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Abstract

To explain the observed stability in real GDP growth, existing endogenous growth theories propose models in which the quantity, quality or variety of the final output increases exponentially in the long run. However, such exponential increases typically require a knife-edge degree of externality, which is not supported by micro-level observations. This paper presents a new theory of long-term growth in which a constant number of new goods are introduced per unit of time and focuses on the movement of prices and quantities after introduction. We show that if the quality-adjusted prices and quantities of individual goods follow a typical pattern of the product lifecycle, then the long-term rate of real GDP growth, as measured by SNA statistics, becomes positive without exponential growth in the quantity, quality or variety of final outputs. We develop a prototype model and its extensions, showing that the conditions for positive real GDP growth are less restrictive than typical knife-edge assumptions. We also demonstrate that the long-term real GDP growth rate in the non-exponential model is closely related to the rate of increase in the money-metric utility.

Keywords: endogenous growth theory, balanced growth, knife-edge condition, product lifecycle, money-metric utility.

JEL Classification Codes: O41, O31.

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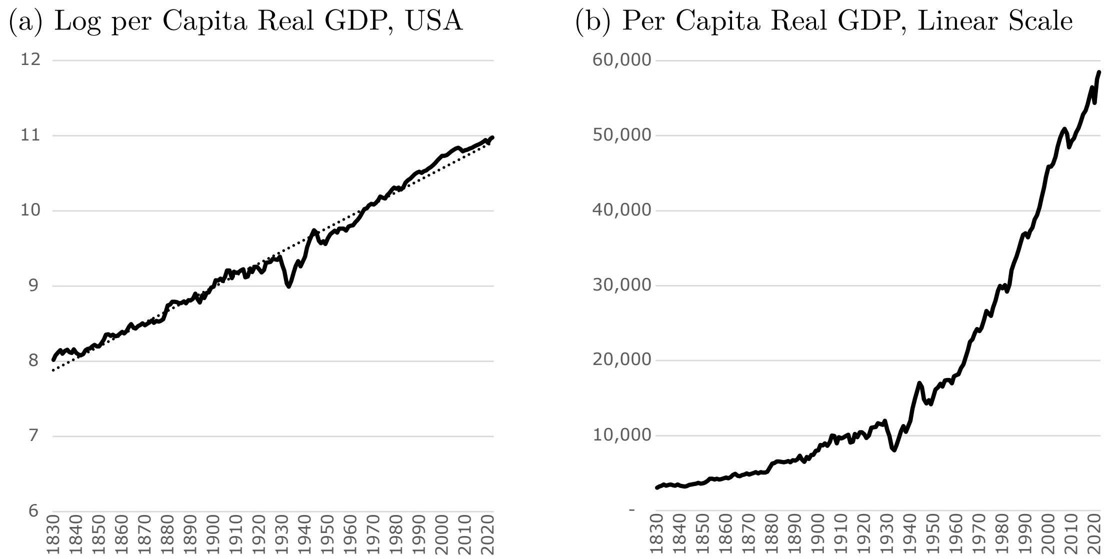


Figure 1: Long-term Evolution of Real GDP per Capita in the United States Since 1830 (2011 International Dollar). Source: Madison Project, Bolt and van Zanden (2025).

1 Introduction

Since around the time the First Industrial Revolution was completed, the growth in real GDP per capita in the United States has been remarkably stable. Figure 1(a) depicts the time series of the real GDP on a log scale, where the slope of the series represents the growth rate. Although there have been short- to midterm fluctuations, the figure clearly shows that the log of the real GDP per capita closely follows a linear trend, implying that the long-term rate of per capita GDP growth is almost constant. Figure 1(b) shows the time path of the U.S. real GDP per capita on a linear scale without taking the logarithm. Given that the GDP growth rate is stable, it is well known that the level of real GDP per capita is increasing exponentially in the long run.

Given these findings, it is natural for existing studies on endogenous growth to explain the phenomenon of long-term growth via models in which the per capita output continues to grow exponentially. Initially, this was an extremely challenging task because reproducible inputs are subject to diminishing returns, which implies that the accumulation of those factors alone cannot explain the exponential growth. Seminal studies in endogenous growth theory thus overcame this challenge by assuming the presence of strong intertemporal knowledge spillovers.

In variety-expansion models (e.g. Romer, 1990; Grossman and Helpman, 1991a), the productivity of new R&D is assumed to increase as knowledge accumulates with the past stock of R&D. To sustain economic growth, the elasticity of this spillover ϕ needs to equal one. Similarly, in quality ladder models (e.g. Grossman and Helpman, 1991b; Aghion and Howitt, 1992), the increment in quality due to successful new R&D depends on the quality of the existing good, which is a result of the past stock of R&D. Sustained growth requires the increments to be proportional to the existing quality, which means that the elasticity of the externality should again be one. Finally, in AK-type growth models (e.g. Romer, 1986; Rebelo, 1991), the elasticity of production with respect to all reproducible factors and the elasticity of their externality effects must add up to one.[1] In almost all endogenous growth models, long-term growth can be sustained only when one such knife-edge condition is satisfied.[2]

Nevertheless, a puzzle remains. Indeed, the externality and nonrivalry of knowledge play essential roles in improving productivity (e.g. Griliches, 1998). However, if we look at the spillover process more precisely, no concrete evidence supports any of these assumptions. Klenow and Rodriguez-Clare (2005, Section 3) reviewed various AK-type models. They concluded that such models are empirically implausible because of the

[1]When there are multiple sectors, at least one sector that produces a reproducible factor (typically physical capital or human capital) must satisfy this restriction. For example, Lucas (1988) initially introduced a human capital accumulation function $\dot{h}_t = h_t^{\phi} G(1 - u_t)$ and then made the assumption $\phi = 1$, following Uzawa (1965). After doing so, he wrote, "the feature that recommends his formulation to us is that it exhibits sustained per capita income growth," which gives a clear example of a case where such a knife-edge assumption is justified not by microlevel observations but rather by the aggregate outcome. Lucas noted that "human capital accumulation is a *social* activity," which suggests that the elasticity $\phi = 1$ includes the effect of externalities.

[2]Growiec (2007, 2010) formally proved that, with any generalization in functional form, exponential growth cannot be explained without imposing at least one knife-edge assumption in the model. An exception is Peretto (2018), who showed that sustained growth can be obtained when $\phi \geq 1$ in the Schumpeterian growth model if the excessive portion of ϕ (i.e., $\phi - 1$) is diluted by the proliferation of products.

lack of a tight enough relationship between investment rates and growth rates in cross-country data. For the elasticity of spillover ϕ in R&D-driven growth models, Jones (1995) clearly stated, *"$\phi = 1$ represents a completely arbitrary degree of increasing returns and... is inconsistent with a broad range of time series data on R&D and TFP growth."* He convincingly stated that $\phi = 0$ is the most natural case, and while ϕ can either be negative by the "fishing out effect" or positive by the "better tools effect," it is reasonable to assume that $\phi < 1$. Bloom et al. (2020) estimated the degree of diminishing returns $(1 - \phi)$ in research productivity in various industries and reported that ϕ is significantly less than one (even negative) for almost all industries. They concluded that improving the quality of goods at a constant exponential rate is becoming more difficult.

A possible answer to this puzzle is *semi-endogenous* growth theory with $\phi \in (0, 1)$, where the long-term rate of growth is ultimately driven by population growth. However, Jones (2022) predicted that economic growth will eventually come to an end, given that there are upper limits on population, research intensity, and education attainment. This paper presents an alternative possibility, i.e., that the measured economic growth can continue indefinitely with a constant population under the natural assumption of $\phi = 0$.

Overview of the mechanism

This paper presents a theory that explains the stability of the observed real GDP growth rate by considering the vintages of products and their product lifecycle. In this setting, we will show that the measured GDP growth rate becomes positive under more agreeable conditions than a knife-edge level of externality, as assumed in existing endogenous growth models.

Recall that we first presented the (log) level of GDP in Figure 1, and then discussed real GDP growth. However, in the System of National Accounts (SNA) statistics (the NIPA in the U.S.), the GDP data are constructed in reverse order. Statistical agencies first calculate the real GDP growth rate by comparing the quantities of various product groups in adjacent years, using the same set of prices for both years. Then, they construct the aggregate level of real GDP via the chain rule:

$$\Big[\text{Real GDP at year } T\Big] = \Big[\text{Real GDP at reference year } t_0\Big] \times \prod_{t=t_0+1}^{T} (1 + g_{t,t-1}),$$

where $g_{t,t-1}$ is the measured real GDP growth rate between year t and year $t-1$.[3] Therefore, the fact that the time series of measured per capita real GDP exhibits exponential growth only means that the series of $g_{t,t-1}$, from which the real GDP is calculated, is positive and stationary. Because the composition of final goods differs across time, it is not evident whether the stationarity in the $g_{t,t-1}$ series implies exponential growth in the quantity or quality of any particular final good. In Appendices A.1 and A.2, we provide two simple examples in which consumer expenditure gradually shifts to newer final goods. In both examples, the $g_{t,t-1}$ series is sustained at a positive constant level even though the quantity or quality of no particular good grows exponentially.

Given that there is no need to explain the exponential increase in any good, less restrictive assumptions are sufficient to explain the fact that the measured real GDP has been growing steadily. To replicate the environment where the real GDP growth rate is calculated by statistical agencies, we consider a stylized model in which new final goods are gradually introduced and explicitly focus on their prices and quantities over their lifecycle. In this multiproduct setting, we show that the measured GDP growth rate becomes a positive constant when the following is true: (i) new goods (or services) are continually introduced to the market; (ii) the quality-adjusted prices of each good decrease as they become older compared to newer goods; and (iii) the expenditure share for very old goods is limited. Condition (i) does not require the number of goods to increase exponentially. Conditions (ii) and (iii) state that the price and quantity for each good should follow the well-observed pattern of the product lifecycle.[4] This type of economic movement does not require a knife-edge level of externality. This contrasts

[3] The real GDP in reference year t_0 can be set arbitrarily because this is simply an index. An often-used method is to set it to the nominal GDP at time t_0.

[4] Appendix A.3 shows that the two examples in Appendices A.1 and A.2 satisfy the three conditions. In addition, we explain how these two simple examples connect to the general equilibrium models in Sections 3 and 4.

with existing endogenous growth models, which require some variables to grow exponentially. Nevertheless, knowledge externalities are crucial for growth, as they often work behind the product lifecycle, which includes quality improvements and cost reductions. Our prototype endogenous model incorporates these, but the decline in quality-adjusted prices does not need to occur at an exponential speed. As a result, a weaker externality is sufficient for sustaining measured real GDP growth.

Our results may still seem paradoxical. Although the output, in terms of quantity, quality, or variety, does not increase exponentially, the measured real GDP is increasing exponentially. Do the real GDP statistics overestimate the actual growth? Not necessarily. Like in recent studies (Baqaee and Burstein, 2023; Jaravel and Lashkari, 2024), the measured real GDP in our model is closely related to the money-metric utility, which evaluates the change in the utility of consumers using the equivalent variation. In the prototype general equilibrium model, we show that the money-metric utility can increase exponentially or even faster depending on how consumers value the arrival of new goods. In particular, if there is an upper bound on the utility obtainable from existing goods (measured in the unit of the utility function), the benefits of being able to buy new goods can surpass the benefits of having an exponentially larger budget.[5] In fact, the measured real GDP tends to underestimate the growth of money-metric utility because it captures only a part of the utility gains related to the introduction of new goods.

Some recent studies view long-term growth differently than an exponential increase in final output at the rate of measured GDP growth. León-Ledesma and Moro (2020) considered a two-sector model and calculated the growth rate via the methodology employed by the NIPA. They showed that the shift in the expenditure share from goods to services explains cross-country growth. In this paper, we propose that continual shifts in expenditure shares from old goods and services to new goods and services are behind the

[5]For example, let us consider the change in money-metric utility from 200 years ago to the present. At the beginning of the 19th century, most industrial goods were absent; thus, it was impossible to live a life as convenient as that today, however rich one was. Effective medical services or drugs were almost nonexistent, and as a result, mortality was high, even among wealthy people. Would you prefer to live 200 years ago if you were given an arbitrarily larger budget at that time? If the answer is no, the change in the money-metric utility over 200 years is infinity. See Section 3.8 for a formal analysis.

stability in measured GDP growth. Aghion et al. (2019) examined the possibility that the measured GDP growth rate underestimates the welfare gains from creative destruction, which is consistent with our findings. A notable difference is that we do not require an exponential increase in the quality of individual products because the expenditure shifts to the newer variety of goods. Philippon (2022) suggested that a linear trend fits the TFP data better than an exponential trend for periods ranging from several decades to a few centuries. According to his theory, long-term GDP growth can be sustained only when there are occasional changes in the linear trend (e.g., by the arrival of general-purpose technologies), and the slope of the linear trend needs to increase exponentially. In this paper, we explore a mechanism that does not require exponential increases or a knife-edge degree of externalities, even in the very long term.

The rest of the paper is constructed as follows. Section 2 presents a stylized but fairly general theory that provides the conditions under which measured real GDP growth can be sustained in a setting without exponential expansion. On the basis of this theory, Section 3 develops a prototype R&D-based endogenous growth model. Without requiring knife-edge conditions, the model shows that innovation continues and that the measured GDP growth remains positive. Section 4 introduces the obsolescence of goods, and Section 5 considers multiple sectors. In these two sections, we generalize the theory and the prototype model to demonstrate that we can obtain a positive constant real GDP growth rate in wider (even less restrictive) situations. Section 6 concludes the paper.

2 Theory

In this section, we theoretically derive the condition under which the real GDP growth rate, as measured by the SNA, can be sustained. In a setting where new goods are continually introduced but not at an exponential speed, we show that the sustainability of measured GDP growth depends on the pattern of changes in prices and quantities in the product lifecycle. The results suggest various possibilities for constructing general equilibrium models in which measured GDP growth can be sustained under less restrictive assumptions than those found in typical endogenous growth models. A simple prototype model is presented in Section 3.

2.1 Measuring GDP Growth with Vintages of Goods

Let us consider an economy with a constant population and many goods. While we follow a convention in the variety expansion model by calling them goods, it is more suitable to think of each good in theory as a group of products or services based on the same technology. Each good is indexed by $i \in [0, N_t]$, where $i = 0$ is the oldest, and $i = N_t$ is the most recently introduced good. The number of goods N_t increases whenever new goods are introduced.[6]

Let $\widetilde{p}_t(i)$ and $\widetilde{x}_t(i)$ denote the price and quantity, respectively, of each good i at time t. We normalize the price level and the quantity unit of each good so that the price and quantity of the newest good are unchanged over time. As in SNA statistics, we define $\widetilde{p}_t(i)$ and $\widetilde{x}_t(i)$ as quality-adjusted values. For example, if the quality of good i is doubled (so that consumers receive the same utility from half the quantity), then our measure of $\widetilde{x}_t(i)$ is doubled, whereas that of $\widetilde{p}_t(i)$ is halved.

In this stylized environment, we follow the method of the SNA statistics to calculate the real GDP growth rate. This can be done by comparing the values of all final outputs between two consecutive years, e.g., year $t-1$ and year t. Their values are measured via the common set of prices, which is usually the set of observed prices in a given base year. Because the base year is frequently updated in official statistics and because we are interested in long-term dynamics, we suppose that there is no gap between the base year and the year in which the growth rate is computed.[7] Then, the real GDP growth rate between years $t-1$ and t can be written as follows:

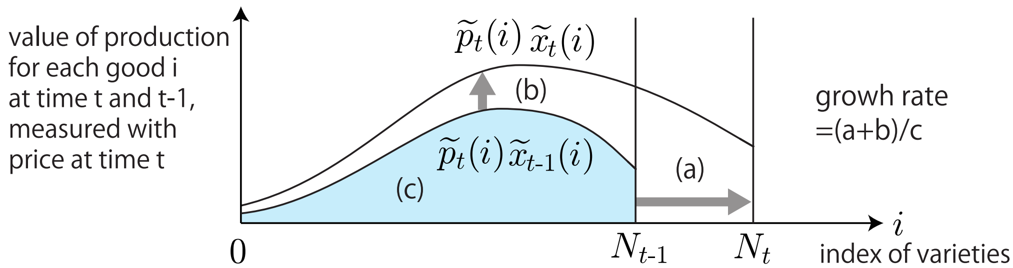
$$g_{t,t-1} = \frac{\int_{N_{t-1}}^{N_t} \widetilde{p}_t(i)\widetilde{x}_t(i)di + \int_0^{N_{t-1}} \widetilde{p}_t(i)\left(\widetilde{x}_t(i) - \widetilde{x}_{t-1}(i)\right)di}{\int_0^{N_{t-1}} \widetilde{p}_t(i)\widetilde{x}_{t-1}(i)di}. \tag{1}$$

This equation is composed of the integrals of two functions: $\widetilde{p}_t(i)\widetilde{x}_t(i)$ and $\widetilde{p}_t(i)\widetilde{x}_{t-1}(i)$. Figure 2 depicts the curves of these two functions against the index of varieties i for two

[6] N_t includes the number of goods that are no longer produced.

[7] In the U.S., the NIPA computes the growth rate in two ways, i.e., by setting the base year to t and by setting it to $t-1$. Then, the agency calculates the geometric average of the two values. For ease of exposition, here, we show only the growth rate in which the base year is t. In appendix B, we explain the calculation of the real GDP growth rate when the base year is $t-1$. The difference between the two cases disappears at the limit where the period length approaches 0, as we consider in the next subsection.

Case 1: When $\widetilde{x}_t(i)$ is always increasing in t.



Case 2: When $\widetilde{x}_t(i)$ decreases with t sometime after introduction.

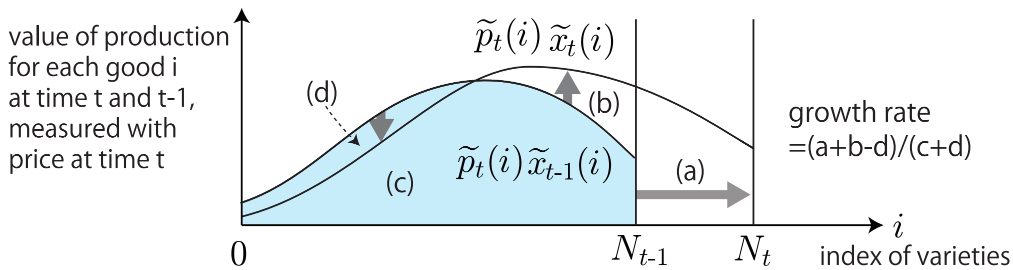


Figure 2: Calculation of the Real GDP Growth Rate: Two Cases.

cases, i.e., when the quantity of existing goods always increases with time (Case 1) and when the quantity of existing goods decreases in some part of their lifecycle (Case 2). In Case 1, area (a) represents the sum of the values of new goods introduced between time $t-1$ and time t, evaluated by the prices at time t. Similarly, area (b) represents the value of the increased production of goods that already existed at time $t-1$. These two areas measure how economic activity has increased from time $t-1$ to time t and correspond to the two terms in the numerator of Equation (1). Area (c) represents the value of total production at time $t-1$, evaluated again by the prices at time t. This area corresponds to the denominator of Equation (1). In this way, the real GDP growth rate can be understood as the ratio of area (a)+(b) to area (c), which measures the rate at which the economic activity at time t increases from time $t-1$.

This procedure can be generalized to the case where the output quantity $\widetilde{x}_t(i)$ is not monotonic in t. Case 2 in Figure 2 illustrates an example where the production of a certain range of goods declines between periods $t-1$ and t. A portion of curve $\widetilde{p}_t(i)\widetilde{x}_t(i)$

then falls below curve $\widetilde{p}_t(i)\widetilde{x}_{t-1}(i)$. In this case, the real GDP growth rate is given by the ratio of area (a)+(b)−(d) to area (c)+(d).

2.2 Non-Exponential Steady State with a Product Lifecycle

The fact that the measured U.S. real GDP growth rate has been stable for almost two centuries suggests that N_t, $\widetilde{p}_t(i)$, and $\widetilde{x}_t(i)$ in Equation (1) may have some steady-state properties in the long run. This subsection presents a simple notion of a steady state in the environment explained thus far. In particular, we focus on the steady-state dynamics where neither variety, quantity, nor quality expands exponentially. For ease of analysis, we describe the economy in continuous time throughout the rest of the paper.

Suppose that, in the long run, N_t increases by a positive constant n per unit of time as follows:

$$\dot{N}_t \to n > 0 \quad \text{as} \quad t \to \infty. \tag{2}$$

Recall that existing variety expansion models require a strong and exact degree of knowledge spillover to maintain the exponential expansion of varieties, where $\dot{N}_t/N_t$ is constant. In contrast, the linear increase in N_t in Equation (2) does not require such strong knowledge spillovers within the R&D sector, as we will see in the general equilibrium model in Section 3.

Let $s(i)$ denote the time when good i is developed. It is convenient to label each good by its age, $\tau = t - s(i)$, i.e., the time passed from its introduction. Given that n new goods are introduced per unit of time, an age τ good is the $n\tau$th newest good. This means that the index of a good i and its age τ are related by the following:

$$i = N_t - n\tau, \text{ or equivalently, } \tau \equiv t - s(i) = \frac{N_t - i}{n}. \tag{3}$$

With this notation, let us say that the economy has reached a steady state if every good's price and quantity follow the same time evolution with respect to τ. Formally, the economy can be said to be converging to a steady state if time-invariant functions $p(\tau)$ and $x(\tau)$ exist such that

$$\widetilde{p}_t(i) \to p(t - s(i)) \equiv p(\tau), \quad \widetilde{x}_t(i) \to x(t - s(i)) \equiv x(\tau) \quad \text{as} \quad t \to \infty. \tag{4}$$

Let $T > 0$ denote the age beyond which the product is never produced. In typical variety-expansion endogenous growth models, goods never retire from the market. In this case, $T = \infty$. However, in practice, many products disappear after some time. Our theory can be applied to both cases, where T is finite or infinite. We assume that $p(\tau)$ and $x(\tau)$ satisfy the following properties:

Assumption 1.
(i) Both $p(\tau)$ and $x(\tau)$ are nonnegative and continuous for all $0 \leq \tau \leq T$, where T is such that $x(\tau) = 0$ for all $\tau > T$. Additionally, they are differentiable for all $0 < \tau < T$.
(ii) T can be infinite, but $p(\tau)$ and $x(\tau)$ do not increase exponentially: $\lim_{\tau\to\infty} p'(\tau)/p(\tau) \leq 0$ and $\lim_{\tau\to\infty} x'(\tau)/x(\tau) \leq 0$ if $T = \infty$.[8]
(iii) The newest good's price and quantity are both positive: $p(0) > 0$ and $x(0) > 0$.[9]

With Assumption 1(i), the present paper focuses on the continuous setting because it is mathematically less demanding and does not sacrifice intuitions. Since $x(\tau)$ represents the quality-adjusted quantity, Assumption 1(ii), combined with Equation (2), guarantees that neither quantity, quality, nor variety grows exponentially in this economy. Assumption 1(iii) is an obvious assumption. When a new good appears in the market, it should imply that the expenditure for the good, $p(0)x(0)$, is positive.

Definition 1. *A non-exponential asymptotic steady state is a situation in which the number of goods follows Equation (2), while the paths of quality-adjusted prices and quantities of goods, i.e., $\widetilde{p}_t(i)$ and $\widetilde{x}_t(i)$, respectively, satisfy Condition (4) and Assumption 1.*

In the remainder of the paper, we use the term "steady state" unless doing so leads to confusion. Figure 3 intuitively depicts the evolution of the quality-adjusted prices and quantities in the above definition of the steady state. The graphs can be viewed in two ways, i.e., drawn against the i-axis (index of goods) running from left to right or drawn against the τ-axis (age of goods) running in the opposite direction. The two variables,

[8] Note that the time derivative of the quantity in the steady state is $\dot{\widetilde{x}}_t(i) = \frac{d}{dt}x(t-s(i)) = x'(t-s(i)) = x'(\tau)$. Therefore, $x'(\tau)/x(\tau) = \dot{\widetilde{x}}_t(i)/\widetilde{x}_t(i)$ represents the growth rate of the quantity of age τ good, or equivalently, that of index $i = N_t - n\tau$ good. Similarly, $p'(\tau)/p(\tau) = \dot{\widetilde{p}}_t(i)/\widetilde{p}_t(i)$ in the steady state.

[9] In this paper, we use the term "positive" to mean greater than (not including) zero.

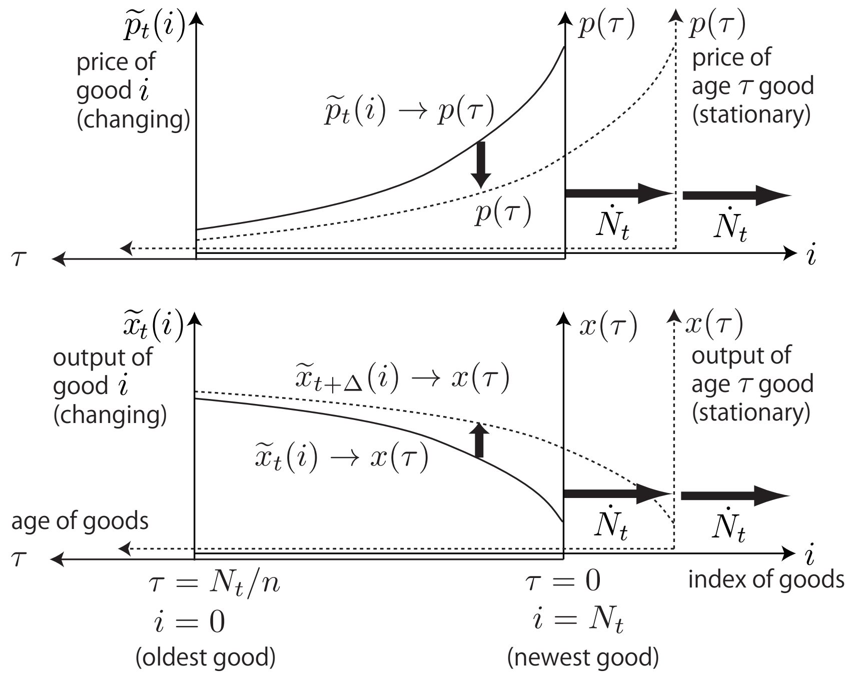


Figure 3: Evolution of Prices and Quantities in a Non-Exponential Steady State.

i and τ, are related according to Equation (3); however, the relationship changes over time as N_t increases. At time t, the origin of the τ-axis coincides with the point of $i = N_t$ on the i-axis because the newest good $i = N_t$ is age $\tau = 0$ at time t. Over time, the origin of the τ-axis moves to the right with the speed of the introduction of new goods, $\dot{N}_t = n$, as does the position of the graph drawn against τ.

The upper panel of Figure 3 illustrates the schedule of quality-adjusted price $p(\tau)$, assuming that it decreases with age τ because a product either becomes cheaper or becomes higher quality over time after its introduction. Then, $\widetilde{p}_t(i)$ is increasing in i at any given time t since the newer goods have a larger index i. The figure also explains the movement of the price of each good $\widetilde{p}_t(i)$ over time. Even in the steady state where function $p(\tau)$ is stationary, the price of individual good $\widetilde{p}_t(i)$ shifts downward to the dotted curve because the position of function $p(\tau)$ continues to move to the right as new goods are developed.[10]

[10]Although this is a convenient way to explain the steady-state dynamics, note that the economic environment, such as technology, preference, and market structure, first determines the evolution of the

The lower panel of Figure 3 explains the evolution of quality-adjusted quantities of goods over time. The panel is drawn under the assumption that $x(\tau)$ is increasing in τ, which naturally matches our example in which older goods have lower quality-adjusted prices. In this case, the demand for each good $\widetilde{x}_t(i)$ increases over time as the $x(\tau)$ function shifts to the right. However, note that Assumption 1(ii) rules out exponential growth in the quantity of any good. Even when $T = \infty$, the growth rate of $x(\tau)$ must be either zero or negative, as $\tau \to \infty$.

Similar to Case 2 of Figure 2, we can also consider a steady state in which the quantity may decrease with age, even though older goods are less expensive. Such a pattern emerges when consumers do not like outdated goods or if newer goods replace parts of functions that are provided by older goods, as we discuss later in Section 4.

2.3 Measured Real GDP Growth Rate in the Steady State

Now, we examine whether the non-exponential steady state explained in Section 2.2 implies a positive and constant real GDP growth rate. Note that the conventional definition of real GDP growth in Equation (1) gives the average growth rate between two discrete periods. To map this definition to a continuous-time growth model, it is convenient to consider the instantaneous growth rate g_t at time t. This can be obtained by replacing $t-1$ in Equation (1) with $t-\Delta$ and taking the limit of $\Delta \to 0$ in $g_{t,t-\Delta}/\Delta$.[11]

$$g_t = \lim_{\Delta \to 0} \frac{g_{t,t-\Delta}}{\Delta} = \frac{\dot{N}_t \cdot \widetilde{p}_t(N_t)\widetilde{x}_t(N_t) + \int_{i \in X_t} \widetilde{p}_t(i)\dot{\widetilde{x}}_t(i)di - \int_{i \in \Omega_t} \widetilde{p}_t(i)\widetilde{x}_t(i)di}{\int_{i \in X_t} \widetilde{p}_t(i)\widetilde{x}_t(i)di}, \qquad (5)$$

where X_t and Ω_t represent the set of goods that are in production and the set of goods that reach the end of life, respectively, at time t. Suppose that the economy converges to a steady state, as defined in Definition 1. The number of goods grows linearly, and the evolution of prices and quantity in terms of age becomes stationary. The long-term growth rate can be obtained by substituting Equations (2)-(4) into Equation (5). If T

price of individual goods $\widetilde{p}_t(i)$ in equilibrium. Then, the long-term pattern of movement in $\widetilde{p}_t(i)$ shapes the stationary $p(\tau)$ function.

[11] See Appendix B for the derivation of (5) when the prices of $t-1$ are used to evaluate $g_{t,t-1}$.

is finite,[12]

$$g_t \to g \equiv \frac{np(0)x(0) + n\int_0^T p(\tau)x'(\tau)d\tau - np(T)x(T)}{n\int_0^T p(\tau)x(\tau)d\tau} \quad \text{as} \quad t \to \infty. \tag{6}$$

When T is infinite,[13]

$$g_t \to g \equiv \lim_{T\to\infty} \frac{np(0)x(0) + n\int_0^T p(\tau)x'(\tau)d\tau}{n\int_0^T p(\tau)x(\tau)d\tau} \quad \text{as} \quad t \to \infty. \tag{7}$$

The interpretations of the growth rates in Equations (6) and (7) are essentially the same as that in Equation (1), except that growth is now represented in terms of age and in continuous time. In the numerator, $np(0)x(0)$ represents the value of newly introduced goods, whereas $n\int_0^T p(\tau)x'(\tau)d\tau$ represents the value of changes in quantities of existing goods given price function $p(\tau)$. When T is finite, $-np(T)x(T)$ represents the loss of the value of goods that retire from the market at the end of their life. All terms are multiplied by n because there are n goods per unit of age. The sum of these terms reflects the speed of increase in economic activity. The denominator of Equations (6) and (7), $n\int_0^T p(\tau)x(\tau)d\tau$, gives the total value of existing production, i.e., the nominal GDP of the economy given prices $p(\tau)$. The ratio of the two yields the real GDP growth rate.

The following proposition provides a simpler formula for the long-term GDP growth rate in the steady state.

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[12] Equation (6) can be obtained from Equation (5) as follows. First, we substitute $p(\tau)$ and $x(\tau)$ for $\widetilde{p}_t(i)$ and $\widetilde{x}_t(i)$. Similarly, $\dot{\widetilde{x}}_t(i)$ can be written as $x'(\tau)$ (see footnote 8). Next, we change the integration variable from di in Equation (5) to $d\tau$. By differentiating Equation (3) for a given t, we obtain $di = -nd\tau$. We also need to change the integration interval. If T is finite, then $X_t = [N_t - nT, N_t]$. From Equation (3), $i = N_t - nT$ and $i = N_t$ correspond to $\tau = T$ and $\tau = 0$, respectively, as illustrated in Figure 3. From these, the denominator of Equation (5) is $\lim_{t\to\infty}\int_{i\in X_t}\widetilde{p}_t(i)\widetilde{x}_t(i)di = \lim_{t\to\infty}\int_T^0 p(\tau)x(\tau)(-n)d\tau \to n\int_0^T p(\tau)x(\tau)d\tau$. Similarly, the second term in the numerator becomes $n\int_0^T p(\tau)x'(\tau)d\tau$. The first and third terms become $np(0)x(0)$ and $-np(T)x(T)$, respectively. Therefore, the limit of the numerator of Equation (5) is $np(0)x(0) + n\int_0^T p(\tau)x'(\tau)d\tau - np(T)x(T)$.

[13] When T is infinite, $X_t = [0, N_t]$. Note that $i = 0$ and $i = N_t$ correspond to $\tau = t$ and $\tau = 0$, respectively. As $t \to \infty$, $\tau = t$ approaches ∞; therefore, the denominator approaches $n\int_0^\infty p(\tau)x(\tau)d\tau$. Note that the third term in the numerator disappears because $\Omega_t = \emptyset$ when $T = \infty$. Note also that in the RHS of (7), we first consider the integration from 0 to (a finite value) T and then take the limit of $T \to \infty$ because $\int_0^\infty p(\tau)x(\tau)d\tau$ can be infinite.

Proposition 1. *Suppose that the economy converges to a non-exponential asymptotic steady state, as defined by Definition 1. Then, the real GDP growth rate g_t asymptotes to g in the long run, where g is given as follows:*
(i) If $\int_0^T p(\tau)x(\tau)d\tau$ is finite (which is always true when T is finite), then[14]

$$g = \frac{-\int_0^T x(\tau)dp(\tau)}{\int_0^T p(\tau)x(\tau)d\tau}. \tag{8}$$

(ii) If $\int_0^T p(\tau)x(\tau)d\tau = \infty$, then $g = 0$.

Proof. (i) First, we consider the case of finite T. In the numerator of (6), integration by parts implies that $\int_0^T p(\tau)x'(\tau)d\tau = p(T)x(T) - p(0)x(0) - \int_0^T p'(\tau)x(\tau)d\tau$. Since $p(0)x(0)$ and $p(T)x(T)$ cancel out, we obtain (8).

Next, we consider the case of $T = \infty$. Given that $\int_0^T p(\tau)x(\tau)d\tau$ is finite, we can write the RHS of Equation (7) as follows: $(p(0)x(0) + \int_0^\infty p(\tau)x'(\tau)d\tau)/(\int_0^\infty p(\tau)x(\tau)d\tau)$. Additionally, the finiteness of $\int_0^\infty p(\tau)x(\tau)d\tau$ implies that $\lim_{\tau\to\infty} p(\tau)x(\tau) = 0$ (i.e., $p(\infty)x(\infty) = 0$). Therefore, integration by parts implies that $\int_0^\infty p(\tau)x'(\tau)d\tau = -p(0)x(0) - \int_0^\infty p'(\tau)x(\tau)d\tau$, from which we obtain (8).

(ii) In this case, T is necessarily ∞. If $\int_0^\infty p(\tau)x'(\tau)d\tau$ is finite, then the result directly follows from Equation (7). Now, suppose that $\int_0^\infty p(\tau)x'(\tau)d\tau$ is either $+\infty$ or $-\infty$. Since both the numerator and the denominator in Equation (7) are infinite, we apply L'Hôpital's rule to Equation (7) to obtain the following:

$$g = \lim_{T\to\infty} \frac{p(T)\,x'(T)}{p(T)\,x(T)} = \lim_{T\to\infty} \frac{x'(T)}{x(T)} \leq 0, \tag{9}$$

where the last inequality follows from Assumption 1(ii). In the following, we show that $g < 0$ does not occur by contradiction. For g to be strictly negative, $x(\tau)$ needs to shrink exponentially, which also means that $x'(\tau)$ must shrink exponentially. However, from $\lim_{\tau\to\infty} p'(\tau)/p(\tau) \leq 0$ in Assumption 1(ii), $\int_0^T p(\tau)x'(\tau)d\tau$ is finite since $p(\tau)x'(\tau)$ should shrink exponentially. Therefore, $g < 0$ contradicts the initial assumption that $\int_0^T p(\tau)x'(\tau)d\tau$ is either $+\infty$ or $-\infty$. □

Although Equation (8) has a simple form, it includes the contributions from the new goods and disappearing goods since it is mathematically equivalent to Equations (6) and

[14] Note that $\int_0^T x(\tau)dp(\tau)$ is equivalent to $\int_0^T p'(\tau)x(\tau)d\tau$ given that $p'(\tau)$ exists.

(7) as long as $\int_0^T p(\tau)x(\tau)d\tau < \infty$. Proposition 1 immediately implies the requirements for positive long-term GDP growth.

Corollary 1. *The long-term real GDP growth rate g is a positive and finite constant if and only if the following two conditions are satisfied:*[15]

$$-\int_0^T x(\tau)dp(\tau) \text{ is positive and finite, and} \tag{10}$$

$$\int_0^T p(\tau)x(\tau)d\tau \text{ is finite.} \tag{11}$$

The expression $\int_0^T p(\tau)x(\tau)d\tau$ in Condition (11) is the denominator of Equation (8). It is the cumulative expenditure that one product attracts over its lifecycle. Note that the nominal GDP in the steady state is $n\int_0^T p(\tau)x(\tau)d\tau$. Therefore, Condition (11) also means that the nominal GDP is constant, given our price normalization. The expression in Condition (10), $-\int_0^T x(\tau)dp(\tau)$, is the numerator of Equation (8). It represents the cumulative reduction in the quality-adjusted price of a good during its product lifecycle. Given that the prices in the model are normalized so that nominal GDP in the steady state is constant, the decline in the quality-adjusted price in the model means that the growth of the quality-adjusted price of goods measured in currencies is slower than the growth of nominal GDP.[16] When this happens, consumers have more purchasing power, which improves their utility. This income effect is more significant when the quantity of the good is greater. Therefore, in Condition (10), the price reduction $-dp(\tau)$ is weighted by quantity $x(\tau)$ and then integrated. The integrated sum gives the total income effect that one product generates over its product lifecycle.

If both conditions are satisfied in a non-exponential steady state, as defined in Definition 1, the real GDP growth rate is strictly positive in the long run, even though no variable grows exponentially. In Appendix C, we discuss the implications of Conditions (10) and (11) in more detail.

[15] Note that $\int_0^T p(\tau)x(\tau)d\tau$ is always positive from Assumption 1; therefore, we require only finiteness in Condition (11).

[16] See Appendix C for a more detailed explanation.

Another way to interpret the formula (8) is to rewrite it as follows:

$$g = -\int_0^T \sigma(\tau) d\ln p(\tau), \text{ where } \sigma(\tau) = \frac{p(\tau)x(\tau)}{\int_0^T p(\tau')x(\tau')d\tau'} \tag{12}$$

is the expenditure share given to age τ goods, and $d\ln p(\tau) = dp(\tau)/p(\tau) = (p'(\tau)/p(\tau))d\tau$ is the growth rate of the price of age τ goods. In this version of the formula, the integral $\int_0^T \sigma(\tau)d\ln p(\tau)$ represents the growth rate of the Divisia price index.[17] Recall that the prerequisite for the formula (8) implies that the nominal GDP ($n\int_0^T p(\tau)x(\tau)d\tau$) is constant under our price normalization. Given this, the formula (12) indicates that the real GDP growth rate can be obtained by subtracting the growth of this Divisia price index from the nominal GDP growth rate (i.e., zero).

2.4 Graphical Examples

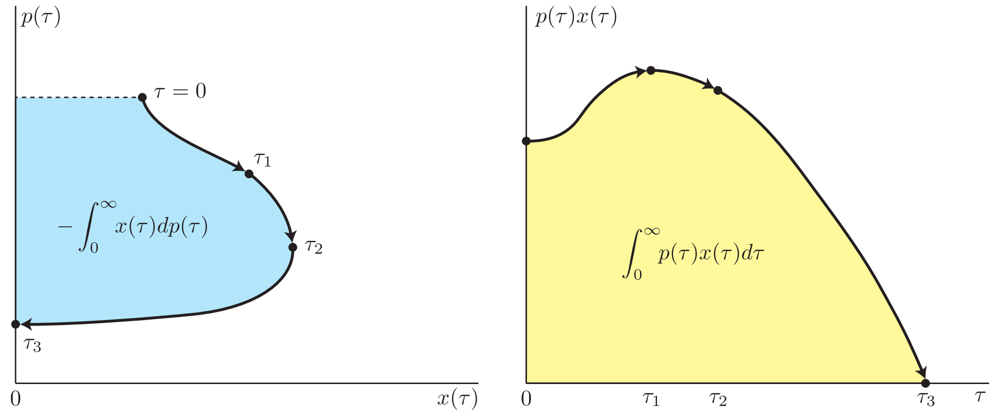
Proposition 1 shows that the real GDP growth rate depends only on functions $p(\tau)$ and $x(\tau)$. Therefore, we can represent the growth rate graphically via the shapes of these two functions. Figure 4 provides three examples.

Example 1 shows the simplest case, where the quality-adjusted price (weakly) decreases with age throughout the product lifecycle. The left panel depicts the evolution of $\{x(\tau), p(\tau)\}$ in the x-p diagram. T is finite in this example. The good enters the market at point $\{x(0), p(0)\}$ and continues to be produced until its age reaches $T = \tau_3$. Then, the numerator, $-\int_0^T x(\tau)dp(\tau)$, can be expressed by the area that is encompassed by the locus of $\{p(\tau), x(\tau)\}$ and the vertical axis in the x-p diagram (shown in blue). This graphical representation can be interpreted as follows. Whenever the quality-adjusted price falls by $dp(\tau)$, either through cost reductions or through quality improvements, consumers can save their purchasing power by the amount $-x(\tau)dp(\tau)$. The blue area shows the cumulative benefits of this good throughout its lifetime. The area is positive and finite as long as $p(0) > p(T)$.[18]

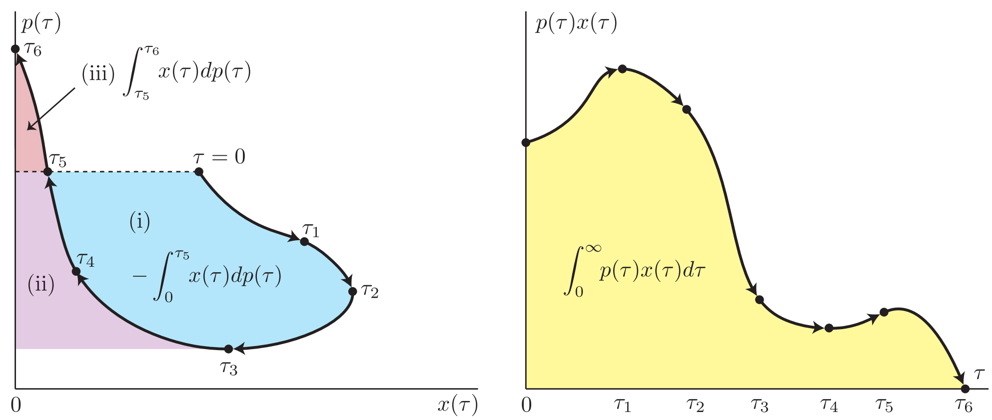
[17] See Hulten (1973) for explanations of the Divisia index numbers. Jorgenson and Griliches (1971) discusses the benefit of using Divisia index numbers in measuring productivity growth. Oulton (2025) argues that Divisia indices represent the ideal to which real-world, discrete indices are an approximation.

[18] $p(0) > p(T)$ requires the price to fall strictly with age at some point in a good's life.

Example 1: When T is finite and $p(\tau)$ is weakly decreasing



Example 2: When T is finite and $p(\tau)$ is nonmonotonic



Example 3: When $T = \infty$ and $p(\tau)$ is decreasing

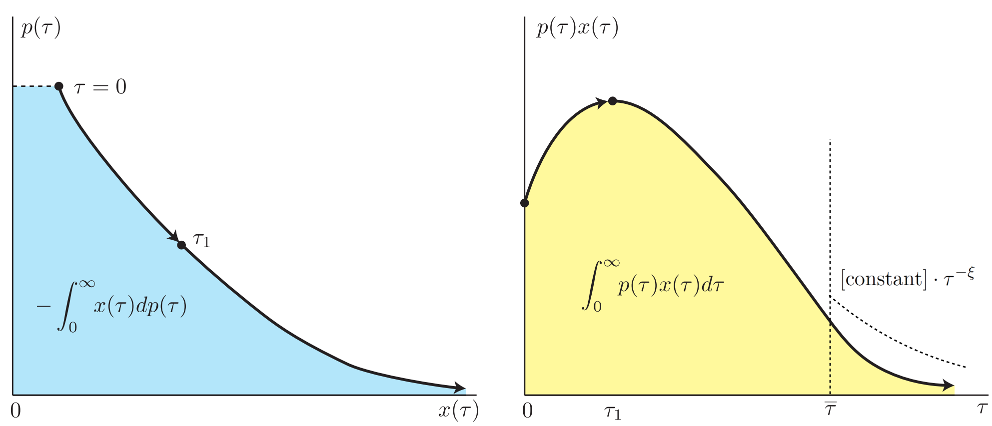


Figure 4: Graphical Representation of the Real GDP Growth Rate.

The right panel shows the evolution of expenditure for a good against its age, $p(\tau)x(\tau)$. The area below the curve (shown in yellow) gives the denominator, $\int_0^T p(\tau)x(\tau)d\tau$. According to Assumption 1, the expenditure for the good is positive at the time of introduction, and it evolves within the nonnegative region during its lifetime. Since expenditure $p(\tau)x(\tau)$ falls to zero at finite $T = \tau_3$, this area is positive and finite. According to Proposition 1, the ratio of the blue area to the yellow area represents the real GDP growth rate. Therefore, we can conclude that the real GDP growth rate in this example is positive and finite.

Next, Example 2 considers a case where $p(\tau)$ is not monotonic. As shown in the left panel, the quality-adjusted price begins to increase after τ_3 years. When the price of the good (relative to the newest good) increases during a part of its lifecycle (from $\tau = \tau_3$ to τ_6), the area between this part of the x-p locus and the vertical axis (marked as (ii) and (iii)) represents the loss of the purchasing power of consumers. This area needs to be deducted from the benefits of the fall in quality-adjusted prices from $\tau = 0$ to τ_3. Therefore, the numerator, $-\int_0^T x(\tau)dp(\tau)$, is given by area (i) minus area (iii) because area (ii) cancels out. It can be either positive or negative but is always finite since $T = \tau_6$ is finite. The yellow area in the right panel gives the denominator, $\int_0^T p(\tau)x(\tau)d\tau$, which is positive and finite. Therefore, the real GDP growth rate is finite and is given by the ratio of the blue area minus the red area to the yellow area. Additionally, note that the growth rate becomes zero only by coincidence, when the blue and red areas are the same size.

Finally, Example 3 shows a case in which the good remains in the market forever ($T = \infty$). The price $p(\tau)$ (relative to the newest good) falls throughout the lifecycle, and the quantity $x(\tau)$ remains positive as $\tau \to \infty$. For the yellow area to be finite, the expenditure on very old goods has to decrease. More concretely, Condition (11) is satisfied if the expenditure on old goods is bounded by a polynomial function of age with a power of less than -1:[19]

$$p(\tau)x(\tau) \leq [\text{constant}] \cdot \tau^{-\xi} \text{ for all } \tau \geq \overline{\tau}, \tag{13}$$

for some $\xi > 1$ and $\overline{\tau} > 0$. The dotted curve in the right panel gives an example of such an upper bound. While we need a concrete model to determine whether Condition (13) is satisfied, let us note that the condition does not require an exponential decrease in expenditure. The RHS of Equation (13) decreases with age at the rate of ξ/τ for $\tau > \overline{\tau}$. The rate of decline in the quality-adjusted price, ξ/τ, can be arbitrarily close to zero when we choose a large $\overline{\tau}$. Therefore, there is no minimum rate at which the expenditure needs to decrease.

The blue area is positive, given that the quality-adjusted price falls throughout the product lifecycle. Combined with Condition (13), the GDP growth rate is also positive. The growth rate is finite if $p(\tau)$ is bounded away from 0 as $\tau \to \infty$.[20] If $p(\tau)$ falls to 0 as $\tau \to \infty$, then the finiteness depends on the relationship between $p(\tau)$ and $x(\tau)$. Specifically, if the quantity depends only on price, then the area becomes finite if the price elasticity of the demand is less than one as the price approaches 0 from above.[21]

3 A Prototype Non-Exponential Growth Model

This section presents a general equilibrium model that yields non-exponential steady-state dynamics. While the theory in the previous section suggests many ways to construct a model that achieves non-exponential growth while capturing various aspects of reality,

[19] Suppose that Condition (13) is satisfied. Then, the denominator of Equation (8) is $\int_0^\infty p(\tau)x(\tau)d\tau \leq \int_0^{\overline{\tau}} p(\tau)x(\tau)d\tau + \int_{\overline{\tau}}^\infty [\text{constant}] \cdot \tau^{\xi} d\tau$. The first term is finite, and the second term becomes $[\text{constant}] \cdot \overline{\tau}^{1-\xi}/(\xi - 1)$, which is also finite.

[20] In this case, $x(\tau)$ must be finite as $\tau \to \infty$ since otherwise, $p(\tau)x(\tau)$ becomes infinite, contradicting Condition (13). Given this, the blue area is finite.

[21] Suppose that we can define a static inverse demand function $P(x)$. Focusing on the case of $x \to \infty$ and $P(x) \to 0$, the blue area can be written as $p(0)x(0) + \int_{x(0)}^\infty P(x)dx$. If the price elasticity of the demand as $p \to 0$ is less than one, then the elasticity of $P(x)$ with respect to x as $x \to \infty$ is greater than one. This means that $P(x)$ is bounded by $[\text{constant}] \cdot x^{-\xi' x}$ for some $\xi' > 1$ for large x. Therefore, the integral is finite.

this section presents the simplest prototype model to convey the substance of the non-exponential growth theory as clearly as possible. We generalize the prototype model in Sections 4 and 5.

3.1 Consumers

Consider an economy with infinitely lived representative consumers of constant population L. At each point in time, each consumer supplies one unit of labor. The wage level is normalized to one.[22] The lifetime utility function of the representative consumer is given by

$$\int_0^\infty \left[\int_0^{N_t} u(\widetilde{c}_t(i))di\right] e^{-\rho t}dt, \tag{14}$$

which is separable across both time and goods. Note that the sub-utility function is symmetric across goods; thus, we do not consider the obsolescence of older goods in this simplest prototype model.

We assume that the sub-utility function $u(c)$ is an increasing, continuous, differentiable, and concave function of c with $u(0) = 0$.[23] In addition, we aim to model consumers so that their demand behavior is reasonable when the price approaches zero and infinity. In particular, we assume that the price elasticity of demand for individual goods is less than one when the price is close to zero or, equivalently, when the quantity is large.[24] Otherwise, the expenditure for a single good becomes infinite such that $p \to 0$, which is unrealistic. At the same time, it is reasonable to assume that the price elasticity is greater than one when the price is very high or, equivalently, when the quantity is small. Otherwise, the expenditure for a single good increases without bound as $p \to \infty$, which is also unrealistic.[25] To satisfy these properties in the simplest way, we consider

[22] We later confirm that the price of the newest good is unchanged over time ($p(0) = (1 + \mu)/q(0) = 1 + \mu$). Therefore, this price normalization is consistent with the theory in the previous section.

[23] In a variety-expansion model, where the range of the integration (0 to N_t) changes endogenously, the utility from a nonexistent good should be zero; i.e., $u(0) = 0$.

[24] As explained in Example 3 of Section 2.4, this condition also implies that the blue area is finite given that the demand depends only on price. Therefore, this condition is crucial for obtaining $g > 0$ given that goods are symmetric. We relax this assumption when obsolescence is introduced in Section 4.

[25] If the elasticity of $u(c)$ were less than one for all $c \geq 0$ (i.e., when $\widehat{c} = 0$ in Equation 15), then the

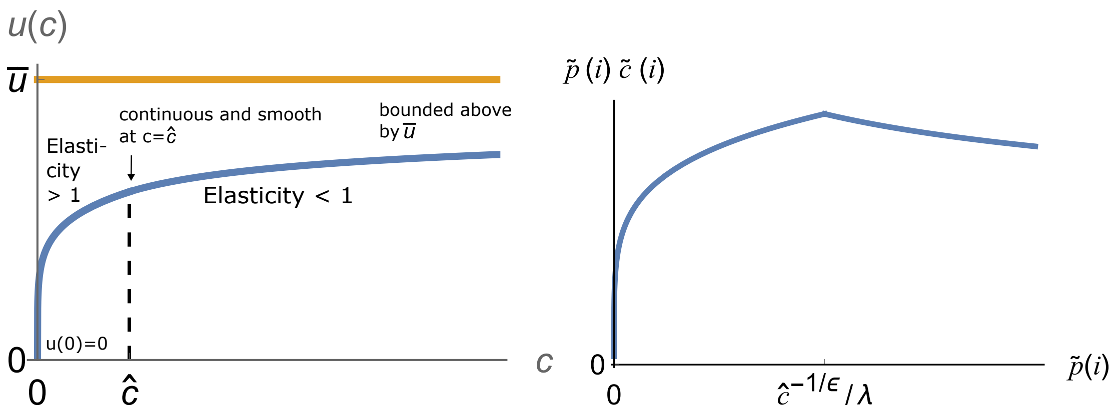


Figure 5: Utility from Each Good (Left) and Expenditure for Each Good (Right)

a sub-utility function in which the elasticity changes at a threshold level $\widehat{c} > 0$:

$$u(\widetilde{c}_t(i)) = \begin{cases} \frac{\widetilde{c}_t(i)^{1-1/\varepsilon}}{1-1/\varepsilon} + \overline{u} & \text{for } \widetilde{c}_t(i) \geq \widehat{c} \quad (0 < \varepsilon < 1), \\ \underline{u}\frac{\widetilde{c}_t(i)^{1-1/\widehat{\varepsilon}}}{1-1/\widehat{\varepsilon}} & \text{for } 0 \leq \widetilde{c}_t(i) < \widehat{c} \quad (\widehat{\varepsilon} > 1), \end{cases} \tag{15}$$

where we specify constants by $\underline{u} = \widehat{c}^{1/\widehat{\varepsilon}-1/\varepsilon} > 0$ and $\overline{u} = (1/(1-1/\widehat{\varepsilon})+1/(1/\varepsilon-1))\widehat{c}^{1-1/\varepsilon} > 0$ so that both $u(c)$ and $u'(c)$ are continuous at $c = \widehat{c}$. The shape of $u(c)$ is shown in the left panel of Figure 5.

The dynamic budget constraint of the representative consumer is given by

$$\dot{k}_t = r_t k_t + 1 - \int_0^{N_t} \widetilde{p}_t(i)\widetilde{c}_t(i)di. \tag{16}$$

In equilibrium, the aggregate asset holding, Lk_t, should equal the value of all the firms in the economy. Consumers maximize their lifetime utility (14) subject to the budget constraint (16), given interest rate r_t, prices of goods $\widetilde{p}_t(i)$ for $i \in [0, N_t]$, initial asset holding k_0, and the standard non-Ponzi game condition.

From the above, we obtain a piecewise isoelastic demand function for individual goods by the representative consumer:

$$\widetilde{c}_t(i) = \begin{cases} \lambda_t^{-\varepsilon}\widetilde{p}_t(i)^{-\varepsilon} & \text{if } \widetilde{p}(i) \leq \widehat{c}^{-1/\varepsilon}/\lambda_t, \\ (\lambda_t/\underline{u})^{-\widehat{\varepsilon}}\widetilde{p}_t(i)^{-\widehat{\varepsilon}} & \text{if } \widetilde{p}(i) > \widehat{c}^{-1/\varepsilon}/\lambda_t. \end{cases} \tag{17}$$

first line of (15) would imply that $u(0) = -\infty$, which is inconsistent with our assumption of $u(0) = 0$.

As shown in the right panel of Figure 5, the expenditure for each good, $\widetilde{p}_t(i)\widetilde{c}_t(i)$, has a tent-shaped curve against its price, $\widetilde{p}_t(i)$, which means that the expenditure for an individual good never explodes when the price approaches either zero or infinity. The shadow price of the budget constraint λ_t evolves according to the Euler equation $\dot{\lambda}_t = (\rho - r_t)\lambda_t$. Its initial value is determined so that the transversality condition $\lim_{t\to\infty} e^{-\rho t}\lambda_t k_t = 0$ is satisfied given the evolution of k_t in Equation (16).

3.2 R&D and Production Technologies

Each consumer works either as a production worker or as a researcher. A researcher succeeds in developing a new good with a Poisson probability of a per unit of time. Let L_t^R denote the number of researchers in the economy, which is to be determined in equilibrium. Over time, the number of goods increases according to

$$\dot{N}_t = aL_t^R. \tag{18}$$

Equation (18) is similar to standard variety expansion models, except that there is no spillover term from the stock of past R&D.

Once developed, each individual good is produced with a linear production technology that requires only labor. The output of good i is given by

$$\widetilde{x}_t(i) = \widetilde{q}_t(i)\widetilde{l}_t(i), \tag{19}$$

where $\widetilde{l}_t(i)$ is the labor input and $\widetilde{q}_t(i)$ is the marginal product of labor in producing good i. Alternatively, we can interpret $\widetilde{x}_t(i)$ as the quality-adjusted output and $\widetilde{q}_t(i)$ as the quality of good i. In this case, one unit of labor produces one unit of good i with quality $\widetilde{q}_t(i)$. In either interpretation, we call $\widetilde{q}_t(i)$ the productivity for good i.

When any good is first developed, the productivity is normalized to 1. Then, as the production of this good proceeds, the productivity increases according to[26]

$$\dot{\widetilde{q}}_t(i) = I(\widetilde{x}_t(i)) \cdot \beta \widetilde{q}_t(i)^{\psi}, \quad 0 < \psi < 1, \tag{20}$$

where $I(\widetilde{x}_t(i))$ is an indicator function that takes a value of 1 when $\widetilde{x}_t(i) > 0$ and 0 otherwise. This means that productivity increases as long as production takes place. The specification in Equation (20) is similar to those in the quality ladder models. There are knowledge spillovers from the past productivity of technology to the current productivity increments. Parameter $\psi \in (0, 1)$ specifies the degree of such spillovers. While quality ladder models need to assume that $\psi = 1$ to achieve an exponential increase in productivity (or quality), we do not make this knife-edge assumption. For the moment, we consider the case of $\psi \in (0, 1)$ and later compare the result to the case of $\psi = 1$. The parameter $\beta > 0$ represents other possible factors that affect the speed at which productivity increases.

As long as $\widetilde{x}_t(i) > 0$, then Equation (20) is an autonomous differential equation in $\widetilde{q}_t(i)$. Similar to Section 2, let $\tau \equiv t - s(i)$ denote the age of the good. Then, the solution to the differential Equation (20) can be written as follows:

$$q(\tau) = \kappa_1 (\tau + \kappa_0)^{\theta}, \tag{21}$$

where $\theta \equiv 1/(1 - \psi) > 1$, $\kappa_0 \equiv \theta/\beta > 0$, and $\kappa_1 \equiv (\beta/\theta)^{\theta} > 0$. Given that $\psi \in (0, 1)$, the productivity improvement is less than exponential. The rate of increase in productivity is given by

$$g_q(\tau) = \frac{q'(\tau)}{q(\tau)} = \frac{\theta}{\tau + \kappa_0} = \frac{\beta}{(1 - \psi)\beta\tau + 1}. \tag{22}$$

In this specification, $g_q(\tau)$ takes the highest value at the time of introduction ($g_q(0) = \beta$) and then then falls to 0 as a good becomes older ($g_q(\infty) = 0$). This rules out the trivial possibility that the exponential increase in the productivity of individual goods explains the sustained GDP growth.

[26] For simplicity, we assume that only experience in terms of time matters for productivity improvement. Alternatively, we can consider experience in terms of the cumulative production amount. Horii (2012) analyzed a model in the latter setting and derived a GDP growth rate defined in the same way as in Equation (1); however, it is a semi-endogenous growth model that requires an exponentially growing population (c.f. Jones, 1995):

3.3 Behavior of Firms

Let us now turn to the behavior of production firms. While any product is protected by a patent forever, the patent *breadth* is limited (e.g. O'Donoghue, Scotchmer, and Thisse, 1998). This means that while other producers are prohibited from using the same technology as the original inventor, they are allowed to produce similar products if they use a technology that is sufficiently different from the original. Alternatively, we may suppose that a part of the technology is kept secret by the inventor and that outsiders need to rely on less efficient technologies. In either case, outsiders face lower productivity than the original firm does.

To formalize this idea, let us assume that there are potentially many outside firms. These firms have partial access to the technology of the original inventor $\widetilde{q}_t(i)$ to produce the same good i. However, their productivity is $1/(1+\mu)$ times lower, where parameter μ represents the patent breadth or the strength of the trade secret. For simplicity, we assume that $0 < \mu < 1/(\widehat{\varepsilon} - 1)$. In this case, the profit-maximizing strategy is to set the limit price, which is $(1+\mu)$ times higher than the marginal cost.[27] Given the production function (19) and the fact that the wage is normalized to one, the pricing by a firm that has τ years of experience is

$$p(\tau) = \frac{1+\mu}{q(\tau)}. \tag{23}$$

3.4 Steady-State Equilibrium

Now, we derive the long-term property of the equilibrium dynamics in this prototype model. The following defines a notion of long-term equilibrium suitable for our model.

Definition 2. *An equilibrium path that satisfies the following properties as $t \to \infty$ is called the asymptotic steady-state equilibrium (ASSE).*

1. *The speed of the introduction of new goods converges to a positive and finite constant: $\dot{N}_t \to n^* > 0$.*

[27] If the patent breadth were infinite, then the firms would choose monopoly pricing. In that case, the profit-maximizing markup would be $1/(\widehat{\varepsilon} - 1)$ if the demand elasticity were $\widehat{\varepsilon} > 1$ and infinity if the elasticity were $\varepsilon < 1$. Since μ is lower than both, the firms set the limit price.

2. *The Lagrange multiplier of the budget constraint, λ_t, converges to a positive and finite constant: $\lambda_t \to \lambda^* > 0$.*

In the steady state, the equilibrium output of a good of age τ is determined by Equations (17) and (23) with $\lambda_t = \lambda^*$ and does not depend on t:

$$x(\tau) = \begin{cases} D(\lambda^*)q(\tau)^{\varepsilon} & \text{if } q(\tau) \geq (1+\mu)\lambda^* \widehat{c}^{1/\varepsilon}, \\ \widehat{D}(\lambda^*)q(\tau)^{\widehat{\varepsilon}} & \text{if } q(\tau) < (1+\mu)\lambda^* \widehat{c}^{1/\varepsilon}, \end{cases} \tag{24}$$

where demand shifters $D(\lambda) = L((1+\mu)\lambda)^{-\varepsilon}$ and $\widehat{D}(\lambda) = L((1+\mu)\lambda/\underline{u})^{-\widehat{\varepsilon}}$ are decreasing functions of λ. The following lemma gives the condition under which the production of all existing goods is determined by the first line of Equation (24), where the price elasticity of demand is $\varepsilon < 1$.

Lemma 1. *Suppose that $\widehat{c}$ is smaller than $\left(a\mu L \int_0^\infty q(\tau)^{\varepsilon-1}e^{-\rho\tau}d\tau\right)^{-1}$. Then, in the ASSE, $q(\tau) \geq (1+\mu)\lambda^*\widehat{c}^{1/\varepsilon}$ for all $\tau \geq 0$.*
Proof: In Appendix D.1.

In the main text, we focus on the simple case where $\widehat{c}$ is sufficiently small so that the assumption in Lemma 1 is satisfied. We leave the analysis of the general case for Appendix D.2. Then, from (23), (24), and the fact that the markup rate is μ, the profit of an age-τ firm is

$$\pi(\tau) = \mu D(\lambda^*)q(\tau)^{\varepsilon-1}. \tag{25}$$

The equilibrium values of n^* and λ^* are determined by the free entry condition for R&D and the labor market clearing condition. Let us first focus on the R&D condition. Recall that the Euler equation is $\dot{\lambda}_t/\lambda_t = \rho - r_t$. Since λ_t is stationary in the ASSE, the interest rate necessarily converges to $r_t \to \rho$. Using interest rate $r_t = \rho$ and the profit function (25), we can calculate the present value of a new firm just after it has succeeded in developing a new good:

$$V(\lambda^*) = \mu D(\lambda^*) \int_0^\infty q(\tau)^{\varepsilon-1}e^{-\rho\tau}d\tau. \tag{26}$$

From the R&D function (18), the expected cost of developing a new good is $1/a$. Therefore, given that there is a positive flow of R&D, $n > 0$, and given that the financial market is complete, the value of the new firm (26) should be equalized to the expected cost of development: $V(\lambda^*) = 1/a$. This condition gives the equilibrium value of $D(\lambda^*)$ in the ASSE:

$$D(\lambda^*) = \frac{1}{a\mu}\left(\int_0^\infty q(\tau)^{\varepsilon-1} e^{-\rho\tau} d\tau\right)^{-1} \equiv D^*. \tag{27}$$

By substituting Equation (21) into Equation (27), we can calculate the value of D^*, which is always positive and finite.[28] We also obtain $\lambda^* = \frac{1}{1+\mu}\left(L/D^*\right)^{1/\varepsilon}$ from the definition of $D(\lambda) = L((1+\mu)\lambda)^{-\varepsilon}$.

Next, let us turn to the labor market. First, Equation (18) implies that the number of research workers in the ASSE is $L^{R*} = n^*/a$. Second, according to functions (19) and (24), the aggregate demand for production workers in the ASSE is[29]

$$L^{P*} = \lim_{t\to\infty} \int_0^{N_t} \widetilde{l}_t(i) di \to n^* \int_0^\infty \frac{x(\tau)}{q(\tau)} d\tau = n^* D^* \int_0^\infty q(\tau)^{\varepsilon-1} d\tau. \tag{29}$$

The labor supply is given by population L. Therefore, the labor market clearing condition is

$$L = L^{R*} + L^{P*} = \frac{n^*}{a} + n^* D^* \int_0^\infty q(\tau)^{\varepsilon-1} d\tau. \tag{30}$$

From Equation (21), the integral in the RHS, $\int_0^\infty q(\tau)^{\varepsilon-1} d\tau$, becomes finite if and only if $\theta(1-\varepsilon) > 1$. Using the definition $\theta \equiv 1/(1-\psi)$, the condition is reduced to $\psi \in (\varepsilon, 1)$, where ψ is the degree of knowledge spillover from past productivity to its increments. If $\psi < \varepsilon$, then the integral is infinite; therefore, Equation (30) implies that $n^* = 0$. Since we are interested in the ASSE with $n^* > 0$, the remaining analysis focuses on the case of $\psi \in (\varepsilon, 1)$.

Then, from Equation (30), we obtain the equilibrium research intensity in the ASSE:

$$n^* = \frac{aL}{1 + aD^* \int_0^\infty q(\tau)^{\varepsilon-1} d\tau}. \tag{31}$$

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[28] Let $\Gamma(\cdot,\cdot)$ denote the upper incomplete Gamma function, defined as $\Gamma(s,z) \equiv \int_z^\infty t^{s-1} e^{-t} dt$. The values of $\Gamma(s,z)$ are available in most programming platforms. The function $\Gamma(s,z)$ is positive and finite for all $s \in (-\infty,\infty)$ and $z \in (0,\infty)$. By changing the variable of integration from τ to $\widetilde{\tau} = (\tau+\kappa_0)/\rho$ and utilizing Equation (21), Equation (27) implies the following:

$$D^* = \frac{\kappa_1^{1-\varepsilon} \rho^{1+\theta(1-\varepsilon)}}{a\mu e^{\rho\kappa_0} \Gamma(1-\theta(1-\varepsilon), \rho\kappa_0)} > 0, \tag{28}$$

[29] In Equation (29), the variable of integration is changed from i to τ via Equation (3).

From (31), $L^{R*} = n^*/a$ and $L^{P*} = L - L^{R*}$ are also obtained. We can calculate the explicit value of n^* as follows. Using Equation (27) and then Equation (21), the equilibrium ratio of the two types of labor is

$$\left(\frac{L^P}{L^R}\right)^* = \frac{\int_0^\infty q(\tau)^{\varepsilon-1} d\tau}{\mu \int_0^\infty q(\tau)^{\varepsilon-1} e^{-\rho\tau} d\tau}, \tag{32}$$

the value of which can be expressed via the Gamma function.[30] Using $\left(L^P/L^R\right)^*$, the ASSE research intensity can be written as

$$n^* = aL^{R*} = \frac{aL}{1 + (L^P/L^R)^*}, \tag{34}$$

which becomes a positive and finite constant given that $\psi \in (\varepsilon, 1)$.

The pair of $D^* = D(\lambda^*)$ in Equation (27) and n^* in Equation (34) characterizes the long-term equilibrium of this economy. These equations also explain how parameters affect long-term dynamics. For example, a larger μ means that the breadth of patents is wider (or that trade secrets are better maintained). A higher value of a means that R&D requires less labor. In these cases, innovation intensity n^* increases because of greater profitability, whereas the output of each good, proportional to D^*, decreases because there are more production firms to which the aggregate labor needs to be distributed.[31] The opposite occurs when the time preference ρ is greater because it increases the interest rate, reducing the present value of profits.

When population L is larger, the research intensity n^* is multiplied proportionally to L. However, the production of each good (proportional to D^*) does not change because both the number of products introduced each year and the number of total production workers are multiplied by the same factor. This outcome resembles the mechanism of the

[30]Using Equation (28), the value of (32) can be calculated as follows:

$$\left(\frac{L^P}{L^R}\right)^* = \frac{\kappa_0^{1-\theta(1-\varepsilon)} \rho^{1+\theta(1-\varepsilon)}}{\mu(\theta(1-\varepsilon)-1)e^{\rho\kappa_0}\Gamma(1-\theta(1-\varepsilon), \rho\kappa_0)} \text{ if } \psi > \varepsilon, \quad \left(\frac{L^P}{L^R}\right)^* = \infty \text{ otherwise.} \tag{33}$$

[31]The derivative of the upper incomplete Gamma function with respect to the second argument, $\partial\Gamma(s,z)/\partial z = -z^{s-1}e^{-z}$, is always negative. Using this property, the properties in the text can be confirmed from Equations (28), (33) and (34).

second-generation endogenous growth models, where the horizontal number of sectors is adjusted proportionally to the total population.[32]

Before closing this subsection, let us briefly compare those results against the case of $\psi = 1$. When $\psi = 1$, the solution to the differential equation (20) is exponential: $q(\tau) = e^{\beta\tau}$. Then, we can calculate n^* and D^* in the ASSE as follows:

$$n^* = \frac{\mu(1-\varepsilon)\beta a L}{(1+\mu)(1-\varepsilon)\beta + \rho}, \quad D^* = \frac{(1-\varepsilon)\beta + \rho}{a\mu}. \tag{35}$$

The comparative static properties with respect to μ, ρ, L and a are the same as those in the case of $\psi \in (\varepsilon, 1)$. Therefore, the exponential growth in productivity ($\psi = 1$) can be viewed as a particular case of our model, although we do not focus on it because it is a knife-edge case.

3.5 Measured Real GDP Growth Rate

Now, we are ready to examine the long-term GDP growth rate, as measured according to the SNA, in this prototype model.[33] In this subsection, we assume that $\psi \in (\varepsilon, 1)$ so that the economy has an ASSE with finite $n^* > 0$ and $\lambda^* > 0$. In addition, using Equations (21), (23) and (24), we can confirm that $p(\tau)$ and $x(\tau)$ satisfy Condition (11) given that $\psi \in (\varepsilon, 1)$.[34] Therefore, we can apply Formula (8) in Proposition 1, or equivalently (12), to calculate the measured real GDP growth rate in the ASSE.

Given that the markup ratio μ is constant, the growth formula (12) becomes

$$g^* = \int_0^\infty g_q(\tau)\sigma(\tau)d\tau. \tag{36}$$

[32]However, note that the long-term growth in these models is typically maintained by the exponential increase in productivity (or quality) in each sector, whereas this paper focuses on the case where such exponential improvements cannot be sustained ($\psi < 1$ in Equation 20).

[33]We continue to focus on the case where $\widehat{c}$ is sufficiently small so that Lemma 1 holds. We examine the general case in Appendix D.3 and show that the measured GDP growth rate becomes positive under the same conditions as in the main text.

[34]Using the definitions of $q(\tau)$ in Equation (21) and $\theta \equiv 1/(1-\psi) > 1$, we find that the denominator of the formula is $\int_0^\infty p(\tau)x(\tau)d\tau = D^*(1+\mu)(1-\psi)\kappa_0^{1-(\psi-\varepsilon)/(1-\psi)}/(\psi-\varepsilon)$. It is positive and finite given that $\psi \in (\varepsilon, 1)$.

The growth formula in this form clarifies that real GDP growth is the weighted average of the rate of productivity increase among goods of various ages, $g_q(\tau)$, where the weights are the expenditure shares, $\sigma(\tau)$. This result is known as Hulten's theorem (Hulten, 1978; Baqaee and Farhi, 2019). Using the formula, we obtain the GDP growth rate in the ASSE as[35]

$$g^* = \frac{\psi - \varepsilon}{1 - \varepsilon}\beta \quad \text{for } \varepsilon < \psi \leq 1. \tag{37}$$

Recall that, in our specification of the technology, the newest goods have the fastest rate of productivity improvement, β, whereas the rate of improvement is lower for older goods because $g_q'(\tau) < 0$ (see Equation 22). In particular, the rate of productivity improvement $g_q(\tau)$ is almost zero for very old goods with large τ. Therefore, it is natural that the aggregate GDP growth rate in Equation (37) is between zero and β.

The growth rate g^* in Equation (37) is decreasing in the price elasticity of demand, ε. Recall that ε also represents the elasticity of substitution across goods. With a higher ε, consumers spend more on old and low-priced goods and less on new and expensive goods. Since the rate of productivity increase in Equation (22) is lower for older goods (with high age τ), the weighted average is also low.

Equation (37) shows that the measured growth rate takes a positive and finite value when the degree of knowledge spillover in production, ψ, is greater than ε. The requirement $\psi > \varepsilon$ can be understood in terms of Condition (11) in Corollary 1. Given that $\psi < 1$, the expenditure for an age-τ good in the ASSE can be written as follows: $p(\tau)x(\tau) = [\text{constant}] \cdot (\tau + \kappa_0)^{-(1-\varepsilon)\theta}$. For $\int_0^\infty p(\tau)x(\tau)d\tau$ to be finite, the power of $(\tau + \kappa_0)^{-(1-\varepsilon)\theta}$ must be less than -1. This is a particular case of Condition (13) in Section 2. Intuitively, for the expenditure on existing goods to be finite, the expenditure for a single good must decline reasonably fast with age. In this prototype model environment, the condition is met if the degree of spillover in the productivity increase, ψ, is greater than ε. Otherwise, $\int_0^\infty p(\tau)x(\tau)d\tau$ becomes infinite, and Proposition 1 implies

[35] Using $p'(\tau) = -(1+\mu)g_q(\tau)/q(\tau)$ and Equation (22), we find $-\int_0^\infty p'(\tau)x(\tau)d\tau = D^*(1+\mu)\kappa_0^{-(\psi-\varepsilon)/(1-\psi)}/(1-\varepsilon)$. Combined with calculations from footnote 34, we obtain $-\int_0^\infty p'(\tau)x(\tau)d\tau / \int_0^\infty p(\tau)x(\tau)d\tau = (\psi-\varepsilon)/(1-\varepsilon)(1-\psi)\kappa_0$. Using definitions $\kappa_0 \equiv \theta/\beta$ and $\theta \equiv 1/(1-\psi)$ gives Equation (37).

that the long-term GDP growth rate is zero.

Given that $\psi > \varepsilon$, growth rate g^* increases with ψ. As ψ increases, the schedule of the $g_q(\tau)$ function in Equation (22) increases, as does the real GDP growth rate because it is a weighted average of $g_q(\tau)$. When ψ reaches 1, the long-term growth rate increases to β. This is an anticipated result; when $\psi = 1$, the productivity of all goods, both the new and the old, increases with a common constant exponential rate of β. Therefore, the case of $\psi = 1$ corresponds to conventional growth theory, where labor productivity increases exponentially and uniformly. However, the main finding is that even when the productivity of each product does not increase exponentially (i.e., with $\psi < 1$), the economy as a whole can exhibit a constant measured growth rate, although it is lower than β.

3.6 Comparative Dynamics and Transition

In the simple prototype setting, the long-term rate of growth in Equation (37) does not depend on the equilibrium values of n^* and D^*, as long as they are positive.[36] When the research intensity n^* is high, more economic activity is added per unit of time. However, in the long run, there is also proportionally more "stock" of existing activities. The real GDP growth rate expresses the ratio between the two, which is unchanged.[37] Similarly, when D^* is larger, each good will have more demand. This means that the production of new goods, as well as the increase in the production of other goods over time, will be greater. In the long run, however, the total value of existing products will also be higher, exactly canceling out the effects on g^*.[38] As a result, even when changes in population

[36]This property depends on the simplistic settings in this prototype model. For example, when the aggregate R&D intensity n^* has some positive spillovers on the rate of productivity increases in individual goods $g_q(\tau)$, then n^* will affect g^*. Additionally, when the amount of production has some effect on $g_q(\tau)$, g^* will depend on D^*.

[37]Nonetheless, it is essential that there is a positive flow of new innovations $n^* > 0$, since otherwise, g^* becomes 0.

[38]This can also be seen in Example 3 of Figure 4. When D^* is increased, the left panel is stretched horizontally (along the $x(\tau)$ axis), whereas the right panel is stretched vertically (along the $p(\tau)x(\tau)$ axis) by the same magnification ratio. As a result, the growth rate, given by the ratio of the two areas,

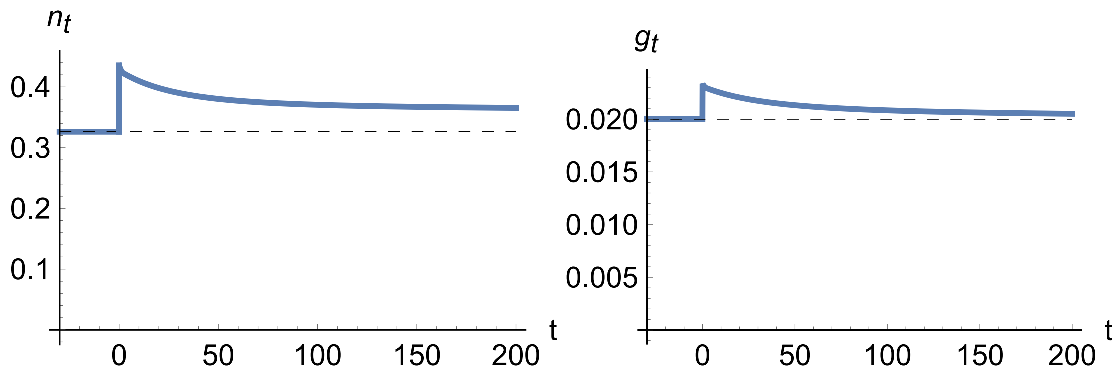

Figure 6: Response of innovation per unit time (n_t: Left panel) and the GDP growth rate (g_t: Right panel) after a permanent increase in R&D productivity a.

L, R&D productivity a, or patent policy μ affect n^* and D^*, they do not affect the long-term real GDP growth rate.

However, those parameter changes affect the GDP growth rate in the short run. In Appendix F, we explain the transitional dynamics of this economy. Figure 6 depicts the response of the economy when R&D productivity a is increased permanently by 10%.[39] Equations (32) and (33) imply that innovation per unit time n_t will increase by the same 10% in the long run, which can be confirmed from the left panel. However, in the short run, there is an overshoot in n_t. This can be interpreted as follows. In the long run, the number of competitors (except for very old and negligible firms) is also increased by 10% because of increased innovations. The number of new innovations in the new steady state (with increased a) is 10% higher than that in the old steady state despite this increased competition. Now, let us consider what happens immediately after the increase in a. The R&D productivity is increased by 10%, but the number of existing firms is not yet affected. Therefore, the new firms enjoy more favorable conditions in the short run than in the long run. This is why there are more entries immediately after the parameter change than in the long run.

is unaffected.

[39]In this numerical example, we assume that the economy is initially in a steady state with parameters $a = 1$, $\mu = 0.2$, $L = 1$, $\beta = 0.04$, $\psi = 0.9$, $\varepsilon = 0.8$, and $\rho = 0.01$. At $t = 0$, the parameter a is increased from 1 to 1.1, while the other parameters are unchanged.

A similar mechanism operates for the evolution of g_t, which is depicted in the right panel. The short-term response of g_t is positive. This can be interpreted from the definition of the instantaneous GDP growth rate in equation (5).[40] Immediately after the increase in a, the introduction of new goods ($n_t \equiv \dot{N}_t$) in the numerator increases, whereas the denominator changes only gradually. Over time, g_t reverts to the original value, as discussed above. Although these results depend on the simplified specification of the prototype model, they provide a possible interpretation of why the measured GDP growth rates in the U.S. and some other developed countries have been relatively stable, even though the underlying parameters seem to have significantly changed over long periods.

3.7 Aggregate Variables and Balanced Growth

The ASSE in this model works very differently from the balanced growth path (BGP) in existing growth models. Nonetheless, we show that when aggregate variables are measured in a conventional way, this model exhibits balanced growth in those measured aggregate variables.

Note that the total labor income for production is L^{P*} since the wage rate is normalized to one. All goods are sold at $(1 + \mu)$ times the labor cost, as shown in Equation (23). Therefore, the aggregate value of production, which equals the aggregate value of consumption, is $C^* = (1 + \mu)L^{P*}$. In our model, investments take the form of R&D, and the total value of R&D outputs is $I^* = n^* V(\lambda^*) = L^{R*}$. The GDP in our model can be calculated as the sum of the value of production and the value of investments: $Y^* = C^* + I^* = (1 + \mu)L^{P*} + L^{R*}$. Similarly, we can derive the steady-state value of aggregate capital, K^*, which is defined as the value of all firms in the economy (knowledge capital).[41]

[40] Note that formula (8) in Proposition 1 applies only in the steady state.

[41] K^* can be calculated as the sum of the present value of the future profits of all firms that exist today. In v years from now, the present value of the profit from those firms will be $e^{-\rho v} \int_v^\infty \pi(\tau) n^* d\tau$, since the profits of firms less than v years old at that time will not be part of the value of today's firms. By aggregating all v and using the profit function (25), we have $K^* = \mu n^* D^* \int_0^\infty e^{-\rho v} \int_v^\infty q(\tau)^{\varepsilon - 1} d\tau dv$, which is constant under the price normalization in the model.

Note that those aggregate variables are measured under the price normalization of our model, in which the nominal wage is set to 1. We now calculate their real values in the same spirit as the SNA.[42] Let $\bar{t}$ be the reference year, and let $Y_{\bar{t}}^{\$}$ be the dollar value of the GDP in year $\bar{t}$, which we assume is known to the researcher. Since the real GDP growth rate is constant at g^* in the ASSE, the real GDP level in t is as follows: $Y_t^{\text{real}} = Y_{\bar{t}}^{\$} e^{g^*(t-\bar{t})}$. Since the ratios among Y^*, C^*, I^* and K^* are constant, their real values increase in the same proportion. Specifically,

$$C_t^{\text{real}} = \frac{C^*}{Y^*} Y_t^{\text{real}} = \frac{1+\mu}{1+\mu+\left(L^R/L^P\right)^*} Y_{\bar{t}}^{\$} e^{g^*(t-\bar{t})}, \tag{38}$$

$$I_t^{\text{real}} = \frac{I^*}{Y^*} Y_t^{\text{real}} = \frac{1}{(1+\mu)\left(L^P/L^R\right)^*+1} Y_{\bar{t}}^{\$} e^{g^*(t-\bar{t})}, \tag{39}$$

where $\left(L^R/L^P\right)^*$ is given by the inverse of Equation (32).

The interest rate $r^* = \rho$ is also defined under our normalization of prices. Since the nominal GDP growth rate in the steady state is zero, the steady-state inflation rate is $-g^*$ in our price normalization. Then, the real interest rate in the steady state is $r^{\text{real}} = r^* + g^* = \rho + g^*$. We can also derive other real aggregate variables in similar ways, and their growth rates are constant. Therefore, if the statistical agency were to measure the aggregate variables in our model economy, then those observed variables would grow exponentially along the BGP, even though neither the quantity, quality, nor variety of individual goods were growing exponentially.

3.8 Welfare Changes

In this subsection, we discuss the changes in the welfare (utility) of the representative consumer over time and its relationship with the measured GDP growth rate. As shown by Equation (14), the lifetime utility of the consumer is $\int_0^\infty U_t e^{-\rho t} dt$, where $U_t =$

[42] The NIPA publishes two series of real GDP. One is the quantity index, which is 100 in the reference year (2012 as of the time of writing). The values for other years are obtained by chaining the real GDP growth rate. The other is the chained (2012) dollar series, the values of which are calculated as the product of the quantity index and the 2012 current dollar value of the corresponding series divided by 100. See U.S. Bureau of Economic Analysis, "Table 1.1.6. Real Gross Domestic Product, Chained Dollars." We use the latter method in this paper.

$\int_0^{N_t} u(\widetilde{c}_t(i))di$ is the instantaneous utility. Using Equations (15), (24) and $c(\tau) = x(\tau)/L$, the instantaneous utility can be written as follows:

$$U_t = N_t \overline{u} - (\overline{u} - u(c(0))) \int_0^t n_{t-\tau} q(\tau)^{\varepsilon - 1} d\tau. \tag{40}$$

In the ASSE, the first term increases $n^* \overline{u}$ per unit time, and the second term converges to a finite value as $t \to \infty$.[43] Therefore, asymptotically, the instantaneous utility linearly increases with time, with a slope of $n^* \overline{u}$.

However, the growth of the instantaneous utility in (40) is measured in units of the utility function in the model (thereafter, utils), which has no clear interpretation. An appropriate way to measure the changes in welfare over time is to focus on the money-metric utility. Following Baqaee and Burstein (2023) and Jaravel and Lashkari (2024), we define the money-metric utility as follows.[44] The change in welfare between t and $t + \Delta$ measured using the equivalent variation is $\zeta_t(\Delta)$, where $\zeta_t(\Delta)$ is defined by

$$v(\{\widetilde{p}_{t+\Delta}(i)\}_{i=0}^{N_{t+\Delta}}, I_{t+\Delta}) = v(\{\widetilde{p}_t(i)\}_{i=0}^{N_t}, I_t \exp \zeta_t(\Delta)). \tag{41}$$

Here, $v(\cdot)$ is the indirect utility function, $\{\widetilde{p}_t(i)\}_{i=0}^{N_t}$ is the set of prices for available goods at time t, and I_t is the expenditure at time t. The definition (41) can be interpreted as follows: $\zeta_t(\Delta)$ is the change in expenditure in logs under the initial prices $\{\widetilde{p}_t(i)\}_{i=0}^{N_t}$ that the representative consumer would need to be indifferent between the budget set defined by initial prices $(\{\widetilde{p}_t(i)\}_{i=0}^{N_t}, I_t \exp \zeta_t(\Delta))$ and the new budget set defined by new prices and expenditure $(\{\widetilde{p}_{t+\Delta}(i)\}_{i=0}^{N_{t+\Delta}}, I_{t+\Delta})$. Note that the change in the budget set includes the change in the range of goods available, from $[0, N_t]$ to $[0, N_{t+\Delta}]$.

In Appendix E, we derive the change in welfare as defined by (41). In the ASSE, this does not depend on the starting time t and can be written as

$$\zeta_t(\Delta) = -\frac{\varepsilon}{1-\varepsilon} \log \left(1 - \frac{1-\varepsilon}{\varepsilon} \Lambda g^* \Delta \right), \text{ where } \Lambda = \frac{(1-\varepsilon)\overline{u}}{c(0)^{1-1/\varepsilon}} > 1. \tag{42}$$

Here, g^* is the measured GDP growth rate in the ASSE, given by (37), and $\Lambda > 1$ is a correction term, which we discuss below. As shown in the left panel of Figure 7, function

[43] When the ASSE exists (i.e., when $\psi \in (\varepsilon, 1)$), $\int_0^\infty n_{t-\tau} q(\tau)^{\varepsilon-1} d\tau = n^*/\beta(\psi - \varepsilon)$.

[44] This metric is called micro welfare in Baqaee and Burstein (2023) and real consumption in Jaravel and Lashkari (2024).

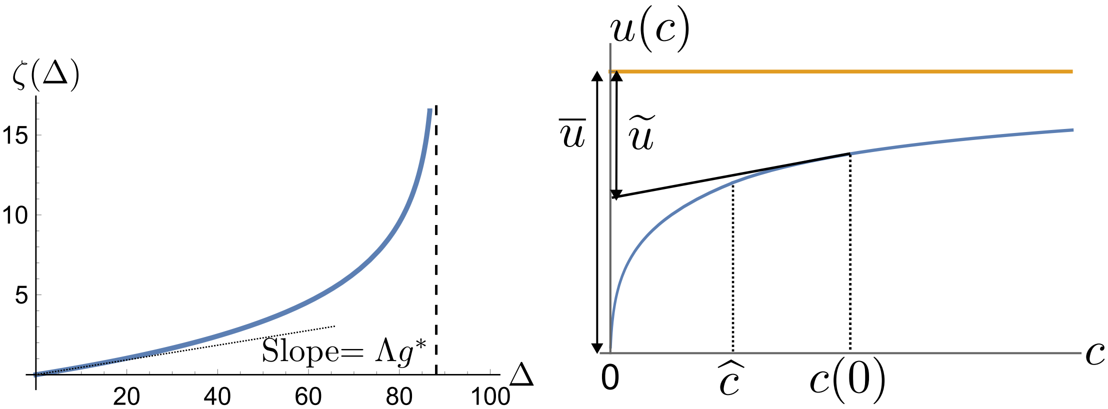

Figure 7: Money-metric of Utility (Left) and the Correction Term (Right).
The value of Λ is given by $\widetilde{u}/\overline{u} > 1$ in the right panel.

$\zeta(\Delta)$ increases more than linearly with Δ. Since $\zeta(\Delta)$ in (41) is defined in logs, this fact means that the money metric of utility (or real consumption as defined by Jaravel and Lashkari, 2024) increases more than exponentially with time. Moreover, function $\zeta(\Delta)$ explodes in a finite duration at $\Delta = \varepsilon/((1-\varepsilon)\Lambda g^*)$. The explosion can be interpreted as follows. After $\varepsilon/(1-\varepsilon)\Lambda g^*$ years, the economy has substantially more varieties than it does today, and the utility from the increased varieties is so high that it cannot be compensated by increasing the expenditure of the representative consumer today. This is because there is an upper bound in the utility obtainable from each individual good (i.e., $\overline{u}$); therefore, given the available range of goods today, there is an upper bound in the overall utility (measured in utils) that can be achieved from increasing the expenditure.

Next, let us discuss the relationship between the change in the money-metric of utility (real consumption) and the measured real GDP growth rate, g^*. Note that $\zeta(\Delta)$ is the equivalent variation between two distant times, t and $t+\Delta$, while g^* represents the instantaneous rate of change in real GDP. To align them, we consider the instantaneous rate of change in the money-metric of utility,

$$\zeta'(0) = \Lambda g^*. \tag{43}$$

Since the correction term Λ is greater than 1, as shown in (42), Equation (43) indicates that the change in the money-metric of utility (real consumption) is greater than the measured real GDP growth rate. In other words, the real GDP statistics underestimate the growth in the money-metric utility. The difference comes from the fact that the marginal utility of goods is observed only after introduction. When a new good is introduced, the consumer purchases $c(0)$ units of it and obtains the utility of $u(c(0))$ (in utils). To accurately measure the benefit from new varieties, we need to incorporate $u(c(0))$ in the measurement. However, there is no way to measure $u(c(0))$, and the GDP statistics in effect replace it with $c(0) \cdot u'(c(0))$, where $u'(c(0))$ can be indirectly observed from $p(0)$. In effect, the GDP statistics are calculated with the assumption that consumers have a utility function that is linear from 0 to $c(0)$ with a slope of $c'(0)$, as shown by the tangent line in the right panel of Figure 7. From this perspective, the highest amount of utility that can be obtained from one variety of goods is viewed as $\widetilde{u} = u(\infty) - u(c(0)) + c(0) \cdot u'(c(0)) = (1/(1-\varepsilon))c(0)^{1-1/\varepsilon}$, while the true value is $\overline{u}$. Since $u(\cdot)$ is concave, $c(0) \cdot u'(c(0))$ is smaller than $u(c(0))$, which implies that $\widetilde{u} < \overline{u}$; therefore, the GDP statistics underestimate the benefit of new varieties. The correction term $\Lambda = \overline{u}/\widetilde{u} > 1$ represents the ratio between the two.

4 Obsolescence

In the prototype model of Section 3, we considered an environment where goods stay in the market forever ($T = \infty$) and where consumers have symmetric preferences across goods (14). Sustained growth in the measured GDP then required the price elasticity of demand ε to be less than one as the quality-adjusted price falls to zero. The condition $\varepsilon < 1$ was necessary to induce consumers to spend less on older (and cheaper) goods. In reality, however, consumers may spend more on new goods simply because they prefer them to older ones, even without the assumption of $\varepsilon < 1$. Here, we show that this assumption can be relaxed once we include obsolescence.

We now consider a generalized version of the lifetime utility function (14):

$$\int_0^\infty \left[\int_0^{N_t} \left[\delta(t - s(i)) u(\widetilde{c}_t(i)) + (1 - \delta(t - s(i)))\widehat{u} \right] di \right] e^{-\rho t} dt, \tag{44}$$

where $t - s(i) = \tau$ is the time after introduction.[45] The function $\delta(\tau)$ is weakly decreasing

[45] Alternatively, we may specify obsolescence as a function of $N_t - i$, i.e., the number of goods newer

in τ with $\delta(0) = 1$ and $\lim_{t\to T} \delta(t) = 0$, where $T > 0$ can be finite or infinite. If T is finite, it defines the lifespan of individual goods. If T is infinite, $\delta(t)$ converges to zero as $t \to \infty$. The steepness of function $\delta(\tau)$ represents the speed of obsolescence, or equivalently, consumers' taste for recently developed goods. Obsolescence may occur for different reasons and has varied effects on the utility of individuals. The constant $\widehat{u} \in [0, \overline{u}]$, where $\overline{u} = u(\infty)$, controls for those differences. One example is the case when newer products replace some of the functionalities of older goods. Suppose that a portion $1 - \delta(\tau)$ of an age-τ good's functionality can be fulfilled by newer goods for free.[46] In this case, we can assume that consumers receive the utility of $(1 - \delta(\tau))\widehat{u}$ for free, with $\widehat{u} > 0$, and the consumption of age-τ good affects only part $\delta(\tau)u(c(\tau))$ of the period utility. Another example is when consumers value the newness of products. The most extreme case is $\widehat{u} = 0$.[47] This specification is suitable, for example, when considering fashion cycles, where outdated and cheaper items are replaced by newer and more expensive ones. As we will discuss in Section 5, the economy can be composed of several sectors, and obsolescence can occur for different reasons across sectors. While the reasons for obsolescence have different implications for welfare, the measured GDP growth rate does not capture those differences, as we see below.

We keep all other settings in Section 3 except that we allow any $\varepsilon > 0$ in the sub-utility function (15). If $\varepsilon > 1$, we can simply assume that $u(c) = c^{1-1/\varepsilon}/(1 - 1/\varepsilon)$ for all $c > 0$.[48] If $\varepsilon \leq 1$, the sub-utility function is the same as (15), and we again assume

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than i. In the ASSE, where n^* new goods are developed per unit time, $\delta(N_t - i)$ becomes $\delta(n^*\tau)$, which shows that obsolescence is faster when R&D is more active. An additional implication in this setting is that policies that promote horizontal R&D may increase the measured GDP growth rate. A higher n^* will make function $\delta(n^*\tau)$ steeper as a function of τ. As we show below, faster obsolescence accelerates measured growth.

[46] For example, suppose that the newest good is smartphones and that the age-τ good is a calculator. When we have smartphones, a large part of the functionality of calculators (which corresponds to $1-\delta(\tau)$) is fulfilled without additional cost.

[47] In this case, U_t will remain constant in the ASSE even when $g^* > 0$. See the discussion on utility in the latter half of this section.

[48] This means that the second line of (15) applies for all $c > 0$ with $\widehat{\varepsilon} = \varepsilon > 1$ and $\underline{u} = 1$.

that $\widehat{c}$ is small enough that all existing goods satisfy $\widetilde{c}_t(i) \geq \widehat{c}$. In this setting, the ASSE exists if and only if $\int_0^T \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1} d\tau$ is finite.[49] In the ASSE, the expenditure for an age-τ good is $e(\tau) = p(\tau)x(\tau) = (1+\mu)D^*\delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1}$. This equation illustrates that even when $\varepsilon > 1$, expenditures for older goods decrease with age if obsolescence is fast enough. Proposition 1 continues to apply in an environment with obsolescence. Given that $\int_0^T e(\tau)d\tau$ is finite, which is equivalent to the finiteness of $\int_0^T \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1} d\tau$, the formula for the GDP growth rate (12) gives

$$g^* = \frac{\int_0^T \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1} g_q(\tau) d\tau}{\int_0^T \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1} d\tau}. \tag{45}$$

When goods retire from the market at a certain age (i.e., when T is finite), $\int_0^T \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1} d\tau$ is obviously finite. Therefore, we always obtain a positive long-term GDP growth rate. When T is infinite and the rate of obsolescence is constant at $\overline{\delta} > 0$ per year, function $\delta(\tau)$ can be expressed as $\exp(-\overline{\delta}\tau)$. In this case, $\int_0^\infty \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1} d\tau$ becomes finite because $\delta(\tau)^\varepsilon$ decreases exponentially with τ and $q(\tau)^{\varepsilon-1}$ does not increase exponentially. Therefore, a constant rate of obsolescence always sustains positive measured GDP growth regardless of ε. Positive GDP growth can also be maintained with slower, non-exponential obsolescence. Consider an example where $\delta(\tau)$ is a negative power function of τ: $\delta(\tau) = \delta_0^\omega(\tau+\delta_0)^{-\omega}$ where ω and δ_0 are positive constants.[50] Then,

[49] This result is obtained in a similar way as the derivation of Equations (27) and (31) in Section 3.4. In both cases, $\varepsilon > 1$ and $\varepsilon < 1$, the consumption of good i becomes $\widetilde{c}_t(i) = \lambda_t^{-\varepsilon}\widetilde{p}(i)^{-\varepsilon}\delta(t-s(i))^\varepsilon$. From this, we can write the equilibrium output of age-τ good in the ASSE as $x(\tau) = D^*\delta(\tau)^\varepsilon q(\tau)^\varepsilon$. Given that the markup is μ, the present discounted value of a new firm is $V^* = \mu D^* \int_0^\infty \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1}$. By substituting V^* into the free entry condition $V^* = 1/a$, we obtain $D^* = \left(a\mu \int_0^T \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1} e^{-\rho\tau} d\tau\right)^{-1}$, which is always positive and finite because of the $e^{-\rho\tau}$ term. Using this value of D^*, the labor market equilibrium implies that the speed of innovation is $n^* = aL\left(1 + aD^* \int_0^T \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1} d\tau\right)^{-1}$. The value of n^* is positive if and only if $\int_0^T \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1} d\tau$ is finite.

[50] We need a constant $\delta_0 > 0$ in $(\tau+\delta_0)^{-\omega}$ because otherwise, $\tau^{-\omega}$ cannot be defined when $\tau = 0$ and $\omega > 0$. The δ_0^ω term normalizes the $\delta(\tau)$ function so that $\delta(0) = 1$.

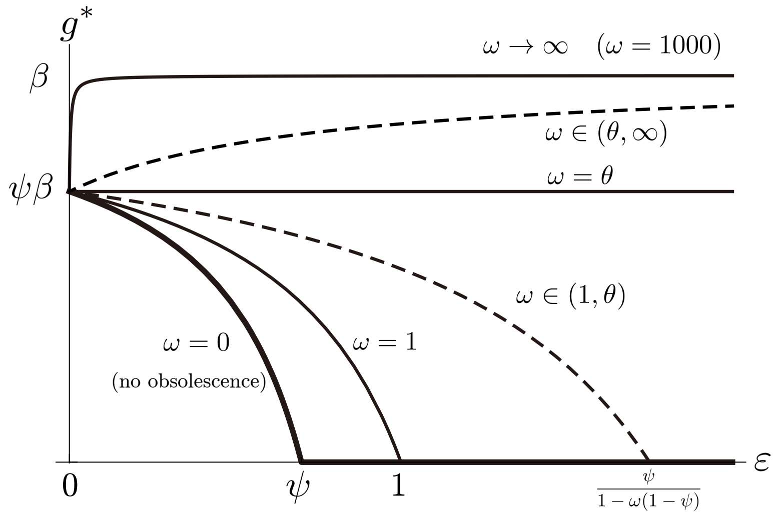

Figure 8: Price Elasticity of Individual Goods and the Measured Long-term GDP Growth Rate Under Different Speeds of Obsolescence.

$\int_0^\infty \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1} d\tau$ becomes finite if and only if[51]

$$\varepsilon < \begin{cases} \frac{\psi}{1-\omega(1-\psi)} & \text{if } \omega < \frac{1}{1-\psi} \; (\equiv \theta), \\ \infty & \text{if } \omega \geq \frac{1}{1-\psi}. \end{cases} \tag{46}$$

In a particular case of $\delta_0 = \kappa_0$, where κ_0 is defined in Equation (21), the long-term GDP growth rate (45) becomes[52]

$$g^* = \frac{\psi - \varepsilon + (1-\psi)\varepsilon\omega}{1 - \varepsilon + (1-\psi)\varepsilon\omega}\beta, \tag{47}$$

which is positive when Condition (46) holds. Figure 8 depicts the relationship between ε and g^* for various values of ω. As we have seen in Section 3, sustained GDP growth

[51]Using Equation (21), $\int_0^\infty \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1} d\tau = \delta_0^{\varepsilon\omega} \kappa_1^{\varepsilon-1} \int_0^\infty (\tau + \delta_0)^{-\omega\varepsilon} (\tau + \kappa_0)^{\theta(\varepsilon-1)} d\tau$. The integral becomes finite if and only if the sum of the powers of the integrand, $-\omega\varepsilon + \theta(\varepsilon - 1)$, is less than minus one. From $\theta = 1/(1-\psi)$, this condition is equivalent to Condition (46).

[52]When ψ is larger, $g_q(\tau)$ is higher given age τ. Nevertheless, Equation (47) shows that g^* is decreasing in ψ if $\varepsilon > 1$. When $\varepsilon > 1$, a larger ψ will induce consumers to spend more on cheaper, older goods. As a result, the expenditure is skewed more toward older goods, where $g_q(\tau)$ is small, reducing the expenditure-weighted average of $g_q(\tau)$:

requires $\varepsilon < \psi$ when obsolescence is not present.[53] When obsolescence is faster (ω is higher), Condition (46) becomes easier to satisfy. In particular, when $\omega > 1$, the first line of Condition (46) is greater than one, which means that $\varepsilon < 1$ is not necessary for $g^* > 0$. When ω is greater than $1/(1-\psi)$, the long-term GDP growth rate g^* is positive regardless of ε.[54]

Figure 8 also shows that when ω is increased, the entire curve for g^* moves upward. Faster obsolescence not only makes sustained GDP growth more likely but also accelerates the measured rate of economic growth. Intuitively, obsolescence skews expenditures toward newer goods. Since newer goods have greater margins for productivity increases than older goods do, the overall growth rate increases with obsolescence. These results have important policy implications. When the government attempts to protect obsolete companies (or industries), it reduces the GDP growth rate not only because of efficiency loss but also because of how expenditure is allocated across firms and industries. Conversely, advertisements and marketing practices that attract consumers to newer goods increase GDP growth, even when the attractiveness of the newer goods is illusory.

Below, we discuss the relationship between the measured GDP growth rate and the change in the welfare of consumers. In appendix G, we show that the instantaneous utility (measured in utils) increases linearly with time, with a slope of $n^*\widehat{u}$ in the ASSE. Therefore, if the obsolescence is caused entirely by the change in consumer tastes (i.e., $\widehat{u} = 0$), the instantaneous utility is stationary, even if the measured GDP growth is positive. To bridge this gap, we consider money-metric utility as in Section 3.8, but this time, we account for the change in taste, again following Baqaee and Burstein (2023) and Jaravel and Lashkari (2024). The change in welfare between t and $t + \Delta$ measured using the equivalent variation is $\zeta_t(\Delta)$, where $\zeta_t(\Delta)$ solves

$$v(\{\widetilde{p}_{t+\Delta}(i)\}_{i=0}^{N_{t+\Delta}}, I_{t+\Delta}, t+\Delta) = v(\{\widetilde{p}_t(i)\}_{i=0}^{N_t}, I_t \exp \zeta_t(\Delta), t+\Delta). \tag{48}$$

The only difference between (41) and the above equation is that there is a third argument

[53] When $\omega = 0$, Condition (46) and Equation (47) reduce to Equation (37).

[54] Interestingly, the measured GDP growth rate increases with ε when $\omega > 1/(1-\psi)$. A higher ε means that consumers are more willing to move from old and obsolete goods to newer goods, thus enhancing the positive effect of obsolescence on growth.

in the indirect utility function $v(\cdot)$, which represents the time at which the preference of consumers is used for evaluation. Following the literature, we use the preference of consumers at $t+\Delta$ to evaluate both sides of (48).[55]

The instantaneous rate of growth in money-metric utility is given by $\zeta'(0)$. In appendix G, we obtain its value in the ASSE as

$$\zeta'(0) = \begin{cases} g^* + \left(\frac{1}{\varepsilon-1} + \frac{\widehat{u}}{c(0)^{1-1/\varepsilon}}\right)\left(\int_0^T \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1}\right)^{-1} & \text{if } \varepsilon > 1, \\ g^* + \left(\frac{1}{1-\varepsilon}(\Lambda-1) + \frac{\widehat{u}}{c(0)^{1-1/\varepsilon}}\right)\left(\int_0^T \delta(\tau)^\varepsilon q(\tau)^{\varepsilon-1}\right)^{-1} & \text{if } \varepsilon \in (0,1), \end{cases} \tag{49}$$

where g^* is the measured GDP growth rate in (45) and $\Lambda > 1$ is a correction term defined in (42). Equation (49) shows that the measured GDP growth rate, g^*, is a part of the growth in money-metric utility, $\zeta'(0)$. This part of $\zeta'(0)$ comes from the changes in the prices of existing goods. In fact, $\zeta'(0)$ is greater than g^* because $\frac{1}{\varepsilon-1}$ when $\varepsilon > 1$ and $\frac{1}{1-\varepsilon}(\Lambda-1)$ when $\varepsilon \in (0,1)$ are both positive. These terms represent the benefits of new goods that are not measured by g^*.[56] If $\widehat{u} > 0$, there are also external effects of new goods, $\widehat{u}/c(0)^{1-1/\varepsilon} > 0$, which are again not measured in g^* but are included in $\zeta'(0)$.

When $\widehat{u} = 0$, the money-metric utility (real consumption) is growing even though the instantaneous utility (measured in utils) is asymptotically constant, as we have seen above. To see this point, suppose that the availability and price of goods do not change from t to $t+\Delta$ but that only the preference changes. Then, at time $t+\Delta$, the instantaneous utility (measured in utils) will be lower because the consumers more deeply discount the utility from existing goods that are now older. To regain the same level of instantaneous utility (in utils), which is constant in the ASSE, either (i) the availability and prices of goods need to change to those of time $t+\Delta$, or (ii) the consumers need to be given more budget. Equation (48) makes this comparison and determines how much more expenditure is needed in the latter case, which is the growth in the money-metric utility between t and $t+\Delta$. In other words, if the availability of goods and their prices are fixed, the welfare of consumers will decline given that their preferences change over time. Relative to this situation, the introduction of new goods and a decrease in quality-

[55] This means that we are calculating the equivalent variations rather than compensating variations.

[56] See the discussion in Section 3.8.

adjusted prices improve the welfare of consumers (as a result, the instantaneous utility measured in utils becomes constant). The growth in the money-metric utility measures these benefits, and the measured GDP growth rate captures a part of it.

5 Multiple sectors

In the non-exponential growth theory, we define the steady state as the situation in which the paths of quality-adjusted prices and quantities, $p(\tau)$ and $x(\tau)$, follow the same pattern in terms of their age (see Definition 1 in Section 2.2). This definition allows the prices and quantities of individual goods at a given time to differ depending on their age. In this sense, our definition of the steady state is more flexible than that in most endogenous growth models, where goods are symmetric in the steady state. Nevertheless, once we look at the data, it is immediately apparent that goods in different categories follow distinct lifecycle patterns. For example, while the product lifecycle is relatively fast in electronics, some basic goods (e.g., grains) show little sign of lifecycle movements.

In this section, we further extend the notion of the steady state by allowing $p(\tau)$ and $x(\tau)$ to follow different patterns. We categorize goods into groups (which we call sectors) so that goods in a sector have the same pattern of movements in terms of quality-adjusted price and quantity with respect to their age, at least in the long run. More specifically, suppose that there are $J > 0$ sectors (or categories) of goods and label each by $j \in \{1, \ldots, J\}$. $N_{j,t}$ denotes the index of the newest good in sector $j \in \{1, \ldots, J\}$. The number of new goods introduced per unit time, $\dot{N}_{j,t} \geq 0$, can differ across sectors. The quality-adjusted price of the ith good in sector j and its quality-adjusted quantity are denoted by $\widetilde{p}_{j,t}(i)$ and $\widetilde{x}_{j,t}(i)$. In this setting, we define the asymptotic steady state as follows.

Definition 3. *A non-exponential asymptotic steady state with multiple sectors is the situation where $\dot{N}_{j,t}$, $\widetilde{p}_{j,t}(i)$ and $\widetilde{x}_{j,t}(i)$, for all $j \in \{1, \ldots, J\}$, satisfy the following conditions:*

(a) $\dot{N}_{j,t}$ converges to a constant; i.e., $\dot{N}_{j,t} \to n_j \geq 0$.

(b) $\widetilde{p}_{j,t}(i)$ and $\widetilde{x}_{j,t}(i)$ converge to time-invariant functions of $\tau = t - s(i)$; i.e., $\widetilde{p}_{j,t}(i) \to$

$p_j(\tau)$ and $\widetilde{x}_{j,t}(i) \to x_j(\tau)$.

(c) Assumption 1 holds, where $p(\tau)$, $x(\tau)$ and T are replaced by $p_j(\tau)$, $x_j(\tau)$ and T_j, respectively.

(d) The expenditure share of the sector,

$$\alpha_{j,t} = \frac{\int_{i \in X_{j,t}} \widetilde{p}_{j,t}(i)\widetilde{x}_{j,t}(i)di}{\sum_{j'=1}^{J} \int_{i \in X_{j',t}} \widetilde{p}_{j',t}(i)\widetilde{x}_{j',t}(i)di}, \tag{50}$$

where $X_{j,t}$ is the set of goods in production in sector j, converges to a constant value, i.e., $\alpha_{j,t} \to \alpha_j \geq 0$.

Definition 3 says that the economy is in a steady state if the composition of sectors in terms of expenditure share is stationary, and each sector satisfies the requirement for the steady state in Definition 1. In addition, Definition 3 does not require n_j to be positive, and therefore includes the possibility where the introduction of goods eventually stops in some sectors. Additionally, it allows α_j to be zero for some j, which means that some sectors may disappear in the long run.

Like Equation (5), the instantaneous GDP growth rate in this multisector economy at any given time t can be defined as follows:

$$g_t = \frac{\sum_{j=1}^{J} \left(\dot{N}_{j,t}\widetilde{p}_{j,t}(N_{j,t})\widetilde{x}_{j,t}(N_{j,t}) + \int_{i \in X_{j,t}} \widetilde{p}_{j,t}(i)\dot{\widetilde{x}}_{j,t}(i)di - \int_{i \in \Omega_{j,t}} \widetilde{p}_{j,t}(i)\widetilde{x}_{j,t}(i)di \right)}{\sum_{j=1}^{J} \int_{i \in X_{j,t}} \widetilde{p}_{j,t}(i)\widetilde{x}_{j,t}(i)di}. \tag{51}$$

Here, the denominator gives the expenditure for all the goods, the first term in the numerator is the value of all the new goods introduced at time t, the second term is the value of the changes in the production of existing goods, and the third term is the value of the disappearing goods ($\Omega_{j,t}$ is the set of goods in sector j that disappear at time t). Using the sectoral expenditure share defined by Equation (50), Equation (51) can be expressed as the share-weighted average of the sectoral GDP growth rate.

$$\begin{aligned} g_t &= \sum_{j=1}^{J} \alpha_{j,t} g_{j,t}, \text{ where,} \\ g_{j,t} &= \frac{\dot{N}_{j,t}\widetilde{p}_{j,t}(N_{j,t})\widetilde{x}_{j,t}(N_{j,t}) + \int_{i \in X_{j,t}} \widetilde{p}_{j,t}(i)\dot{\widetilde{x}}_{j,t}(i)di - \int_{i \in \Omega_{j,t}} \widetilde{p}_{j,t}(i)\widetilde{x}_{j,t}(i)di}{\int_{i \in X_{j,t}} \widetilde{p}_{j,t}(i)\widetilde{x}_{j,t}(i)di}. \end{aligned} \tag{52}$$

Since Equation (52) takes the same form as Equation (5), we can utilize Proposition 1 to obtain the long-term GDP growth rate in a steady state.

Proposition 2. *Suppose that the multisector economy converges to an asymptotic steady state, as defined by Definition 3. Then, the real GDP growth rate g_t asymptotes to*

$$g = \sum_{j=1}^{J} \alpha_j g_j, \tag{53}$$

where g_j is given by Proposition 1, in which $p(\tau)$, $x(\tau)$ and g are replaced by $p_j(\tau)$, $x_j(\tau)$ and g_j, respectively.

Proposition 2, combined with Proposition 1, implies that if there is a category of goods (a sector) with a positive GDP share where Conditions (10) and (11) hold in the long run, the economy-wide long-term GDP growth rate can be positive and finite. Similar to Figure 4 in Section 2.4, we can draw the evolution of $\{x_j(\tau), p_j(\tau)\}$ in the quantity–price space and the evolution of $p_j(\tau)x_j(\tau)$ against τ. The numerator and denominator of g_j are then graphically represented as the blue and yellow areas, respectively. If $\alpha_j > 0$ and both areas are positive and finite, then sector j contributes positively to the long-term GDP growth rate. As in Example 2 of Figure 4, g_j can be negative if the prices of older and disappearing goods in that sector are higher than those of new goods in the same sector. Nonetheless, aggregate GDP growth becomes zero only by coincidence; therefore, nonzero long-term growth rates are the norm rather than the exception. This result contrasts with existing endogenous growth models, where the growth rate can be nonzero only under strict knife-edge conditions.

As a final note, observe that g_js in Proposition 2 are the sectoral output growth rates measured according to their own sectoral price indices. They do not coincide with the sectoral output growth calculated using the general price levels (e.g., the GDP deflator). In the long run, the expenditures to all the surviving sectors (those with positive α_j values) increase at the same rate. Even the sectors with $g_j = 0$ record real income growth of g.

6 Concluding Remarks

Non-exponential growth theory provides a novel interpretation of observed stability in the measured GDP growth rate by focusing on the movement of the quantities and prices of individual goods and calculating the GDP growth rate on the basis of SNA statistics (e.g., the NIPA). It shows that the observed sustained GDP growth is consistent with a less-than-exponential increase in the variety and quality-adjusted output of each good. This finding enables researchers to construct endogenous growth models under less restrictive assumptions than the knife-edge conditions that are required in existing full endogenous growth models. As a result, this paper suggests that an endogenous growth theory can be applied to data with significantly weaker restrictions than previously required.[57]

The readers may still wonder whether we should describe economic growth, as explained in this paper, as exponential or not. The answer depends on how we evaluate economic growth. More specifically, this paper demonstrates that economic growth can be viewed from four distinct perspectives when we explicitly consider multiple final goods that are not necessarily symmetric. First, we can view growth in terms of how the vector of production changes over time, where each entry in the vector represents the output of an individual product. In our model, the dimension of the vector increases linearly over time, and each entry of the vector increases less than exponentially (the rate of growth decreases to zero). In this sense, output growth is not exponential. Nevertheless, the economy can continue the growth process because the expenditure for older goods decreases as goods age, and newly introduced goods receive a constant proportion of the total expenditure. Therefore, the incentive to innovate can be maintained without strong externalities.[58]

[57]Nevertheless, we make simplifying assumptions for the sake of expositional simplicity and ease of understanding. Notably, while existing variety-expansion endogenous growth models assume that the elasticity of spillover from R&D activity is exactly $\phi = 1$, we assume that there is none, i.e., $\phi = 0$. Additionally, we assume that the population is constant. In a working paper, we demonstrate that the intuitions from non-exponential growth theory remain applicable when population growth and decline are incorporated, as well as when R&D externality is present but weaker than $\phi = 1$.

[58]In standard variety expansion models, all goods are symmetric and receive the same expenditure share. Therefore, as the number of goods increases, the share of the expenditure given to a single new

The other three perspectives attempt to map the change in the output vector into a scalar measure. Of these three, two evaluate economic growth in terms of the change in consumer utility. A crude approach is to examine the change in instantaneous utility in the model. In our baseline prototype model, the instantaneous utility (measured in units of the utility function) increases only linearly over time. In an extended model with obsolescence, the change in utility can be zero in an extreme case. However, interpreting these results is difficult because the unit of utility in the model lacks a clear economic meaning.

An alternative approach is to look at the change in the money-metric utility, which represents the equivalent variation between two time points. We have shown that the money-metric utility can increase more than exponentially, especially when consumers highly value the benefits of newer goods in comparison with the benefits of consuming larger quantities of existing goods. In this sense, we could say that growth measured in terms of the utility of consumers is more than exponential.

The fourth way of measuring growth is to focus on real GDP as measured by the SNA. The real GDP growth rate measures the value of change in economic activity, including the appearance and disappearance of goods, as well as changes in the quantity of production of existing goods, divided by the value of existing economic activity. In our model, the real GDP growth rate asymptotically becomes a finite and positive constant. This result stems from the fact that there is a constant (but not exponential) flow of new goods, the value of new goods is higher than that of older, disappearing goods, and the value of existing economic activity is bounded because the expenditure for individual goods decreases as the goods age. The level of real GDP is obtained by chaining the real GDP growth rate and therefore increases exponentially. The real GDP growth rate is meaningful in the sense that it captures an important portion of the instantaneous change in the money-metric utility, although it misses some of the benefits from new

good dilutes. This means that profits obtained from a single successful R&D also decrease. Therefore, to provide firms with sufficient incentives to engage in R&D in equilibrium, these models require a strong degree of externality in the R&D process so that the cost of inventing new goods declines exponentially. Moreover, GDP growth can be maintained only when the number of goods increases exponentially because the contribution of each new good to the economic growth rate decreases toward 0.

goods, as well as possible benefits from external effects.

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Online Appendix for "Non-Exponential Growth Theory"

October 6, 2025

Appendix A Simplest Examples of Non-Exponential Growth

A.1 When Goods Become Free in Two Periods

Consider an economy in discrete time with overlapping generations of products. One new good is introduced every period. When the good is introduced, the price is 2; it falls to 1 in the next period and to 0 thereafter. The output quantity is 1 when it is introduced, 2 in the next period, and 3 thereafter. For example, we can consider each good as a medication for a particular disease. In this example, after two periods, generic drugs with the same effect become available (almost) for free. The pattern of movements of quantities and prices is summarized below.

index	$i = N_t - 3$	$i = N_t - 2$	$i = N_t - 1$	$i = N_t$	$i = N_t + 1$
$\widetilde{x}_{t-1}(i)$	3	2	1	N/A	N/A
$\widetilde{x}_t(i)$	3	3	2	1	N/A
$\widetilde{x}_{t+1}(i)$	3	3	3	2	1
$\widetilde{p}_t(i)$	0	0	1	2	N/A
$\widetilde{p}_{t+1}(i)$	0	0	0	1	2

In the table, i is the index of goods, and N_t is the index of the newest good in period t. The second row shows the amount of production of each good at time t, $\widetilde{x}_t(i)$. The price of each good at time t, $\widetilde{p}_t(i)$, is shown in the fourth row. Note that the newest good in period $t+1$ is $i = N_{t+1} = N_t + 1$. Therefore, the values in the $\widetilde{x}_{t+1}(i)$ and $\widetilde{p}_{t+1}(i)$ rows are shifted to the right by one column. The opposite holds for the $\widetilde{x}_{t-1}(i)$ row.

In SNA statistics, the real GDP growth rate from period $t-1$ to t is defined as the growth in the value of output when the value is evaluated by the prices in the base year. In practice, the base year is frequently updated; thus, we assume that the base year is updated every period to the year of evaluation (i.e., period t). Then, the real GDP growth rate (per period) between period $t-1$ and period t is

$$g_{t,t-1} = \frac{\sum_{i=0}^{N_t} \widetilde{p}_t(i)\widetilde{x}_t(i) - \sum_{i=0}^{N_t-1} \widetilde{p}_t(i)\widetilde{x}_{t-1}(i)}{\sum_{i=0}^{N_t-1} \widetilde{p}_t(i)\widetilde{x}_{t-1}(i)}. \tag{A.1}$$

Using the numbers in the table, the value of the output in t using the prices in t is $\sum_{i=0}^{N_t} \widetilde{p}_t(i)\widetilde{x}_t(i) = 1 \times 2 + 2 \times 1 = 4$. Similarly, the value of the output in $t-1$ using the prices in t is $\sum_{i=0}^{N_t-1} \widetilde{p}_t(i)\widetilde{x}_{t-1}(i) = 1 \times 1 = 1$. Therefore, the GDP growth rate is $g_{t,t-1} = (4-1)/1 = 3 = 300\%$. Similarly, we can calculate the GDP growth rate between periods t and $t+1$ as $g_{t,t+1} = \left(\sum_{i=0}^{N_t+1} \widetilde{p}_{t+1}(i)\widetilde{x}_{t+1}(i) - \sum_{i=0}^{N_t} \widetilde{p}_{t+1}(i)\widetilde{x}_t(i)\right) / \sum_{i=0}^{N_t} \widetilde{p}_{t+1}(i)\widetilde{x}_t(i) = (4-1)/1 = 300\%$. We always obtain the same growth rate as long as this pattern of quantities and prices continues. Therefore, the measured GDP growth in this steady state is constant and positive, even though the output of any good does not grow at an exponential rate.

A.2 When Goods Become Obsolete in Two Periods

Similar to the previous example, one new good is introduced every period. When the new good is introduced, its price is 2, and it falls to 1 thereafter. The good is produced only for the period when it is introduced and for one period afterward. The output quantity is 1 for both periods and then 0 thereafter. One can think of each good as a medication for a particular infectious disease. Owing to medication, the disease is eradicated in two periods, and the good is no longer in demand. The pattern is summarized below.

index	$i = N_t - 3$	$i = N_t - 2$	$i = N_t - 1$	$i = N_t$	$i = N_t + 1$
$\widetilde{x}_{t-1}(i)$	0	1	1	N/A	N/A
$\widetilde{x}_t(i)$	0	0	1	1	N/A
$\widetilde{x}_{t+1}(i)$	0	0	0	1	1
$\widetilde{p}_t(i)$	1	1	1	2	N/A
$\widetilde{p}_{t+1}(i)$	1	1	1	1	2

We can again calculate the GDP growth rate via Equation (A.1). The value of the output in t using the prices in t is $\sum_{i=0}^{N_t} \widetilde{p}_t(i)\widetilde{x}_t(i) = 1\times1+2\times1 = 3$. Similarly, the value of the output in $t-1$ using the prices in t is $\sum_{i=0}^{N_t-1} \widetilde{p}_t(i)\widetilde{x}_{t-1}(i) = 1\times1+1\times1 = 2$. Therefore, the GDP growth rate between periods $t-1$ and t is $g_{t,t-1} = (3-2)/2 = 1/2 = 50\%$.

We can also calculate $g_{t,t+1}$, which is again 50%. The measured growth rate remains constant as long as the same pattern persists.

When we compare the output quantities in periods t and $t-1$, the difference is that we have one unit of the newest good (whose value is 2), and we lose one unit of the 2-period-old good (whose value is 1). Since the price of the new good is higher than that of the old, disappearing good, the numerator is positive. On the basis of the observed prices, the GDP growth rate attributes a greater value to newly appearing goods than to old, disappearing goods.

A.3 Interpretation and Connection to Later Sections

Both examples satisfy the required conditions for positive GDP growth explained in the Introduction: (i) new goods are introduced over time, (ii) the price of goods decreases with age, and (iii) the expenditure for old goods is limited. The two examples differ in terms of how condition (iii) is accomplished. In the first example, the price of a good becomes zero, while in the second example, the quantity becomes zero. In Sections 3 and 4, we provide two general equilibrium models that are essentially similar to these examples. Section 3 considers an economy where new goods are introduced by R&D and the quality-adjusted productivity of a good increases less than exponentially through the learning-by-doing process. There is no externality in the R&D process, and the population is constant. Therefore, the flow of new goods introduced to the market is constant. Then, given that the utility function of consumers is such that the expenditure on a good decreases as its quality-adjusted price approaches zero (which means that the price elasticity of demand is less than one as $p \to 0$), the measured GDP growth rate becomes positive in the long run, similar to the first example. In Section 4, which essentially corresponds to the second example, we introduce obsolescence to the utility function of consumers so that the marginal utility from a good declines as the good ages. We then show that positive GDP growth is obtained under weaker conditions for the price elasticity of demand.

In Sections 3 and 4, we also analyze the relationship between the measured GDP growth rate and the utility of consumers. In both examples, there is no component of consumption that grows exponentially, while the measured real GDP is growing at a positive (exponential) rate. In Section 3.8, we fill this gap by considering the money-metric utility. In the first example, over time, consumers become better off because more diseases can be cured with newly developed medications. The money-metric utility evaluates this benefit in terms of the amount of the budget (expenditure) that is given to a consumer at a given time. If an individual is given more budget in the first example at time t_0, she will be able to buy more medications that are already developed but have not yet become free (at each time, there are two kinds of such goods). However, after two periods, these medications become free, and it becomes possible to cure more diseases, which could not be cured at time t_0 regardless of the individual's budget. Therefore, in this simplest example, the individual's utility at time t_0 cannot surpass that of $t_0 + 2$, however rich she is. This means that the money-metric utility increases quite rapidly (indeed, more than exponentially) between the two periods. In Section 3.8 and Appendix E, we show that the measured real GDP growth rate captures a main part of the instantaneous rate of change in money-metric utility (in the above example, it corresponds to the increase in the money-metric utility between t_0 and $t_0 + 1$). The relationship between real GDP growth and utility in the second example is more complex because the benefit of individual goods changes over time. Section 4 and Appendix G explain the money-metric utility under taste changes, demonstrating that the GDP growth rate captures a portion of it.

Appendix B Real GDP Growth with Previous Year's Prices

In subsection 2.1, we explained the calculation of the real GDP growth rate between $t-1$ and t using the prices at t. However, in many countries, the real GDP growth rate is calculated using the prices at $t-1$ rather than t. Here, we explain how this can be achieved and demonstrate that the result does not change significantly if the period length is short.

A practical problem of using prices at period $t-1$, denoted by $\widetilde{p}_{t-1}(i)$, is that the prices for the new goods that appear after $t-1$ are not defined. Therefore, the real GDP growth rate, $g_{t,t-1}$, must be defined without using $\widetilde{p}_{t-1}(i)$ for $i \in (N_{t-1}, N_t]$. We cannot simply use $\widetilde{p}_{t-1}(i)$ in the formula similar to (1).

In practice, the change in real output (volume) is often calculated by dividing the change in nominal output (value) by the change in the price level. The change in the nominal output is easier to measure than the real output, and the change in the price level can be estimated by available data. More precisely, the change in the price level between periods $t-1$ and t is usually estimated by using only the prices of goods and services that are available for both $t-1$ and t. In our context, this means that the price of the newest goods, which appear between $t-1$ and t, is not used to estimate the change in the price level between $t-1$ and t. Similarly, if some goods disappear from the market between $t-1$ and t, these goods are not used.

Let X_t be the set of goods in production at time t. Then, $X_t \cap X_{t-1}$ indicates the set of goods that are produced both in periods t and $t-1$. Let $\Omega_{t,t-1}$ denote the set of goods that disappear between $t-1$ and t. Then, $X_{t-1} - \Omega_{t,t-1} = X_t \cap X_{t-1} = X_t - (N_{t-1}, N_t]$ holds. Using these notations, the growth factor of the nominal output measured between times $t-1$ and t is

$$1 + g_{t,t-1}^{\text{nominal}} = \frac{\int_{X_t} \widetilde{p}_t(i)\widetilde{x}_t(i)di}{\int_{X_{t-1}} \widetilde{p}_{t-1}(i)\widetilde{x}_{t-1}(i)di}.$$

The growth factor of the price level between time $t-1$ and t, using the goods that are available at both time $t-1$ and t, is

$$1 + \pi_{t,t-1} = \frac{\int_{X_t \cap X_{t-1}} \widetilde{p}_t(i)\widetilde{x}_t(i)di}{\int_{X_t \cap X_{t-1}} \widetilde{p}_{t-1}(i)\widetilde{x}_t(i)di}.$$

Note that this is a Paasche price index in the sense that quantities at time t, i.e., $\widetilde{x}_t(i)$, are used as weights. Then, the real GDP growth rate (factor) is obtained by dividing the growth factor of nominal output by the growth factor of the price level.

$$
\begin{aligned}
1+g_{t,t-1} &= \frac{1+g_{t,t-1}^{\text{nominal}}}{1+\pi_{t,t-1}} \\
&= \frac{\int_{X_t} \widetilde{p}_t(i)\widetilde{x}_t(i)di}{\int_{X_t\cap X_{t-1}} \widetilde{p}_t(i)\widetilde{x}_t(i)di} \cdot \frac{\int_{X_t\cap X_{t-1}} \widetilde{p}_{t-1}(i)\widetilde{x}_t(i)di}{\int_{X_{t-1}} \widetilde{p}_{t-1}(i)\widetilde{x}_{t-1}(i)di} \\
&= \frac{\int_{X_t} \widetilde{p}_t(i)\widetilde{x}_t(i)di}{\int_{X_t-(N_{t-1},N_t]} \widetilde{p}_t(i)\widetilde{x}_t(i)di} \cdot \frac{\int_{X_t\cap X_{t-1}} \widetilde{p}_{t-1}(i)\widetilde{x}_t(i)di}{\int_{X_t\cap X_{t-1}} \widetilde{p}_{t-1}(i)\widetilde{x}_{t-1}(i)di} \cdot \frac{\int_{X_{t-1}-\Omega_{t,t-1}} \widetilde{p}_{t-1}(i)\widetilde{x}_{t-1}(i)di}{\int_{X_{t-1}} \widetilde{p}_{t-1}(i)\widetilde{x}_{t-1}(i)di} \\
&= \frac{1}{1-\sigma_{t,t-1}^{\text{new}}}(1+g_{t,t-1}^{\text{exist}})(1-\sigma_{t,t-1}^{\text{out}}) \\
&\simeq 1+g_{t,t-1}^{\text{exist}}+\sigma_{t,t-1}^{\text{new}}-\sigma_{t,t-1}^{\text{out}},
\end{aligned}
$$

where $g_{t,t-1}^{\text{exist}}$ is the growth rate of the production of goods that exists at both times t and $t-1$, $\sigma_{t,t-1}^{\text{new}}$ is the expenditure share at time t given to new goods that appear between $t-1$ and t, and $\sigma_{t,t-1}^{\text{out}}$ is the expenditure share at time $t-1$ given to old goods that disappear between $t-1$ and t. This version of the formula divides real GDP growth g_{t-1} into its intensive margin g_{t-1}^{exist} and extensive margin $\sigma_{t,t-1}^{\text{new}}-\sigma_{t,t-1}^{\text{out}}$. Note also that in the evaluation of the intensive margin g_{t-1}^{exist}, the formula uses the prices of period $t-1$, $\widetilde{p}_{t-1}(i)$.

Now, we consider the continuous-time limit of this real GDP growth. By letting the period length be Δ and taking the limit of $\Delta\to 0$, we obtain

$$
\begin{aligned}
\lim_{\Delta\to 0}\frac{g_{t,t-\Delta}}{\Delta} &= \lim_{\Delta\to 0}\frac{dg_{t,t-\Delta}}{d\Delta} \\
&= \lim_{\Delta\to 0}\left(\frac{dg_{t,t-\Delta}^{\text{exist}}}{d\Delta}+\frac{d\sigma_{t,t-\Delta}^{\text{new}}}{d\Delta}-\frac{dg_{t,t-\Delta}^{\text{out}}}{d\Delta}\right) \\
&= \frac{\int_{X_t} \widetilde{p}_t(i)\dot{\widetilde{x}}_t(i)di+\dot{N}_t\widetilde{p}_t(N_t)\widetilde{x}_t(N_t)di-\int_{\Omega_t} \widetilde{p}_t(i)\widetilde{x}_t(i)di}{\int_{X_t} \widetilde{p}_t(i)\widetilde{x}_t(i)di},
\end{aligned}
$$

where the first equality comes from L'Hopital's rule, and Ω_t in the last line is the set of goods that disappear from the market exactly at time t. The above result is the same as the RHS of (5).

Appendix C Two Conditions for Sustained GDP Growth

Corollary 1 shows that the measured GDP growth rate becomes positive when the two conditions (10) and (11) are satisfied. Here, we discuss these conditions in more detail.

Condition (10): the quality-adjusted price falls during the product lifecycle

For this condition to be satisfied, $p(\tau)$ must decrease with τ at least for a portion of the product lifecycle. Recall that we normalize the price level so that the price of the newest goods when they appear does not change over time in the steady state. Therefore, Condition (10) only requires the quality-adjusted prices of older goods to decrease relative to those of newer goods, and it is not essential for the prices of individual goods measured in a currency to decrease.

In terms of actual currencies, we can determine that $p(\tau)$ is decreasing if the quality-adjusted currency prices of individual goods lag behind the growth of the nominal per capita GDP. To see this point, suppose that the per capita nominal GDP growth rate in dollars is $g^{\$}$. Note that, given that $\int_0^T p(\tau)x(\tau)d\tau$ is finite, nominal expenditure in our theory's price normalization is constant, which means that there is a $g^{\$}$ difference in the inflation rate between the prices in theory and in dollars. Then, in dollars, the rate of price change for age-τ good is $p'(\tau)/p(\tau) + g^{\$}$. Therefore, we can determine that $p'(\tau)$ is negative if the quality-adjusted dollar prices of individual goods are increasing less rapidly than $g^{\$}$.

With this definition, the quality-adjusted price of a good may decrease with the age of the good for several reasons. For example, the cost of production falls through learning-by-doing and knowledge spillovers. In this case, time and production experience contribute to price reduction. In addition to cost reduction, changes in the form of competition may lower prices because older goods are typically less protected from competition by patents and trade secrets than newer goods are.

Price reductions also occur in the form of quality improvements. For example, the effective price of computers has been declining for decades, not only because computers have become cheaper but also because the average performance of computers has drastically improved. SNA statistics record such changes as a decline in the quality-adjusted price.

Notably, our theory does not require an exponential decrease in the quality-adjusted price. If the quality improvements are exponential, then economic growth can easily be maintained, e.g., as in usual quality-ladder models. According to "Moore's law," the quality of computers has been improving at a constant rate; however, this trend of exponential improvement is expected to slow. In fact, computers are a remarkable exception in terms of continued improvements in performance. Most other products experience a tapering in the rate of productivity improvement as they mature. Our theory shows that slowdowns in productivity increases in individual goods are consistent with a sustained rate of measured GDP growth, as long as a constant number of new products are introduced per unit time.

Finally, let us discuss the case in which the quality-adjusted price of the good increases for some part of its lifecycle, as we present in Example 2 of Figure 4. Although we need a concrete model to analyze how this happens and whether Condition (10) is satisfied, we discuss two possibilities here. One possibility is when products have antique or scarce value as they become very old. In this scenario, $p(\tau)$ increases only when $x(\tau)$ becomes considerably smaller than it is when the good is newer. Another possibility is that producing a good in small lots costs more. This happens, for example, when a particular good continues to be produced to meet a niche demand, typically near the end of the product lifecycle.

The numerator of the equation, $-\int_0^T x(\tau)dp(\tau)$, is the weighted sum of the price changes, $dp(\tau)$, where the weights are the quantities, $x(\tau)$. Therefore, if the quantity $x(\tau)$ tends to be small when $p(\tau)$ increases, then the negative effect of such movements on the GDP growth rate is likely to be limited. Therefore, even when the price at the end of the lifecycle $p(T)$ is higher than the initial price $p(0)$, the lifetime contribution of this good to the real GDP growth rate may well be positive, as in the case of Example 2.

Condition (11): The cumulative expenditure for a single good is finite

This condition requires the expenditure on older goods, $p(\tau)x(\tau)$, to decrease as the goods age so that they are effectively retired from the market in terms of expenditure share. The condition is always satisfied if the representative good ceases to be produced at a finite age T. Even when the good stays in the market forever ($T = \infty$), the condition is satisfied if the expenditure decreases reasonably quickly as the good ages (condition

13). Notably, the speed of the decline in expenditure does not need to be exponential.

The expenditure for the good can decrease as the good ages for several reasons. One possibility is that the price decreases when the price elasticity of demand is less than one, at least for older goods. To illustrate this possibility, suppose that the demand for a good is determined solely by its price $p(\tau)$, and the price falls toward zero. Even when the good becomes almost free, it is unrealistic to expect consumers to demand an infinite amount of any particular product. This consideration suggests that the price demand elasticity of a product tends to be less than one when the price becomes sufficiently low, and the expenditure for the good eventually vanishes as $p(\tau) \to 0$. Section 3 presents a full endogenous growth model on the basis of this idea.

The expenditure for older goods can also decrease for other reasons. Sometimes, consumers are attracted by the novelty of new goods, but they become less interested over time. Advertisements for newer goods increase the speed of the obsolescence of older goods. Changes in the underlying economic environment may also make older goods useless. When these effects are present, Condition (11) may be satisfied regardless of the elasticity of demand. We extend the model to include obsolescence in Section 4.

Appendix D General Case in the Baseline Prototype Model

D.1 Proof of Lemma 1

The proof goes by a "guess and verify" method. Suppose that $\lambda^* < ((1+\mu)\widehat{c}^{1/\varepsilon})^{-1}$, which means that $(1+\mu)\lambda^*\widehat{c}^{1/\varepsilon} < 1$. Then, $q(\tau) > (1+\mu)\lambda^*\widehat{c}^{1/\varepsilon}$ holds for all $\tau \geq 0$, since $q(0) = 1$ and $q'(0) > 0$ for all $\tau > 0$.

Below, we verify that the initial guess is correct under the assumption in the lemma. Since $q(\tau) > (1+\mu)\lambda^*\widehat{c}^{1/\varepsilon}$ holds for all $\tau \geq 0$, we can calculate the steady-state value of λ^* as in Equation (27). Using the assumption of the lemma, $\widehat{c} < \left(a\mu L \int_0^\infty q(\tau)^{\varepsilon-1} e^{-\rho\tau} d\tau\right)^{-1}$, Equation (27) implies

$$\lambda^* = \frac{1}{1+\mu}\left(a\mu L \int_0^\infty q(\tau)^{\varepsilon-1} e^{-\rho\tau} d\tau\right)^{1/\varepsilon} \leq \frac{1}{1+\mu}\widehat{c}^{-1/\varepsilon}, \tag{D.1}$$

which confirms that the initial guess is correct.

In Appendix D.2, we show that the steady-state value of λ^* is unique. Therefore, we are assured that the unique value of λ^* satisfies $\lambda^* < ((1+\mu)\widehat{c}^{1/\varepsilon})^{-1}$; thus, $q(\tau) > (1+\mu)\lambda^*\widehat{c}^{1/\varepsilon}$ for all $\tau \geq 0$.

D.2 Steady-state Equilibrium when $\widehat{c}$ is not Small

In Section 3.4, we assume that $\widehat{c}$ is sufficiently small that $q(\tau) \geq (1+\mu)\lambda^*\widehat{c}^{1/\varepsilon}$ holds for all τ. Here, we analyze the steady-state equilibrium without this assumption. The threshold age of goods is defined as follows:

$$\widehat{\tau}(\lambda^*) = \max\left[0, \frac{\theta}{\beta}\left(\left((1+\mu)\lambda^*\widehat{c}^{1/\varepsilon}\right)^{1/\theta} - 1\right)\right]. \tag{D.2}$$

Then, from Equation (21), $q(\tau) \geq (1+\mu)\lambda^*\widehat{c}^{1/\varepsilon}$ if and only if $\tau \geq \widehat{\tau}(\lambda^*)$.

Using Equation (24), the profit of an age-τ firm in the steady state can be written as follows:

$$\pi(\tau) = \begin{cases} \mu D(\lambda^*) q(\tau)^{\varepsilon-1} & \text{for } \tau \geq \widehat{\tau}(\lambda^*), \\ \mu \widehat{D}(\lambda^*) q(\tau)^{\widehat{\varepsilon}-1} & \text{for } \tau \leq \widehat{\tau}(\lambda^*). \end{cases} \tag{D.3}$$

Using Equations (D.2) and (D.3), the value of a new firm in the steady state can be written as a function of λ^*:

$$V(\lambda^*) = \mu\widehat{D}(\lambda^*)\int_0^{\widehat{\tau}(\lambda^*)} q(\tau)^{\widehat{\varepsilon}-1}e^{-\rho\tau}d\tau + \mu D(\lambda^*)\int_{\widehat{\tau}(\lambda^*)}^{\infty} q(\tau)^{-(1-\varepsilon)}e^{-\rho\tau}d\tau. \tag{D.4}$$

The equilibrium value of λ^* is determined by the free entry condition, $V(\lambda^*) = 1/a$. From $D(\lambda) = L((1+\mu)\lambda)^{-\varepsilon}$ and $\widehat{D}(\lambda) = L((1+\mu)\lambda/\underline{u})^{-\widehat{\varepsilon}}$, we can confirm that function $V(\lambda)$ is continuous and strictly decreasing in λ.[59] Additionally, $\lim_{\lambda\to 0} V(\lambda) = \infty$ and $\lim_{\lambda\to\infty} V(\lambda) = 0$. Therefore, there is a unique value of positive and finite λ^* that solves the free entry condition. This is the steady-state value of λ^*.

Next, let us turn to the labor market. From functions (19) and (24), the total number of production workers in the ASSE can be written as $L^{P*} = n^*\ell(\lambda^*)$, where

$$\ell(\lambda^*) \equiv D(\lambda^*)\int_0^{\widehat{\tau}(\lambda^*)} q(\tau)^{\widehat{\varepsilon}-1}d\tau + D(\lambda^*)\int_{\widehat{\tau}(\lambda^*)}^{\infty} q(\tau)^{-(1-\varepsilon)}d\tau. \tag{D.5}$$

[59] To calculate $V'(\lambda)$, we need to use Leibniz's rule because the range of the integration depends on λ. However, at $\tau = \widehat{\tau}(\lambda)$, we can confirm that $\widehat{D}(\lambda)q(\widehat{\tau}(\lambda))^{\widehat{\varepsilon}-1} = D(\lambda)q(\widehat{\tau}(\lambda))^{\varepsilon-1}$. Therefore, a marginal change in $\widehat{\tau}(\lambda)$ does not affect $V'(\lambda)$.

Note that the first integral in Equation (D.5) is finite because $\widehat{\tau}(\lambda^*)$ is finite. The second integral is finite if the power of $q(\tau)^{-(1-\varepsilon)} \propto (\tau+\kappa_0)^{-\theta(1-\varepsilon)}$ is less than 1, which means that $\theta(1-\varepsilon) > 1$, or equivalently $\psi > \varepsilon$. In the following, we assume that $\psi > \varepsilon$ holds. The function $\ell(\lambda^*)$ is a decreasing and continuous function of λ^*, with $\lim_{\lambda\to 0}\ell(\lambda) = \infty$ and $\lim_{\lambda\to\infty}\ell(\lambda) = 0$. Since λ^* is positive and finite, $\ell(\lambda^*)$ is also positive and finite. Using Equation (D.5), the equilibrium condition for the labor market is written as $n^*\ell(\lambda^*) + (n^*/a) = L$. From this, we obtain

$$n^* = \frac{aL}{1+a\ell(\lambda^*)}. \tag{D.6}$$

Since $\ell(\lambda^*)$ is positive and finite, n^* is also positive and finite.

D.3 Measured Real GDP Growth Rate when $\widehat{c}$ is not Small

As shown in Appendix D.2, the economy has an ASSE with a positive and finite pair of n^* and λ^* whenever $\psi \in (\varepsilon, 1)$. In this ASSE, we now calculate the real GDP growth rate, as measured by the SNA. From Equations (23) and (24), the expenditure for an age τ good can be written as follows:

$$p(\tau)x(\tau) = \begin{cases} (1+\mu)D(\lambda^*)q(\tau)^{-(1-\varepsilon)} & \text{for } \tau \geq \widehat{\tau}(\lambda^*), \\ (1+\mu)\widehat{D}(\lambda^*)q(\tau)^{1-\widehat{\varepsilon}} & \text{for } \tau < \widehat{\tau}(\lambda^*). \end{cases} \tag{D.7}$$

Using Equation (D.7), we can calculate the expenditure shares for the goods of each age:

$$\sigma(\tau) = \begin{cases} D(\lambda^*)q(\tau)^{-(1-\varepsilon)}/\ell(\lambda^*) & \text{for } \tau \geq \widehat{\tau}(\lambda^*), \\ \widehat{D}(\lambda^*)q(\tau)^{1-\widehat{\varepsilon}}/\ell(\lambda^*) & \text{for } \tau < \widehat{\tau}(\lambda^*). \end{cases} \tag{D.8}$$

The measured real GDP growth rate is obtained via the growth formula (36):

$$g^* = \frac{1}{\ell(\lambda^*)}\left(\widehat{D}(\lambda^*)\int_0^{\widehat{\tau}(\lambda^*)} q(\tau)^{\widehat{\varepsilon}-1}g_q(\tau)d\tau + D(\lambda^*)\int_{\widehat{\tau}(\lambda^*)}^{\infty} q(\tau)^{-(1-\varepsilon)}g_q(\tau)d\tau\right). \tag{D.9}$$

Using Equations (21) and (22), the growth rate can be written as follows:

$$\begin{aligned} g^* = \frac{\theta}{\ell(\lambda^*)}\Bigg(&\widehat{D}(\lambda^*)\kappa_1^{\widehat{\varepsilon}-1}\int_0^{\widehat{\tau}(\lambda^*)}(\tau+\kappa_0)^{\theta(\widehat{\varepsilon}-1)-1}d\tau \\ &+ D(\lambda^*)\kappa_1^{-(1-\varepsilon)}\int_{\widehat{\tau}(\lambda^*)}^{\infty}(\tau+\kappa_0)^{-\theta(1-\varepsilon)-1}d\tau\Bigg). \end{aligned} \tag{D.10}$$

The two integrals in Equation (D.10) are both finite, and their sum is positive. Additionally, as discussed in Section D.2, $\ell(\lambda^*)$ is positive and finite. Therefore, given $\psi \in (\varepsilon, 1)$, the measured real GDP growth rate is positive and finite.

Appendix E Money-Metric Utility

In this Appendix, we derive the money-metric utility $\zeta_t(\Delta)$ that satisfies equation (41), which is shown again below:

$$v(\{\widetilde{p}_{t+\Delta}(i)\}_{i=0}^{N_{t+\Delta}}, I_{t+\Delta}) = v(\{\widetilde{p}_t(i)\}_{i=0}^{N_t}, I_t \exp \zeta_t(\Delta)). \tag{41}$$

We focus on the ASSE and continue to assume that the condition for Lemma 1 is satisfied. Then, $\widetilde{c}(i) \geq \widehat{c}$ holds for all $i \in [0, N_t]$ in the ASSE, and therefore,

$$\widetilde{c}(i) = \lambda_t^{-\varepsilon} \widetilde{p}_t(i)^{-\varepsilon} \text{ for } i \in [0, N_t]. \tag{E.1}$$

From this demand function, the expenditure at time t is given by

$$I_t = \int_0^{N_t} \widetilde{p}_t(i)\widetilde{c}_t(i)di = \lambda_t^{-\varepsilon} \int_0^{N_t} \widetilde{p}_t(i)^{1-\varepsilon} di. \tag{E.2}$$

Now, let us represent the consumption of each good as a function of the expenditure (budget) at time t, i.e., I_t, rather than as a function of the Lagrange multiplier, λ_t. By eliminating λ_t in (E.1) using (E.2), we obtain

$$\widetilde{c}_t(i) = \widetilde{p}_t(i)^{-\varepsilon} P_t^{-(1-\varepsilon)} I_t \text{ for } i \in [0, N_t], \tag{E.3}$$

where P_t is a price index defined by

$$P_t = \left(\int_0^{N_t} \widetilde{p}(i)^{1-\varepsilon} di \right)^{1/(1-\varepsilon)}. \tag{E.4}$$

Using the first line of (15) and (E.3), we can express the instantaneous utility (in utils) as a function of I_t as follows:

$$U_t = \int_0^{N_t} u(\widetilde{c}_t(i))di = N_t \overline{u} - \frac{\varepsilon}{1-\varepsilon} \left(\frac{P_t}{I_t} \right)^{(1-\varepsilon)/\varepsilon}.$$

Note that this property also holds when the expenditure (budget) I_t is multiplied by $\exp \zeta_t(\Delta) \geq 1$.[60] Therefore, Equation (41) can be written as follows:

$$N_{t+\Delta}\overline{u} - \frac{\varepsilon}{1-\varepsilon}\left(\frac{P_{t+\Delta}}{I_{t+\Delta}}\right)^{(1-\varepsilon)/\varepsilon} = N_t\overline{u} - \frac{\varepsilon}{1-\varepsilon}\left(\frac{P_t}{I_t \exp \zeta_t(\Delta)}\right)^{(1-\varepsilon)/\varepsilon}. \tag{E.5}$$

Note that, in the ASSE, P_t and I_t in (E.2) and (E.4) converge to constant values.

$$I_t \to (\lambda^*)^{-\varepsilon} n^* \int_0^\infty p(\tau)^{1-\varepsilon} d\tau \equiv I^*, \tag{E.6}$$

$$P_t \to \left(n^* \int_0^\infty p(\tau)^{1-\varepsilon} d\tau\right)^{1/(1-\varepsilon)} \equiv P^*. \tag{E.7}$$

Additionally, $N_{t+\Delta} - N_t = n^*\Delta$ holds. Using these, Equation (E.5) can be solved for $\zeta_t(\Delta)$ as follows.

$$\zeta_t(\Delta) = -\frac{\varepsilon}{1-\varepsilon}\log\left(1 - \frac{1-\varepsilon}{\varepsilon} n^*\overline{u}\left(\frac{I^*}{P^*}\right)^{(1-\varepsilon)/\varepsilon}\Delta\right). \tag{E.8}$$

By differentiating (E.8) by Δ and taking the limit of $\Delta \to 0$, we obtain

$$\zeta_t'(0) = n^*\overline{u}\left(\frac{I^*}{P^*}\right)^{(1-\varepsilon)/\varepsilon}, \tag{E.9}$$

which is the instantaneous rate of increase in the money-metric utility in the ASSE.

Below, we examine the relationship between $\zeta_t'(0)$ and the measured real GDP growth rate in the ASSE, g^*, given by (37). From (E.6) and (E.7),

$$\left(\frac{I^*}{P^*}\right)^{(1-\varepsilon)/\varepsilon} = \left((\lambda^*)^{1-\varepsilon} n^* \int_0^\infty p(\tau)^{1-\varepsilon} d\tau\right)^{-1}. \tag{E.10}$$

Using (E.1) and (E.10), Equation (E.9) becomes

$$\zeta_t'(0) = \frac{\overline{u}}{\int_0^\infty c(\tau)^{-(1-\varepsilon)/\varepsilon} d\tau}. \tag{E.11}$$

Note that (E.1), (23), and (21) imply

$$\frac{c(\tau)}{c(0)} = \left(\frac{p(0)}{p(\tau)}\right)^\varepsilon = \left(\frac{q(\tau)}{q(0)}\right)^\varepsilon = \left(\frac{\tau+\kappa_0}{\kappa_0}\right)^{\varepsilon/(1-\psi)}.$$

[60] When the expenditure (budget) is increased, the consumption of every good increases. Thus, $\widetilde{c}(i) \geq \widehat{c}$ still applies.

Therefore, the integral in (E.11) becomes

$$\begin{aligned}\int_0^\infty c(\tau)^{-(1-\varepsilon)/\varepsilon} d\tau &= c(0)^{-(1-\varepsilon)/\varepsilon} \int_0^\infty \left(\frac{\tau+\kappa_0}{\kappa_0}\right)^{-(1-\varepsilon)/(1-\psi)} d\tau \\ &= c(0)^{-(1-\varepsilon)/\varepsilon} \frac{1-\psi}{\psi-\varepsilon}\kappa_0 \\ &= c(0)^{-(1-\varepsilon)/\varepsilon} \frac{1}{\psi-\varepsilon}\frac{1}{\beta},\end{aligned} \tag{E.12}$$

where the second equality holds under the assumption of $\varepsilon < \psi$, and the last equality is from the definition of $\kappa_0 \equiv 1/(1-\psi)\beta$. By substituting (E.12) into (E.11), we obtain Equation (43) in the main text:

$$\zeta_t'(0) = \frac{(1-\varepsilon)\overline{u}}{c(0)^{-(1-\varepsilon)/\varepsilon}} \frac{\psi-\varepsilon}{1-\varepsilon}\beta = \Lambda g^*, \tag{E.13}$$

where g^* is the measured real GDP growth rate in (37) and $\Lambda > 1$ is the correction term defined in (42). (See Figure 7 for a graphical explanation for Λ). Additionally, note that (E.9) and (E.13) imply that (E.8) is identical to (42) in the main text.

Appendix F Transitional Dynamics

In the first subsection of this appendix, we derive the dynamics of the prototype model of Section 3 without assuming that the economy is in the steady state (ASSE). In the second subsection, we explain how to calculate the rate of GDP growth in the transition.

F.1 Dynamics of the Economy outside the Steady State

Following the main text, we continue to assume that $\widehat{c}$ is sufficiently small so that $\widehat{c}_t(i) \geq \widehat{c}$ holds for all $i \in [0, N_t]$ and t. Then, similar to (24), the demand for an age-τ good is

$$x_t(\tau) = D(\lambda_t)q(\tau)^\varepsilon = D_t q(\tau)^\varepsilon, \tag{F.1}$$

where $D_t = D(\lambda_t) = L((1+\mu)\lambda_t)^{-\varepsilon}$ is the demand shifter and λ_t is the Lagrange multiplier of the consumer's problem. Since the markup rate is μ, the profit of an age-τ firm at time t is $\pi_t(\tau) = \mu D_t q(\tau)^{\varepsilon-1}$. From this, the value of a new firm at time t is

$$\begin{aligned}V_t &= \int_0^\infty \pi_{t+\tau}(\tau) \exp\left[-\int_t^{t+\tau} r_v dv\right] d\tau \\ &= \mu \int_0^\infty D_{t+\tau} q(\tau)^{\varepsilon-1} \exp\left[-\int_t^{t+\tau} r_v dv\right] d\tau,\end{aligned} \tag{F.2}$$

where the last line shows that V_t is a function of the paths of $D_{t+\tau}$ and $r_{t+\tau}$ in the future. Since λ_t follows the Euler equation $\dot{\lambda}_t/\lambda_t = \rho - r_t$, the two variables r_t and $D_t = L((1+\mu)\lambda_t)^{-\varepsilon}$ are related by

$$\frac{\dot{D}_t}{D_t} = \varepsilon(r_t - \rho). \tag{F.3}$$

Since the cost of creating a new firm is $1/a$, the free entry condition is

$$\mu \int_0^\infty D_{t+\tau} q(\tau)^{\varepsilon-1} \exp\left[-\int_t^{t+\tau} r_v dv\right] d\tau \le \frac{1}{a} \text{ with equality if } n_t > 0. \tag{F.4}$$

We first consider the case where the free entry condition (F.4) holds with equality. In this case, using (F.2), this condition can be written as follows:

$$\mu \int_0^\infty D_{t+\tau} q(\tau)^{\varepsilon-1} \exp\left[-\int_t^{t+\tau} r_v dv\right] d\tau = \frac{1}{a}. \tag{F.5}$$

The paths of D_t and λ_t are determined so that the differential equation (F.3) and the integral equation (F.5) simultaneously hold for all t in the future. We solve this system of equations by the guess-and-verify method. Suppose that $D_{t+\tau} = D^*$ and $r_{t+\tau} = \rho$ hold for all $\tau \ge 0$, where D^* is the value of D_t in the ASSE, defined by (27). Because D^* is constant, this guess naturally satisfies the differential equation (F.3). Additionally, the integral equation (F.5) is satisfied because, with $D_{t+\tau} = D^*$ and $r_{t+\tau} = \rho$, Equation (F.5) becomes identical to (27). Therefore, the pair of $D_{t+\tau} = D^*$ and $r_{t+\tau} = \rho$ for all $\tau \ge 0$ is a solution to (F.3) and (F.5).

We also need to consider the equilibrium of the labor market. From (F.1), the amount of labor hired by an age-τ firm is $x(\tau)/q(\tau) = D_t q(\tau)^{-(1-\varepsilon)}$. Since there are $n_{t-\tau}$ firms of age τ at time t, the total employment for production is

$$L_t^P = D_t \int_0^t q(\tau)^{-(1-\varepsilon)} n_{t-\tau} d\tau. \tag{F.6}$$

The employment for R&D at that time is n_t/a, and the labor supply is L. Therefore, the equilibrium of the labor market requires $L_t^P + n_t/a = L$, which can be solved for n_t as follows:

$$n_t = a\left(L - D_t \int_0^t q(\tau)^{-(1-\varepsilon)} n_{t-\tau} d\tau\right). \tag{F.7}$$

When the free entry condition (F.4) holds with equality at time t, the equilibrium path of this economy after t is given by $D_{t+\tau} = D^*$ and $r_{t+\tau} = \rho$ for all $\tau \geq 0$ and

$$n_t = a\left(L - D^* \int_0^t q(\tau)^{-(1-\varepsilon)} n_{t-\tau} d\tau\right). \tag{F.8}$$

Note that the only unknown in the integral equation (F.8) is n_t. Given the history of n_t before t (i.e., $n_{t-\tau}$ for $\tau \in (0, t]$), it is easy to numerically calculate the value of n_t that satisfies (F.8). We can confirm that the free entry condition (F.4) holds with equality if n_t in (F.7) is positive (if it is negative, we need to consider the case of $n_t = 0$ as explained below). Additionally, by comparing (F.8) and (31), we see that $n_t = n^*$ satisfies the labor market equilibrium condition as $t \to \infty$. As shown in the left panel of Figure 6, we numerically confirm that the economy converges to the steady state where $n_t = n^*$.

Next, we consider the case where the free entry condition (F.4) holds with strict inequality. In this case, there is no R&D at time t, which means that $n_t = 0$. From the labor market equilibrium condition (F.7), we obtain the equilibrium value of D_t as follows:

$$D_t = \frac{L}{\int_0^t q(\tau)^{-(1-\varepsilon)} n_{t-\tau} d\tau}. \tag{F.9}$$

Since $n_{t-\tau}$ is predetermined, it is possible to solve for the path of D_t numerically. Note that while $n_t = 0$ holds, the denominator of (F.9) gradually decreases because $q(\tau)^{-(1-\varepsilon)}$, where $\varepsilon < 1$, is decreasing in τ. Therefore, D_t eventually reaches D^*. After this point, the free entry condition holds with equality, and n_t is determined by (F.8). As explained above, the economy then converges to the ASSE. Finally, we explain the dynamics of r_t when $n_t = 0$. Given the path of D_t, Equation (F.3) implies that the interest rate becomes

$$r_t = \rho + \frac{1}{\varepsilon}\frac{\dot{D}_t}{D_t}.$$

Since $\dot{D}_t > 0$ before D_t reaches D^*, r_t should be greater than ρ when the free-entry condition is satisfied with strict inequality.

F.2 Measured GDP Growth in the Transition

In the transition, the real GDP growth rate is given by Equation (5). In the model of Section 3, $T = \infty$ means that Ω_t is an empty set. Therefore, the third term in the numerator of (5) is eliminated. Note that $\widetilde{p}_t(i) = p(\tau) = (1+\mu)/q(\tau)$, where $\tau = t - s(i)$. Additionally, $\widetilde{x}_t(i)$ can be written as $x_t(\tau) \equiv D_t q(\tau)^{\varepsilon}$. Then,

$$\dot{\widetilde{x}}_t(i) = \dot{x}_t(\tau) + x_t'(\tau) = \dot{D}_t q(\tau)^{\varepsilon} + D_t \varepsilon q(\tau)^{\varepsilon-1} q'(\tau).$$

Using (F.3) and (22), the above equation becomes

$$\dot{\widetilde{x}}_t(i) = \varepsilon(r_t - \rho + g_q(\tau)) x_t(\tau).$$

Since $s(i) \equiv t - \tau$ is the date at which good i was the newest good, $i = N_{s(i)} = N_{t-\tau}$ holds. By fixing t, the total differentiation of $i = N_{t-\tau}$ yields $di = -n_{t-\tau} d\tau$. Finally, the set of goods in production is $X_t = [0, N_t]$, which transforms to $[t, 0]$ when represented in terms of age. Using these, Equation (5) can be written as

$$g_t = \frac{n_t p(0) x_t(0) + \varepsilon \int_0^t (r_t - \rho + g_q(\tau)) p(\tau) x_t(\tau) n_{t-\tau} d\tau}{\int_0^t p(\tau) x_t(\tau) n_{t-\tau} d\tau}, \tag{F.10}$$

When $n_t > 0$, the analysis in Section F.1 shows that $D_t = D^*$ and $r = \rho$ hold. In this case, $x_t(\tau)$ can be written as $x(\tau) \equiv D^* q(\tau)^{\varepsilon}$. Using this, (F.10) becomes

$$g_t = \begin{cases} \frac{n_t + \varepsilon \int_0^t g_q(\tau) q(\tau)^{\varepsilon-1} n_{t-\tau} d\tau}{\int_0^t q(\tau)^{\varepsilon-1} n_{t-\tau} d\tau} & \text{if } n_t > 0, \\ \frac{\varepsilon \int_0^t g_q(\tau) q(\tau)^{\varepsilon-1} n_{t-\tau} d\tau}{\int_0^t q(\tau)^{\varepsilon-1} n_{t-\tau} d\tau} + \varepsilon(r_t - \rho) & \text{if } n_t = 0. \end{cases} \tag{F.11}$$

Once we obtain the paths of n_t and r_t as explained in Section F.1, Equation (F.11) allows us to calculate the path of g_t numerically.

Appendix G Money-Metric Utility with Obsolescence

In both cases, $\varepsilon > 1$ and $\varepsilon < 1$, the consumption of good i at time t is

$$\widetilde{c}_t(i) = \lambda_t^{-\varepsilon} \widetilde{p}_t(i)^{-\varepsilon} \delta(t - s(i))^{\varepsilon}, \tag{G.1}$$

where λ_t is a Lagrange multiplier. Then, the total expenditure at time t is

$$I_t = \lambda_t^{-\varepsilon} \int_0^t \widetilde{p}_t(i)^{1-\varepsilon} \delta(t - s(i))^{\varepsilon} di. \tag{G.2}$$

Using (G.2), we can represent the consumption of individual goods $\widetilde{c}(i)$ in terms of I_t rather than λ_t. By eliminating λ_t in (G.1) using (G.2), we obtain

$$\widetilde{c}_t(i) = \widetilde{p}_t(i)^{-\varepsilon}\delta(t - s(i))^{\varepsilon} P_t^{\varepsilon-1} I_t, \tag{G.3}$$

where P_t is a price index when obsolescence is present,

$$P_t = \left(\int_0^{N_t} \widetilde{p}_t(i)^{1-\varepsilon}\delta(t - s(i))^{\varepsilon} di\right)^{1/(1-\varepsilon)}. \tag{G.4}$$

Note that in the ASSE, I_t and P_t converge to finite constants.

$$I_t \to (\lambda^*)^{-\varepsilon} n^* \int_0^T p(\tau)^{1-\varepsilon}\delta(\tau)^{\varepsilon} d\tau \equiv I^*, \tag{G.5}$$

$$P_t \to \left(n^* \int_0^T p(\tau)^{1-\varepsilon}\delta(\tau)^{\varepsilon} d\tau\right)^{1/(1-\varepsilon)} \equiv P^*. \tag{G.6}$$

We can follow the same procedure to obtain the consumption at $t + \Delta$ in the ASSE:

$$\widetilde{c}_{t+\Delta}(i) = \widetilde{p}_{t+\Delta}(i)^{-\varepsilon}\delta(t + \Delta - s(i))^{\varepsilon} P_{t+\Delta}^{\varepsilon-1} I_{t+\Delta}, \tag{G.7}$$

where $I_{t+\Delta} \to I^*$ and $P_{t+\Delta} \to P^*$ as $t \to \infty$. Using these results, we later calculate the instantaneous utility consumers in the ASSE at time $t + \Delta$, which is represented by the LHS of Equation (48); i.e., $v(\{\widetilde{p}_{t+\Delta}(i)\}_{i=0}^{N_{t+\Delta}}, I_{t+\Delta}, t + \Delta)$.

Next, let us consider the situation of a consumer at time t whose preference is that of time $t + \Delta$, while the prices and availability of goods are still those of time t (this situation corresponds to the RHS of Equation 48, $v(\{\widetilde{p}_t(i)\}_{i=0}^{N_t}, I_t \exp\zeta_t(\Delta), t+\Delta)$). This means that the consumer discounts the utility from individual goods by $\delta(t + \Delta - s(i))$, while the prices are $\widetilde{p}_t(i)$. Then, the consumption of good i is

$$\widetilde{c}_t(i; \Delta) = \lambda_t^{-\varepsilon}\widetilde{p}_t(i)^{-\varepsilon}\delta(t + \Delta - s(i))^{\varepsilon}, \tag{G.8}$$

where the term $\delta(t + \Delta - s(i))^{\varepsilon}$ signifies that the preference is that of ahead of time by Δ. Then, the total expenditure at time t is

$$I_t = \lambda_t^{-\varepsilon} \int_0^t \widetilde{p}_t(i)^{1-\varepsilon}\delta(t + \Delta - s(i))^{\varepsilon} di. \tag{G.9}$$

By eliminating λ_t in (G.8) using (G.9), we obtain

$$\widetilde{c}_t(i; \Delta) = \widetilde{p}_t(i)^{-\varepsilon}\delta(t + \Delta - s(i))^{\varepsilon} \widehat{P}_t(\Delta)^{\varepsilon-1} I_t, \tag{G.10}$$

where $\widehat{P}_t(\Delta)$ is a price index at time t given that the preference is that of $t+\Delta$,

$$\widehat{P}_t(\Delta) = \left(\int_0^{N_t} \widetilde{p}_t(i)^{1-\varepsilon} \delta(t+\Delta - s(i))^{\varepsilon} di \right)^{1/(1-\varepsilon)}.$$

On the ASSE, this price index converges to

$$\widehat{P}_t(\Delta) \to \left(n^* \int_{\Delta}^{T} p(\tau - \Delta)^{1-\varepsilon} \delta(\tau)^{\varepsilon} d\tau \right)^{1/(1-\varepsilon)} \equiv \widehat{P}^*(\Delta). \tag{G.11}$$

Now, we are ready to compare both sides of (48). Since the utility function changes its form depending on whether $\varepsilon > 1$ or $\varepsilon < 1$, in the following, we consider the two cases separately.

G.1 Case of $\varepsilon > 1$

In this case, the sub-utility function is given by $u(c) = c^{1-1/\varepsilon}/(1-1/\varepsilon)$ for all $c > 0$. We first derive the change in instantaneous utility (in utils) in the ASSE. From (G.3) and $u(c) = c^{1-1/\varepsilon}/(1-1/\varepsilon)$, the instantaneous utility at t in (44) is

$$\begin{aligned} U_t &= \int_0^{N_t} \left[\delta(t - s(i)) u(\widetilde{c}_t(i)) + (1 - \delta(t - s(i))) \widehat{u} \right] di \\ &= \frac{\varepsilon}{\varepsilon - 1} \left(\frac{I_t}{P_t} \right)^{(\varepsilon-1)/\varepsilon} + \widehat{u} N_t - \widehat{u} \int_0^{N_t} \delta(t - s(i)) di. \end{aligned}$$

From (G.5), (G.6), and $\dot{N}_t \to n^*$, the above expression asymptotically becomes

$$U_t^* = \frac{\varepsilon}{\varepsilon - 1} \left(\frac{I^*}{P^*} \right)^{(\varepsilon-1)/\varepsilon} + \widehat{u} N_t - \widehat{u} n^* \int_0^t \delta(\tau) di. \tag{G.12}$$

Similarly, we can calculate the instantaneous utility at $t+\Delta$ in the ASSE.

$$U_{t+\Delta}^* = \frac{\varepsilon}{\varepsilon - 1} \left(\frac{I^*}{P^*} \right)^{(\varepsilon-1)/\varepsilon} + \widehat{u} N_{t+\Delta} - \widehat{u} n^* \int_0^{t+\Delta} \delta(\tau) di. \tag{G.13}$$

By comparing (G.12) and (G.13) and using $\dot{N}_t = n^*$ and $\delta(t) \to 0$ as $t \to \infty$, we obtain the speed at which the instantaneous utility increases in the ASSE.

$$\dot{U}_t^* = \lim_{\Delta \to 0} \frac{U_{t+\Delta}^* - U_t^*}{\Delta} = \left. \frac{dU_{t+\Delta}^*}{d\Delta} \right|_{\Delta = 0} = \widehat{u} n^* (1 - \delta(t)) \to \widehat{u} n^* \text{ as } t \to \infty.$$

Therefore, the instantaneous utility (in utils) linearly increases with time, with a slope of $\widehat{u} n^*$.

In the following, we consider the money-metric utility, $\zeta_t(\Delta)$, as defined by (48). The LHS of (48), $v(\{\widetilde{p}_{t+\Delta}(i)\}_{i=0}^{N_{t+\Delta}}, I_{t+\Delta}, t+\Delta)$ is $U^*_{t+\Delta}$ in (G.13). In RHS of (48), i.e., $v(\{\widetilde{p}_t(i)\}_{i=0}^{N_t}, I_t \exp\zeta_t(\Delta), t+\Delta)$, I_t represents the expenditure at time t in the ASSE. Since the expenditure in the ASSE is asymptotically constant, as shown in (G.5), this I_t can be equated to I^*. Therefore, this expression represents the instantaneous utility of the representative consumer when (i) the prices and availability of goods are those of time t, (ii) the instantaneous expenditure is $I^* \exp\zeta_t(\Delta)$, and (iii) the preference is that of $t+\Delta$. In this situation, Equation (G.10) implies that the consumption of individual goods is

$$\widetilde{c}_t(i;\Delta) = \widetilde{p}_t(i)^{-\varepsilon}\delta(t+\Delta-s(i))^{\varepsilon}\widehat{P}_t(\Delta)^{\varepsilon-1}I^*\exp\zeta_t(\Delta). \tag{G.14}$$

Additionally, the instantaneous utility in this situation is

$$\widehat{U}_t(\Delta) = \int_0^{N_t}\Big[\delta(t+\Delta-s(i))u(\widetilde{c}_t(i;\Delta)) + (1-\delta(t-s(i)))\widehat{u}\Big]di.$$

In the above equation, the range of the integral is from 0 to N_t because the availability of goods is that of time t. Additionally, the externality term $(1-\delta(t-s(i)))\widehat{u}$ is that of time t. However, sub-utility $u(\widetilde{c}_t(i;\Delta))$ is multiplied by $\delta(t+\Delta-s(i))$, which indicates that consumers have a preference of time $t+\Delta$.

By substituting $u(c) = c^{1-1/\varepsilon}/(1-1/\varepsilon)$ and (G.14) into the above equation and then using (G.11), the instantaneous utility asymptotically becomes

$$\widehat{U}^*(\Delta) = \frac{\varepsilon}{\varepsilon-1}\left(\frac{I^*}{\widehat{P}^*(\Delta)}\right)^{(\varepsilon-1)/\varepsilon}\exp\left[\frac{\varepsilon-1}{\varepsilon}\zeta_t(\Delta)\right] + \widehat{u}N_t - \widehat{u}n^*\int_0^t\delta(\tau)d\tau. \tag{G.15}$$

The money-metric utility (in logs), $\zeta_t(\Delta)$, is defined so that (G.13) coincides with (G.15). We can solve this definition for $\exp\zeta_t(\Delta)$ as follows:

$$\exp\zeta_t(\Delta) = \frac{\widehat{P}^*(\Delta)}{P^*}\left\{1+\frac{\varepsilon-1}{\varepsilon}\left(\frac{P^*}{I^*}\right)^{\frac{\varepsilon-1}{\varepsilon}}\widehat{u}n^*\left(\Delta-\int_t^{t+\Delta}\delta(\tau)d\tau\right)\right\}^{\frac{\varepsilon}{\varepsilon-1}}. \tag{G.16}$$

Note that (G.16) implies $\zeta_t(0) = 0$ because $\widehat{P}^*(0) = P^*$. Equation (G.16) can be interpreted as follows. When consumers have the preference of time $t+\Delta$ (i.e., when consumers discount the utility of goods more heavily than at time t) while the prices and availability of goods are still those of time t, they in effect face higher prices in that they can only achieve lower instantaneous utility with a given budget. This effect is represented by $\widehat{P}^*(\Delta)/P^* > 1$. To compensate for this, $\zeta_t(\Delta)$ must be increased. In addition, if $\widehat{u}$ is positive (i.e., when there are positive externalities from old goods), consumers at time $t+\Delta$ enjoy more externalities than at time t because $n^*\Delta$ more goods are available at time $t+\Delta$. The difference (in utils) is $\widehat{u}n^* \int_t^{t+\Delta}(1-\delta(\tau))d\tau = \widehat{u}n^*\left(\Delta - \int_t^{t+\Delta}\delta(\tau)d\tau\right)$. The second term in the braces of (G.16) represents this effect.

In the following, we derive $\zeta_t'(0)$, the slope of the money-metric utility with respect to Δ at $\Delta = 0$. Note that differentiating the LHS of (G.16) and then substituting $\Delta = 0$ yields $\zeta_t'(0)\exp\zeta_t(0) = \zeta_t'(0)$. Applying the same operation to the RHS of (G.16) and utilizing $\widehat{P}^*(0) = P^*$ yields

$$\zeta_t'(0) = \frac{1}{P^*}\left.\frac{d\widehat{P}^*(\Delta)}{d\Delta}\right|_{\Delta=0} + \left(\frac{P^*}{I^*}\right)^{\frac{\varepsilon-1}{\varepsilon}}\widehat{u}n^*(1-\delta(t)). \tag{G.17}$$

Differentiating (G.11) with respect to Δ and then substituting $\Delta = 0$ yields

$$\left.\frac{d\widehat{P}^*(\Delta)}{d\Delta}\right|_{\Delta=0} = n^*(P^*)^{\varepsilon}\left\{\int_0^T p(\tau)^{-(\varepsilon-1)}\delta(\tau)\left(-\frac{p'(\tau)}{p(\tau)}\right)d\tau + (\varepsilon-1)p(0)^{-(\varepsilon-1)}\right\}$$

Then, using (22), (23) and (G.6), the first term of (G.17) becomes

$$\frac{1}{P^*}\left.\frac{d\widehat{P}^*(\Delta)}{d\Delta}\right|_{\Delta=0} = \frac{\int_0^T q(\tau)^{\varepsilon-1}\delta(\tau)^{\varepsilon}g_q(\tau)d\tau}{\int_0^T q(\tau)^{\varepsilon-1}\delta(\tau)^{\varepsilon}d\tau} + \frac{1}{\varepsilon-1}\frac{1}{\int_0^T q(\tau)^{\varepsilon-1}\delta(\tau)^{\varepsilon}d\tau} \tag{G.18}$$

Note that the first term of (G.18) is the weighted average of the rate of price reduction of existing goods, and it coincides with the measured GDP growth rate g^* in (45). For the consumer with time $t+\Delta$ preference facing time t prices and availability, this amount of expenditure needs to be compensated (in $\zeta'(0)$) so that she has the same utility as the consumer at time $t+\Delta$ in the ASSE. The second term of (G.18) comes from the fact that the consumer with time $t+\Delta$ preference facing time t prices and availability does not have access to $i \in [N_t, N_{t+\Delta}]$ goods. This consumer also needs to be compensated for this fact.

Let us turn to the second term of (G.17). Recall that we assumed that $\delta(t)$ is asymptotically zero as $t \to \infty$.[61] Note also that, in the ASSE, Equations (G.3), (G.5)

[61] If T is finite, $\delta(t)$ becomes zero when $t \geq T$. If T is infinite, $\delta(t)$ converges to zero as $t \to 0$.

and (G.6) imply that the consumption for age-0 good is $c(0) = p(0)^{-\varepsilon}(P^*)^{\varepsilon-1}I^*$, which means that $I^* = p(0)^{\varepsilon}(P^*)^{-(\varepsilon-1)}c(0)$. By using I^* and $\delta(t) \to 0$, the second term of (G.17) becomes $n^*(P^*/p(0))^{\varepsilon-1}\widehat{u}/c(0)^{(\varepsilon-1)/\varepsilon}$. Then, from (23) and (G.6), it becomes

$$\frac{\widehat{u}}{c(0)^{1-1/\varepsilon}} \frac{1}{\int_0^T q(\tau)^{\varepsilon-1}\delta(\tau)^{\varepsilon} d\tau}. \tag{G.19}$$

This term derives from the fact that the sum of positive externalities increases with time as newer goods are developed. Combining (G.18) and (G.19) yields the first line of (49).

G.2 Case of $\varepsilon \in (0, 1)$

In this case, the sub-utility function is given by $u(c) = c^{1-1/\varepsilon}/(1-1/\varepsilon) + \overline{u}$ for $c > \widehat{c}$. As mentioned in the main text, we assume that $\widehat{c}$ is sufficiently small that $\widetilde{c}_t(i) > \widehat{c}$ holds for all t and i. Similar to the previous subsection, the instantaneous utility at times t and $t + \Delta$ in the ASSE asymptotically becomes

$$U_t^* = -\frac{\varepsilon}{1-\varepsilon}\left(\frac{P^*}{I^*}\right)^{(1-\varepsilon)/\varepsilon} + \widehat{u}N_t + (\overline{u} - \widehat{u})n^* \int_0^t \delta(\tau) di, \text{ and} \tag{G.20}$$

$$U_{t+\Delta}^* = -\frac{\varepsilon}{1-\varepsilon}\left(\frac{P^*}{I^*}\right)^{(1-\varepsilon)/\varepsilon} + \widehat{u}N_{t+\Delta} - (\overline{u} - \widehat{u})n^* \int_0^{t+\Delta} \delta(\tau) di. \tag{G.21}$$

By comparing (G.20) and (G.21) and using $\dot{N}_t = n^*$ and $\delta(t) \to 0$ as $t \to \infty$, we obtain the speed at which the instantaneous utility increases in the ASSE.

$$\dot{U}_t^* = \lim_{\Delta \to 0} \frac{U_{t+\Delta}^* - U_t^*}{\Delta} = \left.\frac{dU_{t+\Delta}^*}{d\Delta}\right|_{\Delta=0} = \widehat{u}n^* + (\overline{u} - \widehat{u})n^*\delta(t) \to \widehat{u}n^* \text{ as } t \to \infty.$$

Therefore, similar to the case of $\varepsilon > 1$, the instantaneous utility (in utils) linearly increases over time, with a slope of $\widehat{u}n^*$.

Note that $U_{t+\Delta}^*$ in (G.21) corresponds to the LHS of (48) for the case of $\varepsilon \in (0, 1)$. By a similar procedure that leads to (G.15) in the previous subsection, we obtain the utility of consumers that corresponds to the RHS of (48) for the case of $\varepsilon \in (0, 1)$.

$$\begin{aligned}
\widehat{U}^*(\Delta) &= \int_0^{N_t} \left[\delta(t + \Delta - s(i))u(\widetilde{c}_t(i; \Delta)) + (1 - \delta(t - s(i)))\widehat{u}\right] di \\
&= -\frac{\varepsilon}{1-\varepsilon}\left(\frac{\widehat{P}^*(\Delta)}{I^*}\right)^{(1-\varepsilon)/\varepsilon} \exp\left[-\frac{1-\varepsilon}{\varepsilon}\zeta_t(\Delta)\right] + \widehat{u}N_t \\
&\quad + \overline{u}n^* \int_{\Delta}^{t+\Delta} \delta(\tau) d\tau - \widehat{u}n^* \int_0^t \delta(\tau) d\tau.
\end{aligned} \tag{G.22}$$

The money-metric utility (in logs), $\zeta_t(\Delta)$, is defined so that (G.21) coincides with (G.22). We can solve this definition for $\exp\zeta_t(\Delta)$ as follows:

$$\begin{aligned}\exp\zeta_t(\Delta) = \frac{\widehat{P}^*(\Delta)}{P^*}\Bigg\{1 - \frac{1-\varepsilon}{\varepsilon}\left(\frac{I^*}{P^*}\right)^{\frac{1-\varepsilon}{\varepsilon}} n^* \\ \left(\widehat{u}\Delta - \widehat{u}\int_t^{t+\Delta}\delta(\tau)d\tau + \overline{u}\int_0^{\Delta}\delta(\tau)d\tau\right)\Bigg\}^{-\frac{\varepsilon}{1-\varepsilon}}.\end{aligned} \tag{G.23}$$

Differentiating both sides of (G.23) by Δ and then substituting $\Delta = 0$ yields

$$\zeta_t'(0) = \frac{1}{P^*}\left.\frac{d\widehat{P}^*(\Delta)}{d\Delta}\right|_{\Delta=0} + \left(\frac{I^*}{P^*}\right)^{\frac{1-\varepsilon}{\varepsilon}} n^*\left(\widehat{u} - \widehat{u}\delta(t) + \overline{u}\delta(0)\right). \tag{G.24}$$

The first term of (G.24) is given by (G.18), which does not depend on the value of ε. Additionally, $I^* = p(0)^{\varepsilon}(P^*)^{-(\varepsilon-1)}c(0)$ holds as in the case of $\varepsilon > 1$. Then, using $\delta(0) = 1$ and $\delta(t) \to 0$ as $t \to \infty$, Equation (G.24) becomes

$$\zeta_t'(0) = g^* + \left\{\frac{1}{1-\varepsilon}\left(\frac{(1-\varepsilon)\overline{u}}{c(0)^{1-1/\varepsilon}} - 1\right) + \frac{\widehat{u}}{c(0)^{1-1/\varepsilon}}\right\}\left(\int_0^T q(\tau)^{\varepsilon-1}\delta(\tau)^{\varepsilon}d\tau\right)^{-1}.$$

Note that $(1-\varepsilon)\overline{u}/c(0)^{1-1/\varepsilon}$ is the same as Λ in (42), and $\Lambda > 1$ holds given that $\varepsilon \in (0, 1)$. Therefore, we obtain the second line of (49).