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New Assets, Limits to Arbitrage, and Bubbles in Dividend-Yielding Assets

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Abstract

The successive creation of new assets generates bubbles in dividend-yielding assets. Because assets that have not yet been created cannot be traded in advance, arbitrage between existing and future assets is limited. Consequently, the prices of newly created assets may deviate from their fundamental values, resulting in asset bubbles. However, if no new assets are created, or if future assets can be traded, arbitrage trading eliminates these bubbles.

Keywords: rational bubbles, technological and financial innovation, deregulation, balanced growth.

JEL classification numbers: E1, G1, G12, N2

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1 Introduction

Motivation: Asset price bubbles often follow episodes of innovation and deregulation. Historical examples include the railway mania of the 1840s, the boom in automobile and radio stocks in the 1920s, the Japanese bubble of the 1980s, the dot-com episode of the late 1990s, and the housing boom of the 2000s. Scheinkman (2014) observes that bubbles tend to arise amid excitement about innovation, while Aliber, Kindleberger, and McCauley (2023) place innovation within the broader category of “displacements” – shocks that alter profit opportunities and often precede speculative booms. A common feature of these episodes is the rapid creation of new dividend-paying assets: technological innovation creates new industries and products, financial innovation transforms existing assets into new tradable securities, and deregulation makes previously non-traded assets tradable.

The role of new asset creation in *rational* bubbles remains difficult to understand because newly created assets have historically taken the form of dividend-paying claims. Conventional asset-pricing models typically imply that the price of a dividend-paying asset equals its fundamental value, defined as the present discounted value of its future dividends, ruling out rational bubbles in such assets (see, e.g., Santos and Woodford, 1997). As we have already seen, however, bubbles routinely arise in stocks, land, and housing, all of which generate positive dividends in the form of profits or rents. This disconnect points to the need for a better understanding of the relationship between new asset creation and rational bubbles in dividend-paying assets.

This paper shows how the continual creation of new assets can support rational bubbles in dividend-paying assets within a standard general-equilibrium asset-pricing framework. The key to this result is that arbitrage operates among existing assets but not between existing and unborn assets, because claims on the latter cannot be traded before creation, as emphasized by Tirole (1985). This limitation allows existing assets to trade above their fundamental values. The mechanism applies broadly because new assets arise throughout the economy, including patents, startup bonds, newly built housing, and securities created through IPOs or securitization. It may also help explain the historical association of bubbles with technological and financial innovation and deregulation, all of which can accelerate the creation of tradable assets.

Model and Results: Our benchmark is an overlapping-generations economy with Lucas trees, in which new trees are continually created. We refer to a tree that has not yet been created as an *unborn tree*. We show that the benchmark model admits an equilibrium in which the

prices of existing trees exceed the present discounted value of their dividends, which constitutes a rational bubble. Importantly, the bubble arises in a simple deterministic setting built on a standard model, and the resulting equilibrium lies on a balanced growth path. This result points to the broad applicability of our mechanism, which we demonstrate in models with endogenous investment, infinitely lived agents, or endogenous growth.

The result rests on three assumptions. First, new trees are born every period, either as exogenous endowments or through production, generating a continual flow of new assets. Second, trees are distinguished by their birth dates, so that trees born in different periods can trade at different prices even when they have identical future dividend streams. This assumption is closely related to Galí (2014, 2021), who allows pure-bubble prices to depend on birth year by relaxing symmetry, and Olivier (2000), who allows stock prices to depend on firm age by relaxing symmetric equilibrium. Third, unborn trees cannot be traded because no current market exists for claims on their future dividends. As Tirole (1985, p. 1508) observes, “a painting to be created by a 21st century master cannot be sold in advance by the painter’s forebears. Similarly, patents cannot be granted for future inventions.”

Intuition and Mechanism: The intuition behind our result becomes clear by considering what happens when each of the three assumptions is removed. In each case, fundamental pricing is restored. Without the continual birth of new trees, all trees already exist and are tradable, so the restriction on arbitrage arising from the non-tradability of unborn trees no longer applies. Without distinguishability by birth date, new and existing trees with identical dividend streams constitute a single homogeneous asset and must trade at the same price. If claims on unborn trees can be traded, arbitrage between existing trees and claims on unborn trees becomes possible, eliminating bubbles and restoring fundamental pricing.¹ Taken together, these results show how the continual creation of distinct assets and the non-tradability of unborn assets jointly limit arbitrage between existing and unborn assets, allowing bubbles to persist in our framework. This mechanism provides a possible explanation for the historical association between bubbles and transformative technologies and can be embedded naturally in endogenous growth models of research and development (R&D) investment.

Note that the limit to arbitrage concerns trades between existing and unborn assets, while arbitrage among existing assets remains operative. For example, two existing trees with identical

¹The non-tradability of unborn trees can be interpreted as a restriction on short sales of claims on unborn trees.

dividend streams can trade at different prices in a bubbly steady state while yielding the same rate of return. Because the trees have the same fundamental value, the price gap reflects a difference in their bubble components. This gap grows at the equilibrium rate of return, so the higher resale value of the more expensive tree compensates for its higher purchase price.

Contribution: This paper makes three contributions. First, we show that when continually created assets are distinguished by vintage and unborn assets are non-tradable, the prices of individual dividend-paying assets can contain rational bubble components. This result departs from standard growth models, which typically treat capital as homogeneous or restrict attention to symmetric equilibria across differentiated assets, and clarifies why such models rule out these bubbles. Second, we provide theoretical support for historical studies that link the emergence of new assets to asset price bubbles, often in the context of technological innovation, financial innovation, or deregulation. Third, we demonstrate the broader applicability of the mechanism by showing that it operates along a balanced growth path and in models with endogenous investment, infinitely lived agents, or endogenous growth.

Related Literature

Limits to Arbitrage: Our analysis is related to the literature on how limits to arbitrage can sustain bubbles, speculative valuations, and other departures from fundamental pricing (see Gromb and Vayanos, 2010, for a survey). One strand of this literature emphasizes heterogeneous beliefs and short-sale constraints. Harrison and Kreps (1978) show how these forces create a speculative resale option. Building on this mechanism, Scheinkman and Xiong (2003) show how investor overconfidence generates disagreement and gives rise to speculative bubbles. Other studies focus on the risks and constraints faced by arbitrageurs. De Long, Shleifer, Summers, and Waldmann (1990) study noise-trader risk, Shleifer and Vishny (1997) examine agency problems and capital withdrawals, and Abreu and Brunnermeier (2003) analyze synchronization risk.

Empirical studies document persistent price differences between assets with identical or nearly identical dividend streams, including Siamese twin stocks (Rosenthal and Young, 1990; Froot and Dabora, 1999), corporate spin-offs (Lamont and Thaler, 2003a,b), and ETFs (Iwai, 2011). Existing studies attribute such price differences to limits on arbitrage; Scruggs (2007) and Iwai (2011), for example, emphasize how noise-trader risk can prevent arbitrageurs from eliminating them.

Raional Bubbles: Asset price bubbles have been widely studied in connection with economic growth and business cycles. Because different studies use different definitions of asset price bubbles, our review focuses on the literature on rational bubbles, defined as asset prices exceeding the present discounted value of future dividends in equilibrium (see Hirano and Toda, 2025b; Hori, Im, and Nakaota, 2025). Much of this literature focuses on pure bubbles, namely, assets that pay no dividends and have no intrinsic value. Studies in this tradition include Samuelson (1958), Tirole (1985), Grossman and Yanagawa (1993), Futagami and Shibata (2000), Martin and Ventura (2012), Farhi and Tirole (2012), Aoki, Nakajima, and Nikolov (2014), Galí (2014), Aoki and Nikolov (2015), Hirano, Inaba, and Yanagawa (2015), Hirano and Yanagawa (2017), Hori and Im (2023), and Guerron-Quintana, Hirano, and Jinnai (2023). These studies provide important insights into the macroeconomic effects of bubbles, but do not address how bubbles can arise in assets that themselves generate dividends.

Only a small number of theoretical studies allow bubbles to arise in assets with positive fundamental values. Hirano and Toda (2025a) obtain such bubbles by allowing aggregate output to grow faster than dividends. Consequently, dividends become asymptotically negligible relative to the aggregate endowment, thereby relaxing the non-negligibility condition underlying the no-bubble theorem of Santos and Woodford (1997). Our mechanism instead relaxes a different condition underlying their theorem. Santos and Woodford (1997) rely on the premise that currently tradable portfolios can bound the present value of the aggregate endowment. In our model, however, the continual creation of new assets and the non-tradability of unborn assets prevent currently tradable portfolios from holding claims on assets created in the future. Consequently, the present value of the aggregate endowment cannot be bounded by any currently tradable portfolio. The non-tradability of unborn assets limits arbitrage, allowing rational bubbles to arise on dividend-yielding assets in general equilibrium.

Whether current portfolios include claims on future assets can also be understood in terms of who owns newly created assets. As Domeij and Ellingsen (2018) point out, the framework of Santos and Woodford effectively assigns ownership of newly created assets to owners of existing assets, making claims on unborn assets tradable through existing assets. Our model does not impose this ownership structure.

Olivier (2000) and Tanaka (2011) are also closely related to our analysis, as they obtain bubbles in the stocks of firms in growth models with R&D-driven variety expansion. Their primary focus is on the effects of bubbles on economic growth, whereas we identify the assumptions

responsible for the existence of a bubble equilibrium. We also show that the mechanism extends beyond the particular R&D environment to models with endogenous investment or infinitely lived agents.

Historical Evidence on Asset Creation and Stock-Price Booms

Historical bubble episodes often coincide with the creation of new tradable assets through technological innovation, financial innovation, or deregulation. The following famous episodes draw on Quinn and Turner (2020) and Aliber et al. (2023). The Railway Mania followed the spread of steam locomotion and the easing of restrictions on joint-stock formation.² The U.S. stock-market boom of the 1920s followed the spread of electrification and mass production. New technologies brought a wave of new firms, particularly in radio and aviation industries. Japan’s late-1980s boom followed deregulation that expanded access to equity markets, and IPO activity rose sharply.³ Driven by the revolution in information and communication technology, the dot-com boom featured a large surge in technology IPOs.⁴ Financial innovation was a major shock in the 2000s U.S. housing boom. Securitization transformed previously illiquid mortgage claims into widely traded securities.⁵

Organization of paper: The remainder of the paper is organized as follows. In Section 2, we present a simple endowment economy in which assets with identical dividend streams trade at different prices, giving rise to a bubble in a dividend-paying asset. Sections 3 and 4 develop more general Lucas tree economies and establish conditions for the existence and nonexistence of bubbles. In Section 5, we augment the textbook OLG model of Diamond (1965) with the three key assumptions underlying our mechanism and endogenize the creation of new assets through investment. This extension shows that our mechanism continues to operate when the supply of new assets is determined within the model rather than given exogenously. Section 6 extends the framework to R&D-driven variety expansion. By comparing entrant and incumbent R&D, we show that the existence of a bubble equilibrium depends on whether innovation creates new

²The Bubble Act of 1720 prohibited unincorporated companies from acting as corporate bodies with freely transferable shares unless authorized by a royal charter or an act of Parliament. The repeal of the act in 1825 eased the formation of joint-stock companies.

³The number of IPOs increased from 34 in 1984 to 127 in 1989 and peaked at 141 in 1990 (Quinn and Turner (2020)).

⁴The number of technology IPOs rose from fewer than 100 per year between 1984 and 1991 to 274 in 1996 before peaking at 371 in 1999 (Quinn and Turner (2020)). Netscape’s IPO on August 9, 1995, is widely regarded as an important event in the development of the dot-com boom.

⁵Housing starts had risen from about 1.0 million units in 1991 to 1.7 million by 2002, before the peak in house prices in 2006 (U.S. Census Bureau and U.S. Department of Housing and Urban Development, New Privately-Owned Housing Units Started: Total Units (HOUST), retrieved from FRED).

firms or occurs within existing firms. Finally, Section 7 concludes the paper, and the Appendix contains most of the proofs.

2 An Example of Bubbles in Dividend-Paying Assets

This section presents a parsimonious two-period overlapping-generations (OLG) model where new Lucas trees are successively born. We show the existence of a continuum of bubbly equilibria. To highlight the core results, we defer the formal discussion of the implicit assumptions and mechanisms behind the results to the subsequent sections.

Assets are modeled as Lucas trees. The key features are that (i) new trees are born in every period and (ii) trees are distinguishable by their birth period s . Let N_t denote the number of trees at the beginning of period t . New trees are born at a constant rate in each period, $n_t = \gamma N_t$ ($\gamma \geq 0$). Each tree yields a perpetual dividend stream D starting in the period after birth. Because of feature (ii), different vintages may trade at different prices. Let $p_{s,t}$ denote the price in period t of a tree born in period s . Naturally, there is a symmetric equilibrium where $p_{s,t} = p_{s',t}$. However, we explore the possibility of asymmetric equilibria.

Each generation t consists of households with a constant population size of one. Households live for two periods and consume only in the second period. The price of consumption goods is normalized to one. In the first period, households receive an exogenous endowment w_t of consumption goods and purchase $z_{s,t}$ units of trees $s(\leq t)$: $\sum_{s=-\infty}^t p_{s,t} z_{s,t} = w_t$. The resulting second-period consumption is $c_{2,t+1} = \sum_{s=-\infty}^t (p_{s,t+1} + D) z_{s,t}$. The optimization yields

$$R_{s,t+1} = R_{t+1} := \frac{p_{s,t+1} + D}{p_{s,t}} \quad \text{for } s \leq t. \quad (1)$$

The rate of return R_{t+1} is equal across all existing trees of different ages, while prices of different trees need not to be equalized. The fundamental value of a tree is $v_t = \sum_{u=1}^{\infty} \frac{D}{\prod_{u=1}^t R_{t+u}}$.

To close the model with a growing asset supply, we introduce tree growers who exogenously receive $n_t = \gamma N_t$ new trees in each period. We assume that tree growers neither save nor borrow. They sell their entire tree holdings and consume the proceeds.⁶ The consumption of tree grower t , $C_t^x = p_{t,t} n_t$, where $p_{t,t}$ is the price of new trees.

The aggregate supply of consumption goods in period t is $Y_t = N_t D + w_t$. We assume a balanced growth path (BGP) where $Y_{t+1}/Y_t = w_{t+1}/w_t = N_{t+1}/N_t = 1 + \gamma =: G$. Denote

⁶This setting facilitates the extension to production economies. Even if the new trees are allocated to young or old households, our result is unaffected. See Appendix A.

$\alpha := N_t D / Y_t$, which implies $1 - \alpha := w_t / Y_t$. The consumption goods market clears when $c_{2,t} + p_{t,t} n_t = Y_t$. The asset market clears when $w_t = \sum_{s=-\infty}^t n_s p_{s,t}$.

We first consider an economy with a constant asset supply.

Result 1 (Equilibrium without New Trees) *If the number of trees is constant ($\gamma = 0$), there exists a unique steady-state equilibrium where all trees, regardless of their vintage, are priced at their fundamental value: $p_{s,t} = v^*$ for all $s \leq t$. Asset bubbles are excluded.*

(Proof) Because $\gamma = 0$, $Y_t = N_0 D / \alpha$. Because of $c_{2,t+1} = R_{t+1} w_t = R_{t+1} (1 - \alpha) Y$, the good market equilibrium, $c_{2,t} = Y$, yields $R_t = \frac{1}{1-\alpha} =: R_f$. Because of $v^* = 1/(R_f - 1)$, we have $\sum_{s=-\infty}^t n_s (p_{s,t} - v^*) = \hat{s}_t - v^* N_0 = (1 - \alpha) Y - \frac{\alpha Y}{R_f - 1} = 0$. Since $p_{s,t} \geq v^*$, we have $p_{s,t} = v^*$. $\square$

In contrast, when the economy grows through the successive arrival of new assets ($\gamma > 0$), there is a continuum of bubbly equilibria where asset prices exceed their fundamental values.

Result 2 (Equilibrium with New Trees) *Assume $\gamma > 0$ and $\alpha < (1 - 2\alpha)\gamma$. There is a continuum of steady-state equilibria for $R \in \left[\frac{1+\alpha\gamma}{1-\alpha}, 1 + \gamma\right)$. When $R = \frac{1+\alpha\gamma}{1-\alpha}$, asset prices equal their fundamental values. For $R \in \left(\frac{1+\alpha\gamma}{1-\alpha}, 1 + \gamma\right)$, asset prices exceed their fundamental values.*

(Proof) See Appendix B. $\square$

This example suggests that bubbles in dividend-yielding assets arise once new assets are continually created and distinguished by vintage. At bubbly equilibria, trees of different ages are priced differently despite delivering identical dividends. This raises the question of how such price differentiation can be sustained. The following sections discuss essential assumptions and identify the mechanisms underlying this result and demonstrate its robustness under general preferences, flexible dividend processes, and alternative economic environments, including a Bewley-type infinite-horizon model and production economies.

3 New Trees, Arbitrage, and Bubbles

We develop the full model to discuss key assumptions and identify the mechanism behind bubbles in dividend-paying assets, retaining the notation of the previous section. This section shows that the non-tradability of distinguishable unborn trees restricts arbitrage and thus the prices of dividend-paying trees may deviate from their fundamental values. We employ a two-period OLG model for two reasons. First, it is widely used in the literature on pure bubbles. Second, it allows us to separate the role of the birth of new assets from that of the transversality condition

(TVC), which often rules out bubbles in infinite-horizon models. Importantly, the fundamental source of asset bubbles is not the absence of the TVC in the OLG structure, but rather the limits to arbitrage. The next section shows that bubbles are excluded once arbitrage is unrestricted, even without the TVC. The infinitely-lived agent model in Appendix K shows that bubbles arise when arbitrage is restricted, even with the TVC.

3.1 Model Setting

Lucas Trees: Consider a closed economy with Lucas trees. There is no uncertainty. Time is discrete and denoted by $t = 0, 1, 2, \dots$. The number of trees at the beginning of period t (the end of period $t-1$) is N_t , where $N_0 \in (0, +\infty)$ is given. At the end of each period, $n_{t,t}$ units of trees are born. A tree born in period s is called tree s . After delivering dividends at the beginning of period t , each tree dies at the end of the period with probability $\delta \in [0, 1)$. The number of trees born in period s that remain at the beginning of period $t \geq s$ is $n_{s,t} = (1 - \delta)^{t-s} n_{s,s}$, which also applies to trees born in periods $s < 0$. Consequently, N_t evolves according to

$$N_{t+1} = n_{t,t} + (1 - \delta)N_t = \sum_{s=-\infty}^t (1 - \delta)^{t-s} n_{s,s} = \sum_{s=-\infty}^t n_{s,t}.$$

As to the number of new trees, we assume $n_{t,t} = \gamma N_t$, where $\gamma \geq 0$ is constant. Thus, N_t grows exponentially, $N_{t+1} = (1 - \delta + \gamma)N_t$. For trees born before period 0, we have $n_{s,s} = \gamma N_s = \gamma(1 - \delta + \gamma)^s N_0$ ($s \leq -1$). Even if $n_{t,t}$ is constant ($n_{t,t} = n$), all of our results hold.⁷ In period t , trees born before period t are called *incumbent trees* ($s < t$), trees born in period t are *new trees* ($s = t$), and trees to be born in future periods are *unborn trees* ($s > t$). *Existing trees* in period t are incumbent and new trees ($s \leq t$).

Each tree s produces $D_{s,t}$ consumption goods as dividends at the beginning of period $t \geq s$:

$$D_{s,s} = 0, \quad D_{s,t} = A_{s+1} G_D^{t-(s+1)} \quad (t \geq s+1), \quad \text{and} \quad A_s = A G_A^s,$$

where $A > 0$ is a constant. New trees yield no dividends. Their first positive dividend is $D_{s,s+1} = A G_A^{s+1}$, where $G_A > 0$ governs the growth of this initial dividend across vintages (dotted line in Figure 1). Thereafter, each tree's dividend grows at rate $G_D > 0$ (solid lines). If $G_A = G_D$, the dividend stream of a tree does not depend on its vintage. If $\gamma > 0$, the aggregate

⁷In this case, N_t monotonically converges to n/δ . In the long run, $n = \delta N_t$ new trees are born in each period. By replacing γ with δ , the results under $n_{t,t} = \gamma N_t$ can be applied to this case.

dividend in period t and its growth rate are, respectively, given by

$$D_t^A := \sum_{s=-\infty}^{t-1} n_{s,t-1} D_{s,t} = \hat{D} A_t N_t, \quad (2a)$$

$$G_D^A := \frac{D_{t+1}^A}{D_t^A} = (1 - \delta + \gamma) G_A > 0, \quad (2b)$$

where $\hat{D} := \gamma G_A / \{(1 + \gamma - \delta) G_A - (1 - \delta) G_D\}$ (see Appendix C). To ensure $D_t^A < \infty$, we assume that $(1 - \delta + \gamma) G_A > (1 - \delta) G_D$.

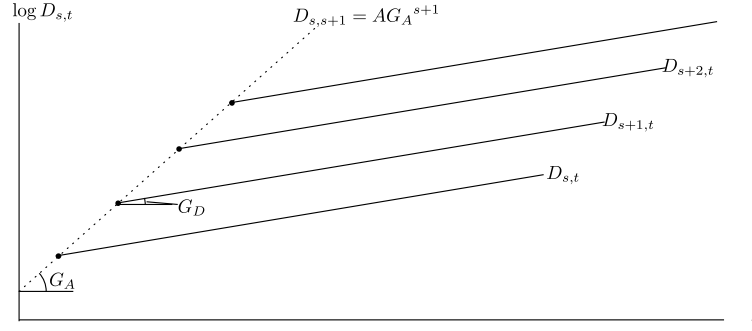


Figure 1 Dividend Dynamics

Crucial Assumptions: We now discuss three key assumptions required for a bubble to arise. First, unlike the standard Lucas tree model, we allow for the birth of new trees.

Assumption 1 *New trees are born in every period ($\gamma > 0$).*

New assets, such as new technologies, newly created firms, and newly built housing, emerge successively. As in Section 2, all new trees are allocated to tree growers who neither save nor borrow. Our results are robust to alternative allocations (see Appendix A). Grower t sells all of its new trees to consume $C_t^x (= p_{t,t} n_{t,t})$ and then exits the economy in period t . After that, these trees continue to yield dividends to their owners. We interpret $C_t^x (= p_{t,t} n_{t,t})$ as investment costs of new trees. We include growers because this setting is applicable to production economies in which the birth of trees is endogenized (see Sections 5 and 6).

Second, we assume the distinguishability of assets.

Assumption 2 *Trees are distinguishable by their birth period.*

Trees of different ages are treated as distinct assets. Thus, new trees are not merely additional units of incumbent trees. Newly created assets, such as new housing, inventions, or

firms, are clearly distinguishable from existing assets. If $G_A \neq G_D$, trees of different ages are distinguishable because they possess distinct dividend streams. If $G_A = G_D$, where all trees yield identical dividends, we assume they remain labeled by their vintage. Galí (2014, 2021) assumes that purely bubble assets are differentiated by their age and thus different vintages can trade at different prices, despite yielding identical zero dividends.⁸ While distinguishability is not essential for the existence of pure bubbles, it plays a central role in generating bubbles in dividend-yielding assets. Section 6 models the distinguishability by assuming that different assets produce differentiated goods, following Olivier (2000). Without Assumption 2, all trees constitute a single homogeneous asset, whose supply changes over time (see Sections 4.1 and 5).

The final assumption is about the trade of unborn trees.

Assumption 3 *Claims to unborn trees are not traded in the market.*

Future assets cannot be traded. As Tirole (1985) points out, trading assets that do not yet exist is difficult (see Introduction of this paper too). In practice, the unpredictability of future events and associated transaction costs often prevent the advance trade of unborn assets. Section 4 relaxes Assumptions 2 and 3 in turn to examine their roles in the existence of bubbles.

New assets naturally include assets created in that period, such as intellectual property rights for newly developed technologies, bonds issued by startup firms, and ownership of newly constructed housing. More broadly, new assets include newly marketized assets that are not previously traded, such as securitized claims on existing buildings and shares issued in the IPOs of existing firms. Therefore, technological innovation, financial innovation, and deregulation promote the creation of new assets, which is represented by an increase in γ .

Households: We use a two-period OLG framework. The population of generation t is $L_t = L_0 G_L^t$, where $G_L > 0$ is the population growth rate. Households of the same generation are homogeneous. Denote the first- and second-period consumption as $c_{1,t}$ and $c_{2,t+1}$, respectively. Generation t has homothetic preferences, $U_t = u(c_{1,t}, c_{2,t+1})$, which is strictly increasing in both arguments and is twice differentiable with indifference curves that are convex to the origin. The marginal rate of substitution u_2/u_1 ($u_i := \partial u / \partial c_i$) satisfies $\lim_{c_2/c_1 \rightarrow 0} u_2/u_1 = +\infty$ and $\lim_{c_2/c_1 \rightarrow +\infty} u_2/u_1 = 0$. Generation t has an endowment of w_t consumption goods in the first

⁸Galí (2021) considers pieces of paper of a cohort-specific color or stamped with the birth period. In Martin and Ventura (2012), new pure bubble assets can have values different from existing pure bubble assets.

period and none in the second. The first- and second-period budget constraints are given by

$$c_{1,t} + \hat{s}_t + a_t = w_t, \quad (3a)$$

$$c_{2,t+1} = \sum_{s=-\infty}^t D_{s,t+1} z_{s,t} + (1 - \delta) \sum_{s=-\infty}^t p_{s,t+1} z_{s,t} + R_{t+1} a_t, \quad (3b)$$

respectively, where $\hat{s}_t := \sum_{s=-\infty}^t p_{s,t} z_{s,t}$ is the holding of trees and a_t is the holding of a bond in net zero supply. Generation t purchase $z_{s,t}$ units of trees $s(\leq t)$ in the first period and receives $D_{s,t+1}$ dividends and sells non-depreciated trees in the second period. Under Assumption 2, the price of each tree, $p_{s,t}$, may differ across vintages. Free disposability guarantees $p_{s,t} \geq 0$. Due to Assumption 3, the unborn trees are not included in (3a) and (3b).

The utility maximization yields $1 = m_{t+1} R_{t+1}$ and the standard pricing function (see Cochrane (2005) and Ljungqvist and Sargent (2012, ch.14)):

$$p_{s,t} = m_{t+1} x_{s,t+1} (= P_t(x_{s,t+1})), \quad s \leq t, \quad (4)$$

where $m_{t+1} := u_2/u_1$ is the stochastic discount factor (SDF) and $x_{s,t+1} := (1 - \delta)p_{s,t+1} + D_{s,t+1}$ is the payoff from tree s . While prices of different vintages need not be equalized, the rates of return on different vintages $R_{s,t+1} := x_{s,t+1}/p_{s,t}$ are equalized as

$$R_{t+1} = R_{s,t+1} := \frac{(1 - \delta)p_{s,t+1} + D_{s,t+1}}{p_{s,t}}, \quad s \leq t. \quad (5)$$

Since the utility function is homothetic, (4) defines

$$\frac{c_{2,t+1}}{c_{1,t}} = \Gamma(R_{t+1}), \quad (6)$$

where $\Gamma(\cdot)$ is an increasing function that satisfies $\lim_{R \rightarrow 0} \Gamma(R) = 0$ and $\lim_{R \rightarrow \infty} \Gamma(R) = \infty$. The intertemporal elasticity of substitution (IES) is $\Gamma'(R)R/\Gamma(R)$. The optimal consumption and saving are $c_{1,t} = C_1(R_{t+1})w_t$, $c_{2,t+1} = C_2(R_{t+1})w_t$, and $\hat{s}_t = S(R_{t+1})w_t$, where

$$C_1(R_{t+1}) := \frac{1}{1 + \frac{\Gamma(R_{t+1})}{R_{t+1}}}, \quad S(R_{t+1}) := 1 - C_1(R_{t+1}), \quad \text{and} \quad C_2(R_{t+1}) := R_{t+1} S(R_{t+1}). \quad (7)$$

If we assume $U_t = u(c_{1,t}) + \beta u(c_{2,t+1})$ with $u(c) = c^{1-\sigma}/(1-\sigma)$, we have $\Gamma(R) = (\beta R)^{\frac{1}{\sigma}}$.

Balanced Growth and Market Equilibrium: The aggregate supply of consumption goods

in period t is $Y_t = D_t^A + w_t L_t$. We assume balanced growth among Y_t , $w_t L_t$, and D_t^A and denote

$$\alpha := \frac{D_t^A}{Y_t} \in (0, 1), \quad \text{or equivalently, } 1 - \alpha := \frac{w_t L_t}{Y_t}, \quad (8)$$

where α is a constant. The economic growth rate $G := Y_{t+1}/Y_t$ is given by (2b) on the BGP. In equilibrium, we have $a_t L_t = 0$ and the following market-clearing conditions:

$$z_{s,t} L_t = n_{s,t}, \quad (9a)$$

$$c_{1,t} L_t + c_{2,t} L_{t-1} + n_{t,t} p_{t,t} = Y_t, \quad (9b)$$

$$\hat{s}_t L_t = \sum_{s=-\infty}^t n_{s,t} p_{s,t}. \quad (9c)$$

Asset Price: From (5), the price of an existing tree ($s \leq t$) is

$$p_{s,t} = \underbrace{\sum_{T=1}^{\infty} \frac{(1-\delta)^{T-1} D_{s,t+T}}{\prod_{u=1}^T R_{t+u}}}_{v_{s,t} > 0} + \underbrace{\lim_{T \rightarrow \infty} \frac{(1-\delta)^T p_{s,t+T}}{\prod_{u=1}^T R_{t+u}}}_{b_{s,t} \geq 0} = \psi_t D_{s,t+1} + b_{s,t}, \quad (10)$$

where

$$\psi_t := \frac{v_{s,t}}{D_{s,t+1}} = \sum_{T=1}^{\infty} \frac{[(1-\delta)G_D]^{T-1}}{\prod_{u=1}^T R_{t+u}} \quad (11)$$

is the fundamental value normalized by the dividend, which is common to all trees. Since $p_{s,t} \geq 0$, the bubble term is nonnegative, $b_{s,t} = p_{s,t} - v_{s,t} \geq 0$. Because $v_{s,t}$ satisfies $v_{s,t} = [(1-\delta)v_{s,t+1} + D_{s,t+1}]/R_{t+1}$ (see (5)), $b_{s,t}$ follows the no-arbitrage condition:

$$b_{s,t+1} = \frac{R_{t+1}}{1-\delta} b_{s,t}. \quad (12)$$

The aggregate bubble is defined by $B_t^A := \sum_{s=-\infty}^t n_{s,t} b_{s,t} \geq 0$.

Definition 1 Tree s has an *asset bubble* in period t if $p_{s,t} > v_{s,t}$. There is an *aggregate asset bubble* in period t if $B_t^A > 0$.

An equilibrium is defined as follows:

Definition 2 An *equilibrium* is a sequence of consumption, savings, and the prices of trees that jointly satisfy the utility maximization conditions and the market-clearing conditions.

To define a steady state, we introduce

$$\phi_{s,t} := \frac{p_{s,t}}{D_{s,t+1}} \quad \text{and} \quad b_t^A := \frac{B_t^A}{Y_t},$$

where $\phi_{s,t}$ is the price-dividend ratio and b_t^A is the aggregate bubble relative to the size of the economy. We use period- $t+1$ dividends to define $\phi_{s,t}$ because tree s pays no dividends in period s . Since $n_{s,t} = (1-\delta)^{t-s}n_{s,s}$ and (12), we have $B_{t+1}^A = n_{t+1,t+1}b_{t+1,t+1} + R_{t+1}B_t^A$. Because (10) holds, the dynamics of b_t^A follows

$$b_{t+1}^A = \frac{\alpha\gamma}{\hat{D}} \frac{b_{t+1,t+1}}{A_{t+1}} + \frac{R_{t+1}}{G} b_t^A = \frac{\alpha\gamma G_A}{\hat{D}} (\phi_{t+1,t+1} - \psi_{t+1}) + \frac{R_{t+1}}{G} b_t^A. \quad (13)$$

The new trees' bubble $n_{t+1,t+1}b_{t+1,t+1}$ affects the dynamics of b_t^A , because bubbles of new trees do not follow the no-arbitrage condition (12) between the periods before and after birth. There may be a symmetric equilibrium where trees of different ages have an identical price-dividend ratio ($\phi_{s,t} = \phi_t$). With Assumption 2, we explore the possibility of asymmetric equilibria where trees of different ages have different price-dividend ratios.

Definition 3 *A steady state equilibrium* is an equilibrium where R_t , $\phi_{t,t}$, ψ_t , and b_t^A are constant.

With $u(c_1, c_2) = c_2$, $G_D = G_A = G_L = 1$, and $\delta = 0$, the model reduces to the example economy in Section 2.

3.2 Low Rate of Return and Limits to Arbitrage

Before deriving the equilibrium, we provide two preliminary but important results.

Low Rate of Return and New Trees: The rational bubble literature argues that pure bubbles occur only if the interest rate is lower than the growth rate in a bubbleless economy. Santos and Woodford (1997) show that asset bubbles may occur only if the present value of the aggregate endowment is infinite. The following theorem shows that the birth of new assets is a necessary condition for a low interest rate and an infinite present value of endowment:

Theorem 1 *Assume that R is constant and that balanced growth condition (8) is satisfied. Only if new trees are born (Assumption 1), $R \leq G$ and $\sum_{t=0}^{\infty} \frac{w_t L_t}{R^t} = \infty$ hold in equilibrium.*

(Proof) Without Assumption 1 ($n_{t,t} = 0$), we have $N_t = (1-\delta)^t N_0$ and $D_t = AG_D^t$. Under balanced growth, Y_t and the aggregate dividend $D_t^A = N_t D_t$ grow at the same rate, $G =$

$(1 - \delta)G_D$. To ensure a finite fundamental value, we must have $R > (1 - \delta)G_D$ (see (11)). This implies the convergence of $\sum_{t=0}^{\infty} \frac{w_t L_t}{R^t}$. The theorem follows by contraposition. $\square$

Theorem 1 holds because individual dividend growth sustains aggregate growth, $G = (1 - \delta)G_D$, if no trees are born. Since the proof uses only the definition of the fundamental value (11), the theorem applies to any balanced growth models.

We next derive a necessary condition for bubbles on dividend-yielding assets:

Proposition 1 *Assume that R is constant and that balanced growth condition (8) is satisfied.*

If $R > G$, the aggregate bubble cannot exist in equilibrium.

(Proof) (13) shows that if $R > G$, b_t^A diverges to infinity, which violates (9c). $\square$

Because the absence of the aggregate bubbles implies that of individual bubbles, Theorem 1 and Proposition 1 implies that when no new trees are born, $R > G$ holds and hence no bubble can arise in dividend-yielding assets. The birth of new trees is necessary for such bubbles.

Limits to Arbitrage and New Trees: Arbitrage trades ensure that asset prices are equal to their fundamental values. Investors sell overpriced assets and buy underpriced ones, which drives prices back to fundamental values. However, when arbitrage is restricted for some reason, asset prices may deviate from their fundamental values, which we call mispricing. We define the law of one price and the notion of no arbitrage, following the standard finance literature (e.g., Cochrane, 2005, ch.4; Ljungqvist and Sargent, 2012, ch.14; LeRoy and Werner, 2014).

Definition 4 Consider a portfolio consisting of I types of assets. Asset i has a payoff of x_i .

The law of one price asserts that the asset price is a linear function of the payoff; $P(\sum_i a_i x_i) = \sum_i a_i P(x_i)$, where a_i is a portfolio weight on asset i . There are *no arbitrage opportunities* if an asset with strictly positive payoff has a strictly positive price.

Intuitively, doubling the payoff doubles the price, and a positive payoff cannot be obtained at zero cost. Since the existing trees are priced linearly, $p_{s,t} = m_{t+1}x_{s,t+1}$ with $m_{t+1} > 0$ ($s \leq t$), the law of one price holds and arbitrage opportunities are absent among them.

Importantly, unborn trees are not priced, although they provide a payoff of $p_{s,s}$ in their birth period if a claim to them is available. Thus, we have the following theorem:

Theorem 2 *Under Assumptions 1 – 3, unborn trees are not priced. Hence, they do not satisfy the law of one price and the no-arbitrage.*

Theorem 2 follows immediately from the definitions. It is nonetheless important because in standard macroeconomic models, assets typically satisfy the law of one price and no-arbitrage. For example, the neoclassical growth model treats newly created capital as additional units of existing capital, which violates Assumption 2. Under Assumptions 1–3, unborn trees are not disciplined by arbitrage. Readers may think that (5) determines the prices of new trees. However, condition (5) only equalizes the rates of return among existing trees and governs relative assets prices over time. It does not determine the initial price of each tree, $p_{t,t}$. Actually, because the prices of unborn trees do not follow (5) or (12), the new trees' bubble affects the dynamics of aggregate bubble (see (13)). Given Theorem 2, therefore, the prices of new trees may deviate from their fundamental values. However, if new trees are indistinguishable from incumbent trees (Section 4.1), or if claims to unborn trees are tradable (Section 4.2), arbitrage pins down the prices of new trees, ruling out bubbles in dividend-yielding assets.

3.3 Bubbleless Equilibrium

To characterize a bubbleless equilibrium, we use (9a) and (9c) because the two equations ensure (9b) by Walras's law (see Appendix D). At a steady state, $v_{s,t} = D_{s,t+1}/[R - (1 - \delta)G_D]$ holds. From $p_{s,t} = v_{s,t}$, (8), (9a), and (9c), we have

$$(1 - \alpha)S(R) = \alpha G \cdot \psi(R), \quad (14)$$

where G and $S(R)$ are defined in (2b) and (7), respectively. The normalized fundamental value $\psi(R) := 1/[R - (1 - \delta)G_D]$ is decreasing in R (see (11)). Although the left-hand side (LHS) may increase or decrease in R depending on the IES, (14) has a unique solution $R = R_L$ (see Figure 2). We can prove the following proposition:

Proposition 2 *Suppose that Assumptions 1–3 and balanced growth condition (8) hold. Then, there exists a unique bubbleless steady state, in which $R = R_L$.*

(Proof) See Appendix E.

3.4 Existence of Bubbly Equilibria

We first outline the existence of bubbles graphically. Since (9a) and (9b) imply (9c) by Walras's law (see Appendix D), we use the former two equations. By using (2a), (2b), (7), (8), (9a), and $\phi_{s,t} = p_{s,t}/D_{s,t+1}$, we rewrite (9b) as $(1 - \alpha)C_1(R_{t+1}) = 1 - (1 - \alpha)C_2(R_t)/G - \gamma\alpha G_A\phi_{t,t}/\hat{D}$.

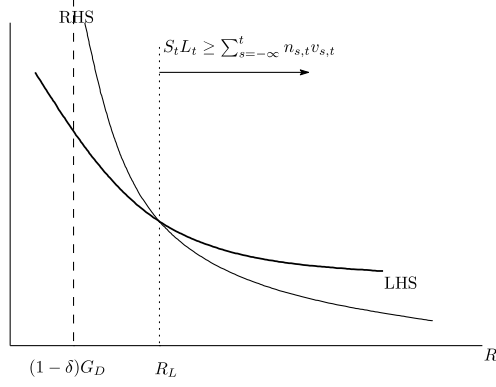


Figure 2 Bubbleless Steady State: Graph of (14)

If R_t and $\phi_{t,t}$ are constant over time, the steady state value of the price-dividend ratio of new trees, $\phi_{t,t}$, is given by

$$\phi_{new}(R) := \frac{\hat{D}}{\gamma \alpha G_A} \left[1 - (1 - \alpha) C_1(R) - \frac{1 - \alpha}{G} C_2(R) \right]. \quad (15)$$

In the bubbleless economy, $\phi_{new}(R) = \psi(R)$ holds, where $\psi(R) = 1/[R - (1 - \delta)G_D]$ is decreasing in R (see (11)). As Appendix F shows, $\phi_{new}(R) = \psi(R)$ has two solutions, $R = R_L$ and $R = G$, where G is given by (2b) (see Figure 3). If $R_L < G$, then $\phi_{new}(R) > \psi(R)$ holds for $R \in (R_L, G)$. Hence, new trees have bubbles. Because an individual bubble follows (12), all trees have bubbles. Therefore, there exists a continuum of bubbly steady states (the thick line in Figure 3).

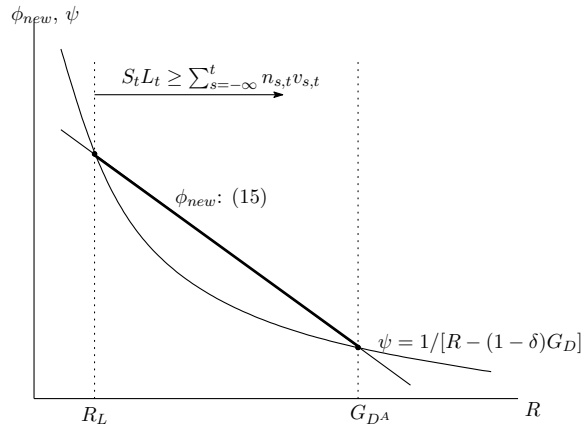


Figure 3 Bubbly Steady States

To see how the existence of bubbly equilibria depends on parameters, we specify the utility function as $U_t = u(c_{1,t}) + \beta u(c_{2,t+1})$, where $u(c) = c^{1-\sigma}/(1-\sigma)$ for $\sigma > 0$ ($\sigma \neq 1$) and $u(c) = \log c$ for $\sigma = 1$. We define the set of parametric conditions $\mathcal{C} \equiv \{C1, C2, C3\}$ depending on the value of the reciprocal of IES, σ :

C1. $0 < \sigma < 1$, $\alpha < 1/2$, and $G > \underline{G}_1 > 0$.

C2. $\sigma = 1$, $\beta/(1 + \beta) > \alpha/(1 - \alpha)$, and $G > \frac{(1-\alpha)\beta}{(1-2\alpha)\beta-\alpha}(1 - \delta)G_D > 0$.

C3. $\sigma > 1$, $(1 - \delta)G_D < \frac{(1-2\alpha)\beta-\alpha}{(1-\alpha)\beta^2}$, and $G \in (\underline{G}_2, \overline{G})$.

Here, $\underline{G}_1$, $\underline{G}_2$, and $\overline{G}$ depend on σ , as defined in Appendix F. The following proposition characterizes the set of steady states:

Proposition 3 (Steady State Equilibria)

1. $\gamma = 0$ (**no new tree**): Under Assumptions 2–3 and (8), independently of any parametric condition in $\mathcal{C}$, the unique equilibrium is the bubbleless steady state, where $R = R_L$.
2. $\gamma > 0$ (**with new trees**): Under Assumptions 1–3 and (8), if one of C1–C3 holds, then $R_L < G$. In this case, the following steady states coexist:

(i) a unique bubbleless steady state, where $R = R_L$ and $\phi_{new}(R_L) = \psi(R_L)$;

(ii) a continuum of bubbly steady states, where $R \in (R_L, G)$, $\phi_{new}(R) > \psi(R)$, and

$$\phi_{s,t} = \psi(R) + [\phi_{new}(R) - \psi(R)]\left[\frac{R}{(1-\delta)G_D}\right]^{t-s} > \psi(R) \text{ for } s \leq t.$$

(Proof) See Appendix F.

Proposition 3-1 restates Theorem 1 and Proposition 1. The main result is Proposition 3-2. As Theorem 1 shows, the birth of new trees is necessary for a low rate of return. However, $R_L < G$ actually holds if one of Conditions C1–C3 holds. These conditions are typically satisfied when the first-period endowment is large relative to Y_t (small α), the saving rate is high (large β), and G is sufficiently high.

Even $R_L < G$ is not sufficient for bubbles in dividend-yielding assets. The limits to arbitrage are essential. As Theorem 2 shows, under Assumptions 1–3, the prices of new trees are not fully disciplined by arbitrage. The prices of new trees may deviate from the prices of incumbent trees and from their own fundamental values. Thus, bubbles in dividend-yielding assets emerge.

Due to the limits to arbitrage, even the fundamental value exhibits indeterminacy. Thus, a continuum of bubbly steady states exists. In the standard model with a single pure bubble asset, there exist a unique bubbly steady state with $R = G$ and multiple bubbly equilibria with $R < G$ that converge to a unique bubbleless steady state. In our model, the birth of new trees supplements the aggregate bubble, sustaining multiple bubbly steady states even when $R < G$. To see this point, we rewrite (13) as

$$b_{t+1}^A = \frac{\alpha\gamma G_A}{\hat{D}}(\phi_{new} - \psi(R)) + \frac{R}{G}b_t^A. \quad (16)$$

The first term is the new bubble. Since it is positive and $R/G < 1$ for any $R \in [R_L, G)$, the aggregate bubble b_t^A monotonically converges to a positive steady-state value.⁹

At a steady state, generation t 's the utility level, $U_t = \frac{w_t^{1-\sigma}}{1-\sigma} \left(1 + \beta^{\frac{1}{\sigma}} R^{\frac{1-\sigma}{\sigma}}\right)^\sigma$, is an increasing function of R . Thus, a bubble in dividend-yielding assets improves household utility.

Numerical example: A graphical illustration of Proposition 3 may help the reader's understanding. Figure 4 shows the (G, σ) -region in which bubbly steady states exist. The horizontal axis shows the annualized gross growth rate $G^{1/30}$, while the vertical axis shows the reciprocal of the IES, σ . As Proposition 3 shows, when $(\sigma \leq 1)$, bubbly steady states exist if the gross growth rate exceeds a threshold $\underline{G}_1$. When $(\sigma > 1)$, bubbly steady states exist for any gross growth rate in an interval $(\underline{G}_2, \overline{G})$ such that bubbly steady states exist for any gross growth rate between them. The U.S. historical average of 30-year rolling real GDP growth rate (1947–2025), which is 3.11%, lies within this region, regardless of the value of the IES.

Innovation, Deregulation, and Bubbles: Unlike pure bubbles, bubbles in dividend-yielding assets necessarily require the creation of new assets. Historically, major asset bubbles frequently coincide with episodes of technological innovation, financial innovation, or deregulation (Aliber et al., 2023; Perez, 2009; Scheinkman, 2014). These episodes share the rapid creation of new dividend-paying assets. Technological innovation creates new products, financial innovation turns existing assets into new securities, and deregulation opens markets to assets that could not be traded. All three raise γ in our model. Proposition 3 suggests a mechanism behind the historical facts. Suppose that γ is initially close to zero. If the shock is large enough for growth $G = (1 - \delta + \gamma)G_A$ to satisfy the relevant condition in $\mathcal{C}$, bubbles in positive-dividend assets may emerge. We theoretically return to this point in Section 6.

Depreciation: Bubbles in dividend-paying assets exist even if $\gamma = \delta > 0$ (for example, Condition C2 holds if $G = G_A$ is large enough). As long as new assets are born, asset bubbles may exist even if the number of assets is constant. Moreover, if $\gamma > 0$, bubbles exist with $\delta = 0$ (see Condition C2). The depreciation of the trees is not essential. Domeij and Ellingsen (2018) show that bubbles in positive-dividend assets may emerge if assets depreciate and then are replaced by new assets. Our results show that what matters is the birth of new assets.

⁹In financial economics, under uncertainty, multiple SDFs support the same asset price level when markets are incomplete, in the sense that the number of assets is smaller than that of states. One might suspect that this result underlies the multiple equilibria in our model. However, this is not the case. Our economy is deterministic, there is a one-to-one relationship between the SDF and asset prices, and the SDF is unique in each equilibrium.

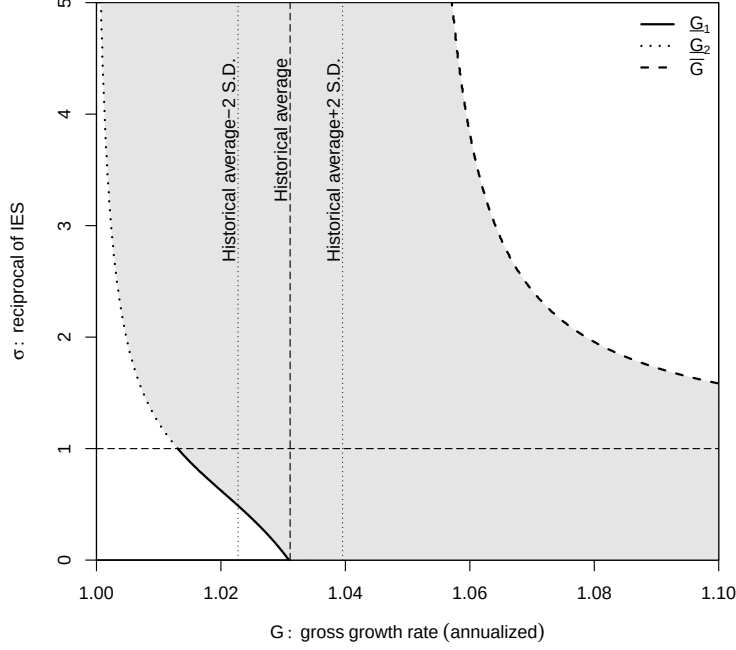


Figure 4 Parameter Region for Asset Bubbles

Note: We set parameter values as follows: According to the National Income and Product Accounts (NIPA), Table 1.12 “National Income by Type of Income,” the combined share of net dividends and rental income in national income was approximately 13.6% from 2023 to 2025. We thus set $\alpha = 0.136$. Because our measure captures only the income share generated by tradable assets, it is smaller than aggregate capital share that is around 0.3. Loveland et al. (2021) report that the annual exit rate of the U.S. listed firms for reasons other than mergers and acquisitions was approximately 2.9% over the period from 1973 to 2017. Since one period in the OLG model is interpreted as about 30 years, we set $\delta = 1 - (1 - 0.029)^{30} \simeq 0.586$. According to Robert Shiller’s data, the average of the 30-year rolling average real dividend growth rates between 1947 and 2025 was approximately 1.0159³⁰. We set $G_D = 1.0159^{30} \simeq 1.606$. Finally, we adopt the conventional value for the time discount factor, $\beta = 0.97^{30} \simeq 0.401$. The minus and plus two standard deviation levels are 2.28% and 3.96%, respectively.

Price Disparity: Since $R > (1 - \delta)G_D$, the price-dividend ratio $\phi_{s,t} = \psi(R) + (\phi_{new} - \psi(R))[\frac{R}{(1-\delta)G_D}]^{t-s}$ grows unboundedly as $t - s$ increases, consistent with rational bubble theory. Thus, trees of different vintages have distinct price-dividend ratios ($\phi_{s,t} \neq \phi_{s',t}$ for $s \neq s'$). When $G_A = G_D$, all trees yield identical dividend streams. Nevertheless, trees of different vintages can have different prices in the bubbly steady state, $p_{s,t} \neq p_{s',t}$. This price disparity stems from the limits to arbitrage across existing and unborn trees. In Galí (2014, 2021), different vintages of purely bubble assets trade at different prices.¹⁰ Cryptocurrencies offer a real-world example. We confirm a similar price disparity even among dividend-paying assets.

One might expect that households could increase their utility by replacing expensive assets with cheaper ones that yield identical dividends. However, this is not the case because rates of

¹⁰Santos and Woodford (1997, Example 4.1) show that with asset bubbles, two types of fiat money exhibit divergent market values, even though they have identical zero dividends.

return are equalized across all existing trees (see (4) or (5)). Consider two assets, A and B. If assets A and B are priced at \$100 and \$1, their payoffs are \$120 and \$1.20, respectively. Selling one unit of A to buy 100 units of B leaves the total payoff unchanged.

This result is related to the empirical literature on the limits to arbitrage and asset bubbles. Because fundamental values are unobservable, identifying bubbles is inherently difficult and hence some studies examine whether assets with seemingly identical payoffs have different prices (e.g., Rosenthal and Young, 1990). The literature finds that mispricing and bubbles are particularly pronounced in newly created assets, such as tech IPOs or carved-out subsidiaries (e.g., Lamont and Thaler, 2003a, b). Our model accounts for these facts.

4 No-Arbitrage, No Bubbles

This section relaxes either Assumption 2 or Assumption 3, maintaining the other settings in Section 3. Our goal here is not to provide realistic descriptions of actual economies but to illustrate how the two assumptions generate bubbles in dividend-yielding assets. The two-period OLG framework has no TVC, $\lim_{T \rightarrow \infty} \frac{(1-\delta)^T p_{s,t+T}}{\prod_{u=1}^T R_{t+u}} = 0$, that excludes bubbles. Nevertheless, bubbles disappear once either assumption is relaxed, because unborn trees satisfy no-arbitrage.

4.1 Trees are Indistinguishable

This subsection removes Assumption 2 by assuming $G_A = G_D$. Since new trees are indistinguishable from incumbent trees, arbitrage between them pins down the price of the former. Now, a single asset with price p_t is traded and its supply increases according to $N_{t+1} = (1 + \gamma - \delta)N_t$. The asset delivers the dividends $D_t = DG_D^t$. The aggregate dividends $D_t^A = N_t D_t$ grow at rate $G = (1 + \gamma - \delta)G_D$. Generation t buys z_t units of assets in the first period. The budget constraints are given by $c_{1,t} + p_t z_t + a_t = w_t$ and $c_{2,t+1} = x_{t+1} z_t + R_{t+1} a_t$, where $x_{t+1} = (1 - \delta)p_{t+1} + D_{t+1}$ is the payoff. Utility maximization yields $p_t = x_{t+1}/R_{t+1}$. The single asset satisfies the law of one price and no-arbitrage in Definition 4. Consequently, mispricing cannot occur:

Proposition 4 *Suppose that $\gamma > 0$ and (8) hold and Assumption 3 is satisfied. Assumption 2 is removed. If one of C1–C3 is satisfied under a time-separable utility with CRRA felicity function, there is a unique bubbleless steady state where $R = R_L < G$ and $\phi_{new} = \psi(R_L)$ holds. Bubbles in dividend-yielding assets do not arise in any steady state.*

(Proof) See Appendix G.

Unlike pure bubbles, low rates of return alone do not generate bubbles in dividend-yielding assets. The limits to arbitrage play a central role.

4.2 Unborn Trees in the Arbitrarily Distant Future are Traded

If unborn trees in the arbitrarily distant future are traded, their present value must be finite. This pins down their prices and rules out bubbles in dividend-yielding assets. Let $\mathbb{I}$ be an arbitrary infinite subset of the positive integers, for example, the set of all positive odd integers. Let $\mathbb{Z}_t = \{t+1, t+2, \dots\}$. Define $\mathbb{S}_t := \mathbb{I} \cap \mathbb{Z}_t$. Then, for any arbitrarily large $s \in \mathbb{S}_t$, there exists an integer $s' > s$ such that $s' \in \mathbb{S}_t$. Relaxing Assumption 3 requires only that some unborn trees be tradable. We replace Assumption 3 with the following assumption:¹¹

Assumption 4 *Unborn trees scheduled to be born in any period $s \in \mathbb{S}_t$ are tradable in period t .*

Consider a representative, infinitely-lived arbitrageur that commits to issuing and settling claims on unborn trees. In period 0, the arbitrageur issues $n_{s,0}$ claims on each unborn tree $s \in \mathbb{S}_0 (= \mathbb{I})$ at price $p_{s,0}$ and lends X_0 in the asset market. This issuance is essentially a short-sale of unborn trees, creating a liability to deliver tree s in period s . In each period $t \geq 1$, the arbitrageur settles the liabilities by purchasing $n_{t,t}$ new trees from grower t and transferring them to claim holders if $t \in \mathbb{I}$; otherwise, growers sell directly to households. The budget constraints for the arbitrageur in period 0 and period $t \geq 1$ are, respectively, given by

$$\sum_{s \in \mathbb{S}_0} p_{s,0} n_{s,0} \geq X_0 \quad \text{and} \quad R_t X_{t-1} \geq \mathbf{1}_{\mathbb{I}}(t) p_{t,t} n_{t,t} + X_t, \quad (17)$$

where $\mathbf{1}_{\mathbb{I}}(t)$ is an indicator function that equals one if $t \in \mathbb{I}$. For all $t \geq 0$, prices and R_t must adjust so that the budget constraints in (17) hold with equality. Otherwise, arbitrageurs earn positive profits. We do not impose the No-Ponzi game (NPG) condition for the arbitrageur.

In period $t \geq 0$, a household of generation t buys $\theta_{s,t}$ claims on tree $s \in \mathbb{S}_t$ at price $p_{s,t}$. Its budget constraints are given by

$$c_{1,t} + \sum_{s=-\infty}^t p_{s,t} z_{s,t} + \sum_{s \in \mathbb{S}_t} p_{s,t} \theta_{s,t} + a_t = w_t, \quad (18)$$

$$c_{2,t+1} = \sum_{s=-\infty}^t x_{s,t+1} z_{s,t} + \sum_{s \in \mathbb{S}_t} p_{s,t+1} \theta_{s,t} + R_{t+1} a_t. \quad (19)$$

¹¹Iraola and Santos (2017) develop a macroeconomic model in which dividend claims arising from expected future inventions are tradable in the current period. This is captured by ξ_t in their Proposition 3.1.

Utility maximization yields $p_{s,t} = p_{s,t+1}/R_{t+1}$ even for unborn tree $s \in \mathbb{S}_t$. Now, unborn trees are priced through intertemporal no-arbitrage both before and after their birth. The market clearing conditions are $\theta_{s,t}L_t = n_{s,s}$ for $s \in \mathbb{S}_t$, $z_{s,t}L_t = n_{s,t}$ for $s \leq t$, $a_tL_t + X_t = 0$ and (9b).

The no-arbitrage for unborn trees ($p_{s,t} = p_{s,t+1}/R_{t+1}$ if $s \in \mathbb{S}_t$) implies

$$\sum_{s \in \mathbb{S}_t} p_{s,t} n_{s,s} = \sum_{s \in \mathbb{S}_t} \frac{p_{s,s} n_{s,s}}{\prod_{u=1}^{s-t} R_{t+u}} \geq \sum_{s \in \mathbb{S}_t} \frac{v_{s,s} n_{s,s}}{\prod_{u=1}^{s-t} R_{t+u}}. \quad (20)$$

In the steady state, we have $v_{s,s} n_{s,s} = \gamma G_A N_t A_t \psi(R) G^{s-t}$ because $v_{s,s} = \psi(R) A_t G_A^{s+1-t}$ and $n_{s,s} = \gamma(1 + \gamma - \delta)^{s-t} N_t$. If $R_t \leq G$ for large t , we have $\sum_{s \in \mathbb{S}_0} p_{s,0} n_{s,s} = \infty$. Then, $\sum_{s \in \mathbb{S}_0} p_{s,0} n_{s,s} = X_0$ cannot hold with any finite X_0 . Setting $X_0 = \infty$ violates the second condition in (17) for $t = 1$. Thus, $R > G$ must hold in steady states.^{12,13}

Proposition 5 *Under Assumptions 1, 2, and 4, the condition $R > G$ holds in a steady-state equilibrium. Bubbles in dividend-yielding assets do not exist. A unique bubbleless steady state is characterized by $R = R_L$ and $\phi_{new} = \psi(R_L)$.*

(Proof) See Appendix H.

The intuition is as follows: Under Assumption 4, trees to be born in the distant future are priced. If they have bubbles, arbitrageurs could earn infinite profits by short-selling these overpriced claims, $\sum_{s \in \mathbb{S}_0} p_{s,0} n_{s,s} = +\infty$. Competition decreases $p_{s,0}$ until such opportunities vanish and the first constraint in (17) holds with equality. This pins down $p_{s,0}$ and requires $R_t > G$ for large t , which rules out bubbles on all trees even without an NPG condition on arbitrageurs.

The above discussion suggests that the key is the non-tradability of sufficiently distant unborn trees. Instead of Assumption 4, let us impose the following assumption:

Assumption 4' *Unborn trees scheduled to be born in any period $s \in \mathbb{S}_t^* := \mathbb{I} \cap \mathbb{Z}_t^*$ are tradable in period t , where $\mathbb{Z}_t^* = \{t+1, t+2, \dots, t+T\}$ and T is a positive integer.*

Unborn trees to be born in the arbitrarily distant future periods $s(> t+T)$ are non-tradable. Because the set of tradable unborn trees $\mathbb{S}_t^*$ is a finite set, the arbitrageur's short selling income is finite even in the presence of bubbles. Thus, we prove the following proposition.

¹²We have $\sum_{s \in \mathbb{S}_t} v_{s,s} n_{s,s} / \prod_{u=1}^{s-t} R_{t+u} \leq \sum_{s=t+1}^{\infty} v_{s,s} n_{s,s} / \prod_{u=1}^{s-t} R_{t+u}$. Thus, the condition $R > G$ is sufficient for the series $\sum_{s \in \mathbb{S}_t} v_{s,s} n_{s,s} / \prod_{u=1}^{s-t} R_{t+u}$ to converge.

¹³Mathematically, the convergence of $\sum_{s \in \mathbb{S}_t} p_{s,t} n_{s,s}$ in (20) is similar to the NPG condition. However, there are differences. While the NPG is an exogenously given constraint on debt, the convergence of $\sum_{s \in \mathbb{S}_t} p_{s,t} n_{s,s}$ is an endogenous equilibrium outcome in our setting.

Proposition 6 *Under Assumptions 1, 2, and 4', Proposition 3 holds. Thus, if any parametric condition in $\mathcal{C}$ holds, a continuum of bubbly steady states exists.*

(Proof) See Appendix I.

We conclude that the trade of unborn trees to be born in the far future pins down the price of unborn trees and excludes bubbles in dividend-yielding assets.

Appendix J considers another way to relax Assumption 3 by assuming that owners of existing trees automatically receive ownership of new trees. As Domeij and Ellingsen (2018) point out, Santos and Woodford (1997) utilize this setting.¹⁴ An example of such ownership is that of a new technology through incumbent R&D (see Section 6.2). In this case, unborn trees are effectively traded as part of existing trees. Thus, bubbles in dividend-yielding assets cannot exist.

5 Neoclassical Growth

In this and the following sections, we endogenize the creation of new assets through investment in physical capital or R&D. We begin with the OLG model of Diamond (1965).¹⁵ In the textbook Diamond model, bubbles in dividend-yielding assets do not exist even when $R \leq G$ holds in a bubbleless steady state. Note that Assumption 2 is not satisfied because capital is treated as a single aggregate asset and hence incumbent and newly produced capital are indistinguishable. In contrast, Assumption 3 is satisfied because ownership of future capital cannot be traded before its creation. This suggests that the absence of Assumption 2 may be responsible for the non-existence of bubbles in dividend-paying capital in the standard Diamond model. We refer to the model incorporating Assumption 2 as the *modified* Diamond model. The validity of Assumption 2 is discussed later.

The notation basically follows Section 3. For simplicity, assume the utility of generation t is $U_t = \log c_{1,t} + \beta \log c_{2,t+1}$. Under Assumption 2, capital of different vintages may have different prices. Under Assumption 3, utility maximization yields $R_{t+1}p_{s,t} = (1 - \delta)p_{s,t+1} + D_{s,t+1}$ only for existing capital $s \leq t$, where $p_{s,t}$ is the period- t price of capital created in period s . The aggregate production function is given by $Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$, where $\alpha \in (0, 1)$ is a constant, Y_t is output, A_t is productivity level, L_t is the size of young population, and the aggregate capital is the sum of all existing vintages, $K_t = \sum_{s=-\infty}^{t-1} k_{s,t-1}$. Here, $k_{s,t} = (1 - \delta)^{t-s} k_{s,s}$ is the amount

¹⁴If the number of rows of $b(s^t)$ is greater than that of the columns in (1.1a) of Santos and Woodford (1997), the owners of existing trees receive the ownership of new trees. The resource constraint (1.3a) shows that the goods are solely used for consumption.

¹⁵In his influential paper, Tirole (1985) examines pure bubbles within the Diamond model.

of capital created in period s that remains in period t . The firm's profit maximization yields $w_t = A_t(1 - \alpha)\kappa_t^\alpha$ and $D_{s,t} = \alpha\kappa_t^{\alpha-1} = D_t$, where $\kappa_t := K_t/(A_t L_t)$. The fundamental value of one unit of capital is $v_t = \sum_{T=1}^{\infty} \frac{(1-\delta)^{T-1} D_{t+T}}{\prod_{u=1}^T R_{t+u}}$. If $p_{s,t} > v_t$, capital s has a bubble.

There are no tree growers. Instead, perfectly competitive investment firms transform one unit of consumption good into one unit of capital goods, $k_{t,t} = I_t$. The investment I_t undertaken by these firms is the counterpart of the grower's consumption. The zero-profit condition implies $p_{new} := p_{t,t} = 1$. Because the price of the new vintage is determined independently of arbitrage, it can exceed the fundamental value. The good and asset markets clear as $Y_t = (1 - \tilde{\beta})w_t L_t + \tilde{\beta}R_t w_{t-1} L_{t-1} + I_t$ and $\tilde{\beta}w_t L_t = \sum_{s=-\infty}^t p_{s,t} k_{s,t}$, respectively, where $\tilde{\beta} := \beta/(1 + \beta)$.

In a steady state, $\kappa := \kappa_t$ and $R := R_t$ remain constant over time. Consequently, aggregate capital K_t grows at the rate $G := G_A G_L$, where $G_A := A_{t+1}/A_t$ and $G_L := L_{t+1}/L_t$.

Proposition 7 *Consider the modified Diamond model. Assume $R_L < G$, where $R_L := (G/\tilde{\beta})[\alpha/(1 - \alpha)] + 1 - \delta (> 1 - \delta)$. Then, there is a continuum of bubbly steady-state equilibria, where $G > 1 - \delta$, $R \in (R_L, G)$, $v = D/[R - (1 - \delta)] < 1$, and $p_{s,t} = v + (p_{new} - v)[R/(1 - \delta)]^{t-s} > v$.*

(Proof) See Appendix L.

Because $G > 1 - \delta$, new capital is created over time and hence Assumption 1 is satisfied. Once Assumption 2 is imposed, the Diamond model admits bubbles on dividend-paying assets. Due to the limits of arbitrage, the fundamental value is not uniquely determined. This result supports our conjecture that the absence of Assumption 2 rules out bubbles on dividend-paying assets in the standard Diamond model. However, imposing Assumption 2 on the Diamond framework requires justifying why homogeneous capital is distinguishable by vintage. Therefore, the next section considers an R&D-based growth model in which differentiated intermediate-good firms issue distinct stocks with positive dividends.

6 Innovation and Bubbles in Dividend-yielding Assets

We now consider R&D investment. Following Olivier (2000) and Tanaka (2011), we employ a variety-expansion framework. Unlike them, we endogenize the bubble size of new firms. The two-period OLG structure and notation remain as before. To clarify the scope of our key assumptions, we distinguish between two types of R&D: entrant R&D and incumbent R&D. Entrant R&D satisfies all three assumptions. In contrast, incumbent R&D violates Assumption 3 because the trade of an incumbent's stock effectively includes the trade of claims on products

not yet invented (see also Appendix J).

6.1 Entrant R&D

We begin with entrant R&D. The final good is the numeraire, and its production is given by $Y_t = L^{1-\alpha} \int_{j=0}^{N_t} y_{j,t}^\alpha dj$ with $\alpha \in (0, 1)$, where L is a constant population of young households and $y_{j,t}$ is the input of variety j . The intermediate goods are differentiated. Intermediate-good firm j transforms one unit of Y_t into one unit of its variety to earn profit $\pi_{j,t}$. Under monopolistic competition, firm optimization yields $q_{j,t} = 1/\alpha = q$, $y_{j,t} = \alpha^{\frac{2}{1-\alpha}} L = y$, and $\pi_{j,t} = (1 - \alpha)\alpha^{\frac{1+\alpha}{1-\alpha}} L = \pi$, where $q_{j,t}$ is the price of intermediate good j .

Perfectly competitive R&D firms invent a mass $n_t = N_{t+1} - N_t$ of new varieties in period t , investing η units of Y_t per variety. New intermediate-good firms start production of these varieties in $t + 1$. To finance these invention costs, firms issue new stocks. Because the intermediate goods are differentiated, Assumption 2 is satisfied. Stocks of new firms become tradable only post-invention, which satisfies Assumption 3. We denote the period- t stock price of firm j by $\hat{p}_{j,t}$. The free entry condition implies $\hat{p}_{j,t} = \eta$ for newly issued stocks $j \in (N_t, N_{t+1}]$ in period t . Because the price of the new vintage is determined independently of arbitrage, it can exceed the fundamental value. The growth rate $\gamma_t = n_t/N_t$ is determined endogenously.

As in Section 5, households have a logarithmic utility function. they save solely by holding the stocks of all existing intermediate-good firms. The no-arbitrage condition $\hat{p}_{j,t} = (\hat{p}_{j,t+1} + \pi)/R_{t+1}$ holds for $j \in [0, N_{t+1}]$. Since all firms pay identical dividends π , they share a common fundamental value v_t . Existing studies typically assume a uniform price $\hat{p}_{j,t} = \hat{p}_t$. Instead of making this assumption, we examine whether different firms trade at different stock prices in equilibrium. If $\hat{p}_{j,t} > v_t$ for some j , stock price bubbles exist.

We focus on steady states, where $R_t = R$ and $\gamma_t = \gamma$ are constant. The fundamental value of intermediate-good firms is $v(R) = \pi/(R - 1)$. The growth rate of economy is $G = 1 + \gamma$. To characterize the equilibrium, let $\gamma_{LL} := \pi/\eta$ and $R_L := 1 + \gamma_{LL}$. We further define $\gamma_L := \frac{(1-\alpha)\beta}{(1+\beta)\eta} \alpha^{\frac{2\alpha}{1-\alpha}} L - 1$ and the function $\Psi(\gamma) := \frac{\alpha(1+\beta)\eta}{\beta\pi} [R_L + \frac{\beta\pi}{\alpha(1+\beta)\eta} - (1 + \gamma)](1 + \gamma)$.

Proposition 8 *Consider the R&D-based growth model described above.*

- (i) *A stock price bubble can exist only if $\gamma_t > 0$.*
- (ii) *If and only if $0 < \eta < (1 - \alpha)\alpha^{\frac{2\alpha}{1-\alpha}} L \left(\frac{\beta}{1+\beta} - \alpha \right)$, there is a continuum of bubbly steady-state equilibria where γ_t is constant at $\gamma_B \in (\gamma_{LL}, \gamma_L)$, $R = \Psi(\gamma_B) (> R_L > 1)$, $v(R) < \eta$, and $\hat{p}_{j,t} = v(R) + (\eta - v(R))R^{t-s_j}$ ($t \geq s_j$ and s_j is the invention period of good j).*

(Proof) See Appendix M.

As Theorem 1 shows, positive growth is necessary for a bubble. A bubble equilibrium further requires η to be low enough to ensure $G > R$. Factors directly lowering η include general-purpose technologies (e.g., the steam engine, the internet), financial innovation (e.g., venture capital), and deregulation (e.g., relaxation of listing criteria). These are the "displacements" of Aliber et al. (2023) and our model shows how they generate bubbles in dividend-yielding assets.

6.2 Incumbent R&D

All R&D is now conducted by incumbents rather than entrants. When an incumbent firm invents a new variety, its ownership automatically accrues to the original shareholders. Consequently, Assumption 3 is not satisfied, as in Appendix J.

There is a unit mass of incumbent firms. Each incumbent $i \in [0, 1]$ operates $N_{i,t}$ varieties in period t . To invent $n_{i,t+1}$ new varieties, the firm must invest $\eta n_{i,t+1}$ units of final goods. The period- t stock price of firm i , $\tilde{p}_{i,t}$, satisfies the no-arbitrage condition:

$$\tilde{p}_{i,t} = \frac{1}{R_{t+1}} \left[\tilde{D}_{i,t+1} + \tilde{p}_{i,t+1} \right], \quad (21)$$

where $\tilde{D}_{i,t+1} := (\pi - \eta\gamma_{i,t+1})N_{i,t+1}$ is the dividend and $\gamma_{i,t+1} := n_{i,t+1}/N_{i,t+1}$ is the investment rate of firm i . In steady state where $\gamma_{i,t+1}$ is constant at γ_i , firm i 's dividend grows at rate $1 + \gamma_i$. For the fundamental value to be finite, $R > 1 + \gamma_i$ must hold for all $i \in [0, 1]$. Since there is no new firm entry, aggregate growth equals the highest firm-level growth in the long run: $G = \max_i \{1 + \gamma_i\}$. This implies $R > G$, which rules out bubbles. Appendix N shows the existence of a unique bubbleless steady state with $\gamma > 0$.

To see how Assumption 3 is violated, let $\tilde{p}_i(N_{i,t+1}, t)$ denote the stock price of firm i . Then, (21) can be rewritten as $\tilde{p}_i(N_{i,t+1}, t) = R_{t+1}^{-1} [\tilde{D}_{i,t+1} + \tilde{p}_i(N_{i,t+2}, t+1)]$, where $N_{i,t+2} = N_{i,t+1} + n_{i,t+1}$. The current stock price depends on the future number of varieties $N_{i,t+2}$, which includes $n_{i,t+1}$ varieties yet to be invented. Thus, trading firm i 's stock effectively involves trading these future varieties, which does not satisfy Assumption 3.

7 Concluding Remarks

This paper shows how the continual creation of new assets and the non-tradability of unborn assets can jointly support rational bubbles in dividend-paying assets. In our benchmark OLG

economy with Lucas trees, the prices of existing trees can exceed the present discounted value of their dividends. These bubbles arise in a simple deterministic economy along a balanced growth path, without relying on uncertainty or irrational expectations.

The mechanism rests on three assumptions. New assets are continually created, assets are distinguished by their dates of creation, and claims on unborn assets cannot be traded. Together, these assumptions limit arbitrage between existing and future assets while preserving arbitrage among existing assets. Existing assets with identical dividend streams can trade at different prices while yielding the same rate of return. Within our framework, removing any one of the three assumptions eliminates the bubble equilibrium. We demonstrate the broad applicability of the mechanism through models with endogenous investment, infinitely lived agents, and endogenous growth. Our model thus provides theoretical support for historical studies that associate asset price bubbles with the creation of new assets following technological innovation, financial innovation, or deregulation.

Appendix

A New trees are distributed to OLG households:

Suppose that there are no tree growers and that all new trees are distribute to the old households. The savings of each household is still $\hat{s}_t = (1 - \alpha)Y_t$. Thus, we have $c_{2,t} = R_t(1 - \alpha)Y_{t-1} + n_t p_{t,t}$. The consumption good markets clear as in the model with growers. This modification leads to the same result as the model in Section 2.

If all new trees are distribute to the young households, their savings is $\hat{s}_t = (1 - \alpha)Y_t + n_t p_{t,t}$. The consumption good market clears

$$\frac{1}{1 + \gamma} \left[(1 - \alpha) + \gamma \alpha \frac{p_{t-1,t-1}}{D} \right] R_t = 1.$$

Assume that $1 + \gamma > 1/(1 - \alpha)$. Then, we can verify that there is an equilibrium where R_t and $p_{new,t}$ are constant at $R(\nu)$ and $p_{new}(\nu)$, respectively, where

$$R(\nu) := (1 - \nu) \frac{1}{1 - \alpha} + \nu(1 + \gamma), \quad \nu \in [0, 1), \quad \text{and} \quad p_{new}(\nu) := \frac{D}{\alpha\gamma} \left[\frac{1 + \gamma}{R(\nu)} - (1 - \alpha) \right].$$

B Proof of Result 2

Because $N_t = (1 + \gamma)N_{t-1}$ and $c_{2,t} = R_t(1 - \alpha)Y_{t-1}$, the consumption good market equilibrium, $c_{2,t} + p_{t,t}n_t = Y_t$, reduces to

$$\frac{1 - \alpha}{1 + \gamma} R_t + \alpha\gamma \frac{p_{t,t}}{D} = 1. \tag{B1}$$

The equilibrium asset prices $p_{s,t}$ must follow condition (1) and satisfy $\sum_{s=-\infty}^t p_{s,t} n_s = w_t$.

Assuming $\alpha < (1 - 2\alpha)\gamma$. The new trees' price is $p_{t,t}$. We guess and verify that there is a steady state equilibrium where R_t and $p_{t,t}$ are constant at $R(\nu)$ and $p_{new}(\nu)$, respectively, where

$$R(\nu) := (1 - \nu) \frac{1 + \alpha\gamma}{1 - \alpha} + \nu(1 + \gamma) \quad \text{and} \quad p_{new}(\nu) := \frac{D}{\alpha\gamma} - \frac{(1 - \alpha)D}{\alpha\gamma(1 + \gamma)} R(\nu), \quad \nu \in [0, 1).$$

Because of $\alpha < (1 - 2\alpha)\gamma$, we have $R(\nu) \in \left[\frac{1 + \alpha\gamma}{1 - \alpha}, 1 + \gamma \right)$. By construction, $R(\nu)$ and $p_{new}(\nu)$ satisfy (B1). With constant R , the fundamental value is given by $v_t = D/(R - 1) =: v^*$. Since $R(\nu) \in \left[\frac{1 + \alpha\gamma}{1 - \alpha}, 1 + \gamma \right)$, $p_{new}(\nu) \geq v^*$ holds (see Figure A1). Define $b_{new}(\nu) := p_{new}(\nu) - v^* \geq 0$. Because v^* is a particular solution of (1), we have $p_{s,t+1} - v^* = R(\nu)(p_{s,t} - v^*)$ and $p_{s,t} =$

$v^* + R(\nu)^{t-s} b_{new}(\nu) \geq \hat{v}$. To confirm that these asset prices are consistent with portfolio allocation, we check if $\sum_{s=-\infty}^t n_s p_{s,t} = w_t$ is satisfied as follows:

$$\begin{aligned} \sum_{s=-\infty}^t n_s p_{s,t} &= v^* N_{t+1} + \gamma N_t \sum_{s=-\infty}^t \left(\frac{R(\nu)}{1+\gamma} \right)^{t-s} b_{new}(\nu) = v^* N_{t+1} + \frac{\gamma N_{t+1}}{1+\gamma - R(\nu)} b_{new}(\nu) \\ &= (1-\alpha) Y_t = w_t. \end{aligned}$$

In the above equation, we use $n_s = \gamma N_s$, $\sum_{s=-\infty}^t n_s = N_{t+1} = (1+\gamma) N_t$, and the definitions of $p_{new}(\nu)$ and v^* . The second equality holds because of $0 < R(\nu) < 1+\gamma$. Then, $R(\nu)$ and $p_{new}(\nu)$ ($\nu \in [0, 1)$) constitute a continuum of steady state equilibrium.

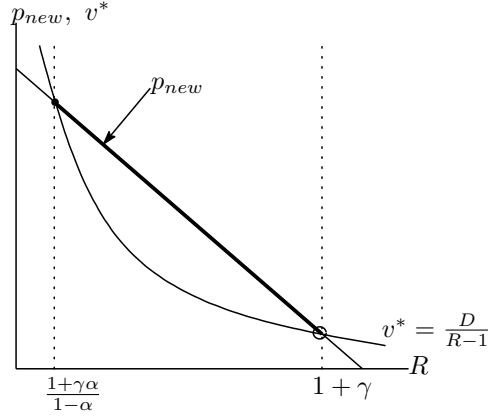


Figure A1 Price of New Trees

C Aggregate Dividends

Assume that $\gamma > 0$. We rewrite the aggregate dividend D_t^A as

$$D_t^A = \sum_{s=-\infty}^{t-1} n_{s,t-1} D_{s,t} = \gamma N_{t-1} A_t \sum_{s=-\infty}^{t-1} \left(\frac{(1-\delta)G_D}{(1-\delta+\gamma)G_A} \right)^{t-1-s}.$$

If $(1-\delta+\gamma)G_A > (1-\delta)G_D$, we have

$$D_t^A = \frac{(1+\gamma-\delta)G_A}{(1-\delta+\gamma)G_A - (1-\delta)G_D} \gamma A_t N_{t-1} = \hat{D} A_t N_t < \infty,$$

where $\hat{D} := \gamma G_A / \{(1+\gamma-\delta)G_A - (1-\delta)G_D\}$. Thus, D_t^A grows at the rate of $(1-\delta+\gamma)G_A$.

With no new trees, we have $D_t^A = N_t D_t$. Thus, D_t^A grows at the rate of $(1-\delta)G_D$.

D Walras's Law

By using $c_{2,t} = \sum_{s=-\infty}^{t-1} \{(1-\delta)p_{s,t} + D_{s,t}\}z_{s,t-1}$, we rearrange (9b) as

$$(9b) \Leftrightarrow c_{1,t}L_t + \sum_{s=-\infty}^{t-1} \{(1-\delta)p_{s,t} + D_{s,t}\}z_{s,t-1}L_{t-1} + n_{t,t}p_{t,t} = Y_t. \quad (D1)$$

Because of (9a) and $Y_t = w_tL_t + D_t^A$, the above equation can be rewritten as

$$(D1) \Leftrightarrow c_{1,t}L_t + \sum_{s=-\infty}^{t-1} \{(1-\delta)p_{s,t} + D_{s,t}\}n_{s,t-1} + n_{t,t}p_{t,t} = w_tL_t + D_t^A. \quad (D2)$$

Since $\hat{s}_t = w_t - c_{1,t}$ and $n_{s,t} = (1-\delta)n_{s,t-1}$ hold, we have

$$(D2) \Leftrightarrow \sum_{s=-\infty}^{t-1} p_{s,t}n_{s,t} + n_{t,t}p_{t,t} = \hat{s}_tL_t + D_t^A - \sum_{s=-\infty}^{t-1} D_{s,t}n_{s,t-1}. \quad (D3)$$

By the definition of D_t^A (see (2a)), (D3) is equivalent to (9c). Thus, (9a) and (9b) ensure (9c).

If we follow the opposite direction, we can derive (9b) from (9a) and (9c). Here, we provide another proof. Define $P_t := \sum_{s=-\infty}^t n_{s,t}p_{s,t}$. Clearly, we have $\hat{s}_tL_t = P_t$ from (9c). Besides, (5) indicates that P_t follows $p_{t,t}n_{t,t} + R_tP_{t-1} = P_t + D_t^A$. If we use $P_t = \hat{s}_tL_t$, $R_t\hat{s}_{t-1}L_{t-1} = c_{2,t}L_{t-1}$, and $\hat{s}_tL_t = w_tL_t - c_{1,t}L_t$ in this equation, we obtain (9b).

E Proof of Proposition 2

In a bubbleless steady state, we have $p_{s,t} = v_{s,t}$ and $b_t^A = B_t^A = 0$ over time. The RHS of (14) decreases from $+\infty$ to zero as R increases from $(1-\delta)G_D$ to $+\infty$. Since $S(R) = \Gamma(R)/(R+\Gamma(R)) \leq 1$, the LHS is bounded from the above. Therefore, we have $\text{LHS}|_{R=(1-\delta)G_D} < +\infty = \lim_{R \rightarrow (1-\delta)G_D} \text{RHS}$. To prove that $\text{LHS} > \text{RHS}$ for large R , we prove the following lemma:

Lemma A1 $\lim_{R \rightarrow \infty} C_2(R) = +\infty$.

(Proof) $C_1(\cdot)$ and $C_2(\cdot)$ are defined in (7). Note that $\lim_{R \rightarrow \infty} C_2(R)/C_1(R) = \lim_{R \rightarrow \infty} \Gamma(R) = \infty$. Suppose that $\lim_{R \rightarrow \infty} C_2(R) < +\infty$. The first-period budget constraint, $C_1(R)w_t + C_2(R)w_t/R = w_t$, implies that $\lim_{R \rightarrow \infty} C_1(R) = 1$, which contradicts $\lim_{R \rightarrow \infty} C_2(R)/C_1(R) = \infty$. $\square$

From $C_2(R) = RS(R)$ and Lemma (A1), we have

$$\frac{LHS}{RHS} = \frac{1-\alpha}{\alpha G} \left[1 - \frac{(1-\delta)G_D}{R} \right] C_2(R) \rightarrow \infty, \text{ as } R \rightarrow \infty.$$

Thus, $LHS > RHS$ for large R . The LHS and RHS of (14) have at least one intersection. Denote R at the intersection by R_L .

We next show that at the intersection(s), the LHS always has a larger slope than the RHS, irrespective of the value of IES. The RHS is downward-sloping for all $R > (1 - \delta)G_D$. If $IES(R_L) \geq 1$, the LHS is upward-sloping at the intersection. Thus, the LHS has a larger slope than the RHS at $R = R_L$.

Next assume that $IES(R_L) < 1$. We evaluate the slopes of the both sides of (14) at $R = R_L$:

$$\left. \frac{\partial RHS}{\partial R} \right|_{R=R_L} = -\frac{1}{R_L - (1 - \delta)G_D} RHS|_{R=R_L} < -\frac{1}{R_L} RHS|_{R=R_L}$$

and

$$\begin{aligned} \left. \frac{\partial LHS}{\partial R} \right|_{R=R_L} &= \left. \frac{\partial}{\partial R} \left(\frac{(1 - \alpha)\Gamma(R)/R}{1 + \Gamma(R)/R} \right) \right|_{R=R_L} \\ &= -\frac{1}{R_L} LHS|_{R=R_L} + (1 - \alpha) \frac{\Gamma'(R_L)/R_L}{1 + \Gamma(R_L)/R_L} - \frac{\frac{\partial}{\partial R} \left(\frac{\Gamma(R_L)}{R_L} \right)}{1 + \Gamma(R_L)/R_L} LHS|_{R=R_L} \\ &> -\frac{1}{R_L} LHS|_{R=R_L} = -\frac{1}{R_L} RHS|_{R=R_L} \\ &> \left. \frac{\partial RHS}{\partial R} \right|_{R=R_L}. \end{aligned}$$

The second term on the second line is positive because of $\Gamma'(R) > 0$. Because $IES(R_L) < 1$, $\frac{\partial}{\partial R} \left(\frac{\Gamma(R_L)}{R_L} \right)$ is negative. Thus, the last term on the second line is positive. Hence, the inequality in the third line holds. Therefore, the LHS always has a larger slope than the RHS at $R = R_L$, meaning that the graph of the LHS intersects with that of the RHS from below, as in Figure 2. The intersection must be unique. The bubbleless steady state is also unique.

F Proof of Proposition 3

(Proof of 1) We prove this part by using (15). If no new trees are born, (15) reduces to $0 = 1 - \alpha + \alpha - (1 - \alpha)C_1(R) - \frac{1 - \alpha}{G}C_2(R)$, where $C_1(R)$ and $C_2(R)$ are defined in (7). We rearrange this equation as follows:

$$\begin{aligned} 0 &= (1 - \alpha) - (1 - \alpha) \frac{1 + \Gamma(R)/G}{1 + \Gamma(R)/R} + \alpha \\ \Leftrightarrow 0 &= (1 - \alpha) \frac{\Gamma(R)/R - \Gamma(R)/G}{1 + \Gamma(R)/R} + \alpha \end{aligned}$$

$$\Leftrightarrow 0 = (1 - \alpha) \frac{\Gamma(R)/R}{1 + \Gamma(R)/R} \frac{G - R}{G} + \alpha.$$

Since $G = (1 - \delta)G_D$ if $n_{t,t} = \gamma N_t = 0$, the above equation is equivalent to (14). Then, a unique bubbleless steady state exists, where b_t^A is constant at zero over time.

(Proof of 2) We prove this part in the following five steps.

Step 1: We first point out that $R \geq R_L$ must hold in equilibrium because the aggregate saving must absorb the aggregate fundamental value, $\hat{s}_t L_t \geq \sum_{s=-\infty}^t n_{s,t} v_{s,t}$ (see Figure 2).

Step 2: We next draw the graphs of ϕ_{new} and $\psi(R)$. For this purpose, We derive the solution(s) of $\phi_{new}(R) = \psi(R)$. Even under general utility functions without time-separable and CRRA assumptions, from (15), we have

$$\begin{aligned} & \gamma \frac{\alpha G_A}{\hat{D}} \frac{1}{R - (1 - \delta)G_D} = 1 - (1 - \alpha)C_1(R) - \frac{1 - \alpha}{G}C_2(R) \\ \Leftrightarrow & \frac{\alpha[(1 + \gamma - \delta)G_A - (1 - \delta)G_D]}{R - (1 - \delta)G_D} = 1 - (1 - \alpha) \frac{1 + \Gamma(R)/G}{1 + \Gamma(R)/R} \\ \Leftrightarrow & \frac{\alpha[(1 + \gamma - \delta)G_A - R]}{R - (1 - \delta)G_D} = (1 - \alpha) - (1 - \alpha) \frac{1 + \Gamma(R)/G}{1 + \Gamma(R)/R} \\ \Leftrightarrow & \frac{G - R}{R - (1 - \delta)G_D} = \frac{G - R}{\alpha G} \frac{(1 - \alpha)\Gamma(R)/R}{1 + \Gamma(R)/R} \\ \Leftrightarrow & (G - R) \left[\frac{1 - \alpha}{\alpha G} S(R) - \frac{1}{R - (1 - \delta)G_D} \right] = 0. \end{aligned}$$

$C_1(R)$ and $C_2(R)$ are defined in (7). The above equation shows that $\phi_{new} = \psi(R)$ has two solutions, $R = R_L$ and $R = G$.

Because ϕ_{new} is determined by (15) and $C_1((1 - \delta)G_D) < +\infty$ and $C_2((1 - \delta)G_D) < +\infty$, we have $\phi_{new}(R)|_{R=(1-\delta)G_D} < +\infty = \lim_{R \rightarrow (1-\delta)G_D} \psi(R)$. From Lemma A1, we also have $\lim_{R \rightarrow +\infty} \phi_{new}(R) < 0 = \lim_{R \rightarrow +\infty} \psi(R)$. Because $\psi(R)$ is a decreasing function of R , Figure 3 gives us the graphs of ϕ_{new} and $\psi(R)$.

Step 3: We show that the bubbly equilibrium exists only if $R_L < G$. Suppose that $G \leq R_L$. $R < R_L$ implies $\hat{s}_t L_t < \sum_{s=-\infty}^t n_{s,t} v_{s,t}$. $R > R_L$ implies $\phi_{new}(R) < \psi(R)$ (see Figure A2). Both the cases cannot be an equilibrium. Thus, if there is a steady state, it must be a unique bubbleless steady state, where $R = R_L$.

Suppose that $R_L < G$. There are two types of equilibrium. See Figure 3. The first one is the bubbleless steady state where $R = R_L$ and $b_t^A = 0$. The second one is the continuum of the

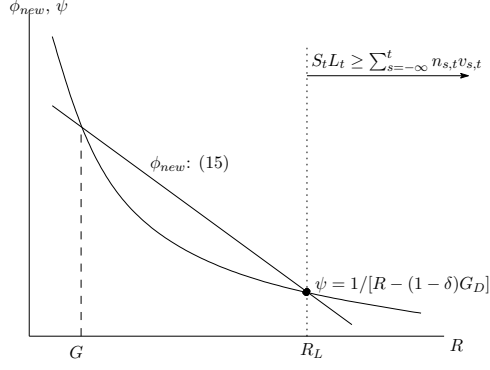


Figure A2 Equilibrium under $G < R_L$

bubbly steady state where $R \in (R_L, G)$ and $\phi_{new}(R) > \psi(R)$. All trees have asset bubbles. The dynamics of b_t^A is given by (16). Because $R < G$, b_t^A monotonically converges to its steady state value in the long run.

If R is equal to G , we have $B_t^A = S_t L_t - \sum_{s=-\infty}^t n_{s,t} v_{s,t} > 0$ (see Figure A3). However, the new trees do not have a bubble, $\phi_{new}(G) = \psi(G)$. These two facts contradict each other. Thus, there are no steady state equilibria with $R = G$.

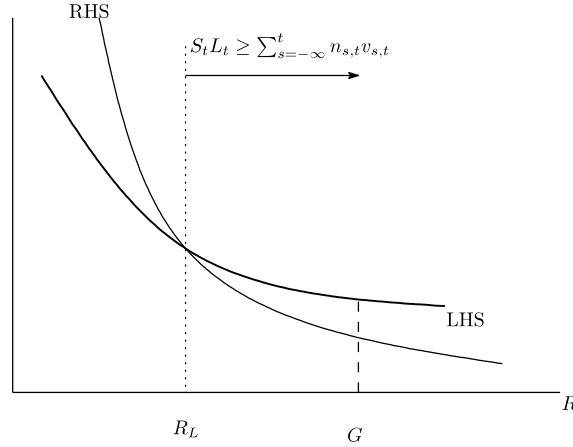


Figure A3 The Condition for $R_L < G$

Step 4: We next derive the condition for $R_L < G$. As Figures 2 and A3, the LHS of (14) is greater than the RHS of (14) for $R > R_L$. Thus, $R_L < G$ holds if and only if

$$(1 - \alpha)S(G) > \frac{\alpha G}{G - (1 - \delta)G_D}. \quad (F1)$$

The RHS decreases from $+\infty$ to α as G increases from $(1 - \delta)G_D$ to $+\infty$. Assume that $u(c) = c^{1-\sigma}/(1 - \sigma)$. Then, we have $\Gamma(R) = (\beta R)^{\frac{1}{\sigma}}$.

(C1) If IES is greater than one ($\sigma < 1$), the LHS of (F1) increases from zero to $1 - \alpha$ as G increases from zero to $+\infty$. If and only if $1 - \alpha > \alpha$, there is a unique $\underline{G}_1$ that is defined by $(1 - \alpha)(\beta \underline{G}_1)^{\frac{1}{\sigma}} / [\underline{G}_1 + (\beta \underline{G}_1)^{\frac{1}{\sigma}}] = \alpha \underline{G}_1 / [\underline{G}_1 - (1 - \delta)G_D]$. Then, (F1) holds for $G = (1 + \gamma - \delta)G_A > \underline{G}_1$. Even when $\gamma = \delta$, $G > \underline{G}_1$ holds if G_A is large enough.

(C2) If IES is one ($\sigma = 1$), the LHS of (F1) is constant at $(1 - \alpha)\beta/(1 + \beta)$. Thus, if and only if $(1 - \alpha)\beta/(1 + \beta) > \alpha$, (F1) holds for

$$G = (1 + \gamma - \delta)G_A > \frac{(1 - \alpha)\beta}{(1 - 2\alpha)\beta - \alpha}(1 - \delta)G_D$$

If G_A/G_D is large enough or δ is close to one, the above equality holds even if $\gamma = \delta$.

(C3) If IES is smaller than one ($\sigma > 1$), the LHS of (F1) is a decreasing and finite function of G that approaches zero as G increases to $+\infty$. Then, if $LHS|_{G=1/\beta} > RHS|_{G=1/\beta}$, which is equivalent to $(1 - \delta)G_D < \frac{(1 - 2\alpha)\beta - \alpha}{(1 - \alpha)\beta^2}$, the LHS and RHS of (F1) have at least two intersections, $\underline{G}_2$ and $\bar{G}$ (see Figure A4). For $G \in (\underline{G}_2, \bar{G})$, (F1) holds.

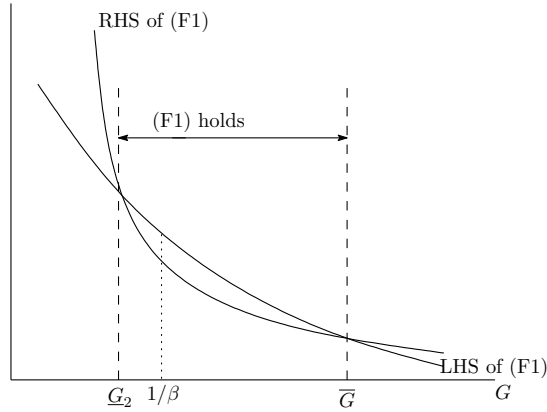


Figure A4 Bubbly Steady State: IES < 1

Step 5: To confirm that the asset prices derived so far are consistent with portfolio allocation, we finally check if (9c) is satisfied in the bubbly steady states. At the steady state, we have $\phi_{s,t} = \psi(R) + (\phi_{new}(R) - \psi(R)) \left[\frac{R}{(1 - \delta)G_D} \right]^{t-s} > \psi(R)$. Because $p_{s,t} = \phi_{s,t}D_{s,t+1}$, we have

$$\begin{aligned} \sum_{s=-\infty}^t p_{s,t} n_{s,t} &= \sum_{s=-\infty}^t \left[\phi_{s,t} D_{s,t+1} + (\phi_{new}(R) - \psi) \left[\frac{R}{(1 - \delta)G_D} \right]^{t-s} n_{s,t} D_{s,t+1} \right] \\ &= \psi D_{t+1}^A + (\phi_{new}(R) - \psi) \frac{\gamma G_A A_{t+1} (1 - \delta + \gamma) N_{t+1}}{G - R} \end{aligned}$$

$$\begin{aligned}
&= \psi D_{t+1}^A + \frac{\phi_{new}(R) - \psi}{G - R} \frac{\gamma G_A}{\hat{D}} D_{t+1}^A \left(= \psi D_{t+1}^A + \underbrace{G \frac{\phi_{new}(R) - \psi}{G - R} \frac{\gamma \alpha G_A}{\hat{D}}}_{b^A} Y_t \right) \\
&= \frac{Y_{t+1}}{G - R} \frac{\alpha \gamma G_A}{\hat{D}} \phi_{new}(R) + \psi D_{t+1}^A \left[1 + \frac{G - (1 - \delta)G_D}{G - R} \right] \\
&= \frac{Y_{t+1}}{G - R} \left[1 - (1 - \alpha)C_1(R) - \frac{1 - \alpha}{G} C_2(R) \right] - \frac{\alpha Y_{t+1}}{G - R} \\
&= \frac{1}{G - R} (\hat{s}_{t+1} L_{t+1} - R \hat{s}_t L_t) \\
&= \hat{s}_t L_t.
\end{aligned}$$

The second equality uses $n_{s,t} = \gamma N_t \left(\frac{1-\delta}{1-\delta+\gamma} \right)^{t-s}$ and $D_{s,t+1} = A_{t+1} \left(\frac{G_D}{G_A} \right)^{t-s}$. The summation converges because $R < G$. The term b^A in the third line is the steady state value of b_t^A and is consistent with (16). The fourth line uses $D_t^A = \alpha Y_t$ and $\psi(R) = 1/[R - (1 - \delta)G_D]$. The fifth line uses (15). The sixth line uses $w_t L_t = (1 - \alpha)Y_t$. Thus, (9c) is satisfied.

G Proof of Proposition 4

The fundamental value is given by

$$v_t = \sum_{\eta=1}^{\infty} \frac{\{(1 - \delta)G_D\}^{\eta-1}}{\prod_{u=1}^{\eta} R_{t+u}} D_{t+1} = \frac{D_{t+1}}{R - (1 - \delta)G_D}.$$

The last equality holds at a steady state where $R > (1 - \delta)G_D$. The aggregate fundamental value must be finite and is given by

$$N_{t+1}v_t = \frac{N_{t+1}D_{t+1}}{R - (1 - \delta)G_D} = \frac{\alpha Y_{t+1}}{R - (1 - \delta)G_D}.$$

The bubble term $b_t = p_t - v_t$ satisfies the no-arbitrage condition $b_{t+1} = \frac{R_{t+1}}{1-\delta} b_t$. The aggregate bubble $B_t^A = N_{t+1}b_t$ follows

$$B_{t+1}^A = \frac{1 + \gamma - \delta}{1 - \delta} R_{t+1} B_t^A.$$

The aggregate savings must be equal to the aggregate value of the trees, $\hat{s}_t L_t = N_{t+1}p_t (= N_{t+1}v_t + B_t^A)$.

At the steady state, this equation can be rewritten as

$$(1 - \alpha)S(R) = \frac{\alpha(1 + \gamma - \delta)G_D}{R - (1 - \delta)G_D} + \frac{B_t^A}{Y_t}. \quad (\text{G1})$$

If $B_t^A > 0$, the growth rate of B_t^A must not be higher than that of Y_t , which means that $\frac{1+\gamma-\delta}{1-\delta}R \leq (1+\gamma-\delta)G_D$. We have $R \leq (1-\delta)G_D$, which contradicts $N_{t+1}v_t \leq \hat{s}_t L_t < \infty$. Hence, we have $B_t^A = 0$ at the steady state. Then, (G1) reduces to $(1-\alpha)S(R) = \alpha G \cdot \psi$, which is the same as (14). Thus, a unique bubbleless steady state exists.

If the parametric condition $\mathcal{C}$ is satisfied under a time-separable utility with CRRA felicity function, $R_L < G$ holds in bubbleless steady state as Proposition 3 shows. $R = R_L$ is a unique bubbleless steady state.

H Proof of Proposition 5

Using $p_{s,t} = m_{t+1}p_{s,t+1}$ for $s \in \mathbb{S}_t$, we derive $\sum_{s \in \mathbb{S}_t} p_{s,t}n_{s,s} = -\mathbf{1}_{\mathbb{I}}(t)p_{t,t}n_{t,t} + R_t \sum_{s \in \mathbb{S}_{t-1}} p_{s,t-1}n_{s,s}$. Comparing this equation with the second one in (17) yields $X_t = \sum_{s \in \mathbb{S}_t} p_{s,t}n_{s,s}$. Using this relation and the market clearing conditions, we aggregate the budget constraints (18) and (19) as $c_{1,t}L_t + \sum_{s=-\infty}^t p_{s,t}n_{s,t} = w_t L_t$ and $c_{2,t+1}L_{t+1} = \sum_{s=-\infty}^t x_{s,t+1}n_{s,t}$, respectively. The model reduces to the benchmark model in Section 3 with the restriction $R > G$. Thus, under certain conditions, a unique bubbleless steady state exists. Under parametric condition $\mathcal{C}$, $R_L < G$ holds and then even bubbleless steady states do not exist.

I Trees scheduled to be born up to T periods ahead are tradable.

We demonstrate that a bubbly steady-state equilibrium can exist if only trees scheduled to be born up to a finite T periods ahead can be traded in the current period. Instead of Assumption 4, we impose Assumption 4'. The budget constraints for the arbitrageur in period 0 and any period $t \geq 1$ are, respectively, given by

$$\sum_{s \in \mathbb{S}_0^*} p_{s,0}n_{s,s} \geq X_0 \quad \text{and} \quad R_t X_{t-1} + \mathbf{1}_{\mathbb{I}}(t+T)p_{t+T,t}n_{t+T,t+T} \geq \mathbf{1}_{\mathbb{I}}(t)p_{t,t}n_{t,t} + X_t. \quad (\text{I1})$$

In period t , trees scheduled to be born in period $t+T$ newly become tradable, provided they belong to the tradable subset $\mathbb{I}$. Consequently, the arbitrageur can short-sell these trees to raise an amount $p_{t+T,t}n_{t+T,t+T}$, which corresponds to the second term on the left-hand side of the second constraint. Again, for all $t \geq 0$, prices and R_t must adjust so that the budget constraints in (I1) hold with equality.

If we replace $\mathbb{S}_t$ in (18) and (19) by $\mathbb{S}_t^*$, we obtain the budget constraints for a representative household of generation t . In the same token, if we replace $\mathbb{S}_t$ by $\mathbb{S}_t^*$, the discussion in Appendix

(H) can be applied to the model in this Appendix.

From $p_{s,t} = p_{s,t+1}/R_{t+1}$ for $s \in \mathbb{S}_t^*$, we have

$$\sum_{s \in \mathbb{S}_t^*} p_{s,t} n_{s,s} = \sum_{s \in \mathbb{S}_t^*} \frac{p_{s,s} n_{s,s}}{\prod_{u=1}^{s-t} R_{t+u}}.$$

Because $\mathbb{S}_t^*$ is a finite set, $\sum_{s \in \mathbb{S}_t^*} p_{s,t} n_{s,s}$ is finite even if $R \leq G$. Similarly, $p_{t+T,t} n_{t+T,t+T}$ is also finite because T is finite. Thus, the two conditions in (I1) holds even if $R \leq G$.

The model reduces to the benchmark model in Section 3 without any restriction on the relative magnitudes of R and G . Consequently, a bubbly steady-state equilibrium continues to exist even under Assumption 4'.

J New Trees Are Distributed to the Owner of Existing Trees

Assume that owners of existing trees automatically have ownership of new trees. Owners of an existing tree receive γ new trees. Denote the price of tree s by $\hat{p}_{s,t}$. Tree s provides the payoff $\hat{x}_{s,t+1} := (1 - \delta)\hat{p}_{s,t+1} + D_{s,t+1} + \gamma\hat{p}_{t+1,t+1}$, which includes the value of the new trees $\gamma\hat{p}_{t+1,t+1}$. The budget constraints of generation t are $c_{1,t} + \sum_{s=-\infty}^t p_{s,t} z_{s,t} + a_t = w_t$ and $c_{2,t+1} = \sum_{s=-\infty}^t \hat{x}_{s,t+1} z_{s,t} + R_{t+1} a_t$. The utility maximization yields $\hat{p}_{s,t} = m_{t+1} \hat{x}_{s,t+1}$ ($s \leq t$). The price of unborn trees enters into the no-arbitrage relationship. Ownership claims to unborn trees are effectively traded in the current period and thus Assumption 3 is relaxed.

The fundamental value of tree $s (\leq t)$ that includes values of unborn trees is a particular solution of $\hat{p}_{s,t} = m_{t+1} \hat{x}_{s,t+1}$:

$$\hat{v}_{s,t} = \underbrace{\sum_{T=1}^{\infty} \frac{(1 - \delta)^{T-1} D_{s,t+T}}{\prod_{u=1}^T R_{t+u}}}_{v_{s,t}} + \gamma \underbrace{\sum_{T=1}^{\infty} \frac{(1 - \delta)^{T-1} \hat{v}_{t+T,t+T}}{\prod_{u=1}^T R_{t+u}}}_{\zeta_t}. \quad (\text{J1})$$

The first term $v_{s,t}$ is the fundamental value based on dividends of the incumbent tree s . The second term ζ_t comes from the intertemporal no-arbitrage of unborn trees between before and after birth. In equilibrium, $\hat{v}_{s,t}$ must be finite because $\hat{p}_{s,t}$ is also finite. We rearrange ζ_t as

$$\zeta_t = \frac{\hat{v}_{t+1,t+1}}{R_{t+1}} + \frac{1 - \delta}{R_{t+1}} \zeta_{t+1} = \frac{v_{t+1,t+1}}{R_{t+1}} + \frac{1 + \gamma - \delta}{R_{t+1}} \zeta_{t+1} = \sum_{T=1}^{\infty} \frac{(1 + \gamma - \delta)^{T-1} v_{t+T,t+T}}{\prod_{u=1}^T R_{t+u}}.$$

The second equality uses $\hat{v}_{t+1,t+1} = v_{t+1,t+1} + \gamma\zeta_{t+1}$ (see (J1)). The last equality comes from recursive substitution. Consider a steady state. If $R > (1 - \delta)G_D$, we have $v_{s,t} = D_{s,t+1}/(R -$

$(1-\delta)G_D) < \infty$. Because $v_{t+T,t+T}$ grows at a rate G_A , ζ_t is finite only if $R > (1-\delta+\gamma)G_A = G$, which rules out asset bubbles.

We decompose price of tree s as $\hat{p}_{s,t} = p_{s,t} + \tilde{p}_{s,t}$, where $p_{s,t}$ is the price based on tree s itself and $\tilde{p}_{s,t}$ is the value that stems from new trees delivered in the subsequent periods. In period 0, the arbitrageur buy incumbent trees and new trees born in period 0 and lend X_0 in the net zero supply asset market. Then, it resells all trees together with rights to receive unborn trees in the subsequent periods, $\sum_{s=-\infty}^0 \hat{p}_{s,0} n_{s,0} \geq \sum_{s=-\infty}^{-1} p_{s,0} n_{s,0} + p_{0,0} n_{0,0} + X_0$ or $\sum_{s=-\infty}^0 \tilde{p}_{s,0} n_{s,0} \geq X_0$. In period $t(\geq 1)$, the arbitrageur receives $R_t X_{t-1}$ and lends X_t , and purchases new trees from grower to deliver them to owners of incumbent trees. The budget constraint is $R_t X_{t-1} \geq X_t + n_{t,t} p_{t,t}$. The competition among arbitrageurs ensures no-arbitrage and hence the budget constraints hold with equality.

Because $\hat{p}_{s,t} = m_{t+1} \hat{x}_{s,t+1}$, they satisfy $p_{s,t} = m_{t+1}[(1-\delta)p_{s,t+1} + D_{s,t+1}]$ and $\tilde{p}_{s,t} = m_{t+1}[(1-\delta)\tilde{p}_{s,t+1} + \gamma\hat{p}_{t+1,t+1}]$. From the second equation, we obtain $R_{t+1} \sum_{s=-\infty}^t \tilde{p}_{s,t} n_{s,t} = \sum_{s=-\infty}^{t+1} \tilde{p}_{s,t+1} n_{s,t+1} + n_{t+1,t+1} p_{t+1,t+1}$. By comparing this equation with $R_t X_{t-1} = X_t + n_{t,t} p_{t,t}$, we have $X_t = \sum_{s=-\infty}^t \tilde{p}_{s,t} n_{s,t}$. If we aggregate budget constraints of households, we obtain $c_{1,t} L_t + \sum_{s=-\infty}^t p_{s,t} n_{s,t} = w_t L_t$ and $c_{2,t+1} L_{t+1} = \sum_{s=-\infty}^t x_{s,t+1} n_{s,t}$. The model returns to the benchmark model in Section 3 with restriction $R > G$. Thus, under a certain condition, there is a unique bubbleless steady state, similarly to Proposition 5.

K An Infinitely Lived Agents Model with Short Sales Constrains

While the TVC typically rules out bubbles on dividend-yielding assets in infinitely lived agent models, Kocherlakota (1992) shows that (pure) bubbles can exist under short-sale constraints (SSC). This appendix demonstrates that bubbles on dividend-yielding assets can exist under Assumptions 1–3 in an infinitely lived agent framework with SSC instead of an OLG structure.

There are two types of homogeneous households, $i \in \{1, 2\}$, each with a population normalized to one. Household i receives an endowment w_t^i , where the sequences are $\{w_t^1\}_{t \geq 0} = \{0, w_1, 0, \dots\}$ and $\{w_t^2\}_{t \geq 0} = \{w_0, 0, w_2, \dots\}$.¹⁶ The aggregate endowment is $w_t = w_t^1 + w_t^2$. Households trade Lucas trees as defined in Section 3. Let $z_{s,t}^i$ denote the share of Lucas tree s held by household i at time t . We assume that at $t = 0$, household 1 owns all existing shares, $z_{s,0}^1 = n_{s,0}$ for all $s < 0$, while household 2 holds no initial shares, $z_{s,0}^2 = 0$. Following the

¹⁶Kocherlakota (1992) assumes income processes of the form $\{\dots, AG^{t-1}, BG^t, AG^{t+1}, \dots\}$ and $\{\dots, BG^{t-1}, AG^t, BG^{t+1}, \dots\}$ with $A > B \geq 0$. We set $B = 0$ to ensure that the equilibrium structure remains comparable to the OLG model presented in the previous sections.

definition in Section 3, the aggregate dividend D_t^A grows at the rate $G = (1 - \delta + \gamma)G_A$. The aggregate consumption goods is $Y_t = w_t + D_t^A$. We assume balanced growth where the income shares $w_t/Y_t = 1 - \alpha$ and $D_t^A/Y_t = \alpha$ remain constant for all t .

Households maximize a lifetime utility function, $\sum_{t=0}^{\infty} \beta^t u(c_t^i)$, subject to the budget constraint $c_t^i + \sum_{s=-\infty}^t p_{s,t} z_{s,t}^i + a_t = \sum_{s=-\infty}^{t-1} x_{s,t} z_{s,t-1}^i + R_t a_{t-1} + w_{i,t}$, the SSC $z_{s,t}^i \geq 0$ and the borrowing constraint (BC) $a_t \geq 0$, where $x_{s,t} = (1 - \delta)p_{s,t} + D_{s,t}$. The period utility function is specified as $u(c) = c^{1-\sigma}/(1 - \sigma)$ for $\sigma \neq 1$ and $u(c) = \log c$ for $\sigma = 1$. There are tree growers who own new trees, sell them to households, and consume the revenue, $p_{t,t} n_{t,t}$. The following market-clearing conditions hold: $Y_t = c_t^1 + c_t^2 + p_{t,t} n_{t,t}$ and $n_{s,t} = z_{s,t}^1 + z_{s,t}^2$.

Household optimization implies that the SSC and BC do not bind for households with a positive endowment, whereas they bind for those with a zero endowment. Therefore, the former households correspond to the young in the OLG model, while the latter correspond to the old. The pricing function and the TVC are given by

$$p_{s,t} = m_{t+1} x_{s,t+1}, \text{ where } m_{t+1} = \max \left\{ \beta \frac{u'(c_{t+1}^1)}{u'(c_t^1)}, \beta \frac{u'(c_{t+1}^2)}{u'(c_t^2)} \right\}, \quad (\text{K1})$$

$$\liminf_{T \rightarrow \infty} \beta^T u'(c_T^i) \sum_{s=-\infty}^T p_{s,T} z_{s,T}^i = 0, \quad (\text{K2})$$

respectively. The TVC requires that the short sales constraint must be binding infinitely often. The optimality conditions (K1) and (K2) are analogous to conditions (c) and (d) in Proposition 2 of Kocherlakota (1992). The only difference in the equilibrium conditions between this infinitely lived agent model and the OLG model in Section 3 is the inclusion of the TVC.

We obtain the following proposition:

Proposition A1 *Consider the infinitely-lived agents economy in this appendix. With Assumptions 1–3, The TVC (K2) is satisfied for all households. Consequently, all theorems and propositions established Sections 3 and 4 remain valid*

(Proof) See Appendix K.1.

Proposition A1 provides a key insight into the mechanism of bubble formation. While Kocherlakota (1992) focuses on pure bubbles (non-dividend-paying assets), we show that dividend-paying assets can also exhibit bubbles if distinguishable new trees are successively born ($\gamma > 0$) and cannot be traded prior to their birth (Assumptions 1–3). Thus, the fundamental source of asset bubbles is not the absence of the TVC in the OLG structure, but rather the limits to

arbitrage arising from Assumptions 1–3.

K.1 Proof of Proposition A1

We prove Proposition A1 in the tree steps.

Step 1: We first show that the TVC (K2) is satisfied for all households. The optimization problem of the households is the same as that in Kocherlakota (1992). Thus, we obtain (K1) and (K2) as the optimality conditions. The household's endowment becomes zero every other period, causing the short-sales constraint to bind at those dates. Hence, in line with Kocherlakota's (1992), the TVC is automatically satisfied.

Step 2: We next show that the equilibrium conditions, other than the TVC, coincide with those in the OLG economy described in the previous sections. As mentioned in Appendix K, households with positive endowment play the same role as the young in the OLG model. Their budget constraints are $c_{i,t} + \hat{s}_{i,t} = w_t$, where $\hat{s}_{i,t} = \sum_{s=-\infty}^t p_{s,t} z_{s,t}^i$, in period t and $c_{i,t+1} = \sum_{s=-\infty}^t x_{s,t+1} z_{s,t}^i$ in period $t+1$, which coincide with those of an OLG household that is young in period t , (3a) and (3b). Similarly, the budget constraints of constrained households coincide with those of an OLG household that is old in period t . Unconstrained households IMRS serves as the SDF and (K1) holds. This is equivalent to (4). Since the rates of returns are equalized among different trees s , (5) holds. Since the utility function is homothetic and time additive, we obtain the equations (6) and (7). Only unconstrained households hold the tree shares, the market-clearing condition for tree s reduces to $n_{s,t} = z_{s,t}^i$, which corresponds to (9a). The market-clearing condition for the goods coincides with (9b). The condition that the aggregate saving equals the aggregate value of the trees (9c) is $\hat{s}_{i,t} = \sum_{s=-\infty}^t p_{s,t} n_{s,t}$. Asset prices are given by (10), (11), and (12). Therefore, even in the model with infinitely lived agents, the equilibrium coincides with that of the OLG model as long as the SSCs bind in alternating periods and the TVC holds.

Step 3: Based on *Step 1* and *Step 2*, the equilibrium in the OLG economy and that in the infinitely-lived agents economy are identical. Therefore, all propositions established in Sections 3 and 4 also hold in the equilibrium with infinitely lived agents.

L Proof of Proposition 7

From $K_t = \sum_{s=-\infty}^{t-1} k_{s,t-1}$, capital accumulation follows $K_{t+1} = I_t + (1 - \delta)K_t$. Therefore, in the steady-state equilibrium, the capital per effective labor satisfies

$$\kappa = \frac{I_t}{A_t L_t} \frac{1}{G - (1 - \delta)}. \quad (\text{L1})$$

Using the fact that $p_{s,t} = v + b_{s,t}$, where the fundamental value is given by $v := D/(R - (1 - \delta))$ and the dividend by $D := \alpha \kappa^{\alpha-1}$, and that the bubble component is $b_{s,t} = b_{new}[R/(1 - \delta)]^{t-s}$ with $b_{new} = b_{t,t} \geq 0$ is a bubble attached to newly created capital, together with $k_{s,t} = k_{s,s}(1 - \delta)^{t-s}$ and $I_s = k_{s,s} = I_t G^{s-t}$, the asset market equilibrium can be rewritten as

$$\tilde{\beta} w_t L_t = \sum_{s=-\infty}^t p_{s,t} k_{s,t} = v K_{t+1} + b_{new} \cdot \frac{I_t}{1 - (R/G)},$$

if $R < G$. Dividing both sides by DK_t , we obtain

$$\tilde{\beta} \left(\frac{1 - \alpha}{\alpha} \right) = \frac{G}{R - (1 - \delta)} + \frac{I_t}{DK_t} \cdot \frac{G}{G - R} \cdot b_{new}. \quad (\text{L2})$$

Thus, in the bubbleless steady state ($b_{new} = 0$), the interest rate is given by

$$R = R_L := \frac{G}{\tilde{\beta}} \frac{\alpha}{1 - \alpha} + 1 - \delta.$$

Using (L1) and $w_t L_t = (1 - \alpha) A_t L_t \kappa^\alpha$, the good market equilibrium condition $Y_t = (1 - \tilde{\beta}) w_t L_t + \tilde{\beta} R_t w_{t-1} L_{t-1} + I_t$ can be rewritten as

$$[G - (1 - \delta)] \kappa^{1-\alpha} - \alpha = \tilde{\beta} (1 - \alpha) \left(1 - \frac{R}{G} \right). \quad (\text{L3})$$

Free-entry condition (zero-profit condition) $1 = p_{new} = v + b_{new}$ is rewritten as

$$1 = \frac{\alpha \kappa^{\alpha-1}}{R - (1 - \delta)} + b_{new}. \quad (\text{L4})$$

Using (L3), (L4) is rewritten as

$$1 = \left(\frac{G - (1 - \delta)}{R - (1 - \delta)} \right) \left(\frac{\alpha}{\tilde{\beta} (1 - \alpha) [1 - (R/G)] + \alpha} \right) + b_{new}. \quad (\text{L5})$$

From (L5), we can show that $b_{new} = 0$ if either of the following conditions holds: $R = R_L := (G/\tilde{\beta})[\alpha/(1-\alpha)] + 1 - \delta$ or $R = G$. Noting that the denominator of the first term on the right-hand side of (L5) is a quadratic function of R , (L5) can be illustrated as in Figure A5 under the condition $R_L < G$.

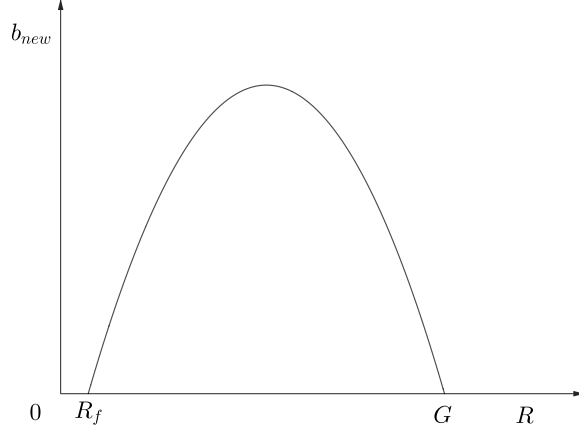


Figure A5 Illustration of (L5) under the condition $R_L < G$.

Then, there exists (a) a bubbleless steady-state equilibrium, where $R = R_L$; and (b) a continuum of bubbly steady-state equilibria, where $R = R(\nu) := (1-\nu)R_L + \nu G$ with $\nu \in (0, 1)$. Note that $R = G$ does not constitute a bubbleless steady-state equilibrium because $R = G$ does not satisfy the asset market equilibrium (L2) if $b_{new} = 0$. At a bubbly steady state, we have

$$\kappa = \left\{ \frac{\alpha + \tilde{\beta}(1-\alpha)(1-R/G)}{G - (1-\delta)} \right\}^{\frac{1}{1-\alpha}},$$

$$b_{new} = 1 - \left(\frac{G - (1-\delta)}{R - (1-\delta)} \right) \left(\frac{\alpha}{\tilde{\beta}(1-\alpha)[1 - (R/G)] + \alpha} \right) = p_{new} - v,$$

from (L3) and (L5).

M Proof of Proposition 8

The goods and financial markets equilibrium conditions are

$$c_{1,t}L + c_{2,t}L + \eta n_t + N_t y = Y_t, \quad (\text{M1a})$$

$$\hat{s}_t L_t \left(= \frac{\beta}{1+\beta} w_t L \right) = \int_0^{N_{t+1}} \hat{p}_{j,t} dj. \quad (\text{M1b})$$

We substitute $c_{1,t} = \frac{w_t}{1+\beta}$, $c_{2,t} = \frac{\beta w_{t-1} R_t}{1+\beta}$, $y = \alpha^{\frac{2}{1-\alpha}} L$, and $w_t = (1-\alpha)\alpha^{\frac{2\alpha}{1-\alpha}} N_t$ into (M1a) and then rewrite the equation as

$$\eta\gamma = \pi \left[1 + \frac{\beta}{\alpha(1+\beta)} \left(1 - \frac{R}{1+\gamma} \right) \right], \quad (\text{M2})$$

where $\gamma = N_{t+1}/N_t$ and $\pi = (1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}} L$.

Proof of (i): We prove the contraposition; if $\gamma = 0$, there are no asset bubbles. From $\gamma = 0$ and (M2), we obtain $R - 1 = \alpha(1+\beta)/\beta$, which implies $v = \pi/(R-1) = \pi\beta/[\alpha(1+\beta)]$ and

$$\int^{N_{t+1}} p_{j,t} dj - N_{t+1}v = \frac{\beta}{1+\beta} w_t L - N_{t+1} \frac{\pi\beta}{\alpha(1+\beta)} = \frac{\beta}{1+\beta} (1-\alpha)\alpha^{\frac{2\alpha}{1-\alpha}} (N_t - N_{t+1}) = 0.$$

Since $p_{j,t} \geq v$, we have $p_{j,t} = v$.

Proof of (ii): We first show that $R = R_L$ and $\gamma = \gamma_L$ hold in a bubbleless steady state. From the definition of R_L , we have $v_t = \pi/(R_L - 1) = \eta$. In a bubbleless steady state, we must have $\hat{p}_{j,t} = \eta$. The stock market equilibrium (M1b) implies $\frac{(1-\alpha)\beta}{1+\beta} \alpha^{\frac{2\alpha}{1-\alpha}} N_t L = N_{t+1}\eta$. The condition $\eta < (1-\alpha)\alpha^{\frac{2\alpha}{1-\alpha}} L \frac{\beta}{1+\beta}$ ensures $N_{t+1}/N_t = 1 + \gamma_L > 1$. The goods market clears because substituting $\gamma = \gamma_L$ and $R = R_L$ into the RHS of (M2) yields

$$\eta\gamma = \pi \left[1 + \frac{\beta}{\alpha(1+\beta)} \left(1 - \frac{1 + \pi/\eta}{\frac{(1-\alpha)\beta}{(1+\beta)\eta} \alpha^{\frac{2\alpha}{1-\alpha}} L} \right) \right] = \frac{\beta\pi}{\alpha(1+\beta)} - \eta = \eta\gamma_L.$$

We next point out that $\gamma \leq \gamma_L$ must hold at a steady state with positive growth. Since $\hat{p}_{j,t} = v + (\eta - v)R^{t-s_j} \geq \eta$, (M1b) can be rewritten as $\frac{\beta}{1+\beta} w_t L \geq N_{t+1}\eta$, which implies that $\gamma \leq \gamma_L$ because $w_t = (1-\alpha)\alpha^{\frac{2\alpha}{1-\alpha}} N_t$. Besides, the fundamental value cannot be greater than η to ensure the finite R&D activities, $n_t < +\infty$. Since $v = \pi/(R-1) \leq \eta$, we must have $R \geq R_L$. If $R > R_L$, we have $\eta > v$, meaning that there is a stock price bubble.

To derive a necessary and sufficient condition for the existence of bubbly steady states, we rearrange (M2) to obtain $R = \Psi(\gamma)$. The equation $R_L = \Psi(\gamma)$ has two solutions, $\gamma = \gamma_L$ and $\gamma = \gamma_{LL}$. We have $\gamma_{LL} < \gamma_L$ if and only if $(1-\alpha)\alpha^{\frac{2\alpha}{1-\alpha}} L \left(\frac{\beta}{1+\beta} - \alpha \right) > \eta$. The RHS of $\Psi(\gamma)$ is a quadratic equation of $1 + \gamma$ with a negative coefficient on $(1 + \gamma)^2$, as shown in Figure A6. If $\gamma_{LL} < \gamma_L$, both $\gamma \leq \gamma_L$ and $R \geq R_L$ are satisfied for $\gamma_B \in (\gamma_{LL}, \gamma_L)$ (see Panel (b) of Figure A6). Thus, the stock price of the new firms has a bubble, $\hat{p}_{j,t} = \eta > v_B$. Because the bubble component $R^{t-s_j}(\eta - v_t)$ increases over time, the stock prices of all firms have bubbles.

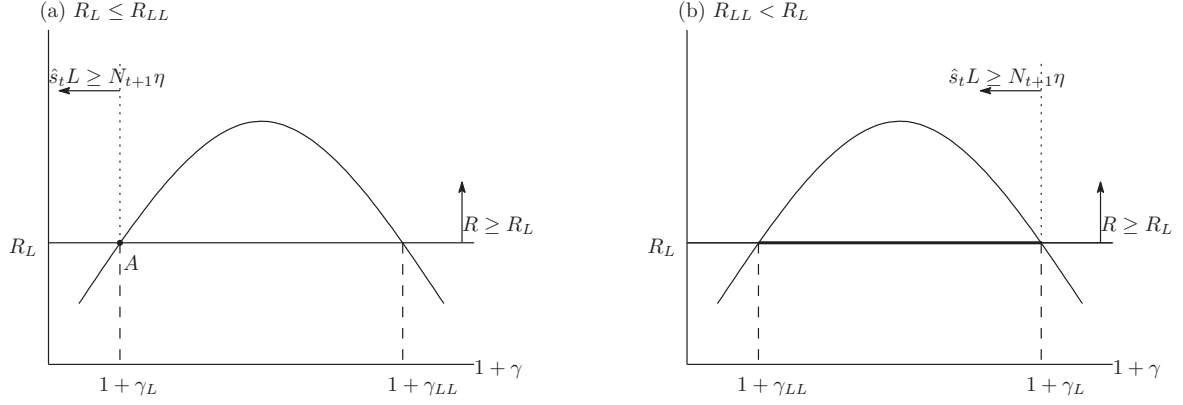


Figure A6 Bubbly Equilibrium in R&D economy

Finally, we check if $\gamma_B \in (\gamma_{LL}, \gamma_L)$ and $R_t = \Psi(\gamma_B)$ satisfy (M1b), which is $s_t L_t = \sum_{s=-\infty}^t n_s p_{s,t}$. Let us rewrite $R = \Psi(\gamma)$ as

$$\frac{R}{1+\gamma} = \frac{\alpha(1+\beta)\eta}{\beta\pi} \left\{ R_L + \frac{\beta\pi}{\alpha(1+\beta)\eta} - (1+\gamma) \right\}.$$

Figure A7 shows that $\frac{R}{1+\gamma} < 1$ holds for $\gamma_B \in (\gamma_{LL}, \gamma_L)$ because $\frac{R}{1+\gamma} = 1$ if $\gamma = \gamma_{LL}$. The stock price of firms invented in period s is given by $\hat{p}_{s,t} = v_B + R_B^{t-s}(\eta - v_B)$, where $R_B = \Psi(\gamma_B)$. Then, we have

$$\begin{aligned} \sum_{s=-\infty}^t n_s p_{s,t} &= N_{t+1} v_B + \gamma_B N_t (\eta - v_B) \sum_{s=-\infty}^t \left(\frac{R_B}{1+\gamma_B} \right)^{t-s} \\ &= N_{t+1} \left\{ v_B + \frac{\gamma_B (\eta - v_B)}{1+\gamma_B - R_B} \right\} \\ &= N_{t+1} \frac{\eta \gamma_B - \pi}{1+\gamma_B - R_B} \\ &= N_{t+1} \frac{\pi}{1+\gamma_B - R_B} \frac{\beta}{\alpha(1+\beta)} \left(1 - \frac{R_B}{1+\gamma} \right) \\ &= N_t \frac{\beta\pi}{\alpha(1+\beta)} \\ &= \frac{\beta}{1+\beta} w_t L. \end{aligned}$$

The first equality uses $n_s = \gamma_B N_s = \gamma_B N_t / (1+\gamma_B)^{t-s}$. The second equality holds because of $R_B < 1+\gamma_B$. The third line uses $v_B = \pi / (R_B - 1)$. The fourth line uses (M2). The fifth line uses $N_{t+1} = N_t(1+\gamma_B)$. Thus, $\gamma_B \in (\gamma_{LL}, \gamma_L)$ and $R_t = \Psi(\gamma_B)$ satisfy (M1b)

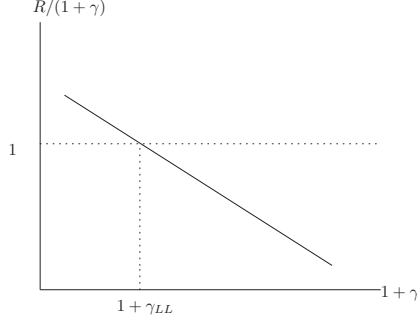


Figure A7 Bubbly Equilibrium in R&D economy

N Incumbent R&D and Bubble

The stock price is composed of the fundamental and bubble terms, $p_i(N_{i,t+1}, t) = v(N_{i,t+1}, t) + b_i(t)$. Only the fundamental term depends on the scale of operation $N_{i,t+1}$. We guess $v(N_{i,t+1}, t) = v_t \cdot N_{i,t+1}$. Then, $p_i(N_{i,t+1}, t)$ satisfies

$$p_i(N_{i,t+1}, t) = \frac{1}{R_{t+1}} [\pi N_{i,t+1} - \eta n_{i,t+1} + v_{t+1}(N_{i,t+1} + n_{i,t+1}) + b_{i,t+1}].$$

The first-order condition for $n_{i,t+1}$ is $v_t = \eta$. The TVC is $\lim_{T \rightarrow \infty} v n_{i,t+T} / \Pi_{s=1}^T R_{t+s} = 0$. Then, at a steady state, growth rate of $n_{i,t+T} (= \gamma_{i,t+T} N_{i,t+T})$ must be smaller than R , which excludes asset bubbles. Because of $p_i(N_{i,t+1}, t) = v_t N_{i,t+1} + b_i(t)$, the above equation reduces to $v_t = (v_{t+1} + \pi) / R_{t+1}$, implying $R_{t+1} = 1 + \pi / \eta = R_L$, where R_L is defined in Section 6.1. The equilibrium conditions for goods and financial markets are given by (M1a) and (M1b), respectively. This incumbent R&D growth model essentially has the same structure as the entrant R&D model in Section 6.1, except for the restriction $R_{t+1} = R_L$. Thus, a unique bubbleless steady state exists if the condition in Proposition 8-(ii) holds.

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