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Abstract

We consider asset-price bubbles with a finite time horizon. Bubbles emerge because of incomplete information on the timing of trades. We analyze investors' decision making on the cognitive investment (C-investment) that restores the complete information and show that the bubbles endogenously collapse. With investors having the option of C-investment, the asset price grows acceleratedly, and explodes with a higher probability as it grows higher. The bubbles collapse when the condition for optimality of the C-investment is satisfied, though the C-investment does not actually take place in equilibrium.

Key words: Finite time horizon, incomplete information, cognitive resources.
JEL Classification: D52, E44

1 Introduction

In various episodes of asset-price bubbles and crashes, we observe that asset prices grow acceleratedly and then burst with higher probability as they grow higher. In this paper we would like to replicate these features in the dynamics of asset-price bubbles in a model where the time horizon is finite and the bubbles emerge due to incomplete information about the timing of trade. In particular, we would like to generate endogenous collapse of the bubbles from optimizations by investors on the costly acquisition of information. This note may be closely related to Awaya, Iwasaki and Watanabe (2022, AIW hereafter), in which investors sequentially buy and sell the bubbly asset with some incomplete information about the true value of the asset. AIW also shows similar dynamics of the bubbles, i.e., the bubble grows acceleratedly and bursts with higher probability as it grows higher. A difference between AIW and our model lies in the reason of the bubble collapses. In AIW model, the collapse occurs depending on the flow of information rather mechanically, while in our model it occurs as a result of optimization by agents.

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Bubbles due to incomplete information: Our model is built on Jiang, Norman, Puzzello, Sultanum and Wright (2024, JNPSW hereafter), in which bubbles emerge due to incomplete information on timing of trade. In the JNPSW model, an investor does not know whether she is the last buyer of the bubbly asset, which is intrinsically useless but traded at a positive value. If she is the last buyer, she pays the price p to buy the asset but cannot sell it to anyone, in which case her payoff is $-p$. If she is not the last buyer, she pays p to buy the asset and gets p' by selling the asset, in which case her payoff is $p' - p$. Given that the investor is the last buyer with probability $\frac{1}{2}$, the expected payoff of buying the asset is $\frac{1}{2}(p' - p) + \frac{1}{2}(-p) = \frac{1}{2}p' - p$. Given that p' is sufficiently large, i.e., $\frac{1}{2}p' - p > 0$, investors are willing to buy the asset without knowing whether they are the last buyers or not, and the bubble continues. This is the JNPSW model of bubbles due to incomplete information on the timing of trade.

Our new ingredient is cognitive investment (C-investment) that enables agents to restore information on the timing of trade. C-investment represents various costly activities of information acquisition, such as research and study on the value of individual corporate stocks. While we focus on the incomplete information on the timing of trade in this model, we would like to emphasize that it could be easily shown that our results may hold in other settings such as the incomplete information about the fundamental value of the asset as in AIW. We will show that optimal decision on C-investment causes the endogenous collapse of the bubbles. We call the bubbles in our model with C-investment as "resource-rational bubbles," following the terminology in the cognitive science (Lieder and Griffiths, 2020). Resource-rationality means the optimal use of limited computational resources according to Lieder and Griffiths (2020). Resource-rational bubbles continue to grow as long as the investors optimally choose not to use their limited computational resources to acquire complete information on the timing of trade.

What we do in this note is the following. In the next section, we extend the two-period JNPSW model to T-period economy, which we call the baseline model. We introduce C-investment in the baseline model in Section 3. We first show that the bubble collapses if the investors make C-investments, because the complete information about the timing of trade is restored in the economy with a finite time horizon and the usual backward induction eliminates the bubble in equilibrium. Next, we show that the bubbly equilibrium ceases to exist if C-investment becomes the optimal choice for the investors, even if they do not actually make C-investments. This is because the existence of the bubbly equilibrium renders the following contradiction: if the bubble is expected to continue, the investors are willing to make C-investments now, which makes the bubble unable to exist now and future, and then the investors become unwilling to make C-investments now. This contradiction is similar to the paradox in Grossman and Stiglitz (1980).[1]

[1] Grossman and Stiglitz (1980) argue that perfectly informationally efficient markets are an impossibility,

2 Baseline model

The model is a T-period economy with varying asset prices, which is an extension of the two-period JNPSW model, in which the asset price is fixed. We introduce incomplete information only in the last two periods, $T-1$ and T.

2.1 Setting

The economy continues for T periods and there are $T+1$ groups of agents in the economy. Each group is a continuum of infinitesimally small individual agents, and has unit mass. The groups are indexed by $t = 0, 1, 2, \cdots, T$. Periods are also indexed by $t = 1, 2, 3, \cdots, T$. Note that periods start from period 1, while groups of agents start from group 0. Group t enters the economy in period t and exits the economy in period $t+1$ for $t = 1, 2, \cdots, T-2$. Group 0 is in the market at the beginning of period 1 and waits for the entry of group 1. Each agent of group t enters the economy as a producer in period t, and becomes a consumer in period $t+1$, who can consume only the goods produced by group $t+1$. At the end of period $t+1$, group t exits the economy. Group 0 is the consumers in period 1, and exits the economy at the end of period 1. An agent of group t can produce consumer goods by spending the unit cost, c_t, in terms of utility in period t for $t = 1, 2, \cdots, T-2$. An agent of group t obtains y units of utility in period $t+1$ by consuming y units of the consumption goods produced by an agent of group $t+1$ for $t = 0, 1, 2, \cdots, T-2$. Agents discount future utility, and the discount factor for period t to $t+1$ is β, where $0 < \beta < 1$ for $t = 1, 2, \cdots, T-2$.

For analytical convenience, we assume

$$c_t = c, \quad \text{for } 1 \leq t \leq T-2. \tag{1}$$

Since the welfare gain of producing one unit of the good in period t is $1-c_t$, the production of the good is welfare-improving if $c \leq 1$, and it is detrimental to welfare if $c > 1$.

At the beginning of period 1, an agent of group 0 holds $k = 1$ unit of the bubbly asset. The bubbly asset is a non-depleting but intrinsically useless object. The total amount of the bubbly asset is unity in every period in this economy. In an equilibrium where the bubbly asset is traded, the consumers in period t (i.e., group $t-1$) can sell the bubbly asset to the producers (i.e., group t) in exchange for consumption goods.

Bubbles due to incomplete information: In periods $T-1$ and T, group $T-1$ and group T are subject to incomplete information about their timing of entry. I will state a detailed description of incomplete information in Assumption 3. Here, I casually describe

since, if prices perfectly reflected available information, there is no profit to gathering information, in which case the prices based on gathered information cannot reflect all available information.

the features of the incomplete information. Nature randomly chooses the entry period of group $T-1$ from periods $T-1$ and T. The other period is designated as the entry period of group T. Groups $T-1$ and T do not know their respective entry periods. They do know that they are entering either period $T-1$ or T, but they do not know exactly in which period they enter the economy, even at the time when they have entered the economy. This ignorance of groups $T-1$ and T about the timing of trade enables the bubble to exist, just as in the JNPSW model. In the entry period, the entering agents, either group $T-1$ or group T, can produce the consumer goods by spending utility cost $c'(=c_{T-1}=c_T)$ per unit, and they become the consumers in the next period. Concerning the discount factor, we assume the following:

Assumption 1. Agents do not discount their utility from period $T-1$ to period T, that is, the discount factor from period $T-1$ to T is 1.

This assumption is necessary to have incomplete information about the entry period, otherwise agents can infer their entry period from how they have discounted their utility in the past.

Definition of equilibrium: We denote the period-t price of the asset in terms of consumption goods by q_t. The equilibrium is defined as the sequence of $\{q_t\}_{t=1}^{T}$, such that (i) agents in period t make decisions optimally, given the expectations of future prices, $\{q_{t+s}\}_{s=1}^{T-t}$, for $t=1,2,\cdots,T$, and (ii) their expectations of prices $\{q_{t+s}\}$ are actually realized in the respective periods $t+s$ for $s=1,2,\cdots,T-t$.

We will argue that there exists a bubbly equilibrium in which $q_t>0$. For the bubbly equilibrium to exist, we need to have the following technical assumption for the parameter values:

$$c'=c_{T-1}=c_T=\frac{1}{2}, \tag{2}$$

as we will argue in Lemma 1. This assumption is technical because what is necessary for the existence of the bubbly equilibrium is $q_{T-1}=q_T$ as we argue below, and equation (2) ensures $q_{T-1}=q_T$ under the specific technical environment of our stylized model.

2.2 Decision-making for the first $T-2$ periods ($t=1,2,\cdots,T-2$)

Agents of group t for $t=1,2,\cdots,T-2$ know that their entry period is period t.

Bubbly asset: Each agent of group 0 (who is the consumes in period 1) waits for the entry of group 1 in period 1 with k_1 units of the bubbly asset, where $k_1=1$. Group $t-1$ in period t has nothing to pay other than the bubbly asset k_t. The total supply of the asset is unity: $k_t=1$. The bubbly asset is intrinsically useless and it provides no dividend. For simplicity we make the restriction that an agent can purchase and hold no greater than one unit of the asset at a time. The price of k_t in period t is q_t in terms of consumption goods. The price q_t is determined, as described below, by bargaining between consumers and producers in period t, given the expectations of future prices $\{q_{t+s}\}_{s=1}^{T-t}$. At the beginning of period t, consumers hold the bubbly asset $k_t = 1$ and obtain $q_t k$ units of goods by selling the bubbly asset k ($\leq k_t$) to producers.

Note on multiplicity of equilibrium: Since the bubbly asset k is intrinsically worthless, there always exists a non-bubbly equilibrium where $q_t = 0$ for all $t = 0, 1, 2, \cdots, T$, and no trade occurs. In the non-bubbly equilibrium, there is no consumption, and all agents have zero utility. For simplicity of analysis, we assume the following:

Assumption 2. Once the economy jumps into the non-bubbly equilibrium, the bubbly equilibrium, in which $q_t > 0$, never reemerge.

Bargaining between group $t-1$ and group t in period t: An agent of group $t-1$ and an agent of group t make a negotiation to determine q_t, given the expectation of the price in the next period (q_{t+1}), whereas they may or may not remember the past prices $\{q_s\}_{s=1}^{t-1}$. We assume that agents of group $t-1$ have full bargaining power and maximize their utility $q_t k$ subject to the participation constraint for agents of group t:

$$\max_{q_t,\ k} q_t k,$$
$$\text{s.t. } V_t \geq 0, \quad \text{and} \quad k \leq k_t,$$

where $V_t \geq 0$ is group t's participation constraint and V_t is group t's expected value, which is given by

$$V_t = -c q_t k + \beta q_{t+1} k,$$

where q_{t+1} is the expected value of the price in period $t+1$. The condition $V_t \geq 0$ implies that q_t is given by the following equation from the expectations on q_{t+1}:

$$q_t = \frac{\beta}{c} q_{t+1}.$$

In equilibrium, $V_t = 0$ and $k = k_{t+1} = 1$. If $c/\beta > 1$, the asset bubble grows, i.e., $q_{t+1} > q_t$. If $c/\beta < 1$, the asset bubble declines, i.e., $q_{t+1} < q_t$.

2.3 Decision-making for the last two periods $T-1$ and T

In the last two periods, the economy is similar to the JNPSW model. Group $T-1$ enters the economy in period $T-1$ or T randomly, and group T enters the economy in the other period. They are subject to incomplete information that they do not know which of the two periods $T-1$ or T they enter.

Assumption 3. Agents of group $T-1$ and group T are subject to the following incomplete information:

- When group $T-1$ or group T enter the economy, they believe that the current period (i.e., their entry period) is period $T-1$ with probability $\frac{1}{2}$ and period T with probability $\frac{1}{2}$.
- When each agent of group T (or $T-1$) enters the economy as the producer, she does not tell the consumers are the agents of group $T-2$ or not. (This assumption is necessary because, if they tell their counterparty is group $T-2$ or not, they can tell the current period is $T-1$ or T.)
- In period t, where $t = 1, 2, \cdots, T-2$, groups $T-1$ and T have rational expectations of future prices $\{q_{t+s}\}_{s=1}^{T-t}$. However, in their entry period, they forget what expectations they had in the past.

Here we consider the decision-making problems in periods $T-1$ and T.

Period $T-1$: Agents of group $T-2$, who know that the current period is period $T-1$, solve the following problem P_{T-1}, given the expectations $\{q_T\}$. Note that Assumption 1 implies that the discount factor is not β, but 1.

$$\mathrm{P}_{T-1}: \max_q q,$$
$$\text{s.t. } V \equiv \frac{1}{2}(-c_{T-1}q + q_T) + \frac{1}{2}(-c_T q + 0) = -c'q + \frac{1}{2}q_T \geq 0,$$

where $V \geq 0$ is the participation constraint for the agents who enter the economy in period $T-1$, whom we call "entrants $T-1$." Entrants $T-1$ are either group $T-1$ or T. We also call the agents who enter the economy in period T "entrants T." V is the value of entrant $T-1$ on the premise that she believes with probability $\frac{1}{2}$ that the current period is period $T-1$ and with probability $\frac{1}{2}$ that it is period T. Note that entrants $T-1$ have the expectation that the price in period T will be q_T, even though they do not know the current period is period $T-1$ or T. If the current period is $T-1$, she pays $c_{T-1}q$ to buy the asset and can sell it at the price q_T in the next period, whereas if the current period is T, she pays $c_T q$ to buy the asset and get nothing in the next period because the current period is the last period. Since $c_{T-1} = c_T = c'$, we have $V = -c'q + \frac{1}{2}q_T$.

Period T: Entrants $T-1$ solve the following problem P_T, given the expectations $\{q_T\}$.

$$\mathrm{P}_T: \max_q q,$$
$$\text{s.t. } V \equiv \frac{1}{2}(-c_{T-1}q + q_T) + \frac{1}{2}(-c_T q + 0) = -c'q + \frac{1}{2}q_T \geq 0,$$

where $V \geq 0$ is the participation constraint for entrants T. Note that entrants $T-1$ already know that the previous period is period $T-1$ and the current period is period T, since they meet new entrants in the current period. On the other hand, entrants T does not know the current period is T, but believes that the current period is period $T-1$ with probability $\frac{1}{2}$ and period T with probability $\frac{1}{2}$.

The following lemma clarifies the condition for the existence of the rational expectations equilibrium with incomplete information:

Lemma 1. The prices in periods $T-1$ and T must be equal in equilibrium, i.e., $q_{T-1} = q_T$. The condition for the existence of the rational expectations equilibrium is equation (2).

Proof. The solution to P_{T-1} is the price of the asset in period $T-1$. The solution to P_T is the price of the asset in period T. Obviously, P_{T-1} and P_T are identical problems and the solutions are the same:

$$q = \frac{1}{2c'} q_T.$$

Since the solution to P_T should be q_T in the rational expectations equilibrium, it should hold that $q_T = \frac{1}{2c'} q_T$, which implies that (2) must be satisfied. □

We have shown that the prices in the rational expectations equilibrium with incomplete information must satisfy

$$q_T = q_{T-1}.$$

2.4 Summary

The features of the baseline model is summarized as follows.

- The bubbly asset can be traded with positive prices for finite periods of time. In the bubbly equilibrium, which is a rational expectations equilibrium, the sequence of prices $\{q_t\}_{t=1}^{T}$ is determined backward as follows.[2]

  · $q_T = q^*$ is given exogenously,

  · $q_{T-1} = q_T = q^*$,

  · $q_{t-1} = \frac{\beta}{c} q_t$ for $2 \leq t \leq T-1$.

  In period t for $1 \leq t \leq T-2$, agents $t, t+1, \cdots, T-2$ have the expectations on the price path $\{q_{t+s}\}_{s=1}^{T-t}$, which is actually realized in equilibrium. Group $T-1$ and group T have the expectation, $\{q_T\}$, which is actually realized in period T in

[2] Alternative assumption may be that q_1, the price in period 1, is given exogenously. For example, we can assume that one unit of the bubbly asset can be transformed to one unit of the consumer good, and vice versa, only in period 1. Given this assumption, we can set $q_1 = 1$. Then q_t for $t \geq 2$ is given forwardly by $q_t = \frac{c}{\beta} q_{t-1}$ for $2 \leq t \leq T-1$, and $q_T = q_{T-1}$.

equilibrium, whereas they do not know whether their entry period is exactly period $T-1$ or period T.

- The bubble grows, $q_{t+1} > q_t$, if $\frac{c}{\beta} > 1$. It declines, $q_{t+1} < q_t$, if $\frac{c}{\beta} < 1$. It is stable, $q_{t+1} = q_t$, if $\frac{c}{\beta} = 1$.
- If $c > 1$, the bubble is detrimental to the social welfare, because the social surplus of trading the bubbly asset in period t is $q_t - cq_t = (1-c)q_t < 0$. If $c < 1$, the bubble is welfare improving.
- There always exists the non-bubbly equilibrium in which $q_t = 0$ for all $t \in \{1, 2, \cdots, T\}$.

3 The model of bubbles with cognitive investments

We add the possibility of *cognitive investment* (C-investment) as an option for each agent of groups $T-1$ and T into the baseline model. We assume the following for the structure of the model:

- All agents of groups t ($t = 0, 1, 2, \cdots, T-1, T$) are born before the beginning of period 1.
- Group t ($t = 1, 2, \cdots, T-2$) waits for their entry until period t.
- Group $T-1$ and group T wait for their entry until period $T-1$, while their entry period is period $T-1$ with probability $\frac{1}{2}$ and period T with probability $\frac{1}{2}$.
- Individual agents of group $T-1$ and group T can make the C-investment in period $t \in \{1, 2, \cdots, T-2\}$, which is described in the following assumption.

Assumption 4. Each individual agent of group j ($j = T-1,\ T$) can learn her entry period (either period $T-1$ or T) by making a C-investment in period t, where $t \in \{1, 2, \cdots, T-2\}$. The C-investment is not publicly observable, but the measure of the agents who make the C-investment is publicly observable. So if the measure λ_t of agents in group j make the C-investment in period t, then all agents learn that the measure λ_t of agents in group j know their entry period beforehand. Meanwhile, when an agent who made the C-investment enters the market in period $T-1$ or T, her trading counterpart cannot know whether this agent made the C-investment or not. The cost of the C-investment in period t for each agent of group $T-1$ or T is identical value, $\tilde{\omega}_t$, which is a random variable that follows a time-invariant normal distribution:

$$\ln \tilde{\omega}_t \sim N(\mu, \sigma).$$

In period t, the agents of groups $T-1$ and T decide whether or not to make the C-investment, after observing the realization of $\tilde{\omega}_t$.

3.1 Cognitive investment makes the bubble burst

Suppose all members of group $T-1$ or group T made the C-investment in period t, where $t \leq T-2$. Then, the bubble collapses in period t.

Proposition 1. *Suppose that all agents of either group $T-1$ or group T make the C-investment in period t $(\leq T-2)$. Then, the bubble bursts in period t, i.e., $q_{t+s} = 0$ for $s = 0, 1, \cdots, T-t$.*

Proof. It is sufficient to examine the cases where both group $T-1$ and group T make C-investment, and where group $T-1$ makes C-investment and group T does not.

- **Case where both groups invest:** Since the measure of agents who made the C-investment is public information, all agents know that all agents know their entry period. In this environment, it is obvious that no one accepts the bubbly asset as payment in period T, because the recipient of the asset has no chance to sell it to anyone. Backward induction implies that the asset is not accepted in periods $t+s$, where $s = 0, 1, 2, \cdots, T-t$. Thus, the bubble collapses in period t, when both group $T-1$ and group T make the C-investment.

- **Case where only one group invests:** Consider the case where only group $T-1$ makes C-investment and group T does not. Suppose that the trade continues until period $T-1$. There are two possibilities: Group T who did not make C-investment enters period $T-1$, or group T enters period T.

  - **Case 1: Group T enters in period $T-1$**
    In this case, group $T-1$ enters in period T and they know that their entry period is the last period, T. So group $T-1$ never accepts the asset as payment in period T. From this reasoning, each agent of group T knows that accepting the asset as payment in her entry period (period $T-1$) is not optimal for her, because she will get nothing in exchange for the asset in the next period (i.e., period T).

  - **Case 2: Group T enters in period T**
    In this case, it is obvious that accepting the asset as payment in the current period (period T) is not optimal because she has no chance to sell it as the economy ends in period T.

  The above arguments demonstrate that it is not optimal for group T to accept the asset, even though they do not know whether the current period is $T-1$ or T. This result arises from the fact that group T knows that group $T-1$ knows the current period is $T-1$ or T, as group $T-1$ made C-investment in period $t \leq T-2$. Thus, group T will rationally refuse to accept the asset in their entry period, no matter whether it is period $T-1$ or T. Then group $T-1$ also never accepts the asset in their entry period. Then the usual backward induction from period $T-2$ to t implies that agents of groups $t+s$ never accept the asset in period $t+s$ for $s \in \{0, 1, 2, \cdots, T-t\}$. Therefore, the bubble collapses in period t, when all agents of group $T-1$ make the C-investment.

We have proven that the bubble collapses in period $t \in \{1, 2, \cdots, T-2\}$, if either group $T-1$ or group T makes the C-investment in period t. □

3.2 Equilibrium

In this subsection, we describe the equilibrium dynamics with the C-investment. With the option of C-investment, there emerges the possibility of endogenous bursting of the bubble in the bubbly equilibrium.

Bubble dynamics for $t \leq T-2$: Suppose, for $t \leq T-2$, that η_t is the probability that the bubble collapses in t conditional on that the bubble continued until $t-1$. The value of η_t is the equilibrium outcome, which is later given by (8) and (9). We will show in Proposition 2 that the equilibrium shifts to the non-bubbly equilibrium when a certain condition about the optimality of the C-investment, which will be specified later, is satisfied, even though no one makes the C-investment. We make the following assumption about $\{\eta_t\}_{t=1}^{T-2}$.

Assumption 5. In period $t \in \{1, 2, \cdots, T-2\}$, agents of group $T-1$ and T have rational expectations $\{q_{t+s}, \eta_{t+s}\}_{s=1}^{T-2-t}$. However, in their entry period, they forget all relevant information from which they can infer whether their entry period is $T-1$ or T. They have the expectation that the price in period T is q_T, though they do not know whether their entry period is period $T-1$ or T.

The decision problem is similar to that in Section 2. In period t, agents of group $t-1$ who are the consumers have the full bargaining power and they maximize their consumption in period t, given the expectations on q_{t+1} and η_{t+1}, subject to the participation constraint for the producers, i.e., agents of group t:

$$\max_{q_t} q_t,$$
$$\text{s.t.} \quad V_t \equiv -cq_t + \beta(1-\eta_{t+1})q_{t+1} \geq 0.$$

Note that this problem is simplified by imposing the condition that the quantity of the asset is always unity: $k_t = 1$. The binding participation constraint for agents of group t, where $t \leq T-2$, is $V_t = 0$, which implies that, for $t \leq T-2$,

$$q_{t+1} = \frac{c}{\beta(1-\eta_{t+1})} q_t. \tag{3}$$

We assume that the expectation $q_T = q^*$ is given exogenously. Since the C-investment does not take place in period $T-1$ or T, the equilibrium is the same as that in Section 2 for periods $T-1$ and T. Therefore, as Lemma 1 shows, $q_T = q_{T-1} = q^*$ and $q_{T-1} = \frac{c}{\beta} q_{T-2}$. The above equation, then, implies the following:

$$q^* = q_T = q_{T-1} = \frac{c}{\beta} q_{T-2} = \frac{c^{T-t-1}}{\beta^{T-t-1}[\prod_{s=t+1}^{T-2}(1-\eta_s)]} q_t, \qquad \text{for} \quad t \leq T-3. \tag{4}$$

Note that $\eta_{T-1} = 0$. This equation defines q_t backwardly from q_T. In period $t \in \{1, 2, \cdots, T-2\}$, the bubble collapses with probability η_t in which case, $q_{t+s} = 0$ for $s = 0, 1, 2, \cdots, T-t$. The probability η_t is also backwardly defined later by (8) and (9).

Determination of probability η_t: We assume that the C-investment can take place only in period $t \leq T-2$, and agents of groups $T-1$ and T have no opportunity to make the C-investment in periods $T-1$ and T. We define $\Lambda \in [0, 1]$ as the total measure of agents in group $T-1$, who make the C-investment during periods $1, 2, \cdots, T-2$. We focus on the symmetric equilibrium where the measure of agents who make the C-investment is also Λ in group T.[3]

Proposition 2. *Suppose that the following inequality is satisfied in period $t \in \{1, 2, \cdots, T-2\}$:*

$$\tilde{\omega}_t < \beta^{T-t-1}[\prod_{s=t+1}^{T-2}(1-\eta_s)]\frac{1}{4}q^*. \tag{5}$$

Then, the variable Λ is nonexistent, meaning that the bubbly equilibrium ceases to exist from period t on, and the equilibrium degenerates to the non-bubbly equilibrium.

Proof. We consider the dynamics in the bubbly equilibrium backward. Note that q_T must be positive (i.e., $q_T > 0$) in the bubbly equilibrium.[4] First, we consider the equilibrium in period T, in which the entrants who made the C-investment never purchase the bubbly asset. They never purchase the bubbly asset because they know that the current period is the last period. Then, the consumers or incumbents in period T maximize their consumption q subject to the participation constraints of the entrants who did not make the

[3] Our results do not change even if we allow groups $T-1$ and T to have different values of the measure of those who made the C-investment.

[4] It can be shown as follows. Suppose $q_T = 0$. In the rational expectations equilibrium, no one accept to buy the bubbly asset in period $T-1$ as it becomes worthless next period. It is true even though group $T-1$ and group T do not know whether their entry periods are either $T-1$ or T. The bubbly asset is worthless today if today is T, and it will become worthless tomorrow if today is $T-1$. Thus $q_{T-1} = 0$ must hold. Then, the backward induction implies there is no bubble from the beginning. We have proven that if $q_T = 0$ then $q_t = 0$ for all t. Thus, it must be the case that $q_T > 0$ in the bubbly equilibrium, where $\exists t,\ q_t > 0$.

C-investment. Their problem is written as follows.

$$\max_{q} q,$$
$$\text{s.t. } V_N \equiv \frac{1}{2}(-\frac{1}{2}q + (1-\Lambda)q_T) + \frac{1}{2}(-\frac{1}{2}q + 0) \geq 0,$$

where V_N is the expected value for the entrants in period T who did not make the C-investment. The first term of V_N is the payoff from entering $T-1$, which reflects the entering agent's expectation that the current period is period $T-1$ with probability $\frac{1}{2}$ and period T with probability $\frac{1}{2}$. If the current period is $T-1$, the entrant pays $\frac{1}{2}q$ to produce q to buy the asset and she can sell it with probability $1-\Lambda$ at price q_T in the next period. Note that the consumer in period T receives q_T with probability $1-\Lambda$ and 0 with probability Λ because her trading counterpart is randomly assigned from the unit mass of the entrants. Note that the incumbent in period T knows which group enters in period T and thus knows Λ of her counterpart. Thus, the first term is $\frac{1}{2}(-\frac{1}{2}q+(1-\Lambda)q_T)$. The second term is the payoff in the case where the current period is T. In this case, the entrant pays $\frac{1}{2}q$ to buy the asset and gets nothing in the next period because there is no next period. The binding participation constraint $V_N = 0$ decides q by

$$q = (1-\Lambda)q_T. \tag{6}$$

Since we are analyzing the decision making in period T, it must be the case in the rational expectations equilibrium that

$$q = q_T. \tag{7}$$

These two equations imply that

$$\Lambda = 0$$

in equilibrium. Now, we will show by contradiction that Λ cannot be zero, if (5) is satisfied. Suppose $\Lambda = 0$. Then, the expected present value of an agent of group $T-1$ or T who did not make the C-investment is as follows, given that the expectation is taken in period t (before the entries of group $T-1$ and T take place).

$$EPV_N = \beta^{T-t-1}[\prod_{s=t+1}^{T-2}(1-\eta_s)]\left\{\frac{1}{2}(-\frac{1}{2}q + q_T) + \frac{1}{2}(-\frac{1}{2}q + 0)\right\},$$

because the value in period $T-1$ or T is discounted by β^{T-t-1} and the probability that the bubble survives from period t to period $T-1$ is $\prod_{s=t+1}^{T-2}(1-\eta_s)$. Since the incumbents in periods $T-1$ and T solve the same problem as in Section 2 because $\Lambda = 0$, we have $q = q_T = q^*$. The expected present value of an agent of group $T-1$ or T who made the C-investment in period t is as follows, given that the measure $\Lambda = 0$ of other agents make the C-investment and that the expectation is taken in period t *before* the C-investment is made.

$$EPV_I = \beta^{T-t-1}[\prod_{s=t+1}^{T-2}(1-\eta_s)]\left\{\frac{1}{2}(-\frac{1}{2}q+q_T)+\frac{1}{2}(0+0)\right\}.$$

The last term $(0+0)$ in EPV_I means that if the agent who made the C-investment enters the market in period T, she can avoid to expend $\frac{1}{2}q$ because she knew in advance that she will enter in the last period. We multiply the last term $(0+0)$ by probability $\frac{1}{2}$ because the expectation is taken before the agent conducts the C-investment. As we explain below, the agent decides whether to make the C-investment by comparing the values of EPV_N and EPV_I. The gain from making the C-investment, which is evaluated in period t before the C-investment is made, is $EPV_I - EPV_N$:

$$EPV_I - EPV_N = \beta^{T-t-1}[\prod_{s=t+1}^{T-2}(1-\eta_s)]\frac{1}{4}q^*,$$

given that $q = q^*$. Now, suppose that $\tilde{\omega}_t$ is realized in period t and condition (5) is satisfied. Condition (5) means that the gain from making the C-investment is strictly bigger than the cost of it. This condition means that making the C-investment is optimal for all agents of groups $T-1$ and T. As all of them make the C-investment in period t, it must be the case that

$$\Lambda = 1,$$

which contradicts our assumption that $\Lambda = 0$. We have shown a contradiction that if $\Lambda = 0$, the decision-making by the agents implies $\Lambda = 1$ in equilibrium, given that (5) is satisfied.

We can also show the following contradiction if we assume $\Lambda = 1$. Suppose that $\Lambda = 1$ in equilibrium. In this case, Proposition 1 implies that $q_{T-1} = q_T = 0$. Then, C-investment is worthless because there is no gain from knowing the entry period. For any t and $\tilde{\omega}_t > 0$, the cost is strictly bigger than the benefit of the C-investment, which is zero. Thus, it must be the case that $\Lambda = 0$, which contradicts the assumption that $\Lambda = 1$.

We have proven that the equilibrium value of Λ is nonexistent if condition (5) is satisfied. In other words, we have shown that the bubbly equilibrium ceases to exist if condition (5) is satisfied. The only equilibrium that can exist is the non-bubbly one where $q_{t+s} = 0$ for all $s = 0, 1, \cdots, T-t$. □

This proposition directly implies the following lemma.

Lemma 2. The probability η_t is backwardly defined as follows, from period $T-2$ to period 1:

$$\eta_{T-2} = \Pr\left[\tilde{\omega} < \beta\frac{1}{4}q^*\right], \quad \text{and} \tag{8}$$

$$\eta_t = \Pr\left[\tilde{\omega} < \beta^{T-t-1}[\prod_{s=t+1}^{T-2}(1-\eta_s)]\frac{1}{4}q^*\right], \qquad \text{for } t \leq T-3. \tag{9}$$

The proof is obvious, because η_t is the probability that condition (5) is satisfied.

Nature of the bubble dynamics: Since the distribution of $\tilde{\omega}_t$ is stationary, the above equations, (8) and (9), imply that η_t increases in t. Since η_t increases in t, the bubble q_t grows at an accelerated rate, as shown in (3), i.e., $\frac{q_{t+1}}{q_t} = \frac{c}{\beta(1-\eta_{t+1})}$ is increasing in t for $t = 1, 2, \cdots, T-2$.

4 Conclusion

In the episodes of asset-price bubbles, we often observe that the bubbles grow acceleratedly and they collapse with higher probability as they grow higher. We reproduce these features from the investors' optimal decisions on costly information acquisition.

Given that the bubbles emerge due to the incomplete information, they collapse when the investors restore the complete information by costly information acquisition that we call the cognitive investment. In this note, we have shown that integrating cognitive investment into the analysis of bubbly dynamics could offer deeper insights into asset price dynamics. We leave further exploration of this avenue for future research.

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