Temporarily Explosive Land Price Dynamics

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Abstract

The land price temporarily rises explosively while it is normally stationary. We construct a dynamic general equilibrium macro-finance model with rational expectations that generates such temporarily explosive land price dynamics. The model generates explosive dynamics only when (i) a production function exhibits a strong spillover of capital so that the production function is linear in capital but the spillover is muted stochastically so that the production function exhibits a decreasing return of capital, and (ii) the economy entails high leverage. We also discuss land price bubbles and bubble detection.

JEL codes: D52, D53, E32, E44, G11, G12

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1 Introduction

Many countries have episodes of land price booms. Examples include, but are not limited to, Japan in the 1980s and the United States in the late 2000s (see Kindleberger, Aliber, and Solow (2005) for other episodes of asset price booms). During those episodes of land price booms, the land price to dividend ratio (P-D ratio) behaves explosively, followed by a sharp fall, while the P-D ratio is stationary before and after the bubbles.

In this paper, we construct a dynamic general equilibrium rational expectation macro-finance model that generates such a temporarily explosive land price dynamics followed by a large reduction, emerging from and returning to a stationary path. A key aspect of our model is that agents expect that the land price will fall back to the stationary level in the future. Hirano and Toda (2025a) have constructed a theoretical model to generate explosive dynamics of a land price. In their model, however, land price is always expected to behave explosively. Our model differs from theirs in that agents in the economy correctly take into account the possibility that the explosiveness will end in the future.

Our model has two key features. First, capital has a spillover effect on labor productivity, and the degree of the spillover is stochastic. Suppose that the spillover is strong (which we consider a rare event) so that a production function becomes linear in capital as in the AK model. In each period, the economy faces the probability that spillover is muted permanently so that the production function becomes a decreasing return in capital. This low-spillover state is an absorbing state, and agents correctly take into account the probability of a decrease in the spillover. Second, our model has an incomplete market setting. Entrepreneurs who have access to the land market face a borrowing constraint governed by leverages as in the standard macro-finance models. This lets us study the interaction of financial regulations and the existence of bubbles along with macroeconomic dynamics.

We first analyze macroeconomic dynamics with our model. Our model generates a temporary explosive path as a unique equilibrium only when the spillover is strong and the leverages are sufficiently high. The land price falls sharply when the spillover is muted (whose probability is correctly taken into account by agents). When the spillover is not strong or the leverages are not high, the unique equilibrium of this economy is stationary. The degree of the spillover and the leverages make a qualitative difference in macroeconomic dynamics. Although some papers have worked on temporarily explosive dynamics of different objects, such as Ascari, Bonomolo, and Lopes (2019) and Bianchi and Melosi (2019) for explosive inflation dynamics, ours is the first to describe the temporarily explosive dynamics of the P-D ratio.[1]

We emphasize that the condition for the explosive dynamics, that is, strong spillover and high leverages, is likely satisfied only in an unusual economic environment. It is natural to consider that, normally, the spillover effect of capital is not so strong or the leverages are not so high. Therefore, explosive land price dynamics are present only as a rare event, and macroeconomic dynamics is typically stationary.

Lastly, we discuss the conditions for the existence of land price bubbles. We show that the condition for explosive land price followed by a sharp fall coincides with the condition for the existence of land price bubbles. When the macroeconomic dynamics are stationary, the land price does not contain bubbles.

We also argue that our theoretical framework is closely related to the empirical literature on bubble detection. Papers such as Phillips, Wu, and Yu (2011), Phillips, Shi, and Yu (2015a), Phillips, Shi, and Yu (2015b), and Phillips and Shi (2019) detect the origin and collapse of bubbles by finding the period of the explosive path of the P-D ratio. Their procedure is consistent with our macro-finance framework, with the assumption that the fundamental value is not explosive. This assumption implies that the reduction of the land price is greater for bubbles that last longer.

Note that we can connect our theoretical framework to the empirical literature because we consider bubbles on a dividend-yielding asset. In models of pure bubbles, we cannot analyze the behavior of the P-D ratio because the dividend is zero and, accordingly, the P-D ratio is undefined. We can connect the theoretical and empirical work on bubbles because our paper considers bubbles on a dividend-yielding asset where the P-D ratio is well-defined. For this reason, it is important to consider bubbles on a dividend-yielding asset.

We contribute to the long-lasting literature on bubbles (see Hirano and Toda (2024) for review). Hirano and Toda (2025a) constructed a deterministic model of bubbles on a dividend-yielding asset, and Hirano, Jinnai, and Toda (2022) studied the link between financial leverages and bubbles. In these works, however, the P-D ratio of the bubbly asset (land) takes a diverging path (unless parameter values change as an event of measure zero), which makes it harder to apply their model to applications because the P-D ratio does not diverge in data. In contrast, our model features temporary bubbles with an anticipation of collapse; the P-D ratio is explosive when an asset price bubble exists but it becomes stationary once a bubble bursts. This feature makes our model more suitable for

[1] Ascari et al. (2019) relies on sunspot shocks to generate temporarily explosive behavior. In contrast, in Bianchi and Melosi (2019), temporarily explosive dynamics is possible because of the possibility of returning to stationary economy. Our modelling strategy is closer to Bianchi and Melosi (2019).

applications including various policy analyses.

The rest of the paper is organized as follows. We begin by providing motivating facts in Section 2. We provide our model and analyze it in Section 3. In Section 4, we discuss land price bubbles. We conclude our paper in Section 5.

2 Motivating Facts

To motivate our analysis, we provide data. Figure 1 plots the P-D ratio in the U.S. housing. We use the S&P CoreLogic Case-Shiller index as the house price and use the CPI for rent of the primary residence in the U.S. city average extracted from FRED as the data for dividends from land.

The P-D ratio shows relatively stable behavior until 2000 and during the 2010s, but an explosive increase during the 2000s followed by a sharp drop in the late 2000s. This mostly stable but temporarily explosive pattern is difficult to explain in a standard model. In such a model, the P-D ratio behaves in a stationary manner, and the explosive paths are generated only by intensifying shocks. If parameters are set to generate an explosive path endogenously in such a model, the stationary path in the long-run is not explained. The existing model cannot jointly explain the long-run stationary path and temporary explosive path.

This type of explosiveness is empirically tested in Phillips et al. (2011), Phillips et al. (2015b), and many others. They focus mainly on Nasdaq, and the explosiveness of the P-D ratio is tested by unit root tests.[2] They also argue that the explosiveness of the P-D ratio indicates the existence of bubbles. We discuss this point in Section 4.

The data also shows a rapid increase in the P-D ratio after the Covid-19 pandemic in 2020. In contrast to the explosive increase in the 2000s, however, the increase after the pandemic is not followed by a sharp fall. As we will see in the following sections, the magnitude of the falls after an explosive increase is important in determining the existence of bubbles. We will also discuss this point in Section 4.

[2] If we apply the method of Phillips et al. (2015a) to the US data, we obtain that the P-D ratio from March 1999 to December 2007 is explosive

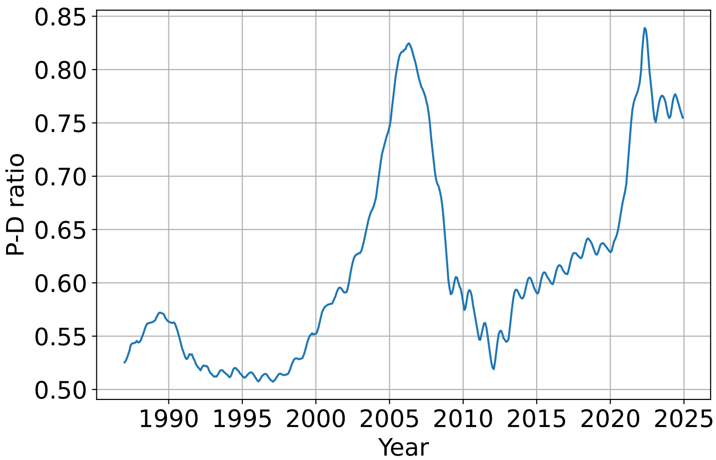

Figure 1: P-D ratio in the United States

We also provide the P-D ratio in Japan in Figure 2. We use land value published by the Ministry of Land, Infrastructure, Transport and Tourism for the land price and use the imputed rent from the System of National Accounts published by the Cabinet Office for the rent. Since we can only obtain the index of the imputed rent, we show the P-D ratio normalized at 1975.

The P-D ratio in Japan shows an explosive movement in the 1980s, while it was normally stable.[3] After the explosive movement until 1991, the P-D ratio fell sharply. This year is the year after the Ministry of Finance regulated the lending to real estate in 1990, which is believed to be one of the causes of the collapse of the bubble.

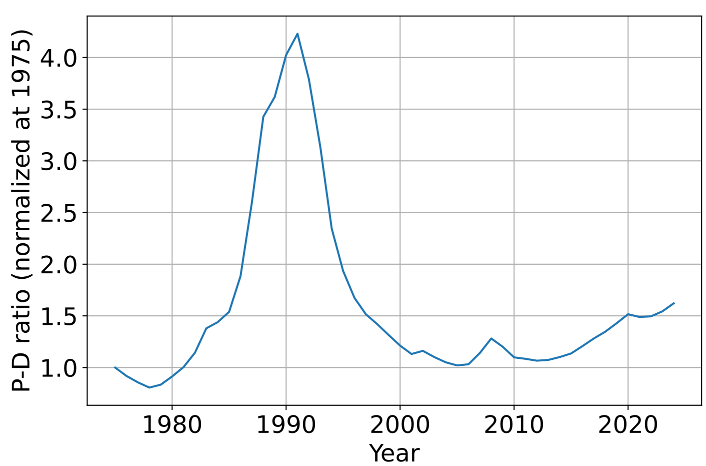

Figure 2: P-D ratio in Japan

[3] If we apply the method of Phillips et al. (2015a) to the annual Japanese data, we obtain that the P-D ratio from 1988 to 1992 is explosive, although it is difficult to get a very accurate identification due to the small sample size.

3 A Model

In this section, we present our model that can generate the temporarily explosive behavior observed in data. The model is similar to the one in Section 6 of Hirano and Toda (2024). The key difference from their model is that the spillover of the capital in the production function changes stochastically, so that the return from capital is stochastic.

3.1 Environment and Equilibrium

The economy consists of infinitesimally small agents with measure one. There are two types of agents: entrepreneurs and savers. They differ in their investment opportunities. Entrepreneurs have an access to capital and land markets while savers do not. Each agent becomes an entrepreneur with probability π and a saver with probability $1-\pi$.

An agent indexed by i has a logarithmic utility over consumption $c_{i,t}$:

$$\mathbb{E}\left[\sum_{t=0}^{\infty}\beta^t \log c_{i,t}\right],$$

where β is a discount factor, i is the index of each agent, and t is the index of time.

Each agent receives a return from capital investment and land investment from the pervious period. Let the capital investment and land investment be $k_{i,t+1}$ and $x_{i,t+1}$, respectively. Also, each agent can borrow $b_{i,t+1}$ with a net interest rate r_{t+1}. Agent i's budget constraint is

$$c_{i,t}+k_{i,t+1}+P_{X,t}x_{i,t+1}-b_{i,t+1}=R^K_t k_{i,t}+\left(P_{X,t}+D\right)x_{i,t}-\left(1+r_t\right)b_{i,t},$$

where $P_{X,t}$ is the price of land, D is the dividend from land[4], and R^K_t is the return from capital. Recall that only entrepreneurs can invest in capital and land, so that saver's budget constraint becomes

$$c_{i,t}-b_{i,t+1}=R^K_t k_{i,t}+\left(P_{X,t}+D\right)x_{i,t}-\left(1+r_t\right)b_{i,t}.$$

Note that $k_{i,t}$ and $x_{i,t}$ are zeros for agents who were savers in the privous period, i.e. at $t-1$.

Each agent can borrow up to the borrowing limit $\theta k_{i,t+1}+\theta^X P_{X,t}x_{i,t+1}$ due to financial

[4] Hirano and Toda (2025b) analyze a model of deterministic bubbles with a growing dividend. In their model, results with a constant dividend are maintained with an exogenously growing dividend unless the dividend grows as fast as output.

friction. The borrowing constraint can be written as

$$b_{i,t+1} \leq \theta k_{i,t+1} + \theta^X P_{X,t} x_{i,t+1}. \tag{1}$$

The parameters, $\theta > 0$ and $\theta^X > 0$, represent the degree of leverage. The larger these parameters are, the more leverages the agents can take.

We can show by guess and verify that the solution of the utility maximization problem is

$$c_{i,t} = (1-\beta)\, w_{i,t}, \tag{2}$$

where $w_{i,t}$ is a wealth of an agent i. The wealth is defined as

$$w_{i,t} := R_t^K k_{i,t} + (P_{X,t} + D)\, x_{i,t} - (1 + r_t)\, b_{i,t}.$$

This linear consumption function makes aggregation straightforward, which is why we focus on the logarithmic utility.

The goods market is perfectly competitive. There are infinitesimally small firms of measure one. Each firm is indexed by $j \in [0, 1]$. A firm j produces output with the following production function:

$$Y_{j,t} = A K_{j,t}^{\alpha} (\vartheta_t)^{1-\alpha}, \tag{3}$$

where $Y_{j,t}$ is the output of firm j, A is the total factor productivity, $K_{j,t}$ is the capital of firm j, α is the capital share, and ϑ_t is the productivity of labor input. Although we do not explicitly model labor market for brevity, this specification of the productivity function can be justified by assuming that inelastic labor input is fixed at unity and that the suppliers of labor spend their entire labor incomes every period as hand-to-mouth consumers.

The labor productivity, ϑ_t, is determined as

$$\vartheta_t = K_t^{\phi_t},$$

where $K_t := \int_0^1 K_{j,t} dj$ is the aggregate capital and ϕ_t captures the spillover effect of capital. The spillover effect of capital, ϕ_t, is a stochastic variable; $\phi_t = 1$ when an economy is in the H-state and takes $\underline{\phi} \in [0, 1)$ when the economy is in the L-state. The transition from each state is assumed as follows:

$$\phi_{t+1} = \begin{cases} \begin{cases} 1 & \text{with probability } \lambda \\ \underline{\phi} & \text{with probability } 1-\lambda \end{cases} & \text{if } \phi_t = 1 \\ \underline{\phi} & \text{if } \phi_t = \underline{\phi} \end{cases}.$$

This assumption implies that the economy stays at the L-state forever once the economy enters the L-state, i.e. the L-state is an absorbing state. Note that agents know that once the economy switches to the L-state with probability $1-\lambda$, the economy never returns back to the H-state.

When $\phi_t = 1$, the production function becomes linear in capital as in the AK model, which entails endogenous growth. when $\phi_t = \underline{\phi}$, the production function is concave in capital, and the economy cannot grow unless the parameter A grows exogenously.

The representative firm rents capital at a rate R_t^K. The profit maximization problem of the representative firm with the production function (3) leads to the capital rental rate given by

$$R_t^K = \alpha A K_t^{\alpha+\phi_t(1-\alpha)-1} = \begin{cases} \alpha A & \text{when } \phi_t = 1 \ \ (H\text{-state}) \\ \alpha A K_t^{\left(1-\underline{\phi}\right)(\alpha-1)} & \text{when } \phi_t = \underline{\phi} \ \ (L\text{-state}) \end{cases}.$$

In this representation of the capital rental rate, we have used the equilibrium condition that $K_t := \int_0^1 K_{j,t} dj$ and the assumption that all firms are homogeneous.

We make a timing assumption that ϕ_{t+1} is realized at the beginning of period t. This assumption lets us avoid the lending contract problem under uncertainty. Because of this assumption, when an agent makes a decision on capital and land investment at period t, the value of R_{t+1}^K is revealed to the agent without uncertainty.

We make this timing assumption for various reasons. First, the assumption allows us to avoid a contract problem under uncertainty. If the return from capital is uncertain, the contract for lending from the saver to the entrepreneur must entail a complicated contract problem with uncertainty. Second, the assumption allows us to avoid the portfolio choice problem. As we will see later, the assumption lets us derive a simple non-arbitrage condition. Lastly, and relatedly to the previous two points, this assumption lets us solve the model with aggregate uncertainty as if the model does not have uncertainty. As a result, we can analytically analyze explosive macroeconomic dynamics that are not easy to analyze numerically.

We focus on an equilibrium where the borrowing constraint (1) binds for all t.[5] As in standard macroeconomic models, the equilibrium in this economy is defined as follows. In a general equilibrium, (1) agents make their consumption-saving decision and portfolio choice optimally, (2) firms make their decision on capital demand optimally, and (3) all markets clear:

$$\int_0^1 k_{i,t} di = K_t := \int_0^1 K_{j,t} dj,$$

$$\int_0^1 b_{i,t} di = 0,$$

$$\int_0^1 x_{i,t} di = \overline{X},$$

where $\overline{X}$ is the aggregate supply of land, which is constant.

In our model, three equations characterize the equilibrium of this economy. The first one is the resource constraint:

$$K_{t+1} + P_{X,t}\overline{X} = \beta \left[R_t^K K_t + (P_{X,t} + D)\,\overline{X} \right], \tag{4}$$

where K_t is aggregate capital at the beginning of period t. The second one is the optimal capital investment by entrepreneurs:

$$K_{t+1} = \frac{1}{1-\theta} \left[\beta\pi \left(R_t^K K_t + (P_{X,t} + D)\,\overline{X} \right) - \left(1 - \theta^X \right) P_{X,t}\overline{X} \right]. \tag{5}$$

The third one is the non-arbitrage condition:

$$\frac{\frac{P_{X,t+1}+D}{P_{X,t}} - (1 + r_{t+1})\,\theta^X}{1-\theta^X} = \frac{R_{t+1}^K - (1 + r_{t+1})\,\theta}{1-\theta}. \tag{6}$$

The left and right sides of the equation (6) are leveraged return on land and capital, respectively. For example, when an agent invests in one unit of land, $1-\theta^X$ unit must be self-financed. This investment on land using the borrowings yields the return from land $\frac{P_{X,t+1}+D}{P_{X,t}}$ and requires $(1 - r_{t+1})\theta^X$ unit of repayment in the next period. Thus, the left-hand side represents the return taking the leverages into account. The same intuition holds for capital investment, and the right-hand side represents the return from capital taking leverage into account.

At the decision making in period t, the economy has no uncertainty between t to $t+1$. At the decision making in the next period, $t+1$, the economy has no uncertainty between

[5] In the propositions below, we numerically check that there exist parameters that yield the described equilibrium. We analytically investigate this point in Appendix A.4 using a special case of our model.

$t+1$ to $t+2$. Therefore, the non-arbitrage equation without the expectation operator (6) holds for all t.

The solution of the model is written as

$$K_{t+1} = \frac{\theta^X - (1-\pi)}{\frac{1}{\beta}\left(\theta^X - \theta\right) - (1-\pi-\theta)} \left[R_t^K K_t + D\overline{X}\right]$$

and

$$P_{X,t} = \frac{1-\pi-\theta}{\frac{1}{\beta}\left(\theta^X - \theta\right) - (1-\pi-\theta)} \frac{1}{\overline{X}} \left[R_t^K K_t + D\overline{X}\right].$$

Combining them, the dynamics of the capital and the land price can be expressed recursively as

$$K_{t+1} = \mathcal{C}_K R_t^K K_t + \mathcal{C}_K D\overline{X}, \tag{7}$$

$$P_{X,0} = \mathcal{C}_P \frac{1}{\overline{X}} \left[R_0^K K_0 + D\overline{X}\right],$$

and

$$P_{X,t+1} = \mathcal{C}_K R_{t+1}^K P_{X,t} + \mathcal{C}_P D \;\text{ for } t > 0, \tag{8}$$

where $\mathcal{C}_P := \frac{1-\pi-\theta}{\frac{1}{\beta}(\theta^X-\theta)-(1-\pi-\theta)}$ and $\mathcal{C}_K := \frac{\theta^X-(1-\pi)}{\frac{1}{\beta}(\theta^X-\theta)-(1-\pi-\theta)}$. We focus on a parameter set of $\theta < 1-\pi < \theta^X$ and $0 < \frac{1}{\beta}\left(\theta^X - \theta\right) - (1-\pi-\theta)$, so that $\mathcal{C}_P$ and $\mathcal{C}_K$ are both positive. By taking derivatives, we can easily see how two coefficients change with θ^X and θ as follows.

Lemma 1. *$\mathcal{C}_P$ is decreasing in θ^X and θ. $\mathcal{C}_K$ is increasing in θ^X and θ.*

Proof.

$$\frac{d\mathcal{C}_P}{d\theta^X} = \frac{-(1-\pi-\theta)\frac{1}{\beta}}{\left(\frac{1}{\beta}\left(\theta^X-\theta\right)-(1-\pi-\theta)\right)^2} < 0.$$

$$\frac{d\mathcal{C}_P}{d\theta} = \frac{-\left(\frac{1}{\beta}\left(\theta^X-\theta\right)-(1-\pi-\theta)\right)-(1-\pi-\theta)\left(\frac{1}{\beta}-1\right)}{\left(\frac{1}{\beta}\left(\theta^X-\theta\right)-(1-\pi-\theta)\right)^2} < 0.$$

$$\frac{d\mathcal{C}_K}{d\theta^X} = \frac{\frac{1}{\beta}\left(\theta^X-\theta\right)-(1-\pi-\theta)-\left(\theta^X-(1-\pi)\right)\frac{1}{\beta}}{\left(\frac{1}{\beta}\left(\theta^X-\theta\right)-(1-\pi-\theta)\right)^2} = \frac{\left(\frac{1}{\beta}-1\right)(1-\pi-\theta)}{\left(\frac{1}{\beta}\left(\theta^X-\theta\right)-(1-\pi-\theta)\right)^2} > 0.$$

$$\frac{d\mathcal{C}_K}{d\theta^X} = \frac{-\left(\theta^X-(1-\pi)\right)\left(-\frac{1}{\beta}+1\right)}{\left(\frac{1}{\beta}\left(\theta^X-\theta\right)-(1-\pi-\theta)\right)^2} > 0.$$

□

Throughout the paper, we focus on an equilibrium in which the borrowing constraint (1) binds and the consumption function is given by (2). Then, the equilibrium dynamics governed by (4), (5), and (6) are uniquely determined.

3.2 Economic Dynamics

We begin our analysis on the dynamics of the economy. We briefly discuss the dynamics of capital first and then move on to the P-D ratio.

We first focus on the dynamics of capital in the H-state. As can be seen from (7), the capital diverges to infinity when $\mathcal{C}_K \alpha A > 1$, approaching a gross growth rate $\mathcal{C}_K \alpha A$. When $\mathcal{C}_K \alpha A < 1$, the capital converges to a steady state $\frac{\mathcal{C}_K D\overline{X}}{1-\mathcal{C}_K \alpha A}$.

Next, we analyze capital in the L-state. Let $K_{L,SS}$ be the steady state capital at the L-state in the long-run. This K_{SS} is implicitly determined by

$$K_{L,SS} = \mathcal{C}_K \left[\alpha A K_{L,SS}^{\alpha+\underline{\phi}(1-\alpha)} + D\overline{X} \right]. \tag{9}$$

The results of comparative statics of $K_{L,SS}$ with respect to θ^X, θ, and A are stated in the following lemma.

Lemma 2. *$K_{L,SS}$ increases with θ^X, θ, and A.*

Proof. The implicit function theorem leads to

$$\frac{dK_{L,SS}}{d\theta^X} = -\frac{\frac{d\mathcal{C}_K}{d\theta^X}\left[\alpha A K_{L,SS}^{\alpha+\underline{\phi}(1-\alpha)} + D\overline{X}\right]}{-1 + \mathcal{C}_K \alpha A \left(\alpha + \underline{\phi}(1-\alpha)\right) K_{L,SS}^{(\alpha-1)(1-\underline{\phi})}}.$$

The denominator is negative because (9) implies

$$\mathcal{C}_K \alpha A \left(\alpha + \underline{\phi}(1-\alpha)\right) K_{L,SS}^{(\alpha-1)(1-\underline{\phi})} = \left(1 - \mathcal{C}_K D\overline{X} K_{L,SS}^{-1}\right) \left(\alpha + \underline{\phi}(1-\alpha)\right),$$

and $1 - \mathcal{C}_K D\overline{X} K_{L,SS}^{-1} < 1$ and $\alpha + \underline{\phi}(1-\alpha) < 1$ hold. Therefore, combined with Lemma 1, $\frac{dK_{L,SS}}{d\theta^X} > 0$.

Similarly, we can derive

$$\frac{dK_{L,SS}}{d\theta} = -\frac{\frac{d\mathcal{C}_K}{d\theta}\left[\alpha A K_{L,SS}^{\alpha+\underline{\phi}(1-\alpha)} + D\overline{X}\right]}{-1+\mathcal{C}_K\alpha A\left(\alpha+\underline{\phi}\left(1-\alpha\right)\right)K_{L,SS}^{(\alpha-1)\left(1-\underline{\phi}\right)}} > 0$$

and

$$\frac{dK_{L,SS}}{dA} = -\frac{\mathcal{C}_K\alpha K_{L,SS}^{\alpha+\underline{\phi}(1-\alpha)}}{-1+\mathcal{C}_K\alpha A\left(\alpha+\underline{\phi}\left(1-\alpha\right)\right)K_{L,SS}^{(\alpha-1)\left(1-\underline{\phi}\right)}} > 0$$

by the same argument. □

Next, we analyze the P-D ratio defined as $\frac{P_{X,t}}{D}$. When the economy is in the H-state, we can see from (8) that $P_{X,t}$ and the P-D ratio diverge to infinitely explosively when $\mathcal{C}_K\alpha A > 1$. When $\mathcal{C}_K\alpha A < 1$, $P_{X,t}$ converges to a steady state $\frac{\mathcal{C}_K D\overline{X}}{1-\mathcal{C}_K\alpha A}$ and the P-D ratio converges to $\frac{\mathcal{C}_K\overline{X}}{1-\mathcal{C}_K\alpha A}$.[6]

When the economy is in the L-state, $P_{X,t}$ converges to a steady state. The steady state is given by $\mathcal{C}_P\frac{1}{\overline{X}}\left[\alpha A K_{L,SS}^{(\alpha-1)\left(1-\underline{\phi}\right)} + D\overline{X}\right]$. The dynamics of $P_{X,t}$ is stationary, so is the P-D ratio.

We summarize the arguments in the following proposition.

Proposition 1. *The land price and the P-D ratio shows explosive dynamics in the H-state if $\mathcal{C}_K\alpha A > 1$. If $\mathcal{C}_K\alpha A < 1$, they are stationary in the H-state. In the L-state, they are stationary irrespective of the parameter values.*

3.3 Numerical Examples

We provide numerical examples of our model to show the macroeconomic dynamics differ qualitatively. Throughout this section, we use $A = 4$, $\underline{\phi} = 0$, $\theta = 0.1$, $\theta^X = 0.7$, $\pi = 0.4$, $\beta = 0.96$, $\overline{X} = 1$, $D = 0.1$, and $\alpha = \frac{1}{3}$ unless otherwise noted. With these parameters, $\mathcal{C}_K = 0.8$ and $\mathcal{C}_K\alpha A = 1.067$, implying that the land price shows explosive dynamics in the H-state.

In Figure 3 I show a result of a simulation where the economy is in the L-state steady state until $t = -1$, the economy enters the Hstate (i.e. ϕ_t switches to 1) at $t = 0$, and the economy goes back to the L-state (i.e. ϕ_t switches back to 0) from $t = 10$.

[6] When $\mathcal{C}_K\alpha A = 1$, the P-D ratio diverges to infinity but its growth is additive. Since this is a knife edge case, we focus on the case of $\mathcal{C}_K\alpha A \neq 1$.

The dynamics of the land price shows explosive behavior when the economy is in the H-state. Along with the land price, capital also accumulates. This leads to a large reduction in the land price at $t = 10$.

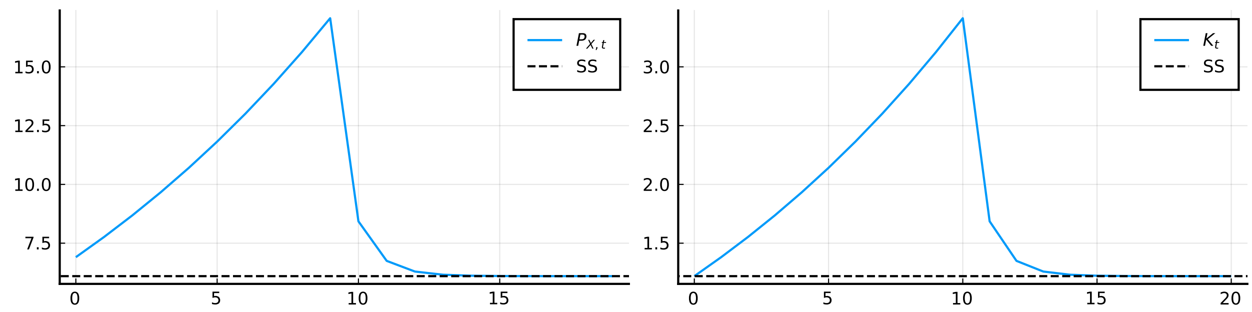

Figure 3: Simulated path of the land price and capital with a sharp drop at $t = 10$ due to a change in ϕ_t

The macroeconomic dynamics of land price due to a change in leverage is qualitatively different from the one due to the change in ϕ_t. Figure 4 plots the simulated path when θ^X temporarily rises in the H-state. Until $t = -1$, the economy is in the H-state steady state with $\theta^X = 0.65$. From $t = 0$ to $t = 9$, θ^X rises to 0.66. From $t = 10$, θ^X goes back to 0.65. For all t, the economy stays in the H-state, and $\phi_t = 1$. With $\theta^X = 0.65$ and $\theta^X = 0.66$, $\mathcal{C}_K \alpha A = 0.914$ and $\mathcal{C}_K \alpha A = 0.96$, respectively, implying that the land price is stationary for both cases.

The stark contrast of the dynamics relative to the change in ϕ_t lies in the dynamics of the land price. The land price initially falls as θ^X rises, starting to rise until $t = 9$. Then, θ^X goes back to 0.65 from $t = 10$, making the economy stationary. This change leads to a rise in the land price, which is counterfactual. While changes in θ^X can create a rise or fall in the land price, the dynamics of the land price generated by changes in ϕ_t is more consistent with data.

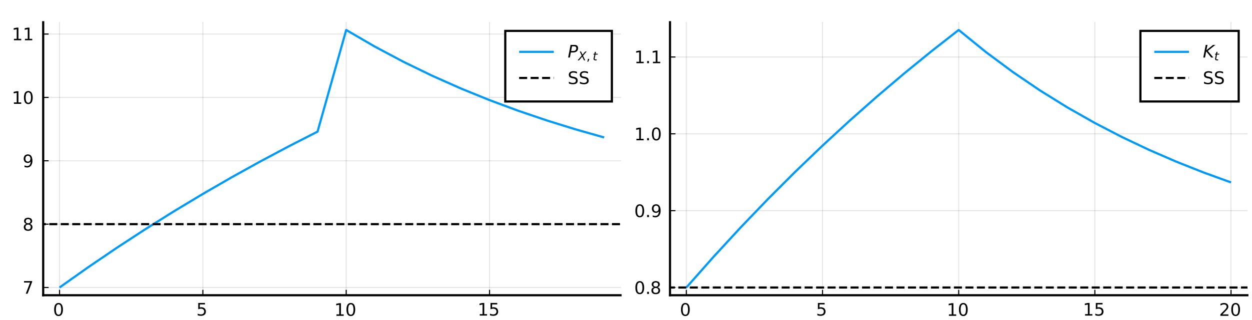

Figure 4: Simulated path of the land price and capital with a sharp drop at $t = 10$ due to a change in θ^X

The two example paths we have shown display temporarily explosive land price dynamics. In Figure 5, we simulate an economy with a temporary leverage hike within L-state. The economy has $\theta^X = 0.7$ until $t = -1$, θ^X rises to 0.75 from $t = 1$ to $t = 9$, and θ^X goes back to 0.7 from $t = 10$. Throughout the simulation, the economy is in the L-state.

The dynamics of the land price and capital in Figure 5 are very different from the ones in Figure 3 and 4. In Figure 5, the land price and capital are both stationary. There is no explosive behavior in land price, which we observe in data shown in Figure 1 and 2.

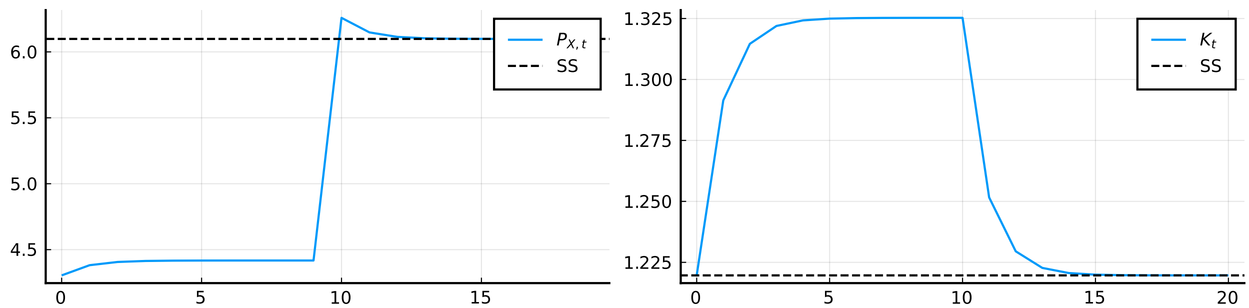

Figure 5: Simulated path of the land price and capital with a sharp drop at $t = 10$ due to a change in θ^X

4 Dicussion

In the empirical literature on bubble detections, explosive land price dynamics are closely related to bubbles (see Phillips et al. (2011) and Phillips et al. (2015a) for example). We discuss the existence of bubbles in this section.

4.1 Existence of Bubbles

In this section, we establish the existence of bubbles in the stochastic economy. Before going into the details, we introduce a new notation of variables to keep track of paths under the potential realization of parameters in the future. For any variable $\mathcal{X}_t$ in the model, let $\mathcal{X}_{(H),t}$ denote the variable under the shock realization where the economy is in H-state, i.e. $\phi_0 = \cdots = \phi_t = 1$. Similarly, with $k \le t$, let $\mathcal{X}_{(L,k),t}$ be the variable under the shock realization where the economy switches from H-state to L-state between period k and $k+1$, i.e. $\phi_0 = \cdots = \phi_k = 1$ and $\phi_{k+1} = \cdots = \phi_t = \underline{\phi}$.

With this notation, we derive the expression to decompose the land price at $t = 0$ in the H-state into its fundamental and non-fundamental parts. Since the non-arbitrage condition (6) holds for each $t \ge 0$, $P_{(H),X,t+1}$, $R^K_{(H),t+1}$, $r_{(H),t+1}$, $P_{(L,t),X,t+1}$, $R^K_{(L,t),t+1}$, and

$r_{(L,t),t+1}$ must satisfy

$$\frac{\frac{P_{(H),X,t+1}+D}{P_{(H),X,t}} - \left(1 + r_{(H),t+1}\right)\theta^X}{1-\theta^X} = \frac{R^K_{(H),t+1} - \left(1 + r_{(H),t+1}\right)\theta}{1-\theta} \tag{10}$$

and

$$\frac{\frac{P_{(L,t),X,t+1}+D}{P_{(H),X,t}} - \left(1 + r_{(L,t),t+1}\right)\theta^X}{1-\theta^X} = \frac{R^K_{(L,t),t+1} - \left(1 + r_{(L,t),t+1}\right)\theta}{1-\theta}. \tag{11}$$

These two imply[7] that

$$\mathbb{E}\left[\frac{\frac{P_{X,t+1}+D}{P_{(H),X,t}} - \left(1 + r_{t+1}\right)\theta^X}{1-\theta^X} | \phi_t = 1\right] = \mathbb{E}\left[\frac{R^K_{t+1} - \left(1 + r_{t+1}\right)\theta}{1-\theta} | \phi_t = 1\right] \tag{12}$$

for all $t \geq 0$. Define $\mathcal{R}_{(H),t,t+1} := \mathbb{E}\left[\left(1-\theta^X\right)\frac{R^K_{t+1}-(1+r_{t+1})\theta}{1-\theta} + \left(1+r_{t+1}\right)\theta^X | \phi_t = 1\right]$.[8] Then, (12) for all $t \geq 0$ can be written as

$$P_{(H),X,t} = \frac{\mathbb{E}\left[P_{X,t+1} + D | \phi_t = 1\right]}{\mathcal{R}_{(H),t,t+1}} = \frac{\lambda P_{(H),X,t+1} + (1-\lambda) P_{(L,t),X,t+1} + D}{\mathcal{R}_{(H),t,t+1}}. \tag{13}$$

By iterating (13) forward, $P_{(H),X,0}$ can be expressed as

$$\begin{aligned} P_{(H),X,0} = {} & \frac{(1-\lambda) P_{(L,0),X,1} + D}{\mathcal{R}_{(H),0,1}} + \lambda \frac{(1-\lambda) P_{(L,1),X,2} + D}{\mathcal{R}_{(H),1,2}\mathcal{R}_{(H),0,1}} + \lambda^2 \frac{(1-\lambda) P_{(L,2),X,3} + D}{\mathcal{R}_{(H),2,3}\mathcal{R}_{(H),1,2}\mathcal{R}_{(H),0,1}} \\ & + \cdots + \lambda^{T-1}\frac{(1-\lambda) P_{(L,T-1),X,T} + D}{\prod_{t=1}^{T}\mathcal{R}_{(H),t-1,t}} + \frac{\lambda^T P_{(H),X,T}}{\prod_{t=1}^{T}\mathcal{R}_{(H),t-1,t}} \text{ for some } T. \end{aligned}$$

Notice that we do not condition on ϕ_1 when evaluating $P_{(H),X,0}$ because the land price at $t = 0$ is determined before the state for $t = 1$ is revealed.

[7](10) and (11) also imply $\mathbb{E}\left[\frac{\frac{P_{X,t+1}+D}{P_{(H),X,t}}}{\left(1-\theta^X\right)\frac{R^K_{t+1}-(1+r_{t+1})\theta}{1-\theta} + (1+r_{t+1})\theta^X} | \phi_t = 1\right] = 1$. While the discounting based on this alternative equation is also possible, we focus on the discounting based on the equation (12) because our procedure can incorporate the case of pure bubble, $D = 0$, as a special case while the alternative discounting procedure cannot consider the case of $D = 0$.

[8]Note that our procedure of discounting by $\mathcal{R}_{(H),t,t+1}$ is consistent with the agent's non-arbitrage conditions in the portfolio choice problem for each period. We discuss this point in Appendix C.

In the limit of $T \to \infty$,

$$P_{(H),X,0} = \underbrace{\sum_{T=1}^{\infty} \lambda^{T-1} \frac{(1-\lambda) P_{(L,T-1),X,T} + D}{\prod_{t=1}^{T} \mathcal{R}_{(H),t-1,t}}}_{\text{Fundamental term}} + \underbrace{\lim_{T\to\infty} \frac{\lambda^T P_{(H),X,T}}{\prod_{t=1}^{T} \mathcal{R}_{(H),t-1,t}}}_{\text{Bubble term}}. \tag{14}$$

We define the sum of the first two terms as the fundamental term. This fundamental term represents the fundamental value of the land, i.e. the present discounted value of the future stream of the dividend. We define the last term as the bubble term. We call that the land price contains a bubble or that a land price bubble exists when the bubble term is positive:

$$\lim_{T\to\infty} \frac{\lambda^T P_{(H),X,T}}{\prod_{t=1}^{T} \mathcal{R}_{(H),t-1,t}} > 0. \tag{15}$$

This corresponds to the case when the land price is higher than the fundamental value of the land. If the bubble term is zero, the transversality condition for the land price holds, and the land price does not contain a bubble.

The next proposition establishes the existence of a bubble under a certain parameter set.

Proposition 2. *(Existence of Bubbles) Suppose $\phi_0 = \phi_1 = 1$. When $1 < \mathcal{C}_K \alpha A$, the unique equilibrium contains a bubble. When $\mathcal{C}_K \alpha A < 1$, the unique equilibrium does not contain a bubble.*

Proof. We first prove the first part of the proposition. Using (13), the limit in (15) can be expressed as

$$\begin{aligned}
\lim_{T\to\infty} \frac{\lambda^T P_{(H),X,T}}{\prod_{t=1}^{T} \mathcal{R}_{(H),t-1,t}} &= P_{(H),X,0} \lim_{T\to\infty} \prod_{t=1}^{T} \left(\frac{\frac{\lambda P_{(H),X,t}}{P_{(H),X,t-1}}}{\frac{\lambda P_{(H),X,t}+(1-\lambda)P_{(L,t-1),X,t}+D}{P_{(H),X,t-1}}} \right) \\
&= P_{(H),X,0} \lim_{T\to\infty} \prod_{t=1}^{T} \left(\frac{\lambda P_{(H),X,t}}{\lambda P_{(H),X,t} + (1-\lambda) P_{(L,t-1),X,t} + D} \right) \\
&= P_{(H),X,0} \lim_{T\to\infty} \prod_{t=1}^{T} \left(1 - \frac{(1-\lambda) P_{(L,t-1),X,t} + D}{\lambda P_{(H),X,t} + (1-\lambda) P_{(L,t-1),X,t} + D} \right) \\
&= P_{(H),X,0} \lim_{T\to\infty} \prod_{t=1}^{T} \left(1 - \frac{(1-\lambda) \frac{P_{(L,t-1),X,t}}{P_{(H),X,t}} + \frac{D}{P_{(H),X,t}}}{\lambda + (1-\lambda) \frac{P_{(L,t-1),X,t}}{P_{(H),X,t}} + \frac{D}{P_{(H),X,t}}} \right).
\end{aligned}$$

This limit is non-zero if and only if

$$\sum_{t=1}^{T} \log\left(1 - \frac{(1-\lambda)\frac{P_{(L,t-1),X,t}}{P_{(H),X,t}} + \frac{D}{P_{(H),X,t}}}{\lambda + (1-\lambda)\frac{P_{(L,t-1),X,t}}{P_{(H),X,t}} + \frac{D}{P_{(H),X,t}}}\right) \tag{16}$$

converges with $T \to \infty$.

Let $Z_t := \frac{(1-\lambda)\frac{P_{(L,t-1),X,t}}{P_{(H),X,t}} + \frac{D}{P_{(H),X,t}}}{\lambda+(1-\lambda)\frac{P_{(L,t-1),X,t}}{P_{(H),X,t}} + \frac{D}{P_{(H),X,t}}}$. Then, by Taylor expansion around $Z_t = 0$,

$$\log(1 - Z_t) = -\left(Z_t + \frac{Z_t^2}{2} + \frac{Z_t^3}{3} + \dots\right).$$

Notice that

$$\begin{aligned}\frac{P_{(L,t-1),X,t}}{P_{(H),X,t}} &= \frac{\mathcal{C}_K\left(\alpha + \underline{\phi}(1-\alpha)\right) A\left(K_{(L,t-1),t}\right)^{(1-\underline{\phi})(\alpha-1)} P_{(H),X,t-1} + \mathcal{C}_P D}{\mathcal{C}_K A P_{(H),X,t-1} + \mathcal{C}_P D} \\ &= \frac{\mathcal{C}_K\left(\alpha + \underline{\phi}(1-\alpha)\right) A\left(K_{(L,t-1),t}\right)^{(1-\underline{\phi})(\alpha-1)} + \mathcal{C}_P \frac{D}{P_{(H),X,t-1}}}{\mathcal{C}_K A + \mathcal{C}_P \frac{D}{P_{(H),X,t-1}}}.\end{aligned}$$

Since the law of motion for capital is expressed recursively as (7), $K_{(L,t-1),t}$ can be expressed as

$$\begin{aligned} K_{(L,t-1),t} &= \mathcal{C}_K R^K_{t-1} K_{(H),t-1} + \mathcal{C}_K D\overline{X} \\ &= \mathcal{C}_K \alpha A K_{(H),t-1} + \mathcal{C}_K D\overline{X} \\ &= \mathcal{C}_K \alpha A\left(\mathcal{C}_K \alpha A K_{(H),t-2} + \mathcal{C}_K D\overline{X}\right) + \mathcal{C}_K D\overline{X} \\ &= \dots \\ &= (\mathcal{C}_K \alpha A)^t K_{(H),0} + \sum_{s=1}^{t} (\mathcal{C}_K \alpha A)^{s-1} \mathcal{C}_K D\overline{X} \\ &= (\mathcal{C}_K \alpha A)^t K_{(H),0} + \sum_{s=1}^{t} (\mathcal{C}_K \alpha A)^{s-1} \mathcal{C}_K D\overline{X} \\ &= (\mathcal{C}_K \alpha A)^t K_{(H),0} + \mathcal{C}_K D\overline{X}\frac{(\mathcal{C}_K \alpha A)^t - 1}{\mathcal{C}_K \alpha A - 1}. \end{aligned} \tag{17}$$

From (17) and $1 < \mathcal{C}_K \alpha A$, we can see that $\lim_{t\to\infty} K_{(L,t-1),t} = \infty$, $K_{(L,t-1),t} > 0$ for all t, and

$$K_{(L,t-1),t} = \mathcal{O}\left(\left(\mathcal{C}_K \alpha A\right)^t\right) \text{ as } t \to \infty.$$

Similarly, from (8) and $1 < \mathcal{C}_K \alpha A$, we can see that $\lim_{t\to\infty} P_{(H),X,t} = \infty$, $P_{(H),X,t} > 0$ for all t, and

$$P_{(H),X,t} = \mathcal{O}\left(\left(\mathcal{C}_K A\right)^t\right) \text{ as } t \to \infty.$$

Therefore, we can see that $\lim_{t\to\infty} Z_t = 0$, $Z_t > 0$ for all t, and

$$Z_t = \mathcal{O}\left(\left(\mathcal{C}_K A\right)^{-t}\right) \text{ as } t \to \infty.$$

Since an infinite sum of a geometric series with the terms converging to zero is convergent, $\sum_{t=0}^{\infty} Z_t$ is convergent, so are $\sum_{t=0}^{\infty} Z_t^2, \sum_{t=0}^{\infty} Z_t^3, \ldots$. Thus, $\sum_{t=0}^{\infty} \log\left(1 - Z_t\right)$ and (16) are convergent. Hence, the bubble term is non-zero and the bubble exists.

For the second part of the proposition, the arguments until (17) in the proof hold. From (17) and $\mathcal{C}_K A < 1$, we can see that $\lim_{t\to\infty} K_{(L,t-1),t} = \mathcal{C}_K D \overline{X} \frac{1}{1-\mathcal{C}_K A} > 0$. Also, from the discussion in Section 3.2, $\lim_{t\to\infty} P_{(H),X,t-1} < \infty$. Thus, $\lim_{t\to\infty} Z_t > 0$. Hence, $\sum_{t=0}^{\infty} \log\left(1 - Z_t\right)$ and (16) are not convergent, the bubble term is zero, and the bubble does not exist. □

It can be seen from the proof that the convergence of Z_t to zero as $t \to \infty$ is a necessary condition for a bubble to arise. For Z_t to converge to zero, it is required that $\frac{P_{(L,t-1),X,t}}{P_{(H),X,t}}$ converges to zero as can be seen from equation (16). This requirement means that the gap between the land price of a survived bubble and the one of a collapsed bubble at the time of collapse must be larger as the burst is later. In other words, long-lasting bubbles is necessarily accompanied by a large reduction in land price.

We would like to point out that the property of the bubble highlighted above, that is, a bubble that lasts longer must be accompanied by a larger reduction of the land price at the collapse, holds in a more general setting of the model. In our model, when the economy switches to the L-state, the return from capital becomes decreasing in capital. Since the long-lasting bubble leads to large capital accumulation, the return from capital becomes low, and the non-arbitrage implies that the land price growth rate is low.

Notice that there are alternative changes that make the bubble arise. For example, if θ^X in the L-state approaches to $1 - \pi$ sufficiently fast as $t \to \infty$ while keeping θ^X in the H-state constant, this also makes the bubble arise even when $\phi_t = 1$ in both states.

The next proposition makes a comparative statics of leverages.

Proposition 3. *(Comparative Statics of Leverages) The higher θ or θ^X is, the more likely the bubble may arise when the economy is in the H-state.*

Proof. Take a derivative of $\mathcal{C}_K$ with respect to θ and θ^X to get

$$\frac{\partial \mathcal{C}_K}{\partial \theta} = \frac{\left(\theta^X - (1-\pi)\right)\left(\frac{1}{\beta} - 1\right)}{\left(\frac{1}{\beta}\left(\theta^X - \theta\right) - (1 - \pi - \theta)\right)^2} > 0,$$

and

$$\frac{\partial \mathcal{C}_K}{\partial \theta^X} = \frac{(1 - \pi - \theta)\left(\frac{1}{\beta} - 1\right)}{\left(\frac{1}{\beta}\left(\theta^X - \theta\right) - (1 - \pi - \theta)\right)^2} > 0.$$

A rise in θ or θ^X raises $\mathcal{C}_K$, and $1 < \mathcal{C}_K A$ is more likely to be satisfied. □

Let us point out that Proposition 2 and Proposition 3 imply that bubbles exist only in a certain economic environment. First, the economy must be in H-state in which the spillover effect of capital makes the production function linear in capital as in the AK endogenous growth model. We did not model how the economy enters the H-state because we consider it as an uncommon event, and the switch from L-state to H-state does not occur frequently. For the bubble to exist, this uncommon event must occur.

Second, when the economy enters H-state, the leverages, θ^X and θ, must be high. If the economy enters H-state with low leverage, $\mathcal{C}_K A$ is lower than one, and the economy does not contain a bubble, and the economy behaves stationarily. A bubble does not always exist, but rather, it exists only when the leverages, typically determined by policies, are high.

This leads to a policy implication that setting sufficiently high leverages lead to bubbles when the economy enters the H-state. If one wants to avoid bubbles, leverages cannot be too high. Differently from Hirano et al. (2022), though closely related, high leverages during the strong spillover followed by weak spillover is required for the bubble to exist.

4.2 Connection to the Literature on Bubble Detection

Let us discuss the connection of our analysis to the literature on bubble detection. To obtain the intuition, we revisit the condition for the existence of the bubble (15). The bubble condition states that the bubble term defined in (14) must be positive. From (13), we can rewrite the denominator of the bubble term, $R_{(H),t,t+1}$, as

$$R_{(H),t,t+1} = \frac{\lambda P_{(H),X,t+1}}{P_{(H),X,t}} + \frac{(1-\lambda)P_{(L),X,t+1}}{P_{(H),X,t}} + \frac{D}{P_{(H),X,t}}.$$

The first term is λ times the gross growth rate of the land price when the bubble continues. The numerator of the bubble term grows at this rate. The bubble term is positive when this first term equals to $R_{(H),t,t+1}$ in the limit of $t \to \infty$. If any of the second and the third term are positive, the denominator of the bubble term is larger than the numerator, and the bubble term is zero. Given the structure of our model (specifically, the land price can be written recursively as (8)), the existence of a bubble is equivalent to the convergence of the second and the third term to zero as $t \to \infty$.

The second term is the size of the reduction in the land price at the time of the burst. The convergence of this second term to zero means that the size of the reduction in the land price becomes larger when the bubble lasts longer. When the second term does not converge to zero, the fundamental price of the land grows explosively along with the land price at the H-state, and a change in the state from H to L does not lead to a larger reduction in land price even when a bubble lasts longer.

The third term is the inverse of the P-D ratio. Papers on the bubble detections, such as Phillips et al. (2011) and Phillips et al. (2015a), check the explosiveness of the P-D ratio. Their detection procedures consider the case where the fundamental term is not explosive and check the explosiveness of the inverse of the third term. In other words, they focus on the case where the second term converges to zero and check the convergence of the third term to zero.

We summarize the connection to the literature on bubble detections.

Proposition 4. *(Signal of Bubbles) When the P-D ratio behaves explosively, we may be warned that the land price contains a bubble. This explosiveness of the P-D ratio does not necessarily mean the existence of the bubbles because the fundamental term may be explosive and the change in the state may not lead to a large reduction in land price. If we can assume that the fundamental term is not explosive, the explosiveness of the P-D ratio implies the bubble.*

The proposition provides a key insight on bubble detection. If an exponential growth of the P-D ratio is observed, we should be warned that the economy may contain bubbles because the explosive dynamics is necessary for the existence of bubbles. Ideally, we should check the explosiveness of the fundamental term as the explosiveness is one of the important necessary conditions for the bubble existence which corresponds to a larger reduction of the land price for bubbles that last longer. In practice, however, the fundamental term is not observable, and it is difficult to make an inference on the dynamics of the fundamental term. If we assume that the fundamental term is not explosive, the explosiveness of the bubble correctly identifies the bubble as in the literature.

Not only detecting the timing of bubbles, we can speak to the causes of the bubbles thanks to our theoretical model. In Section 3, with our theoretical model, we have found that a large spillover and high leverages are required for bubbles to exist. We can check the changes and their timings in data counterpart of θ^X and θ to infer if the cause of a bubble is a policy to raise these leverage parameters given high spillover or a technological change that raises the spillover given high leverages.

5 Conclusion

We provided a macro-finance model that can generate temporary explosive dynamics. We have also shown that the land price contains a bubble. We draw an implication of bubble detection and find that not only the dynamics during the explosive period but also the dynamics after the explosive period are critical in determining the existence of bubbles.

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Appendix

A Discussion

A.1 Land Price to GDP Ratio

Here, we analyze the land price to GDP ratio. The land price to GDP ratio in our economy is given by

$$\frac{P_{X,t}\overline{X}}{Y_t + D\overline{X}} = \frac{\mathcal{C}_P\left[R_t^K K_t + D\overline{X}\right]}{AK_t^{\alpha+\phi_t(1-\alpha)} + D\overline{X}} = \mathcal{C}_P\left(1 + \frac{\alpha - 1}{1 + \frac{D}{AK_t^{\alpha+\phi_t(1-\alpha)}}\overline{X}}\right). \tag{18}$$

When the economy is in the H-state, i.e. $\phi_t = 1$, the land price to GDP ratio decreases in A. Also, since $\mathcal{C}_P$ is decreasing in θ^X and θ, the land price to GDP ratio decreases in θ^X or θ when the economy is in the H-state.

When the economy is in the L-state, i.e, $\phi_t = \underline{\phi} < 1$, we can see from (18) that $\frac{P_{X,t}\overline{X}}{Y_t + D\overline{X}} < \mathcal{C}_P$. In other words, the land price to GDP ratio is higher in the H-state.

Note that the land price to GDP ratio is decreasing in K_t in the L-state too. Since K_t converges to $K_{L,SS}$ in the long-run, combined with Lemma 1 and Lemma 2, we can see that a rise in θ^X or θ decreases the long-run land price to GDP ratio at the L-state.

The arguments in this section on the land price to GDP ratio are summarized in the following proposition.

Proposition 5. *Given parameters, the land price to GDP ratio in the H-state is higher than the one in the L-state. Given a state and capital, a rise in A, θ^X, or θ decreases the land price to GDP ratio.*

A.2 Credit to GDP Ratio

Lastly, we analyze the credit to GDP ratio. The credit to GDP ratio in our economy is given by

$$\frac{B_{t+1}}{Y_{t+1}+D\overline{X}} = \frac{\theta\mathcal{C}_K + \theta^X \mathcal{C}_P}{A\left(\mathcal{C}_K\right)^{\alpha+\phi_{t+1}(1-\alpha)}\left(R_t^K K_t + D\overline{X}\right)^{(\alpha-1)(1-\phi_{t+1})} + D\overline{X}\left(R_t^K K_t + D\overline{X}\right)^{-1}}.$$

In the H-state, the ratio is simplified to

$$\frac{B_{t+1}}{Y_{t+1}+D} = \frac{\theta\mathcal{C}_K + \theta^X \mathcal{C}_P}{A\mathcal{C}_K + D\left(\alpha A K_t + D\overline{X}\right)^{-1}}.$$

This credit to GDP ratio in the H-state is increasing in K_t. When $\alpha A\mathcal{C}_K > 1$ and the economy stays at the H-state, $K_t \to \infty$ and, therefore, the credit to output ratio converges to $\frac{1}{A}\frac{\left(\theta^X-\theta\right)(1-\pi)}{\theta^X-(1-\pi)}$, which is decreasing in A, θ^X, and θ.

In the L-state where $\phi_{t+1} = \underline{\phi} < 1$, the credit to GDP ratio is increasing in K_t. In the long-run, K_t converges to $K_{L,SS}$ implicitly determined by (9) and, therefore, we can derive that the ratio converges to

$$\frac{B_{L,SS}}{Y_{L,SS}+D\overline{X}} = \frac{\left(\theta^X-\theta\right)(1-\pi)}{\theta^X-(1-\pi)}\left[A\left(K_{L,SS}\right)^{(\alpha-1)\left(1-\underline{\phi}\right)} + D\overline{X}\left(K_{L,SS}\right)^{-1}\right]^{-1}.$$

This ratio at the limit in the L-state is smaller than the ratio in the limit in the H-state if and only if

$$\left(K_{L,SS}\right)^{(\alpha-1)\left(1-\underline{\phi}\right)} + \frac{D}{AK_{L,SS}}\overline{X} - 1 > 0. \tag{19}$$

Because of Lemma 2, the inequality is likely to be satisfied when A, θ^X, and θ are small.

We summarize the findings of this section in the following proposition.

Proposition 6. *Given parameters, the credit to GDP ratio in the H-state is higher than the one in the L-state if and only if (19) is satisfied. Given that the economy stays in the H-state, a rise in A, θ^X, or θ decreases the credit to GDP ratio in the limit.*

A.3 Interest Rate

When the economy stays in the H-state, the gross growth rate of capital converges to $\mathcal{C}_K R^K_{t+1} = \mathcal{C}_K \alpha A$. Note that

$$\mathcal{C}_K = \frac{\theta^X - (1-\pi)}{\frac{1}{\beta}\left(\theta^X - \theta\right) - (1-\pi-\theta)} = 1 + \frac{\left(1-\frac{1}{\beta}\right)\left(\theta^X - \theta\right)}{\frac{1}{\beta}\left(\theta^X - \theta\right) - (1-\pi-\theta)} < 1. \tag{20}$$

Therefore, the gross growth rate of capital and output is lower than αA in the limit.

When the economy stays in the H-state, we can also analyze the interest rate analytically. From the non-arbitrage condition (6),

$$1 + r_{t+1} = \frac{P_{X,t+1} + D}{P_{X,t}} \frac{1-\theta}{\theta^X - \theta} - R^K_{t+1} \frac{1-\theta^X}{\theta^X - \theta}.$$

Since $\frac{P_{X,t+1}+D}{P_{X,t}} \to \mathcal{C}_K \alpha A$,

$$1 + r_{t+1} \to \mathcal{C}_K \alpha A \left(1 + \frac{\left(1-\frac{1}{\beta}\right)\left(1-\theta^X\right)}{\theta^X - (1-\pi)}\right).$$

Since $1 - \frac{1}{\beta} < 0$, $1 - \theta^X > 0$, and $\theta^X - (1-\pi) > 0$, the right hand side is less than $\mathcal{C}_K \alpha A$.

Hence, the following relation holds in the H-state in the limit:

$$1 + r < 1 + g_K = 1 + g_Y = \mathcal{C}_K \alpha A < \alpha A = R^K, \tag{21}$$

where g_K and g_Y are the net growth rate of capital and output, respectively, and r is the net interest rate in the limit. In other words, the growth rate of the economy is higher than the interest rate. This phenomenon is consistent with the historical pattern of the U.S. (Blanchard, 2019)

A.4 Binding Borrowing Constraint

We discuss when the borrowing constraint (1) binds, paying particular attention to a special case of $D \to 0$.

For the constraint to bind, the gross interest rate $1 + r_{t+1}$ must be lower than the return from capital R^K_{t+1} and the return from land $\frac{P_{X,t+1}+D}{P_{X,t}}$. From (6), we can obtain

$$R^K_{t+1} - (1 + r_{t+1}) = \frac{1-\theta}{1-\theta^X}\left[\frac{P_{X,t+1}+D}{P_{X,t}} - (1 + r_{t+1})\right].$$

Thus, with $\theta < 1$ and $\theta^X < 1$, if the return from capital is greater than the gross interest rate, so is the return from land, and vice versa. Also from (6), we can obtain

$$R^K_{t+1} - (1 + r_{t+1}) = \frac{1-\theta}{\theta^X - \theta}\left(R^K_{t+1} - \frac{P_{X,t+1}+D}{P_{X,t}}\right).$$

This implies that when $\theta < \theta^X < 1$, the borrowing constraint binds if and only if $R^K_{t+1} > \frac{P_{X,t+1}+D}{P_{X,t}}$

When the economy in H-state at periot $t+1$, $R^K_{t+1} = A$. Since the dynamics of $P_{X,t}$ is characterized by 8, the return from land is

$$\frac{P_{X,t+1}+D}{P_{X,t}} = \frac{\mathcal{C}_K R^K_{t+1} P_{X,t} + \mathcal{C}_P D + D}{P_{X,t}} = \mathcal{C}_K A + \frac{(1+\mathcal{C}_P)\,D}{P_{X,t}}.$$

The borrowing constraint binds if and only if

$$(1-\mathcal{C}_K)\,A - \frac{(1+\mathcal{C}_P)\,D}{\mathcal{C}_P\left(AK_t\frac{1}{\overline{X}} + D\right)} > 0.$$

Note that K_t increases over time when $\mathcal{C}_K A > 1$. In this case, the borrowing constraint binds for all t if the inequality holds at $t = 0$.

In the long run, the economy converges to a steady state under L-state. In L-state, $R^K_{t+1} = \alpha A K_{t+1}^{(1-\underline{\phi})(\alpha-1)}$, and

$$K_{t+1} = \mathcal{C}_K \alpha A K_t^{(1-\underline{\phi})(\alpha-1)} K_t + \mathcal{C}_K D \overline{X}.$$

In the limit of $D \to 0$, the steady state capital, K_{SS}, is given by

$$K_{SS} = (\mathcal{C}_K \alpha A)^{\frac{1}{(1-\underline{\phi})(1-\alpha)}}.$$

The return from capital in the steady state with the limit of $D \to 0$ is

$$R^K_{SS} = \mathcal{C}_K^{\frac{(1-\underline{\phi})(\alpha-1)}{(1-\underline{\phi})(1-\alpha)}}\,(\alpha A)^{\frac{(1-\underline{\phi})(\alpha-1)}{(1-\underline{\phi})(1-\alpha)}+1} = \frac{1}{\mathcal{C}_K}.$$

Since the return from land in the steady state with $D \to 0$ is one, the borrowing constraint binds in the steady state if and only if $\mathcal{C}_K < 1$. This always holds because of (20). Hence, the borrowing constraint always binds in the steady state at the limit of $D \to 0$.

B An Alternative Model

We provide the existence of a bubble in an alternative model where savers also have access to land investment. We consider an equilibrium where entrepreneurs borrow from savers and invest only in capital, and savers invest in land and lend to entrepreneurs. The equations characterizing the equilibrium are

$$K_{t+1} + P_{X,t}\overline{X} = \beta \left[R_t^K K_t + \left(P_{X,t} + D \right) \overline{X} \right],$$

$$K_{t+1} = \frac{1}{1-\theta} \beta \pi \left[R_t^K K_t + \left(P_{X,t} + D \right) \overline{X} \right],$$

and

$$\frac{P_{X,t+1} + D}{P_{X,t}} = 1 + r_{t+1}.$$

The first equation is the resource constant. This resource constraint is unchanged from the model in Section 3. The second equation represents the optimal saving decision by entrepreneurs. In the equilibrium we consider with this alternative model, entrepreneurs invest only in capital because the return from capital investment is strictly greater than the return from land. The third equation is the non-arbitrage condition of savers. The return from land investment equals to the retun from lending, so that the savers engage in both lending and land investment. With this third equation and the return from capital is greater than the interest rate, it is optimal for entrepreneurs to borrow up to the borrowing limit without investing in land, leading to the second equation.

We can write the solution of the alternative model in a very similar way as we did in Section 3. The solution for $P_{X,t}$ and K_t is

$$P_{X,t} = \hat{\mathcal{C}}_P \left[R_t^K K_t + D\overline{X} \right] \frac{1}{\overline{X}}$$

and

$$K_{t+1} = \frac{\beta\pi}{(1-\beta)(1-\theta) + \beta\pi} \left[R_t^K K_t + D\overline{X} \right],$$

where $\hat{\mathcal{C}}_K := \frac{\beta\pi}{(1-\beta)(1-\theta)+\beta\pi}$ and $\hat{\mathcal{C}}_P := \frac{\beta(1-\theta-\pi)}{(1-\beta)(1-\theta)+\beta\pi}$. In a recursive form, the solution for

$P_{X,t}$ is

$$P_{X,t+1} = \hat{\mathcal{C}}_K R^K_{t+1} P_{X,t} + \hat{\mathcal{C}}_P D.$$

Thus, this alternative model has a very similar structure to the model in Section 3. The only difference is that $\mathcal{C}_K$ becomes $\hat{\mathcal{C}}_K$ and $\mathcal{C}_P$ becomes $\hat{\mathcal{C}}_P$. This observation leads to the following proposition.

Proposition 7. *When $1 < \hat{\mathcal{C}}_K A$, the unique equilibrium in the alternative model contains a bubble. When $\hat{\mathcal{C}}_K A < 1$, the unique equilibrium in the alternative model does not contain a bubble.*

In this alternative model, entrepreneurs, who borrow from savers, do not invest in land. As a result, leverage for land, θ^X, has no effect. In reality, θ^X seems crucial in determining the dynamics. For θ^X to have an effect, we need to consider the model presented in Section 3.

If this economy permanently stays at H-state, the following relation holds in the limit:

$$1 + r = 1 + g_Y = \hat{\mathcal{C}}_K A < A = R^K.$$

In other words, the interest rate is equal to the growth rate of the economy. This is different from the relation (21) in Section 3 and inconsistent with the U.S. data where the interest rate is lower than the economic growth rate. The model presented in Section 3 gives a better prediction than the alternative model in this aspect.

C Discounting

Since (10) and (11) hold for all $t > 0$, the following equation holds:

$$\begin{aligned}
&\mathbb{P}\left(\phi_1 = 1, \ldots, \phi_{t+1} = 1\right) \frac{R^K_{(H),t+1} - \left(1 + r_{(H),t+1}\right)\theta}{1-\theta} \mathbb{E}\left[\frac{1}{c_{(H),i,t+1}}\right] \\
&\quad + \mathbb{P}\left(\phi_1 = 1, \ldots, \phi_t = 1, \phi_{t+1} = \underline{\phi}\right) \frac{R^K_{(L,t),t+1} - \left(1 + r_{(L,t),t+1}\right)\theta}{1-\theta} \mathbb{E}\left[\frac{1}{c_{(L,t),i,t+1}}\right] \\
&\quad + \cdots + \mathbb{P}\left(\phi_1 = 1, \phi_2 = \underline{\phi}\right) \frac{R^K_{(L,1),t+1} - \left(1 + r_{(L,1),t+1}\right)\theta}{1-\theta} \mathbb{E}\left[\frac{1}{c_{(L,1),i,t+1}}\right] \\
&= \mathbb{P}\left(\phi_1 = 1, \ldots, \phi_{t+1} = 1\right) \frac{\frac{P_{(H),X,t+1} + D}{P_{(H),X,t}} - \left(1 + r_{(H),t+1}\right)\theta^X}{1-\theta^X} \mathbb{E}\left[\frac{1}{c_{(H),i,t+1}}\right]
\end{aligned}$$

$$
+ \mathbb{P}\left(\phi_1 = 1, \ldots, \phi_t = 1, \phi_{t+1} = \underline{\phi}\right) \frac{\frac{P_{(L,t),X,t+1}+D}{P_{(H),X,t}} - \left(1 + r_{(L,t),t+1}\right)\theta^X}{1-\theta^X} \mathbb{E}\left[\frac{1}{c_{(L,t),i,t+1}}\right]
$$

$$
+ \cdots + \mathbb{P}\left(\phi_1 = 1, \phi_2 = \underline{\phi}\right) \frac{\frac{P_{(L,1),X,t+1}+D}{P_{(H),X,t}} - \left(1 + r_{(L,1),t+1}\right)\theta^X}{1-\theta^X} \mathbb{E}\left[\frac{1}{c_{(L,1),i,t+1}}\right],
$$

where $\mathbb{P}$ is a probability operator and the expectation is over idiosyncratic type shock. This equality is equivalent to

$$
\mathbb{E}\left[\frac{R^K_{t+1} - (1 + r_{t+1})\,\theta}{1-\theta} \frac{1}{c_{i,t+1}} | \phi_0 = 1, \phi_1 = 1\right] = \mathbb{E}\left[\frac{\frac{P_{X,t+1}+D}{P_{X,t}} - (1 + r_{t+1})\,\theta^X}{1-\theta^X} \frac{1}{c_{i,t+1}} | \phi_0 = 1, \phi_1 = 1\right],
$$

where the expectation is over idiosyncratic type shock and aggregate spillover shock. This equality is the non-arbitrage condition for the portfolio choice at period t solved at period 0.