0, ϕ) plane. One is the basic research curve, which slopes downward in the range of $A_0 \in (\overline{A}, \widehat{A})$ and becomes flat for $A_0 \geq \widehat{A}$. Above this curve, if the government is to maximize the long-term growth rate, it should increase funding for basic research.[68] Intuitively, the basic research should be included in the growth-maximizing policy when its effectiveness, ϕ, is larger. In particular, observe that the curve includes the region of $A_0 \in (\overline{A}, \widehat{A})$, where R&D subsidies are not effective for promoting growth (See Proposition 3). This means that even when the economy's initial research productivity is so low that direct subsidies to firms will not improve growth performance, funding basic research could still improve the situation.

In the region of $A_0 > \widehat{A}$, there is another upward-sloping curve, which we call the R&D subsidy curve. Below this curve, the growth-maximizing policy includes a positive R&D subsidy. Therefore, when the economy lies between the two curves, the growth-maximizing policy is a mix of basic research enhancement and R&D subsidies for private firms. If ϕ is above the R&D subsidy curve, however, focusing only on basic research will eventually maximize the long-term growth.[69] Therefore, contrary to the finding in Proposition 3, the fact of $A_0 \geq \widehat{A}$ does not automatically justify the use of R&D subsidies for private firms. Because the funding pool is common and limited, there is a case for prioritizing basic research, even if it seems less directly connected to economic growth.

The following proposition formally establishes these findings.

Proposition 9 (Basic Research)

(a) Suppose that $\overline{A} < A_0 \leq \widehat{A}$. If $\phi > \underline{\phi}(A_0)$, long-run growth is maximized with additional basic research, that is, at some $L^B > L_0^B$. The threshold $\underline{\phi}(A_0)$ is decreasing in A_0 and satisfies $\underline{\phi}(A_0) \to \infty$ as $A_0 \downarrow \overline{A}$.

(b) Suppose that $A_0 > \widehat{A}$. Define $\underline{\Phi} \equiv (L_0^B/L)\widehat{A}/(\widehat{A} - 2\alpha\overline{A})$.

(i) If $\phi \leq \underline{\Phi}$, long-run growth is maximized by a positive R&D subsidy with $L^B = L_0^B$.

(ii) If $\phi > \underline{\Phi}$, long-run growth is maximized with additional basic research, that is, at some $L^B > L_0^B$.

(iii) If, in addition, $\underline{\Phi} < \phi < \widehat{\phi}(A_0)$, long-run growth is maximized by a combination

[68] In the region marked as "No policy", we confirmed that a marginal increase in basic research or R&D subsidy from the no-policy BGP will not increase the long-run growth rate (Appendix D.4).

[69] Strictly speaking, the proposition does not formally establish that the growth-maximizing policy never includes an R&D subsidy whenever $\phi > \widehat{\phi}(A_0)$. Still, we numerically confirmed that this statement is true for numerous combinations of parameters, finding no exceptions.

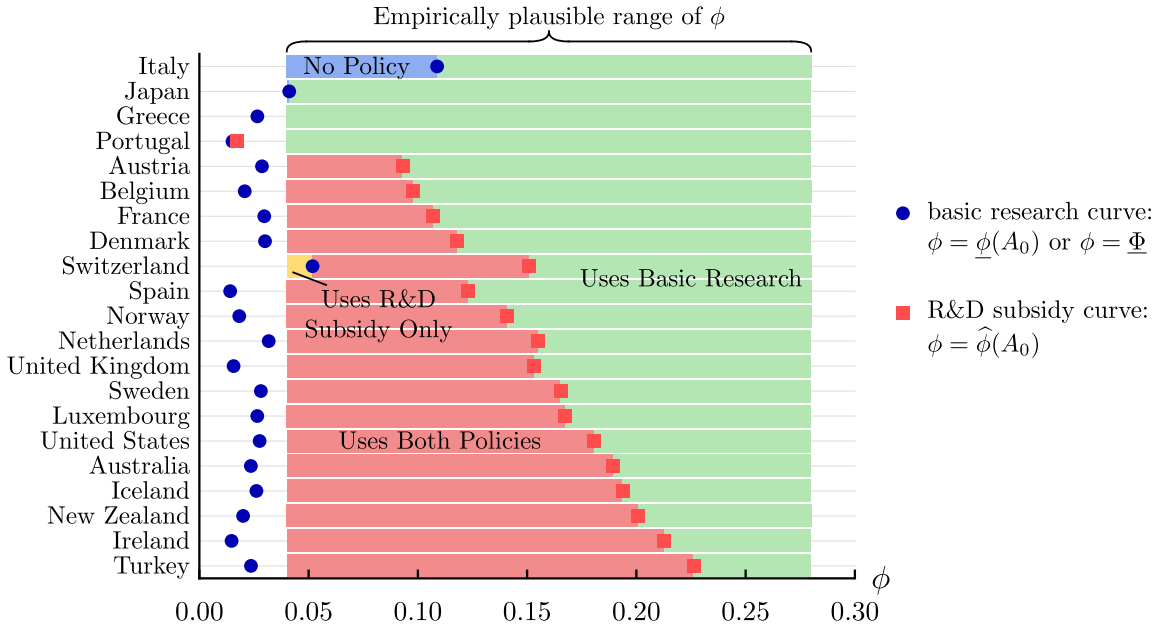


Figure 13: Growth-maximizing policy by country

of additional basic research and a positive R&D subsidy.
The definitions of $\underline{\phi}(A_0)$ *and* $\widehat{\phi}(A_0)$ *are given in Appendix D.4.*
Proof: See Appendix D.4.

In Section 4.4 (see Figure 7), we presented which countries can enhance long-term growth by debt-financed subsidies and which cannot. Now, with an additional policy instrument, we examine how the result changes for each country. There are two new parameters that potentially influence the result. The first is the initial share of labor employed in public basic research, L_0^B/L. For each country, this value is calibrated from basic research expenditure as a share of GDP over the period 2000–2023. As explained in Appendix D.5, this yields $L_0^B/L = 0.545\%$ for the United States and 0.361% for Italy. Another crucial parameter is the elasticity of private R&D productivity with respect to public basic research, ϕ. Jaffe (1989) reports estimates ranging from 0.04 to 0.28 across industries for the elasticity of corporate patenting with respect to university research expenditures in the same state. Using innovation counts rather than patents, Acs, Audretsch and Feldman (1992) obtain an estimate of 0.241. Since the country-specific value for this parameter is not available, we present the result for the plausible range of $\phi \in [0.04, 0.28]$. The remaining parameters are the same as those in Table 1.

Figure 13 shows the result for the 21 countries.[70] Recall that, in Section 4.4, we concluded that Japan, Italy and Greece cannot enhance long-term growth when they are not

[70] Among the 24 countries we studied at baseline, we needed to exclude Canada, Germany, and Finland because their basic research expenditure is not available in the MSTI database.

able to increase the tax rate. Now we have a different picture. Japan and Greece can enhance growth by financing additional basic research expenditures through debt issuance.[71] The result for Italy is mixed. Only when ϕ is larger than 0.109 does our theory imply that such a policy enhances growth, while the plausible range for this parameter is $[0.04, 0.28]$. Therefore, for Italy, further studies to estimate the parameter ϕ will help determine whether it can enhance growth through this route.

The baseline model implies that all the other countries can maximize their growth by relying on debt-financed R&D subsidies. Now, the extended model has a different implication. With the exception of Switzerland, using only R&D subsidies is never growth-maximizing. Instead, it is desirable to use at least a part of the debt-financed funds to expand the basic research. In particular, if the value of ϕ is close to the upper end of the plausible range (for example, in the case of the U.S., $\phi > 0.181$), allocating all the funds to public basic research, rather than R&D subsidies for private firms, leads to the highest growth.

Figure 14 shows the long-term growth rate for different combinations of basic research expansion and R&D subsidy in the United States and Italy, under three values of ϕ in the plausible range. The growth-maximizing policies, shown by the black squares, are as predicted by Figure 13. For the United States, the growth-maximizing policy uses both instruments when $\phi = 0.04$ or 0.13, and only basic research when $\phi = 0.28$. For Italy, it uses neither instrument when $\phi = 0.04$, and only basic research when $\phi = 0.13$ or 0.28. The figure also shows that a policy pair whose total expenses for the two policies are excessive cannot be implemented. If the government tried to choose such a pair, there would be no equilibrium path (Appendix D.3), and lenders would not lend to the government. This illustrates the trade-off between the instruments.

7.2 Unsustainable Subsidies

The benchmark analysis shows that there is a maximum sustainable subsidy rate $\overline{\theta}(A)$, above which a permanent subsidy cannot be implemented. In practice, however, governments often allow fiscal deficits to grow beyond what is consistent with long-run debt sustainability.[72] This section examines what happens to the economy when the government implements such a policy. Such a policy necessarily entails a regime change, because the

[71] Japan's threshold is 0.041, which almost coincides with Jaffe (1989)'s lower bound.

[72] The political economy literature attributes this tendency to a war of attrition among socio-economic groups that disagree over how the burden of fiscal adjustment should be shared, thereby systematically delaying necessary stabilizations (Alesina and Drazen, 1991; Alesina and Passalacqua, 2016).

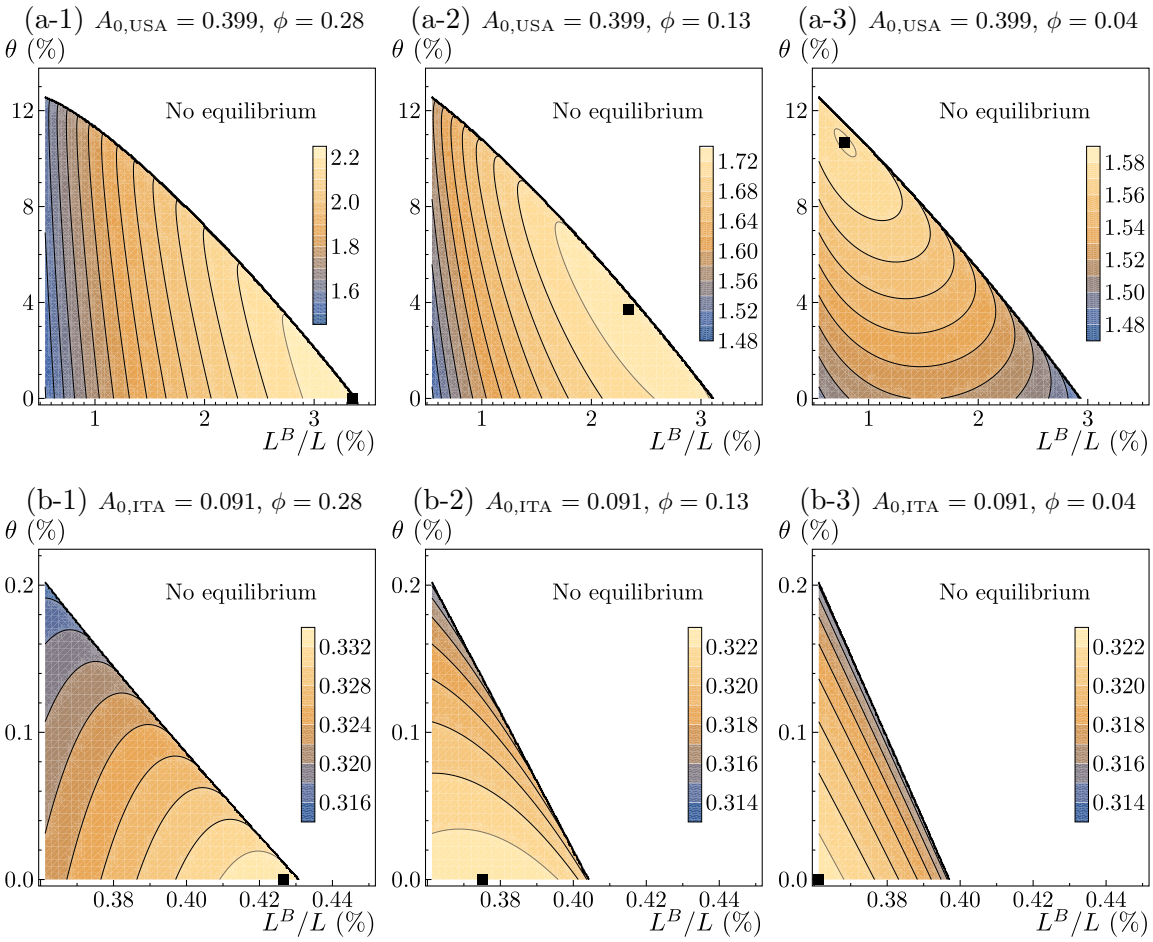


Figure 14: Long-run growth effects of basic research expansion and the R&D subsidy. The colored area shows the long-term growth rate in % as indicated by the color chart. The black square in each panel shows the growth-maximizing policy combination. The economy has no equilibrium path when the policy pair is in the white area (i.e., the expense cannot be financed). The parameters other than A_0 and ϕ are the same as in Figure 13.

"splurge" regime can only be sustained for a finite duration. We assume that when the debt-GDP ratio eventually becomes too high, fiscal consolidation is imposed on the government, either as self-correction to avoid sovereign default or through external pressure from creditors and international institutions such as the IMF.[73] We compare the outcome from such a policy with a sustainable policy that maximizes the long-term growth rate. The following analysis shows whether such an unsustainable policy is actually beneficial to the economy, and if so, how long the benefit lasts.

In both cases, the economy starts from the no-policy BGP with $D_0 = 0$ at $t = 0$. The sustainable policy sets a permanent subsidy rate that maximizes the long-term growth rate, $\theta_t = \theta^s \equiv \widehat{\theta}(A)$ for all t. Under this policy, the economy converges to the long-term BGP, on which the debt-GDP ratio is $D^*(A, \theta^s)$. The unsustainable policy initially sets the

[73] In an R&D-based growth model, Ferraro and Peretto (2020) examine the macroeconomic consequences of consolidations imposed after a period of unsustainable deficit accumulation.

subsidy rate higher than the highest sustainable rate, $\theta^u > \overline{\theta}(A)$, in what we call the splurge regime, and then switches to the austere regime, where the subsidy is permanently canceled ($\theta = 0$).[74] We assume that the regime change occurs when the post-switch debt-GDP ratio is 0.9 times the unsustainable threshold without subsidies, $D_U(A, 0) = \alpha(1/\overline{A} - 1/A)$, beyond which the economy has no equilibrium path (Proposition 1(b-ii)).[75] The timing of the regime change depends on the subsidy rate θ^u. In our calculations, the duration of the splurge regime, denoted by T, is shorter when θ^u is higher. In the examples below, we use the same parameters as the baseline model (see Table 1 in Section 4.4), again considering two values of A that are calibrated to the U.S. and Japan. For each A, we set θ^u so that the duration of the splurge regime is $T = 100$. This pins down the subsidy rate in the splurge regime as $\theta^u = 37.0\%$ in the high-research-productivity country and 16.4% in the low-A country.

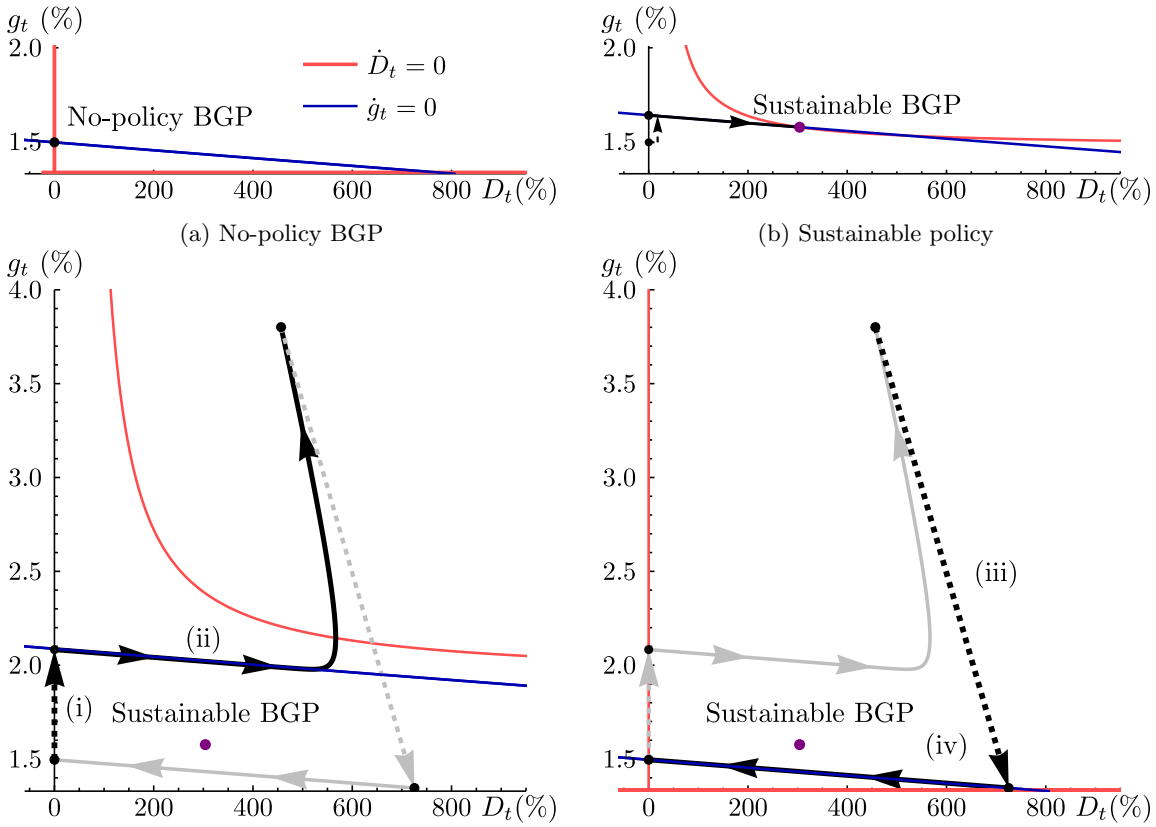
The four panels in Figure 15 show the phase diagram and the transitional dynamics under two policies, numerically calculated using the parameter values for the high-A economy and shown on the same scale. Panel (a) shows the BGP of the no-policy economy. Both policies start from this point. Panel (b) shows the dynamics under the sustainable policy. At the time of the introduction of the policy, the phase diagram changes from (a) to (b), and the growth rate jumps up to the equilibrium path under the new policy. After the jump, the economy evolves along this path, on which the debt-GDP ratio gradually increases and the growth rate declines. In the long run, the economy converges to the sustainable BGP, on which the growth rate is higher than that on the no-policy BGP.

Panels (c) and (d) explain the dynamics of the economy under the unsustainable policy. At the time of the introduction of the unsustainable subsidy, the phase diagram changes from (a) to (c). Since the subsidy rate is higher than the case of panel (b), the shift of the $\dot{g}_t = 0$ locus is more significant. Given this, the size of the jump in the growth rate, as shown by (i) and representing the short-run growth-stimulating effect, is larger under the unsustainable policy than under the sustainable policy.

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[74] In the main text, we keep the assumption that the government cannot raise taxes even in the future. Therefore, the austere regime only means that the subsidy is terminated. In a separate simulation, however, we computed the case in which the government, after terminating the subsidy, taxes all asset income at the rate of 20% until the debt-GDP ratio returns to zero. We set the subsidy rate and the debt-GDP ratio just after the termination the same as below. Even though the taxes on dividends and capital gains are anticipated, the effect of the last-minute R&D investment is larger, and R&D and the growth rate still rise toward the termination. Due to the opposing effects, the rise is smaller. In the high-A country, the growth rate just before the termination is 3.43%, compared with 3.80% reported in the main text.

[75] The value of 0.9 is illustrative. If we instead set it to 0.8, the subsidy rate that can be maintained for $T = 100$ years becomes slightly lower (from 37.0% to 34.9% in the high-A country and from 16.4% to 15.0% in the low-A country), and the period in which full-production GDP is higher under the unsustainable policy than under the sustainable policy becomes slightly longer (from 293 to 297 years and from 228 to 234 years, respectively).



(a) No-policy BGP

(b) Sustainable policy

(c) Splurge regime under unsustainable policy

(d) Austere regime under unsustainable policy

Figure 15: Phase diagram under sustainable and unsustainable policy ($A = 0.399$). In the case of unsustainable policy, the economy transitions from panel (c) to panel (d), following arrows (i) to (iv).

Technically speaking, the size of the jump is determined not by an equilibrium path that converges to a long-term BGP, which does not exist under the splurge regime, but by the path shown by (ii) in panel (c). In 100 years, this path reaches a point where the post-regime-change value of the debt-GDP ratio is exactly 0.9 times $D_U(A, 0)$. Appendix E.2 explains how this path is computed. It is notable that the growth rate just before the regime change is quite high at 3.80% due to strong last-minute investment demand as the splurge (high-subsidy) regime is about to end. To establish the financial market equilibrium before and after the regime change, the interest rate needs to spike at the time of the regime change. This spike induces consumption to jump up (due to individuals' dynamic optimization) and investment to fall, which creates a downward jump in the growth rate, as represented by (iii) in Panel (d). Also, the debt-GDP ratio jumps up due to this spike in the interest rate. After the jump, the economy gradually converges to the no-policy BGP, as shown by (iv). Appendix E.1 derives the conditions that connect the two regimes at the time of the regime change.

This analysis suggests that a limited-time R&D subsidy could stimulate economic

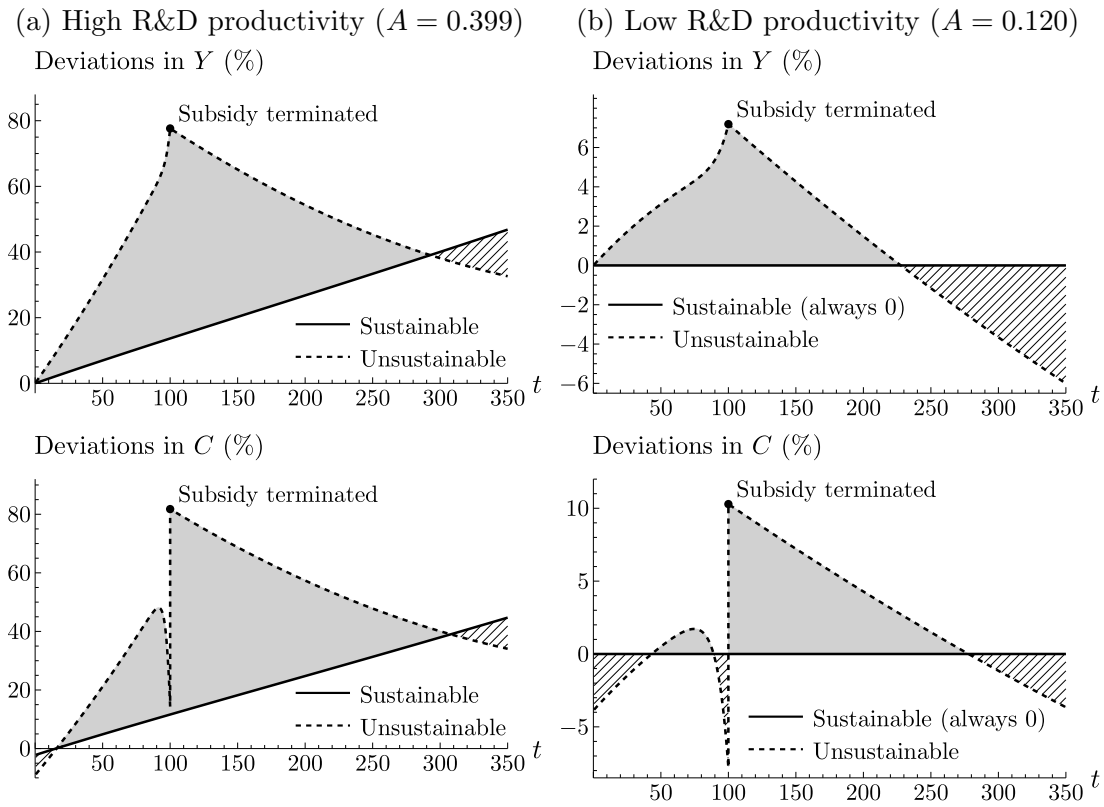


Figure 16: The time path of the full-production GDP (top) and the aggregate consumption (bottom) relative to the no-policy BGP. The gray area indicates the difference between the unsustainable and sustainable subsidies when the former is higher. The hatched area shows the difference when the latter is higher.

growth more than a permanent one. Not only is it possible to dole out a higher subsidy rate, but the fact that the subsidy will be terminated in a finite time also creates a last-minute surge in investment. Anticipating the high return available at the time of the regime change, individuals postpone consumption, which frees labor for R&D before the termination. However, in our numerical example, path (ii) in panel (c) is quite close to the $\dot{g}_t = 0$ locus, which suggests that this anticipation effect is minimal at the time of introduction, given that the termination is 100 years away. As the time nears the point of termination, the anticipation effect becomes larger, and the growth rate rises.[76]

Figure 16 shows the time path of full-production GDP Y_t and aggregate consumption C_t under two policies, as percentage deviations from the no-policy baseline.[77] Under the

[76]In the high-A country with $T = 100$, the growth rate exceeds the value on the $\dot{g}_t = 0$ locus by one percentage point only within the last two years before the termination, and the gap reaches 1.8 percentage points just before it. We also conducted numerical simulations with $T = 10$ and $T = 30$. The gap then reaches one percentage point four years and three years before the termination, and 2.7 and 2.6 percentage points just before it.

[77]For the case of the low-A country, the growth-maximizing sustainable policy is actually $\theta = 0$, given that a tax on R&D is not available (See Proposition 3). Therefore, the deviation of the sustainable policy from the no-policy BGP is zero.

unsustainable policy, full-production GDP is higher than under the sustainable policy for about 290 years in the high-A country and about 230 years in the low-A country. This result holds even in the low-A country, where $\theta = 0$ is recommended when only the long-term growth is considered.[78] Aggregate consumption is initially lower than the no-policy BGP because the high R&D subsidy encourages R&D rather than consumption, but it surpasses the no-policy path in a few decades thanks to higher productivity, and then peaks just after the subsidy is terminated. There is another dip in aggregate consumption just before the subsidy's termination due to a surge in last-minute R&D investment, which crowds out consumption.

We also computed the expected lifetime utility of each cohort alive at the introduction of the policy, under both the sustainable and the unsustainable policies (Figure E.1 in Appendix E.3). If each individual supports the policy that gives the higher expected utility, virtually all individuals (younger than 426 years old) support the unsustainable policy under the U.S. parameters. Under the Japanese parameters, in contrast, only individuals younger than 45.8 support it. The elderly in Japan do not support it because the subsidy makes entry easier, and the expected increase in future competition lowers the value of the shares of existing firms.

The analysis in this section has shown that unsustainable policies can be beneficial to the economy and individuals for extended periods, and that the population at the time of voting may support such policies if they decide based on their own benefits. However, this does not necessarily mean that these policies are recommendable. After the finite beneficial period, both output and consumption fall below the sustainable policy path, and the gap widens over time. Therefore, the question of whether subsidy policies should be sustainable boils down to the intergenerational trade-off.[79]

7.3 Asset Taxation

Thus far, we have assumed that the government cannot raise any tax. This assumption allows us to examine the implications of the political constraint in a simple way, but it may be too strong. The literature on taxes and political economy has suggested that opposition

[78] The right bottom panel of Figure 9 in Section 4.5 shows that a sustainable positive subsidy can also temporarily raise Y and C relative to the no policy case. However, the magnitude of deviation is much smaller (less than 1%) than the unsustainable subsidy considered in this section (up to around 7% in Y_t and 10% in C_t).

[79] Actual politics may not weigh the welfare of future generations, but it might still prevent such a policy. The elderly in Japan would oppose such a policy not because they value future generations, but because they want to protect their vested interest in the shares of existing firms. The same could happen in the U.S. if wealthy individuals have disproportionately strong political power.

to raising capital income taxes might be weaker than opposition to raising labor income taxes.[80] This section relaxes the assumption of the baseline model and examines how our results change when the government can use an asset income tax.

Allowing asset income taxation raises two questions. First, can the government attain a higher long-term growth rate than in Proposition 3? Second, can the tax revenue eliminate the need for government debt when the government maximizes the growth rate? The answers to both questions are negative. The highest attainable growth rate is unchanged. Moreover, attaining the maximum growth rate requires a higher research subsidy and a higher debt-GDP ratio when the asset income tax rate is higher.

To examine these questions, suppose that the government taxes income from all assets at the uniform rate $\tau_r \in [0, 1)$. We write the values on the long-term BGP with τ_r as an additional argument, such as $g^*(A, \theta, \tau_r)$, so that $g^*(A, \theta, 0)$ equals $g^*(A, \theta)$ in the baseline model. To simplify the analysis, we only consider the case where the research subsidy is growth-promoting ($A > \widehat{A}$) and the interest rate is positive on the long-term BGP without the subsidy and the tax ($\theta = \tau_r = 0$).[81] We treat τ_r as given and examine how the government chooses the research subsidy under this tax. Individuals then receive the after-tax return $(1 - \tau_r)r_t$, while the government obtains the tax revenue $\tau_r r_t K_t$. Since the same tax rate applies to government bonds and the shares of intermediate goods firms, the no-arbitrage condition and the free entry condition are unchanged. The other parts of the model are the same as in Section 3. The equilibrium dynamics are derived in Appendix F.

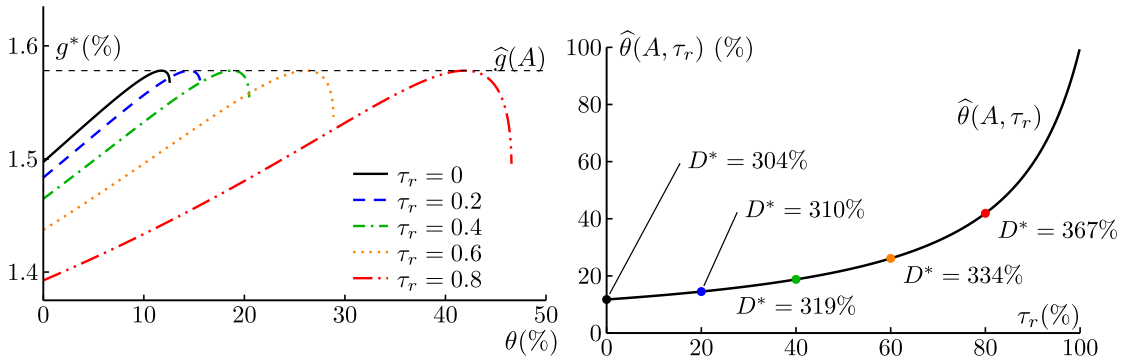
The asset income tax has two opposing effects on the possibility of raising the growth rate by debt-financed research subsidies. First, the additional revenue enables the government to sustain a higher research subsidy. Second, the lower after-tax return weakens the life-cycle saving motive of households. The resulting reduction in the supply of funds to the capital market raises the pre-tax interest rate and lowers the value of firms.[82] Therefore, the asset income tax discourages R&D, for a given rate of research subsidy. Surprisingly,

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[80] Statutory tax rates on capital income have been repeatedly revised in OECD countries (Harding and Marten, 2018). This observation suggests that the political constraint on this tax may be weaker than that on labor income taxation. Dynamic optimal-taxation studies also allow capital income taxes to be revised over time (e.g., Farhi, Sleet, Werning and Yeltekin, 2012; Scheuer and Wolitzky, 2016). In an R&D-based growth model, Peretto (2011) considers a debt-financed cut in the tax on dividends.

[81] If the interest rate is negative on the growth-maximizing BGP and the government reimburses a fraction τ_r of the loss from the investment, parts (b)–(d) of Proposition 10 below hold except that the signs in parts (c) and (d) are reversed, as long as $\widehat{\theta}(A, \tau_r) \in [0, 1)$.

[82] The asset income tax itself is neutral to the value of the firm given the pre-tax interest rate, because the tax proportionally affects the required rate of return and the post-tax dividends. The results do not change if government bonds are exempt from the tax. Taxing the interest on government bonds only returns part of the interest to the government itself. If the bonds were exempt, their interest rate would equal the after-tax return $(1 - \tau_r)r_t$, and the government would pay the same net interest.



(a) Long-run growth rate under different τ_r and θ

(b) Growth-maximizing subsidy rate and debt-GDP ratio

Figure 17: The long-term growth rate and the debt-GDP ratio with asset income taxation and research subsidy, computed with the U.S. calibration ($A = 0.399$).

the two opposing effects exactly cancel out if the government adjusts the research subsidy rate across different asset income tax rates.

Proposition 10 (Asset Income Taxation) *Suppose $A > \widehat{A}$ and $r^*(A, 0, 0) > 0$. Then, for any $\tau_r \in [0, 1)$, the following results hold.*

(a) $r^*(A, \theta, \tau_r) > 0$ holds whenever the long-term BGP exists.

(b) The highest attainable long-term growth rate is the same as $\widehat{g}(A)$ in Proposition 3.

(c) The subsidy rate $\widehat{\theta}(A, \tau_r)$ that attains this growth rate is positive and strictly increasing in τ_r, with $\widehat{\theta}(A, 0) = \widehat{\theta}(A)$.

(d) The long-term debt-GDP ratio at this subsidy rate is strictly increasing in τ_r.

Proof: See Appendix F.3.

As shown by Figure 17(a), when the government levies a higher asset income tax rate, it can subsidize research at a higher rate to maximize long-term growth. However, the growth-reducing effect of the asset income tax is exactly offset by the higher research subsidy rate made possible by the asset income tax rate. Still, not everything comes down to the same. As shown by Figure 17(b), the debt-GDP ratio is higher when the maximum growth is attained by the pair of a high asset income tax rate and a correspondingly high research subsidy rate, when compared to both being low (in particular, when $\tau_r = 0$).

The intuitive reason is as follows. When the government provides a research subsidy and then levies an asset income tax, it effectively gives money at the firm's founding and then takes it back by taxing the returns from the firm (dividends and capital gains), which continues forever. The fact that the growth rate is unchanged (i.e., the incentives for innovation are unaffected) implies that the present value of the taxes on the returns from the firm, discounted at the after-tax interest rate, equals the increase in the subsidy

compared with the case of $\tau_r = 0$. Still, from the government's perspective, there is a time difference between paying the subsidy and taking it back. The difference shows up as increased government debt. When τ_r is higher, the government has to pay a larger subsidy at the beginning, although it can levy a higher tax rate on the returns from the firm. This leads to a higher debt-GDP ratio.

7.4 Country Premium in a Small Open Economy

In reality, the capital market of a country is not closed to the rest of the world. Government bonds can be absorbed by foreign funds as well as by the savings of domestic households. In our baseline model, the crowding out comes entirely from the competition for those savings between the government and the intermediate goods firms, as shown by (18). Access to foreign funds weakens this competition, and it disappears in a small open economy that takes the world interest rate as given.

However, a larger government debt tends to raise the country premium, and the firms in that country then face a higher interest rate.[83] Since a higher interest rate reduces the present value of the future profits realized by R&D, it discourages research even when the competition for domestic savings is absent. The crowding-out channel thus moves from the domestic asset market to the country premium. In the baseline model, the relative size of this channel and the direct effect of the subsidy determined whether the policy can be financed at all (Proposition 2) and whether it raises the long-term growth rate (Proposition 3). This section shows that there are similar conditions in a small open economy once the country premium is taken into account.

We describe this country-premium channel in the simplest possible setting. The capital market is connected to the world market, and the interest rate of each country is determined by

$$r_t = \overline{r} + \eta \max\{D_t, 0\}, \qquad \eta > 0. \tag{60}$$

Here $\overline{r} > 0$ is the world safe interest rate, which is the rate faced by a country that has no government debt and hence no country premium. The parameter η measures how strongly the government debt affects the country premium.[84]

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[83] Ağca and Celasun (2012) document that corporate borrowing costs are higher in emerging market economies where the public external debt is higher, and Li, Magud and Werner (2023) find that the long-run pass-through from sovereign to corporate bond yields in these economies is close to one. While the response of the country premium to the debt level could differ among countries, Proposition 11 depends only on the slope of the premium at zero debt.

[84] A debt-elastic country premium is a common specification in small open economy models. See, for example, Schmitt-Grohé and Uribe (2003), where the premium is an exponential function of the debt. We let the interest rate respond linearly to a positive debt-GDP ratio for simplicity.

The market for final goods is also connected to the world market. Intermediate goods, on the other hand, are used in the domestic supply chain and are sold only to the domestic final goods sector, and we do not consider international migration of labor. The R&D sector, the intermediate goods sector, and the allocation of labor are therefore the same as in the baseline model, and the supply-side relations given by (16), (19), and (25) continue to hold. Since the interest rate is given by (60) and final goods are traded in the world market, the consumption and saving of households do not affect the dynamics of the debt-GDP ratio and the growth rate. We therefore do not consider the household side in this section. As shown in Appendix G, combining (60) with the supply-side relation (27) gives the growth rate for a given level of government debt D_t:[85]

$$g_t = \psi_1(A,\theta) - \psi_2(\theta)D_t, \text{ where} \tag{61}$$
$$\psi_1(A,\theta) \equiv \frac{(1-\alpha)\left(\overline{g}(A) - (1-\theta)\overline{r}\right)}{\alpha - (2\alpha-1)\theta}, \quad \psi_2(\theta) \equiv \frac{(1-\alpha)(1-\theta)\eta}{\alpha-(2\alpha-1)\theta}.$$

Equation (61) shows that a larger government debt raises the country premium and the interest rate faced by firms, and that it therefore lowers the equilibrium growth rate.

In the small open economy, r_t and g_t are both determined by (60) and (61) as functions of D_t. Substituting them into the debt dynamics (29) gives a single differential equation in D_t (see Appendix G). In this equation, D_t increases when the subsidy expenditure exceeds the surplus from government debt, $(g_t - r_t)D_t$, and decreases otherwise. Since r_t rises and g_t falls with D_t, the gap $g_t - r_t$ narrows as D_t rises, and the surplus first increases and then decreases with D_t. As long as the subsidy rate is not too large, the surplus equals the subsidy expenditure at two positive levels of the debt-GDP ratio. Below the lower level, the subsidy expenditure exceeds the surplus, and the debt-GDP ratio rises. Between the two levels, the surplus exceeds the subsidy expenditure, and the ratio falls. Hence, the lower level is the long-term BGP of this economy, and the higher level is the unsustainable threshold, $D_U(A,\theta)$. We write the debt-GDP ratio and growth rate on the long-term BGP as $D^*(A,\theta)$ and $g^*(A,\theta)$, and its interest rate as $r^*(A,\theta)$. As in the baseline model, the economy converges to the long-term BGP if the initial debt-GDP ratio is below $D_U(A,\theta)$, and there is no equilibrium path if it is above $D_U(A,\theta)$ (Appendix G).

Proposition 11 (Small Open Economy) *(a) Suppose $\theta = 0$. The economy has a long-term BGP with a nonnegative debt-GDP ratio if and only if $A > \overline{A}^{\text{open}} \equiv \alpha\overline{r}/(1-\alpha)^2$. On this BGP, $D^*(A,0) = 0$ and the growth rate $g^*(A,0)$ is higher than*

[85] For simplicity, we consider only the situation in which R&D is positive in equilibrium.

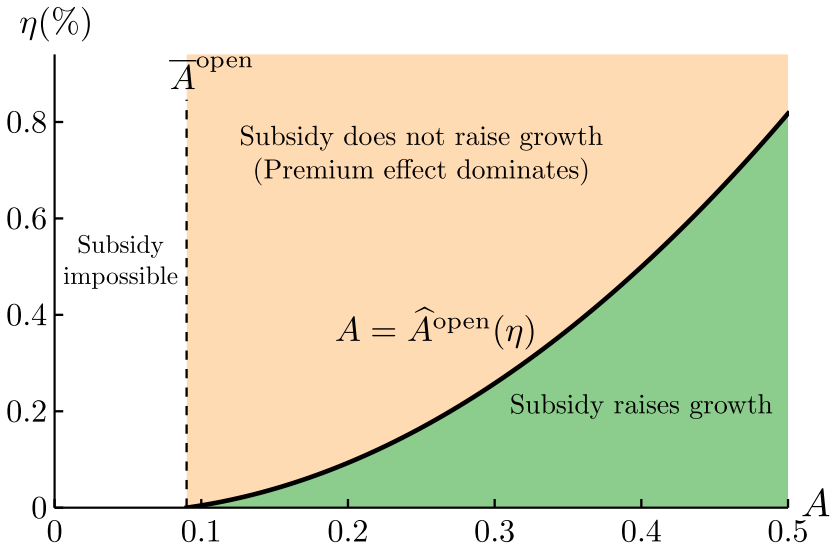


Figure 18: Country premium and the growth effect of debt-financed R&D subsidies. The figure illustrates the qualitative results of Proposition 11, using $\overline{r} = 0.3\%$ as an example (other parameters are the same as in the baseline model). The regions show the effect of the subsidy when its rate is marginally increased from zero.

the interest rate $r^*(A, 0)$.

(b) Suppose $A > \overline{A}^{\text{open}}$ *holds. Then, a marginal increase in the research subsidy rate* θ *from 0 raises the long-term growth rate* $g^*(A, \theta)$ *if and only if* $A > \widehat{A}^{\text{open}}(\eta)$*, where* $\widehat{A}^{\text{open}}(\eta)$ *is the inverse function of*

$$\eta = \frac{(1-\alpha)A\left((1-\alpha)^2 A - \alpha\overline{r}\right)\left((2\alpha-1)A + \alpha\overline{r}\right)}{\alpha^4\left((1-\alpha)A - \alpha\overline{r}\right)}, \tag{62}$$

which is well defined, strictly increasing in η*, and satisfies* $\widehat{A}^{\text{open}}(0) = \overline{A}^{\text{open}}$.

Proof: See Appendix G.

The two parts of Proposition 11 show that the intuition behind Propositions 2 and 3 carries over to the small open economy. As Figure 18 illustrates, the growth rate exceeds the interest rate in the long run only when the research productivity A is above the threshold $\overline{A}^{\text{open}}$. This condition alone does not make a debt-financed research subsidy effective. For a marginal subsidy to raise the growth rate, A must exceed the higher threshold $\widehat{A}^{\text{open}}(\eta)$. The condition becomes more demanding as η becomes larger, because a stronger response of the interest rate to government debt strengthens the crowding out and offsets the direct effect of the subsidy.

8 Conclusion

In the literature on endogenous growth, it is widely agreed that economic growth is driven by R&D, and growth can be promoted by subsidizing R&D. However, it is often difficult to raise tax rates to finance new R&D subsidies. Given this reality, we considered two questions. The first question is whether governments can obtain the funds required to implement R&D subsidies by issuing debt, without raising tax rates now or in the future. Put another way, the question asks whether the government could run a Ponzi scheme, in which it perpetually rolls over its debts. The second question is whether, if the answer to the first question is yes, such a policy can actually promote economic growth.

We find the result depends on each country's characteristics. In particular, feasibility crucially depends on R&D productivity, which measures the relationship between the proportion of R&D labor and the rate of new product development. In the calibration, we find that the answer to the first question is yes for all the countries in our sample. The condition for the second question is tighter, but most countries, except Japan, Italy and Greece, satisfy it. Those countries can indeed promote long-term growth by R&D even when taxes cannot be raised. In addition, we confirmed that the result is not due to the economy being dynamically inefficient. Even though the private return on R&D can be lower than the growth rate, we find that the social rate of return of R&D is higher than the growth rate on any balanced growth path.

We re-evaluated this result in alternative settings. When population growth is explicitly introduced in the model, we find that the condition becomes even tighter for countries with slow population growth. Japan, Italy and Greece cannot now pass the condition for the first question; i.e., they cannot roll over their debts without raising taxes. In addition, Portugal fails to meet the second threshold. It cannot enhance growth by debt-financed subsidies.

The findings so far are ironic, in the sense that countries most in need of reviving growth could not be saved by this scheme. These countries' low growth implies that their R&D productivity is low, which makes it harder for them to meet the conditions. However, another specification gives those slow-growing countries some hope. We extend the baseline model to include public basic research, which the literature finds helps improve the private R&D productivity. Then, all the countries, with the possible exception of Italy, seem to be able to enhance growth by appropriately combining the public basic research expansion and R&D subsidies, using the funds obtained by debt rollover.

It is worth noting that even when countries fail to meet the second condition, they could

enhance growth for a limited time by providing unsustainable levels of subsidies. We find that the benefit could continue for a long time, to the extent that the current generation and their government may well support it, potentially explaining why such policies are implemented in some countries. We also extended the model to the cases where political constraints are relaxed and where the country is open to the international capital market. Both extensions confirm that the main qualitative results of this study continue to hold.

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Online Appendix for
Growth Promotion Policies when Taxes Cannot be Raised

Katsunori Minami Ryo Horii

This Version: November 10, 2026

A Analysis of the Baseline Model

A.1 Derivation of the Consumption of Newborns

In this section, we derive the consumption of a newly born individual (7). As explained in the main text, each individual maximizes the expected utility (3) subject to the budget constraint (4), initial condition $k_{t,t} = 0$, and the non-Ponzi game condition:

$$\lim_{T\to\infty} e^{-\int_t^T (r_{t''}+\mu)dt''} k_{s,T} \geq 0. \tag{A.1}$$

At the optimum, the non-Ponzi game condition (A.1) holds with equality, because otherwise the individual could consume more without violating it. From (4), (A.1) with equality, and the Euler equation for individuals, $\dot{c}_{s,t} = (r_t - \rho)c_{s,t}$, we obtain

$$c_{s,t} = (\rho + \mu)(k_{s,t} + h_{s,t}), \tag{A.2}$$

where $h_{s,t}$ is the present value of future labor income, or "human wealth," of an individual of generation s evaluated at time t. It is defined by

$$h_{s,t} \equiv \int_t^\infty e^{-\delta(t'-s)} w_{t'} e^{-\int_t^{t'} (r_{t''}+\mu)dt''} dt'. \tag{A.3}$$

By averaging (A.2) for all individuals, we obtain the average consumption as follows:

$$\overline{c}_t = (\rho + \mu)\left(\overline{k}_t + \overline{h}_t\right), \tag{A.4}$$

where $\overline{h}_t$ is the average human wealth, defined by

$$\overline{h}_t \equiv \frac{1}{N_t}\int_{-\infty}^t N_{s,t} h_{s,t} ds. \tag{A.5}$$

Note that, from (A.3), (A.5) and (2), $\overline{h}_t$ and the human wealth of a newborn $h_{t,t}$ are related

as follows:

$$\overline{h}_t = \frac{1}{N_t}\int_{-\infty}^{t} N_{s,t} e^{-\delta(t-s)} \int_{t}^{\infty} e^{-\delta(t'-t)} w_{t'} e^{-\int_t^{t'}(r_{t''}+\mu)dt''} dt' ds = \frac{L_t}{N_t} h_{t,t} = \frac{\beta}{\beta+\delta} h_{t,t}. \quad \text{(A.6)}$$

This relationship means that $h_{t,t} = ((\beta+\delta)/\beta)\overline{h}_t$; i.e., a newborn has $(\beta+\delta)/\beta > 1$ times the human wealth of an average individual. Equation (A.2) with $s = t$ and $k_{t,t} = 0$ implies that the consumption of a newborn is $c_{t,t} = (\rho+\mu)h_{t,t} = (\rho+\mu)(\beta+\delta)\overline{h}_t/\beta$. Eliminating $\overline{h}_t$ from (A.4) yields (7).

A.2 Derivation of the Equilibrium Dynamics

We first derive the interest rate (21). From (14), (20), and (13), the profit of a firm is $\pi_t = (1-\alpha)n_t^{(1-2\alpha)/\alpha} L_t^P$. When R&D is positive, the value of a firm is $v_t = \alpha(1-\theta)Ln_t^{(1-2\alpha)/\alpha}/A$ by (16). Its growth rate is $-(2\alpha-1)AL_t^R/(\alpha L)$ by (15). Substituting these expressions into (19) yields (21).

Next, consider the debt-GDP ratio $D_t = B_t/Y_t$. Since Y_t grows at the rate g_t, the time derivative of D_t is $\dot{D}_t = \dot{B}_t/Y_t - g_t D_t$. The government budget constraint (17) then gives

$$\dot{D}_t = (r_t - g_t)D_t + \frac{\theta\alpha}{\overline{g}(A)} g_t,$$

where we use $w_t = \alpha Y_t/L$ and $L_t^R = Lg_t/\overline{g}(A)$ from (20), (24), and (25). The first term is the change in the ratio when the primary balance is zero. The second is subsidy expenditure relative to full-production GDP. Substituting (27) yields (29).

Finally, we derive the dynamics of g_t. The goods market clears when $C_t = X_t$. Hence, (14), (24), and the labor-market condition imply $C_t/Y_t = L_t^P/L = 1 - g_t/\overline{g}(A)$. Differentiating gives $\dot{g}_t = -(\overline{g}(A) - g_t)(\dot{C}_t/C_t - g_t)$. If consumption grows more slowly than full-production GDP, fewer workers are needed in production, and the labor released into research raises the growth rate. The assets of individuals consist of government bonds and shares of firms. Using (18) and the value of a firm obtained above, we have

$$\frac{K_t}{Y_t} = D_t + \frac{\alpha(1-\theta)}{A}, \qquad \frac{C_t}{Y_t} = \frac{\overline{g}(A) - g_t}{\overline{g}(A)}. \quad \text{(A.7)}$$

Substituting these ratios and the aggregate Euler equation (8) into the expression for $\dot{g}_t$ yields (28). A higher interest rate raises consumption growth and reduces the labor available for research. A larger amount of aggregate assets has the opposite effect through the replacement of existing individuals by newborns without financial wealth, as explained in Section 3.1.

A.3 The Conventional Debt-GDP Ratio

The equilibrium dynamics of the model involve three variables, government debt B_t, the number of goods n_t, and R&D labor L_t^R, of which B_t and n_t are predetermined. In the main text, we solve them in two steps. First, we construct $D_0 = B_0/Y_0$ from n_0 and B_0, and, given D_0, we solve the dynamics of g_t and D_t in (28) and (29). Second, given n_0, we obtain the path of n_t from that of g_t, which gives the paths of $B_t = D_t Y_t$ and $L_t^R = Lg_t/\overline{g}(A)$. In this subsection, we show that the qualitative results are the same if we use the conventional debt-GDP ratio, $D_t^{\text{conv}} \equiv B_t/\text{GDP}_t$, instead of D_t in the first step.

By (22), (24), and (25), actual GDP is $\text{GDP}_t = \big(1 - (1-\alpha)g_t/\overline{g}(A)\big)Y_t$, so the two ratios are related by

$$D_t^{\text{conv}} = \frac{\overline{g}(A)D_t}{\overline{g}(A) - (1-\alpha)g_t}, \tag{A.8}$$

and they correspond one to one for a given g_t. Differentiating (A.8) with respect to time gives

$$\dot{D}_t^{\text{conv}} = \frac{\overline{g}(A)\dot{D}_t + (1-\alpha)D_t^{\text{conv}}\dot{g}_t}{\overline{g}(A) - (1-\alpha)g_t}. \tag{A.9}$$

We substitute (29) into (A.9). Eliminating D_t from the result and from (28) by (A.8) and using (26), we obtain

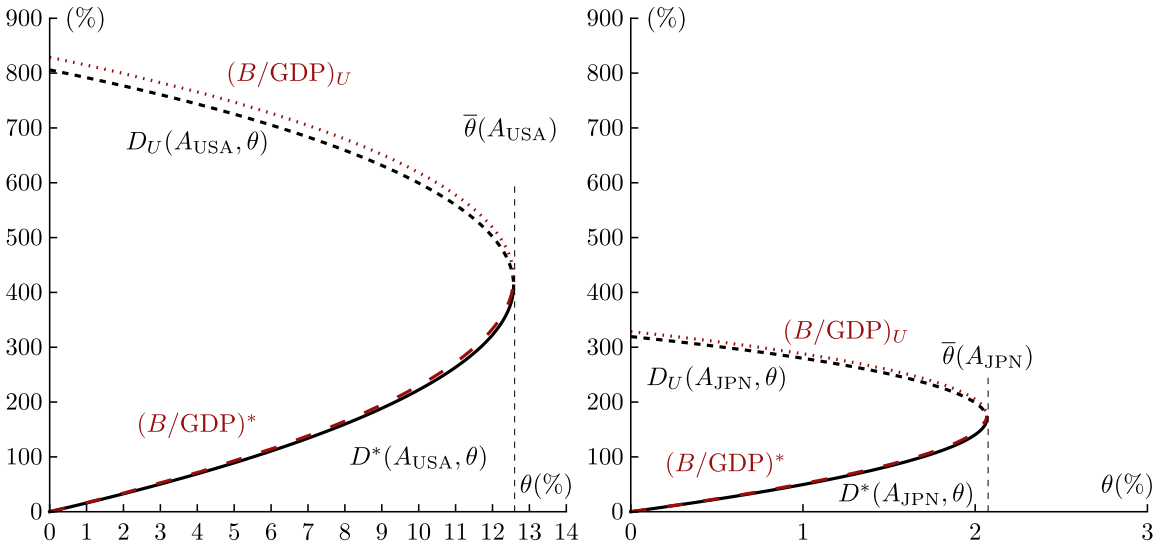
$$\begin{aligned}\dot{g}_t = {} & \big(\overline{g}(A) - g_t\big)\big(s(\theta)(g_t - \widetilde{g}(A,\theta)) - \delta + \rho\big) \\ & + (\rho+\mu)(\delta+\mu)\Big(\big(\overline{g}(A) - (1-\alpha)g_t\big)D_t^{\text{conv}} + (1-\alpha)(1-\theta)\Big),\end{aligned} \tag{A.10}$$

$$\dot{D}_t^{\text{conv}} = -s(\theta)\big(g_t - \widetilde{g}(A,\theta)\big)D_t^{\text{conv}} + \frac{\theta\alpha g_t + (1-\alpha)D_t^{\text{conv}}\,\dot{g}_t}{\overline{g}(A) - (1-\alpha)g_t}. \tag{A.11}$$

Substituting (A.10) for $\dot{g}_t$ in (A.11), we obtain a system of two first-order differential equations in g_t and D_t^{conv}.

Setting $\dot{D}_t = \dot{g}_t = 0$ in (A.9) gives $\dot{D}_t^{\text{conv}} = 0$. That is, a steady state of the system in D_t and g_t is also a steady state of the system in D_t^{conv} and g_t. The converse holds in the same way. Since the relation (A.8) maps each path of one system to a path of the other, Proposition 1 holds in either system, with the long-term BGP and the unsustainable threshold expressed in the corresponding ratio, once the initial condition is also converted, as we explain below. The thresholds of A in Propositions 2 and 3, $\overline{A}$ and $\widehat{A}$, and the subsidy rates $\overline{\theta}(A)$ and $\widehat{\theta}(A)$ are defined independently of government debt. Since the two systems describe the same economy with different variables, these propositions do not depend on the choice between D_t and D_t^{conv} either.

We use D_t in the main text for two reasons. First, the dynamics of D_t in (29) do



(a) When $A > \widehat{A}$ ($A_{\text{USA}} = 0.399$) (b) When $A < \widehat{A}$ ($A_{\text{JPN}} = 0.120$)

Figure A.1: The long-term conventional debt-GDP ratio $(B/\text{GDP})^*$ and the unsustainable threshold $(B/\text{GDP})_U$, compared with $D^*(A,\theta)$ and $D_U(A,\theta)$.

not involve $\dot{g}_t$ on the RHS, which makes the analysis much simpler. Second, the initial condition is the vertical line $D_t = D_0$ in the phase diagram. In contrast, n_0 and B_0 do not determine D_0^{conv}, because GDP_0 depends on the jump variable g_0. When the system is described in terms of g_t and D_t^{conv}, the initial condition is defined by a curve in the phase diagram. From (22), (23), (25), and (26), we can write the initial condition as

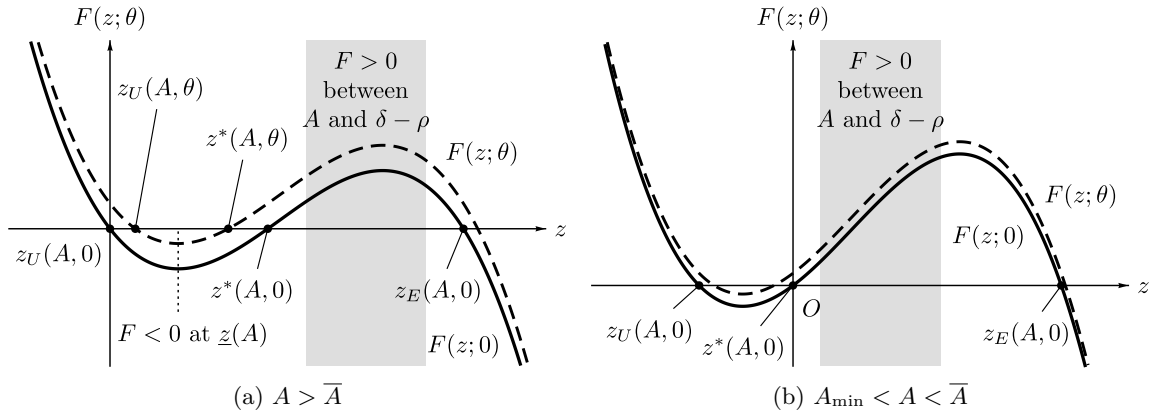
$$g_0 = \frac{A}{\alpha}\left(1 - \frac{B_0}{n_0^{(1-\alpha)/\alpha} L\, D_0^{\text{conv}}}\right), \quad g_0 \in [0, \overline{g}(A)].$$

When B_0 is zero, D_0^{conv} must also be zero from the definition $D_0^{\text{conv}} = B_0/\text{GDP}_0$. Then, the RHS becomes indeterminate (involving 0/0), which means g_0 can take any value in $[0, \overline{g}(A)]$.

The levels of the two ratios are also close. Figure A.1 compares $D^*(A,\theta)$ and $D_U(A,\theta)$ in the lower panels of Figure 8 with the corresponding values of D_t^{conv}, $(B/\text{GDP})^*$ and $(B/\text{GDP})_U$. The latter are higher because actual GDP is smaller than full-production GDP, but only by at most 3.4%. Along the transition paths in the third row of Figure 9, D_t^{conv} is higher than D_t by at most 3.6%.

A.4 Steady States with R&D

We now examine the properties of steady states in the region with R&D. Let z denote the $g - r$ gap at a candidate steady state in this region. By (27), we have $z = s(\theta)\big(g - \widetilde{g}(A,\theta)\big)$. Eliminating D from the two steady-state loci (30) and (31) then shows that the steady-state



(a) $A > \overline{A}$ (b) $A_{\min} < A < \overline{A}$

Figure A.2: The steady-state condition $F(z;\theta) = 0$ for different levels of R&D productivity. The shaded area is the interval between A and $\delta - \rho$, on which $F(z;\theta)$ is positive.

gap must satisfy $F(z;\theta) = 0$, where[86]

$$F(z;\theta) \equiv (\rho+\mu)(\delta+\mu)\frac{\theta\alpha\overline{g}(A)}{1-\theta} - z\Big((A-z)(\delta-\rho-z) - (\rho+\mu)(\delta+\mu)\Big). \tag{A.12}$$

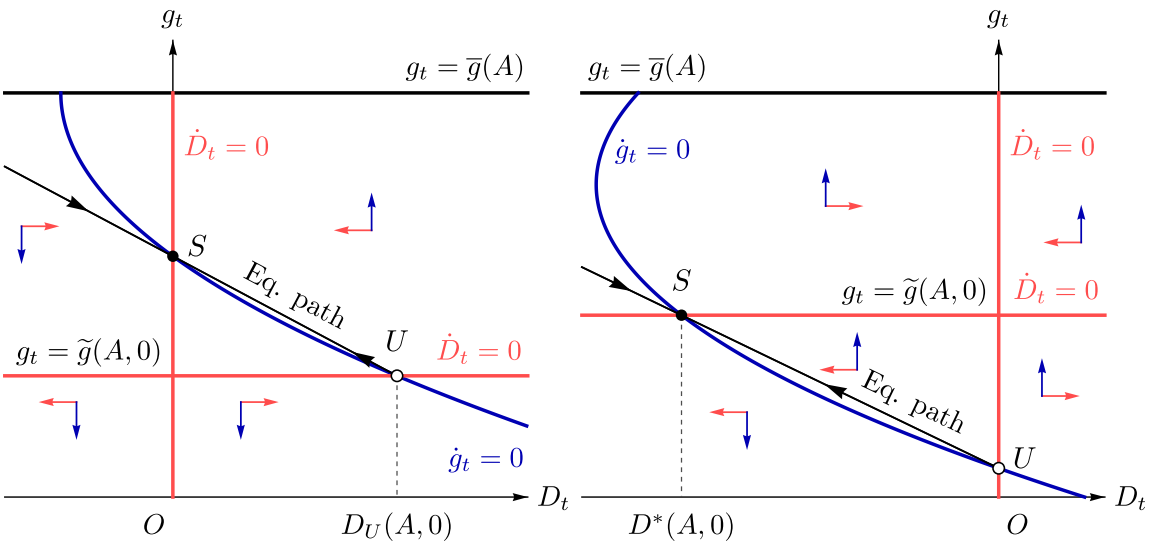
A root of $F(z;\theta) = 0$ corresponds to a steady state with R&D only if the associated growth rate $g = \widetilde{g}(A,\theta) + z/s(\theta)$ satisfies $0 < g \leq \overline{g}(A)$, which implies $z \leq A$. Moreover, the aggregate Euler equation (8) implies that any feasible steady state satisfies $z < \delta - \rho$.[87] The product $(A-z)(\delta-\rho-z)$ in (A.12) is nonpositive between A and $\delta-\rho$, so $F(z;\theta)$ is positive throughout this interval. Since $F(z;\theta)$ is strictly concave for $z > \max\{A, \delta-\rho\}$ and tends to minus infinity as $z \to \infty$, there is exactly one root $z_E > \max\{A, \delta-\rho\}$. It corresponds to $E = (D_E, g_E)$, which is excluded by the labor resource constraint. Thus, at most two roots can correspond to steady states with R&D.

As illustrated in Figure A.2, there can be two roots of $F(z;\theta) = 0$ in the R&D region. When two steady states exist, the one with the larger z has the higher growth rate, since g is increasing in z. It also has the lower debt-GDP ratio. For $\theta > 0$, $F(0;\theta) > 0$ implies that the two roots other than z_E have the same sign. On each side of $z = 0$, the $\dot{D}_t = 0$ locus given by (31) is decreasing in g, as illustrated in Figure 5(a) for positive debt and Figure A.7(c) for negative debt. The case of $\theta = 0$ is examined below. We denote the higher-growth, lower-debt steady state by $S = (D^*, g^*)$ and the other by $U = (D_U, g_U)$, and write z_U for the gap at U. Appendix A.7 shows why S is the long-term BGP and D_U is the unsustainable threshold.

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[86]Strictly speaking, the $\dot{D}_t = 0$ locus (31) is undefined at $z = 0$ when $\theta = 0$, because this case corresponds to $g = \widetilde{g}(A,0)$. However, this case also satisfies the original steady-state conditions $\dot{g}_t = \dot{D}_t = 0$ for an appropriate value of D_t and is included among the roots of $F(z;0) = 0$.

[87]On a BGP, $\dot{C}/C = g$, so (8) gives $z = \delta - \rho - (\rho+\mu)(\delta+\mu)K/C$. When $z > 0$, (29) implies $D \geq 0$, and hence $K > 0$ by (A.7). When $z \leq 0$, the inequality follows from $\delta > \rho$.



(a) $A > \overline{A}$ and $\theta = 0$ (b) $A_{\min} < A < \overline{A}$ and $\theta = 0$

Figure A.3: Stationary loci without subsidies. Panels (a) and (b) compare the two sides of $A = \overline{A}$.

Without subsidies, the $\dot{D}_t = 0$ locus consists of the two lines $D = 0$ and $g = \widetilde{g}(A, 0)$, and $F(z; 0) = 0$ has the root $z = 0$, which corresponds to the second line. The other roots satisfy $(A - z)(\delta - \rho - z) = (\rho + \mu)(\delta + \mu)$. The intersection on the line $g = \widetilde{g}(A, 0)$ has $D = \alpha(1/\overline{A} - 1/A)$ by (30). Thus, S is the intersection on $D = 0$ when $A > \overline{A}$, as in Figure A.3(a), whereas U is the intersection on $D = 0$ when $A < \overline{A}$, as in panel (b).

To express the growth rate at S in terms of the $g - r$ gap, let $z^* \equiv g^* - r^* = s(\theta)\big(g^* - \widetilde{g}(A, \theta)\big)$. Substituting z^* into $F(z^*; \theta) = 0$ and using (27) to eliminate θ, we obtain

$$g^* = \overline{g}(A)\frac{z^*(\delta - \rho - z^*) + (1 - \alpha)(\rho + \mu)(\delta + \mu)}{z^*(\delta - \rho - z^*) + (\rho + \mu)(\delta + \mu)}. \tag{A.13}$$

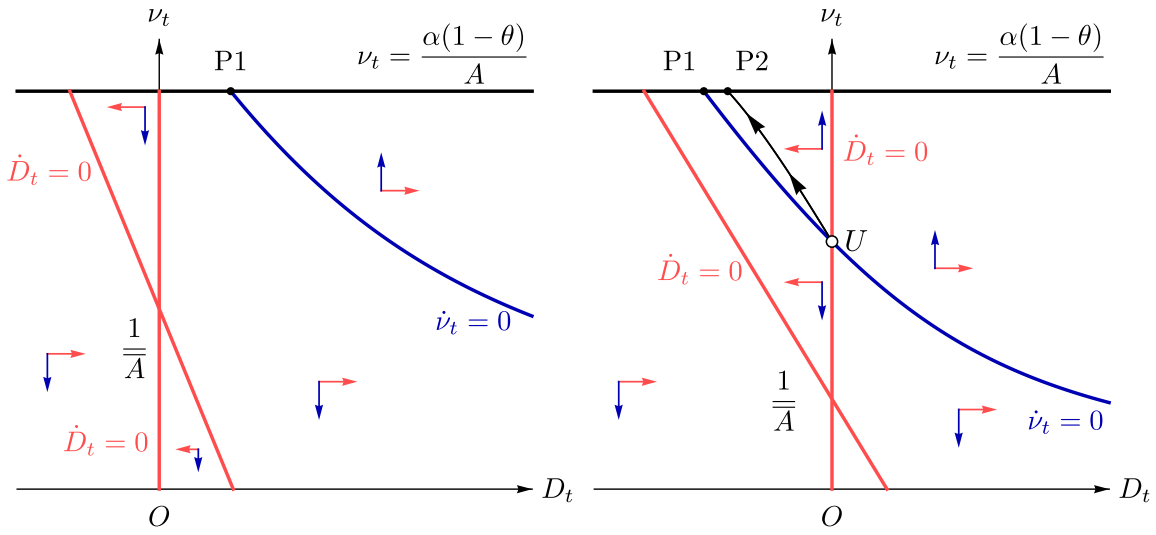
Since (A.13) is increasing in $z^*(\delta - \rho - z^*)$ for feasible steady states, $z^* \leq 0$ implies $g^* \leq (1 - \alpha)\overline{g}(A)$, so at least a fraction α of labor is allocated to production at S.

When research productivity is too low for that intersection to have positive growth, the corresponding steady state is in the region without R&D, examined next.

A.5 Dynamics in the Region without R&D

When research is absent, $g_t = 0$, $L_t^P = L$, and the free-entry condition need not hold with equality. Let $\nu_t = v_t n_t / Y_t$ be the aggregate value of firms relative to full-production GDP. The region is $0 < \nu_t \leq \alpha(1 - \theta)/A$, and free entry holds with equality at its upper boundary. The subsidy affects this boundary but does not affect the dynamics inside, because the government pays no subsidy.

Consumption and full-production GDP are constant in this region. Setting $\dot{C}_t = 0$ in



(a) $A/(1-\theta) > A_{\min}$ (b) $A/(1-\theta) < A_{\min}$

Figure A.4: Phase diagram in D_t and $\nu_t \equiv v_t n_t / Y_t$ in the region without R&D. In panel (a), $A/(1-\theta) > A_{\min}$ and there is no steady state. In panel (b), $A/(1-\theta) < A_{\min}$ and U lies inside the region.

(8) gives

$$r_t = (\rho+\mu)(\delta+\mu)(D_t+\nu_t) - (\delta-\rho). \tag{A.14}$$

The government budget constraint and the no-arbitrage condition then imply

$$\dot{D}_t = \big((\rho+\mu)(\delta+\mu)(D_t+\nu_t) - (\delta-\rho)\big)D_t, \tag{A.15}$$

$$\dot{\nu}_t = \big((\rho+\mu)(\delta+\mu)(D_t+\nu_t) - (\delta-\rho)\big)\nu_t - (1-\alpha). \tag{A.16}$$

The $\dot{D}_t = 0$ locus consists of $D_t = 0$ and $D_t + \nu_t = 1/\overline{A}$. The latter is the line on which $r_t = 0$, where $\dot{\nu}_t = -(1-\alpha) < 0$. Therefore, a steady state has $D_t = 0$. Solving $\dot{\nu}_t = 0$ on this axis gives $U = (0, \nu_U)$, where $\nu_U = \alpha / A_{\min}$ and

$$A_{\min} \equiv \frac{-(\delta-\rho) + \sqrt{(\delta-\rho)^2 + 4(1-\alpha)(\rho+\mu)(\delta+\mu)}}{2(1-\alpha)/\alpha}. \tag{A.17}$$

Thus, U lies inside the region without R&D if $A/(1-\theta) < A_{\min}$, as in Figure A.4(b), whereas no steady state exists in this region if $A/(1-\theta) > A_{\min}$. The interest rate at U is positive by $r_U \nu_U = 1 - \alpha$.

The two regions meet at $g_t = 0$ and $\nu_t = \alpha(1-\theta)/A$. The point P1, where the $\dot{g}_t = 0$ locus reaches this boundary, is also the intersection of the $\dot{\nu}_t = 0$ locus with the boundary. Its debt-GDP ratio is positive if and only if $A/(1-\theta) > A_{\min}$, by (30) and (A.17).

A.6 Properties of Equilibrium Paths

Two regions in the phase diagram. This subsection combines the dynamics with R&D in Appendix A.4 and those without R&D in Appendix A.5. An equilibrium path may move between these regions, with debt, consumption, and firm value remaining continuous at their common boundary. We therefore draw the two regions in one phase diagram, Figure A.7 in Appendix A.7. The upper half of each panel describes the R&D region in the (D_t, g_t) plane, while the lower half describes the region without R&D in the (D_t, ν_t) plane. The common boundary is given by $g_t = 0$ in the former and $\nu_t = \alpha(1-\theta)/A$ in the latter, with the same D_t. P1 is the point where the $\dot{g}_t = 0$ and $\dot{\nu}_t = 0$ loci reach this boundary, as established in Appendix A.5.

Definition of an equilibrium path. An equilibrium path is a path along which households and firms optimize with perfect foresight, the government budget constraint holds, and all markets clear for all $t \geq 0$. At time $t = 0$, D_0 is predetermined. In addition, once the initial growth rate $g_0 \in [0, \overline{g}(A)]$ is given (if $g_0 = 0$, the initial value of firms ν_0 must also be specified), the equilibrium conditions determine the entire path. However, this path may not continue forever. In finite time, the implied growth rate may exceed $\overline{g}(A)$, implying negative consumption; the value of firms may become negative, which contradicts the usual assumption of limited shareholder liability; or the amount of government debt may become infinite, which means there is no continuing path of D_t after that. We can rule out such a path as an equilibrium path because it is incompatible with households' optimization. Once they know their plan will fail in a finite time, they should revise the consumption plan backward from that time, thereby changing the initial consumption level (which determines g_0). Therefore, the initial value g_0 was not compatible with rational expectations. If there is no g_0 that is compatible with rational expectations, then the economy has no equilibrium.

Linear representation and the candidate path. The quadratic terms of (28) and (29) are $-s(\theta)g_t^2$ and $-s(\theta)g_t D_t$. Define the three-dimensional vector $\boldsymbol{\gamma}_t = (Y_t/Y_0)^{s(\theta)}(D_t, g_t, 1)$. Then

$$\dot{\boldsymbol{\gamma}}_t = \mathbf{M}\boldsymbol{\gamma}_t, \quad \text{where}$$

$$\mathbf{M} = \begin{pmatrix} \frac{\overline{g}(A)}{1-\theta} & \frac{\theta\alpha}{\overline{g}(A)} & 0 \\ \overline{g}(A)(\rho+\mu)(\delta+\mu) & A+\delta-\rho+\frac{2\overline{g}(A)}{1-\theta} & \overline{g}(A)\Big(\frac{\alpha(1-\theta)(\rho+\mu)(\delta+\mu)}{A} - \frac{\overline{g}(A)}{1-\theta} - \delta + \rho\Big) \\ 0 & s(\theta) & 0 \end{pmatrix}. \tag{A.18}$$

Lemma A.1 (Linear Representation of the Dynamics) *A path of* (28) *and* (29) *yields*

a solution of (A.18). *Conversely, if* $\boldsymbol{\gamma}_t$ *solves* (A.18) *and* $\gamma_{3,t} > 0$*, its component ratios* $(\gamma_{1,t}/\gamma_{3,t}, \gamma_{2,t}/\gamma_{3,t})$ *solve the original system.*

Proof. From $\dot{Y}_t/Y_t = g_t$, the third component satisfies $\dot{\gamma}_{3,t} = s(\theta)\gamma_{2,t}$. Differentiating the first two components and substituting (29) and (28) gives the first two rows of (A.18). The converse follows by differentiating the component ratios.

The general solution is $\boldsymbol{\gamma}_t = e^{\mathbf{M}t}\boldsymbol{\gamma}_0$. An eigenvector can be normalized to $(D, g, 1)$, because the second and third rows of (A.18) rule out a nonzero eigenvector whose third component is zero. For such an eigenvector, the third row gives the eigenvalue $s(\theta)g$, while the first two rows yield the steady-state conditions. Thus, if z is a root of $F(z;\theta) = 0$, the corresponding growth rate is $g = \widetilde{g}(A,\theta) + z/s(\theta)$, and the associated eigenvalue is $s(\theta)g = z + s(\theta)\widetilde{g}(A,\theta)$. One of these roots corresponds to the excluded intersection E. The eigenvalue associated with E has the largest real part, since the three roots of $F(z;\theta) = 0$ sum to $A + \delta - \rho$, whereas the root corresponding to E exceeds both A and $\delta - \rho$ by Appendix A.4.

When both steady states have R&D, the three eigenvalues are distinct, and diagonalizing $\mathbf{M}$ gives

$$\boldsymbol{\gamma}_t = \chi^* e^{s(\theta)g^* t}(D^*, g^*, 1) + \chi_U e^{s(\theta)g_U t}(D_U, g_U, 1) + \chi_E e^{s(\theta)g_E t}(D_E, g_E, 1), \tag{A.19}$$

where the coefficients sum to one and are uniquely determined by (D_0, g_0). These coefficients are the coordinates of the initial point relative to the three vertices in Figure A.5. Since E has the largest eigenvalue, a path with $\chi_E > 0$ eventually approaches $g_E > \overline{g}(A)$ and violates the labor resource constraint. If $\chi_E < 0$, the third component of $\boldsymbol{\gamma}_t$ eventually becomes negative, which is incompatible with its definition. Hence, a path remaining feasible in the R&D region indefinitely must satisfy $\chi_E = 0$ and therefore lie on the straight line through S and U.

The lower end of the candidate path. This line cannot be extended indefinitely toward negative debt without violating the labor resource constraint. Its slope is $-\overline{g}(A)(\rho + \mu)(\delta + \mu)/z_E$, which extends to $\theta = 0$ by continuity. For $\theta > 0$, this follows from (31) at S and U and from the product $z^* z_U z_E$ of the three roots of $F(z;\theta) = 0$, which equals the constant term of $F(z;\theta)$. The line reaches $g_t = \overline{g}(A)$ at

$$\underline{D}(A,\theta) = -\frac{\alpha(1-\theta)}{A} - \frac{\alpha}{z_E - A}, \tag{A.20}$$

which follows from (30) at S, the sum of the three roots, and the value of $F(z;\theta)$ at $z = A$.

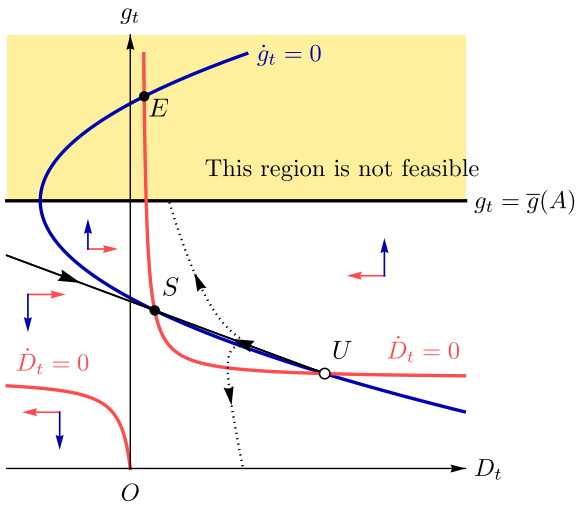


Figure A.5: The steady states S and U, the excluded intersection E, and the straight line through S and U. Paths off the line leave the R&D region through one of its boundaries.

This value is below $-\min\{\alpha/(\delta-\rho), \alpha/(\rho+\mu)\}$, so the initial condition in Proposition 1 lies to its right.[88]

Exclusion of alternative paths. By the definition of an equilibrium path, a path that violates the labor resource constraint, makes firm value negative, or makes debt infinite in finite time is not an equilibrium path. The following lemma shows where paths lead in each region. The proof of Proposition 1 combines its parts to exclude all paths except those that converge to S.

Lemma A.2 (Paths in the Two Regions)

(a) Suppose that both steady states have R&D. The saddle path is a straight line that starts at U and passes through S, which we call the equilibrium line. Unless the initial point is exactly at U, all the paths that start from the equilibrium line converge to S.

(b) A path with R&D that does not converge to a steady state in that region leaves the region in finite time. Either g_t exceeds $\overline{g}(A)$, implying negative consumption, or g_t falls to zero and the path enters the region without R&D. If both steady states have R&D, this occurs for every path with R&D that starts neither on the equilibrium line nor at U.

(c) Along a path without R&D that does not start at U, one of the following occurs in

[88] If the growth rates at both other intersections are at least $\widetilde{g}(A,\theta)$, their sum implies $z_E - A \leq \delta - \rho$, giving the first bound. Otherwise, the gap at S and D^* are both nonpositive. The production labor share at S is then at least α, by (A.13). At E, the condition $F(z;\theta) = 0$ implies that $(g-r)(g-r-\delta+\rho)$ exceeds $(\rho+\mu)(\delta+\mu)$. The positive solution at equality is $\delta+\mu$, so z_E exceeds $\delta+\mu$. Multiplying this quantity by the production labor share and dividing by $(\rho+\mu)(\delta+\mu)$ gives the distance from D^* to the lower end of the line. The second bound follows.

finite time: the path enters the R&D region at P1 or to its right, firm value becomes negative, or debt becomes infinite.

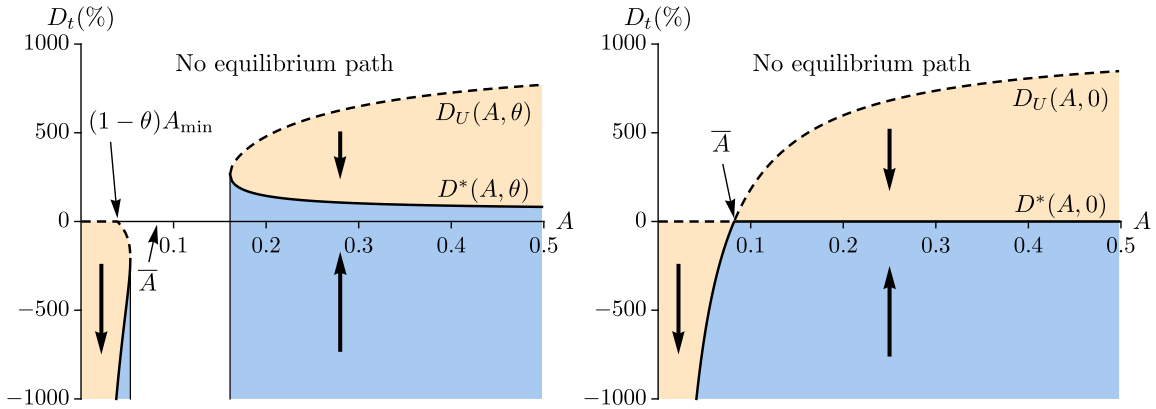
Proof. Part (a) follows from (A.19). On the straight line through S and U, χ_E is zero, and the coefficient χ^* of the higher-growth component is positive precisely when $D_t < D_U$, by $g^* > g_U$ and $D^* < D_U$. A path that starts on the equilibrium line other than at U therefore moves monotonically toward S.

For part (b), the linear representation shows that a path remaining in the R&D region indefinitely must exclude the eigenvector associated with E. If the other two eigenvalues are complex, the third component of their real combination changes sign, so the path cannot remain feasible. If they are distinct and real, the path converges to the higher-growth solution unless it stays at the lower-growth solution or reaches a boundary. For a path reaching $g_t = \overline{g}(A)$ from below, $D_t \geq -\alpha(1-\theta)/A$ must hold at that point by (28). If the inequality is strict, $\dot{g}_t$ is positive, and the labor resource constraint is immediately violated. At equality, $\dot{D}_t = \alpha > 0$ holds, so $\dot{g}_t$ becomes positive immediately afterward. If both steady states have R&D, a path that starts off the straight line through S and U does not converge to a steady state, because its coefficient χ_E is not zero. A path that starts on the extension of the equilibrium line beyond U does not converge to a steady state either, because its coefficient χ^* is negative, so that the third component of $\boldsymbol{\gamma}_t$ eventually becomes negative.

For part (c), differentiating (A.14) with respect to time and substituting (A.15) and (A.16) give

$$\dot{r}_t = r_t(r_t + \delta - \rho) - (1-\alpha)(\rho+\mu)(\delta+\mu).$$

The positive stationary interest rate r_U satisfies $r_U(r_U + \delta - \rho) = (1-\alpha)(\rho+\mu)(\delta+\mu)$. If $r_t < r_U$, the interest rate becomes negative in finite time and stays negative. Then $\dot{\nu}_t$ is smaller than $-(1-\alpha)$ by (A.16), so ν_t becomes negative in finite time unless the path leaves the region earlier. If $r_t = r_U$, the interest rate remains constant, and $D_t + \nu_t$ stays at ν_U. When $D_t > 0$, debt increases and firm value becomes negative, whereas $D_t < 0$ implies that firm value rises until the free-entry boundary is reached. If $D_t = 0$, the path starts at $U = (0, \nu_U)$ and stays there. If $r_t > r_U$, the interest rate rises and becomes infinite in finite time, and so does $D_t + \nu_t$ by (A.14). When $D_t \leq 0$, debt does not increase, so firm value rises until the free-entry boundary is reached. When $D_t > 0$, the ratio ν_t/D_t satisfies $d(\nu_t/D_t)/dt = -(1-\alpha)/D_t$ by (A.15) and (A.16). As long as the path stays in the region, ν_t is bounded, so that debt becomes infinite at that time and ν_t/D_t becomes zero. Since debt increases, integrating the last equation shows that ν_t is at most $1-\alpha$ times the



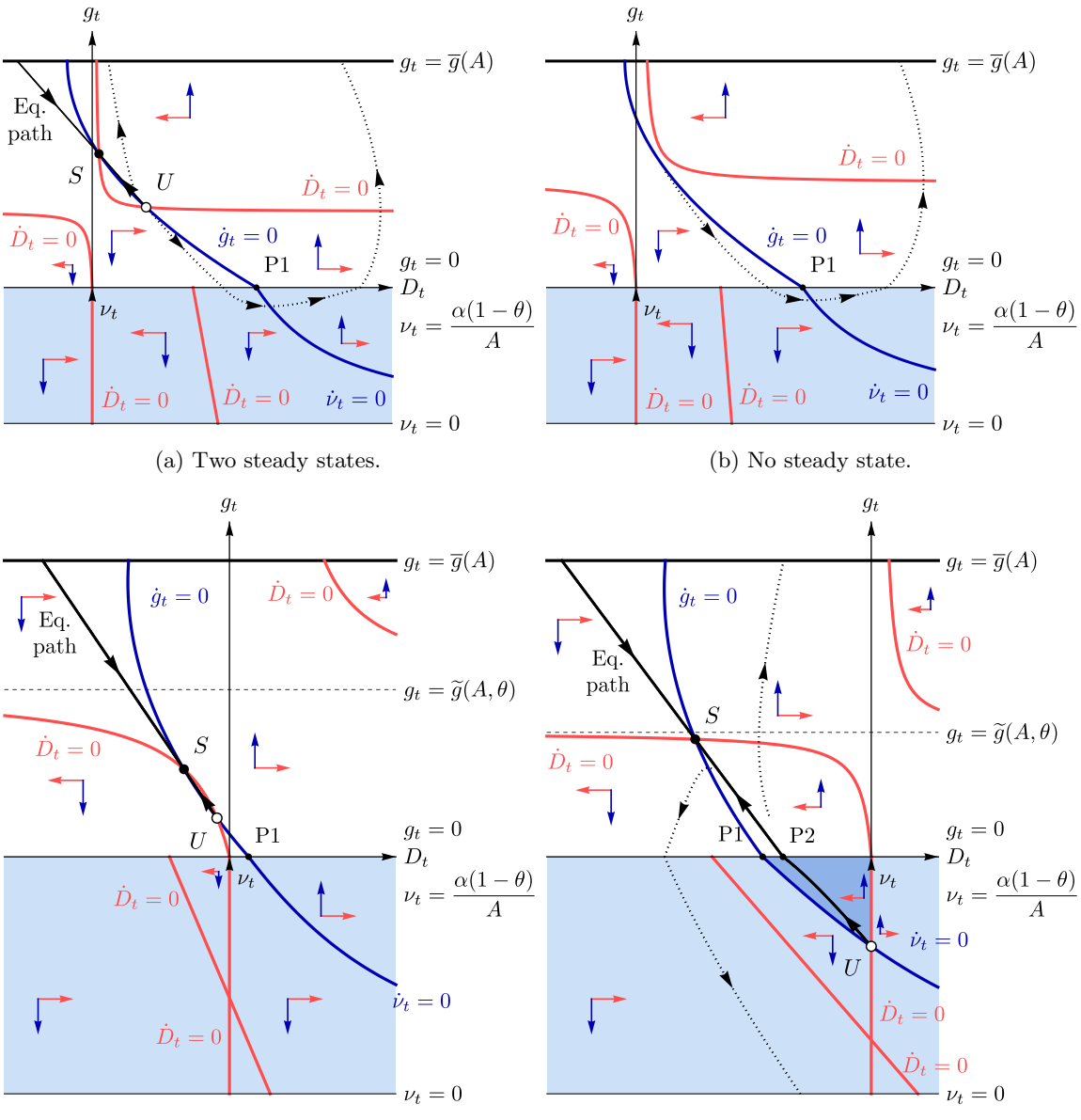
(a) $\theta = 5\%$ (b) $\theta = 0$

Figure A.6: The long-term BGP $D^*(A,\theta)$ (solid) and the unsustainable threshold $D_U(A,\theta)$ (dashed) against research productivity. D_t falls if $D^*(A,\theta) < D_t < D_U(A,\theta)$ and rises if $D_t < D^*(A,\theta)$. Given A and θ, there is no equilibrium path if D_t lies in the white area. The other parameters are those in Table 1.

time remaining until debt becomes infinite. Hence, unless the path reaches the free-entry boundary, firm value becomes negative before debt becomes infinite, or it becomes zero at that time. At the free-entry boundary, $\dot{\nu}_t$ has the same sign as $\dot{g}_t$ at $g_t = 0$, so entry into the R&D region is possible only at P1 or to its right.

A.7 Proof of Proposition 1

Figure A.6(a) illustrates the initial debt-GDP ratios that admit an equilibrium path for each level of R&D productivity A, holding $\theta = 5\%$ fixed. For sufficiently large A, the economy corresponds to Figure A.7(a), where $0 < D^* < D_U$ and equilibrium paths exist for initial debt-GDP ratios in the shaded area. As A decreases, D^* and D_U merge when $\overline{\theta}(A) = \theta$, and no steady state exists for intermediate values of A, including $\overline{A}$, as in panel (b). When A decreases further, two steady states reappear with $D^* < D_U \leq 0$, so no equilibrium path exists for $D_0 \geq 0$. Both steady states have R&D if $A > (1-\theta)A_{\min}$, as in panel (c), whereas U lies in the region without R&D and $D_U = 0$ if $A < (1-\theta)A_{\min}$, as in panel (d). Figure A.6(b) shows the limit as θ tends to zero. Both ends of the white area converge to $\overline{A}$, so the two parts of the shaded area join. For $A > \overline{A}$, D_U rises and D^* falls to zero. For $A < \overline{A}$, D_U converges to zero, even when $A > A_{\min}$ and U has R&D. The rest of this subsection proves what the arrows in Figure A.6 show. Given A and θ, the economy has a unique equilibrium path that converges to D^* from any initial debt-GDP ratio below D_U. It has no equilibrium path if the initial ratio is above D_U, or if there is no steady state. We can see where each path leads in Figure A.7 from the directions in which paths can cross the loci and the boundaries.



(a) Two steady states.

(b) No steady state.

(c) Both steady states have R&D.

(d) U is in the region without R&D.

Figure A.7: Equilibrium and excluded paths in the two regions. In panels (c) and (d), $A < \overline{A}$, and negative debt represents government net assets. In panel (d), the path from U enters the R&D region at P2. In the darker set, D_t falls and ν_t rises.

Proof of part (a). We count the steady states in the two regions. On the $\dot{g}_t = 0$ locus in the R&D region, let $z_t = g_t - r_t = s(\theta)(g_t - \widetilde{g}(A, \theta))$. Substituting (30) into (29) shows that $F(z_t; \theta)$ equals $s(\theta)\overline{g}(A)(\rho + \mu)(\delta + \mu)\dot{D}_t$, so F has the sign of $\dot{D}_t$ along the locus. At P1, where $g_t = 0$, the interest rate r_t equals $\overline{g}(A)/(1-\theta)$, which is positive, and the debt dynamics (29) reduce to $\dot{D}_t = r_t D_t$. The sign of F at P1 therefore coincides with that of D_t. At the upper boundary $g_t = \overline{g}(A)$, we have $z_t = A$ and $F(A; \theta) > 0$. Exactly one root of $F(z; \theta) = 0$ lies above $z = A$ and is excluded (Appendix A.4). Hence, apart from the knife-edge case in which two solutions coincide, the R&D region contains zero or two steady states when P1 has positive debt, and exactly one when P1 has negative debt.

By Appendix A.5, P1 has positive debt if and only if $A/(1-\theta) > A_{\min}$, and the region without R&D then has no steady state. If $A/(1-\theta) < A_{\min}$, P1 has negative debt, so the R&D region contains one steady state, S, and the region without R&D contains the other, $U = (0, \nu_U)$. Therefore, except for knife-edge cases, the economy has two steady states or none.

Proof of part (b). First suppose both steady states have R&D, as in Figure A.7(a) or (c). For $D_0 < D_U$, the equilibrium line selects a unique initial growth rate, and the resulting path converges to S. By (A.20) and the initial condition of the proposition, the labor resource constraint holds along this path. The transversality condition and non-Ponzi game condition of households hold along this path, because the path converges to S, where the gap $g^* - r^*$ is below $\delta - \rho$ (Appendix A.4), so that the return on annuities $r^* + \mu$ exceeds the growth rate $g^* - \delta$ of each individual's labor income. By Lemma A.2, along every other path, the labor resource constraint is violated, firm value becomes negative, or debt becomes infinite, unless the path enters the R&D region at P1 or to its right. As illustrated in Figure A.7(a) and (c), such a returning path crosses the upper labor boundary, which violates the resource constraint. Hence, the selected path is the unique equilibrium for $D_0 < D_U$, and no equilibrium exists for $D_0 > D_U$.

Next suppose $U = (0, \nu_U)$ lies in the region without R&D, as in Figure A.7(d). This occurs when $A < (1-\theta)A_{\min}$, so that D_U is zero. In this case, the smallest root of $F(z;\theta) = 0$ corresponds to a negative growth rate, and the straight line through the associated intersection and S reaches $g_t = 0$ at P2. Along this line, g_t rises toward S, so $\dot{g}_t$ is positive at P2, and P2 lies to the right of P1. P2 also has negative debt, by $\dot{D}_t \geq 0$ at $D_t = 0$ in the R&D region, whereas debt falls along the selected line toward S. Hence, P2 lies on the upper edge of the darker set in Figure A.7(d). No path enters this set from outside, so the path through P2 originates from U and eventually converges to S.

Along this path, debt decreases from zero to the debt ratio at P2, so each initial debt ratio between these values selects exactly one point. For lower initial debt ratios satisfying the proposition's lower bound, the economy begins directly on the R&D segment of the same equilibrium path. As in the first case, the transversality condition and non-Ponzi game condition of households hold along this path, because the path converges to S. All other paths violate firm valuation, or enter the R&D region away from P2 and therefore cannot remain feasible, as shown by Lemma A.2 and Figure A.7(d). No equilibrium exists for $D_0 > 0$. This completes the proof of part (b).

Proof of part (c). Suppose the economy has no steady state, as in Figure A.7(b). Then $\dot{D}_t$ is positive along the entire $\dot{g}_t = 0$ locus, as at P1. By Lemma A.2, a path with R&D

must cross the upper labor boundary or enter the region without R&D. In the latter case, firm value becomes negative, debt becomes infinite in finite time, or the path returns to the R&D region at P1 or to its right. Such a returning path eventually crosses the upper labor boundary, as illustrated in Figure A.7(b). Hence, no path can satisfy all equilibrium conditions indefinitely, proving part (c).

A.8 Proof of Proposition 2

We use the stationary condition (A.12) to determine when a long-term BGP with $g^* > r^*$ exists and to derive the upper bound on the subsidy rate. Since θ enters $F(z;\theta)$ only through its constant term, an increase in θ shifts the cubic upward without changing the locations of its extrema.

The condition on research productivity. We first show that a long-term BGP with $g^* > r^*$ exists for some $\theta \geq 0$ if and only if $A > \overline{A}$. When $\theta = 0$, $F(z;0) = 0$ has a root at $z = 0$, and the other two roots satisfy $(A - z)(\delta - \rho - z) = (\rho + \mu)(\delta + \mu)$. The largest root lies above A and is infeasible, as shown in Appendix A.4. The smaller root is positive if and only if $A > \overline{A}$, because the left-hand side is strictly decreasing on $0 < z < \min\{A, \delta - \rho\}$ and exceeds the right-hand side at $z = 0$ precisely when $A > \overline{A}$. Thus, when $A > \overline{A}$, the long-term BGP has $g^* > r^*$ without subsidies, and this property persists for sufficiently small positive subsidies.

When $A < \overline{A}$, the smaller root is negative. As illustrated in Figure A.2(b), increasing θ shifts $F(z;\theta)$ upward, so the two remaining real roots, when they exist, move toward each other in the negative-z region Hence, no positive subsidy can generate a long-term BGP with $g^* > r^*$.

The upper bound on the subsidy. Suppose that $A > \overline{A}$, as illustrated in Figure A.2(a). When $\theta = 0$, the two feasible roots are $z = 0$ and a positive root corresponding to S. As θ increases, the upward shift of $F(z;\theta)$ moves the smaller root rightward and the larger root leftward. They eventually merge at the local minimum of $F(z;\theta)$ when $\theta = \overline{\theta}(A)$.

Since $F_z(z;\theta)$ does not depend on θ, the location of this minimum is the smaller solution of

$$3z^2 - 2(A + \delta - \rho)z + (\delta - \rho)(A - \overline{A}) = 0. \tag{A.21}$$

We denote this solution by $\underline{z}(A)$. It is independent of θ and lies between 0 and $(\delta - \rho)/2$, because the left-hand side of (A.21) is positive at $z = 0$ and negative at $z = (\delta - \rho)/2$. The upper bound is determined by $F(\underline{z}(A); \overline{\theta}(A)) = 0$. Using (A.21) to simplify this condition and solving for $\overline{\theta}(A)$ yields (34). For $\theta < \overline{\theta}(A)$, the two feasible roots remain distinct, and

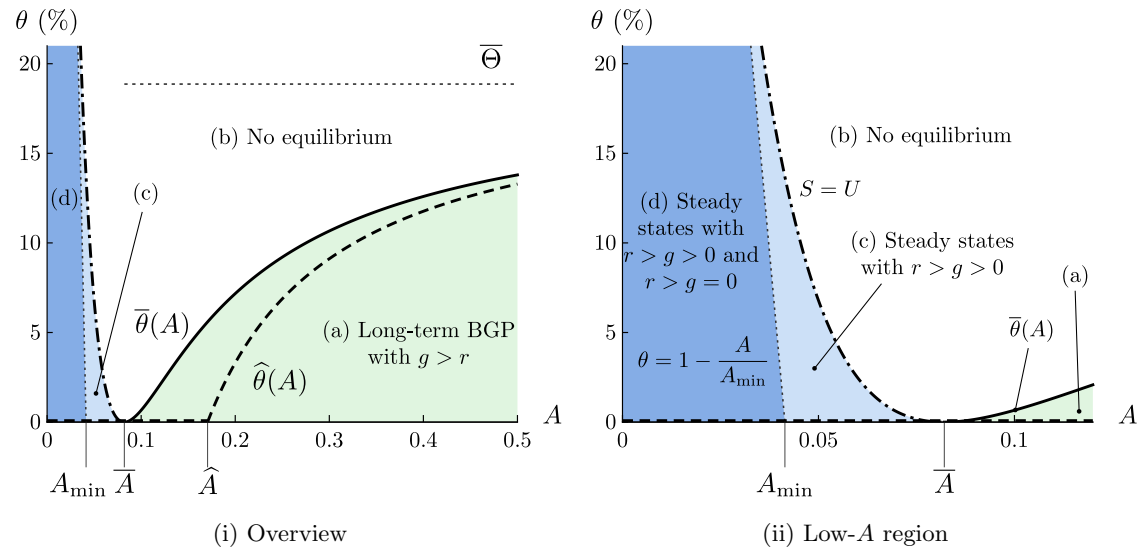


Figure A.8: Steady states in the (A, θ) plane, with the low-A range enlarged.

no feasible root remains when θ exceeds $\overline{\theta}(A)$. At $\theta = \overline{\theta}(A)$, the two roots coincide. This is the knife-edge case excluded in Proposition 1(a), so the coincident steady state is not a long-term BGP.

Comparative statics and the limiting upper bound. We next show that $\overline{\theta}(A)$ is strictly increasing in A. At the contact, $F(\underline{z}(A); \overline{\theta}(A)) = 0$ and $F_z(\underline{z}(A); \overline{\theta}(A)) = 0$. Totally differentiating the first condition with respect to A gives $F_A + F_z \underline{z}'(A) + F_\theta \overline{\theta}'(A) = 0$. From $F_z = 0$ at the local minimum, we obtain $\overline{\theta}'(A) = -F_A/F_\theta$. By (A.12), $F_\theta = (1-\alpha)(\rho+\mu)(\delta+\mu)A/(1-\overline{\theta}(A))^2 > 0$. Using $F = 0$, we also obtain $F_A = -\underline{z}(A)\big((\rho+\mu)(\delta+\mu) + \underline{z}(A)(\delta-\rho-\underline{z}(A))\big)/A < 0$. Therefore, $\overline{\theta}'(A) > 0$.

To find the limit, divide (A.21) by A. Since $\underline{z}(A)$ stays between 0 and $(\delta-\rho)/2$, the terms divided by A vanish as $A \to \infty$, yielding $\underline{z}(A) \to (\delta-\rho)/2$. Substituting this limit into $F(\underline{z}(A); \overline{\theta}(A)) = 0$ after dividing by A gives

$$\lim_{A\to\infty} \overline{\theta}(A) = \overline{\Theta} \equiv \frac{\delta-\rho}{\delta-\rho+4(1-\alpha)\overline{A}}. \tag{A.22}$$

When research productivity is lower. If $A < \overline{A}$, no steady state with $g^* > r^*$ exists for any $\theta \geq 0$, as established above. When R&D is conducted and $\theta > 0$, any steady state must have $g^* < r^*$, which implies a negative debt-GDP ratio by (29). When both steady states have R&D, $D^* < D_U < 0$, as in Figure A.7(c). If $A/(1-\theta) < A_{\min}$, U is instead the steady state without R&D and $D_U = 0$, as in panel (d). The two steady states with R&D may also merge when the subsidy rate reaches their contact condition. These configurations are summarized in Figure A.8. By Proposition 1, an economy with positive initial debt cannot follow an equilibrium path in these cases.

A.9 Proof of Proposition 3 and Corollary 1

Proof of Corollary 1. As illustrated in Figure A.2(a), an increase in θ shifts $F(z;\theta)$ upward, reducing the larger feasible root z^*. By (A.13), long-run growth increases with θ when $z^* > (\delta-\rho)/2$ and decreases when $z^* < (\delta-\rho)/2$.

Proof of Proposition 3. By (A.13), the subsidy affects long-run growth only through $z^* = g^* - r^*$. Growth is increasing in $z^*(\delta-\rho-z^*)$ and is highest at $z^* = (\delta-\rho)/2$. Evaluating (A.13) at this gap gives $\widehat{g}(A)$ in (36). The supply-side relation (27) then yields $\widehat{\theta}(A)$ in (35). By (33) and (35), this subsidy rate is positive if and only if $A > \widehat{A}$. If $A \leq \widehat{A}$, the gap reaches $(\delta-\rho)/2$ only at $\theta = \widehat{\theta}(A) \leq 0$, so that no positive subsidy raises long-run growth by Corollary 1.

We next verify that $\widehat{\theta}(A)$ is feasible when $A > \widehat{A}$. By Appendix A.8, $\underline{z}(A)$ lies below $(\delta-\rho)/2$. Moreover, by (A.12), the derivative $F_z((\delta-\rho)/2;\theta)$ equals $(\rho+\mu)(\delta+\mu)+(\delta-\rho)^2/4$, which is positive. Thus, $F(z;\theta)$ is increasing at $z=(\delta-\rho)/2$, so this gap is the larger of the two feasible roots at the subsidy rate $\widehat{\theta}(A)$, and the two roots are distinct at this rate. Hence, $\widehat{\theta}(A) < \overline{\theta}(A)$, and the growth-maximizing subsidy is attainable on the long-term BGP. Finally, differentiating (35) shows that $\widehat{\theta}(A)$ is strictly increasing in A. As $A \to \infty$, it converges to $\overline{\Theta}$ in (A.22).

A.10 Estimation of A

This appendix derives the estimator for A. From (25) and (26),

$$A = \frac{\alpha}{1-\alpha}\frac{g_t}{L_t^R/L} = \frac{\alpha}{1-\alpha}\frac{w_t L/\mathrm{GDP}_t}{w_t L_t^R/\mathrm{GDP}_t} g_t. \tag{A.23}$$

Using (20), (22), and (23), we obtain

$$\frac{w_t L}{\mathrm{GDP}_t} = \frac{\alpha A}{A-\alpha g_t}. \tag{A.24}$$

Substituting (A.24) into (A.23) and solving for A yields (37).

A.11 Proof of Proposition 4

Under each subsidy rate, the part of an equilibrium path with positive R&D lies on a straight line (Appendix A.7).

Let $\theta_1 < \theta_2$ be two subsidy rates. Since debt does not jump, it suffices to compare the lines with θ_1 and θ_2 at the debt ratio when the rate changes, as in Figure A.9.

The slope of each line, given in Appendix A.6, is inversely proportional to the gap z_E

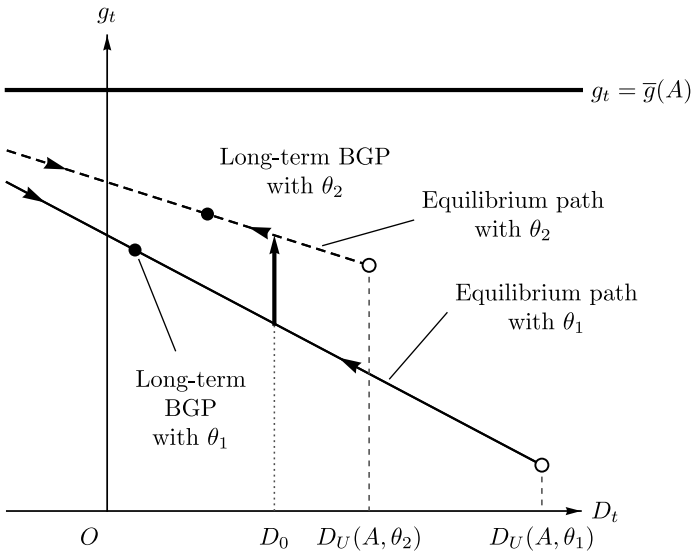


Figure A.9: The equilibrium lines with two subsidy rates. The line with $\theta_2 > \theta_1$ lies above the line with θ_1.

at E. As Figure A.2(a) shows, a higher subsidy shifts $F(z;\theta)$ upward and raises z_E, so that the line with θ_2 is flatter. Each line reaches $g_t = \overline{g}(A)$ at $\underline{D}(A,\theta)$ in (A.20), whose two terms rise toward zero with the subsidy. The line with θ_2 therefore reaches the upper labor boundary closer to the vertical axis. Since it is also flatter, it lies above the line with θ_1 wherever it is below this boundary. This range includes the debt ratio when the rate changes, because the economy has an equilibrium path under θ_2. An unanticipated increase in the subsidy therefore raises g_t, and the new path has positive R&D. The same comparison proves the result for a decrease, in which g_t may fall to zero.

A.12 Transition Paths and Deviations from the Balanced Growth Path without Subsidies

When the subsidy changes permanently and unexpectedly at $t = 0$, D_t does not jump, whereas g_t jumps to the new equilibrium path. We first consider the case in which both S and U have R&D, as in the simulations in Section 4.5. Omitting the arguments (A,θ) of their coordinates, the line through them is

$$g_t = g^* + \frac{g_U - g^*}{D_U - D^*}(D_t - D^*). \tag{A.25}$$

On the line, $\chi_E = 0$ in (A.19), and the initial weights are fixed by D_0. Taking component ratios gives

$$D_t - D^* = \frac{(D_U - D^*)(D_0 - D^*)}{(D_U - D_0)e^{s(\theta)(g^* - g_U)t} + D_0 - D^*}, \tag{A.26}$$

$$\frac{Y_t}{Y_0} = e^{g^* t}\left(\frac{D_U - D_0 + (D_0 - D^*)e^{-s(\theta)(g^* - g_U)t}}{D_U - D^*}\right)^{1/s(\theta)}. \tag{A.27}$$

The second expression follows from $\gamma_{3,t} = (Y_t/Y_0)^{s(\theta)}$. Thus, convergence eventually occurs at the rate $s(\theta)(g^* - g_U)$, which tends to zero as the two steady states coincide. Under the parameters of Section 4.5, this rate is 0.0027 for $A = 0.399, \theta = 11.74\%$, and 0.0007 for $A = 0.120, \theta = 2\%$. If U is in the region without R&D, the matrix solution in Appendix A.6 gives the segment with research and is joined to the path in (D, ν) at P2.

We derive the percentage deviations shown in Figure 9. The economy starts from $D_0 = 0$, so Y_t is given by (A.27) with this initial condition. Without subsidies, $Y_t^{\theta=0} = Y_0 e^{g^*(A,0)t}$. Hence,

$$\frac{Y_t - Y_t^{\theta=0}}{Y_t^{\theta=0}} = e^{(g^*(A,\theta) - g^*(A,0))t}\left(\frac{D_U(A,\theta) - D^*(A,\theta)e^{-s(\theta)(g^* - g_U)t}}{D_U(A,\theta) - D^*(A,\theta)}\right)^{1/s(\theta)} - 1, \tag{A.28}$$

where g^* and g_U in the exponent are evaluated under the subsidy θ.

Some labor is used for research rather than consumption. By (A.7), consumption is $C_t = (1 - g_t/\overline{g}(A))Y_t$, so

$$\frac{C_t - C_t^{\theta=0}}{C_t^{\theta=0}} = \frac{\overline{g}(A) - g_t}{\overline{g}(A) - g^*(A,0)}\frac{Y_t}{Y_t^{\theta=0}} - 1. \tag{A.29}$$

Similarly, GDP is $\mathrm{GDP}_t = \big(1 - (1-\alpha)g_t/\overline{g}(A)\big)Y_t$ by (22), so

$$\frac{\mathrm{GDP}_t - \mathrm{GDP}_t^{\theta=0}}{\mathrm{GDP}_t^{\theta=0}} = \frac{\overline{g}(A) - (1-\alpha)g_t}{\overline{g}(A) - (1-\alpha)g^*(A,0)}\frac{Y_t}{Y_t^{\theta=0}} - 1. \tag{A.30}$$

The factor in front of $Y_t/Y_t^{\theta=0}$ in this last expression lies between one and the corresponding factor for consumption. Thus, the GDP deviation lies between those of full-production GDP and consumption.

B Social Rate of Return of R&D and Dynamic Efficiency

B.1 Proof of Proposition 5

Here, we derive the social rate of return of R&D. As explained in Definition 1, we discretize time in the model with a step size of Δt. We later take the limit of $\Delta t \to 0$. We assume that the economy is on the BGP before time t. The state variable n_t at time t is also on the BGP because it is predetermined by time t. Then, we marginally increase L_t^R by $dL_t^R > 0$,

which reduces the production of consumption goods at time t (denoted by $dX_t < 0$) but increases the number of goods at $t + \Delta t$ (denoted by $dn_{t+\Delta t} > 0$). Then, at $t + \Delta t$, we marginally decrease $L^R_{t+\Delta t}$ by $dL^R_{t+\Delta t} < 0$ so that $n_{t+2\Delta t}$ returns to the original BGP. The production of consumption goods at $t+\Delta t$ will be higher than that on the BGP. We denote the difference by $dX_{t+\Delta t} > 0$.

In the following, we derive the relationships among $dn_{t+\Delta t}$, dL^R_t, and $dL^R_{t+\Delta t}$. The discrete version of (15) at the time between t and $t + \Delta t$ is

$$n_{t+\Delta t} = n_t + aL^R_t n_t \Delta t, \tag{B.1}$$

where $a = A/L$. Taking the total derivative of Equation (B.1) while keeping n_t constant yields

$$dn_{t+\Delta t} = an_t \Delta t \, dL^R_t. \tag{B.2}$$

Similar to (B.1), the discrete version of (15) from time $t + \Delta t$ to time $t + 2\Delta t$ is

$$n_{t+2\Delta t} = n_{t+\Delta t} + aL^R_{t+\Delta t} n_{t+\Delta t} \Delta t. \tag{B.3}$$

By taking the total derivative of (B.3), while keeping $n_{t+2\Delta t}$ unchanged from the BGP, we obtain

$$0 = \left(1 + aL^R_{t+\Delta t}\Delta t\right) dn_{t+\Delta t} + an_{t+\Delta t}\Delta t \, dL^R_{t+\Delta t}. \tag{B.4}$$

We substitute $n_{t+\Delta t}$ in (B.1) and $dn_{t+\Delta t}$ in (B.2) into (B.4), which yields

$$0 = (1 + aL^R_{t+\Delta t}\Delta t)dL^R_t + (1 + aL^R_t \Delta t)dL^R_{t+\Delta t}. \tag{B.5}$$

Recall that the economy is on the BGP before the manipulation of L^R_t, where L^R_t is constant for all t. Let this constant value be L^{R*}. Then, $L^R_t = L^{R*} + dL^R_t$ and $L^R_{t+\Delta t} = L^{R*} + dL^R_{t+\Delta t}$. Substituting these values into (B.5) gives

$$0 = (1 + aL^{R*}\Delta t)(dL^R_t + dL^R_{t+\Delta t}) + 2a\Delta t \, dL^R_t \, dL^R_{t+\Delta t}. \tag{B.6}$$

This equation implies that the ratio of the changes in L^R_t and $L^R_{t+\Delta t}$ is

$$-\frac{dL^R_{t+\Delta t}}{dL^R_t} = 1 + \frac{2a\Delta t \, dL^R_{t+\Delta t}}{1 + aL^{R*}\Delta t}. \tag{B.7}$$

Now, we consider the changes in the final output X_t in periods t and $t + \Delta t$. Note that in period t, n_t is still on the BGP. Therefore, from (14), the changes in X_t and L^R_t are

related by

$$dX_t = \frac{\partial X_t}{\partial L_t^R} dL_t^R = n_t^{\frac{1-\alpha}{\alpha}} \left(-dL_t^R\right). \tag{B.8}$$

In period $t + \Delta t$, $n_{t+\Delta t}$ deviates from the BGP by $dn_{t+\Delta t}$. Therefore, the change in $X_{t+\Delta t}$ can be expressed as

$$\begin{aligned} dX_{t+\Delta t} &= \frac{\partial X_{t+\Delta t}}{\partial n_{t+\Delta t}} dn_{t+\Delta t} + \frac{\partial X_{t+\Delta t}}{\partial L_{t+\Delta t}^R} dL_{t+\Delta t}^R \\ &= \frac{1-\alpha}{\alpha} n_{t+\Delta t}^{\frac{1-\alpha}{\alpha}-1} (L - L_{t+\Delta t}^R) dn_{t+\Delta t} - n_{t+\Delta t}^{\frac{1-\alpha}{\alpha}} dL_{t+\Delta t}^R \\ &= n_{t+\Delta t}^{\frac{1-\alpha}{\alpha}} \left[\frac{1-\alpha}{\alpha} (L - L_{t+\Delta t}^R) \frac{a\Delta t \, dL_t^R}{1 + aL_t^R \Delta t} - dL_{t+\Delta t}^R \right]. \end{aligned} \tag{B.9}$$

where we use (B.1) and (B.2) in the third line.

We obtain the social return of R&D investment by substituting (B.8) and (B.9) into Definition 1:[89]

$$\begin{aligned} r^s \Delta t &= \frac{dX_{t+\Delta t}}{-dX_t} - 1 \\ &= \left(\frac{n_{t+\Delta t}}{n_t} \right)^{\frac{1-\alpha}{\alpha}} \left(\frac{1-\alpha}{\alpha} (L - L_{t+\Delta t}^R) \frac{a\Delta t}{1 + aL_t^R \Delta t} - \frac{dL_{t+\Delta t}^R}{dL_t^R} \right) - 1. \end{aligned} \tag{B.10}$$

On the RHS of the above equation, we can use the Taylor expansion as follows:

$$\left(\frac{n_{t+\Delta t}}{n_t} \right)^{\frac{1-\alpha}{\alpha}} = (1 + a\, L_t^R \Delta t)^{\frac{1-\alpha}{\alpha}} = 1 + \frac{1-\alpha}{\alpha} aL_t^R \Delta t + O((\Delta t)^2),$$

where $O((\Delta t)^2)$ is the collection of terms that are of order $(\Delta t)^2$ and higher. We can also eliminate $dL_{t+\Delta t}^R / dL_t^R$ via (B.7). Applying both to (B.10), we obtain

$$r^s = \frac{1-\alpha}{\alpha} aL_t^R + \frac{\frac{1-\alpha}{\alpha} a(L - L_{t+\Delta t}^R) + 2a\, dL_{t+\Delta t}^R}{1 + aL_t^R \Delta t} + O(\Delta t). \tag{B.11}$$

Recall that Δt is infinitesimally small. In the continuous-time limit, $\Delta t \to 0$. Additionally, we consider marginal perturbations of dL_t^R and $dL_{t+\Delta t}^R$ from the BGP. Therefore, $dL_{t+\Delta t}^R \to 0$. Applying these to (B.11) and $aL = A$, we obtain

$$r^s = \frac{1-\alpha}{\alpha} A = \overline{g}(A).$$

Since L_t^R cannot exceed L, the growth rate g_t in (25) is at most $\overline{g}(A)$, which equals r^s.

[89] In Definition 1, r^s is defined as the limit of the first line of (B.10) as $\Delta t \to 0$. We calculate the limit after we rewrite (B.10) as (B.11).

B.2 Proof of Proposition 6

Let g^* be the growth rate in aggregate consumption on the original BGP, and let L^{R*} and L^{P*} be the constant amounts of R&D labor and production labor, respectively. Suppose that from this BGP, the path of aggregate consumption, $C_t = X_t$, is increased by dX_t from time t to time $t + \Delta t$, where Δt is a small time interval. In the following, we show that the aggregate consumption from $t + \Delta t$ cannot be maintained on the original BGP.

As explained in the proof of Proposition 5 in Appendix B.1, the only way to increase aggregate consumption at time t is to reduce R&D labor and increase production labor. Then, at $t + \Delta t$, $n_{t+\Delta t}$ should be lower than the BGP value, which means that the production technology of the final goods is inferior to that of the original BGP case. To keep $X_{t+\Delta t}$ at the original BGP value despite a smaller $n_{t+\Delta t}$, we need to allocate more workers to production, which means that $L^P_{t+\Delta t} > L^{P*}$.

Below, we examine the dynamics of the economy from $t + \Delta t$ in continuous time. For $t' \geq t + \Delta t$, the aggregate consumption should be the same as that of the original BGP, which is growing at the rate of g^*. By taking the log of (14) and then differentiating with respect to t', we obtain

$$\frac{\dot{X}_{t'}}{X_{t'}} = \frac{1-\alpha}{\alpha}\frac{\dot{n}_{t'}}{n_{t'}} + \frac{\dot{L}^P_{t'}}{L^P_{t'}} = g^* \text{ for all } t' \geq t + \Delta t.$$

Using (15), (25) and (23), the above equation can be rewritten as

$$\begin{aligned}\frac{\dot{L^P_{t'}}}{L^P_{t'}} &= g^* - \frac{1-\alpha}{\alpha}A\frac{L^R_{t'}}{L} \\ &= \frac{1-\alpha}{\alpha}A\frac{L^{R*} - L^R_{t'}}{L} \\ &= \frac{1-\alpha}{\alpha}A\frac{L^P_{t'} - L^{P*}}{L}.\end{aligned}$$

Therefore, we have the following autonomous differential equation for $L^P_{t'}$ for $t' \geq t+\Delta t$:

$$\dot{L^P_{t'}} = \frac{1-\alpha}{\alpha}A\frac{L^P_{t'} - L^{P*}}{L}L^P_{t'}. \tag{B.12}$$

As shown in Figure B.1, the RHS of (B.12) is positive when $L^P_{t'} > L^{P*}$. Recall that $L^P_{t+\Delta t} > L^{P*}$. Therefore, $\dot{L^P_{t'}} > 0$ for all $t' \geq t + \Delta t$, and $L^P_{t'}$ exceeds the total labor supply L in a finite period. Therefore, such a path is not feasible.

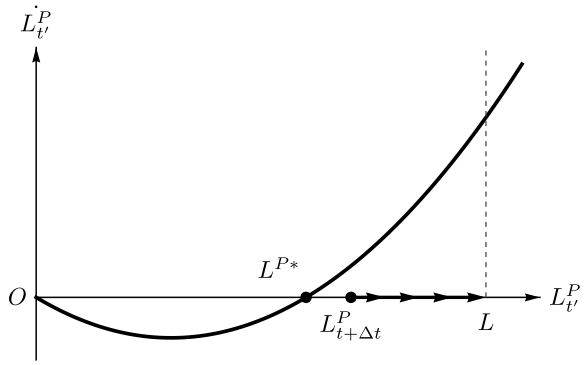


Figure B.1: Movement of L^P_t after $t' = t + \Delta t$

C Analysis of the Second-Generation Model

This appendix provides the details of the analysis of the second-generation model in Section 6. We first prove Lemma 1 and derive the equilibrium dynamics. We then examine the steady states and prove Propositions 7 and 8. Finally, we describe the details of the calibration in Section 6.4 and show the relationship with the baseline model discussed in Section 6.5.

C.1 Proof of Lemma 1

Preliminaries

We first derive three expressions used in the proof. First, the price of the good of industry j is $p_s(j) = (w_s - f v_s^{\text{new}})/\alpha$, and its demand is $x_s(j) = q_s(j)^{\alpha/(1-\alpha)} p_s(j)^{-1/(1-\alpha)} X_s$ (Section 6.1). Hence, the profit of the leader is

$$\pi_s(j) = (1-\alpha)\left(\frac{w_s - f v_s^{\text{new}}}{\alpha}\right)^{-\frac{\alpha}{1-\alpha}} X_s\, q_s(j)^{\frac{\alpha}{1-\alpha}}. \tag{C.1}$$

Second, from (43), the quality at $s \geq t$ is

$$q_s(j) = q_t(j) \exp\left(\int_t^s \left(\omega_{\text{in}} \ell_u^R(j) + \omega_{\text{out}} \frac{L_u^R}{n_u}\right) du\right), \tag{C.2}$$

where the spillovers from other industries are common to all industries. Third, the leader loses its position when an outsider obtains a new core technology, which occurs at the rate $a\ell_u^R(j)$, or when the industry becomes obsolete, which occurs at the rate ε. Therefore, the value of the leader is

$$v_t(j) = \int_t^\infty \pi_s(j) \exp\left(-\int_t^s \left(r_u + a\ell_u^R(j) + \varepsilon\right) du\right) ds. \tag{C.3}$$

Proof of Part (a)

Suppose that the leader of industry j uses a unit of effective labor for research. This research does not change the rate at which the leader is replaced, because this rate depends only on the research of outsiders by (C.3). It raises the quality of the good at the rate ω_{in}, and by (C.1) and (C.2), all future profits rise in the same proportion. Hence, the value of the leader increases at most at the rate $\alpha\omega_{\text{in}}v_t(j)/(1-\alpha)$, and it increases less if outsiders respond by conducting more research in the industry. The cost of this research is $(1-\theta)w_t - fv_t^{\text{new}}$, because the leader also obtains the ideas of the worker. An outsider who pays the same cost obtains $av_t(j)$, so the free-entry condition (42) implies $av_t(j) \leq (1-\theta)w_t - fv_t^{\text{new}}$. By the assumption of the lemma, $\alpha\omega_{\text{in}}/(1-\alpha)$ is less than a, so the benefit of the research is smaller than its cost. Therefore, the leader does not conduct research, regardless of whether outsiders conduct research in the industry.

Proof of Part (b)

By part (a), only outsiders conduct research in each industry, and their research $\ell_u^R(j)$ appears both in the rate of replacement in (C.3) and in the growth of quality in (C.2). Substituting (C.1) and (C.2) into (C.3) gives

$$\begin{aligned} v_t(j) = q_t(j)^{\frac{\alpha}{1-\alpha}} \int_t^\infty (1-\alpha) \left(\frac{w_s - fv_s^{\text{new}}}{\alpha}\right)^{-\frac{\alpha}{1-\alpha}} X_s \\ \times \exp\left(\int_t^s \Big(\frac{\alpha\omega_{\text{out}}}{1-\alpha}\frac{L_u^R}{n_u} - r_u - \varepsilon\Big)du\right) \\ \times \exp\left(-\Big(a - \frac{\alpha\omega_{\text{in}}}{1-\alpha}\Big)\int_t^s \ell_u^R(j)du\right) ds. \end{aligned} \tag{C.4}$$

Only $q_t(j)$ and the last exponential factor differ across industries. In the last exponential factor, the research of outsiders lowers the value of the leader through replacement at the rate a and raises it through quality growth at the rate $\alpha\omega_{\text{in}}/(1-\alpha)$. Under the assumption $\omega_{\text{in}} < a(1-\alpha)/\alpha$, the former effect dominates, so that this factor is at most one.

We use this property to compare two industries j and j' with $q_t(j) \geq q_t(j')$. We show that $v_t(j)/q_t(j)^{\alpha/(1-\alpha)} \leq v_t(j')/q_t(j')^{\alpha/(1-\alpha)}$, with strict inequality if outsiders conduct research in industry j before they start research in industry j'. Consider the first date at or after t at which outsiders conduct research in industry j'. Until this date, $q_s(j) \geq q_s(j')$ continues to hold by (C.2), and the last exponential factor in (C.4) is one for j' and at most one for j. If there is no such date, the inequality follows from (C.4). Otherwise, at this date, the free-entry condition (42) holds with equality in industry j', so that $v_s(j') \geq v_s(j)$, and hence the inequality holds at this date. Splitting the integral in (C.4) at this date, we obtain the inequality at t. In particular, if $q_t(j) = q_t(j')$, the values of the two leaders are

equal.

Now suppose that $q_0(j)$ is the same for all industries and that two industries j and j' have different qualities at some date. Consider the last date until which their qualities are equal. Right after this date, outsiders conduct research in both industries, in only one of them, or in neither.[90] If they conduct research in neither industry, the qualities remain equal by (C.2). If they conduct research in only one of them, the strict inequality above contradicts the equality of the values at this date. If they conduct research in both, the free-entry condition (42) holds with equality in both industries, and the values of the two leaders and their time derivatives are equal. Since the value of each leader satisfies $\dot{v}_t(j) = \big(r_t + a\ell_t^R(j) + \varepsilon\big)v_t(j) - \pi_t(j)$ by (C.3), we obtain $a\big(\ell_t^R(j) - \ell_t^R(j')\big)v_t(j) = \pi_t(j) - \pi_t(j')$. Hence, by (C.1) and (C.2), the ratio $q_t(j)/q_t(j')$ follows a differential equation whose right-hand side is zero when the ratio is one, so that the ratio remains one. In the other two cases, the qualities remain equal for a while after this date, which contradicts the definition of this date. Therefore, $\ell_t^R(j)$ and $q_t(j)$ are the same for all industries at every t, including new industries, which start with the average quality of the existing industries.

C.2 Derivation of Equilibrium Dynamics

First suppose vertical R&D is conducted. The free-entry condition (44) and the effective subsidy rate (52) give $av_t = (1-\theta^e)(w_t - fv_t)$. By (46), $v_t = \alpha(1-\theta^e)n_t^{(1-\alpha)/\alpha}q_t/a$, and its growth rate is $g_t - g_L$. The profit-value ratio is $(1-\alpha)aL_t^P/((1-\theta^e)\alpha n_t)$. Substituting these expressions and $aL_t^P/n_t = \kappa(g_L+\varepsilon) - g_{q,t}/\lambda$ into (50) yields the supply-side relation (51).

Differentiating $D_t = B_t/Y_t$ gives $\dot{D}_t = (r_t - g_t)D_t + \theta w_t L_t^R/Y_t$. Since the subsidy $\theta w_t L_t^R$ equals θ^e times the cost of vertical R&D, $(w_t - fv_t)L_t^R$, the expenditure ratio is $\alpha\theta^e g_{q,t}/\overline{g_q}(\kappa, g_L)$. Using (51) then gives (54). For the growth rate, the goods market and the production equations imply $C_t/Y_t = 1 - g_{q,t}/\overline{g_q}(\kappa, g_L)$. Hence, $\dot{g}_{q,t} = -(\overline{g_q} - g_{q,t})(\dot{C}_t/C_t - g_t)$. The aggregate Euler equation is unchanged, except that $\beta - \mu = g_L$. In addition, the asset market gives

$$\frac{K_t}{Y_t} = D_t + \frac{\alpha(1-\theta^e)}{\kappa(g_L+\varepsilon)}. \tag{C.5}$$

[90] We consider equilibria in which, for each industry, the set of dates with research consists of finitely many intervals in any bounded period.

Substituting these expressions yields (53). For reference, the two stationary loci are

$$\dot{D}_t = 0: \quad D_t = \frac{\alpha\theta^e}{\overline{g_q}(\kappa, g_L)} \frac{g_{q,t}}{g_t - r_t}, \tag{C.6}$$

$$\dot{g}_{q,t} = 0: \quad D_t = \frac{\overline{g_q}(\kappa, g_L) - g_{q,t}}{\overline{g_q}(\kappa, g_L)(\rho + \mu)(\delta + \beta)} \big(\delta - \rho + g_L - (g_t - r_t)\big) - \frac{\alpha(1 - \theta^e)}{\kappa(g_L + \varepsilon)}, \tag{C.7}$$

where $g_t - r_t$ is the affine function of $g_{q,t}$ in (51). The first expression is used when $g_t \neq r_t$. At $\theta = 0$, the debt locus consists of $D_t = 0$ and $g_t = r_t$.

Next suppose vertical R&D is absent. Then $g_{q,t} = 0$, $g_t = g_L/\alpha$, and $C_t = Y_t$. Define $\nu_t = n_t v_t / Y_t$ as in Appendix A.5. The free-entry condition requires

$$0 < \nu_t \leq \frac{\alpha(1 - \theta^e)}{\kappa(g_L + \varepsilon)}, \tag{C.8}$$

with equality on the boundary. The aggregate Euler equation gives $g_t - r_t = \delta - \rho + g_L - (\rho + \mu)(\delta + \beta)(D_t + \nu_t)$. Substituting this into the government budget and the valuation of firms gives

$$\dot{D}_t = \big((\rho + \mu)(\delta + \beta)(D_t + \nu_t) - (\delta - \rho + g_L)\big) D_t, \tag{C.9}$$

$$\dot{\nu}_t = \big((\rho + \mu)(\delta + \beta)(D_t + \nu_t) - \delta + \rho + \varepsilon\big)\nu_t - (1 - \alpha). \tag{C.10}$$

The subsidy affects only the boundary, because none is paid inside this region. The R&D dynamics have the same common quadratic term as in Appendix A.6, with coefficient $(1-\alpha)s(\theta^e)/(\alpha\lambda)$ in place of $s(\theta)$. Thus, the linear representation applies to $(D_t, g_{q,t})$. The dynamics without vertical R&D must instead be described in (D_t, ν_t), because $g_{q,t}$ is fixed at zero.

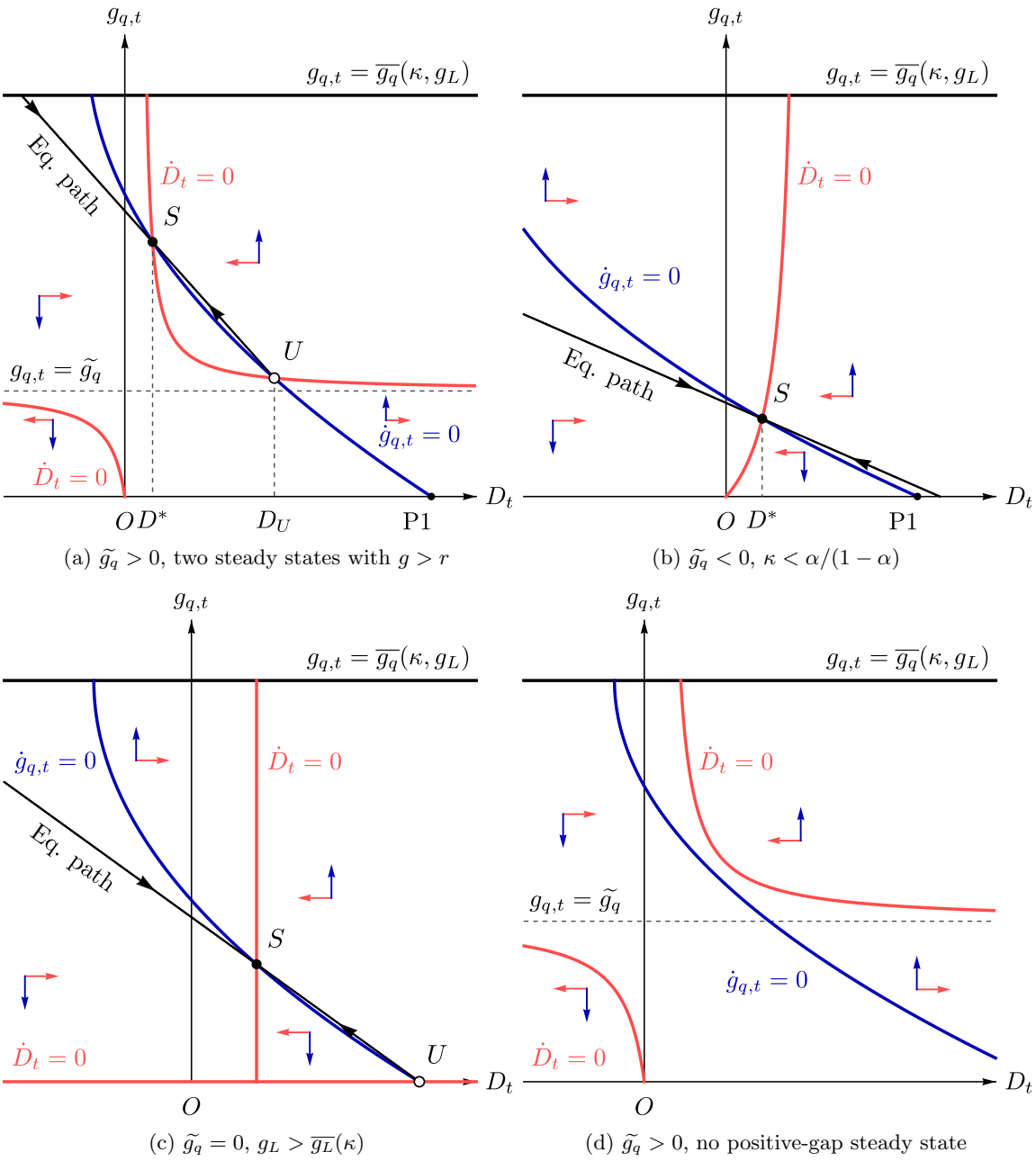
C.3 Steady States with Vertical R&D

Figure C.1 shows the stationary loci in the (D, g_q) plane. The $\dot{g}_q = 0$ locus opens to the right, and the $\dot{D} = 0$ locus is a hyperbola unless its asymptote is at $g_q = 0$. We first examine the economy without subsidies, where the debt locus consists of $D = 0$ and $g_q = \widetilde{g_q}(\kappa, g_L, 0)$.

The steady state on $D = 0$ satisfies $g > r$ if the growth locus crosses the axis above the $g = r$ line. Substituting this line into (C.7) gives the condition (55). Equivalently,

$$\kappa > \overline{\kappa}(g_L) \equiv \frac{(\rho + \mu)(\delta + \mu + g_L)}{(g_L + \varepsilon)(\delta - \rho + g_L)} - 1. \tag{C.11}$$

This threshold is strictly decreasing in g_L, and its inverse is $\overline{g_L}(\kappa)$.



(a) $\widetilde{g}_q > 0$, two steady states with $g > r$

(b) $\widetilde{g}_q < 0$, $\kappa < \alpha/(1-\alpha)$

(c) $\widetilde{g}_q = 0$, $g_L > \overline{g_L}(\kappa)$

(d) $\widetilde{g}_q > 0$, no positive-gap steady state

Figure C.1: The stationary loci with vertical R&D. Panels (a) and (d) show two intersections and no intersection when $\widetilde{g}_q > 0$. In panel (b), $\widetilde{g}_q < 0$, and the second economic steady state is in the region without vertical R&D. In panel (c), $\widetilde{g}_q = 0$ and U is on the boundary.

As in Figure A.7 for the baseline model, let P1 denote the point where the $\dot{g}_{q,t} = 0$ locus meets the boundary $g_{q,t} = 0$. A positive-debt intersection with vertical R&D also requires P1 to have positive debt. Evaluating (C.7) at P1 gives

$$\frac{\kappa}{1-\theta^e} > \underline{\kappa}(g_L), \tag{C.12}$$

$$\underline{\kappa}(g_L) = \alpha \frac{\sqrt{(\delta-\rho-\varepsilon)^2 + 4(1-\alpha)(\rho+\mu)(\delta+\mu+g_L)} - (\delta-\rho-\varepsilon)}{2(1-\alpha)(g_L+\varepsilon)}. \tag{C.13}$$

The threshold $\underline{\kappa}(g_L)$ is decreasing in g_L. The two thresholds evaluate the growth locus on the $g = r$ line and at $g_q = 0$. These two lines coincide when $\kappa/(1-\theta^e) = \alpha/(1-\alpha)$, as in Figure C.1(c), and the value of g_L at which $\overline{\kappa}(g_L) = \alpha/(1-\alpha)$ is $\widetilde{g_L}$ in (56). Hence, at $g_L = \widetilde{g_L}$, the two thresholds coincide at $\alpha/(1-\alpha)$. Below this population-growth rate, $\overline{\kappa} > \underline{\kappa}$, and above it the ordering is reversed. Thus, horizontal innovation may sustain $g > r$ without vertical R&D. Appendix C.4 examines that possibility.

The shape of the $\dot{D} = 0$ locus depends on the sign of $\widetilde{g_q}$, which, by (51), satisfies

$$\widetilde{g_q}(\kappa, g_L, \theta) < 0 \quad \Longleftrightarrow \quad \frac{\kappa}{1-\theta^e} < \frac{\alpha}{1-\alpha}. \tag{C.14}$$

When $\widetilde{g_q} < 0$, $g > r$ even at $g_q = 0$, giving the upward-sloping debt locus in Figure C.1(b). When $\kappa < \alpha/(1-\alpha)$, a higher subsidy brings $\widetilde{g_q}$ to zero at $\theta = \alpha - (1-\alpha)\kappa$. If $g_L > \overline{g_L}(\kappa)$, the higher-growth intersection survives, as in Figure C.1(c), and eventually disappears at a contact inside the research region. If $\widetilde{g_L} < g_L \leq \overline{g_L}(\kappa)$, it instead reaches the boundary with $g_q = 0$ and $g = r$. These are the two ways in which the upper subsidy bound is reached.

To check these intersections, we use the gap $z \equiv g - r$ at a candidate steady state, which is an increasing affine function of g_q by (51). Eliminating D from the two stationary loci gives $F^{2G}(z; \theta^e) = 0$, where

$$\begin{aligned} F^{2G}(z;\theta^e) \equiv & -\alpha(\rho+\mu)(\delta+\beta)\theta^e z_{\min}(\kappa, g_L, \theta^e) \\ & - z\Big(\big((1+\kappa)(g_L+\varepsilon) - z\big)\big(\delta - \rho + g_L - z\big) - (\rho+\mu)(\delta+\beta)\Big), \end{aligned} \tag{C.15}$$

and

$$z_{\min}(\kappa, g_L, \theta^e) \equiv (g_L + \varepsilon)\left(1 - \frac{(1-\alpha)\kappa}{\alpha(1-\theta^e)}\right). \tag{C.16}$$

By (51), $z_{\min}(\kappa, g_L, \theta^e)$ is the gap at $g_q = 0$, whose sign is opposite to that of $\widetilde{g_q}(\kappa, g_L, \theta)$ by (C.14). Hence, the region with positive vertical R&D corresponds to $z_{\min}(\kappa, g_L, \theta^e) < z \leq (1+\kappa)(g_L + \varepsilon)$.

As in the baseline model, the graph of $F^{2G}(z; \theta^e)$ has the qualitative shape illustrated in Figure A.2, with A and $\delta - \rho$ replaced by $(1+\kappa)(g_L+\varepsilon)$ and $\delta - \rho + g_L$, respectively. Unlike the baseline case, its constant term can be negative. Nevertheless, at the upper labor boundary $z = (1+\kappa)(g_L+\varepsilon)$, (C.15) gives $F^{2G}(z;\theta^e) = (\rho+\mu)(\delta+\beta)\big((1+\kappa)(g_L + \varepsilon) - \alpha\theta^e z_{\min}(\kappa, g_L, \theta^e)\big) > 0$. If $\delta - \rho + g_L$ exceeds this boundary, F^{2G} is also positive there. Hence, as in the baseline model, exactly one root z_E exceeds both $(1+\kappa)(g_L+\varepsilon)$ and $\delta - \rho + g_L$. The corresponding intersection E is excluded by the labor resource constraint.

The other two roots, when they exist, represent steady states with vertical R&D only if they exceed $z_{\min}(\kappa, g_L, \theta^e)$. As in the baseline model, we denote the higher-growth steady state by S and the other by U when both are feasible. If the smaller root does not exceed $z_{\min}(\kappa, g_L, \theta^e)$, it is excluded from the region with vertical R&D. Steady states without vertical R&D are examined in Appendix C.4.

C.4 Steady States without Vertical R&D

The dynamics (C.9) and (C.10) do not depend on the subsidy. The policy changes only the boundary (C.8). A steady state in this region has either $D = 0$ or $g = r$.

At $D = 0$, the firm-value ratio is the positive solution of $(\rho + \mu)(\delta + \beta)\nu^2 - (\delta - \rho - \varepsilon)\nu - (1 - \alpha) = 0$. When $g_L > \widetilde{g_L}$, this point is $S = (0, \nu^*)$, where

$$\nu^* = \frac{\delta - \rho - \varepsilon + \sqrt{(\delta - \rho - \varepsilon)^2 + 4(1 - \alpha)(\rho + \mu)(\delta + \beta)}}{2(\rho + \mu)(\delta + \beta)}. \tag{C.17}$$

The gap there is $g_L + \varepsilon - (1 - \alpha)/\nu^*$. It is positive if and only if $g_L > \widetilde{g_L}$.[91] The point lies in the region without vertical R&D if and only if $\kappa/(1 - \theta^e) < \underline{\kappa}(g_L)$, by (C.8). Thus, the same threshold that locates P1 relative to $D = 0$ also determines whether this steady state conducts vertical research.

At $g = r$, the other steady state is

$$\nu_U = \frac{1 - \alpha}{g_L + \varepsilon}, \qquad D_U = \frac{\delta - \rho + g_L}{(\rho + \mu)(\delta + \beta)} - \frac{1 - \alpha}{g_L + \varepsilon}. \tag{C.18}$$

Its debt ratio is positive if and only if $g_L > \widetilde{g_L}$. It lies in the region without vertical R&D if $\kappa/(1 - \theta^e) < \alpha/(1 - \alpha)$. When $g_L > \widetilde{g_L}$, the steady state U remains in this region after S begins vertical research, because firm value is lower at U than at S, as in Figure C.2(b).

If $g_L < \widetilde{g_L}$, the roles are reversed. The steady state at $g = r$ has negative debt and is S, whereas the steady state on $D = 0$ is U. We then denote the positive value in (C.17) by ν_U. Neither point sustains a positive gap in this case.

The two regions meet at $g_q = 0$ and the upper boundary of ν. If a steady state is on this boundary, the stationary debt condition requires $D = 0$ or $g = r$. The former occurs at $\kappa/(1 - \theta^e) = \underline{\kappa}(g_L)$, and the latter at $\kappa/(1 - \theta^e) = \alpha/(1 - \alpha)$. At these values, the two descriptions give the same steady state, so that a steady state moves continuously from one region to the other as θ rises. The first value determines $\underline{\theta}(\kappa, g_L)$ in Appendix C.6, and

[91] Since ν^* is the positive solution of $(\rho + \mu)(\delta + \beta)\nu^2 - (\delta - \rho - \varepsilon)\nu - (1 - \alpha) = 0$, the gap is positive if and only if the left-hand side is negative at $\nu = (1 - \alpha)/(g_L + \varepsilon)$. This condition is equivalent to $(g_L + \varepsilon)(\delta - \rho + g_L) > (1 - \alpha)(\rho + \mu)(\delta + \mu + g_L)$, which holds if and only if $g_L > \widetilde{g_L}$ by (56).

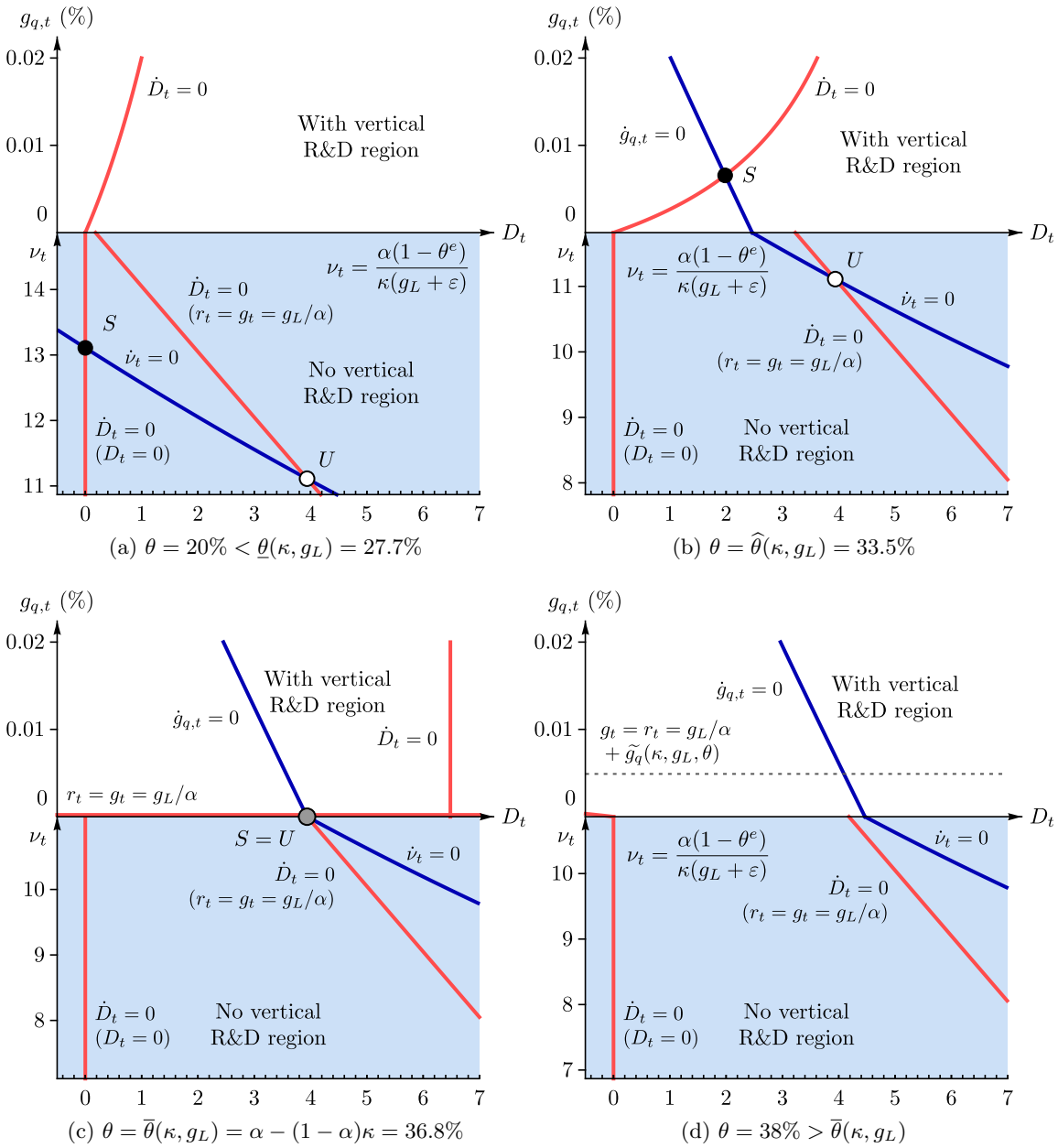


Figure C.2: Phase diagrams when small subsidies have no effect. The parameters are those in Table 1 with $\varepsilon = 0.01$ and $\lambda = (1-\alpha)/\alpha$. The upper and lower halves of each panel show the regions with and without vertical R&D, respectively. Black circles indicate the long-term BGP S, white circles the steady state U at the unsustainable threshold, and gray circles coincident steady states.

the second determines the upper bound $\alpha - (1-\alpha)\kappa$ in Proposition 7.

C.5 Uniqueness and Nonexistence of Equilibrium

We show that Proposition 1 holds for $D_0 \geq 0$, with g_t replaced by $g_{q,t}$ in the R&D region. The lower-debt steady state is S, and the debt ratio at U is the unsustainable threshold, whichever region contains these points.

Except for knife-edge cases, the count across both regions is two or zero, as in Proposi-

tion 1(a). At P1, $\dot{D} = (r - g)D$, where $r - g$ has the sign of $\widetilde{g_q}$ by (51). Together with the sign of the debt ratio at P1 and the excluded upper solution of (C.15), this gives the number of positive-g_q intersections. If $\widetilde{g_q} > 0$, the count is the same as in the baseline model (Appendix A.7). If $\widetilde{g_q} < 0$ and both steady states without vertical R&D lie inside their region, there is no positive-g_q intersection: the growth locus has negative debt throughout the feasible range, whereas the stationary budget with $g > r$ requires nonnegative debt. Once the no-debt steady state crosses the boundary, there is one intersection with vertical R&D and the other steady state remains without it.

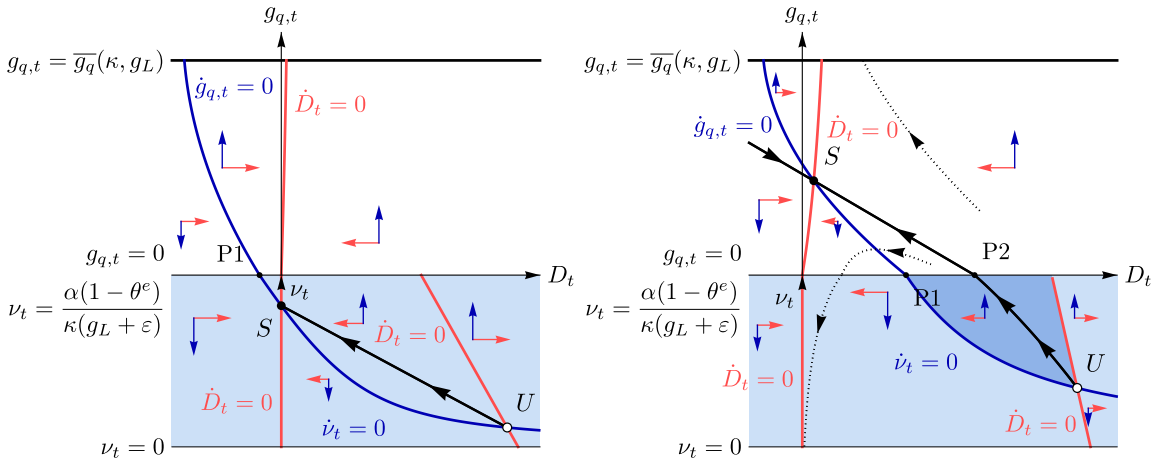
First suppose $\kappa/(1-\theta^e) > \alpha/(1-\alpha)$, so that $\widetilde{g_q} > 0$. The configurations in Figure C.1(a) and (d) are the same as those in Figure 5. The common quadratic terms of (53) and (54) give the linear representation of Lemma A.1. The excluded intersection E has the largest eigenvalue, as in Appendix A.6, because its gap exceeds both $(1+\kappa)(g_L+\varepsilon)$ and $\delta-\rho+g_L$ (Appendix C.3). The same eigenvalue argument therefore excludes paths off the equilibrium line.

The proof for the region without vertical R&D requires three further checks. By (C.8), $(g_L + \varepsilon)\nu_t$ is smaller than $1 - \alpha$. By (C.10), firm value therefore falls on the line $g_t = r_t$, on which debt is constant, as on the line $r_t = 0$ in Figure A.7. If both steady states have vertical R&D, P1 has positive debt and the region without vertical R&D has no steady state. Finally, at the boundary, $\dot{\nu}_t$ has the sign of $\dot{g}_{q,t}$ at $g_{q,t} = 0$. These properties give the same directions in which paths can cross the loci and the boundaries as in Figure A.7, and hence the same exclusion of paths. If U instead lies in the region without vertical R&D, D_U is zero, and there is no equilibrium path with positive initial debt.

Next suppose $\kappa/(1 - \theta^e) \leq \alpha/(1 - \alpha)$ and $g_L < \widetilde{g_L}$. In this case, S has negative debt and D_U is zero by Appendix C.4. P1 is to the left of the vertical axis, and the growth locus has negative debt throughout the R&D region. A path with nonnegative debt and vertical R&D therefore crosses the labor boundary. Without vertical R&D, a path above U reaches the upper boundary, whereas along a path with positive debt below U, firm value becomes negative or debt becomes infinite in finite time, because $(g_L + \varepsilon)\nu_U$ is smaller than $1 - \alpha$. Hence, an economy with positive initial debt has no equilibrium path.

Finally, suppose $\kappa/(1-\theta^e) < \alpha/(1-\alpha)$ and $g_L > \widetilde{g_L}$. The two configurations are shown in Figure C.3. The steady state S is without vertical R&D when $\kappa/(1 - \theta^e) \leq \underline{\kappa}(g_L)$, and otherwise it is in the R&D region. In both cases, U remains without vertical R&D and its debt ratio is given by (C.18).

In the region without vertical R&D, both quadratic terms are $(\rho + \mu)(\delta + \beta)(D_t + \nu_t)$ times the corresponding variable. Multiplying $(D_t, \nu_t, 1)$ by the integrating factor



(a) $\kappa/(1-\theta^e) < \underline{\kappa}(g_L)$ (b) $\underline{\kappa}(g_L) < \kappa/(1-\theta^e) < \alpha/(1-\alpha)$

Figure C.3: Equilibrium paths when U is in the region without vertical R&D. In panel (a), S is also in that region. In panel (b), the equilibrium path enters the R&D region at P2, and D_t falls and ν_t rises in the darker set.

$\exp(-(\rho+\mu)(\delta+\beta)\int_0^t (D_u+\nu_u)du)$ gives a linear system. Its third stationary solution has negative firm value and is excluded. Among the two feasible points, the eigenvalue of the no-debt point is higher precisely when its gap is positive. Thus, the argument of Lemma A.2 applies in this region as well.

In Figure C.3(a), the equilibrium line joins U and $S = (0, \nu^*)$. Every path on the line with $0 \leq D_0 < D_U$ remains within the region and converges to S. A path with vertical R&D is excluded because P1 is to the left of the axis. Along every other path without vertical R&D, firm value becomes negative, debt becomes infinite in finite time, or the path reaches that R&D region. This proves existence and uniqueness below D_U, and nonexistence above it.

In panel (b), the equilibrium line reaches $g_q = 0$ at P2, to the right of P1. It meets the boundary before the line on which $g = r$.[92] The unique path through P2 in the region without vertical R&D converges to U backward in time, as in Figure A.7(d). Along a path below both loci, firm value becomes negative. A path above enters the R&D region off the equilibrium line, or crosses downward and then makes firm value negative. Thus, the diagram selects exactly one path to P2 for every debt ratio between its value there and

[92] On the boundary, the point where $g = r$ lies to the right of P1 by the gap at $g_q = 0$ divided by $(\rho+\mu)(\delta+\beta)$. This follows from (C.7) at $g_q = 0$ and the gap in the region without vertical R&D (Appendix C.2). Write g_E for the full-output growth rate at the excluded intersection E, so that its vertical growth rate is $g_E - g_L/\alpha$. Substituting the equilibrium line into (53) at $g_q = 0$ shows that the debt ratio at P2 is $1 - \widetilde{g}_q/(g_E - g_L/\alpha)$ times the debt ratio at P1. This factor exceeds one because $\widetilde{g}_q$ is negative. Hence, the distance from P1 to P2 is the debt ratio at P1 times $-\widetilde{g}_q/(g_E - g_L/\alpha)$. This factor equals the gap at $g_q = 0$ divided by the gap at E minus the gap at $g_q = 0$. By (C.7), $(\rho+\mu)(\delta+\beta)$ times the debt ratio at P1 is less than $\delta - \rho + g_L$ minus the gap at $g_q = 0$. This is less than the gap at E minus the gap at $g_q = 0$, because the gap at E exceeds $\delta - \rho + g_L$. Therefore, the distance from P1 to P2 is smaller than the distance from P1 to the point where $g = r$. This gives the ordering drawn in panel (b).

D_U.

As in Appendix A.7, this path, followed by the straight line to S, is the unique equilibrium path for $0 \leq D_0 < D_U$, and there is none for $D_0 > D_U$. The transversality condition and non-Ponzi game condition of households hold as in Appendix A.7, since each candidate path continues to the feasible steady state S.

C.6 When Small Subsidies Have No Effect

The preceding two subsections showed that horizontal innovation alone can sustain a long-term BGP with $g > r$, even without vertical R&D. At such a steady state, the private return to vertical R&D is insufficient to cover its cost, so the free-entry condition holds with strict inequality. A small subsidy therefore need not induce vertical R&D, leaving long-run growth unchanged. This subsection examines how a sufficiently large subsidy can overcome this barrier and induce vertical R&D.

We focus on the case where $\kappa < \alpha/(1-\alpha)$ and $\widetilde{g_L} < g_L < \underline{g_L}(\kappa)$. The threshold $\underline{g_L}(\kappa)$ is defined by $\kappa = \underline{\kappa}(g_L)$, where $\underline{\kappa}(g_L)$ is given in (C.13). Figure C.2 illustrates how the steady states change as the subsidy rate increases, using $\kappa = 2.79$ and $g_L = 0.5\%$.

In Panel (a), the long-term BGP S lies in the region without vertical R&D, with $D^* = 0$ and $g > r$. As shown in Appendix C.4, the dynamics in this region are independent of θ because no subsidies are paid. Thus, increasing θ leaves S unchanged, but shifts the boundary of the region without vertical R&D. The steady state reaches this boundary when $\kappa/(1-\theta^e) = \underline{\kappa}(g_L)$, by (C.8) and (C.13). Using (52), the minimum subsidy rate required to induce vertical R&D is therefore

$$\underline{\theta}(\kappa, g_L) = \frac{\underline{\kappa}(g_L) - \kappa}{1 + \underline{\kappa}(g_L)}. \tag{C.19}$$

Below this threshold, S remains unchanged, so the subsidy has no effect on long-run growth. Once θ exceeds this threshold, S enters the region with vertical R&D, and g_q^* becomes positive.

As θ increases further, g_q^* rises until θ reaches $\widehat{\theta}(\kappa, g_L)$, as shown in Figure C.2(b), and then declines. In this numerical example, S eventually reaches $g_q^* = 0$ and $g^* = r^*$ at $\theta = \alpha - (1-\alpha)\kappa$, as shown in Panel (c). This is the upper bound $\overline{\theta}(\kappa, g_L)$ established in Proposition 7. Beyond this rate, no steady state with $g > r$ exists, as illustrated in Panel (d).

When $\overline{g_L}(\kappa) < g_L < \underline{g_L}(\kappa)$, increasing θ initially induces vertical R&D at S, as in Figure C.2(b). Unlike the numerical example, however, S remains in the region with

positive vertical R&D as the phase diagram moves from Panel (b) through (c) to (a) of Figure C.1, where U also enters this region. With a further increase in θ, the two steady states merge and disappear while $g_q^* > 0$ and $g^* > r^*$, leading to the configuration in Figure C.1(d).

C.7 Proof of Proposition 7

Proof of part (i). Suppose first that $\kappa > \alpha/(1-\alpha)$. Then $z_{\min}(\kappa, g_L, \theta^e) < 0$ for every subsidy rate, so a steady state with $g > r$ must conduct vertical R&D by Appendix C.4. By (C.15), a feasible positive root requires $(1+\kappa)(g_L+\varepsilon)(\delta-\rho+g_L) > (\rho+\mu)(\delta+\beta)$. Conversely, if this inequality holds, $F^{2G}(z;0) = 0$ has a feasible positive root, which persists under sufficiently small positive subsidies. By (55), the condition is equivalent to $g_L > \overline{g_L}(\kappa)$.

Next suppose $\kappa \leq \alpha/(1-\alpha)$. If $g_L \leq \widetilde{g_L}$, neither steady state without vertical R&D has a positive gap by Appendix C.4. A steady state with vertical R&D cannot have one either. If $z_{\min}(\kappa, g_L, \theta^e) \geq 0$, then $\kappa/(1-\theta^e) \leq \alpha/(1-\alpha) \leq \underline{\kappa}(g_L)$, so the necessary condition (C.12) fails. If $z_{\min}(\kappa, g_L, \theta^e) < 0$, the constant term of $F^{2G}(z;\theta^e)$ is positive, while $(1+\kappa)(g_L+\varepsilon)(\delta-\rho+g_L)$ does not exceed $(\rho+\mu)(\delta+\beta)$ by $\kappa \leq \alpha/(1-\alpha)$ and (56). Hence, $F^{2G}(z;\theta^e) > 0$ for every feasible $z > 0$. Conversely, if $g_L > \widetilde{g_L}$, the no-subsidy economy has a long-term BGP with $g > r$, with or without vertical R&D as described in Appendices C.3 and C.4. This proves the two necessary and sufficient population-growth conditions.

Proof of part (ii). Suppose these conditions hold. If $\widetilde{g_L} < g_L \leq \overline{g_L}(\kappa)$, the long-term BGP reaches $g_q = 0$ and $g = r$ at $\theta = \alpha - (1-\alpha)\kappa$, as shown in Figure C.2(c). Beyond this rate, $z_{\min}(\kappa, g_L, \theta^e) < 0$ and the constant term of $F^{2G}(z;\theta^e)$ is positive. Since κ does not exceed $\overline{\kappa}(g_L)$, the equation $F^{2G}(z;\theta^e) = 0$ has no feasible positive root. Hence, this rate is the upper bound.

If $g_L > \overline{g_L}(\kappa)$, the higher-growth intersection remains inside the region with positive vertical R&D when $z_{\min}(\kappa, g_L, \theta^e) = 0$. As the subsidy increases further, the two intersections eventually merge at the local minimum of $F^{2G}(z;\theta^e)$. Since the subsidy affects only the constant term, the location of this local minimum is independent of θ^e. Differentiating (C.15) with respect to z gives the contact condition

$$\begin{aligned} 3z^2 - 2\big((1+\kappa)(g_L+\varepsilon) + \delta - \rho + g_L\big)z \\ + (1+\kappa)(g_L+\varepsilon)(\delta-\rho+g_L) - (\rho+\mu)(\delta+\beta) = 0. \end{aligned} \tag{C.20}$$

Let $\underline{z}(\kappa, g_L)$ denote the smaller solution of (C.20). It lies between 0 and $(\delta-\rho+g_L)/2$,

because the left-hand side of (C.20) is positive at $z = 0$ by $g_L > \overline{g_L}(\kappa)$ and negative at $z = (\delta - \rho + g_L)/2$. The upper subsidy rate $\overline{\theta}(\kappa, g_L)$ is determined by $F^{2G}(\underline{z}(\kappa, g_L); \theta^e) = 0$, together with (52). At this contact, both steady states have positive vertical R&D and $g > r$.

To establish uniqueness and the comparative static, divide (C.15) by $(\rho+\mu)(\delta+\beta)(g_L+\varepsilon)$ to obtain

$$\theta^e \left(\frac{(1-\alpha)\kappa}{1-\theta^e} - \alpha \right) = z \left(\left(1 + \kappa - \frac{z}{g_L + \varepsilon} \right) \frac{\delta - \rho + g_L - z}{(\rho+\mu)(\delta+\beta)} - \frac{1}{g_L + \varepsilon} \right). \tag{C.21}$$

Over the feasible interval $0 < z \leq (1+\kappa)(g_L + \varepsilon)$, the right-hand side of (C.21) attains its positive maximum at $z = \underline{z}(\kappa, g_L)$. The left-hand side is strictly increasing in θ whenever it is positive, because θ^e increases with θ by (52), and it tends to infinity as θ approaches one. Hence, the upper subsidy rate is unique.

For any fixed feasible positive gap below both $(1+\kappa)(g_L+\varepsilon)$ and $\delta - \rho + g_L$, the right-hand side of (C.21) is strictly increasing in g_L. In particular, $(\delta - \rho + g_L - z)/(\delta + \beta)$ increases with g_L, while increasing g_L also raises $(1+\kappa)(g_L+\varepsilon)$. Therefore, its positive maximum increases with g_L. Since the left-hand side does not depend on g_L, $\overline{\theta}(\kappa, g_L)$ is strictly increasing in this case.

C.8 Proof of Proposition 8

For the calculations in this subsection, write $z \equiv g^* - r^*$. Unlike the baseline case, the stationary growth rate depends on the subsidy both through this gap and through the value of the industries created without vertical R&D. Combining (C.6), (C.7), and (51) gives

$$g_q^* = \overline{g_q}(\kappa, g_L) \left(1 - \frac{\alpha(\rho+\mu)(\delta+\beta)(1 + (1-\theta^e)/\kappa)}{z(\delta - \rho + g_L - z) + (\rho+\mu)(\delta+\beta)} \right). \tag{C.22}$$

The term involving $(1-\theta^e)/\kappa$ is the effect of horizontal entry. The gap in this expression is determined by (C.15), with $g - r = z$. These two equalities are sufficient for the comparative statics.

Proof of part (i). We first consider a long-term BGP with positive vertical R&D without subsidies. Since g_L is fixed and θ^e increases with θ, the sign of the growth effect is the sign of $dg_q^*/d\theta^e$. Differentiating (C.22) gives

$$\frac{dg_q^*}{d\theta^e} = \frac{\alpha\lambda(g_L+\varepsilon)(\rho+\mu)(\delta+\beta)}{z(\delta - \rho + g_L - z) + (\rho+\mu)(\delta+\beta)} \left(1 + \frac{(1+\kappa-\theta^e)(\delta - \rho + g_L - 2z)}{z(\delta - \rho + g_L - z) + (\rho+\mu)(\delta+\beta)} \frac{dz}{d\theta^e} \right). \tag{C.23}$$

The first term in parentheses is the positive effect at a given gap. The second is the effect

through the change in the gap. Differentiating the stationary condition (C.15) at $\theta^e = 0$ gives

$$\left.\frac{dz}{d\theta^e}\right|_{\theta^e=0} = -\frac{(\rho+\mu)(\delta+\beta)(g_L+\varepsilon)((1-\alpha)\kappa-\alpha)}{z((1+\kappa)(g_L+\varepsilon)+\delta-\rho+g_L-2z)}.$$

At this rate, the stationary condition also gives $((1+\kappa)(g_L+\varepsilon)-z)(\delta-\rho+g_L-z) = (\rho+\mu)(\delta+\beta)$. Substituting these expressions into (C.23) shows that the initial growth effect is positive if and only if

$$(g_L+\varepsilon)(\delta-\rho+g_L-z)^2 > (1-\alpha)(\rho+\mu)(\delta+\beta)(\delta-\rho+g_L-2z). \tag{C.24}$$

The factor removed from the derivative is positive, by $0 < z < \min\{(1+\kappa)(g_L+\varepsilon), \delta-\rho+g_L\}$. Completing the square in $\delta-\rho+g_L-z$ shows that the inequality (C.24) holds for every feasible positive z when $g_L > \widetilde{g_L}$.

If $g_L < \widetilde{g_L}$, let $\Xi = \sqrt{1-(g_L+\varepsilon)(\delta-\rho+g_L)/((1-\alpha)(\rho+\mu)(\delta+\beta))}$, as in the threshold calculation. Solving the quadratic inequality gives $z > \Xi(\delta-\rho+g_L)/(1+\Xi)$. Along the no-subsidy stationary condition, the gap increases with κ. Substituting the threshold gap gives

$$\widehat{\kappa}(g_L) = (1+\Xi)\overline{\kappa}(g_L) + \Xi\left(1+\frac{\delta-\rho+g_L}{(g_L+\varepsilon)(1+\Xi)}\right). \tag{C.25}$$

Thus, the initial effect is positive if and only if $\kappa > \widehat{\kappa}(g_L)$. This threshold exceeds $\overline{\kappa}(g_L)$ and is strictly decreasing in g_L.[93] Its inverse defines $\widehat{g_L}(\kappa)$ for $\kappa > \alpha/(1-\alpha)$, with $\overline{g_L}(\kappa) < \widehat{g_L}(\kappa) < \widetilde{g_L}$.

It remains to show that a negative initial growth effect cannot become positive at a higher subsidy. Consider any interior rate at which $dg_q^*/d\theta^e = 0$. The expression for g_q^* from (51) increases with both the gap and θ^e, so $dz/d\theta^e < 0$ there. By (C.23), $z < (\delta-\rho+g_L)/2$ must also hold.

At the higher-growth intersection, $F^{2G}(z;\theta^e)$ is increasing in the gap. Its derivative with respect to θ^e is $(\rho+\mu)(\delta+\beta)(g_L+\varepsilon)((1-\alpha)\kappa/(1-\theta^e)^2-\alpha)$, which strictly increases with θ^e. Once the gap starts falling, it therefore continues to fall. Substituting its derivative into (C.23) shows that the marginal growth effect is positive precisely when

$$\left(\frac{1-\theta^e}{1+\kappa-\theta^e}\right)^2\big(z(\delta-\rho+g_L-z)+(\rho+\mu)(\delta+\beta)\big)^2 > (1-\alpha)(g_L+\varepsilon)(\rho+\mu)(\delta+\beta)(\delta-\rho+g_L-2z). \tag{C.26}$$

[93] Both Ξ and $\overline{\kappa}$ decrease with g_L. More precisely, $d\overline{\kappa}/dg_L = -(1+\overline{\kappa})/(g_L+\varepsilon) - (\rho+\mu)^2/((g_L+\varepsilon)(\delta-\rho+g_L)^2)$, while the derivative of $(\delta-\rho+g_L)/(g_L+\varepsilon)$ is smaller than $1/(g_L+\varepsilon)$. In differentiating (C.25), all terms involving $d\Xi/dg_L$ are negative. The remainder is smaller than $(-(1+\Xi)(1+\overline{\kappa})+\Xi/(1+\Xi))/(g_L+\varepsilon) < 0$. As g_L approaches $\widetilde{g_L}$ from below, Ξ tends to zero and $\widehat{\kappa}$ tends to $\alpha/(1-\alpha)$. The defining relation has a unique inverse above this value.

At a stationary growth rate, the two sides are equal. Beyond it, the left-hand side decreases because θ^e rises and z falls below $(\delta - \rho + g_L)/2$, while the right-hand side increases. Hence, every interior stationary point of long-run growth is a maximum, and the marginal effect cannot turn from negative to positive.

When $\kappa > \alpha/(1-\alpha)$, any long-term BGP with $g > r$ already has vertical research without subsidies. The initial-effect condition therefore gives $g_L > \widehat{g_L}(\kappa)$. When $\kappa \leq \alpha/(1-\alpha)$, such a BGP exists if and only if $g_L > \widetilde{g_L}$. If it already has vertical research, the initial effect is positive. Otherwise, a sufficiently large subsidy induces research at the threshold in Appendix C.6 and raises growth. This proves part (i).

Proof of part (ii). Under the condition of part (i), some subsidy raises growth and there is at most one interior maximum. Growth is not maximized at the upper bound. If $g_L > \overline{g_L}(\kappa)$, the intersections meet at a gap below $(\delta - \rho + g_L)/2$. The derivative of $F^{2G}(z;\theta^e)$ with respect to the gap tends to zero from above, while its derivative with respect to θ^e is positive. Hence, $dz/d\theta^e$ tends to minus infinity, and the negative indirect effect in (C.23) dominates. If $\widetilde{g_L} < g_L \leq \overline{g_L}(\kappa)$, vertical research vanishes at the upper bound, so its growth rate is lower there than at rates immediately below. Thus, the growth-maximizing subsidy lies strictly between zero and the upper bound.

Proof of part (iii). In the stated range, S remains in the region without vertical R&D until θ reaches $\underline{\theta}(\kappa, g_L)$, as shown in Figure C.2(a). Since S has no vertical R&D, the output growth rate remains g_L/α throughout this interval.

C.9 Calibration Details

This subsection describes the data and the three steps of the estimation of κ in Section 6.4. For each of the 24 countries, the growth rates of real GDP and population and the investment-GDP ratio are the averages over 2000–2023 in the Penn World Table version 11.0 (Feenstra et al., 2015). For the history of population, we use the annual population series in the Maddison Project Database (Bolt and van Zanden, 2025) from 1820 through 2022 and extend it to 2023 using the population growth in the Penn World Table.

In the first step, we consider $\xi_t \equiv n_t/(fL_t)$, the number of industries relative to the rate at which new industries are created. By (39), it evolves according to $\dot{\xi}_t = 1 - (g_{L,t} + \varepsilon)\xi_t$, which does not depend on f, where $g_{L,t}$ is the growth rate of the population, and hence of L_t, at time t. If $g_{L,t}$ stays at g_L, ξ_t converges to $1/(g_L + \varepsilon)$, which corresponds to (40). For each country, we start from the first year of the longest uninterrupted population series that ends in 2023, which is 1820 except for Canada and New Zealand (1870), Turkey (1923), and Iceland and Luxembourg (1950). The initial value of ξ_t is the value to which

it would converge if the average population growth rate over the first ten years continued. Using the population growth rate in each year as $g_{L,t}$, we update ξ_t by

$$\xi_{t+1} = e^{-(g_{L,t}+\varepsilon)}\xi_t + \frac{1 - e^{-(g_{L,t}+\varepsilon)}}{g_{L,t} + \varepsilon}. \tag{C.27}$$

We then compute the average level of ξ_t and its average log growth rate over 2000–2023.

In the second step, we compute the growth rate implied by the model for a given value of κ, using the data other than the growth rate of real GDP. By (47), the growth rate of full-production output is the sum of the population growth rate, $(1-\alpha)/\alpha$ times the rate of horizontal innovation, $\dot{n}_t/n_t$, and the rate of vertical innovation. The rate of horizontal innovation is $\dot{n}_t/n_t = g_{L,t} + \dot{\xi}_t/\xi_t$, which is obtained in the first step. By Lemma 1, the rate of vertical innovation is $\lambda a L_t^R/n_t = \lambda\kappa(L_t^R/L_t)/\xi_t$. Hence,

$$g_t = g_{L,t} + \frac{1-\alpha}{\alpha}\Big(g_{L,t} + \frac{\dot{\xi}_t}{\xi_t}\Big) + \frac{\lambda\kappa}{\xi_t}\frac{L_t^R}{L_t}. \tag{C.28}$$

For the last term, we infer the R&D labor share from the investment-GDP ratio, as in Section 4.4. In the current model, GDP is the sum of the wages and the profits, $\mathrm{GDP}_t = w_t L_t + n_t \pi_t$. Since the wage includes the value of new businesses, $f v_t$, per unit of labor, GDP can also be written as $\mathrm{GDP}_t = C_t + w_t L_t^R + f v_t L_t^P$, where the last term is the value of the ideas of production workers. In the model, firms pay wages for two kinds of labor, production labor and research labor. If these expenditures were classified according to the current national accounting standards, which treat R&D expenditure as capital formation, only the cost of research labor, $w_t L_t^R$, would be classified as investment. We therefore regard only $w_t L_t^R$ as investment. Without subsidies, $f v_t/(w_t - f v_t) = 1/\kappa$ holds by the free-entry condition (44). Since labor income net of the value of ideas, $(w_t - f v_t)L_t$, equals αY_t by (46) and (47), we have $w_t L_t = \alpha(1+1/\kappa)Y_t$. Moreover, $n_t\pi_t = (1-\alpha)X_t = (1-\alpha)Y_t(1-L_t^R/L_t)$ by (45) and (47). Hence, the R&D labor share is

$$\frac{L_t^R}{L_t} = \frac{(1+\alpha/\kappa)\, w_t L_t^R/\mathrm{GDP}_t}{\alpha(1+1/\kappa) + (1-\alpha)\, w_t L_t^R/\mathrm{GDP}_t}, \tag{C.29}$$

where $w_t L_t^R/\mathrm{GDP}_t$ is measured by the investment-GDP ratio. Substituting this share into (C.28), we obtain the growth rate implied by the model for a given κ.

In the third step, we choose κ so that the growth rate implied by the model equals the growth rate of real GDP. In this step, we evaluate each term of (C.28) by its average over 2000–2023. For g_t and $g_{L,t}$, we use the average growth rates of real GDP and population.[94]

[94]We measure growth rates by averages of annual log differences, because in the continuous-time model

For $\dot{\xi}_t/\xi_t$ and ξ_t, we use the averages over 2000–2023 of the log growth rate and the level of ξ_t obtained in the first step. For L_t^R/L_t, we use (C.29) with the average investment-GDP ratio. This condition is a quadratic equation in κ, and it has a unique positive solution, because the growth rate of real GDP exceeds the sum of the first two terms on the right-hand side of (C.28) in all 24 countries.

If we also regard the value of the ideas of production workers, $fv_tL_t^P$, as investment, the investment-GDP ratio measures $(w_tL_t^R + fv_tL_t^P)/\text{GDP}_t$. In this alternative specification, the estimates of κ are higher in all 24 countries. The classification of the countries is then the same as in the baseline model (Section 4.4), except that Italy lies in the region where the government cannot sustain the subsidies by rolling over its debt.

C.10 Nesting of the Baseline Model

As in Section 6.5, set $g_L = 0$ and $\lambda = (1-\alpha)/\alpha$, and let f and ε tend to zero while $\varepsilon/f = L$ stays fixed. Then $\kappa = a/f$ tends to infinity, whereas $\kappa\varepsilon = aL$ stays fixed.

First consider the production side. No industries are created or become obsolete in the limit, and $n_t = 1$ by (40). The term fv_t disappears from wages and the private cost of research. Hence, the free-entry condition becomes $av_t \leq (1-\theta)w_t$, and the effective subsidy rate equals θ. The no-arbitrage condition becomes $r_t = (\pi_t + \dot{v}_t)/v_t - aL_t^R$, with $\pi_t = (1-\alpha)X_t$. These are the conditions of the quality-ladder model of Grossman and Helpman (1991b), where aL_t^R is the replacement rate of leaders. Each replacement raises log quality by λ, so $g_t = \lambda aL_t^R$.

The pricing difference comes from the final-goods technology. Our CES aggregator gives the monopoly price w_t/α when fv_t vanishes, whereas their Cobb–Douglas aggregator gives a limit markup equal to the quality step e^λ. The markups coincide approximately when $\lambda = (1-\alpha)/\alpha$ and exactly if $\lambda = \ln(1/\alpha)$. We use the former value to obtain the exact correspondence with the baseline variety-expansion model.

In this limit, $A^* = aL$ by (58). Both models then have $g_t = \overline{g}(A^*)L_t^R/L$. They also have the same aggregate profit and the same value of firms from free entry. Net of the loss from replacement, the change in leader value in the quality-ladder model is the same as the change in firm value in the variety-expansion model. Thus, the interest rate is the same function of research labor, given by (21) with $A = A^*$.

Since households and the government are unchanged, (53) and (54) converge to (28)

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the average growth rate of a variable over a year equals the difference in its logarithm. Section 4.4 uses simple growth rates instead. If we use the simple growth rate of real GDP, keeping the other inputs unchanged, the classification changes only for Greece, which moves to the region where the subsidies are sustainable but do not raise growth.

and (29). The dynamics without vertical research converge to (A.15) and (A.16). The stationary loci, their contacts, and the equilibrium paths therefore coincide in the limit. This is a correspondence between the reduced-form equilibrium conditions, not an identification of horizontal expansion with quality improvement.

Finally, setting $g_L = 0$ in the threshold formulas gives $\varepsilon(1 + \overline{\kappa}(0)) = \overline{A}$ and $\varepsilon\underline{\kappa}(0) \to A_{\min}$. In (C.25), $\Xi \to 1$, so $\varepsilon(1 + \widehat{\kappa}(0)) \to \widehat{A}$. Thus, the conditions for sustainable rollover, the onset of research, and growth-promoting subsidies converge to those of the baseline model. The contact condition (C.20) and the stationary condition (C.15) also converge to their baseline counterparts, giving the subsidy upper bound $\overline{\theta}(A^*)$ when $A^* > \overline{A}$.

In the stationary growth relation (C.22), the term $(1 - \theta^e)/\kappa$ vanishes because no new industry is created without research. The relation becomes (A.13). Hence, for $A^* > \widehat{A}$, the growth-maximizing subsidy and the maximized growth rate converge to $\widehat{\theta}(A^*)$ and $\widehat{g}(A^*)$.

D Analysis of the Basic Research Model

This appendix analyzes the model with basic research in Section 7.1. We derive the equilibrium dynamics, examine the steady states, and prove Proposition 9. Finally, we describe the calibration of L_0^B/L.

D.1 Derivation of Equilibrium Dynamics

Write $b = (L^B - L_0^B)/L_0^B \geq 0$ for the proportional increase in basic research employment, so $A = A_0(1 + b)^\phi$. The initial basic researchers are paid by the existing labor tax. This tax multiplies each individual's labor income by the same constant, and therefore leaves the aggregate consumption equation unchanged. Only the additional wages are financed by debt.

The labor available to private production and research is $L - bL_0^B$. Thus,

$$L_t^P + L_t^R = L - bL_0^B, \tag{D.1}$$

$$\overline{G}(b) = \left(1 - b\frac{L_0^B}{L}\right)\overline{g}\big(A_0(1 + b)^\phi\big). \tag{D.2}$$

The latter is the maximum growth rate when all remaining private labor is used for R&D. The private technologies and free-entry condition are unchanged. Substituting the new labor allocation into (21) gives

$$g_t - r_t = s(\theta)\big(g_t - \widetilde{g}(\theta, b)\big), \qquad \widetilde{g}(\theta, b) = \frac{1 - \alpha}{1 - \alpha\theta}\overline{G}(b). \tag{D.3}$$

The government pays for research subsidies and the additional basic researchers. Dividing its budget by Y_t gives

$$\dot{D}_t = (r_t - g_t)D_t + \frac{\theta\alpha}{\overline{g}(A)}g_t + \alpha b\frac{L_0^B}{L}. \tag{D.4}$$

The value of firms relative to output is still $\alpha(1-\theta)/A$, whereas consumption relative to output is now $(\overline{G}(b) - g_t)/\overline{g}(A)$. Repeating the derivation in Appendix A.2 gives

$$\dot{g}_t = (\overline{G}(b) - g_t)(g_t - r_t - \delta + \rho) + \overline{g}(A)(\rho+\mu)(\delta+\mu)\left(D_t + \frac{\alpha(1-\theta)}{A}\right). \tag{D.5}$$

Together with (D.3), these are two equations in D_t and g_t. Their stationary loci are

$$\dot{D}_t = 0: \quad D_t = \frac{\alpha}{g_t - r_t}\left(b\frac{L_0^B}{L} + \frac{\theta g_t}{\overline{g}(A)}\right), \tag{D.6}$$

$$\dot{g}_t = 0: \quad D_t = \frac{\overline{G}(b) - g_t}{\overline{g}(A)(\rho+\mu)(\delta+\mu)}\big(\delta - \rho - (g_t - r_t)\big) - \frac{\alpha(1-\theta)}{A}, \tag{D.7}$$

with the budget condition used in its undivided form if $g_t = r_t$. At $b = 0$, the model coincides with the baseline model with $A = A_0$.

D.2 Steady States

For a given policy pair (θ, b), steady states are determined by the intersections of (D.6) and (D.7). As in the baseline model, the $\dot{g}_t = 0$ locus opens to the right. However, expenditure on additional basic research shifts the $\dot{D}_t = 0$ locus, which no longer passes through the origin when $b > 0$, even if $\theta = 0$.

To characterize the steady states, write $z \equiv g - r$. Eliminating D from the stationary conditions using (D.3) gives

$$\begin{aligned} z\Big(\big(A(1 - bL_0^B/L) - z\big)(\delta - \rho - z) - (\rho+\mu)(\delta+\mu)\Big) \\ = (\rho+\mu)(\delta+\mu)A\left(\frac{(1-\alpha)\theta}{1-\theta} + b\frac{L_0^B}{L}\right). \end{aligned} \tag{D.8}$$

The labor constraint $g \leq \overline{G}(b)$ is equivalent to $z \leq A(1 - bL_0^B/L)$. As in Appendix A.4, exactly one root lies above this bound and is excluded by the resource constraint. The remaining roots describe steady states with private R&D only if $g > 0$. When two such steady states exist, the higher-growth, lower-debt one is denoted by S, and the other by U.

The quadratic terms of the dynamic equations are unchanged, so the linear representation and the equilibrium-line argument in Appendix A.6 continue to apply. Additional

basic research introduces a constant term in the debt equation, modifying the first-row, third-column entry of the linear system without changing the common quadratic term used in the transformation.

D.3 Uniqueness and Nonexistence of Equilibrium

For $b > 0$, Proposition 1 holds for $D_0 \geq 0$, with $\overline{G}(b)$ as the growth limit. In the region without private research, $C_t/Y_t = 1 - bL_0^B/L$, and the free-entry boundary remains $\nu_t = \alpha(1-\theta)/A$. The debt and firm-value equations are $\dot{D}_t = r_t D_t + \alpha b L_0^B/L$ and $\dot{\nu}_t = r_t \nu_t - (1-\alpha)(1 - bL_0^B/L)$. A steady state with positive firm value requires $r_t > 0$, whereas the stationary debt condition then requires negative debt. Hence, this region contains no steady state with nonnegative debt.

If $\alpha + bL_0^B/L < 1$, there is a steady state U with $D_U = -\alpha b L_0^B/(L r_U)$ and $\nu_U = (1-\alpha)(1 - bL_0^B/L)/r_U$. The interest rate is the positive solution of

$$r_U^2 + (\delta - \rho) r_U = (\rho + \mu)(\delta + \mu) \frac{1 - \alpha - bL_0^B/L}{1 - bL_0^B/L}.$$

This point lies in the region without private research if and only if $r_U \geq \overline{G}(b)/(1-\theta)$, the rate on its boundary at $g = 0$. If $\alpha + bL_0^B/L \geq 1$, there is no positive solution for r_U and hence no steady state in this region. At P1, $\dot{D}$ has the sign of $\overline{G}(b)/(1-\theta) - r_U$ when the positive root exists, and is positive otherwise. The sign test in the proof of Proposition 1(a) in Appendix A.7 therefore gives two intersections with positive growth or none when the steady state without private research lies outside, and one when it lies inside. Except for knife-edge cases, counting both regions gives two steady states or none, as in Proposition 1(a).

The proof of Proposition 1 now follows the geometry of Appendix A.7. The excluded upper intersection has the largest eigenvalue, and the only paths remaining in the R&D region lie on the equilibrium line and converge to S. At $D_t = 0$, debt rises in either region because basic research spending is positive. Without private research, the interest rate again depends only on $D_t + \nu_t$, and its time derivative depends only on the interest rate, vanishing at the solutions of the equation for r_U above. The proof of Lemma A.2(c) therefore applies, and firm value becomes negative or debt becomes infinite in finite time, unless the path enters the R&D region. The transversality condition and non-Ponzi game condition of households hold along the candidate path to S, as in that proof.

D.4 Proof of Proposition 9

We prove the two parts of the proposition separately. In part (a), Proposition 3 shows that R&D subsidies alone cannot increase long-run growth when $A_0 \leq \widehat{A}$. We therefore examine whether a marginal increase in basic research from the initial BGP raises long-run growth. In part (b), where $A_0 > \widehat{A}$, we characterize the growth-maximizing combination of basic research and R&D subsidies. In the following, write $z^* \equiv g^* - r^*$.

Proof of part (a). Suppose $\overline{A} < A_0 \leq \widehat{A}$. We examine whether a marginal increase in basic research from the initial BGP with $\theta = b = 0$ raises long-run growth. With $\theta = 0$, (D.3) implies $g^* = (1-\alpha)(\overline{G}(b) + z^*)$. Differentiating with respect to b at $b = 0$ gives

$$\left.\frac{dg^*}{db}\right|_{b=0} = (1-\alpha)\left(\frac{1-\alpha}{\alpha}A_0\left(\phi - \frac{L_0^B}{L}\right) + \left.\frac{dz^*}{db}\right|_{b=0}\right). \tag{D.9}$$

To determine dz^*/db, let $z_0^*(A_0)$ denote the steady-state gap when $\theta = b = 0$, determined by

$$\big(A_0 - z_0^*(A_0)\big)\big(\delta - \rho - z_0^*(A_0)\big) = (\rho+\mu)(\delta+\mu). \tag{D.10}$$

The relevant solution satisfies $0 < z_0^*(A_0) \leq (\delta-\rho)/2$ for $\overline{A} < A_0 \leq \widehat{A}$, as shown in Appendix A.9. Keeping $\theta = 0$, we differentiate (D.8) with respect to b at $b = 0$. Using $A'(0) = \phi A_0$ and (D.10), we obtain

$$\left.\frac{dz^*}{db}\right|_{b=0} = \frac{A_0\big(\delta - \rho - z_0^*(A_0)\big)\big(\phi z_0^*(A_0) - A_0 L_0^B/L\big)}{z_0^*(A_0)\big(A_0 + \delta - \rho - 2z_0^*(A_0)\big)}. \tag{D.11}$$

We now derive the condition under which a marginal increase in basic research raises long-run growth. Substituting (D.11) into (D.9) and simplifying using (D.10), we obtain $dg^*/db|_{b=0} > 0$ if and only if

$$\phi > \underline{\phi}(A_0) \equiv \frac{L_0^B}{L}\left(1 + \frac{\alpha(\rho+\mu)(\delta+\mu)\big(\delta-\rho-z_0^*(A_0)\big)}{z_0^*(A_0)\big((\delta-\rho-z_0^*(A_0))^2 + (1-\alpha)(\rho+\mu)(\delta+\mu)\big)}\right). \tag{D.12}$$

Since R&D subsidies alone cannot improve growth in this parameter range, this improvement requires additional basic research. Moreover, the threshold $\underline{\phi}(A_0)$ is strictly decreasing in A_0. To see this, the initial gap $z_0^*(A_0)$ is strictly increasing in A_0.[95] The fraction in (D.12) is strictly decreasing in $z_0^*(A_0)$, since its derivative with respect to z has the opposite sign of $(1-\alpha)(\rho+\mu)(\delta+\mu)(\delta-\rho) + (\delta-\rho-z)^2(\delta-\rho-2z)$, which is positive for

[95] Solving (D.10) for A_0 gives $A_0 = (\rho+\mu)(\delta+\mu)/(\delta-\rho-z)+z$. The right-hand side is strictly increasing in z for $0 < z \leq (\delta-\rho)/2$ and equals $\overline{A}$ at $z = 0$. Hence, its inverse $z_0^*(A_0)$ is strictly increasing in A_0 and approaches zero as $A_0 \downarrow \overline{A}$.

$0 < z \leq (\delta - \rho)/2$. Therefore, $\underline{\phi}(A_0)$ is strictly decreasing in A_0 and diverges as $A_0 \downarrow \overline{A}$.

Proof of part (b). Suppose $A_0 > \widehat{A}$. We first derive an upper bound on long-run growth for a given level of basic research and then examine whether this bound can be attained through R&D subsidies. Eliminating D^* and θ from the stationary conditions, as in Appendix A.4, gives

$$g^* = \overline{g}(A)\left(1 - b\frac{L_0^B}{L} - \frac{\alpha(\rho+\mu)(\delta+\mu)}{z^*(\delta-\rho-z^*)+(\rho+\mu)(\delta+\mu)}\right), \qquad A = A_0(1+b)^{\phi}. \quad \text{(D.13)}$$

Compared with (A.13), basic research increases R&D productivity through $A = A_0(1+b)^{\phi}$ but reduces the labor available for private production and R&D by bL_0^B. For a given b, the dependence of growth on z^* is therefore unchanged from the baseline model.

We first fix b and consider how long-run growth depends on the steady-state gap z^*. Equation (D.13) is maximized with respect to z^* at $z^* = (\delta - \rho)/2$, yielding the upper bound

$$\widehat{g}(b) = \overline{g}(A_0)(1+b)^{\phi}\left(1 - b\frac{L_0^B}{L} - 2\alpha\frac{\overline{A}}{\widehat{A}}\right). \quad \text{(D.14)}$$

The remaining question is whether this upper bound can be attained at the saddle-stable steady state S through an R&D subsidy. Substituting $g^* = \widehat{g}(b)$ and $z^* = (\delta - \rho)/2$ into (D.3) and solving for θ yields the candidate subsidy rate

$$\widehat{\theta}(b) = \frac{A(1 - bL_0^B/L - 2\overline{A}/\widehat{A}) - (\delta-\rho)/2}{A(1 - bL_0^B/L - 2\alpha\overline{A}/\widehat{A}) - (\delta-\rho)/2}. \quad \text{(D.15)}$$

The denominator of (D.15) exceeds its numerator by $2(1-\alpha)A\overline{A}/\widehat{A} > 0$. Hence, $\widehat{\theta}(b) \in [0, 1)$ if and only if its numerator is nonnegative. At $z = (\delta - \rho)/2$, the derivative of the left-hand side of (D.8) with respect to z is $-(\delta-\rho)^2/4 - (\rho+\mu)(\delta+\mu) < 0$. Hence, whenever the candidate subsidy rate is feasible, the corresponding steady state is S.

We now determine the level of basic research that maximizes $\widehat{g}(b)$ and examine whether this maximum is attainable. Recall that $\underline{\Phi} \equiv (L_0^B/L)\widehat{A}/(\widehat{A} - 2\alpha\overline{A})$, as defined in Proposition 9. Differentiating (D.14) gives

$$\widehat{g}'(b) = \overline{g}(A_0)(1+b)^{\phi-1}\frac{L_0^B}{L}\left(\frac{\phi - \underline{\Phi}}{\underline{\Phi}} - (1+\phi)b\right). \quad \text{(D.16)}$$

The sign of (D.16) is determined by the term in parentheses, which is strictly decreasing in b. If $\phi \leq \underline{\Phi}$, $\widehat{g}(b)$ is maximized at $b = 0$. Since $A_0 > \widehat{A}$, the subsidy rate $\widehat{\theta}(0)$ is positive and less than one, so this upper bound is attainable. Hence, long-run growth is maximized without additional basic research, proving part (b-i).

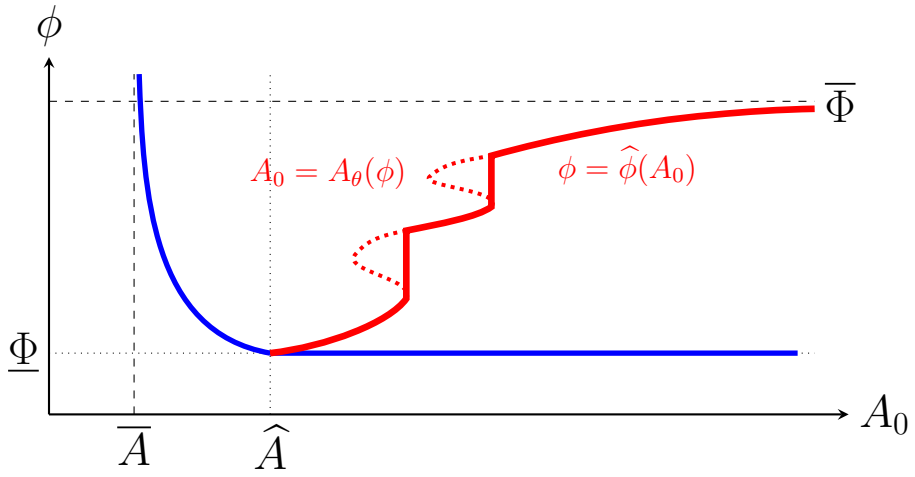


Figure D.1: Construction of the threshold $\widehat{\phi}(A_0)$ from the boundary $A_\theta(\phi)$ (schematic). For each A_0, $\widehat{\phi}(A_0)$ is the smallest value of ϕ on the boundary. The dotted part of the boundary is therefore not used.

If $\phi > \underline{\Phi}$, (D.16) implies that a small increase in b from zero raises $\widehat{g}(b)$. Since $\widehat{\theta}(0) \in (0,1)$, this higher growth rate is attainable for sufficiently small $b > 0$. Hence, any growth-maximizing policy requires $b > 0$, proving part (b-ii).

For part (b-iii), we first maximize the upper bound $\widehat{g}(b)$ without imposing its attainability. When $\phi > \underline{\Phi}$, (D.16) gives

$$\widehat{b} = \frac{\phi - \underline{\Phi}}{(1+\phi)\underline{\Phi}} > 0. \tag{D.17}$$

We then determine whether this upper bound can actually be attained at $b = \widehat{b}$. Substituting (D.17) into the numerator of (D.15), we find that $\widehat{\theta}(\widehat{b}) > 0$ if and only if $\phi < \overline{\Phi}$ and $A_0 > A_\theta(\phi)$, where

$$A_\theta(\phi) = \frac{\widehat{A}(\delta-\rho)(1+\phi)}{4(1-\alpha)\overline{A}(\overline{\Phi}-\phi)}(1+\widehat{b})^{-\phi}, \qquad \overline{\Phi} \equiv \frac{\widehat{A}(1+L_0^B/L) - 2\overline{A}}{2(1-\alpha)\overline{A}}. \tag{D.18}$$

At $\phi = \underline{\Phi}$, $\widehat{b} = 0$ and $A_\theta(\underline{\Phi}) = \widehat{A}$. The boundary tends to infinity as ϕ approaches $\overline{\Phi}$. For each $A_0 > \widehat{A}$, let $\widehat{\phi}(A_0)$ be the first value at which this boundary reaches A_0, as shown in Figure D.1. Before this crossing, $A_0 > A_\theta(\phi)$ and $\widehat{\theta}(\widehat{b}) > 0$. Thus, for $\underline{\Phi} < \phi < \widehat{\phi}(A_0)$, growth is maximized by positive basic research expansion and a positive research subsidy.

D.5 Calibration of L_0^B/L in the Basic Research Model

The GDP of this economy is defined as the sum of consumption expenditures $C_t = X_t$, private investment expenditures for R&D, $(1-\theta)w_t L_t^R$, and government expenditures, $\theta w_t L_t^R + w_t L^B$. Since the production of final goods (14) and the wage (20) are unchanged,

GDP becomes $\text{GDP}_t = n_t^{\frac{1-\alpha}{\alpha}}(L_t^P + \alpha L_t^R + \alpha L^B)$. Therefore, the ratio of basic research expenditure to GDP is given by

$$\frac{w_t L^B}{\text{GDP}_t} = \frac{\alpha L^B}{L_t^P + \alpha L_t^R + \alpha L^B} \equiv \sigma_t^B. \tag{D.19}$$

At the benchmark $b = 0$, we have $L^B = L_0^B$ and $L_t^P + L_t^R = L$ from (D.1). Eliminating L_t^P from (D.19) and then solving for L_0^B/L yields $L_0^B/L = (\sigma_t^B/(\alpha(1-\sigma_t^B)))\big(1-(1-\alpha)L_t^R/L\big)$. Using (25), this equation becomes

$$\frac{L_0^B}{L} = \frac{\sigma_t^B}{\alpha(1-\sigma_t^B)}\left(1 - \frac{\alpha}{A_0}g_t\right).$$

For σ_t^B, we use the average value of the basic research expenditures as a percentage of GDP in the 2000–2023 period from the OECD Main Science and Technology Indicators, over the years for which it is available. This series includes basic research funded by firms, and we use it as a proxy for publicly funded basic research. Accordingly, for g_t, we use the average value of the real GDP growth rate in the same period from the Penn World Table version 11.0, as in Section 4.4. Then, we obtain the estimate of the average value of L^B/L in the 2000–2023 period, which we treat as L_0^B/L. This procedure yields $L_0^B/L = 0.545\%$ for the United States and $L_0^B/L = 0.361\%$ for Italy.

E Analysis of the Sustainable and Unsustainable Subsidies

E.1 Conditions at the Termination of the Subsidy

This subsection derives the conditions that connect the equilibrium path before and after the regime change under the unsustainable policy, as discussed in Section 7.2.

Below, we show that this transition creates discontinuities in both g_t and D_t, even though the regime change is fully anticipated. The discontinuities originate from the discontinuous change of θ from $\theta^u > 0$ to 0 at time T. From (16) and (20), the cost of R&D net of the subsidy before T is $(1-\theta^u)\alpha L n_t^{(1-2\alpha)/\alpha}/A$. Therefore, at T, the cost of entry, net of the subsidy, jumps up by a factor of $1/(1-\theta^u)$. As long as there are positive amounts of entry both before and after T, the value of firms equals the cost of entry both before and after T. By writing the value of the firm just before T by v_{T-} (formally, $v_{T-} = \lim_{t\uparrow T} v_t$), the value of the firm before and after the regime change is connected by $v_{T-} = (1-\theta^u)v_T$. Note that the no-arbitrage condition (19) is derived assuming that the firm value is differentiable with respect to time. When the firm value changes discretely,

we need to construct the no-arbitrage condition using the gross interest factor (the inverse of the discount factor), as we explain below.

Let us define the gross interest factor $R(t,t')$ by the amount of final goods obtained at time t' by investing one unit of final goods in government bonds at time $t \leq t'$. $1/R(t,t')$ represents the discount factor between t and t'; i.e., the value at t of one unit of final goods delivered at t'. The gross interest factor $R(t,t')$ has the following properties. Under perfect foresight, $R(t,t'') = R(t,t')R(t',t'')$ holds for $t \leq t' \leq t''$. Also, if the interest rate is finite between t and t', then $R(t,t') = \exp\left(\int_t^{t'} r_{t''}dt''\right)$ holds.

Note that within each policy regime, the interest rate is given by (21), which is finite. Therefore, $R(t,t') = \exp\left(\int_t^{t'} r_{t''}dt''\right)$ holds if $t < t' < T$ or $T \leq t < t'$. However, this relationship may not hold on an interval that spans $t < T$ and $t' \geq T$ because Equation (21) depends on the no-arbitrage condition (19), which is not well defined at T when $\dot{v}_t$ does not exist. In this case, we need to consider the gross interest factor between just before T and at T, denoted by $R(T-,T) \equiv \lim_{t \uparrow T} R(t,T)$.

Given that investors know that the subsidy is only available before T, they are willing to pay a discrete interest premium if they can obtain funds before the subsidy terminates (i.e., $t < T$) and repay them at $t' \geq T$, because funds at $t < T$ can be used to take advantage of the subsidy while those at $t' \geq T$ cannot. What matters is whether they can get funds before T, not how long before T. They are willing to pay this premium even as the duration between $t < T$ and $t' \geq T$ approaches zero. When interpreted in terms of interest rate (which is defined per unit time), there would be an unbounded spike in interest rate at time T. Expressed in terms of the gross interest factor, the spike can be represented as $R(T-,T) > 1$.

We now formally derive the value of $R(T-,T)$. From (20), the cost of developing a new variety before subsidies, $w_t L/(An_t)$, depends only on n_t. Since n_t always follows (15), it is continuous at T. However, the cost borne by firms rises from $(1-\theta^u)w_t L/(An_t)$ to $w_t L/(An_t)$ at T. Since the free entry condition (16) holds with equality both before and after T, the value of firms satisfies $v_{T-} = (1-\theta^u)v_T$. Note that the value of a firm at $t < T$ can be divided into two parts,

$$v_t = \int_t^{T-} \pi_{t'} e^{-\int_t^{t'} r_{t''}dt''} dt' + \frac{v_T}{R(t,T)}, \tag{E.1}$$

where the first term is the present discounted value of the profits before T, and the second term is the value of the firm at T, discounted by $R(t,T)$. Using $v_{T-} = (1-\theta^u)v_T$, we can

rewrite (E.1) as

$$R(t,T) = \frac{1}{1-\theta^u} \frac{v_{T-}}{v_t - \int_t^{T-} \pi_{t'} e^{-\int_t^{t'} r_{t''} dt''} dt'}. \tag{E.2}$$

The integral in the denominator is the present value of the profits paid between t and $T-$. From Appendix A.2, the profit is $\pi_{t'} = (1-\alpha) n_{t'}^{(1-2\alpha)/\alpha} L_{t'}^P$, which is at most $(1-\alpha) L\, n_t^{(1-2\alpha)/\alpha}$ for $t' \geq t$ because the number of goods n_t does not decrease. Also $e^{-\int_t^{t'} r_{t''} dt''}$ is finite because r_t is finite between t and $T-$. This implies that the integral is at most $T-t$ times a bounded value, so that it converges to zero as t approaches T from below. Since v_t converges to $v_{T-} = (1-\theta^u) v_T$, taking the limit of (E.2) gives

$$R(T-,T) = \frac{v_T}{v_{T-}} = \frac{1}{1-\theta^u} > 1. \tag{E.3}$$

In terms of the instantaneous interest rate, we can interpret (E.3) as r_t having an unbounded spike at time T.[96] Because individuals know the subsidy will be terminated at T, there is a huge demand for last-minute R&D just before T. Since the investment is done by borrowing, the interest rate needs to spike at T to maintain equilibrium in the financial market. The government bond also needs to provide the same rate of return. This means that the outstanding debt will be multiplied by $R(T-,T) = 1/(1-\theta^u) > 1$ at the time T.[97] Since $Y_t = n_t^{(1-\alpha)/\alpha} L$ cannot jump, $D_t = B_t/Y_t$ is also multiplied by $1/(1-\theta^u)$:

$$D_T = \frac{D_{T-}}{1-\theta^u}. \tag{E.4}$$

The spike in the interest rate also affects the consumption. The dynamic optimization of individuals implies that the intertemporal marginal rate of substitution between t and t' should always be equal to the gross interest factor between t and t'. Given the logarithmic utility function (3), this requirement reduces to $(c_{s,t'}/c_{s,t}) e^{\rho(t'-t)} = R(t,t')$.[98] Taking the limit $t \uparrow T$ and using (E.3), we obtain $c_{s,T} = c_{s,T-}/(1-\theta^u)$. Intuitively, the individuals who

[96] We can also interpret this as a result, by analogy with the relationship between the distribution function and the density function in statistics. In this analogy, $\ln R(t,t')$ corresponds to the mass of distribution between t and t', and r_t corresponds to the density function at point t. If $R(t,t')$ is continuous and differentiable at t, the derivative of $\ln R(t,t')$ with respect to t would give $-r_t$. However, when $R(t,t')$ is discontinuous, r_t is not well defined. This happens when the distribution has a mass point. In that case, the distribution's density has an unbounded spike at this point. In this analogy, we can think of the interest rate as having an unbounded spike at time T.

[97] In the perfect foresight model of capital accumulation with an investment subsidy studied by Judd (1985), the return on bonds contains the term $-\dot{\theta}/(1-\theta)$, where θ is the subsidy rate. When the subsidy rate falls discretely from θ^u to zero, the integral of this term is $\ln\bigl(1/(1-\theta^u)\bigr)$, which coincides with the mass point obtained here. The same paper also notes that the subsidy rate need not be differentiable if the equilibrium conditions are interpreted in terms of generalized functions.

[98] When $R(t,t')$ is differentiable, this relationship gives the usual Euler equation.

know that there is a spike in the interest rate at T are willing to wait until T to consume more. Thus, individual consumption jumps up at T. Since the number of individuals born between t and T converges to zero as $t \uparrow T$, aggregating over generations by (5) gives $C_T = C_{T-}/(1-\theta^u)$. The aggregate consumption also jumps up by the factor of $1/(1-\theta^u)$.

Since final goods are used only for consumption, $C_t = X_t = n_t^{(1-\alpha)/\alpha} L_t^P$ holds from (14). Because n_t is continuous, the jump in consumption implies $L_T^P = L_{T-}^P/(1-\theta^u)$. Substituting $L_t^P/L = 1 - g_t/\overline{g}(A)$, which follows from (25) and (23), we obtain

$$g_{T-} = \theta^u \, \overline{g}(A) + (1-\theta^u) \, g_T. \tag{E.5}$$

As explained above, investors are willing to do R&D before T, which creates a spike in the interest rate at T. This induces consumers to wait until T to consume more. The labor freed up in the production sector can thus be allocated to R&D. At time T, this frenzy ends, R&D investment falls, and the growth rate jumps down.

E.2 Method of Numerical Simulation

We use the linear system in Appendix A.6 and the closed forms in Appendix A.12. Under the sustainable policy, (A.25), (A.26), and (A.27), evaluated at $\theta = \theta^s$ and $D_0 = 0$, give the equilibrium path.

For the unsustainable policy, first compute the path after termination under $\theta = 0$. The post-switch debt ratio is $D_T = 0.9 D_U(A, 0)$, and g_T is the height of the equilibrium line at this debt ratio. The closed forms apply with initial condition D_T and time $t - T$. The pre-switch point is then determined by (E.4) and (E.5).

Next, fix a candidate subsidy $\theta^u > \overline{\theta}(A)$ and evaluate $\mathbf{M}$ in (A.18) at that rate. Before termination, the solution is obtained from

$$\frac{\boldsymbol{\gamma}_t}{\gamma_{3,T-}} = e^{\mathbf{M}(t-T)}(D_{T-}, g_{T-}, 1), \qquad t < T.$$

Taking component ratios yields (D_t, g_t). This expression applies even though the economy has no feasible steady state under a permanent subsidy at that rate.

The subsidy begins at the latest earlier time at which $D_t = 0$, found by bisection. We check that the third component remains positive and that $0 < g_t < \overline{g}(A)$ throughout this interval. We then adjust θ^u by bisection until the interval has the prescribed length T. For the calibrations reported in the paper, the interval becomes shorter as θ^u rises.

Within either regime, output ratios follow from $Y_t/Y_{t'} = (\gamma_{3,t}/\gamma_{3,t'})^{1/s(\theta)}$. Output is

continuous at termination. The consumption and GDP paths then follow from the relations in Appendix A.12.

E.3 Expected Lifetime Utility of Each Cohort

This subsection explains how we compute the expected lifetime utility of the individuals alive at $t = 0$, reported in Section 7.2. Before $t = 0$, the economy is on the balanced growth path without subsidies, on which $r_t = r^*(A, 0)$ and the wage grows at $g^*(A, 0)$. For an individual born at $s < 0$, human wealth at birth is $h_{s,s} = w_s/\big(r^*(A,0)+\mu+\delta-g^*(A,0)\big)$ by (A.3), and consumption is $c_{s,t} = (\rho+\mu)h_{s,s}e^{(r^*(A,0)-\rho)(t-s)}$ by (A.2) and the Euler equation for individuals. Substituting these into (4) and integrating from $k_{s,s} = 0$ gives the asset holding just before $t = 0$,

$$k_{s,0-} = \frac{w_0\big(e^{-(r^*(A,0)-\rho-g^*(A,0))s} - e^{\delta s}\big)}{r^*(A,0) + \mu + \delta - g^*(A,0)}. \tag{E.6}$$

At $t = 0$, the subsidy rate rises from zero to θ^s or θ^u unexpectedly. Since the government has no debt at $t = 0$, all assets are shares of firms, and the value of a firm falls from $w_0L/(An_0)$ to $(1-\theta)w_0L/(An_0)$ by (16), where θ is the new subsidy rate. Hence, $k_{s,0} = (1-\theta)k_{s,0-}$. Human wealth at $t = 0$ is $h_{s,0} = e^{\delta s}h_{0,0}$ by (A.3), and $h_{0,0} = \big((\beta+\delta)/\beta\big)\overline{h}_0$ by (A.6), where $\overline{h}_0 = C_0/\big((\rho+\mu)N_0\big) - K_0/N_0$ by (A.4), with $C_0/Y_0 = 1 - g_0/\overline{g}(A)$ and $K_0/Y_0 = \alpha(1-\theta)/A$ from (A.7). Consumption at $t = 0$ is $c_{s,0} = (\rho+\mu)(k_{s,0} + h_{s,0})$ by (A.2). For $t > 0$, $c_{s,t} = c_{s,0}\exp\big(\int_0^t (r_{t'} - \rho)dt'\big)$ by the Euler equation for individuals, multiplied by $1/(1-\theta^u)$ for $t \geq T$ under the unsustainable policy (Appendix E.1). The interest rate is $r_t = \overline{g}(A)/(1-\theta) - \big(s(\theta)-1\big)g_t$ by (27), and $\int_0^t g_{t'}dt' = \ln(\gamma_{3,t}/\gamma_{3,0})/s(\theta)$ holds within each regime by the definition of $\boldsymbol{\gamma}_t$, where g_t and $\gamma_{3,t}$ are given by the closed forms in Appendices A.12 and E.2. By (3), the expected utility of the individual from $t = 0$ onward is proportional to $\int_0^\infty e^{-(\rho+\mu)t}\ln c_{s,t}\,dt$, which equals $\ln c_{s,0}/(\rho+\mu)$ plus a term that is common to all cohorts under a given policy. We evaluate the integral numerically using the closed form of r_t. An individual supports the unsustainable policy if this expected utility is higher than under the sustainable policy, and the boundary age reported in Section 7.2 is the age at which the two are equal.

Figure E.1 shows the gain from each policy relative to the case without subsidies, by the age of the individual at $t = 0$. We measure the gain by the proportional increase in consumption at all remaining ages that would give the same expected utility as the policy. Since utility is logarithmic, this proportion is the exponential of $\rho + \mu$ times the difference in $\int_0^\infty e^{-(\rho+\mu)t}\ln c_{s,t}\,dt$, minus one. Under a positive subsidy, the gain falls with

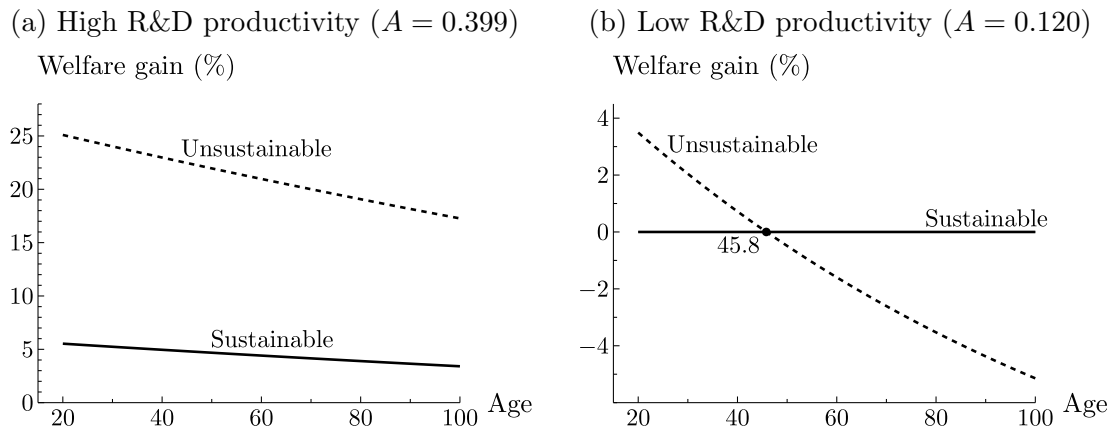


Figure E.1: Welfare gains from the two policies by the age of the individuals alive at the introduction of the policy. The gain is the percentage increase in consumption at all remaining ages that is equivalent to the policy, relative to the case without subsidies. In Panel (b), the sustainable policy is $\theta = 0$, so its gain is zero.

age, because older individuals hold more shares of existing firms, whose value falls when the subsidy is introduced. In Panel (b), the sustainable policy is $\theta = 0$, and the gain from the unsustainable policy turns negative at the age of 45.8.

F Analysis of the Asset Taxation Extension

This appendix analyzes the model with asset taxation in Section 7.3. We derive the equilibrium dynamics, examine the steady states, and prove Proposition 10.

F.1 Derivation of Equilibrium Dynamics

The asset income tax changes the return received by households to $(1-\tau_r)r_t$. Hence, with $\beta = \mu$, the aggregate consumption equation is

$$\dot{C}_t = \big((1-\tau_r)r_t - \rho + \delta\big)C_t - (\rho+\mu)(\delta+\mu)K_t. \tag{F.1}$$

The government receives $\tau_r r_t K_t$ and pays the research subsidy. Its budget is $\dot{B}_t = r_t B_t + \theta w_t L_t^R - \tau_r r_t K_t$. The supply side is unchanged, because the tax applies equally to interest, dividends, and capital gains. Thus, the pre-tax relation (27) still holds, and the after-tax gap is

$$g_t - (1-\tau_r)r_t = (1-\tau_r)s(\theta)\big(g_t - \widetilde{g}(A,\theta)\big) + \tau_r g_t. \tag{F.2}$$

The consumption and asset ratios in (A.7) also remain unchanged. Substituting them into the government budget and the consumption equation gives

$$\dot{D}_t = -\big(g_t - (1-\tau_r)r_t\big)D_t + \frac{\theta\alpha}{\overline{g}(A)}g_t - \tau_r r_t \frac{\alpha(1-\theta)}{A}, \tag{F.3}$$

$$\dot{g}_t = (\overline{g}(A) - g_t)\big(g_t - (1-\tau_r)r_t - \delta + \rho\big) + \overline{g}(A)(\rho+\mu)(\delta+\mu)\left(D_t + \frac{\alpha(1-\theta)}{A}\right). \tag{F.4}$$

The tax on bond interest reduces the effective interest cost of government debt. The last term of (F.3) is the tax revenue from firm shares. Since r_t is determined by (27), the system is again in D_t and g_t.

F.2 Steady States

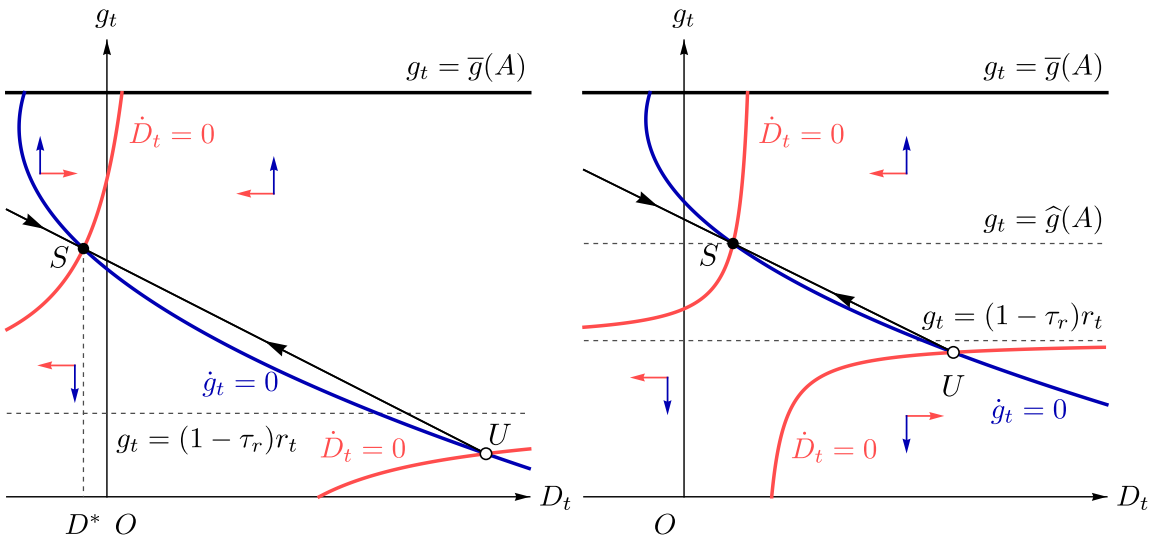
The steady states are the intersections of the two stationary loci obtained from (F.3) and (F.4). Eliminating D gives

$$\begin{aligned}\big(g-(1-\tau_r)r\big)\Big(&\big((1-\tau_r)A + \tau_r\overline{g}(A) - (g-(1-\tau_r)r)\big)\\ &\times\big(\delta-\rho-(g-(1-\tau_r)r)\big) - (\rho+\mu)(\delta+\mu)\Big)\\ = (\rho+\mu)(\delta+\mu)(1-\alpha)A&\left(\frac{(1-\tau_r)\theta}{1-\theta} - \frac{(1-\alpha)\tau_r}{\alpha}\right).\end{aligned} \tag{F.5}$$

The return r is the function of g in (27), so this is a cubic condition in g. There is one extra intersection above $g = \overline{g}(A)$, denoted by E and excluded by the labor constraint. The higher-growth feasible intersection is S, as in the baseline model. The common quadratic coefficient of the dynamic equations is now $(1-\tau_r)s(\theta) + \tau_r$, so the linear representation and the equilibrium-line argument in Appendix A.6 continue to apply.

Unlike the baseline case, the budget may have a surplus without subsidies. With positive pre-tax interest, a positive after-tax gap and $\theta = 0$ require $D < 0$ in (F.3). This is the configuration shown in Figure F.1(a). The subsidy shifts the stationary loci, and the rate that attains $\widehat{g}(A)$ selects the higher-growth intersection in panel (b), as proved below.

At $g = \overline{g}(A)$, the after-tax gap is $(1-\tau_r)A + \tau_r\overline{g}(A)$, and the difference between the two sides of (F.5) is negative. Without subsidies it equals $-(\rho+\mu)(\delta+\mu)(1-\alpha\tau_r)A$, and a higher subsidy makes it smaller. Hence, the excluded intersection is unique by the argument in Appendix C.3.



(a) $\theta = 0$ (b) $\theta = \widehat{\theta}(A, \tau_r)$

Figure F.1: Stationary loci with asset taxation. Panel (a) shows the no-subsidy economy, and panel (b) shows the subsidy that attains the highest long-run growth rate.

F.3 Proof of Proposition 10

For the calculations here, write g^*, r^*, and D^* for $g^*(A, \theta, \tau_r)$, $r^*(A, \theta, \tau_r)$, and $D^*(A, \theta, \tau_r)$, and $z \equiv g^* - (1-\tau_r)r^*$ for the after-tax gap. The stationary budget gives $zD^* = \theta\alpha g^*/\overline{g}(A) - \tau_r r^*\alpha(1-\theta)/A$. Substituting this into the growth condition and using the pre-tax supply relation (27) yields

$$\big(\overline{g}(A) - g^*\big)z(\delta - \rho - z) = (\rho + \mu)(\delta + \mu)\big(g^* - (1-\alpha)\overline{g}(A)\big).$$

Thus, the growth rate is given by (A.13) with this after-tax gap. Neither tax nor subsidy appears separately in that relation.

Proof of part (a). Suppose a long-term BGP with positive research has $r^* \leq 0$. By (27), this requires $g^* \geq (1-\alpha)\overline{g}(A)/(\alpha - (2\alpha - 1)\theta)$, a threshold increasing in θ. Since the no-policy BGP has positive interest, $g^* > g^*(A, 0, 0)$. The after-tax gap is at least g^*, so it exceeds the original no-policy gap, which is above $(\delta - \rho)/2$ by $A > \widehat{A}$. But (A.13) is decreasing in the gap above this value, implying $g^* < g^*(A, 0, 0)$, a contradiction. If research is absent on the BGP, the stationary valuation of firms gives $r^*\nu^* = 1 - \alpha > 0$. Hence, interest is positive whenever a long-term BGP exists.

Proof of part (b). The highest growth rate allowed by (A.13) is attained at $z = (\delta - \rho)/2$ and equals $\widehat{g}(A)$. Substituting this gap and growth rate into (F.2) gives the required subsidy:

$$\widehat{\theta}(A, \tau_r) = \frac{(1 - \alpha\tau_r)\widehat{g}(A) - (1-\alpha)\big((1-\tau_r)\overline{g}(A) + (\delta - \rho)/2\big)}{(\alpha + \tau_r - 2\alpha\tau_r)\widehat{g}(A) - (1-\alpha)(\delta - \rho)/2}. \tag{F.6}$$

Since the interest rate $r^*(A,0,0)$ is positive, the growth rate $g^*(A,0,0)$ exceeds the gap without policy, which is above $(\delta-\rho)/2$ because A is larger than $\widehat{A}$. Together with $\widehat{g}(A) > g^*(A,0,0)$ by Proposition 3, this gives $\widehat{g}(A) > g^*(A,0,0) > (\delta-\rho)/2$. The denominator is therefore positive, because the coefficient $\alpha+\tau_r-2\alpha\tau_r$ is at least $1-\alpha$. The numerator is linear in τ_r and positive at both zero and one, so it is positive for all tax rates under consideration. The denominator exceeds the numerator by $(1-\alpha)(1-\tau_r)(\overline{g}(A)-\widehat{g}(A)) > 0$. Hence, the required subsidy lies in $(0,1)$.

The resulting intersection is S, as in Figure F.1(b). Its growth rate is below the labor limit, and the derivative of the left-hand side of (F.5) with respect to the after-tax gap at $z=(\delta-\rho)/2$ is $-(\delta-\rho)^2/4-(\rho+\mu)(\delta+\mu) < 0$. It is therefore the higher-growth feasible intersection. The upper growth bound is attained, proving part (b).

Proof of part (c). Setting $\tau_r = 0$ in (F.6) gives $\widehat{\theta}(A,0) = \widehat{\theta}(A)$. Differentiating shows that the derivative with respect to τ_r has the sign of $(\overline{g}(A)-\widehat{g}(A))(2\widehat{g}(A)-\delta+\rho)$, which is positive by the preceding inequalities.

Proof of part (d). At the optimum, the growth rate and after-tax gap do not change with the tax rate. The stationary growth condition therefore gives

$$D^*\big(A,\widehat{\theta}(A,\tau_r),\tau_r\big) = \frac{(\delta-\rho)(\overline{g}(A)-\widehat{g}(A))}{2\overline{g}(A)(\rho+\mu)(\delta+\mu)} - \frac{\alpha(1-\widehat{\theta}(A,\tau_r))}{A}.$$

Only the second term changes with τ_r, and its derivative with respect to τ_r is $(\alpha/A)\,\partial\widehat{\theta}(A,\tau_r)/\partial\tau_r$, which is positive by part (c).

G Analysis of the Small Open Economy Extension

This appendix analyzes the small open economy in Section 7.4. We derive the equilibrium dynamics, examine the steady states, and prove Proposition 11.

G.1 Derivation of Equilibrium Dynamics

The supply side is the same as in the baseline model. Solving (27) for the interest rate gives $r_t = \overline{g}(A)/(1-\theta) - (s(\theta)-1)g_t$. Substituting the country-premium relation (60) yields the growth rate (61). Substituting (60) and (61) into the debt dynamics (29) gives

$$\dot{D}_t = (\eta+\psi_2(\theta))D_t^2 + \left(\overline{r} - \psi_1(A,\theta) - \frac{\theta\alpha\psi_2(\theta)}{\overline{g}(A)}\right)D_t + \frac{\theta\alpha\psi_1(A,\theta)}{\overline{g}(A)}. \tag{G.1}$$

Thus, the system reduces to one differential equation in nonnegative debt.

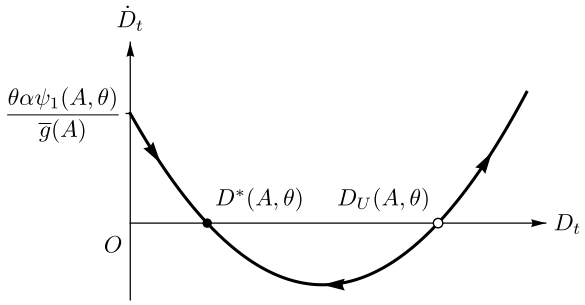


Figure G.1: Debt dynamics in the small open economy with a positive subsidy.

G.2 Steady States

The right-hand side of (G.1) is an upward-opening quadratic, because its leading coefficient $\eta+\psi_2(\theta)$ is positive. At zero debt it equals subsidy expenditure, which is positive for $\theta > 0$ in the research region. When the subsidy is not too large, the quadratic has two positive zeros $D^* < D_U$, and

$$\dot{D}_t = (\eta + \psi_2(\theta))(D_t - D^*)(D_t - D_U).$$

The arrows in Figure G.1 show that debt converges to D^* from every nonnegative initial value below D_U. The lower-debt point is the long-term BGP, and the higher one is the unsustainable threshold.

Above D_U, debt rises at a positive rate that increases with debt. Within finite time it becomes large enough to stop research by (61). Thereafter, output is constant, and $\dot{D}_t = (\overline{r}+\eta D_t)D_t$ exceeds ηD_t^2. Hence, $1/D_t$ falls at least at the rate η, and debt becomes infinite within a period of $1/(\eta D_t)$. By the definition of an equilibrium path in Appendix A.6, there is no equilibrium path from $D_0 > D_U$.

G.3 Proof of Proposition 11

Proof of part (a). When $\theta = 0$, zero debt is a steady state and $r^*(A, 0) = \overline{r}$. By (61), $g^*(A, 0) = ((1-\alpha)/\alpha)(\overline{g}(A) - \overline{r})$. Its gap is positive if and only if $A > \overline{A}^{\text{open}}$. The other zero of (G.1) is $\alpha(g^*(A, 0) - \overline{r})/\eta$. If $A > \overline{A}^{\text{open}}$, this zero is positive, and zero debt is the lower steady state. If $A < \overline{A}^{\text{open}}$, it is negative, and debt rises from every positive initial value. This proves part (a).

Proof of part (b). At the no-subsidy BGP, $D^* = 0$. Differentiating the stationary government budget with respect to the subsidy gives

$$\left.\frac{dD^*}{d\theta}\right|_{\theta=0} = \frac{\alpha g^*(A,0)}{\overline{g}(A)(g^*(A,0) - \overline{r})}.$$

The numerator is additional subsidy spending per unit of the subsidy rate, and the gap in

the denominator is the rate at which growth reduces the debt ratio. Differentiating (61) then gives

$$\left.\frac{dg^*}{d\theta}\right|_{\theta=0} = \frac{1-\alpha}{\alpha}\left(\overline{g}(A) - g^*(A,0) - \eta \left.\frac{dD^*}{d\theta}\right|_{\theta=0}\right).$$

The first effect encourages research directly. The second crowds it out through debt and the country premium. Growth rises if and only if

$$\eta < \widehat{\eta}(A) \equiv \frac{\overline{g}(A)(g^*(A,0) - \overline{r})(\overline{g}(A) - g^*(A,0))}{\alpha g^*(A,0)}.$$

Substituting the no-policy growth rate gives the right-hand side of (62).

To obtain its inverse, write this threshold as $\widehat{\eta}(A) = (\overline{g}(A)/\alpha)(\overline{g}(A) - g^*(A,0))(1 - \overline{r}/g^*(A,0))$. Each factor is positive and increasing when $A > \overline{A}^{\text{open}}$. In particular, the second equals $(1-\alpha)((2\alpha-1)A + \alpha\overline{r})/\alpha^2$, which increases because α exceeds $1/2$. The threshold is zero at $\overline{A}^{\text{open}}$ and tends to infinity with A. Therefore, its inverse $\widehat{A}^{\text{open}}(\eta)$ is well defined and strictly increasing, with $\widehat{A}^{\text{open}}(0) = \overline{A}^{\text{open}}$. This proves part (b).