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Tomohiro Hirano[†] Alexis Akira Toda[‡]

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Abstract

Since McCallum (1987), it has been well known that in an overlapping generations (OLG) economy with land, the equilibrium is Pareto efficient because with balanced growth, the interest rate exceeds the growth rate ($R > G$), precluding infinite debt rollover (a Ponzi scheme). We show that, once we remove knife-edge restrictions on the production function and allow unbalanced growth, under some conditions an efficient equilibrium with land bubbles necessarily emerges and infinite debt rollover becomes possible, a markedly different insight from the conventional view derived from the Diamond (1965) landless economy. We also examine the possibility of Pareto inefficient equilibria.

Keywords: infinite debt rollover, land, necessity of land bubble, Pareto (in)efficiency, Ponzi scheme, unbalanced growth.

JEL codes: D53, D61, E60, G12.

1 Introduction

In recent years, there has been a heated debate in academic and policy circles about the sustainability of public debt, in particular, the possibility of infinite debt rollover (Ponzi scheme) in a low interest rate environment (Blanchard, 2019; Mian et al., 2022; Ball and Mankiw, 2023; Kocherlakota, 2023; Bloise and Vailakis, 2024; Abel and Panageas, 2025). The theoretical basis for such arguments is the

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influential paper by Diamond (1965). He considers an overlapping generations (OLG) economy with capital and labor and shows that, under certain conditions, the oversaving problem ($R < G$, where R is the interest rate and G is the economic growth rate) can arise, leading to Pareto inefficient equilibria. In such circumstances, the perpetual rollover of public debt may eliminate the inefficient equilibrium, thereby restoring Pareto efficiency.

However, these recent papers and the policy debate miss a critical point raised a long time ago by McCallum (1987) and Homburg (1991) that in an OLG economy with a productive, non-reproducible, and non-depreciating asset like land, such inefficiencies with $R < G$ cannot arise in the first place. Land resolves the oversaving problem and makes the equilibrium Pareto efficient with $R > G$, implying that the presence of land dramatically changes the condition of R versus G . The intuition is that, along a balanced growth path, the land price and rent grow at the same rate as the output, which necessarily makes the rate of return (interest rate) higher than the growth rate of the economy ($R > G$).¹ In fact, an even earlier paper by Nichols (1970) states:

Because the price of land must increase proportionately at the growth rate of output while the rental yield is also positive, the rate of return on land must exceed the growth rate. [...] Oversaving or inefficient accumulation of excess capital in the sense of Phelps is shown to be impossible. (p. 339)

If this view is correct, the debate on infinite debt rollover would be moot in the real world where land is present, because $R > G$ would preclude such a Ponzi scheme.

The purpose of this paper is to study the efficiency properties of an OLG economy with land and examine whether sustainable infinite debt rollover is possible or not in the first place. We draw three messages from our analyses. (i) One cannot ignore the presence of land in analyzing infinite debt rollover: the mere existence of land fundamentally changes the economic conditions under which sustainable infinite debt rollover is possible and produces a markedly different insight from the conventional view derived from the Diamond (1965) landless economy. That is, if the land economy falls into a sufficiently low interest rate environment, land price bubbles necessarily emerge in equilibrium, which is Pareto efficient. Contrary to common understanding, despite Pareto efficiency, infinite debt rollover becomes

¹This is not limited to OLG models with land: the same intuition applies to models with infinitely-lived agents, i.e., $R > G$ holds on a balanced growth path. Otherwise, land prices would be infinite, which cannot be an equilibrium.

possible. (ii) Contrary to McCallum (1987)’s claim, even in an OLG model with land, a Pareto inefficient equilibrium may exist. In addition, a Pareto efficient equilibrium can exist in which land price bubbles arise. (iii) In a land economy, the dynamic path with infinite debt rollover requires unbalanced growth, with the productivities of different production factors growing at different rates. This is because $R > G$ necessarily holds with balanced growth in a land economy, which precludes infinite debt rollover.

We derive these findings with the simplest possible two-period overlapping generations model. The only departure from the standard models such as Samuelson (1958) and Diamond (1965) is the introduction of land—a productive, non-reproducible, and non-depreciating asset. In each period, agents arrive and live for two periods (young and old age). Agents are endowed with consumption goods when young and old, which captures relative income levels over the life cycle. There is a fixed supply of land, which produces some output but also serves as a means of saving. There is no aggregate uncertainty. This is all that is required to obtain our main results. Even in this simplest setting, the mere existence of land yields a markedly different insight. That is, under certain conditions concerning the relative income levels, the interest rate becomes sufficiently low. Once the economy falls into such a situation, the only possible equilibrium is one featuring land price bubbles. In other words, in an environment with a sufficiently low interest rate, land price bubbles are necessary for the existence of equilibrium, and Pareto efficiency is simultaneously achieved. Despite Pareto efficiency, this situation makes infinite debt rollover possible. We can obtain this insight only by considering the existence of land. We establish this result as Theorem 1, with an example in Proposition 3.1 where labor and land are inputs for production and the production function takes the constant elasticity of substitution (CES) form.

Moreover, we study the possibility of a Pareto inefficient equilibrium in an OLG economy with land and show that McCallum (1987)’s result also requires revision. To illustrate this point, we employ the CES production function and assume that the growth rate of labor productivity (which includes the population growth rate) is higher than that of land productivity. We also assume that the elasticity of substitution between land and non-land factors is greater than one, which is empirically supported (Footnote 7). In this case, under certain conditions on relative income levels in the young and old ages, interest rates will not fall enough to generate the inevitable emergence of land bubbles but enough so that $R < G$. More precisely, while the land price grows at the same rate as land rents, aggregate endowments grow faster. As a result, in the long run, the land price

becomes negligible relative to aggregate endowments, so the long-run equilibrium asymptotically behaves like a landless economy as if no trade in land takes place. If the economy falls into such a circumstance, the long-run interest rate falls below the economic growth rate ($R < G$). In addition to such an inefficient equilibrium, there can also exist a Pareto efficient equilibrium in which land bubbles occur. Again, despite Pareto efficiency, infinite debt rollover is possible. We derive these insights in such a way that goes beyond McCallum (1987), whose analysis turns out to be a knife-edge case of the elasticity of substitution being just equal to one and/or of equal productivity growth rates across different production factors, e.g., in our model, labor and land have the same productivity growth rate.

Our results not only challenge the conventional wisdom on sustainable infinite debt rollover but also have important implications for its macro-theory construction. Unlike the standard analysis of infinite debt rollover in a landless economy, which typically entails balanced growth, the dynamic path with infinite debt rollover in a land economy entails unbalanced growth. Put simply, as long as there is land, which is always the case in reality, and the standard no-arbitrage condition holds, the rate of return on government bonds (the safe interest rate) equals the rate of return on holding land. We obtain $R > G$ on a balanced growth path, regardless of whether the model features overlapping generations or infinitely-lived agents. If $R > G$, by model construction, infinite debt rollover is impossible, even if in the corresponding landless economy, $R < G$ holds and a Ponzi scheme appears to be possible along a balanced growth path: the mere presence of land breaks the result. Therefore, we cannot ignore the existence of land in the analysis of infinite debt rollover, and we should be careful in deriving policy implications of infinite debt rollover in a model without land. Of course, it should be noted that incorporating various realistic factors may make $R < G$ possible under balanced growth, even in models with land. However, in such cases, as well as the fact that the key characteristics of land raised by McCallum (1987) and Homburg (1991) are violated, the conditions for indefinite rollover of government bonds would no longer be reduced to a simple relationship of G versus R in the first place. Depending on the types of friction/market segmentation, the conditions would change accordingly. This suggests that linking the rollover of government bonds to G versus R in a model without land would not provide a coherent theoretical framework.

2 Endowment economy

We start the discussion with an overlapping generations endowment economy with land.

2.1 Model

We consider a standard two-period overlapping generations (OLG) model without uncertainty. The only departure from Samuelson (1958) and Diamond (1965) is the introduction of land, which is a productive, non-reproducible, and non-depreciating asset. Time is denoted by $t = 0, 1, \dots$.

Demography, preferences, and endowments At time t , a mass $N_t > 0$ of new agents are born, who live for two dates. In addition, at $t = 0$ there is a mass N_{-1} of initial old agents.² The utility function of generation $t \geq 0$ is denoted by $U_t(c_t^y, c_t^o)$, where c_t^y and c_t^o denote the consumption of young and old at time t . We assume $U_t : \mathbb{R}_{++}^2 \rightarrow \mathbb{R}$ is continuously differentiable, quasi-concave, has positive partial derivatives, and satisfies the Inada condition. The initial old care only about their consumption c_0^o . At time t , each young and old are endowed with $e_t^y > 0$ and $e_t^o \geq 0$ units of the consumption good, which capture the relative income levels over the life cycle.

Asset There is a unit supply of land, which pays exogenous dividend (rent) $r_t \geq 0$ at time t in units of the consumption good. Let P_t be the (endogenous) ex-dividend land price at time t . We assume $r_t > 0$ infinitely often so that $P_t > 0$ for all t . At $t = 0$, land is initially owned by the old.

Equilibrium Generation t maximizes the utility $U_t(c_t^y, c_t^o)$ subject to the budget constraints

$$\text{Young:} \quad c_t^y + P_t x_t = e_t^y, \quad (2.1a)$$

$$\text{Old:} \quad c_{t+1}^o = e_{t+1}^o + (P_{t+1} + r_{t+1})x_t, \quad (2.1b)$$

where x_t denotes land holdings. A *rational expectations equilibrium* consists of a sequence $\{(P_t, c_t^y, c_t^o, x_t)\}_{t=0}^\infty$ such that (i) (c_t^y, c_{t+1}^o, x_t) solves the utility maximiza-

²Diamond (1965) and Tirole (1985) assume exogenous population growth, while fixing the per capita labor supply (corresponding to endowment here) constant. Hirano and Toda (2025a,c) assume exogenous endowment or technology growth, while fixing the population constant. Our setting subsumes both cases.

tion problem and (ii) the commodity and land markets clear, so

$$N_t c_t^y + N_{t-1} c_t^o = N_t e_t^y + N_{t-1} e_t^o + r_t, \quad (2.2a)$$

$$N_t x_t = 1. \quad (2.2b)$$

Under the maintained assumptions, an equilibrium always exists.³ Using the budget constraints (2.1) to eliminate c_t^y, c_{t+1}^o from the utility function, the individual optimization reduces to maximizing

$$U_t(e_t^y - P_t x_t, e_{t+1}^o + (P_{t+1} + r_{t+1})x_t)$$

over land holdings x_t . The first-order condition for optimality is

$$-U_{t,1}(y_t, z_{t+1})P_t + U_{t,2}(y_t, z_{t+1})(P_{t+1} + r_{t+1}) = 0, \quad (2.3)$$

where the subscripts 1, 2 denote partial derivatives with respect to the first and second arguments, and we write $(c_t^y, c_{t+1}^o) = (y_t, z_{t+1})$ to simplify the notation. In equilibrium, the land market clearing condition (2.2b) implies $x_t = 1/N_t$, so

$$(y_t, z_{t+1}) := (e_t^y - P_t/N_t, e_{t+1}^o + (P_{t+1} + r_{t+1})/N_t). \quad (2.4)$$

2.2 Inevitable emergence of land bubbles

In this section, we provide economic conditions under which land price bubbles inevitably emerge in equilibrium.

From the first-order condition (2.3), we obtain the gross risk-free rate between time t and $t + 1$ as

$$R_t := \frac{P_{t+1} + r_{t+1}}{P_t} = \frac{U_{t,1}}{U_{t,2}}(y_t, z_{t+1}). \quad (2.5)$$

Let $q_t > 0$ be the date-0 price (Arrow-Debreu price) of the date- t good defined by $q_0 = 1$ and $q_t = 1/\prod_{s=0}^{t-1} R_s$ for $t \geq 1$. We define the *fundamental value* of land at time t by the present value of dividends

$$V_t := \frac{1}{q_t} \sum_{s=t+1}^{\infty} q_s r_s. \quad (2.6)$$

Appendix A shows that either $P_t = V_t$ for all t or $P_t > V_t$ for all t . If $P_t = V_t$

³See, for example, Balasko and Shell (1980, Proposition 3.10) without the asset and Hirano and Toda (2025a, Theorem 1) with the asset. These papers assume a constant population, but it is straightforward to adapt the proof for an arbitrary population.

for all t , there is no land bubble and we say that the equilibrium is *fundamental* or *bubbleless*; if $P_t > V_t$ for all t , there is a land bubble and we say that the equilibrium is *bubbly*.

Among bubbly equilibria, it is of interest to focus on those in which the land bubble does not vanish in the long run. To make this statement precise, let $\{a_t\}$ be a positive sequence such that

$$0 < \liminf_{t \rightarrow \infty} \frac{N_t e_t^y + N_{t-1} e_t^o + r_t}{a_t} \leq \limsup_{t \rightarrow \infty} \frac{N_t e_t^y + N_{t-1} e_t^o + r_t}{a_t} < \infty, \quad (2.7)$$

that is, the ratio of the aggregate resources $N_t e_t^y + N_{t-1} e_t^o + r_t$ to a_t remains bounded above and bounded away from zero. There is no shortage of such sequences, for instance $a_t = N_t e_t^y + N_{t-1} e_t^o + r_t$. With such a sequence, we say that a bubbly equilibrium is *asymptotically bubbly* if the detrended asset price is bounded away from zero, or $\liminf_{t \rightarrow \infty} P_t/a_t > 0$ (Hirano and Toda, 2025a, Definition 2). Clearly, the notion of asymptotically bubbly equilibrium does not depend on the choice of the sequence $\{a_t\}$ as long as it satisfies (2.7). When all equilibria are asymptotically bubbly, we say “necessity of bubbles”, implying that the emergence of bubbles is inevitable (Hirano and Toda, 2025a).

To illustrate the inevitable emergence of land price bubbles in the OLG model with land, we introduce several assumptions.

Assumption 1. *The utility function $U_t(c^y, c^o) = U(c^y, c^o)$ does not depend on time, has positive partial derivatives, and is differentially strictly quasi-concave. Furthermore, $c \gg 0$ and $U(c') = U(c)$ implies $c' \gg 0$, i.e., indifference curves do not touch the boundary of \mathbb{R}_+^2 .*

The utility function being “differentially strictly quasi-concave” means that indifference curves are strictly convex (see Appendix B). We next assume population grows at a constant rate and per capita endowments converge.

Assumption 2. *Population satisfies $N_t = G^t$ for some $G > 0$. Endowments satisfy $\lim_{t \rightarrow \infty} (e_t^y, e_t^o) = (e^y, e^o)$ for some $e^y, e^o \geq 0$.*

The following theorem shows that if the dividend growth rate exceeds the natural interest rate but is below the economic growth rate, then land price bubbles necessarily emerge and the economy is Pareto efficient.

Theorem 1. *If Assumptions 1, 2 hold, $e^y > 0$, and*

$$R := \frac{U_1}{U_2}(e^y, e^o) < G_r := \limsup_{t \rightarrow \infty} r_t^{1/t} < G, \quad (2.8)$$

then all equilibria are asymptotically bubbly ($\liminf_{t \rightarrow \infty} P_t/G^t > 0$) and Pareto efficient.

In Theorem 1, the necessity of land bubbles directly follows from the Bubble Necessity Theorem of Hirano and Toda (2025a). If the land price P_t does not grow at the same rate as the economy, then noting that $N_t = G^t$, individual savings $P_t/N_t = P_t/G^t$ converges to zero. Then individual consumption (2.4) converges to the long-run endowment (e^y, e^o) , and hence the gross risk-free rate (2.5) converges to the natural interest rate $R = (U_1/U_2)(e^y, e^o)$ in (2.8). But by assumption $R < G_r$, so the present value of dividends (the fundamental value of land) is infinite, which is impossible. Intuitively, without a land bubble, interest rates become excessively low and lower than the dividend growth rate; under such conditions, land prices would become infinite, which cannot happen in equilibrium. In other words, under (2.8), the emergence of land price bubbles is necessary for equilibrium existence.⁴

The interesting and novel part of Theorem 1 is Pareto efficiency. It is well known in OLG models with low natural interest rates like Samuelson (1958) and Tirole (1985) that the introduction of money (a pure bubble asset without dividends) can restore efficiency. While Theorem 1 somewhat resembles these earlier contributions, it is substantially different. What the earlier literature shows is that the introduction of a pure bubble asset *can* make the equilibrium efficient: it does not rule out inefficient equilibria (e.g., autarky), so bubbles and efficiency are a possibility. In contrast, Theorem 1 shows that in the presence of land with dividend growth rate exceeding the natural interest rate, all equilibria *must* be asymptotically bubbly and efficient: bubbles and efficiency are a necessity.

2.3 Bubble necessity and infinite debt rollover

So far, we have considered an economy with a single long-lived asset, i.e., land. This is without loss of generality, because in deterministic economies, by the absence of arbitrage, the rate of return on all assets must be equalized, so we can aggregate the market capitalization of all assets into just one and treat the economy as if there were a single asset. Hence, it is straightforward to extend the model to include multiple assets, in which case aggregate bubbles attached to multiple assets emerge as the equilibrium outcome and the claims in Theorem 1 stand without change.

⁴Note that Theorem 1 does not imply the uniqueness of equilibrium; it merely claims that *all equilibria* of the economy satisfy the stated conditions.

We now explicitly introduce public debt and consider the case where the fiscal authority rolls it over indefinitely by issuing new bonds and redeeming existing bonds every period. Theorem 1 implies that such a Ponzi scheme is possible. The following corollary clarifies this point.

Corollary 2. *Let $\{(P_t, c_t^y, c_t^o, x_t)\}_{t=0}^\infty$ be an equilibrium of the economy \mathcal{E} in Theorem 1 and let $R_t = (P_{t+1} + r_{t+1})/P_t$ be the corresponding equilibrium interest rate. Let $\theta \in (0, 1]$ and $B_t := \theta(P_t - V_t) > 0$, where V_t is the fundamental value of land (2.6), and let $\tilde{\mathcal{E}}$ be an economy in which the fiscal authority issues debt B_t at time t . If for all t we let*

$$\begin{aligned}\tilde{P}_t &:= V_t + (1 - \theta)(P_t - V_t), \\ (\tilde{c}_t^y, \tilde{c}_{t+1}^o) &:= (c_t^y, c_{t+1}^o), \\ \tilde{x}_t &:= x_t = 1/N_t,\end{aligned}$$

and $\tilde{c}^o := c_0^o - B_0/N_{-1}$, then $\{(\tilde{P}_t, \tilde{c}_t^y, \tilde{c}_t^o, \tilde{x}_t, B_t)\}_{t=0}^\infty$ is an equilibrium of $\tilde{\mathcal{E}}$ and infinite debt rollover is possible: $B_{t+1} = R_t B_t$ for all t .

Proof. The proof is immediate from Theorem 1, so we only provide a sketch. By (2.5) and the definitions of the date-0 price q_t and the fundamental value (2.6), both P_t and V_t satisfy the same no-arbitrage condition

$$\begin{aligned}P_t &= \frac{1}{R_t}(P_{t+1} + r_{t+1}), \\ V_t &= \frac{1}{R_t}(V_{t+1} + r_{t+1}).\end{aligned}$$

Taking the difference and using the definition of B_t , we obtain $B_t = B_{t+1}/R_t$. Hence infinite debt rollover is possible if this is an equilibrium. Clearly, \tilde{P}_t satisfies $\tilde{P}_t + B_t = P_t$ and $\tilde{P}_t = (\tilde{P}_{t+1} + r_{t+1})/R_t$. Therefore, under the interest rate R_t , generation $t \geq 0$ optimally demands the entire asset \tilde{P}_t and the public debt B_t , resulting in the same consumption (c_t^y, c_{t+1}^o) as in \mathcal{E} . Only the consumption of the initial old needs to be modified to account for the issuance of initial public debt B_0 . \square

The intuition for Corollary 2 is straightforward. In any bubbly equilibrium, the bubble component of the asset must grow at the rate of interest. Thus, we can arbitrarily divide the sequence of bubbles proportionally in two parts, call one part public debt, and we can construct an observationally equivalent equilibrium. If the equilibrium is asymptotically bubbly as in Theorem 1, then the bubble

does not vanish relative to the economy and the fiscal authority is able to sustain deficits indefinitely by rolling over debt.

The insight derived from Theorem 1 and Corollary 2 is markedly different from the conventional view derived from Diamond (1965) and the large subsequent literature. The conventional approach considers a hypothetical world in which land does not exist. In such an economy, equilibrium usually exists, regardless of the level of interest rates; if interest rates become sufficiently low and lower than the growth rate of the economy, a Pareto inefficient equilibrium may arise, in which case infinite debt rollover is possible. Since Diamond (1965), this has been the standard argument: it is often claimed that the existence of a Pareto inefficient equilibrium and the possibility of infinite debt rollover are two sides of the same coin. However, the mere existence of land fundamentally changes this view. That is to say, when there is land, which is always the case in reality, if interest rates become sufficiently low and lower than the dividend growth rate, there is no equilibrium without a land price bubble in the first place. The land bubble inevitably emerges, which achieves Pareto efficiency (Theorem 1). Pareto inefficient equilibria do not exist, so dynamic inefficiency is irrelevant to whether infinite debt rollover is possible or not. Rather, despite Pareto efficiency, it is precisely the inevitable land bubble that makes infinite debt rollover possible.⁵

3 Production economy

We next present a production economy where Theorem 1 and Corollary 2 hold. The modern economy is transitioning towards a knowledge-based economy where human capital plays an important role for economic growth. Taking this into account, we consider a simple production economy where human capital (labor) is the engine for economic growth.⁶

⁵The same applies to monetary (pure bubble) models. Conventionally, monetary (pure bubble) equilibria exist if and only if the economy has a Pareto inefficient equilibrium in the moneyless economy, i.e., $R < G$. A representative paper is Tirole (1985), and the subsequent large literature builds on the same argument (see Hirano and Toda (2024) for developments in monetary (pure bubble) models). In contrast, in a land economy in which a land price bubble inevitably emerges, the existence of a Pareto inefficient equilibrium is irrelevant for the emergence of land price bubbles.

⁶We abstract from capital, as it is not essential for our main results and complicates the analysis by increasing the dimension of the dynamical system. See Hirano, Jinnai, and Toda (2022), Hirano and Toda (2024, §6), and Hirano, Kishi, and Toda (2025) for macroeconomic models with capital including intangible capital that show the inevitable emergence of asset price bubbles.

3.1 Model

As in §2, we consider a two-period OLG model. Per capita endowments of the good are denoted by $e_t^y, e_t^o \geq 0$. In addition, each agent is endowed with one unit of labor only when young and earns the wage w_t . The aggregate supply of land is $X = 1$, which is initially owned by the old. We introduce e_t^y and e_t^o to capture the relative income levels over the life cycle. For instance, if $e_t^o = 0$, the young have a strong incentive to save to smooth consumption, which would lead to low interest rates. If e_t^o is large, the young have less incentive to save, so interest rates will be higher. In the examples below, we characterize the equilibrium property with e_t^y and e_t^o together with other underlying parameters of the economy.

To simplify the analysis, we introduce parametric assumptions on preferences and technology.

Assumption 3. *The utility function $U(c^y, c^o) = u(c^y) + \beta u(c^o)$ is additively separable, where $\beta > 0$ governs time preference and the period utility function u exhibits constant relative risk aversion γ :*

$$u(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma} & \text{if } 0 < \gamma \neq 1, \\ \log c & \text{if } \gamma = 1. \end{cases} \quad (3.1)$$

As we can see from §2, the CRRA assumption is not essential, but it is useful to obtain closed-form solutions.

A representative firm produces the consumption good by using human capital H and land X as inputs for production.

Assumption 4. *The production function takes the constant elasticity of substitution (CES) form*

$$F(H, X) = \begin{cases} A (\alpha H^{1-1/\sigma} + (1-\alpha)X^{1-1/\sigma})^{\frac{1}{1-1/\sigma}} & \text{if } 0 < \sigma \neq 1, \\ AH^\alpha X^{1-\alpha} & \text{if } \sigma = 1, \end{cases} \quad (3.2)$$

where $\sigma > 0$ is the elasticity of substitution between labor and land and $A > 0$, $\alpha \in (0, 1)$ are parameters.

The CES assumption is not essential: see Appendix C.2 for the general case. Taking partial derivatives, the wage and rental rate are given by

$$w = F_H(H, X) = A (\alpha H^{1-1/\sigma} + (1-\alpha)X^{1-1/\sigma})^{\frac{1}{\sigma-1}} \alpha H^{-1/\sigma}, \quad (3.3a)$$

$$r = F_X(H, X) = A (\alpha H^{1-1/\sigma} + (1-\alpha)X^{1-1/\sigma})^{\frac{1}{\sigma-1}} (1-\alpha)X^{-1/\sigma}. \quad (3.3b)$$

Following Hirano and Toda (2025c), assume $\sigma > 1$.⁷ As will become clear in Proposition 3.1 and Corollary 3, in an OLG economy with land, $\sigma > 1$ is essential for infinite debt rollover. We normalize land supply to $X = 1$ and set $H = N_t = G^t$ with $G > 1$: N_t now represents labor productivity including population growth. Note that G^t corresponds to the relative productivity between labor and land. Hence, $G > 1$ implies that labor productivity grows faster than land productivity, which is natural.

It follows from (3.3), $G > 1$, and $\sigma > 1$ that the long-run behavior of wage and rent are given by

$$w_t = A \left(\alpha G^{(1-1/\sigma)t} + 1 - \alpha \right)^{\frac{1}{\sigma-1}} \alpha G^{-t/\sigma} \sim A \alpha^{\frac{\sigma}{\sigma-1}} =: w, \quad (3.4a)$$

$$r_t = A \left(\alpha G^{(1-1/\sigma)t} + 1 - \alpha \right)^{\frac{1}{\sigma-1}} (1 - \alpha) \sim A \alpha^{\frac{1}{\sigma-1}} (1 - \alpha) G^{t/\sigma} =: r G^{t/\sigma}. \quad (3.4b)$$

Using (3.4b), the asymptotic rent growth rate is

$$G_r := \limsup_{t \rightarrow \infty} r_t^{1/t} = G^{1/\sigma}. \quad (3.5)$$

3.2 Bubble necessity and infinite debt rollover

Applying Theorem 1 and Corollary 2, we obtain the following proposition, with the same intuition as Theorem 1.

Proposition 3.1. *Suppose Assumptions 2–4 hold with $G, \sigma > 1$. Let $w := A \alpha^{\frac{\sigma}{\sigma-1}}$ and $r := A \alpha^{\frac{1}{\sigma-1}} (1 - \alpha)$. If*

$$R := \frac{1}{\beta} \left(\frac{e^o}{e^y + w} \right)^\gamma < G^{1/\sigma}, \quad (3.6)$$

then all equilibria are asymptotically bubbly and Pareto efficient. Furthermore, infinite debt rollover is possible.

Proof. Immediate from Theorem 1 and (3.5). □

As we can see from (3.6), the parameter space in which land price bubbles inevitably emerge gets larger as the relative income of the old $e^o/(e^y + w)$ gets smaller. The intuition is that when the relative income of the old is small, the

⁷Hirano and Toda (2025c) cite Epple et al. (2010) and Ahlfeldt and McMillen (2018) for empirical evidence that the elasticity of substitution between land and non-land factors for producing real estate is greater than 1. The condition $\sigma > 1$ is also natural because the GDP share of land-intensive industries have declined along with economic development (Hirano and Toda, 2025c, Figs. 1, 2).

young have a strong incentive to save to smooth consumption over the life cycle, which suppresses the natural interest rate.

To illustrate Proposition 3.1, we present an example based on a closed-form solution. To construct such an example, we start from the price sequence $\{P_t\}$ and reverse-engineer endowments $\{(e_t^y, e_t^o)\}$. Let $p > 0$ and set $P_t = pG^t$. Let $e_t^o = e^o$ be constant. Then the gross risk-free rate (2.5) reduces to

$$\frac{1}{\beta}(z_{t+1}/y_t)^\gamma = G + \frac{r_{t+1}}{p}G^{-t}. \quad (3.7)$$

Solving for y_t and using the budget constraint yields

$$e_t^y = z_{t+1} \left[\beta \left(G + \frac{r_{t+1}}{p}G^{-t} \right) \right]^{-1/\gamma} - w_t + p. \quad (3.8)$$

Letting $t \rightarrow \infty$, we obtain

$$e_t^y \rightarrow e^y := (e^o + Gp)(\beta G)^{-1/\gamma} - w + p, \quad (3.9)$$

where we note that $z_t \rightarrow e^o + Gp$ by (2.4), $G > 1$, $\sigma > 1$, $P_t = pG^t$, and $r_t \sim rG^{t/\sigma}$ by (3.4b). Therefore, we may construct an equilibrium for large enough t if p is sufficiently large. Furthermore, this equilibrium is asymptotically bubbly and Pareto efficient, as the following proposition shows.

Proposition 3.2. *Let everything be as in Proposition 3.1. For any $p > 0$, let $P_t = pG^t$ and choose $e^o \geq 0$ so that the right-hand side of (3.9) is positive, or*

$$e^o > (\beta G)^{1/\gamma}(w - p) - Gp. \quad (3.10)$$

Then there exists $t_0 \geq 0$ such that e_t^y in (3.8) is positive for $t \geq t_0$. The sequence $\{(w_t, r_t, P_t, c_t^y, c_t^o, x_t)\}_{t=t_0}^\infty$ defined by $(c_t^y, c_{t+1}^o) = (y_t, z_{t+1})$ in (2.4) and $x_t = 1/G^t$ is an equilibrium for the economy starting at t_0 with endowment $\{(e_t^y, e^o)\}_{t=t_0}^\infty$, which is asymptotically bubbly and Pareto efficient.

If in addition

$$e^o < \frac{(\beta G)^{1/\gamma} + G}{(G^{1-1/\sigma})^{1/\gamma} - 1} p, \quad (3.11)$$

then all equilibria are asymptotically bubbly and Pareto efficient.

Again, despite the achieving Pareto efficiency, the land bubble makes infinite debt rollover possible, as Corollary 2 demonstrates. Note that by choosing p

sufficiently large, namely

$$p \geq \frac{(\beta G)^{1/\gamma}}{(\beta G)^{1/\gamma} + G} w,$$

the right-hand side of (3.10) becomes negative, so any $e^o \geq 0$ will satisfy (3.10).

The distinctive feature of a model with the inevitable emergence of land bubbles and infinite debt rollover is that the dynamics necessarily exhibits unbalanced growth, i.e., given $G > 1$ and $\sigma > 1$, the productivity growth rates of labor and land are different and hence wages and land rents grow at different rates. Suppose, instead, that $G = 1$ or $\sigma = 1$. Then wages and land rents will grow at the same rate, achieving balanced growth, where the wage-output and rent-output ratios are positive and constant. Since $R > G$, we have $P_t = V_t$, the equilibrium is Pareto efficient, and infinite debt rollover is impossible. The intuition is that, if $R > G$ but infinite debt rollover is possible, then debt grows faster than the economy, which is not sustainable. We make this result the following proposition.

Proposition 3.3. *Suppose Assumptions 2–4 hold. If (i) $G = 1$ or $\sigma = 1$ and (ii) $(e_t^y, e_t^o)/G^{(\alpha-1)t}$ is constant, then there exists a unique balanced growth path equilibrium. In this equilibrium, the interest rate exceeds the economic growth rate, land prices reflect fundamentals, Pareto efficiency is achieved, and infinite debt rollover is impossible.*

Proposition 3.3 implies that in a land economy, there is a fundamental difficulty in generating a Ponzi scheme with balanced growth. The standard analysis of infinite debt rollover typically considers an economy without land and focuses on a balanced growth path or a single dynamic path converging to it. However, as McCallum (1987)’s analysis precisely demonstrates, once we consider land, we necessarily obtain $R > G$ on a balanced growth path, which makes infinite debt rollover impossible. In other words, the mere introduction of land reverses the relationship from $R < G$ to $R > G$, even though in the landless economy $R < G$ could arise on a balanced growth path and a Ponzi scheme appears to be possible. Therefore, in analyzing infinite debt rollover, one cannot ignore the presence of land, and we should be careful in deriving policy implications of infinite debt rollover, including quantitative analyses, in a model without land.

3.3 Pareto inefficient equilibrium

Proposition 3.1 states that, in a low interest rate environment (the endowment of the old is sufficiently small so that (3.6) holds), land bubbles necessarily arise and

all equilibria are Pareto efficient. We next provide an example in which a Pareto inefficient equilibrium may occur when this condition is not satisfied.⁸

To construct a Pareto inefficient equilibrium, we start from the price sequence $\{P_t\}$ and reverse-engineer endowments $\{(e_t^y, e_t^o)\}$. Let $p > 0$ and set $P_t = pG^{t/\sigma}$, where $\sigma > 1$ is the elasticity of substitution in the CES production function (3.2). Let $e_t^o = e^o$ be constant. Then the gross risk-free rate (2.5) reduces to

$$\frac{1}{\beta}(z_{t+1}/y_t)^\gamma = G^{1/\sigma} \left(1 + \frac{r_{t+1}}{p} G^{-\frac{t+1}{\sigma}}\right). \quad (3.12)$$

Solving for y_t and using the budget constraints yields

$$e_t^y = z_{t+1} \left[\beta G^{1/\sigma} \left(1 + \frac{r_{t+1}}{p} G^{-\frac{t+1}{\sigma}}\right) \right]^{-1/\gamma} - w_t + pG^{(1/\sigma-1)t}. \quad (3.13)$$

Letting $t \rightarrow \infty$, we obtain

$$e_t^y \rightarrow e^y := e^o [\beta G^{1/\sigma} (1 + r/p)]^{-1/\gamma} - w, \quad (3.14)$$

where we note that $z_t \rightarrow e^o$ by (2.4), $G > 1$, $\sigma > 1$, $P_t = pG^{t/\sigma}$, and $r_t \sim rG^{t/\sigma}$ by (3.4b). Therefore, we may construct an equilibrium for large enough t if e^o is sufficiently large. The following proposition makes this statement precise and constructs Pareto efficient or inefficient equilibria.

Proposition 3.4. *Let everything be as in Proposition 3.1. For any $p > 0$, let $P_t = pG^{t/\sigma}$. Choose $e^o > 0$ so that the right-hand side of (3.14) is positive, or*

$$e^o > w [\beta G^{1/\sigma} (1 + r/p)]^{1/\gamma}. \quad (3.15)$$

Then there exists $t_0 \geq 0$ such that e_t^y in (3.13) is positive for $t \geq t_0$. The sequence $\{(w_t, r_t, P_t, c_t^y, c_t^o, x_t)\}_{t=t_0}^\infty$ defined by $(c_t^y, c_{t+1}^o) = (y_t, z_{t+1})$ in (2.4) and $x_t = 1/G^t$ is a fundamental equilibrium for the economy starting at t_0 with endowment $\{(e_t^y, e^o)\}_{t=t_0}^\infty$. Furthermore, the equilibrium is efficient if and only if

$$R := G^{1/\sigma} \left(1 + \frac{r}{p}\right) \geq G \iff p \leq \frac{r}{G^{1-1/\sigma} - 1}. \quad (3.16)$$

The intuition for the condition e^o satisfying (3.15) is that with e^o large enough, the young do not have a strong incentive to save because they expect a significant

⁸See also Abel et al. (1989) and Geerolf (2018), who empirically examine whether Pareto inefficiency arises or not.

income when old. Then interest rates will not become low enough to generate the inevitable emergence of land bubbles, but they may fall enough so that $R < G$. More precisely, under (3.15), with a labor productivity growth rate of $G > 1$ and a CES production function with elasticity of substitution $\sigma > 1$, the wage (3.4a) converges, and land rent (3.4b) and price grow at rate $G^{1/\sigma}$. Since $\sigma > 1$ implies $G^{1/\sigma} < G$, the land price becomes negligible relative to aggregate endowments, so the equilibrium asymptotically behaves like the landless economy as if no trade in land takes place. Once the economy falls into such a circumstance, the long-run interest rate falls below the economic growth rate, i.e., (3.16) fails to hold. Therefore, even in an OLG model with land, a Pareto inefficient equilibrium with $R < G$ can arise,⁹ as in Samuelson (1958), Diamond (1965), and Tirole (1985).

Two remarks are in order. First, such a Pareto inefficient equilibrium may arise in a nonstationary process characterized by unbalanced growth, but not in a stationary state characterized by balanced growth like in ordinary OLG models without land (Proposition 3.3). Second, as we discuss later in §4.1, the conclusion of McCallum (1987) itself is correct in that the presence of land can achieve Pareto efficiency, but his analysis turns out to be a knife-edge case.

4 Discussion

In this section, we first argue that McCallum (1987)’s conclusion itself is correct, but the analysis turns out to be a knife-edge case. Then, we analyze from a more general perspective and discuss the literature.

4.1 Comparison with McCallum (1987)

Concerning the possibility of a Pareto inefficient equilibrium in an OLG model with land, McCallum (1987, p. 333) explicitly claims “in an economy with land—a nonreproducible, nondepreciating, and productive asset—this possibility [Pareto inefficiency] cannot obtain, for the real exchange value of land can and will be as large as is needed to accommodate desired private saving at an efficient rate of interest”. McCallum (1987, p. 333) focuses on an equilibrium with steady state growth by stating “it is assumed that the economy (i.e., [production function] f

⁹Take large enough T , and for $t \geq T$, tax the young by $\epsilon > 0$, which generates a tax revenue of $N_t \epsilon = G^t \epsilon$ at time t . If we transfer this tax revenue to the old, each old receives $N_t \epsilon / N_{t-1} = G \epsilon$. This transfer effectively lets the young save at the rate of return G , which exceeds the market interest rate R_t (because R_t converges to $R < G$ as $t \rightarrow \infty$). Hence, the equilibrium is Pareto inefficient. Also, when $R < G$, issuing and rolling over public debt infinitely may restore Pareto efficiency if we set the initial bond price appropriately.

and [utility function] u) is capable of attaining a unique steady state and that its dynamics are such that this steady state will be approached as time passes". Moreover, McCallum (1987, Footnote 20) states that the conjectured steady state growth condition holds if the production function is Cobb-Douglas.

As in McCallum (1987)'s analysis, consider the Cobb-Douglas production function with $\sigma = 1$ in (3.2). Then, as we have shown in Proposition 3.3, we have $R > G$, $P_t = V_t$, and the equilibrium is Pareto efficient. Therefore, infinite debt rollover is impossible. This result also holds if we assume that labor and land have the same productivity growth rates ($G = 1$ in our setting). However, in either case, it requires a knife-edge restriction. If we deviate from those knife-edge cases and consider the case where labor productivity grows faster than land productivity and $\sigma > 1$, we obtain Proposition 3.4, i.e., a Pareto inefficient equilibrium can arise.¹⁰

In the rest of this section, we revisit the analysis of McCallum (1987) from a more general perspective. On p. 335, McCallum (1987) assumes that the labor share (aggregate labor income divided by output) converges to a positive value as the economy grows, which essentially restricts the production function to be Cobb-Douglas. In fact, we have the following proposition.

Proposition 4.1. *Let $F(H, X)$ be a neoclassical production function and $f(x) = F(x, 1)$. Then the land share $F_X(H, 1)/F(H, 1)$ converges to $1 - \alpha \in (0, 1)$ as $H \rightarrow \infty$ if and only if there exist a constant $A > 0$ and a function ε with $\lim_{x \rightarrow \infty} \varepsilon(x) = 0$ such that*

$$f(x) = Ax^\alpha \exp\left(\int_1^x \frac{\varepsilon(t)}{t} dt\right). \quad (4.1)$$

Proof. Since $F_X(x, 1) = f(x) - xf'(x)$, the land share converges to $1 - \alpha \in (0, 1)$ if and only if

$$\frac{tf'(t)}{f(t)} = \alpha + \varepsilon(t)$$

with $\varepsilon(t) \rightarrow 0$ as $t \rightarrow \infty$. Dividing both sides by t , integrating from $t = 1$ to $t = x$, and taking the exponential, we obtain (4.1) with $A = f(1)$. \square

Theorem 1.3.1 of Bingham et al. (1987) implies that the exponential part in (4.1) is slowly varying, so Proposition 4.1 implies that the production function must be essentially restricted to Cobb-Douglas with elasticity of substitution being exactly equal to one, which is a knife-edge case.

¹⁰O'Connell and Zeldes (1988, Footnote 19) and Abel et al. (1989, Footnote 6) also cast doubt on the robustness of McCallum (1987)'s claim. However, they did not prove the existence of a Pareto inefficient equilibrium.

The following proposition shows that we can recover McCallum (1987)’s result without assuming the convergence of dynamic paths.¹¹

Proposition 4.2. *Let $F(H, X)$ be a neoclassical production function, $n = \inf_t N_t$, and $N = \sup_t N_t$. If the endowment-output ratio $(N_t e_t^y + N_{t-1} e_t^o)/F(N_t, 1)$ is bounded above and the land share satisfies*

$$\inf_{H \in [n, N]} \frac{F_X(H, 1)}{F(H, 1)} > 0, \quad (4.2)$$

then all equilibria are Pareto efficient and have no land bubble.

Applying l’Hôpital’s rule, any $f(x) = F(x, 1)$ with $\lim_{x \rightarrow \infty} f'(x) = w > 0$ (such as the CES production function with elasticity of substitution $\sigma > 1$) satisfies $xf'(x)/f(x) \rightarrow 1$ as $x \rightarrow \infty$, and since

$$\frac{F_H(x, 1)}{F(x, 1)} = \frac{f(x) - xf'(x)}{f(x)} = 1 - \frac{xf'(x)}{f(x)},$$

it violates (4.2). If we restrict F to be the CES production function (3.2), by log-differentiating $f(x) = F(x, 1)$, we obtain

$$\frac{xf'(x)}{f(x)} = \frac{\alpha x^{1-1/\sigma}}{\alpha x^{1-1/\sigma} + 1 - \alpha}.$$

Hence if $0 < n < N = \infty$, (4.2) holds if and only if $\sigma \leq 1$, which leads to the following corollary.

Corollary 3. *In an economy with land and economic growth, $\sigma > 1$ is necessary for infinite debt rollover.*

4.2 Related literature

The “land efficiency” result proposed by McCallum (1987) and Homburg (1991) seems to be well known. However, the literature on infinite debt rollover, including the recent papers we cited at the beginning of the introduction, has not given sufficient attention to the result. In fact, none of those recent papers cite McCallum (1987) or Homburg (1991). To our knowledge, our paper would be the first

¹¹McCallum (1987) assumed $e_t^y = e_t^o = 0$ and steady state growth to prove Pareto efficiency. Homburg (1991, Theorem, p. 456) proves efficiency assuming $e_t^y = e_t^o = 0$ and condition (4.2) (which is Equation (8) of Homburg (1991)) by applying Proposition 5.3 of Balasko and Shell (1980). Proposition 1 of Rhee (1991) is the same as Homburg (1991), which is acknowledged in Footnote 3. Proposition 4.2 is more general than these results.

to study the possibility of infinite debt rollover, i.e., a Ponzi scheme, in an OLG economy with land.

Regarding the possibility of a Pareto inefficient equilibrium in an OLG model with land, our paper is related to Rhee (1991), though there are three substantial differences. First, Rhee (1991)'s main economic question is dynamic inefficiency in an economy with land, which is indeed the title of his paper, not the possibility of infinite debt rollover in an economy with land. Hence, he does not examine whether infinite debt rollover is possible by incorporating public debt. Second, Rhee (1991) never studies the case where land price bubbles inevitably emerge, nor the possibility of infinite debt rollover under such circumstances. In other words, our analyses in §2 and Proposition 3.1 in §3, which are the central results in the present paper, are absent in Rhee (1991). Third, Rhee (1991)'s analysis is limited to providing a counterexample to McCallum (1987) under high-level assumptions: he directly makes an assumption (Assumption A) on the growth rate on land rent (and endogenous object); he simply claims that the proofs of his Propositions 2 and 3 are equivalent to Tirole (1985), which itself contains several issues as recently pointed out by Pham and Toda (2026). In this sense, it would be fair to say that Rhee (1991) made a good conjecture on the possibility of a Pareto inefficient equilibrium. Our analysis in §3.3 provides a firmer footing for his conjecture and makes the economic intuition clear for why a Pareto inefficient equilibrium may arise.¹²

5 Conclusion

This paper has studied whether infinite debt rollover is possible in an OLG economy with land. We can draw three messages from our analyses.

Our central message is that one cannot ignore land in the analysis of infinite debt rollover because its mere existence fundamentally changes the economic condition under which infinite debt rollover is possible or not. Even if the circumstance of $R < G$ arises and a Ponzi scheme appears to be feasible on a balanced growth path in a landless economy, the relationship reverses from $R < G$ to $R > G$ once we introduce land. More generally, when the condition $R < G$ matters for

¹²Note that an equilibrium with land could be inefficient with property taxes (Kim and Lee, 1997; Hellwig, 2020) or transaction costs (Hellwig, 2022). However, these frictions effectively make land a depreciating asset like physical capital (with depreciation rate corresponding to the property tax rate) and change one of the key characteristics of land McCallum (1987) raises, i.e., non-depreciation. Our Proposition 3.4 provides an example of a Pareto inefficient equilibrium, while keeping the key characteristics of land.

the economic analysis, we need to be careful whether the model includes land or not. Moreover, the existence of land fundamentally alters the conventional view of infinite debt rollover. In a land economy, if the economy falls into a sufficiently low interest rate environment, land price bubbles necessarily emerge as the equilibrium outcome, and Pareto efficiency is simultaneously achieved. Contrary to the conventional wisdom, despite Pareto efficiency, infinite debt rollover is possible.

Our second message is that even in an OLG model with land, a Pareto inefficient equilibrium may arise. Furthermore, a Pareto efficient equilibrium with land price bubbles can exist. Again, despite Pareto efficiency, infinite debt rollover is possible. We have derived these results in such a way that goes beyond McCallum (1987)'s analysis; his analysis is correct in that the presence of land can generate $R > G$, Pareto efficiency, and $P_t = V_t$, but his analysis turns out to require a knife-edge restriction.

Our third message is that in the economy with land, infinite debt rollover entails unbalanced growth dynamics, where the productivities of different production factors grow at different rates. The approach with unbalanced growth provides a new perspective on macro-theory construction in the analysis of infinite debt rollover.

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A Fundamental results on asset price bubbles

In this appendix, we review fundamental results on asset price bubbles (Santos and Woodford, [1997](#); Hirano and Toda, [2024](#), [2025a,b](#)).

Consider a deterministic infinite-horizon economy with time indexed by $t = 0, 1, \dots$. Let $q_t > 0$ be the date-0 price (Arrow-Debreu price) of the date- t good with normalization $q_0 = 1$. Suppose an asset pays dividend D_t and trades at price $P_t > 0$ at time t . The absence of arbitrage implies

$$q_t P_t = q_{t+1} (P_{t+1} + D_{t+1}).$$

Iterating over t , for any $T > t$ we obtain

$$q_t P_t = \sum_{s=t+1}^T q_s D_s + q_T P_T. \quad (\text{A.1})$$

Since $q_t > 0$, $P_t \geq 0$, and $D_t \geq 0$ for all t , the partial sum $\left\{ \sum_{s=t+1}^T q_s D_s \right\}$ is increasing in T and bounded above by $q_t P_t$, so it is convergent. Therefore, letting $T \rightarrow \infty$ in [\(A.1\)](#) and dividing both sides by $q_t > 0$, we obtain

$$P_t = \frac{1}{q_t} \sum_{s=t+1}^{\infty} q_s D_s + \frac{1}{q_t} \lim_{T \rightarrow \infty} q_T P_T. \quad (\text{A.2})$$

Comparing (A.2) and the fundamental value V_t in (2.6), we have $P_t \geq V_t$ for all t . We say that the asset price contains a *bubble* at time t if $P_t > V_t$, or in other words, the asset price P_t exceeds its fundamental value V_t defined by the present value of dividends. Note that $P_t = V_t$ for all t if and only if

$$\lim_{T \rightarrow \infty} q_T P_T = 0. \quad (\text{A.3})$$

We refer to (A.3) as the *no-bubble* condition.

The following Bubble Characterization Lemma plays a fundamental role in determining the existence of bubbles.

Lemma A.1 (Bubble Characterization, Montrucchio, 2004, Proposition 7). *In any equilibrium, the asset price exhibits a bubble if and only if*

$$\sum_{t=0}^{\infty} \frac{D_t}{P_t} < \infty. \quad (\text{A.4})$$

See Hirano and Toda (2025a, Lemma 1) for a simple proof of Lemma A.1. Lemma A.1 states that a bubble emerges if and only if the price-dividend ratio P_t/D_t grows sufficiently fast.

B Fundamental results on Pareto efficiency

In this appendix, we review fundamental results that characterize Pareto efficiency in infinite-horizon economies.

Consider an infinite-horizon economy with time indexed by $t = 0, 1, \dots$ and agents indexed by $i \in I$, where I is either a finite or countably infinite set. To ease the burden of notation, suppose that there is no uncertainty and let $q_t > 0$ be the Arrow-Debreu price (with normalization $q_0 = 1$), although this assumption is inessential. Let $q = (q_t)_{t=0}^{\infty}$ be the price vector, P_t the asset price, and $D = (D_t)_{t=0}^{\infty}$ the dividend stream. Let $e_i = (e_{it})_{t=0}^{\infty}$ be the endowment vector of agent i and x_{i0} be the endowment of the asset, where we normalize $\sum_{i \in I} x_{i0} = 1$. Agent i has a locally nonsatiated utility function U_i over consumption $c_i = (c_{it})_{t=0}^{\infty}$. A competitive equilibrium consists of state prices q , initial asset price P_0 , and consumption allocation (c_i) such that c_i maximizes utility U_i subject to the date-0 budget constraint $q \cdot c_i \leq q \cdot e_i + (P_0 + D_0)x_{i0}$ and markets clear, so $\sum_i c_i = \sum_i e_i + D$.

The following result is an adaptation of the bubble impossibility result of Santos and Woodford (1997) (see also Hirano and Toda (2024, §3.4)) and the standard

proof of the first welfare theorem.

Proposition B.1. *If in equilibrium the present value of aggregate endowment $q \cdot \sum_i e_i$ is finite, then there is no bubble (the asset price P_0 equals the present value of dividends $\sum_{t=1}^{\infty} q_t D_t$) and the allocation (c_i) is Pareto efficient.*

Proof. Since U_i is locally nonsatiated, the budget constraint holds with equality:

$$q \cdot c_i = q \cdot e_i + (P_0 + D_0)x_{i0}.$$

Summing across i and noting that $\sum_i x_{i0} = 1$, we obtain

$$q \cdot \sum_i c_i = q \cdot \sum_i e_i + P_0 + D_0.$$

Using the market clearing condition $\sum_i c_i = \sum_i e_i + D$, we obtain

$$q \cdot \left(\sum_i e_i + D \right) = q \cdot \sum_i e_i + P_0 + D_0.$$

Noting that $q_0 = 1$ and hence $q_0 D_0 = D_0$, we obtain

$$q \cdot \sum_i e_i + \sum_{t=1}^{\infty} q_t D_t = q \cdot \sum_i e_i + P_0.$$

Since $q \cdot \sum_i e_i < \infty$, we have $P_0 = \sum_{t=1}^{\infty} q_t D_t$, so there is no bubble.

To show that (c_i) is Pareto efficient, suppose to the contrary that there is a feasible allocation (c'_i) that Pareto dominates it. Then by the principle of revealed preference, we have

$$q \cdot c'_i \geq q \cdot e_i + (P_0 + D_0)x_{i0}$$

for all i , with at least one strict inequality. Summing across i , we obtain

$$q \cdot \sum_i c'_i \geq q \cdot \sum_i e_i + P_0 + D_0,$$

with strict inequality if $q \cdot \sum_i c'_i < \infty$. But since (c'_i) is feasible and the present value of aggregate endowment is finite, it follows that

$$\infty > q \cdot \left(\sum_i e_i + D \right) > q \cdot \sum_i e_i + P_0 + D_0.$$

Canceling $q \cdot \sum_i e_i < \infty$ from both sides and using $P_0 = \sum_{t=1}^{\infty} q_t D_t$, we get the

contradiction $0 > 0$. □

In the context of the model in §2, we obtain the following corollary.

Corollary 4. *Let q_t be the date-0 price and $\{a_t\}$ be as in (2.7). If $\sum_{t=0}^{\infty} q_t a_t < \infty$, then the equilibrium is fundamental and Pareto efficient.*

Proof. Condition (2.7) implies

$$\sum_{t=0}^{\infty} q_t a_t < \infty \iff \sum_{t=0}^{\infty} q_t (N_t e_t^y + N_{t-1} e_t^o + r_t) < \infty.$$

The rest of the proof is essentially the same as Proposition B.1 by letting the set of agents be $I = \{-1, 0, \dots\}$, letting agent $i = -1$ be the initial old, agent $i = t \geq 0$ be generation t , and accounting for population growth. □

Proposition B.1 is often sufficient to establish the efficiency of an equilibrium without bubbles. However, it is useless for studying bubbly equilibria that are potentially efficient, because it rules out inefficiency and bubbles simultaneously. For this purpose, we need more powerful tools.

In overlapping generations models, there is a well-known efficiency criterion called the *Cass criterion*

$$\sum_{t=0}^{\infty} \frac{1}{q_t} = \infty, \tag{B.1}$$

first introduced by Cass (1972) for the neoclassical growth model and adapted to other settings by Benveniste (1976), Balasko and Shell (1980), Okuno and Zilcha (1980), among others.

Let U_i be the utility function of agent i , which we assume to be strictly quasi-concave. Let (c_i) and (c'_i) be two feasible allocations and $d_i := c'_i - c_i$ be the difference. If (c'_i) Pareto dominates (c_i) , then by definition we have $U_i(c_i + d_i) = U_i(c'_i) \geq U_i(c_i)$ for all i with a strict inequality for at least one i . Since (c_i) and (c'_i) are both feasible and U_i is strictly quasi-concave, for any $\epsilon \in (0, 1]$, the allocation $(c_i + \epsilon d_i)$ is feasible and $U_i(c_i + \epsilon d_i) \geq U_i(c_i)$ for all i with a strict inequality for at least one i . This argument shows that to find a Pareto improvement over (c_i) , we may focus on feasible allocations that are arbitrarily close to (c_i) .

Let $U : \mathbb{R}_{++}^L \rightarrow \mathbb{R}$ be a general utility function, where L is the number of commodities ($L = 2$ in the two-period OLG model with homogeneous goods). We recall that U is called *differentiably strictly increasing* if it has positive partial derivatives, i.e., $\nabla U(c) \gg 0$ for all $c \in \mathbb{R}_{++}^L$. Furthermore, if U is twice continuously differentiable, we say that U is *differentiably strictly quasi-concave* if for all

$c \in \mathbb{R}_{++}^L$ and $0 \neq v \in \mathbb{R}^L$, we have

$$\langle \nabla U(c), v \rangle = 0 \implies \langle v, \nabla^2 U(c) v \rangle < 0, \quad (\text{B.2})$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in \mathbb{R}^L . Intuitively, differentiability strictly quasi-concavity implies that upper contour sets are convex and their boundaries are nowhere “flat”: see Villanacci et al. (2002, p. 14, Definition 29) and Toda (2025, p. 155, Proposition 11.11(ii)).

In the two-good case, such utility functions imply that indifference curves are downward sloping and have convex graphs (i.e., the second derivative is strictly positive). To see why, write $c = (y, z)$ and consider the indifference curve $U(y, z) = \text{constant}$. Setting $z = \phi(y)$ and differentiating, we obtain

$$U_1 + U_2 \phi'(y) = 0 \iff \phi'(y) = -\frac{U_1}{U_2} < 0.$$

Differentiating once more, we obtain

$$\begin{aligned} U_{11} + 2U_{12}\phi'(y) + U_{22}(\phi'(y))^2 + U_2\phi''(y) &= 0 \\ \iff \phi''(y) &= -\frac{U_{11} - 2U_{12}(U_1/U_2) + U_{22}(U_1/U_2)^2}{U_2} = -\frac{\langle v, \nabla^2 U v \rangle}{U_2} > 0 \end{aligned} \quad (\text{B.3})$$

for $v = (1, -U_1/U_2)$ using (B.2).

We can show that if we can uniformly bound the “elasticity” of indifference curves, then any equilibrium that satisfies the Cass criterion is Pareto efficient.

Proposition B.2. *Consider the model of §2 with constant population ($N_t = 1$) and bounded endowments ($\sup_t (e_t^y + e_t^o + r_t) < \infty$). Suppose each utility function U_t is twice continuously differentiable, differentiability strictly increasing, and differentiability strictly quasi-concave. Let $\{(c_t^y, c_t^o)\}_{t=0}^\infty$ be an interior allocation such that $c_t^y + c_t^o = e_t^y + e_t^o + r_t$ and q_t be the corresponding date-0 price. Let $z = \phi_t(y)$ be the indifference curve of generation t obtained by solving $U_t(y, z) = U_t(c_t^y, c_{t+1}^o)$ and suppose there exists $\mu > 0$ such that the elasticity satisfies*

$$-\frac{y\phi_t''(y)}{\phi_t'(y)} > 2\mu \quad (\text{B.4})$$

uniformly over t and y in a neighborhood of c_t^y . If the Cass criterion (B.1) holds, then $\{(c_t^y, c_t^o)\}_{t=0}^\infty$ is Pareto efficient.

Proof. The proof is essentially the same as Theorem 3A of Okuno and Zilcha (1980), which builds on Benveniste (1976).

We prove the contrapositive. Suppose $\{(c_t^y, c_t^o)\}_{t=0}^\infty$ is Pareto inefficient and let $\{(y_t, z_t)\}_{t=0}^\infty$ be a feasible Pareto improvement. By the definition of ϕ_t and applying Taylor's theorem, we have

$$z_{t+1} \geq \phi_t(y_t) = c_{t+1}^o + \phi_t'(c_t^y)(y_t - c_t^y) + \frac{1}{2}\phi_t''(c_t^y + \theta(y_t - c_t^y))(y_t - c_t^y)^2$$

for some $\theta \in (0, 1)$. By an earlier remark, without loss of generality we may assume that (y_t, z_t) is arbitrarily close to (c_t^y, c_t^o) . Hence using (B.4), we obtain

$$z_{t+1} \geq c_{t+1}^o + \phi_t'(c_t^y)(y_t - c_t^y) - \frac{\phi_t'(c_t^y)}{c_t^y} \mu(y_t - c_t^y)^2.$$

Individual optimality implies $\phi_t'(c_t^y) = -q_t/q_{t+1}$, so

$$q_t(y_t - c_t^y) + q_{t+1}(z_{t+1} - c_{t+1}^o) \geq \frac{\mu}{q_t c_t^y} [q_t(y_t - c_t^y)]^2. \quad (\text{B.5})$$

Equation (B.5) corresponds to Equation (1) in Okuno and Zilcha (1980).¹³

Note that because generation t weakly prefers (y_t, z_{t+1}) to (c_t^y, c_{t+1}^o) , by revealed preference the left-hand side of (B.5) is nonnegative. The fact that indifference curves have uniformly bounded elasticities implies that we can bound the left-hand side from below by a quadratic term. Noting that endowments are bounded from above, by redefining μ if necessary, we obtain

$$q_t(y_t - c_t^y) + q_{t+1}(z_{t+1} - c_{t+1}^o) \geq \frac{\mu}{q_t} [q_t(y_t - c_t^y)]^2, \quad (\text{B.6})$$

i.e., we may drop c_t^y in the denominator of (B.5). Define $(\delta_t, \epsilon_t) := (q_t(y_t - c_t^y), q_t(z_t - c_t^o))$. By feasibility, we have $y_t + z_t \leq e_t^y + e_t^o + r_t = c_t^y + c_t^z$, so

$$\delta_t + \epsilon_t \leq 0. \quad (\text{B.7})$$

Since $\{(y_t, z_t)\}_{t=0}^\infty$ Pareto improves over $\{(c_t^y, c_t^o)\}_{t=0}^\infty$, by revealed preference we have $\epsilon_0 \geq 0$ and

$$\delta_t + \epsilon_{t+1} \geq 0, \quad (\text{B.8})$$

¹³This equation contains a typographical error: $p_t^*(t)(c_t - c_t^*)$ on the right-hand side should be $p_t^*(t)(c_t^* - c_t)$.

with at least one strict inequality. Combining these inequalities, we have

$$\begin{aligned}\delta_{t+1} - \delta_t &= (\delta_{t+1} + \epsilon_{t+1}) - (\delta_t + \epsilon_{t+1}) \leq 0, \\ \epsilon_{t+1} - \epsilon_t &= (\delta_t + \epsilon_{t+1}) - (\delta_t + \epsilon_t) \geq 0,\end{aligned}$$

so

$$\delta_t \leq \dots \leq \delta_0 \leq -\epsilon_0 \leq 0 \leq \epsilon_0 \leq \dots \leq \epsilon_t. \quad (\text{B.9})$$

If $\epsilon_t = 0$ for all t , then $0 \geq \delta_t = \delta_t + \epsilon_{t+1} \geq 0$ for all t , which is a contradiction to a Pareto improvement. Therefore $\epsilon_t > 0$ for at least one t , and because $\{\epsilon_t\}$ is monotone, we have $\epsilon_t > 0$ for large enough t , say $t \geq t_0$. Similarly, $\delta_t \leq -\epsilon_t < 0$ for $t \geq t_0$. Using (B.6) and $-\delta_t \geq \epsilon_t \geq 0$ from (B.7) and (B.9), we obtain

$$\epsilon_{t+1} \geq -\delta_t + \frac{\mu}{q_t} \delta_t^2 \geq \epsilon_t + \frac{\mu}{q_t} \epsilon_t^2,$$

which is positive for $t \geq t_0$. Let $e_t := \epsilon_t/q_t = z_t - c_t^o > 0$ for $t \geq t_0$. Then

$$q_{t+1}e_{t+1} \geq q_t e_t + \mu q_t e_t^2.$$

Taking the reciprocal yields

$$\frac{1}{q_{t+1}e_{t+1}} \leq \frac{1}{q_t e_t (1 + \mu e_t)} = \frac{1}{q_t e_t} - \frac{\mu}{1 + \mu e_t} \frac{1}{q_t}.$$

Noting that $e_t = z_t - c_t^o$ is bounded from above (because aggregate endowment is bounded), we can take $e > 0$ such that $e_t \leq e$ for all t . Therefore,

$$\frac{1}{q_{t+1}e_{t+1}} \leq \frac{1}{q_t e_t} - \frac{\mu}{1 + \mu e} \frac{1}{q_t}.$$

Taking the sum from $t = t_0$ to $t = T$, we obtain

$$\frac{\mu}{1 + \mu e} \sum_{t=t_0}^T \frac{1}{q_t} \leq \frac{1}{q_{t_0} e_{t_0}} - \frac{1}{q_{T+1} e_{T+1}} \leq \frac{1}{q_{t_0} e_{t_0}}.$$

Letting $T \rightarrow \infty$, we obtain $\sum_{t=t_0}^{\infty} (1/q_t) < \infty$, so the Cass criterion (B.1) fails. \square

Remark 1. The condition (B.4) can be easily satisfied in special settings. For instance, it suffices that generations have identical preferences (ϕ_t does not depend on t), the equilibrium allocation is uniformly bounded away from zero, and indifference curves through the equilibrium allocation does not approach the boundary

of \mathbb{R}_+^2 when restricted to feasible allocations. (See the proof of Theorem 1.) Okuno and Zilcha (1980, p. 805) provide a counterexample to Proposition B.2 if condition (B.4) is violated.

Remark 2. Although we assumed a two-period OLG model with a single commodity in each period, this assumption is not essential. Benveniste (1986) explains how we can obtain an analog of (B.5) in a general setting.

C Proofs

C.1 Proof of Theorem 1

To derive the Cass criterion for the model in §2 with arbitrary population, we detrend the original economy (denoted \mathcal{E}) and reduce to a hypothetical economy (denoted $\tilde{\mathcal{E}}$) with no population growth and bounded endowments. Let the sequence $\{a_t\}$ satisfy (2.7). Define the endowment and consumption in $\tilde{\mathcal{E}}$ by

$$\begin{aligned}(\tilde{e}_t^y, \tilde{e}_t^o) &:= (N_t c_t^y / a_t, N_{t-1} c_t^o / a_t), \\(\tilde{c}_t^y, \tilde{c}_t^o) &:= (N_t c_t^y / a_t, N_{t-1} c_t^o / a_t).\end{aligned}$$

Dividing the resource constraint (2.2a) by a_t yields

$$\tilde{c}_t^y + \tilde{c}_t^o = \tilde{e}_t^y + \tilde{e}_t^o + d_t,$$

where $d_t := r_t / a_t$ is detrended dividend. Eliminating (c_t^y, c_{t+1}^o) from the utility function yields

$$U_t(c_t^y, c_{t+1}^o) = U_t((a_t / N_t) \tilde{c}_t^y, (a_{t+1} / N_t) \tilde{c}_{t+1}^o) =: \tilde{U}_t(\tilde{c}_t^y, \tilde{c}_{t+1}^o).$$

Using (2.5), the gross risk-free rates in \mathcal{E} and $\tilde{\mathcal{E}}$ are related as

$$\tilde{R}_t := \frac{\tilde{U}_{t,1}}{\tilde{U}_{t,2}}(\tilde{c}_t^y, \tilde{c}_{t+1}^o) = \frac{a_t}{a_{t+1}} \frac{U_{t,1}}{U_{t,2}}(c_t^y, c_{t+1}^o) = \frac{a_t}{a_{t+1}} R_t.$$

Therefore, the date-0 prices are related as

$$\tilde{q}_t := \frac{1}{\tilde{R}_0 \cdots \tilde{R}_{t-1}} = \frac{a_t}{a_0} \frac{1}{R_0 \cdots R_{t-1}} = \frac{a_t}{a_0} q_t.$$

Consequently, the Cass criterion (B.1) for $\tilde{\mathcal{E}}$ is equivalent to

$$\sum_{t=0}^{\infty} \frac{1}{q_t a_t} = \infty, \quad (\text{C.1})$$

which we refer to as the Cass criterion for \mathcal{E} . See Geanakoplos and Polemarchakis (1991, Theorems 5a, 5b) for condition (C.1).

With this definition, we can connect the Cass criterion (hence efficiency) to asset price bubbles as follows.

Lemma C.1. *Let q_t be the date-0 price and $\{a_t\}$ be as in (2.7). In any asymptotically bubbly equilibrium, the Cass criterion (C.1) holds.*

Proof. Take any asymptotically bubbly equilibrium. By definition (see Appendix A), $\lim_{t \rightarrow \infty} q_t P_t > 0$ exists and $\liminf_{t \rightarrow \infty} P_t/a_t > 0$. Therefore

$$\liminf_{t \rightarrow \infty} \frac{1}{q_t a_t} = \left(\lim_{t \rightarrow \infty} \frac{1}{q_t P_t} \right) \left(\liminf_{t \rightarrow \infty} \frac{P_t}{a_t} \right) > 0,$$

so (C.1) holds. □

While the Cass criterion is not sufficient for Pareto efficiency, it is almost so (Proposition B.2). Thus, Lemma C.1 implies that asymptotically bubbly equilibria are almost efficient.

Proof of Theorem 1. The claim that all equilibria are asymptotically bubbly follows from Hirano and Toda (2025a, Theorem 2). However, because population grows but endowments converge in our setting, while population is constant but endowments grow in Hirano and Toda (2025a), we provide a formal correspondence.

Let $\tilde{\mathcal{E}}$ be an economy with constant population (mass 1) with endowment, consumption, asset holdings, and utility

$$\begin{aligned} (\tilde{e}_t^y, \tilde{e}_{t+1}^o) &= (G^t e_t^y, G^t e_{t+1}^o), \\ (\tilde{c}_t^y, \tilde{c}_{t+1}^o) &= (G^t c_t^y, G^t c_{t+1}^o), \\ \tilde{x}_t &= G^t x_t, \\ \tilde{U}_t(y, z) &= U(y/G^t, z/G^t). \end{aligned}$$

Multiplying the budget constraints (2.1) of the original economy \mathcal{E} by G^t , we obtain the budget constraints of $\tilde{\mathcal{E}}$. Using $N_t = G^t$, the market clearing condition

(2.2) for \mathcal{E} is identical to that for $\tilde{\mathcal{E}}$. By the definition of \tilde{U}_t , we have

$$\tilde{U}_t(\tilde{c}_t^y, \tilde{c}_{t+1}^o) = U(\tilde{c}_t^y/G^t, \tilde{c}_{t+1}^o/G^t) = U(c_t^y, c_{t+1}^o).$$

Since the utility maximization problems in \mathcal{E} and $\tilde{\mathcal{E}}$ are equivalent, there is a one-to-one correspondence between equilibria of \mathcal{E} and $\tilde{\mathcal{E}}$.

To prove that all equilibria of \mathcal{E} are asymptotically bubbly, it suffices to prove it for $\tilde{\mathcal{E}}$. We verify the assumptions of Theorem 2 of Hirano and Toda (2025a) (henceforth HT). By Assumption 1, \tilde{U}_t is continuous, quasi-concave, continuously differentiable, and has positive partial derivatives, so Assumption 1 of HT holds. By Assumption 2, young's endowment growth satisfies $\tilde{e}_{t+1}^y/\tilde{e}_t^y \rightarrow G$ as $t \rightarrow \infty$ and the endowment ratio satisfies $\tilde{e}_t^o/\tilde{e}_t^y \rightarrow e^o/(Ge^y) \geq 0$, so Assumption 2 of HT holds. The forward rate function (reciprocal of marginal rate of substitution) satisfies

$$\tilde{f}_t(y, z) := \frac{\tilde{U}_{t,1}}{\tilde{U}_{t,2}}(G^t y, G^t z) = \frac{U_1}{U_2}(y, z),$$

which is independent of t , so Assumption 3 of HT clearly holds.¹⁴ Condition (2.8) corresponds to condition (20) of HT. Therefore, by Theorem 2 of HT, all equilibria are asymptotically bubbly.

We next show the Pareto efficiency of the equilibrium. By Lemma C.1, the Cass criterion (C.1) holds. Hence by Proposition B.2, it suffices to check the elasticity condition (B.4). Let $p_t = P_t/G^t$ and $d_t = r_t/G^t$ be the detrended asset price and dividend. In equilibrium, the young holds $1/N_t = 1/G^t$ shares of the asset, so p_t equals savings. The consumption of generation t is therefore

$$(c_t^y, c_{t+1}^o) = (e_t^y - p_t, e_{t+1}^o + G(p_{t+1} + d_{t+1})).$$

By Assumption 2, we have $(e_t^y, e_t^o) \rightarrow (e^y, e^o)$ and $d_t \rightarrow 0$. Since the equilibrium is asymptotically bubbly, we can take $p > 0$ such that $\liminf_{t \rightarrow \infty} p_t \geq p$. The budget constraint of the young also implies $\limsup_{t \rightarrow \infty} p_t \leq e^y$. Now let the young purchase half of the equilibrium asset holdings, which is obviously feasible (and suboptimal). This trade will result in consumption

$$(e_t^y - p_t/2, e_{t+1}^o + G(p_{t+1} + d_{t+1})/2),$$

which is asymptotically bounded below by $(e^y/2, e^o + Gp/2) \gg 0$. By Assumption

¹⁴In HT, a_t is young's endowment, while we scale by G^t here, but the distinction is clearly unimportant.

1, indifference curves do not touch the boundary of \mathbb{R}_+^2 . Since consumption is bounded above, any Pareto improving allocation over (c_t^y, c_{t+1}^o) is contained in a compact subset of \mathbb{R}_{++}^2 . By minimizing the elasticity (the left-hand side of (B.4)) over this compact set, we obtain a strictly positive number, so condition (B.4) holds. \square

C.2 General production function

Let $F(H, X)$ be a general neoclassical production function. The elasticity of substitution is defined by the percentage change in relative inputs with respect to a percentage change in relative price:

$$\sigma := -\frac{\partial \log(H/X)}{\partial \log(w/r)},$$

where $w = F_H$ and $r = F_X$. Let $h = \log(H/X)$ and $\sigma(h)$ be the elasticity of substitution corresponding to h . Then

$$\frac{\partial}{\partial h} \log \frac{w}{r} = -\frac{1}{\sigma(h)}.$$

Integrating h_0 to h and applying the intermediate value theorem for integrals, we obtain

$$\log \frac{w}{r}(h) - \log \frac{w}{r}(h_0) = -\frac{1}{\sigma((1-\theta)h_0 + \theta h)}(h - h_0)$$

for some $\theta \in (0, 1)$. Suppressing the dependence of σ on h , we obtain

$$\frac{w}{r}(h) = \frac{w}{r}(h_0)(H/H_0)^{-1/\sigma}, \quad (\text{C.2})$$

where we recall $h = \log(H/X)$ and $h_0 = \log(H_0/X)$. Therefore, if

$$w := \lim_{H \rightarrow \infty} F_H(H, 1) = \lim_{H \rightarrow \infty} \frac{F(H, 1)}{H} > 0,$$

it follows from (C.2) shows that

$$r = F_X(H, 1) \sim F_X(H_0, 1)(H/H_0)^{1/\sigma}$$

for large H . Thus, if the elasticity of substitution is bounded away from 1 at high input level H , then the rent r grows slower than the input H , which generalizes the analysis in §3.1. For an analysis along these lines, see Hirano and Toda (2025c, §3).

C.3 Proof of Proposition 3.2

If (3.10) holds, then we have $e^y > 0$ in (3.9). Therefore, there exists $t_0 \geq 0$ such that $e_t^y > 0$ for $t \geq t_0$. We then obtain an equilibrium starting at t_0 by the argument preceding Proposition 3.2. Since $r_t \sim rG^{t/\sigma}$ and $P_t = pG^t$ with $\sigma > 1$, we have $\sum_{t=t_0}^{\infty} r_t/P_t < \infty$. By Lemma A.1, the equilibrium is bubbly, and it is asymptotically bubbly because $P_t/G^t = p > 0$. Pareto efficiency follows from Lemma C.1 and Proposition B.2.

The natural interest rate in (3.6) is given by

$$R := \frac{1}{\beta} \left(\frac{e^o}{e^y + w} \right)^\gamma = \frac{1}{\beta} \left(\frac{e^o}{(e^o + Gp)(\beta G)^{-1/\gamma} + p} \right)^\gamma,$$

where we have used (3.9). Making R less than $G^{1/\sigma}$ and solving for e^o , we obtain (3.11). By Proposition 3.1, all equilibria are asymptotically bubbly and Pareto efficient. \square

C.4 Proof of Proposition 3.3

Regardless of $\sigma = 1$ or $G = 1$, under the maintained assumptions, using (3.3), there exist constants $e^y, e^o \geq 0$ and $w, r > 0$ such that

$$(e_t^y, e_t^o, w_t, r_t) = (e^y G^{(\alpha-1)t}, e^o G^{(\alpha-1)t}, w G^{(\alpha-1)t}, r G^{\alpha t}).$$

In a balanced growth path equilibrium, we must have $P_t = pG^{\alpha t}$ for some $p > 0$. Using (2.4), the equilibrium consumption of generation t is

$$(y_t, z_{t+1}) = (e^y + w - p, e^o G^{\alpha-1} + (p+r)G^\alpha) G^{(\alpha-1)t}.$$

Therefore, the gross risk-free rate (2.5) reduces to

$$R = \frac{1}{\beta} \left(\frac{e^o G^{\alpha-1} + (p+r)G^\alpha}{e^y + w - p} \right)^\gamma = \left(1 + \frac{r}{p} \right) G^\alpha. \quad (\text{C.3})$$

Let

$$f(p) := \frac{1}{\beta} \left(\frac{e^o G^{\alpha-1} + (p+r)G^\alpha}{e^y + w - p} \right)^\gamma - \left(1 + \frac{r}{p} \right) G^\alpha.$$

Then f is continuous and strictly increasing in p . Since $f(0) = -\infty$ and $f(e^y + w) = \infty$, by the intermediate value theorem, there exists a unique $p \in (0, e^y + w)$ such that $f(p) = 0$. This p satisfies (C.3), so there exists a unique balanced growth path equilibrium.

In this equilibrium, by (C.3) we have $R = (1+r/p)G^\alpha > G^\alpha$, so the interest rate exceeds the economic (output) growth rate and infinite debt rollover is impossible. Setting $q_t = 1/R^t$ and $a_t = G^{\alpha t}$ in Corollary 4, the equilibrium is fundamental and Pareto efficient. \square

C.5 Proof of Proposition 3.4

If (3.15) holds, then we have $e^y > 0$ in (3.14). Therefore, there exists $t_0 \geq 0$ such that $e_t^y > 0$ for $t \geq t_0$. We then obtain an equilibrium starting at t_0 by the argument preceding Proposition 3.4. Since $r_t/G^{t/\sigma} \rightarrow r$ and $P_t/G^{t/\sigma} = p$, we have $r_t/P_t \rightarrow r/p > 0$ as $t \rightarrow \infty$. Since $\sum_{t=t_0}^{\infty} r_t/P_t = \infty$, by Lemma A.1 the equilibrium is fundamental.

We next show that the equilibrium is Pareto efficient if and only if (3.16) holds. By (3.4b), we have

$$r_t G^{-t/\sigma} = A \left(\alpha + (1 - \alpha) G^{-(1-1/\sigma)t} \right)^{\frac{1}{\sigma-1}} (1 - \alpha) \geq r$$

always. If (3.16) holds, then

$$R_t = G^{1/\sigma} \left(1 + \frac{r_{t+1}}{p} G^{-\frac{t+1}{\sigma}} \right) \rightarrow G^{1/\sigma} \left(1 + \frac{r}{p} \right) = R \quad (\text{C.4})$$

and $R_t \geq R \geq G$ for all t . Since $1/q_t = R_0 \cdots R_{t-1} \geq R^t \geq G^t$, it follows that

$$\sum_{t=0}^{\infty} \frac{1}{q_t G^t} \geq \sum_{t=0}^{\infty} (R/G)^t = \infty$$

and the Cass criterion (C.1) holds. By the same argument as the proof of Theorem 1, the equilibrium is Pareto efficient.

Conversely, suppose (3.16) fails and hence $R < G$. Suppose we tax the young at time t by $\epsilon > 0$, which generates a tax revenue of $N_t \epsilon = G^t \epsilon$. If we transfer this tax revenue to the old, each old receives $N_t \epsilon / N_{t-1} = G \epsilon$. As $t \rightarrow \infty$, the equilibrium consumption allocation $(c_t^y, c_{t+1}^o) = (y_t, z_{t+1})$ converges to $(e^y + w, e^o) \gg 0$, so this transfer is feasible for small enough $\epsilon > 0$. Using (3.12), the first-order effect of this transfer on generation t 's utility is

$$\begin{aligned} \left. \frac{\partial}{\partial \epsilon} U(c_t^y - \epsilon, c_{t+1}^o + G\epsilon) \right|_{\epsilon=0} &= -U_1(y_t, z_{t+1}) + G U_2(y_t, z_{t+1}) \\ &\rightarrow (G - R) U_2(e^y + w, e^o) := D > 0 \end{aligned}$$

as $t \rightarrow \infty$, where the last line follows from $(y_t, z_{t+1}) \rightarrow (e^y + w, e^o)$ and (C.4).

Let $f_t(\epsilon) := U(c_t^y - \epsilon, c_{t+1}^o + G\epsilon)$. Take $\delta \in (0, D)$. Since $f'_t(0) \rightarrow D$ as $t \rightarrow \infty$ and f_t is continuously differentiable, we can take $T \geq t_0$ large enough and $\epsilon > 0$ small enough such that $f'_t(\epsilon) > D - \delta > 0$ for $t \geq T$. Since f_t is concave, we have

$$f_t(0) - f_t(\epsilon) \leq f'_t(\epsilon)(0 - \epsilon) \iff \frac{f_t(\epsilon) - f_t(0)}{\epsilon} \geq f'_t(\epsilon) > D - \delta,$$

so $f_t(\epsilon) > f_t(0)$ for $t \geq T$. Therefore, by implementing the transfer of ϵ for dates $t \geq T$ only, we can make a Pareto improvement. \square

C.6 Proof of Proposition 4.2

Since F is neoclassical (increasing, concave, and homogeneous of degree 1), we have

$$F(H, X) = HF_H(H, X) + XF_X(H, X),$$

so $F_H(x, 1) = f'(x) > 0$ and $F_X(x, 1) = f(x) - xf'(x) > 0$. Therefore, $xf'(x)/f(x) \in (0, 1)$. By (4.2), we can take $\delta \in (0, 1)$ such that

$$\sup_{x \in [n, N]} \frac{xf'(x)}{f(x)} \leq 1 - \delta. \quad (\text{C.5})$$

Take any equilibrium. Let

$$\begin{aligned} w_t &= F_H(N_t, 1) = f'(N_t) > 0, \\ r_t &= F_X(N_t, 1) = f(N_t) - N_t f'(N_t) > 0 \end{aligned}$$

be the wage and rent. Let $q_t > 0$ be the date-0 price, $V_0 = \sum_{t=1}^{\infty} q_t r_t \leq P_0 < \infty$ the fundamental value of land, and $a_t = f(N_t)$ the output. By assumption,

$$\frac{N_t e_t^y + N_{t-1} e_t^o + f(N_t)}{a_t} = 1 + \frac{N_t e_t^y + N_{t-1} e_t^o}{f(N_t)}$$

is bounded above and bounded away from 0, so (2.7) holds. Since F is neoclassical, we have $a_t = w_t N_t + r_t$. By (C.5), we have

$$1 = \frac{w_t N_t + r_t}{a_t} \geq \frac{r_t}{a_t} \geq \delta.$$

Therefore,

$$\sum_{t=0}^{\infty} q_t a_t \leq \frac{1}{\delta} \sum_{t=0}^{\infty} q_t r_t = \frac{1}{\delta} (r_0 + V_0) < \infty.$$

By Corollary 4, the equilibrium is fundamental and Pareto efficient.

□