A Perturbational Approach for Approximating Heterogeneous-Agent Models*

Anmol Bhandari
University of Minnesota

Thomas Bourany
University of Chicago

David Evans
University of Oregon

Mikhail Golosov
University of Chicago

March 2023

Abstract

We develop a new perturbational technique to approximate equilibria of a broad class of stochastic heterogeneous-agent models with complex state spaces, such as multi-dimensional distributions of endogenous variables. A key insight of our approach is that it is possible to analytically characterize first, second, and higher-order approximations of the stochastic process that governs this distributional state. These characterizations have linear recursive structures, and we derive exact expressions for approximating coefficients as solutions to a small-dimensional linear system of equations. To the first order of approximation, our method is as fast and precise as existing state-of-the-art techniques that linearize heterogeneous agent models using so-called "MIT shocks," but the ability to quickly scale to higher orders enables us to study a broader set of questions, such as the impact of risks, endogenous household portfolio formation, and welfare implications of macroeconomic stabilization policies.

*We thank Adrien Auclert, participants of SITE conference, and seminar participants at Minneapolis Fed, EIEF, EUI, Bocconi, U of Chicago, and U of Oregon for helpful comments. Bhandari, Evans, and Golosov thank the NSF for support (grant #36354.00.00.00).

1 Introduction

A significant body of empirical research highlights that macroeconomic fluctuations affect households differently. Business cycle models with heterogeneous-agent (HA), with their ability to utilize rich micro-level data sets, are quickly becoming the canonical framework for macroeconomic analysis. However, solving these models can be challenging due to their complex state variables, which are high- or even infinite-dimensional objects such as distributions of agents' endogenous choices and exogenous characteristics. These variables are endogenous and can change stochastically over time, making it difficult to find solutions. As a result, researchers often resort to simplified models with limited heterogeneity or use linearization techniques that do not fully account for considerations related to risk, insurance or portfolio choice.

This paper aims to create a numerical method that can manage high-dimensional state variables and is easily scalable for second and higher-order approximations. The focus on scalability is crucial as higher-order approximations are necessary to examine nonlinearities and interaction effects of macroeconomic shocks. Additionally, they are unavoidable when assessing the consequences of risks and asset pricing, household portfolios, and evaluating the welfare effects of macroeconomic stabilization policies.

Our proposed method utilizes a "small-noise" perturbation approach for approximating HA models that have distributional and time-varying state variables. While small-noise expansions have long been used for approximating representative-agent (RA) macroeconomic models[1], such as in Schmitt-Grohé and Uribe (2004), they face challenges when applied to HA models. In RA economies, these expansions approximate equilibrium policy functions around a non-stochastic steady state using various orders of Taylor expansions with respect to a scalar that multiplies all aggregate shocks. The reason for its wide use in RA settings is because of its applicability to a broad class of economies, scalability to any order of approximation, and fast computational speed. However, when applied to HA models, this approach breaks down quickly as the dimension of the state in the recursive representation increases, since its computational complexity grows exponentially. Additionally, this approach struggles to handle kinks in policy functions, which is a natural feature of HA models with occasionally binding borrowing constraints.

A common technique since Reiter (2009) is to use a discrete representation of the equilib-

[1] This approach was developed to solve quantitative macroeconomic models, which originally were based almost exclusively on the representative agent assumption. The same techniques work well if agents are non-identical but heterogeneity is small. For simplicity, we refer to all such environments as RA models.

rium conditions (for instance, using the popular "histogram method", see Young (2010)) and construct a linearization using a variety of numerical methods around the economy without aggregate shocks. Our approach follows Reiter (2009) by utilizing the invariant distribution in the economy without aggregate shocks as the "point-of-approximation." However, then we proceed differently. Rather than differentiating the finite-dimensional system of equations, we approximate the infinite-dimensional derivatives. We demonstrate that changing the order of differentiation and discretization is conceptually similar only to the first order. At higher orders, extensions of the histogram method fail to capture the true derivatives and can give misleading answers. The second difference is that we represent the approximations using derivatives only in a small set of directions—ones in which the endogenous state moves along an equilibrium path— and not all directions as standard implementations of Reiter (2009) do. This matters because calculating and storing derivatives in all directions, especially at higher order, is extremely costly when the state space is large.

Our approach relies on linear operator techniques to derive results. In standard heterogeneous agent (HA) economies, the dependence of any policy function on distributions is captured by Fréchet derivatives. While these derivatives are typically infinite-dimensional objects, they are linear or multi-linear operators, which can be characterized analytically. These properties, combined with the linearity of the operator, enable us to describe any order of approximation of the law of motion of the aggregate distribution through a linear recursive structure that can be explicitly characterized. As a result, we can derive exact analytical characterizations of various orders of approximation of HA models.

Our method has several advantages. Because we rely on exact analytical derivatives, our approach is faster and more stable relative to alternatives, even at first order. The tractable linear recursive structure allows us to collapse the problem of finding any order of approximations into solving small-dimensional linear systems of equations. We show that matrices involved in this linear system can be efficiently constructed by exploiting the sparseness in the pre-computed basis matrices used to store policy functions at the point-of-approximation, i.e., the economy without aggregate shocks. Furthermore, with simple extensions, our method can handle time-varying aggregate risk, portfolio choice, and deterministic transitions across different steady states.

We apply our algorithm to calibrated versions of the canonical real business cycle model with heterogeneity, namely the Krusell and Smith (1998) model. We report diagnostic measures such as accuracy and speed and compare them to alternative methods. For example, we can fully solve for the first-order approximation of a high-dimensional version of the Krusell-

Smith economy in about 0.5 seconds and the second-order approximation within 2-3 seconds on a standard desktop computer. Furthermore, we use extensions of the basic model to analyze several applications designed to illustrate the usefulness of going beyond first-order approximations. These applications include the analysis of fiscal stabilization policies, the aggregate and distributional effects of macroeconomic uncertainty fluctuations, and properties of household portfolios across the asset distribution.

1.1 Related literature

Several recent papers have developed alternative methods to compute equilibria of HA economies. When the underlying state in the HA model can be summarized by sufficiently simple functions, the original approach of Krusell and Smith (1998) works well.[2] Boppart et al. (2018) observed that first-order approximations of the stochastic economy can be fully constructed from first-order approximations of the deterministic response to a one-time unexpected "MIT-style" shock, and study the stochastic properties of HA economies using responses from sequences of MIT shocks.

In an important recent contribution, Auclert et al. (2021), or ABRS for short, developed a fast and efficient sequence-space method to solve a broad class of HA economies to the first-order approximation. There is a close connection between our approach and theirs and we explain that connection in greater detail in Section 3.4; here we quickly summarize main similarities and differences.

ABRS build on the result of Boppart et al. (2018) that knowing responses to MIT shocks allows one to construct first-order approximations to stochastic economies. The key insight of ABRS is that the response of the distribution to an MIT shock can *be* described as a recursive linear system, which allows one quickly linearize the 'MIT' shock. ABRS use this insight to computationally construct the key object in their analysis (that they call "the Jacobian") and convert the approximation problem into a small-dimensional linear system of equations.

Our method is related to ABRS, and in fact directly builds on their insights, as we also exploit the recursively linear structure of Fréchet derivatives to simplify the analysis and collapse our approximation problem to small-dimensional linear systems of equations. When we restrict attention only to the first-order approximation, there is equivalence between the two approaches. While their linear recursive system is different from ours as we use two different representations of the equilibrium conditions, it is possible to prove that as the grid size in their approximation procedure converges to zero their first order approximating coefficients

[2]Some recent work extends global solution methods to more complex environments using machine learning techniques. See Maliar et al. (2021), Kahou et al. (2021), Childers et al. (2022), and Han et al. (2021) for details.

would coincide with ours. Moreover, it takes approximately the same time to compute first-order responses under their method and ours as both techniques ultimately use linear recursive properties of key objects to construct closely related linear systems of equations.

There are two key differences between our approach and theirs that enable us to obtain higher-order approximations. Firstly, we use recursive state space representation of equilibrium conditions, while ABRS use sequence-space formulation. For orders of approximation that are higher than the first, knowing responses to MIT shocks is insufficient to recover all approximation coefficients. Secondly, and perhaps even more importantly, ABRS follow most of the literature and start by approximating the distribution and its law of motion using "histogram method" before taking any derivatives. We instead derive exact analytical expressions for the derivatives first before using any numerical approximations. This distinction is important. In the paper we show that even in the limit, as the grid size used in the histogram method becomes arbitrarily small, constructed approximations do not converge to their exact theoretical counterparts, and that some of the second-order terms get lost. In our applications, we show that this can significantly affect the conclusions one obtains.

The class of economies that we consider in this paper are discrete-time infinite horizon versions of heterogeneous agent models with distributional states. There is a parallel literature that studies continuous-time versions of these economies. See, for instance, Kaplan et al. (2018), Achdou et al. (2020), Ahn et al. (2018) in the context of consumption-savings models; Alvarez and Lippi (2022) and Alvarez et al. (2023a) in the context of price-setting models; and Bigio et al. (2023) for an application to public debt maturity. In related work, Bilal (2023) and Alvarez et al. (2023b) use mean field game techniques to construct approximations in these class of models with aggregate shocks. The approaches share the use of linear operators over infinite-dimensional spaces to characterize the exact derivatives analytically.

Our paper is also related to the approximation method used by Bhandari et al. (2021). Like us, they used a state-space variational approach and Fréchet derivatives to obtain various orders of approximations of HA economies. Their approach does not extend to economies with occasionally binding borrowing constraints. It also changes the point-of-approximation at each node of the aggregate history, which requires many re-computations of the point-of-approximation. Our method instead computes the point-of-approximation only once and can handle kinks in policy functions that emerge due to borrowing constraints. The analytical characterization of the law of motion of the aggregate distribution, which is the central theoretical result that simplifies our approach, is new to our paper.

The rest of the paper is organized as follows. Section 2 presents our baseline environment, Section 3 describes our approximation techniques in that environment, Section 4 show how these techniques can be extended to models of transition dynamics, stochastic volatility, and portfolio problems. Section 5 provides numerical illustration of our techniques. Section 6 concludes.

2 Environment

Many infinite-period heterogeneous agents (HA) models have a representation that is recursive in a vector of exogenous disturbances and a distribution of endogenous state variables chosen by individual agents in previous periods. In this section we describe a broad class of such economies.

Consider an infinite period economy with a unit measure of agents. Let $x_{i,t}$ denote a vector of endogenous variables that are chosen by agent i in period t, and X_t denote a vector of aggregate variables that all individuals take as given. Let $\theta_{i,t}$ and Θ_t denote vectors of idiosyncratic and aggregate exogenous shocks. In a typical application, the optimality conditions describing optimal choice of $x_{i,t}$ depend on a subset of individual choices made in the previous period, $z_{i,t-1} \in x_{i,t-1}$, aggregate variables X_t, idiosyncratic shocks $\theta_{i,t}$, as well as the expectations $\mathbb{E}_{i,t} x_{i,t+1}$. These optimality conditions can be summarized by some mapping F of the form

$$F\left(z_{i,t-1}, x_{i,t}, \mathbb{E}_{i,t} x_{i,t+1}, X_t, \theta_{i,t}\right) = 0 \text{ for all } i, t, \tag{1}$$

where initial $(z_{i,-1}, \theta_{i,0})$ are given. Aggregated variables X_t are determined by some market clearing conditions that depend on aggregate shocks Θ_t and aggregations of individual choices, $\int x_{i,t} di$. We represent these conditions by some mapping G of the form

$$G\left(\int x_{i,t} di, X_t, \Theta_t\right) = 0 \text{ for all } t. \tag{2}$$

The initial conditions of the system are given by Θ_{-1} and Ω_{-1} where Ω_{-1} is the joint distribution over (z, θ). By the equilibrium of the system we mean the solution $\left\{X_t\left(\mathcal{E}^t\right)\right\}_{t,\mathcal{E}^t}$ to (1) and (2) given the initial conditions. We mainly focus on finding equilibrium values of the aggregates X_t, since those are the main focus in many applications, but while doing so we also describe a procedure to recover equilibrium values of $x_{i,t}$. We refer to equations (1) and (2) as the *sequence-space representation* of equilibrium.

To simplify the exposition, in the body of the paper we treat both $\theta_{i,t}$ and Θ_t as scalars that follow AR(1) processes

$$\Theta_t = \rho_\Theta \Theta_{t-1} + \mathcal{E}_t, \tag{3}$$

$$\theta_{i,t} = \rho_\theta \theta_{i,t-1} + \varepsilon_{i,t}, \tag{4}$$

but show in the appendix how our approach extends to the case when $\theta_{i,t}$ and Θ_t are multi-dimensional. $\mathcal{E}_t$ and $\varepsilon_{i,t}$ are mean zero stochastic processes independent across time and agents, coefficients ρ_Θ and ρ_θ satisfy $|\rho_\Theta|, |\rho_\theta| < 1$, and $\mathcal{E}_t$ is bounded. For now, we assume that stochastic processes for $\mathcal{E}_t$ and $\varepsilon_{i,t}$ are time invariant, but we drop this assumption in Section 4. We use μ to denote the probability distribution of $\varepsilon_{i,t}$.

Equations (1) and (2) allow us to have a representation of HA environments that is both parsimonious in terms of notation and general in terms of economic features that it captures. By appropriately defining $x_{i,t}$ and X_t, it includes models in which aggregate shocks affect individual optimality conditions directly (let some subset X'_t of X_t be defined via $X'_t - \Theta_t = 0$ as a part of G), individual choices of agents depend on their expectations of aggregate variables in the future (let a subset of $x'_{i,t}$ of $x_{i,t}$ be defined by $x'_{i,t} = X'_t$ as part of F, then $\mathbb{E}_t x'_{i,t+1} = \mathbb{E}_t X'_{t+1}$)[3], and variance or other higher moments of $\{x_{i,t}\}_i$ affect aggregate variables (for example, if variance of some $x'_{i,t} \in x_{i,t}$ is relevant for equilibrium determination, include additional variables $x''_{i,t}$ as a part of vector $x_{i,t}$ and variables X'_t and X''_t as parts of vector X_t, and add equation $x''_{i,t} - (x'_{i,t} - X'_t)^2 = 0$ to F and equations $X'_t - \int x'_{i,t} di = 0$ and $X''_t - \int x''_{i,t} di = 0$ to G).

2.1 Example using the Krusell-Smith economy

We illustrate how specific economic environments maps into our representation using a simple version of the economy studied by Krusell and Smith (1998). All agents supply inelastically one unit of labor that is subject to idiosyncratic efficiency shocks $\theta_{i,t}$. Agents receive wage W_t and save capital $k_{i,t}$ that earns gross return R_t. Agent's optimization problem is

$$\max_{\{c_{i,t}, k_{i,t}\}_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_{i,t})$$

subject to

$$c_{i,t} + k_{i,t} - R_t k_{i,t-1} - W_t \exp(\theta_{i,t}) = 0, \tag{5}$$

$$k_{i,t} \geq 0, \tag{6}$$

where initial $k_{i,-1}$ and $\theta_{i,0}$ are given. Efficiency $\theta_{i,t}$ follows exogenous stochastic process. We assume that the initial distribution of efficiencies among agents is stationary, so that the aggregate efficiency level is constant, and normalize $\int \exp(\theta_{i,0}) di = 1$.

[3] It is also straightforward to allow expectations of future aggregates to enter in G, i.e. $G\left(\int x_{i,t} di, X_t, \mathbb{E}_t X_{t+1}, \Theta_t\right)$. We opted to present our approach without this term to maintain parsimony without losing any generality.

Agents rent capital and supply efficiency-adjusted labor to firms each period. Firms are competitive and produce output using Cobb-Douglas technology with aggregate productivity $\exp(\Theta_t)$ and capital share of α. Wages W_t and rental rates are determined by the market clearing conditions so that supply of labor and capital by consumers is equal to the demand for those factors by firms. The gross return R_t includes the rental rate and the un-depreciated fraction of capital stock, $1-\delta$.

Let $\zeta_{i,t}$ be the Lagrange multipliers on borrowing constraint (6). Agent i optimality conditions can be summarized by stochastic sequences $\{k_{i,t}, c_{i,t}, u_{i,t}, \zeta_{i,t}, \lambda_{i,t}\}_{i,t}$ that satisfy equations (5) and

$$R_t U_c(c_{i,t}) - \lambda_{i,t} = 0, \quad U_c(c_{i,t}) + \zeta_{i,t} - \beta \mathbb{E}_t \lambda_{i,t+1} = 0, \quad k_{i,t}\zeta_{i,t} = 0, \quad u_{i,t} - k_{i,t-1} = 0, \tag{7}$$

where U_c is the derivative of U. The aggregate capital stock K_t, wages W_t and gross returns R_t are pinned down by

$$W_t - (1-\alpha)\exp(\Theta_t) K_t^{\alpha} = 0, \quad K_t - \int u_{i,t} di = 0, \tag{8}$$

$$R_t + \delta - \alpha \exp(\Theta_t) K_t^{\alpha-1} - 1 = 0. \tag{9}$$

Equations (5), (7), (8) and (9) fully summarize the equilibrium dynamics of this economy.[4]

It is easy to see how these equations map into our sequence-space representation. In particular, let $x_{i,t} = [k_{i,t}, c_{i,t}, u_{i,t}, \lambda_{i,t}, \zeta_{i,t}]^{\mathrm{T}}$, $z_{i,t} = k_{i,t}$, and $X_t = [K_t, W_t, R_t]^{\mathrm{T}}$. Given this definition, equations (5) and (7) define mapping F, and equations (8) and (9) define mapping G.

2.2 The state-space representation

We assume that system (1) and (2) can be written recursively, with shock Θ_t and the joint distribution of $\{(z_{i,t-1}, \theta_{i,t})\}_i$ forming the aggregate state variable.[5] We use Ω_{t-1} to denote this distribution, with $\Omega_{t-1}\langle z, \theta \rangle$ being the mass of agents with $z_{i,t-1} \leq z$ and $\theta_{i,t} \leq \theta$. Let $Z_t = [\Theta_t, \Omega_{t-1}]^{\mathrm{T}}$ be the aggregate state. We use tildes to denote policy functions in the recursive representation. Thus, $\widetilde{X}(Z)$ denotes policy functions for aggregate variables

[4] It is easy to see that some variables, such as $u_{i,t}$ or $\lambda_{i,t}$ are redundant and the sequence problem can be specified without them. We chose this specification to minimize the number of arguments that appear in F and G mappings, which makes our formulas more transparent. Alternative descriptions of the optimality conditions would result in slightly different forms for F and G but all other results would extend directly to those specifications.

[5] In particularly, the Krusell and Smith economy is recursive in aggregate productivity and the joint distribution of individual capital holdings and labor efficiency.

and $\widetilde{x}(z,\theta,Z)$ denotes policy functions for individual variables. Naturally, $\widetilde{z}$ denotes policy functions corresponding to $z_{i,t}$. Since $\widetilde{z}$ is a subject of vector $\widetilde{x}$, we can write

$$\widetilde{z} = \mathsf{P}\widetilde{x}$$

for some selection matrix P.

The *state-space* representation of (1) and (2) is given by equations

$$F\left(z,\widetilde{x},\mathbb{E}\widetilde{x},\widetilde{X},\theta\right) = 0 \text{ for all } (z,\theta,Z), \tag{10}$$

$$G\left(\int \widetilde{x}d\Omega,\widetilde{X},\widetilde{\Theta}\right) = 0 \text{ for all } Z, \tag{11}$$

and

$$\widetilde{\Omega}(Z)\left\langle z',\theta'\right\rangle = \int\int \iota\left(\widetilde{z}(z,\theta,Z) \le z'\right)\iota(\rho_\theta\theta + \epsilon \le \theta')\mu(\epsilon)\,d\epsilon d\Omega\left\langle z,\theta\right\rangle \text{ for all } (\Theta,\Omega), \tag{12}$$

where in equation (10) we use $\mathbb{E}\widetilde{x}$ to denote conditional expectation of $\widetilde{x}$ with respect to $(\epsilon,\mathcal{E})$:

$$\mathbb{E}\widetilde{x} = \mathbb{E}\left[\widetilde{x}\left(\tilde{z}(z,\theta,Z),\rho_\theta\theta+\epsilon,\rho_\Theta\Theta+\mathcal{E},\widetilde{\Omega}(Z)\right)\Big| z,\theta,Z\right].$$

This recursive system is initialized by some Z_0, corresponding to the initial joint distribution and aggregate shock. We refer to $\widetilde{\Omega}(Z)$ in described in equation (12) as the Law of Motion (LoM) for the aggregate distribution.

3 The perturbational approach

One of the most common methods to solve representative agents (RA) macroeconomic models is to use a perturbational approach.[6] Under this approach, one perturbs the aggregate shock process as

$$\Theta_t = \rho_\Theta\Theta_{t-1} + \sigma\mathcal{E}_t, \tag{13}$$

where $\sigma \ge 0$ is a scalar and considers sequences of economies parameterized by σ. The original economy corresponds to $\sigma = 1$. Policy functions must satisfy recursive state-space representation for all σ, and one then uses Taylor expansions of the state-state representation with respect to σ, evaluated as $\sigma = 0$, to recover approximation coefficients of various orders.

This approach is very popular for solving and estimating DSGE models, and it lies in the heart of a commonly used approximation software package DYNARE. In standard RA economies, all computations can be done very quickly, and this approach extends naturally to second- and higher-order approximations.

[6]See Schmitt-Grohé and Uribe (2004) or Judd (1998) for overview

Using the perturbational approach for HA economies faces at least two difficulties. First, the endogenous state Ω is a high- (typically, infinitely-) dimensional object. Brute force application of perturbational techniques requires computing derivatives of policy functions with respect to each of the dimensions of the state variable, which quickly becomes impractical with even moderate amount of heterogeneity. Second, borrowing constraints and other frictions in HA economy introduce kinks in policy functions, which makes them non-differentiable.

In this section, we show that these difficulties can be overcome in a large class of HA economies. In particular, using linear operator techniques, one can sidestep the difficult task of computing all derivatives of policy functions and instead derive *analytically* exact expressions for all coefficients in the first-, second- and, in principle, higher-order expansions of the state-space representation. These analytical expressions then can then be calculated on the computer using sparse matrices. This allows us to find approximate equilibrium dynamics quickly and easily even in cases when heterogeneity is very large.

3.1 The zeroths order approximation

Policy functions must satisfy (10), (11) and (12) for all σ, and so in the perturbed economy they take the form $\widetilde{X}(Z;\sigma)$, $\widetilde{x}(z,\theta,Z;\sigma)$, $\widetilde{\Omega}(Z;\sigma)$. To simplify our exposition, we treat z as a scalar in the body of the paper; in the appendix we give extension to the case when z is a vector.

Let $\overline{X}(Z)$, $\overline{x}(z,\theta,Z)$, $\overline{\Omega}(Z)$ be policy functions in the economy corresponding to $\sigma=0$, and let $\overline{Z}(Z) = \left[\rho_\Theta \Theta, \overline{\Omega}(Z)\right]^{\mathrm{T}}$. Standard applications of perturbation approach consider approximations around a deterministic steady state. The analogue of that steady-state in our HA economy is $Z^* = [0,\Omega^*]^{\mathrm{T}}$, where Ω^* is the invariant distribution of the HA economy in which $\Theta_t = 0$ for all t. We call this economy the zeroth order approximation. Throughout, we assume that the initial condition of our system is given by $\Theta_{-1}=0$ and $\Omega_{-1}=\Omega^*$.

Let $\overline{Z}_t := \underbrace{\overline{Z}(\overline{Z}(....\overline{Z}}_{t\text{ times}}(Z_0))$ be the value of Z_t in the $\sigma=0$ economy. Throughout this section, we maintain the following assumptions.

Assumption 1. *(a).* $\lim_{t\to\infty}\overline{Z}_t(Z_0) = Z^*$ *for all* Z_0 *in a neighborhood of* Z^**;*

(b). $\widetilde{X}$, $\widetilde{\Omega}$ *are smooth with respect to* (Z,σ) *in neighborhood of* $(Z^*,0)$ *and uniquely determined;*

(c). Ω^* *has a finite number of mass-points;*

(d). $\overline{x}$ *continuous and piecewise smooth and* Ω^**-a.e. smooth;*

(e). $\widetilde{x}$ *are* Ω^**-a.e. smooth with respect to* (Z,σ) *in neighborhood of* $(Z^*,0)$ *and uniquely determined;.*

Parts (a) and (b) are direct generalization of stability conditions that are required for perturbational method in RA settings (Blanchard and Kahn (1980), Schmitt-Grohé and Uribe (2004)). Conditions (c) and (d) are new ones and allows us to incorporate occasionally binding borrowing constraints and the resulting distributions with mass points. We allow the stationary distribution to have a finite number of of mass points. We also allow for individual policy functions to have a finite number of kinks but assume that the kinks and mass points do not align so the kinks are of Ω^*-measure zero. Condition (e) is just the analogue of condition (b) for individual policy functions with kinks.

To understand our new conditions (c) and (d), consider again our example of the Krusell-Smith economy. In that application, Ω^* corresponds to the invariant distribution of capital in the economy without aggregate shocks. With continuous shocks Ω^* has one unique mass point at the borrowing constraint. Individual policy functions for capital $\overline{k}(\theta, k)$ are non-differentiable at the level of θ at which the borrowing constraint starts to bind, but for each k there is at most one θ_k when this occurs. Since $\theta_{i,t} = \rho_\theta \theta_{i,t-1} + \varepsilon_{i,t}$ and any realization of $\varepsilon_{i,t}$ is of μ-measure of zero, there must be Ω^*-measure of points (θ_k, k) must be zero as well, satisfying the smoothness requirement in (d). This holds irrespective of whether in the invariant distribution there is a mass of agents with capital holdings at the borrowing constraint or not.

In many applications it is easy to compute the invariant distribution of HA economy when there are no aggregate shocks. Thus, for the purposes of this paper, we treat Ω^*, $\overline{X}(Z^*)$ and $\overline{x}(z, \theta, Z^*)$ as known objects. Since our expansions are around $Z = Z^*$, we drop explicit references to aggregate state and use $\overline{X}$ and $\overline{x}(z, \theta)$ to refer to these policy functions when $Z = Z^*$. We use $\overline{\Lambda}(z', \theta', z, \theta)$ to denote transition probability from (z, θ) to (z', θ') under Ω^*.

Our analytical derivations exploit properties of various linear operators. One of the most important operators for us will be Fréchet derivatives. We use $\overline{X}_Z$, $\overline{x}_Z(z, \theta)$ and $\overline{\Omega}_Z$ to denote the Fréchet derivative of $\overline{X}(Z)$, $\overline{x}(z, \theta, Z)$ and $\overline{\Omega}(Z)$ with respect to Z, evaluated at $Z = Z^*$. We denote the value of that derivative in direction $\hat{Z}$ by $\overline{X}_Z \cdot \hat{Z}$, etc.[7] Similar notation applies to higher orders, e.g., $\overline{X}_{ZZ}$ denotes the second-order Fréchet derivative and $\overline{X}_Z \cdot \left(\hat{Z}', \hat{Z}''\right)$ denotes its value in directions $\hat{Z}', \hat{Z}''$. The Fréchet derivative of $\overline{Z}$ satisfies $\overline{Z}_Z = \begin{bmatrix} \begin{matrix} \rho_\Theta \\ \overline{\Omega}_Z \end{matrix} & \mathbf{0} \end{bmatrix}$.

Economically, $\overline{X}_Z$ captures collection of marginal effects of changes of every dimension of Z, and $\overline{X}_Z \cdot \hat{Z}$ is the first order effect of perturbing the aggregate state from Z^* to $Z^* + \hat{Z}$ on

[7] Our terminology is slightly different from the one used by Luenberger (1997) and is meant to highlight the economic meaning of these objects. Luenberger (Chapter 7) would refer to $\overline{X}_Z \cdot \hat{Z}$ as the "Fréchet differential of $\overline{X}$ (at Z^*) with increment $\hat{Z}$".

$\overline{X}$. In HA economies, Fréchet derivatives are high-dimensional objects and computing them is impractical. This, turns out, to be also not necessary. Fréchet derivatives are linear (or multi-linear for higher orders of derivatives) operators and can be easily manipulated analytically without knowing their specific values. In order to find the approximations, we need to find note the whole Fréchet derivative but only its values in specific directions, and those values always remain small-dimensional objects even if the derivative itself is infinite-dimensional.

We use $\mathsf{F}_x(z,\theta)$, $\mathsf{F}_{x^e}(z,\theta)$, $\mathsf{F}_X(z,\theta)$ to denote derivatives of F in (10) with respect to $\widetilde{x}, \mathbb{E}\widetilde{x}, \widetilde{X}$, all evaluated at (z,θ,Z^*). Similarly, G_x, G_X, G_Θ denote derivatives of G in (11) with respect to its three arguments evaluated at Z^*. Their higher order analogues are denoted by $\mathsf{F}_{xx}(z,\theta)$, G_{xX}, etc. We use subscripts $\overline{x}_z(z,\theta)$, $\overline{x}_{zz}(z,\theta)$, $\overline{x}_\theta(z,\theta)$, etc to denote various derivatives of $\overline{x}(z,\theta)$. All these derivatives can be constructed from the zeroths order economy and we treat them as known for the purposes of our approximations. Finally, $\overline{X}_\sigma$, $\overline{X}_{\sigma\sigma}$, $\overline{x}_\sigma(z,\theta)$, etc denote various orders of derivatives of policy functions $\widetilde{X}(Z;\sigma)$ and $\widetilde{x}(z,\theta,Z;\sigma)$ with respect to σ, evaluated at $(Z,\sigma) = (Z^*,0)$. We refer to these terms as precautionary motives.

3.1.1 Remark on numerical implementation of zeroth order terms

The optimal decision rules, $\bar{x}(z,\theta)$, are approximated using a finite number of basis functions.[8] The coefficients of these basis functions that solve the households optimality conditions F , given a candidate $\overline{X}$, at finite set of interpolation gridpoints are found using the endogenous grid method of Carroll (2006). The stationary distribution Ω^* is approximated with a grid over income and assets that is finer than the one used for the policy functions. Given policy rules $\overline{z}(z,\theta)$ and the process for θ the stationary transition density, $\overline{\Lambda}$, for the individual states is approximated using the method of Young (2010) as a large sparse matrix. Ω^* is recovered by finding the eigenvector associated with the unit eigenvalue of $\overline{\Lambda}$. Finally, $\overline{X}$ is found such that at the induced Ω^* and $\overline{x}(z,\theta)$ to satisfy G.

At the stationary equilibrium $(\overline{x},\overline{X},\Omega^*)$ the derivatives of F (i.e., $\mathsf{F}_x(z,\theta)$, $\mathsf{F}_X(z,\theta)$, etc) can be computed at any point using automatic differentiation by evaluating the F function with the steady state policy rules. In the same manner, the derivatives of G (i.e. $\mathsf{G}_x, \mathsf{G}_X$, and G_Θ) can also be evaluated using automatic differentiation. Finally, derivatives of $\overline{x}$ (i.e. $\overline{x}_z(z,\theta)$, $\overline{x}_\theta(z,\theta)$, etc.) can be evaluated by weighting the appropriate derivatives of the basis functions by the coefficients that approximate $\overline{x}(z,\theta)$.

[8] For our numerical work we use quadratic spline basis functions

3.2 First-order approximations

We now show how linear operator techniques can be used to obtain the first order equilibrium approximation. Define a sequence of directions $\{\hat{Z}_t\}_t$ recursively as follows: $\hat{Z}_0 = [1, \mathbf{0}]^{\mathrm{T}}$ and $\hat{Z}_t := \overline{Z}_Z \cdot \hat{Z}_{t-1}$. Let $\overline{X}_{Z,t}$ be the value of the Fréchet derivative of $\overline{X}$ evaluated in direction $\hat{Z}_t$, i.e. $\overline{X}_{Z,t} := \overline{X}_Z \cdot \hat{Z}_t$. We use similar notation for all other policy functions that depend on Z, e.g., $\overline{x}_{Z,t}(z,\theta) := \overline{x}_Z(z,\theta) \cdot \hat{Z}_t$.

To understand the economic intuition behind this expression, consider the effect of an aggregate shock $\mathcal{E}_0$ in period 0. On the impact, this shock does not affect the aggregate distribution Ω_0 but it changes the aggregate productivity Θ_0 by $\mathcal{E}_0$. Thus, to the first (in fact, any) order of approximation, state Z_0 changes by $\hat{Z}_0 \cdot \mathcal{E}_0$. This, in turn, induces changes in the aggregate states in the future. The first order response of the LoM to changes in the aggregate state is captured by $\overline{\Omega}_Z$. Therefore, $\hat{\Omega}_1 = \overline{\Omega}_Z \cdot \hat{Z}_0$ is the first order effect on the aggregate distribution next period and $\hat{Z}_1 \cdot \mathcal{E}_0 = \left[\rho_\Theta, \hat{\Omega}_1\right]^{\mathrm{T}} \cdot \mathcal{E}_0$ is the first order change in the aggregate state induced by $\mathcal{E}_0$. By induction, $\hat{Z}_t = \left[\rho_\Theta^t, \hat{\Omega}_t\right]^{\mathrm{T}}$ is the first order change in the aggregate state t period after shock $\mathcal{E}_0$. $\overline{X}_{Z,t}$ captures the response of policy functions to this change in the aggregate state. The sequence $\{\overline{X}_{Z,t}\}_t$ represents what is often referred to colloquially in the literature as the impulse response to an "MIT shock".

With this notation in place, we now describe how to recover the first order equilibrium approximation using this constructed values of the Fréchet derivatives.

Lemma 1(FO). To the first order approximation, satisfies

$$X_t\left(\mathcal{E}^t\right) = \overline{X} + \sum_{s=0}^{t} \overline{X}_{Z,t-s}\mathcal{E}_s + O\left(\|\mathcal{E}\|^2\right).$$

This lemma re-formulates in the state-space representation a well-known insight (see, e.g., Boppart et al. (2018) or Auclert et al. (2021)) that one can construct first-order approximations to stochastic economies using impulse responses to MIT shocks. The main take away for our purposes is that finding the first order equilibrium approximation is equivalent to finding values of the Fréchet derivative $\overline{X}_Z$ in a sequence of directions $\{\hat{Z}_t\}_t$. In order to find these values, we take a Fréchet derivative of (11) and evaluate it in some direction $\hat{Z}_t$.

Lemma 2(FO). For any t,

$$\mathsf{G}_x\left(\int \overline{x}_{Z,t} d\Omega^* + \int \overline{x} d\hat{\Omega}_t\right) + \mathsf{G}_X \overline{X}_{Z,t} + \mathsf{G}_\Theta \rho_\Theta^t = 0. \tag{14}$$

In equation (14) terms G_x, G_X, $\mathsf{G}_\Theta\rho_\Theta^t$ are all known from the zeroth order approximation. The expression in the brackets is the Fréchet derivative $\left(\overline{\int x d\Omega}\right)_{Z,t}$ written explicitly. This derivative decomposes the first order changes in this integral into two components: the effect from changes in individual policy functions, the term $\int \overline{x}_{Z,t} d\Omega^*$, and the effect from changes in distribution, the term $\int \overline{x} d\hat{\Omega}_t$. We want to characterize each of these two integrals.

We start $\int \overline{x}_{Z,t} d\Omega^*$. It depends on the Fréchet derivatives of the individual policy functions, $\overline{x}_{Z,t}(z,\theta)$, which at this stage are unknown. It turns out that those derivatives can be replaced with $\{\overline{X}_{Z,s}\}_s$ weighted with coefficients known from the zero order:

Lemma 3(FO). For any t,

$$\overline{x}_{Z,t}(z,\theta) = \sum_{s=0}^{\infty} \underbrace{\mathsf{x}_s(z,\theta)}_{=\partial x_t/\partial X_{t+s}} \overline{X}_{Z,t+s}, \tag{15}$$

where matrices $\mathsf{x}_s(z,\theta)$ are given by

$$\mathsf{x}_0(z,\theta) = -\left(\mathsf{F}_x(z,\theta) + \mathsf{F}_{x^e}(z,\theta)\mathbb{E}\left[\overline{x}_z|z,\theta\right]\mathsf{P}\right)^{-1}\mathsf{F}_X(z,\theta), \tag{16}$$

$$\mathsf{x}_{s+1}(z,\theta) = -\left(\mathsf{F}_x(z,\theta) + \mathsf{F}_{x^e}(z,\theta)\mathbb{E}\left[\overline{x}_z|z,\theta\right]\mathsf{P}\right)^{-1}\mathsf{F}_{x^e}(z,\theta)\mathbb{E}\left[\mathsf{x}_s|z,\theta\right]. \tag{17}$$

Matrix $\mathsf{x}_s(z,\theta)$ captures first order responses of agent (z,θ) to changes in aggregate variables s periods in the future, $\partial x_t/\partial X_{t+s}$, so individual response $\overline{x}_{Z,t}(z,\theta)$ depends on expected future path of aggregate variables $\{\overline{X}_{Z,t+s}\}_s$ weighted with $\mathsf{x}_s(z,\theta)$. Importantly, coefficients $\{\mathsf{x}_s(z,\theta)\}_s$ are known explicitly in closed form. This significantly simplifies their calculation as we describe in more details in Section 3.2.1. Using Lemma 3(FO) we can express the integral $\int \overline{x}_{Z,t} d\Omega^*$ as a sum of $\{\overline{X}_{Z,s}\}_s$ weighted with coefficients known from the zeroth order:

Corollary 1(FO). For any t,

$$\int \overline{x}_{Z,t} d\Omega^* = \sum_{s=0}^{\infty}\left(\int \mathsf{x}_s d\Omega^*\right)\overline{X}_{Z,t+s}.$$

We now turn to characterizing integral $\int \overline{x} d\hat{\Omega}_t$. This requires us to describe the LoM of the aggregate distribution $\hat{\Omega}_t$ which is a high-dimensional object. Two linear operators, $\mathcal{M}$ and $\mathcal{L}$, greatly simplify this description. For any integrable functions $y : z \times \theta \to \mathbb{R}$, these operators return $\mathcal{M} \cdot y$ and $\mathcal{L} \cdot Y$ defined as follows:[9]

$$(\mathcal{M} \cdot y)\left\langle z',\theta'\right\rangle := \int \overline{\Lambda}(z',\theta',z,\theta) y(z,\theta)\, d\Omega^*(z,\theta),$$

$$(\mathcal{L} \cdot y)\left\langle z',\theta'\right\rangle := \int \overline{\Lambda}(z',\theta',z,\theta)\overline{z}_z(z,\theta) y(z,\theta)\, dz d\theta.$$

[9] In the general case, when z can be multi-dimensional vector, $\mathcal{M}$ is defined for a space of integrable functions $Y : z \times \theta \to \mathbb{R}^{\dim z}$. See appendix for details.

Operators $\mathcal{M}$ and $\mathcal{L}$ describe how changes in policy functions and distributions propagate over time to the first order of approximation. To get intuition for what they represent consider the following thought experiment. Suppose that for some reason individual policy functions change in period 0 by some $\hat{z}_0(z,\theta)$. How does this change affect the aggregate distribution over time? To answer this question, differentiate equation (12) to show that, the first order, the change in the distribution next period, $\hat{\Omega}_1$, must satisfy

$$\hat{\Omega}_1 \left\langle z',\theta' \right\rangle = -\iint \delta(\bar{z}(z,\theta)-z')\iota(\rho_\theta\theta+\epsilon\le\theta')\mu(\epsilon)\hat{z}_0(z,\theta)\, d\epsilon d\Omega^*.$$

Differentiate both sides of this expression with respect to θ' to obtain

$$\frac{d}{d\theta}\hat{\Omega}_1 \left\langle z',\theta' \right\rangle = -\int \underbrace{\int \delta(\bar{z}(z,\theta)-z')\delta(\rho_\theta\theta+\epsilon-\theta')\mu(\epsilon)d\epsilon}_{=\overline{\Lambda}(z',\theta',z,\theta)} \hat{z}_0(z,\theta)\, d\Omega^*,$$

or simply

$$\frac{d}{d\theta}\hat{\Omega}_1 = -\mathcal{M}\cdot\hat{z}_0.$$

Thus, operator $\mathcal{M}$ captures the first order effect of changes in individual policy functions on the aggregate distribution next period, holding the aggregate distribution fixed at Ω^*.

How does this change in the distribution percolates over time? One the one hand, changes in aggregate state induce changes in individual policy functions next period, $\hat{z}_1$, that has the distributional consequences captured by the $\mathcal{M}$ operator as discussed above. On the other hand, changes in the aggregate distribution $\hat{\Omega}_1$ would induce changes in this distribution in the future even if individual policy functions remained unchanged. Let $\hat{\Omega}_2$ be this latter component, that can be formally written as

$$\hat{\Omega}_2 \left\langle z',\theta' \right\rangle = \iint \iota\left(\bar{z}(z,\theta)-z'\right)\iota(\rho_\theta\theta+\epsilon\le\theta')\mu(\epsilon)\, d\epsilon d\hat{\Omega}_1.$$

Apply the integration by parts to the integral on the right hand side and then differentiate both sides of this equation with respect to θ' to obtain

$$\frac{d}{d\theta}\hat{\Omega}_2 \left\langle z',\theta' \right\rangle = \int \underbrace{\int \delta(\bar{z}(z,\theta)-z')\delta(\rho_\theta\theta+\epsilon-\theta')\mu(\epsilon)d\epsilon}_{=\overline{\Lambda}(z',\theta',z,\theta)} \bar{z}_z(z,\theta)\frac{d}{d\theta}\hat{\Omega}_1\left\langle z,\theta\right\rangle dzd\theta,$$

or simply

$$\frac{d}{d\theta}\hat{\Omega}_2 = \mathcal{L}\cdot\frac{d}{d\theta}\hat{\Omega}_1.$$

Thus, operator $\mathcal{L}$ captures the effect of the first order change in the aggregate distribution in period t on the aggregate distribution in period $t+1$, holding policy rules fixed.

With these operators, we can succinctly describe the LoM for $\hat{\Omega}_t$. We use notation z_s to refer to the part of vector x_s that corresponds to individual state variables, i.e., $\mathsf{z}_s = \mathsf{P}\mathsf{x}_s$. We will maintain this convention also when we describe higher order analogues of x_s.

Lemma 4(FO). For any t, $\frac{d}{d\theta}\hat{\Omega}_t$ satisfies a recursion

$$\frac{d}{d\theta}\hat{\Omega}_{t+1} = \mathcal{L} \cdot \frac{d}{d\theta}\hat{\Omega}_t - \sum_{s=0}^{\infty} \mathsf{a}_s \overline{X}_{Z,t+s}, \tag{18}$$

where $\mathsf{a}_s = \mathcal{M} \cdot \mathsf{z}_s$ and $\frac{d}{d\theta}\hat{\Omega}_0 = \mathbf{0}$.

Equation (18) shows that the LoM for $\frac{d}{d\theta}\hat{\Omega}_{t+1}$ be can separated into two components: the backward-looking component that captures how changes in the distribution last period percolates over time, $\mathcal{L} \cdot \frac{d}{d\theta}\hat{\Omega}_t$, and the forward-looking component that captures how expected changes in aggregate variables in the future affect individual policy functions today and, therefore, the distribution tomorrow, $\mathcal{M} \cdot \sum_{s=0}^{\infty} \mathsf{z}_s \overline{X}_{Z,t+s}$.

The recursive structure derived in Lemma 4(FO) allows us to simplify $\int \overline{x} d\hat{\Omega}_t$. Observe that, due to integration by parts, this integral can be written as

$$\int \overline{x} d\hat{\Omega}_t = -\int \overline{x}_z \frac{d}{d\theta}\hat{\Omega}_t dz d\theta := -\mathcal{I} \cdot \frac{d}{d\theta}\hat{\Omega}_t,$$

with operator $\mathcal{I}$ providing a convenient shorthand for the last integral. This immediately gives the following corollary.

Corollary 2(FO). For any t,

$$\int \overline{x} d\hat{\Omega}_t = \sum_{s=0}^{\infty} \left(\mathcal{I} \cdot \mathsf{A}_{t,s}\right) \overline{X}_{Z,s},$$

where $\{\mathsf{A}_{t,s}\}_{t,s}$ follow a recursion $\mathsf{A}_{0,s} = 0$, $\mathsf{a}_s = \mathcal{M} \cdot \mathsf{z}_s$, and $\mathsf{A}_{t,s} = \mathcal{L} \cdot \mathsf{A}_{t-1,s} + \mathsf{a}_{s-t-1}$.

Combine Corollaries 1(FO) and 2(FO) with Lemma 2(FO) to obtain the main result of this section:

Proposition 1(FO). $\left\{\overline{X}_{Z,t}\right\}_t$ is the solution to

$$\mathsf{G}_x \sum_{s=0}^{\infty} \mathsf{J}_{t,s} \overline{X}_{Z,s} + \mathsf{G}_X \overline{X}_{Z,t} + \mathsf{G}_\Theta \rho_\Theta^t = 0, \tag{19}$$

where $\{\mathsf{J}_{t,s}\}_{t,s}$ satisfies $\mathsf{J}_{t,s} = \int \mathsf{x}_{s-t} d\Omega^* + \mathcal{I} \cdot \mathsf{A}_{t,s}$.

This proposition provides the exact analytical expressions for the linear system of equation solution to which determines $\{\overline{X}_{Z,t}\}_t$. Since $\{\mathsf{A}_{t,s}\}_{t,s}$ has a simple linear recursive structure, Jacobian $\{\mathsf{J}_{t,s}\}_{t,s}$ it can be quickly constructed numerically and the system (19) inverted to find $\left\{\overline{X}_{Z,t}\right\}_t$.

3.2.1 Remark on numerical implementation of first order terms

We provide an overview of the numerical implementation of the first order algorithm. For details see appendix. The derivatives of the decisions rules, x_s, are approximated using the collocation method with the same basis functions and interpolation gridpoints as $\overline{x}$. At every gridpoint, x_s can be found directly using small dimensional matrix operations since the linear expectation operators in (16) and (17) can be represented using a matrix by pre-evaluating the basis functions at appropriate points.

The functions a_s are approximated by column vectors with length equal to the distribution gridpoints. They are constructed by weighting the density of stationary distribution, Ω^*, with the policy rules z_s evaluated at the distribution gridpoints and then multiplying by the stationary transition matrix $\overline{\Lambda}$. The operator $\mathcal{L}$ is approximated by a large sparse matrix and is constructed by weighting the columns of the stationary transition matrix $\overline{\Lambda}$ with the policy rules $\overline{z}_z$ evaluated at the distribution gridpoints. Finally the operator $\mathcal{I}$ is then approximated by a row-vector constructed by evaluating $\overline{x}_z$ at the distribution gridpoints. The components of $\mathsf{J}_{t,s}$, i.e. $\mathcal{I} \cdot \mathsf{A}_{t,s}$, can then be constructed using simple matrix operations. By truncating at a horizon T, (19) can be represented in matrix form as constant vector plus a matrix multiplying the stacked vector $\{\overline{X}_{Z,s}\}$. Once the equilibrium $\{\overline{X}_{Z,s}\}$ are found the derivatives $\overline{x}_{Z,t}$ and $\frac{d}{d\theta}\hat{\Omega}_t$ can constructed from (15) and (18) and stored as coefficients of the basis functions.

3.3 Second order approximations

Our approach has a key property where the steps used to obtain the first order approximations can be applied to second-order and higher-order approximations with only small adjustments. To better understand why this is the case, it is helpful to provide a simple example before diving into the detailed explanations.

In the recursive representation policy functions (e.g, $\overline{x}$) depend on other policy functions (e.g., $\overline{z}$) that, in turn, depend on aggregate shocks and states. To see implications of this fact for second order expansions, consider an example of mapping $f\left(g\left(a\right)\right)$, where a is a scalar and f and g are uni-dimensional functions. Let g_a, g_{aa} and f_g, f_{gg} be derivatives of $g\left(\cdot\right)$ and $f\left(\cdot\right)$. The first order expansion of f with respect to a is

$$\frac{\partial}{\partial a} f = f_g g_a. \tag{20}$$

This equation can be interpreted as saying that the first order change in f, captured by $\frac{\partial}{\partial a} f$, is equal to the first order response of f, captured by f_g, to the first order change in g, captured by g_a. The first order approximations that we developed in Section 3.2 can be described as finding $\frac{\partial}{\partial a} f$ and g_a. The functional form for f was often explicitly known from the F and G mappings and we obtained the analogue of f_g from the zeroths order economy.

The second order expansion of $f\left(g\left(a\right)\right)$ satisfies

$$\frac{\partial^2}{\partial a^2} f = \underbrace{f_g g_{aa}}_{\text{first order response to second order change}} + \underbrace{f_{gg} g_a g_a}_{\text{second order response to first order change}} . \tag{21}$$

Thus, the second order change in f can be separated into two terms: the first order response of f to the second order change of g and the second order response of f to the first order change in g. This separation will be useful since the two responses will play different roles in approximations. The second order response to the first order change will often be known from the first and zeroth order economies, so that the second term on the right hand side of (21) can be treated as a known constant. The first order response to the second order change is yet unknown but it has the same structure as the first order equation (20), except that the first order change g_a is replaced by the second order change g_{aa}. But this observation implies that essentially the entire procedure we developed for the first order approximations can be recycled for finding second order approximations, modulo adding constants known from the lower-order approximations.

Keeping this insight in mind, we now turn to our general economy and start by constructing directions pertinent for second order approximations. We need two sets of directions, $\{\hat{Z}_{t,s}\}_{t,s}$ and $\{\hat{Z}_{\sigma\sigma,t}\}_t$, that we define recursively as follows:

$$\hat{Z}_{t,k} = \overline{Z}_Z \cdot \hat{Z}_{t-1,k-1} + \overline{Z}_{ZZ} \cdot \left(\hat{Z}_{t-1}, \hat{Z}_{k-1}\right) \text{ for all } t,k > 0,$$

$$\hat{Z}_{\sigma\sigma,t} = \left[0, \overline{\Omega}_{\sigma\sigma}\right]^{\mathrm{T}} + \overline{Z}_Z \cdot \hat{Z}_{\sigma\sigma,t-1} \text{ for all } t > 0,$$

with $\hat{Z}_{0,s} = \hat{Z}_{t,0} = \hat{Z}_{\sigma\sigma,0} = \mathbf{0}$. Direction $\hat{Z}_{t,s}$ captures the second order change in the aggregate state in response to aggregate shocks that occurred t and k periods ago. Similarly to (21), it consists of two terms: the first order response of the LoM to the second order change in the aggregate state last period, $\overline{Z}_Z \cdot \hat{Z}_{t-1,k-1}$, and the second order response of the LoM to the first order changes in the aggregate states, $\overline{Z}_{ZZ} \cdot \left(\hat{Z}_{t-1}, \hat{Z}_{k-1}\right)$. Direction $\hat{Z}_{\sigma\sigma,t}$ captures precautionary motives, i.e. how variables change due to risk. These objections did not appear in the first order approximations as they were all identically equal to zero. Term $\left[0, \overline{\Omega}_{\sigma\sigma}\right]^{\mathrm{T}}$ captures how risk affects the LoM in the current period, term $\overline{Z}_Z \cdot \hat{Z}_{\sigma\sigma,t-1}$ captures how the LoM responds, to the first order, to the changes in the aggregate distribution induced by precautionary motives in the past.

We define $\{\overline{X}_{ZZ,t,k}\}_{t,k}$ and $\{\overline{X}_{\sigma\sigma,t}\}_t$ as $\overline{X}_{ZZ,t,k} = \overline{X}_Z \cdot \hat{Z}_{t,k} + \overline{X}_{ZZ} \cdot \left(\hat{Z}_t, \hat{Z}_k\right)$ and $\overline{X}_{\sigma\sigma,t} = \overline{X}_{\sigma\sigma} + \overline{X}_Z \cdot \hat{Z}_{\sigma\sigma,t}$. They play the same role in the second order approximations as $\{\overline{X}_{Z,t}\}_t$ played in the first order.

Lemma 1(SO). To the second order approximation, X_t satisfies

$$X_t\left(\mathcal{E}^t\right) = ... + \frac{1}{2}\left(\sum_{s=0}^{t}\sum_{m=0}^{t}\overline{X}_{ZZ,t-s,t-m}\mathcal{E}_s\mathcal{E}_m + \overline{X}_{\sigma\sigma,t}\right) + O\left(\|\mathcal{E}\|^3\right),$$

where ... are the first-order terms.

$\overline{X}_{ZZ,t-s,t-m}\mathcal{E}_s\mathcal{E}_m$ is the second-order response in period t to shocks that occurred s and m periods ago and $\overline{X}_{\sigma\sigma,t}$ is the effect of precautionary motives on aggregate variables. In order to find $\{\overline{X}_{ZZ,t,k}\}_{t,k}$ and $\{\overline{X}_{\sigma\sigma,t}\}_t$ we take the second order Fréchet derivative of (11) and evaluate it in the appropriate directions.

Lemma 2(SO). For any t,k

$$\mathsf{G}_x\left(\int \overline{x}_{\sigma\sigma,t}d\Omega^* + \int \overline{x}d\hat{\Omega}_{\sigma\sigma,t}\right) + \mathsf{G}_X\overline{X}_{\sigma\sigma,t} = 0, \tag{22}$$

$$\mathsf{G}_x\left(\int \overline{x}_{ZZ,t,k}d\Omega^* + \int \overline{x}d\hat{\Omega}_{t,k} + \mathsf{D}_{t,k}\right) + \mathsf{G}_X\overline{X}_{ZZ,t,s} + \mathsf{G}_{\Theta\Theta,t,k} = 0, \tag{23}$$

where explicit expression for $\mathsf{G}_{\Theta\Theta,t,k}$ is given in the appendix and $\mathsf{D}_{t,k} = \int \bar{x}_{Z,t}d\hat{\Omega}_k + \int \bar{x}_{Z,k}d\hat{\Omega}_t$.

Equation (22) has structure identical to equation (14), except that first order derivatives are replaced with the second order one. This could have been easily anticipated from our simple example (see equation (21)) by setting the first order changes g_a (i.e., terms corresponding to the first order precautionary motive) to zero. Equation (23) has an additional term $\mathsf{G}_{\Theta\Theta,t,k}$ that captures second order response of the G mapping to the first order changes in aggregate variables. We provide the explicit expression for this term in the appendix. That expression is lengthy but very intuitive, as it depends on the second-order derivatives of G, such as G_{XX}, and the first order changes in equilibrium variables such as $\overline{X}_{Z,t}$. Importantly, since both G_{XX} and $\overline{X}_{Z,t}$ are known from zeroth and first order approximations, $\mathsf{G}_{\Theta\Theta,t,k}$ can be calculated explicitly and we treat it as known for the purpose of this section. Term $\mathsf{D}_{t,k}$ can be written in terms of objects constructed in Section 3.2 using the intergration by parts:

$$\mathsf{D}_{t,k} = -\int \overline{x}_{zZ,t}\frac{d}{d\theta}\hat{\Omega}_k dzd\theta - \int \overline{x}_{zZ,k}\frac{d}{d\theta}\hat{\Omega}_t dzd\theta.$$

Thus, it remains to understand the terms $\int \overline{x}_{\sigma\sigma,t}d\Omega^*$ and $\int \overline{x}d\hat{\Omega}_{\sigma\sigma,t}$ to solve for the response of the economy with respect to risk, $\overline{X}_{\sigma\sigma,t}$, and we must also characterize $\int \overline{x}_{ZZ,t,k}d\Omega^*$ and $\int \overline{x}d\hat{\Omega}_{t,k}$ to find the curvature terms, $\overline{X}_{ZZ,t,k}$.

To characterize $\int \overline{x}_{\sigma\sigma,t} d\Omega^*$ and $\int \overline{x}_{ZZ,t,k} d\Omega^*$ we first extend Lemma 3(SO) to the second order.

Lemma 3(SO). For any t,

$$\overline{x}_{\sigma\sigma,t}(z,\theta) = \sum_{s=0}^{\infty} \mathsf{x}_s(z,\theta)\overline{X}_{\sigma\sigma,t+s} + \mathsf{x}_{\sigma\sigma}(z,\theta), \tag{24}$$

$$\overline{x}_{ZZ,t,k}(z,\theta) = \sum_{s=0}^{\infty} \mathsf{x}_s(z,\theta)\overline{X}_{ZZ,t+s,k+s} + \mathsf{x}_{t,k}(z,\theta), \tag{25}$$

where explicit expressions for $\mathsf{x}_{\sigma\sigma}$ and $\mathsf{x}_{t,k}$ are provided in the appendix.

Expressions (24) and (25) have similar structure to (15). The second order changes in individual policy functions $\overline{x}_{\sigma\sigma,t}$ and $\overline{x}_{ZZ,t,k}$ depend on the first order responses x_s on the second order changes in the aggregates, $\overline{X}_{\sigma\sigma,t+s}$ and $\overline{X}_{ZZ,t+s,k+s}$. In addition, $\overline{x}_{ZZ,t,k}$ depends on the second order response to the first order changes in the aggregates, and it is captured by $\mathsf{x}_{t,k}$. Similarly to the term $\mathsf{G}_{\Theta\Theta,t,k}$ in equation (23), it depends on the second order derivatives of F mapping and first order terms $\{\overline{X}_{Z,t}\}_t$. Its derivation is lengthy but straightforward. The only new element is adjustments for kinks in policy functions which are captured by a δ-function component of $\mathsf{x}_{t,k}$ at the kinks. These adjustments were not necessary in the first order approximations since policy functions $\overline{z}(z,\theta)$ were continuous, but they are needed at higher orders due to discontinuities of $\overline{z}_z(z,\theta)$ at kinks. These adjustments can be constructed from the zeroth order terms and are described in the appendix. Term $\mathsf{x}_{\sigma\sigma}$ captures precautionary effect of risk and it is proportional to $var(\mathcal{E})$.[10]

Using Lemma 3(SO) we obtain the second order analogue of Corollary 1(FO):

Corollary 1(SO). For any t,

$$\int \overline{x}_{\sigma\sigma,t} d\Omega^* = \sum_{s=0}^{\infty} \left(\int \mathsf{x}_s d\Omega^*\right) \overline{X}_{\sigma\sigma,t+s} + \int \mathsf{x}_{\sigma\sigma} d\Omega^*,$$

$$\int \overline{x}_{ZZ,t,k} d\Omega^* = \sum_{s=0}^{\infty} \left(\int \mathsf{x}_s d\Omega^*\right) \overline{X}_{ZZ,t+s,k+s} + \int \mathsf{x}_{t,k} d\Omega^*.$$

In order to characterize integrals $\int \overline{x} d\hat{\Omega}_{\sigma\sigma,t}$ and $\int \overline{x} d\hat{\Omega}_{t,k}$ in equations (24) and (25) we need to describe the LoM. The easiest way to understand the intuition for what is to come is to

[10] Explicit formulas show that, in addition to objects known from the zeroths and first order approximations, $\mathsf{x}_{\sigma\sigma}$ also depend on $\overline{x}_{ZZ,0,0}$, so in practice one needs first to find $\left\{\overline{X}_{ZZ,t,k}\right\}_{t,k}$ and hence $\overline{x}_{ZZ,0,0}$ before finding $\left\{\overline{X}_{\sigma\sigma,t}\right\}_t$.

consider the simple example that we gave in the beginning of this section but allow f to also depend on a independently of $g(a)$, i.e. $f(g(a), a)$. Equation (21) then becomes

$$\frac{\partial^2}{\partial a^2} f = \underbrace{f_g g_{aa} + f_{aa}}_{\text{first order response to second order change}} + \underbrace{f_{gg} g_a g_a + 2 f_{ga} g_a}_{\text{second order response to first order change}}. \tag{26}$$

The key observation is that, for non-linear functions, the second order response to first order changes consists of two terms: the second order response to the interaction of first order changes ($f_{gg} g_a g_a$ in equation (26)) and the change in the first order response to the first order change ($2 f_{ga} g_a$ in equation (26)).

When taking the second derivative of the LoM, the operators capturing the first order responses to second order changes f_g are the $\mathcal{L}$ and $\mathcal{M}$ present in Lemma 4(FO). In addition, we have new operators representing the second order response to first order changes (the f_{gg} and f_{ga} terms of equation (26)). Recall that operator $\mathcal{L}$ depends on policy functions $\overline{z}_z$ and, therefore, it is a function of Z. Let $\mathcal{L}_Z$ be the Fréchet derivative of $\mathcal{L}$ and let $\mathcal{L}_{Z,t}$ denote $\mathcal{L}_{Z,t} = \mathcal{L}_Z \cdot \hat{Z}_t$ so

$$(\mathcal{L}_{Z,t} \cdot y) \langle z', \theta' \rangle := \int \overline{\Lambda}(z', \theta', z, \theta) \overline{z}_{zZ,t}(z,\theta) y(z,\theta)\, dz d\theta,$$

which is the counterpart of the f_{ga} term. Next, operators $\frac{d}{dz}\mathcal{M}$ and $\frac{d}{dz}\mathcal{L}$ will capture the second order response to the interaction of first order changes (f_{gg} term) where we use the notation $\mathcal{M} \cdot (y_1, y_2)$ and $\mathcal{L} \cdot (y_1, y_2)$ to denote $\mathcal{M} \cdot y$ and $\mathcal{L} \cdot y$ where y is a point-wise products of functions y_1 and y_2, i.e. $y(z,\theta) = y_1(z,\theta) y_2(z,\theta)$ for all (z,θ). The second order expansion of the LoM can then be written recursively as

Lemma 4(SO). For all t,

$$\frac{d}{d\theta}\hat{\Omega}_{\sigma\sigma,t+1} = \mathcal{L} \cdot \frac{d}{d\theta}\hat{\Omega}_{\sigma\sigma,t} - \sum_{s=0}^{\infty} \mathsf{a}_s \overline{X}_{\sigma\sigma,t+s} - \mathcal{M} \cdot \mathsf{z}_{\sigma\sigma}, \tag{27}$$

and

$$\begin{aligned} \frac{d}{d\theta}\hat{\Omega}_{t+1,k+1} = {} & \mathcal{L} \cdot \frac{d}{d\theta}\hat{\Omega}_{t,k} - \sum_{s=0}^{\infty} \mathsf{a}_s \overline{X}_{ZZ,t+s,k+s} - \mathcal{M} \cdot \mathsf{z}_{t,k} \\ & + \frac{d}{dz}\mathcal{M} \cdot (\overline{z}_{Z,t+1}, \overline{z}_{Z,k+1}) - \frac{d}{dz}\mathcal{L} \cdot \left(\frac{d}{d\theta}\hat{\Omega}_{t+1}, \overline{z}_{Z,k+1}\right) - \frac{d}{dz}\mathcal{L} \cdot \left(\frac{d}{d\theta}\hat{\Omega}_{k+1}, \overline{z}_{Z,t+1}\right) \\ & + \mathcal{L}_{Z,t} \cdot \frac{d}{d\theta}\hat{\Omega}_k + \mathcal{L}_{Z,k} \cdot \frac{d}{d\theta}\hat{\Omega}_t. \end{aligned} \tag{28}$$

To understand intuition for these expressions, consider first equation (27). As the precautionary savings motive is zero to first order, equation (27) doesn't rely on any of the new operators, using only $\mathcal{L}$ and $\mathcal{M}$. The first two terms on the right hand side of (27) have exactly the same interpretation as equation (18), just applied to the second-order terms that arise due to the precautionary motive: $\mathcal{L} \cdot \frac{d}{d\theta}\hat{\Omega}_{\sigma\sigma,t}$ is the first order response of the LoM to the second order changes in distribution $\hat{\Omega}_{\sigma\sigma,t}$, and $\mathsf{a}_s\overline{X}_{\sigma\sigma,t+s}$ is the first order response of the LoM to the expected second order changes in aggregate prices, with all changes driven by precautionary motives. The precautionary motive also affects policy functions in period t, and $\mathcal{M} \cdot \mathsf{z}_{\sigma\sigma}$ captures the first order response of the LoM of the aggregate distribution to this adjustments in policy functions.

Equation (28) captures the second order changes in the aggregate distribution due to aggregate shocks. The first line of this equation is the exact parallel of equation (27) and it shows first order responses of the LoM to the second order changes in various policy functions. The second and third lines of equation (28) capture the second order responses of the LoM to the first order changes in the policies as captured in equation (26). The second line of (27) consists of the second order response to the interaction of first order changes ($f_{gg}g_a g_a$ in equation (26)), while the third line is the change in the first order response to the first order change ($2f_{ga}g_a$ in equation (26)).

Equations (27) and (28) have a recursive structure that is very similar to that of (18). Now we use that recursive structure like we did in Corollary 2(FO) to get an expression $\int \bar{x} d\hat{\Omega}_{\sigma\sigma,t}$ and $\int \overline{x} d\hat{\Omega}_{t,k}$. To see how this can be done, define

$$\mathsf{b}_{\sigma\sigma} := \mathcal{M} \cdot \mathsf{z}_{\sigma\sigma},$$

$$\mathsf{b}_{t,k} := \mathcal{M} \cdot \mathsf{z}_{t+1,k+1} - \mathcal{L}_{Z,t} \cdot \frac{d}{d\theta}\hat{\Omega}_k - \mathcal{L}_{Z,k} \cdot \frac{d}{d\theta}\hat{\Omega}_t,$$

$$\mathsf{c}_{t,k} = \mathcal{M} \cdot (\overline{z}_{Z,t+1}, \overline{z}_{Z,k+1}) - \mathcal{L} \cdot \left(\frac{d}{d\theta}\hat{\Omega}_{t+1}, \overline{z}_{Z,k+1}\right) - \mathcal{L} \cdot \left(\frac{d}{d\theta}\hat{\Omega}_{k+1}, \overline{z}_{Z,t+1}\right),$$

then the recursive LoMs, (27) and (28), can be written more succinctly as

$$\frac{d}{d\theta}\hat{\Omega}_{\sigma\sigma,t+1} = \mathcal{L} \cdot \frac{d}{d\theta}\hat{\Omega}_{\sigma\sigma,t} - \sum_{s=0}^{\infty} \mathsf{a}_s\overline{X}_{\sigma\sigma,t+s} - \mathsf{b}_{\sigma\sigma} \tag{29}$$

$$\frac{d}{d\theta}\hat{\Omega}_{t+1,k+1} = \mathcal{L} \cdot \frac{d}{d\theta}\hat{\Omega}_{t,k} - \sum_{s=0}^{\infty} \mathsf{a}_s\overline{X}_{ZZ,t+s,k+s} - \mathsf{b}_{t,k} + \frac{d}{dz}\mathsf{c}_{t,k}. \tag{30}$$

Equation (29) follows the same recursive structure as (18) with only the addition of the $\mathsf{b}_{\sigma\sigma}$ term. As such, one would expect it to aggregate in the same way using integration by parts: $\int \bar{x} d\hat{\Omega}_{\sigma\sigma,t} = -\mathcal{I} \cdot \frac{d}{d\theta}\hat{\Omega}_{\sigma\sigma,t}$.

Equation (30) has the same structure as (29) with the addition of a new term $\frac{d}{dz}\mathsf{c}_{t,k}$. To get intuition for how the presence of $\frac{d}{dz}\mathsf{c}_{t,k}$ alters $\int \overline{x} d\hat{\Omega}_{t,k}$ assume that all terms except for $\mathsf{c}_{0,0}$

are zero. Under this assumption $\frac{d}{d\theta}\hat{\Omega}_{1,1} = \frac{d}{dz}\mathsf{c}_{0,0}$, which implies that

$$\int \bar{x} d\hat{\Omega}_{\sigma\sigma,1} = -\int \overline{x}_z \frac{d}{dz}\mathsf{c}_{0,0} dz d\theta = \int \overline{x}_{zz}\mathsf{c}_{0,0} dz d\theta := \mathcal{I}^{(zz)} \cdot \mathsf{c}_{0,0},$$

where the second equality was achieved by an additional application integration by parts. This insight carries over to future periods as $\frac{d}{d\theta}\hat{\Omega}_{2,2} = \mathcal{L} \cdot \frac{d}{dz}\mathsf{c}_{0,0}$. Using integration by parts we get

$$\mathcal{L} \cdot \frac{d}{dz}\mathsf{c}_{0,0} = -\mathcal{L}^{(zz)} \cdot \mathsf{c}_{0,0} + \frac{d}{dz}\mathcal{L}^{(z,z)} \cdot \mathsf{c}_{0,0}$$

where $\mathcal{L}^{(zz)}$ and $\mathcal{L}^{(z,z)}$ are defined as $\mathcal{L}$ with $\overline{z}_z$ being replaced by $\overline{z}_{zz}$ and $\overline{z}_z\overline{z}_z$ respectively. We then observe that the aggregated decisions, $\int \bar{x} d\hat{\Omega}_{\sigma\sigma,2}$, simplify to

$$-\int \overline{x}_z \left(-\mathcal{L}^{(zz)} \cdot \mathsf{c}_{0,0} + \frac{d}{dz}\mathcal{L}^{(z,z)} \cdot \mathsf{c}_{0,0} \right) dz d\theta = \mathcal{I} \cdot \mathcal{L}^{(zz)} \cdot \mathsf{c}_{0,0} + \mathcal{I}^{(zz)} \cdot \mathcal{L}^{(z,z)} \cdot \mathsf{c}_{0,0}.$$

Applying these insights to the full LoMs gives the following corollary

Corollary 2(SO). For all t,

$$\int \bar{x} d\hat{\Omega}_{\sigma\sigma,t} = \sum_{s=0}^{\infty} \left(\mathcal{I} \cdot \mathsf{A}_{t,s}\right) \overline{X}_{\sigma\sigma,s} + \mathcal{I} \cdot \mathsf{B}_{\sigma\sigma,t},$$

where $\{\mathsf{B}_{\sigma\sigma,t}\}_t$ follows a recursion $\mathsf{B}_{\sigma\sigma,t+1} = \mathsf{b}_{\sigma\sigma} + \mathcal{L} \cdot \mathsf{B}_{\sigma\sigma,t}$; and

$$\int \overline{x} d\hat{\Omega}_{t,k} = \sum_{s=0}^{\infty} \left(\mathcal{I} \cdot \mathsf{A}_{t,s}\right) \overline{X}_{t-k+s,s} + \mathcal{I} \cdot \mathsf{B}_{t,k} + \mathcal{I}^{(zz)} \cdot \mathsf{C}_{t,k},$$

where $\{\mathsf{B}_{t,k}, \mathsf{C}_{t,k}\}_{t,k}$ follow recursions

$$\mathsf{C}_{t+1,k+1} = \mathsf{c}_{t+1,k+1} + \mathcal{L}^{(z,z)} \cdot \mathsf{C}_{t,k},$$

$$\mathsf{B}_{t+1,k+1} = \mathsf{b}_{t+1,k+1} + \mathcal{L} \cdot \mathsf{B}_{t,k} + \mathcal{L}^{(zz)} \cdot \mathsf{C}_{t,k}.$$

This recursive structure allows us simplify (22) and (23) and construct linear systems of equations that can be inverted to obtain the second order approximation:

Proposition 1(SO). $\left\{\overline{X}_{ZZ,t,k}\right\}_{t,k}$ and $\left\{\overline{X}_{\sigma\sigma,t}\right\}_t$ are the solutions to linear systems

$$\mathsf{G}_x \sum_{s=0}^{\infty} \mathsf{J}_{t,s}\overline{X}_{\sigma\sigma,s} + \mathsf{G}_x\mathsf{H}_{\sigma\sigma,t} + \mathsf{G}_X\overline{X}_{\sigma\sigma,t} = 0, \tag{31}$$

and

$$\mathsf{G}_x \sum_{s=0}^{\infty} \mathsf{J}_{t,s}\overline{X}_{ZZ,t-k+s,s} + \mathsf{G}_x\mathsf{H}_{t,k} + \mathsf{G}_X\overline{X}_{ZZ,t,k} + \mathsf{G}_{\Theta,t,k} = 0. \tag{32}$$

where $\mathsf{H}_{\sigma\sigma,t} = \int \mathsf{x}_{\sigma\sigma} d\Omega^* + \mathcal{I} \cdot \mathsf{B}_{\sigma\sigma,t}$ and $\mathsf{H}_{t,k} = \int \mathsf{x}_{t,k} d\Omega^* + \mathsf{D}_{t,k} + \mathcal{I} \cdot \mathsf{B}_{t,k} + \mathcal{I}^{(zz)} \cdot \mathsf{C}_{t,k}$.

3.3.1 Remarks on numerical implementation

The approximated objects are similar to those in section (3.2.1) with the details being left to appendix. The derivatives of the decisions rules, $\mathsf{x}_{t,k}$ and $\mathsf{x}_{\sigma\sigma,t}$, are approximated using the collocation method with the same basis functions and gridpoints as $\overline{x}$. The operator $\mathcal{L}_{Z,t}$ is approximated by a sparse matrix which is constructed by weighting the columns of $\overline{\Lambda}$ with the derivatives $\overline{z}_{Z,t}(z,\theta)$ evaluated at the distribution gridpoints. Similarly the operators $\mathcal{L}^{(zz)}$ and $\mathcal{L}^{(z,z)}$ are sparse matrices constructed by weighting the columns of $\overline{\Lambda}$ with the derivatives $\overline{z}_{zz}(z,\theta)$ and $\overline{z}_z(z,\theta)\overline{z}_z(z,\theta)$, respectively, all evaluated at the gridpoints of Ω^*. Finally, $\mathcal{I}^{(zz)}$ is a row vector constructed by evaluating $\overline{x}_{zz}(z,\theta)$ at the gridpoints of Ω^*. All the components of $\mathsf{H}_{t,k}$ and $\mathsf{H}_{\sigma\sigma,t}$ can then be constructed via simple matrix operations.

3.4 Comparison to literature

Our approach builds on the variational techniques such as Schmitt-Grohé and Uribe (2004). Those techniques were originally developed to study representative agent models. When applied to heterogeneous agent economies, such an approach would seek to solve directly for derivatives $\overline{X}_Z$, $\overline{X}_{ZZ}$, which quickly becomes impractical as the dimensionality of Z grows. Our approach shows that this problem can be side-stepped by solving for the values of those derivatives in appropriately chosen directions. These values remain small-dimensional objects even when Z and hence $\overline{X}_Z$ are are large, as in most canonical HA economies.

Our first-order approximation is closely related to the method developed in an important paper by Auclert et al. (2021), or ABRS for short. ABRS focus only on the first order approximations and they work directly with the sequence-space representation (1) and (2). Similarly to us, they also consider approximation around point Z^* and exploit the insight of Boppart et al. (2018) that the first-order approximation of a stochastic economy can be constructed from impulse responses to MIT shocks in deterministic economy. We can describe key ideas behind ABRS method using our notation. ABRS first numerically differentiate (1) to approximate derivatives $\frac{\partial x_0(z,\theta)}{\partial X_s}$ in the first order representation

$$\hat{x}_0\left(z,\theta\right) = \sum_{s=0}^{\infty} \frac{\partial x_0\left(z,\theta\right)}{\partial X_s}\hat{X}_s, \tag{33}$$

where $\{\hat{X}_s\}_s$ are changes in the aggregate variables induced by an MIT shock. ABRS then assume that the LoM for the aggregate distribution, equation (12), can be approximated by the histogram method (see Young (2010)) and under that assumption show that together with

(33) the first order approximation of (2) can be written as

$$\mathsf{G}_x \sum_{s=0}^{\infty} \mathcal{J}_{t,s} \hat{X}_s + \mathsf{G}_X \hat{X}_t + \mathsf{G}_\Theta \rho_\Theta^t = 0, \tag{34}$$

where $\mathcal{J}_{t,s}$ has a recursive linear structure that is easy to compute. Economically, $\mathcal{J}_{t,s}$ captures derivatives $\partial \left(\int x_{i,t} di\right) / \partial X_s$ and thus represents a Jacobian in the sequence-space representation.

There are obvious parallels between equations (33) and (34) in the ABRS approach and equations (15) and (19) in ours. One difference between the two is that all expressions in our equations are exact while $\frac{\partial x_0(z,\theta)}{\partial X_s}$ and $\mathcal{J}_{t,s}$ in (33) and (34) are obtained numerically. Mathematically, this difference is non-essential. In particular, it can be shown that if the approximation error in the derivative $\frac{\partial x_0(z,\theta)}{\partial X_s}$ and the grid side in the histogram method go to zero, then the solution to (34) converges to the exact solution, given by equation (19). Despite this equivalence, we found certain advantages in our approach even at the first order. It is easy to compute derivatives of the state-space representation of F using automatic differentiation and, therefore, $\{\mathsf{x}_s\}_s$ can be constructed computationally quickly and reliably. In contrast, differentiating numerically the infinite (or, at least, large) sequence of F to find $\frac{\partial x_0(z,\theta)}{\partial X_s}$ is more time consuming and less computationally stable. As a result, we found in our experimentation that our first order approximation is somewhat faster that ABRS (we discuss this point in more details in Section 5).

The key difference between our approach and that of ABRS lies in the ability to handle higher order approximations. There are two reasons for why extending ABRS approach to higher order approximations is difficult. First, higher order of approximations of MIT shocks do not allow one to recover terms related to the precautionary motive, i.e. $\{X_{\sigma\sigma,t}\}_t$ terms, and thus obtain full second-order approximation. Second, while the histogram method correctly approximate the LoM to the first order, it fails at higher orders in the sense that it does not converge to the correct expressions even as the grid size goes to zero. We prove it formally in the appendix but the intuition for this result can be easily seen from equation (28) or (26). The histogram method locally linearizes the LoM for the aggregate distribution, and thus it misses terms that capture second order responses of the LoM to the first order changes in policy functions.[11]

To derive analytically properties of the LoM, we used linear operator techniques that were first developed in Bhandari et al. (2021). Approximations considered in that paper scaled

[11]This follows from the fact that the projection functions which assign households to appropriate bins of the histogram are linear in household endogenous states.

both aggregate and idiosyncratic shocks and are not applicable to economies in which policy functions have kinks, e.g., due to the occasionally binding borrowing constraints. Their environment also lacked dynamics in the aggregate distribution, which is one of the key complications that our method is developed to overcome using recursive characterization in Lemmas 4(FO) and 4(SO).

4 Extensions

We now discuss how our approach described in Section 3 can be extended to three classes of problems: models with transition dynamics from some initial distribution to a steady state (as occurs, for examples, in models in which a permanent policy change induces transition from the steady state under one set of policies to a different steady state under a different set of policies), models with stochastic volatility, and portfolio problems.

4.1 Transition dynamics

In this section, we relax assumption of Section 3 that the initial distribution Ω_0 coincides with Ω^*. Instead, we assume that initial state is given by $(0, \Omega_0)$ for some Ω_0 and describe how to characterize the transition dynamics of this economy. We focus on the first order approximations and show that they follow the same structure as Section 3.2.

Let $\hat{\Omega}_0 = \Omega^* - \Omega_0$ and consider a sequence of directions $\{\hat{Z}_{\Omega,t}\}_t$ defined recursively by $\hat{Z}_{\Omega,0} = [0, \hat{\Omega}_0]^{\mathrm{T}}$ and $\hat{Z}_{\Omega,t} = \overline{Z}_Z \cdot \hat{Z}_{\Omega,t-1}$. Similarly, define $\{\overline{X}_{\Omega,t}\}_t$ as $\overline{X}_{\Omega,t} := \overline{X}_Z \cdot \hat{Z}_{\Omega,t}$. This sequence of values of the Fréchet derivatives characterizes transition dynamics to the first order.

Lemma 1(TD). To the first order approximation, X_t satisfies

$$\mathbb{E}_0 X_t = \overline{X} + \overline{X}_{\Omega,t} + O\left(\left\|\mathcal{E}, \hat{\Omega}_0\right\|^2\right).$$

The derivations of Section 3.2 apply with minimal changes and they show that a simple extension of Proposition 1(FO) characterizes $\{\overline{X}_{\Omega,t}\}_t$:

Proposition 1(TD). $\left\{\overline{X}_{\Omega,t}\right\}_t$ is the solution to

$$\mathsf{G}_x \sum_{s=0}^{\infty} \mathsf{J}_{t,s} \overline{X}_{\Omega,s} + \mathsf{G}_X \overline{X}_{\Omega,t} + \mathsf{G}_x \mathsf{J}_{\Omega,t} = 0, \tag{35}$$

where $\mathsf{J}_{\Omega,t} = \mathcal{I} \cdot \mathcal{L}^t \cdot \frac{d}{d\theta} \hat{\Omega}_0$.

The main difference between Proposition 1(TD) and Proposition 1(FO) is the last term in equation (35). This term generalizes equation (19) to account for $\Omega_0 \neq \Omega$, which means that the initial direction is now non-trivial and $\frac{d}{d\theta} \hat{\Omega}_0 \neq 0$.

4.2 Stochastic volatility

For many applications that require realistic modeling of financial markets or uncertainty about government policies it is important to allow for variation in volatility of aggregate shocks. A standard way to approximate such problems is to consider third-order expansions (see, e.g., discussion in Fernández-Villaverde et al. (2011)). While it is possible to use a third-order extension of our approximation for this purposes, in this section we present a much simpler *second-order* approximation that attains the same objective.

We extend our model in Section 2 so that aggregate shock $\mathcal{E}_t$ follows process

$$\mathcal{E}_t = \sqrt{1+\Upsilon_{t-1}}\mathcal{E}_{\Theta,t}, \tag{36}$$

$$\Upsilon_t = \rho_\Upsilon \Upsilon_{t-1} + \mathcal{E}_{\Upsilon,t}, \tag{37}$$

where $|\rho_\Upsilon| < 1$ and $\mathcal{E}_{\Theta,t}$ and $\mathcal{E}_{\Upsilon,t}$ are mean-zero i.i.d. variables with support of $\mathcal{E}_{\Upsilon,t}$ bounded so that Υ_t always remains greater than -1. We assume that initial conditions are such that $\Upsilon_{-1} = 0$. This stochastic process provides a simple way to capture shocks to volatility of aggregate variables. When stochastic process for $\mathcal{E}_{\Upsilon,t}$ is degenerate, i.e., $\mathcal{E}_{\Upsilon,t} = 0$, then this process collapses to our baseline environment and conditional volatility of aggregate shocks $var_{t-1}(\mathcal{E}_t)$ is constant and given by $var(\mathcal{E}_{\Theta,t})$. When $\mathcal{E}_{\Upsilon,t}$ is non-degenerate then $var_{t-1}(\mathcal{E}_t)$ is time-varying and satisfies $(1+\Upsilon_{t-1})var(\mathcal{E}_{\Theta,t})$.

The state in the recursive representation now consists of a triplet $(\Upsilon, \Theta, \Omega)$. One way to approximate this economy is to scale both shocks $\mathcal{E}_{\Theta,t}$ and $\mathcal{E}_{\Upsilon,t}$ with σ and approximate equilibrium around the deterministic point $(0, 0, \Omega^*)$. In order to capture time-varying volatility, this approach would indeed require using third-order approximations. Instead, a much faster and simpler method is to proceed as in Section 2 and scale only the combined shock $\mathcal{E}_t$ with σ, just as we did in equation (13). Since shocks $\mathcal{E}_{\Upsilon,t}$ and $\mathcal{E}_{\Theta,t}$ are not scaled with σ, volatility $1 + \Upsilon_t$ follows a non-trivial stochastic process even when $\sigma = 0$. Thus, our approximations are around $(\Upsilon, 0, \Omega^*)$, where Υ is a non-trivial random variable.

Observe that when $\sigma = 0$, volatility Υ has no effect on any endogenous variable. Therefore, the invariant distribution Ω^* is independent of Υ and coincides with the one we considered before. Similarly, the derivatives of policy functions $\widetilde{X}(\Upsilon, Z; 0)$ and $\widetilde{x}(z, \theta, \Upsilon, Z; 0)$ with respect to $Z = (\Theta, \Omega)$ are also independent of Υ and, in fact, coincide with their values in the baseline economy. Similarly, derivatives $\overline{X}_\sigma$, $\overline{x}_\sigma$ is equal to zero for any Υ, but the second-order derivative, $\overline{X}_{\sigma\sigma}(\Upsilon)$ and $\overline{x}_{\sigma\sigma}(\Upsilon)$ are non-trivial functions of Υ. This dependence of these derivatives on Υ allows us to compute effects of stochastic volatility using second-order approximations.These observations imply that the only modification that our approach requires is in devising a method that generalizes $\left\{\overline{X}_{\sigma\sigma,t}\right\}_t$ and find its values in stochastic volatility settings.

We proceed as follows. Take any history $\mathcal{E}^t_\Upsilon$, with its implied history of volatilities $(\Upsilon_0, ..., \Upsilon_t)$ and construct the sequence of directions $\{\hat{Z}_{\sigma\sigma,t}(\mathcal{E}^t_\Upsilon)\}_{t,\mathcal{E}^t_\Upsilon}$ as follows

$$\hat{Z}_{\sigma\sigma,t}\left(\mathcal{E}^t_\Upsilon\right) = \left[0, \overline{\Omega}_{\sigma\sigma}(\Upsilon_{t-1})\right]^T + \overline{Z}_Z \cdot \hat{Z}_{\sigma\sigma,t-1}\left(\mathcal{E}^{t-1}_\Upsilon\right)$$

with $\hat{Z}_{\sigma\sigma,-1} = \mathbf{0}$ and $\Upsilon_{t-1} = \Upsilon_{t-1}(\mathcal{E}^{t-1}_\Upsilon)$ defined via (37). Similarly, $\{\overline{X}_{\sigma\sigma,t}(\mathcal{E}^t_\Upsilon)\}_{t,\mathcal{E}^t_\Upsilon}$ are defined by

$$\overline{X}_{\sigma\sigma,t}\left(\mathcal{E}^t_\Upsilon\right) := \overline{X}_{\sigma\sigma}(\Upsilon_t) + \overline{X}_Z \cdot \hat{Z}_{\sigma\sigma,t}\left(\mathcal{E}^t_\Upsilon\right).$$

This definition is the same as that of $\left\{\overline{X}_{\sigma\sigma,t}\right\}_t$ in Section 3.3 except that it explicitly recognizes the fact that $\overline{X}_{\sigma\sigma}$ and $\overline{\Omega}_{\sigma\sigma}$ depend on Υ. The definition of $\overline{x}_{\sigma\sigma,t}(\mathcal{E}^t_\Upsilon)$ is modified analogously. The next result extends Lemmas 1(FO) and 1(SO) to the settings with stochastic volatility.

Lemma 1(SV). To the second order approximation, X_t satisfies

$$X_t\left(\boldsymbol{\mathcal{E}}^t\right) = \overline{X} + \sum_{s=0}^{t} \overline{X}_{Z,t-s}\mathcal{E}_s + \frac{1}{2}\left(\sum_{s=0}^{t}\sum_{m=0}^{t} \overline{X}_{ZZ,t-s,t-m}\mathcal{E}_s\mathcal{E}_m + \overline{X}_{\sigma\sigma,t}\left(\mathcal{E}^t_\Upsilon\right)\right) + O\left(\|\mathcal{E}\|^3\right), \quad (38)$$

where sequences $\left\{\overline{X}_{Z,t}\right\}_t$, $\left\{\overline{X}_{ZZ,t,k}\right\}_{t,k}$ are the same as in Sections 3.2 and 3.3.

To find $\{\overline{X}_{\sigma\sigma,t}(\mathcal{E}^t_\Upsilon)\}_{t,\mathcal{E}^t_\Upsilon}$, the analysis proceeds along the same lines as in Section 3.3. The key potential complication is that $\overline{X}_{\sigma\sigma,t}$ is a non-linear function of $\mathcal{E}^t_\Upsilon$, which would make characterization difficult. Stochastic process (36) simplifies this problem as it implies that $\overline{X}_{\sigma\sigma,t}$ is a linear function of $\mathcal{E}^t_\Upsilon$ with a simple characterization of this dependence. The key step towards deriving it is to observe that the analogue of equation (24) in our stochastic volatility economy becomes

Lemma 3(SV). For any t,

$$\overline{x}_{\sigma\sigma,t}(z,\theta,\mathcal{E}^t_\Upsilon) = \sum_{s=0}^{\infty} \mathsf{x}_s(z,\theta)\mathbb{E}\left[\overline{X}_{\sigma\sigma,t+s}|\mathcal{E}^t_\Upsilon\right] + \mathsf{x}_{\sigma\sigma}(z,\theta) + \mathsf{x}_\Upsilon(z,\theta)\Upsilon_t, \quad (39)$$

where x_s, $\mathsf{x}_{\sigma\sigma}$ are the same as in Lemma 3(SO) and the explicit equation for x_Υ is provided in the appendix.

Equation (39) shows that the direct effect of stochastic volatility shocks on $\overline{x}_{\sigma\sigma,t}(z,\theta,\mathcal{E}^t_\Upsilon)$ is captured entirely by the term $\mathsf{x}_\Upsilon(z,\theta)\Upsilon_t$ that is linear in Υ_t (and hence, $\mathcal{E}^t_\Upsilon$) and conditional expectations of $\mathbb{E}[\overline{X}_{\sigma\sigma,t+s}|\mathcal{E}^t_\Upsilon]$ that is a linear operator as well. Equation G, that describes the relationship between $\overline{x}_t$ and $\overline{X}_t$, then implies that $\overline{X}_t$ must be linear in $\mathcal{E}^t_\Upsilon$ as well. Moreover, this dependence has a very simple characterization.

Proposition 1(SV). The stochastic process $\overline{X}_{\sigma\sigma,t}\left(\mathcal{E}^t_\Upsilon\right)$ satisfies

$$\overline{X}_{\sigma\sigma,t}\left(\mathcal{E}^t_\Upsilon\right) = \overline{X}_{\sigma\sigma,t} + \sum_{s=0}^{t} \overline{X}_{\Upsilon,t-s}\mathcal{E}_{\Upsilon,s},$$

where $\{\overline{X}_{\sigma\sigma,t}\}_t$ is the same as in Section 3.3 and $\{\overline{X}_{\Upsilon,t}\}_t$ satisfies

$$\mathsf{G}_x \sum_{j=0}^{\infty} \mathsf{J}_{k,j}\overline{X}_{\Upsilon,j} + \sum_{j=0}^{k} \mathsf{G}_x \mathsf{H}_{\Upsilon,k-j}\rho^j_\Upsilon + \mathsf{G}_X \overline{X}_{\Upsilon,k} = 0 \tag{40}$$

and $\mathsf{H}_{\Upsilon,t} = \mathcal{I}\cdot\mathcal{L}^k\cdot\mathcal{M}\cdot \mathsf{z}_\Upsilon$.

Thus, the proposition shows that in order to extend our approach to the models of stochastic volatility, it is sufficient to solve that one more system of linear equations (40) that has a structure very similar to other second-order terms, equations (31) and (32).

4.3 Portfolio problems

Portfolio problems, in which agents allocate their wealth to multiple assets with stochastic returns that depend on aggregate shocks, do not fit into our framework in Section 3. In such problems, assets become indistinguishable in the economy without aggregate shocks. As a result, each agent's portfolio allocation is indeterminate in the zeroth order economy, which violates the assumption we used in Section 3.1. We now describe how our approach can be modified to overcome this difficulty.

It will be easiest to motivate our approach by using a simple example that allows us to introduce all the key portfolio features that are present in a broad class of portfolio problems. Consider the same Krusell and Smith economy as in Section 2.1 but allow agents to trade, in addition to capital, a risk-free bond that is available in zero net supply. Let R^f_t be the interest rate on risk-free bond between periods $t-1$ and t, and $R^x_t = R_t - R^f_t$ be the excess return of capital.

We use $a_{i,t}$ to denote wealth of agent i at the end of period t. This wealth is allocated between investment in capital $k_{i,t}$ and bonds $a_{i,t} - k_{i,t}$. Assuming for concreteness that the borrowing constraint is on total assets holdings, agents' optimality conditions can be written as the choice over $\{c_{i,t}, a_{i,t}, k_{i,t}\}_t$ to maximize their utility subject to the borrowing constraint $a_{i,t} \geq 0$ and the budget constraint

$$c_{i,t} + a_{i,t} - W_t \exp(\theta_{i,t}) - R^f_t a_{i,t-1} - R^x_t k_{i,t-1} = 0. \tag{41}$$

Market clearing conditions now include the condition that the aggregate demand for bonds is equal to their aggregate supply, which in our example is zero: $\int (a_{i,t} - k_{i,t})di = 0$. Agents' optimality conditions are represented by stochastic sequences $\{a_{i,t}, c_{i,t}, k_{i,t}, u_{i,t}, \zeta_{i,t}, \lambda_{i,t}\}_{i,t}$ that satisfy (41) and

$$R_t^f U_c(c_{i,t}) - \lambda_{i,t} = 0, \quad U_c(c_{i,t}) + \zeta_{i,t} - \beta \mathbb{E}_t \lambda_{i,t+1} = 0, \quad a_{i,t}\zeta_{i,t} = 0, \quad u_{i,t} - a_{i,t-1} = 0, \tag{42}$$

$$\mathbb{E}_{t-1}\left[\lambda_{i,t} R_t^x\right] = 0. \tag{43}$$

Market clearing conditions for aggregate variables $\left\{K_t, W_t, R_t^f, R_t^x\right\}_t$ are given by (8) and

$$R_t^f + R_t^x + \delta - \alpha \exp\left(\Theta_t\right) K_t^{\alpha-1} - 1 = 0, \quad K_t - \int k_{i,t} di = 0. \tag{44}$$

The reader can immediately recognize close parallels between this specification and that in Section 2.1, with wealth $a_{i,t-1}$ rather than capital $k_{i,t-1}$ acting as the individual endogenous state variable.

The direct application of the perturbational approach is difficult because some variables are indeterminate in the zeroth order economy. This can be easily seen in our example. In the absence of aggregate shock, excess returns on capital are zero and capital and bonds are perfect substitutes. This implies that while aggregate capital and bonds are determined in the zeroth order economy (and equal to $\overline{K}_t$ and 0 respectively), each individual optimal portfolio allocation of wealth between capital and bonds is not determined. As is well know from portfolio theory (see, e.g., Devereux and Sutherland (2011)), the perturbational approach extends if one approximates around portfolios that are the limits, as $\sigma \to 0$, of optimal portfolios problem in stochastic economy for $\sigma > 0$. Finding these portfolios and their limits requires second order approximations.

A small extension of our approach from Section 3.2 allows one to simultaneously find these limiting portfolios and the first order equilibrium responses without having to solve the full second order approximation. We show how this extension works for a broad class of portfolio problems. Let $x_{i,t}$ and $k_{i,t}$ be vectors of individual variables that are and are not determined in the zeroth order economy. For the ease of the exposition we assume $k_{i,t}$ to be uni-dimensional but we show in the appendix that all results extend directly when $k_{i,t}$ is an arbitrary vector. We can write individual optimality conditions as

$$F\left(z_{i,t-1}, x_{i,t}, \mathbb{E}_{i,t} x_{i,t+1}, X_t, \theta_{i,t}, R_t^x k_{i,t}\right) = 0 \text{ for all } i, t, \tag{45}$$

and

$$\mathbb{E}_{t-1}\left[m_{i,t} R_t^x\right] = 0 \text{ for all } i, t, \tag{46}$$

where $m_{i,t}$ is an some element of $x_{i,t}$. We can write it as $m_{i,t} = \mathsf{S}x_{i,t}$ and $R_t^x = \mathsf{R}X_t$ for some selection matrices S and R. Aggregate feasibility conditions are given by

$$G\left(\int x_{i,t}di, X_t, \Theta_t\right) = 0 \text{ for all } t \tag{47}$$

and

$$\int k_{i,t-1}di - K_t = 0 \text{ for all } t.$$

This representation naturally nests our motivational example of the Krusell and Smith economy, where assets $a_{i,t-1}$ serve as the individual endogenous state variable $z_{i,t-1}$, and vector $x_{i,t}$ includes all individual choices and appropriate Lagrange multipliers except for $k_{i,t}$. It also includes a broad class of portfolio problems. In the appendix we show how to map small open economy models and models with different types of risky technologies into this representation.

We now write this system recursively. As the examination of the Krusell and Smith example shows, we need to be careful about measurability of different variables. Note, in particular, that $a_{i,t-1}$, $k_{i,t-1}$ and R_t^f in that example are all measurable with respect to period $t-1$ information set, while other variables are measurable with respect to period t information. To capture these different measurability conditions, we include both "previous period" and "current period" realization of shocks in individual and aggregate states. Thus, our distribution Ω will be a measure over a triple (z, θ, θ_-), or $(z, \boldsymbol{\theta})$ for short, where θ_- and θ correspond to the "previous" and "current" period respectively. The aggregate state is $Z = [\Theta, \Theta_-, \Omega]^{\mathrm{T}}$. The recursive representation consists of policy functions $\widetilde{x}(z, \boldsymbol{\theta}, Z)$, $\widetilde{k}(z, \boldsymbol{\theta}, Z)$, $\widetilde{X}(Z)$ that satisfy

$$F\left(z, \widetilde{x}, \mathbb{E}\widetilde{x}, \widetilde{X}, \widetilde{R}^x\widetilde{k}, \boldsymbol{\theta}\right) = 0 \text{ for all } (z, \boldsymbol{\theta}, Z), \tag{48}$$

$$\mathbb{E}\left[\mathsf{S}\widetilde{x}\widetilde{R}^x|\theta\_, \Theta\_, \Omega\right] = 0 \text{ for all } (z, \theta\_, Z), \tag{49}$$

$$G\left(\int \widetilde{x}d\Omega, \widetilde{X}, \widetilde{\Theta}\right) = 0 \text{ for all } Z, \tag{50}$$

$$\widetilde{R}^x(Z) = \mathsf{R}\widetilde{X}(Z), \qquad \int \widetilde{k}d\Omega = \mathsf{K}\widetilde{X} \text{ for all } Z, \tag{51}$$

$$\mathsf{T}\widetilde{X}(Z) \text{ and } \widetilde{k}(z, \boldsymbol{\theta}, Z) \text{ are independent of } \Theta \text{ and } (\theta, \Theta) \text{ for all } Z, \tag{52}$$

and the LoM for the distribution,

$$\widetilde{\Omega}(Z)\left\langle z', \boldsymbol{\theta}'\right\rangle = \int\int \iota\left(\widetilde{z}(z, \boldsymbol{\theta}, Z) \le z'\right)\iota(\rho_\theta\theta + \epsilon \le \theta')\iota\left(\theta \le \theta'_-\right)\mu(\epsilon)\, d\epsilon d\Omega\left\langle z, \boldsymbol{\theta}\right\rangle \text{ for all } Z, \tag{53}$$

Equations (48), (50) and (53) as the exact analogue of (10), (11) and (12) from Section 3. Equation (49) is the state-space representation of (46). It will be convenient to keep it separately from the rest of the optimality conditions in F as this is the equation that will be used to pin down limiting portfolios of agents and the only equation that will require a second order expansion. Equations (51) and (52) are various selections and measurability conditions and R, K and T are some selection matrices.

Much of our analysis from Section 3.2 proceeds with minimal changes. The sequence of directions $\{\hat{Z}_t\}_t$ is defined as in that section but now it is initialized by $\hat{Z}_0 = [1, 0, \mathbf{0}]^{\mathrm{T}}$. Construction of $\{\overline{X}_{Z,t}\}_t$ and Lemma 1(FO) are unchanged. Similarly, integrals in definition of operators $\mathcal{L}$ and $\mathcal{I}$ are now with respect to $d\boldsymbol{\theta}$ rather than $d\theta$. Portfolio problems introduce changes in the analogues of Lemmas 3(FO) and 4(FO) that we discuss in more details next.

Let $\overline{R}_{Z,0} = \mathsf{R}\overline{X}_{Z,0}$ and $\overline{R}_{\sigma\sigma,0} = \mathsf{R}\overline{X}_{\sigma\sigma,0}$. Note that $\overline{R}_{\sigma\sigma,0}$ is a second order order term but, fortunately, it can be found without having to solve for the whole second order approximation. Economically, $\overline{R}_{Z,0}$ is the response of the excess returns of risky assets to aggregate shocks, while $\overline{R}_{\sigma\sigma,0}$ are asset risk premia. Let $\mathfrak{S}(\overline{R}_{Z,0})$ be a mapping defined as

$$\mathfrak{S}\left(\overline{R}_{Z,0}\right) = \frac{1}{(\overline{R}_{Z,0})^2 var\left(\mathcal{E}\right)}.$$

When the portfolio of risky assets is uni-dimensional, as the convention that we use in body of this section, $(\overline{R}_{Z,0})^2 var\left(\mathcal{E}\right)$ is the conditional variance of the returns of the risky assets and $\mathfrak{S}(\overline{R}_{Z,0})$ is the reciprocal of this variance. When k is multi-dimensional, mapping $\mathfrak{S}(\overline{R}_{Z,0})$ becomes the inverse of the covariance matrix of returns of risky assets, which is one of the central objects in the classical portfolio theory.

We can now state the analogue of Lemmas 3(FO) for portfolio problems:

Lemma 3(PF). For $t > 0$, $\overline{x}_{Z,t}\left(z, \boldsymbol{\theta}\right)$ satisfies (15). $\overline{x}_{Z,0}\left(z, \boldsymbol{\theta}\right)$ satisfies

$$\overline{x}_{Z,0}\left(z, \boldsymbol{\theta}\right) = \sum_{s=0}^{\infty} \mathsf{x}_s\left(z, \theta\right)\overline{X}_{Z,s} + \mathsf{x}^k(z,\theta)\overline{k}\left(z, \theta\_\right)\overline{R}_{Z,0},$$

where

$$\mathsf{x}^k(z,\theta) = -\left(\mathsf{F}_x(z,\theta) + \mathsf{F}_{x^e}(z,\theta)\mathbb{E}\left[\overline{x}_z|z,\theta\right]\mathsf{P}\right)^{-1}\mathsf{F}_k(z,\theta)$$

$$\overline{k}(z,\theta\_) = \mathsf{k}_{\sigma\sigma}(z,\theta\_)\mathfrak{S}\left(\overline{R}_{Z,0}\right)\overline{R}_{\sigma\sigma,0} + \mathfrak{S}\left(\overline{R}_{Z,0}\right)var\left(\mathcal{E}\right)\sum_{s=0}^{\infty}\mathsf{k}_s(z,\theta\_)\overline{X}_{Z,s}\overline{R}_{Z,0} \qquad (54)$$

and explicit expressions for $\mathsf{k}_{\sigma\sigma}$ and k_s are given in the appendix.

Lemma 3(PF) shows that the relationship between $\overline{x}_{Z,t}$ and $\{\overline{X}_{Z,s}\}_s$ is the same as in our baseline case in all periods except $t = 0$. In period $t = 0$, portfolio choices introduce an additional term, $\mathsf{x}^k(z,\theta)\overline{k}(z,\theta\_)\overline{R}_{Z,0}$, in the equation describing individual responses $\overline{x}_{Z,0}$. This term has a natural economic interpretation. $\overline{k}(z,\theta\_)$ is the optimal limiting portfolio of agent

$(z, \theta\_)$ and, thus, $\overline{k}(z, \theta\_)\overline{R}_{Z,0}$ is the realized return to that portfolio due to aggregate shocks. Multiplying by $\mathsf{x}^b(z, \theta)$ which has the same structure as equation (16) adjusts the presence of portfolio to all individual choice variables. Equation (54) provides a formula for the optimal limiting portfolio $\overline{k}(z, \theta\_)$. A reader familiar with portfolio theory can easily recognize many elements in this formula.[12] Term $\mathfrak{S}\left(\overline{R}_{Z,0}\right)\overline{R}_{\sigma\sigma,0}$ is asset risk premia multiplied by the inverse of the convariance matrix. This term captures the classical risk-return trade-off in portfolio theory. $\overline{X}_{Z,s}\overline{R}_{Z,0}$ captures how excess returns co-vary with aggregate variables at different time horizons s, and $\mathsf{k}_s(z, \theta\_)\overline{X}_{Z,s}\overline{R}_{Z,0}$ captures how that aggregate covariance translates into individual-level covariances. Thus, $\mathfrak{S}\left(\overline{R}_{Z,0}\right)\mathsf{k}_s(z, \theta\_)\overline{X}_{Z,s}\overline{R}_{Z,0}var\left(\mathcal{E}\right)$ captures how portfolios hedge individual risks. As Lemma 3(PF) shows, weights $\mathsf{k}_{\sigma\sigma}$ and k_s in formula (54) are known from the zeroth order.

We now turn to the portfolio version of Lemma 4(FO). Since $\boldsymbol{\theta}$ is now bi-dimensional, the analogue of $\frac{d}{d\theta}\hat{\Omega}_t$ becomes $\frac{d}{d\boldsymbol{\theta}}\hat{\Omega}_t := \frac{d}{d\theta}\frac{d}{d\theta\_}\hat{\Omega}_t$.

Lemma 4(PF). For $t > 1$, $\frac{d}{d\boldsymbol{\theta}}\hat{\Omega}_t$ satisfies (18). $\frac{d}{d\boldsymbol{\theta}}\hat{\Omega}_1$ satisfies

$$\frac{d}{d\boldsymbol{\theta}}\hat{\Omega}_1 = -\sum_{s=0}^{\infty}\mathsf{a}_s\overline{X}_{Z,s} - \mathsf{w}_{\sigma\sigma}\mathfrak{S}\left(\overline{R}_{Z,0}\right)\overline{R}_{\sigma\sigma,0} - \mathfrak{S}\left(\overline{R}_{Z,0}\right)var\left(\mathcal{E}\right)\sum_{s=0}^{\infty}\mathsf{w}_s\overline{X}_{Z,s}\overline{R}_{Z,0},$$

where explicit expressions for w_s and $\mathsf{w}_{\sigma\sigma}$ are given in the appendix. $\frac{d}{d\boldsymbol{\theta}}\hat{\Omega}_0 = \mathbf{0}$.

As with Lemma 3(PF), portfolio problem introduces additional adjustment terms only in one period. These adjustment terms aggregates portfolios of individual agents and has the same structure and intuition as expression for $\overline{k}(z, \theta\_)$ in equation (54).

We can use these results to characterize the first order approximation of our economy. Differentiate equations (50), (51) and (52) in direction $\hat{Z}_t$ and simplify those derivative using Lemmas 3(PF) and 4(PF) to obtain

Proposition 1(PF). $\left\{\overline{X}_{Z,t}\right\}_t$ and $\overline{R}_{\sigma\sigma,0}$ are the solution to

$$\begin{aligned} 0 = & \mathsf{G}_x\sum_{s=0}^{\infty}\mathsf{J}_{t,s}\overline{X}_{Z,s} + \mathsf{G}_X\overline{X}_{Z,t} + \mathsf{G}_\Theta\rho_\Theta^t \\ & + \mathsf{G}_x\left(\sum_{s=0}^{\infty}\mathsf{J}^w_{t,s}\mathfrak{S}\left(\overline{R}_{Z,0}\right)\overline{R}_{Z,0}\overline{X}_{Z,s}var\left(\mathcal{E}\right) + \mathsf{J}^w_{\sigma\sigma,t}\mathfrak{S}\left(\overline{R}_{Z,0}\right)\overline{R}_{\sigma\sigma,0}\right), \end{aligned} \tag{55}$$

$$\mathsf{W}_{\sigma\sigma}\mathfrak{S}\left(\overline{R}_{Z,0}\right)\overline{R}_{\sigma\sigma,0} + \mathfrak{S}\left(\overline{R}_{Z,0}\right)\sum_{s=0}^{\infty}\mathsf{W}_s\overline{R}_{Z,0}\overline{X}_{Z,s}var\left(\mathcal{E}\right) = \mathsf{K}\overline{X}, \tag{56}$$

[12] See, e.g., Viceira (2001) who derives similar equation using different techniques. Aparisi de Lannoy et al. (2022) derive optimal portfolio formulas like (54) using perturbational techniques.

$$\mathsf{T}\overline{X}_{Z,0} = 0, \qquad \mathsf{R}\overline{X}_{Z,t} = 0 \text{ for } t \geq 1, \tag{57}$$

$$\overline{R}_{Z,0} = \mathsf{R}\overline{X}_{Z,0}, \tag{58}$$

with $\mathsf{J}^w_{\sigma\sigma,t} = \mathcal{I} \cdot \mathcal{L}^{t-1} \cdot \mathsf{w}_{\sigma\sigma}$, $\mathsf{J}^w_{t,s} = \mathcal{I} \cdot \mathcal{L}^{t-1} \cdot \mathsf{w}_s$.

The first line of equation (55) has exactly the same form as equation (19), which characterized the first order approximation to our baseline economy. The second line in (55) captures additional effects from portfolio choices. These portfolio choices must, in turn, also satisfy the asset market clearing condition, equation (56), which also allows us to pin down risk premia $\overline{R}_{\sigma\sigma,0}$ without doing the full second order expansion.

The system of equations (55), (56), (57) and (58) is non-linear in $\{\overline{X}_{Z,t}\}_t$ and $\overline{R}_{\sigma\sigma,0}$ due to the nonlinear operator $\mathfrak{S}(\overline{R}_{Z,0})$. It has, however, a mathematical structure that can be utilized to solve it quickly numerically: holding $\overline{R}_{Z,0}$ fixed, (55), (56) and (57) form a linear system in $\{\overline{X}_{Z,t}\}_t$ and $\overline{R}_{\sigma\sigma,0}$. This observation provides a natural algorithm for solving the system of equations in Proposition 1(PF): guess $\overline{R}_{Z,0}$ and solve the linear system (55) - (57) for $\{\overline{X}_{Z,t}\}_t$ and $\overline{R}_{\sigma\sigma,0}$; verify if the initial guess satisfies equation (58); if necessary, adjust the guess for $\overline{R}_{Z,0}$ and iterate until the fixed point is found. This simple iterative procedure allows one to simultaneously solve for the limiting portfolios of all agents, the first order approximations, and the second order risk premium.

5 Numerical Results

In this section, we apply our algorithm to calibrated versions of the Krusell and Smith (1998) model. First, we use the calibrated model to report diagnostics such as accuracy and speed and compare them to alternative methods. Second, we use extensions of the Krussel-Smith model to study several applications that illustrate the usefulness of going beyond first-order approximations. These applications include analysis of stabilization policies, aggregate and distributional effects of fluctuations in macroeconomic uncertainty, and properties of household portfolios.

5.1 Baseline Model and Calibration

Our baseline model extends the Krusell-Smith framework of Section 2 to include capital adjustment costs. Investment in capital is subject to convex adjustment, which are assumed to take the form:

$$\phi(I_t, K_{t-1}) = \frac{\phi}{2}\left(\frac{I_t}{K_{t-1}} - \delta\right)^2 K_{t-1}.$$

We assume that capital is held by a continuum of perfectly competitive mutual funds with a price per unit capital given by P_t. This intermediary mutual funds holds the individual saving $a_{i,t}$ as deposit, representing the market value. The aggregate saving is invested in capital by the fund:

$$P_t K_t = \int a_{i,t} di$$

The shares held in the mutual funds have a financial return, which is the sum of (i) capital gains net of depreciation and investment and (i) dividends given by the marginal product of capital $r^k = \partial Y / \partial K$, net of capital adjustment costs:

$$R_t = \frac{P_t \left(1 - \delta + \mathcal{I}_t\right) + r_t^k - \phi(I_t, K_{t-1})}{P_{t-1}}$$

The optimal investment choice of the mutual funds implies that the price per unit capital is exactly equal to the marginal cost of investment, namely

$$P_t = 1 + \phi \frac{I_t}{K_{t-1}}.$$

Note that P_t also represents the marginal gain of capital inside in the firm, i.e. the Q-ratio as in the Tobin's Q theory, c.f. Hayashi (1982).

Calibration To calibrate our model, we set the period length to one quarter. The parameter α is set to 0.36 to target the capital share of income. We use an isoelastic period utility $U(c) = \frac{c^{1-\gamma}}{1-\gamma}$ and vary the risk aversion parameter γ between 2 and 7. For each choice for risk aversion, we set adjustment cost parameter ϕ to match a 3% standard deviation of unleveraged quarterly returns to equity. Unless otherwise specified the plots in this section are for risk aversion set to 5. For the parameters governing the aggregate and idiosyncratic labor productivity in (3) and (4), we choose values used by Auclert et al. (2021). We discretize the idiosyncratic labor productivity process using the Rouwenhorst method with a grid of $N_\epsilon = 7$ grid points. Regarding the discretization of the policy functions, we use a set of $N_z = 60$ quadratic spline basis function to approximate the policy $\bar{x}(z)$ with unequally spaced knot points. The distribution over states $\bar{\omega}$ is approximated with unequally spaced grid with $I_z = 1000$ points along the asset dimension. The performance of our algorithm is not sensitive to variations in these two values N_z and I_z. The calibration parameters are summarized in Table 1.

Simulations We begin by applying Lemma 1(SO) and simulate $X_t\left(\mathcal{E}^t\right)$ for different paths of $\mathcal{E}^t$. The first path is $\mathcal{E}^t = (1, 0, 0, \ldots)$ that captures a one-time, one standard deviation

Table 1: CALIBRATION OF THE KRUSELL-SMITH ECONOMY

Parameter	Description	Value
α	Capital share	0.36
β	Discount factor	0.983
γ	Risk aversion	$[2, 7]$
δ	Depreciation rate of capital	1.77%
ϕ	Adjustment cost of capital	$[32, 125]$
ρ_ϵ	Idiosyncratic mean reversion	0.966
$\sigma_\epsilon/\sqrt{1-\rho_\epsilon^2}$	Cross-sectional std of log earnings	0.503
ρ_Θ	Aggregate mean reversion	0.80
σ_Θ	Std of Aggregate TFP shocks	0.014
N_ϵ	Points in Markov chain for ϵ	7
N_z	Grid points for the policy rule $\bar{x}^i(z)$	60
I_z	Grid points for the distribution $\bar{\omega}_i$	1000
T	Time horizon (in quarters) for IRF	400

positive shock to TFP. In the left panel of Figure 1, we plot response of aggregate capital and decompose the response to show the contributions from the zeroth $\overline{X}$; first $\overline{X}+\sum_{s=0}^{t}\overline{X}_{Z,t-s}\mathcal{E}_s$; second-order interaction terms $\overline{X}+\sum_{s=0}^{t}\overline{X}_{Z,t-s}\mathcal{E}_s+\frac{1}{2}\left(\sum_{s=0}^{t}\sum_{m=0}^{t}\overline{X}_{ZZ,t-s,t-m}\mathcal{E}_s\mathcal{E}_m\right)$; and the second-order risk term $\overline{X}+\sum_{s=0}^{t}\overline{X}_{Z,t-s}\mathcal{E}_s+\frac{1}{2}\left(\sum_{s=0}^{t}\sum_{m=0}^{t}\overline{X}_{ZZ,t-s,t-m}\mathcal{E}_s\mathcal{E}_m+X_{\sigma\sigma,t}\right)$. Not surprisingly, we see in the figure that the entire second-order response to the one-time, one standard deviation positive shock to TFP this path is due to the risk $X_{\sigma\sigma,t}$ term. To activate the second-order interaction terms, we simulate $\mathcal{E}^t=(1,1,0,\ldots)$, which contains two consecutive one standard deviation shocks to TFP. In right panel of Figure 1, we see that the interaction terms (green line) now capture a non-trivial part of the second order response.

More generally nonlinearities in the model manifest by making impulse responses history dependent. In particular, a response of aggregates at date $t+k$ to a TFP shock that occurs at date t depends on the history of shocks $\{\mathcal{E}_j\}_{j=0}^{t}$. To see how much previous shocks matter, to second order, we can compute

$$\mathbb{E}\left[X_{t+k}|\mathcal{E}_t,\mathcal{E}_{t-j}\right]-\mathbb{E}\left[X_{t+k}|\mathcal{E}_t,\mathcal{E}_{t-j}=0\right]$$

which is the change in the impulse response resulting from a shock j periods in the past. Applying Lemma 1(SO) it is straightforward to see that this term is exactly given by $\bar{X}_{ZZ,k,k+j}\mathcal{E}_t\mathcal{E}_{t-j}$. In Figure 2 we plot these changes in the impulse responses for various values of j for aggregate capital. In the figure we normalize by a 1 s.d. impulse response so these should be interpreted as the percentage change in the impulse response induced by a shock in the previous period. A sequence of recent positive TFP shocks will result in amplified business cycles: the increase (fall) in output will be larger for a positive (negative) TFP shock. For example, if the economy

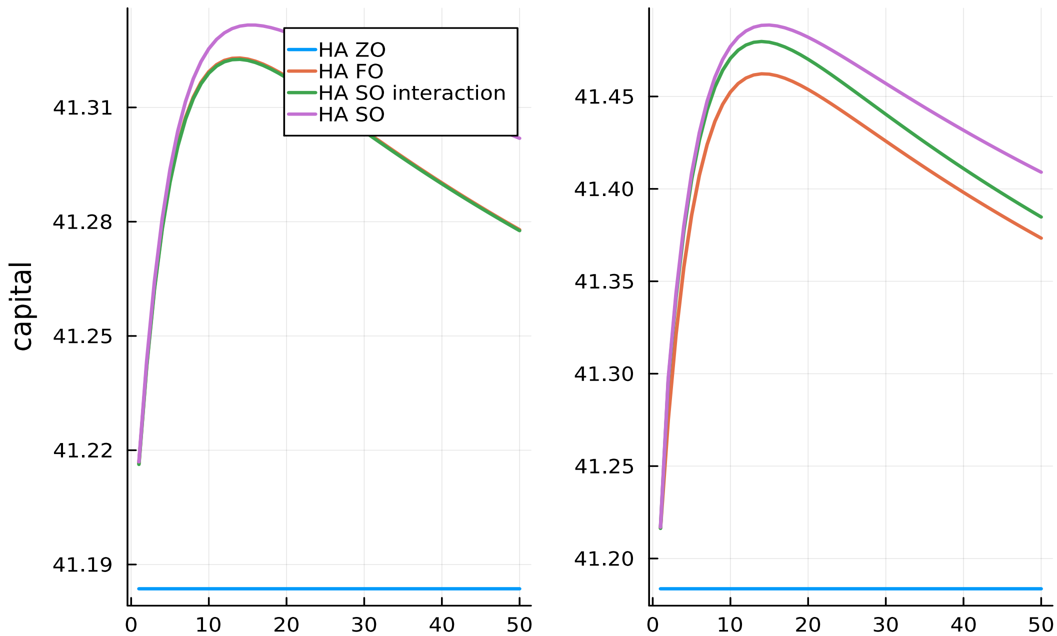

Figure 1: Simulated for K_t. The line labeled “ZO” is the zeroth order approximation; labeled “FO” includes first order terms; labeled “SO interaction” adds the interaction $\mathcal{E}_s\mathcal{E}_m$ terms, and and labeled “SO” is full second order response. The left panel are simulations for $\mathcal{E}^t = (1, 0, 0, \ldots)$ and right panel are simulations for $\mathcal{E}^t = (1, 1, 0, \ldots)$.

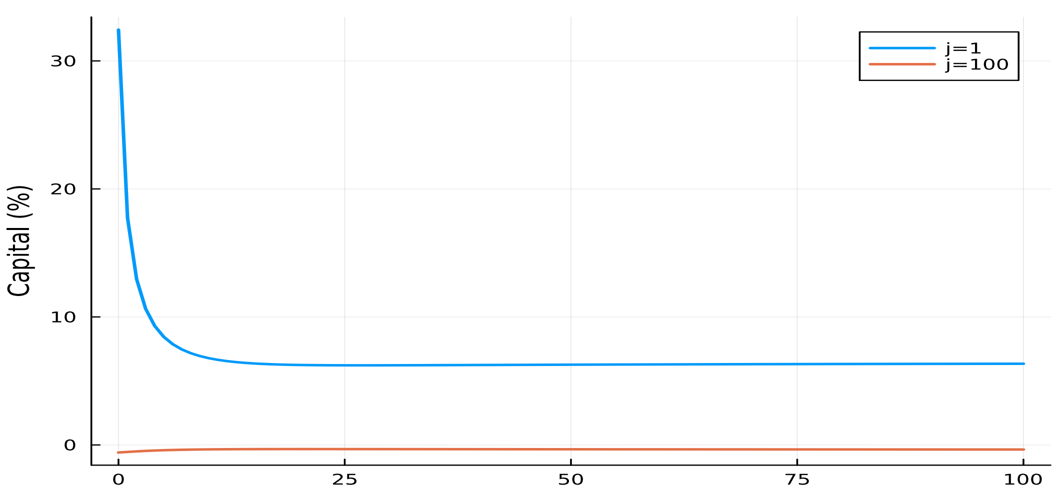

Figure 2: Non-linearities in Impulse Responses. The two lines plot the sequences $\left\{\bar{X}_{ZZ,k,k+j}\mathcal{E}_t\mathcal{E}_{t-j.}\right\}_t$ for two values of $j \in \{1, 100\}$

had received a positive TFP shock in the previous period the response of capital could be up to 30% large in initial periods. This analysis shows that nonlinearities ignored by first-order approximations are salient in the basic Krusell-Smith economy.

5.2 Diagnostics

Accuracy In this section, we study diagnostics related to accuracy and speed. Measuring the numerical errors made in the simulation of heterogeneous agents models with aggregate shocks is notoriously difficult since there are no "reference point" to compare with. As such, we test the accuracy of our method by studying the response to a one-time, one standard deviation positive shock to TFP. All approaches give an approximation $\hat{X}_t$ to the path of all aggregates in response to this shock. Given these approximations, these sequences of aggregate variables are the only thing needed to compute the dynamics of the rest of the model in a fully non-linear way: first using the sequence of prices, the optimal policies $\tilde{x}_t(z)$ of households are computed using standard methods. Second, given these policies, we compute the law of motion of the distribution $\tilde{\omega}_t(z)$, and, third, aggregating up, we compute the law of motion $\tilde{X}_t$. We can then measure the accuracy by comparing $\hat{X}_t$ with the non-linear counterpart $\tilde{X}_t$.

In Figure 3, we plot the % error in the capital stock $\frac{\hat{K}_t - \tilde{K}_t}{\tilde{K}_t}$. For comparison purposes we show errors using 3 different methodologies: our benchmark approach described in Section 3.2, a modification of our approach which approximates the law of motion of the distribution with a histogram as described in Section 3.4, and the Sequence Space Jacobian approach of ABRS As would be anticipated by Figure 3 all three approaches have roughly the same error to first order, with the maximal error being on the order of 0.04% of the capital stock. At higher orders, our baseline approach has errors which remain very small over time, while the histogram approach generates errors which become the same order of magnitude as those of the first order approximation.

Speed Next, we turn to the computational speed of our approach which we break down in Table 2 by each stage of the algorithm. The timings for the first order approximation are reported in the first two columns of the table.[13] All told, once the steady state has been computed, our algorithm takes 0.5 seconds to solve for the $\overline{X}_{Z,t}$ terms with roughly equal time spent in all 4 of the main steps. As Lemma 1(FO) highlights, $\overline{X}_{Z,t}$ are all that is needed to simulate the path of aggregates and to compute ergodic moments from the first order approximation. The other first order terms, $\bar{x}_{Z,t}$ and $\bar{\Omega}_{Z,t}$, are required for the second order

[13] All numbers are reported using a 20 core M1 ultra mac studio.

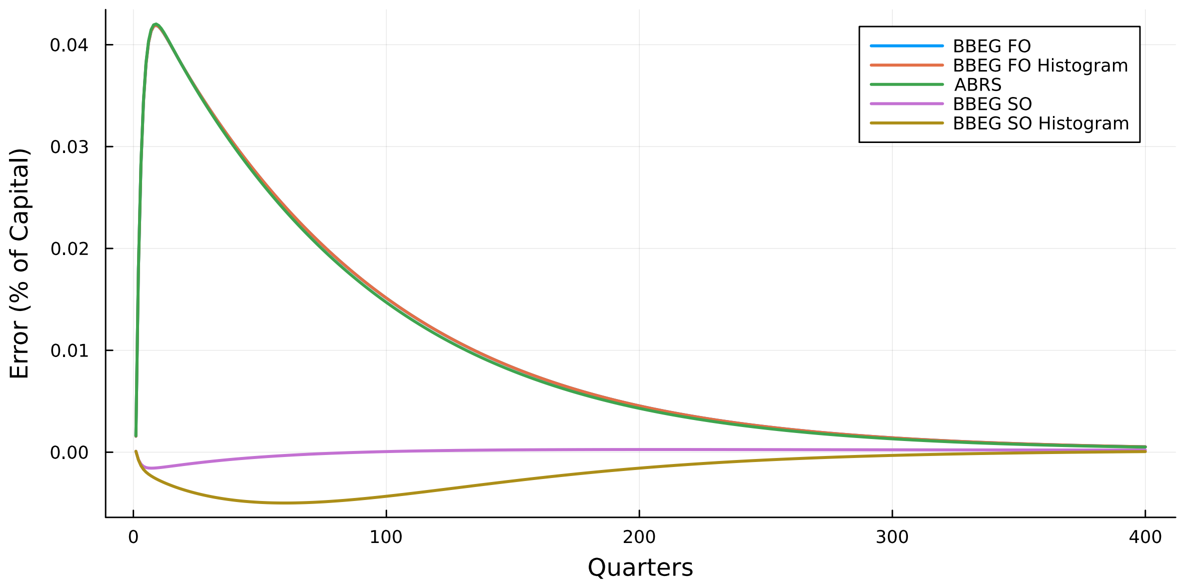

Figure 3: Nonlinear error $\frac{\hat{K}_t - \tilde{K}_t}{\tilde{K}_t}$ for the three first order approximations and the second order approximations.

Table 2: COMPUTATIONAL SPEED: FIRST AND SECOND ORDER

First Order		Second Order		
Step	Time	Step	Time (ZZ)	Time$(\sigma\sigma)$
		Additional First Order Terms	0.70s	
Lemma 3(FO) Terms	0.07s	Lemma 3(SO) Terms	0.64s	0.05s
Lemma 4(FO) Terms	0.13s	Lemma 4(SO) Terms	0.21s	0.45s
Corollary 2(FO) Terms	0.17s	Corollary 2(SO) Terms	0.07s	0.05s
Proposition 1(FO) Terms	0.13s	Proposition 1(SO) Terms	0.19s	0.28s
Total	0.5s		1.81s	0.83s
ABRS	1.51s			

approximation and take an additional 0.7 seconds to compute.

We compare this to our own implementation of the Sequence Space Jacobian of ABRS which takes approximately 1.5 seconds to compute the equivalent on the $\overline{X}_{Z,t}$. Of that time, approximately 1.35 seconds are spent on the backward and forwards iteration steps which are the equivalent of the terms computed in Lemma 3(FO) and 4(FO). As we detailed in Section 3.2.1, once the steady state is solved for our implementation of the algorithm requires only sparse linear operations which can be done quickly and efficiently independently of how the steady state is solved for. The methodology of ABRS generally relies on numerical differentiation of global transition code, and is therefore limited by the efficiency of that global code. Moreover, very careful attention has to be paid to those numeric derivatives in order to ensure that they are accurate, see appendix C.1 of Auclert et al. for details. These numerical issues would be amplified with a second order approximation as numerical second derivatives are more prone to numerical error. By giving explicit expressions for these second derivatives in terms of derivatives of F and G we sidestep these issues. The addition time to compute the second order approximation is broken out in the last two columns of Table 2. As highlighted in Section 3.3 there are two additional types of terms in the second order approximation: the curvature terms, $\bar{X}_{ZZ,t,k}$, and risk correction terms $\bar{X}_{\sigma\sigma,t}$. As they follow the same mathematical structure, we break out the computational time separately for both types. The curvature terms take 1.11 seconds to compute[14] while the risk adjustment terms take 0.83 seconds. The vast majority of the computational time for the curvature terms is spent on Lemma 3(SO) and Proposition 1(SO) which is a result of a large number of quadratic forms required to compute the $\mathsf{x}_{t,k}(z,\theta)$ and $\mathsf{G}_{\Theta,t,k}$ terms. All combined, computing the second order approximation requires an additional 2 to 3 seconds relative to the first order approximation.

5.3 Applications

In this section we study three applications that highlight the usefulness of higher order approximations in heterogeneous agent models.

5.3.1 Welfare from stabilization policy

In addition to studying non-linearities, second order approximations can be used to evaluate the welfare effects of fiscal stabilization policies. We extend the model to include fiscal policies

[14] Here we report only the time required to compute that $\bar{X}_{ZZ,t,t}$ terms. We do this for two reasons. Firstly, for most ergodic moments only the $\bar{X}_{ZZ,t,t}$ are required. Secondly, computing the addition $\bar{X}_{ZZ,t,t+i}$ terms are trivially parallelizable for each i so, with enough processors, computing all the $\bar{X}_{ZZ,t,k}$ terms would not require any additional time.

that vary over the business cycle in form of a time varying labor-tax

$$\tau_t = \overline{\tau} + \tau_\Theta \Theta_t,$$

which is returned lump-sum to the households. Households with labor productivity $\theta_{i,t}$ will receive transfers T_t and $(1 - \tau_t) W_t \exp(\theta_{i,t})$ in after-tax labor income in the current period.

In the baseline, we assumed exogenous labor; therefore, the optimal choice of the intercept $\overline{\tau}$ is not interesting. However, a planner who cares about redistribution has a non-trivial tradeoff with the choice of τ_Θ. We are interested in calculating welfare measures for a given tax policy and the optimal tax cyclicality.

The question of optimal cyclicality is non-trivial with incomplete markets and can be meaningfully answered only with a minimum of second-order expansion. With complete markets (representative agent), Ricardian equivalence holds and allocations do not depend on τ_Θ. With incomplete markets and borrowing constraints, Ricardian equivalence fails, and a planner would face a tradeoff between redistribution across agents and insurance across states. However, to the first-order approximation, certainty equivalence holds, and insurance concerns are absent, so there is no meaningful answer to optimal τ_Θ. We therefore use second-order expansions from Section 3.3 to find the optimal cyclicality.

For a given τ_Θ, define utilitarian welfare as $\mathcal{W}(\Omega, \Theta; \tau_\Theta) = \int V(\theta, k, \Theta, \Omega; \tau_\Theta)\, d\Omega$ where V is the value of an individual with who starts with idiosyncratic states (θ, k) when the aggregate state is (Θ, Ω) under policy indexed by τ_Θ. Observe that if we extend x and X to include V and $\mathcal{W}$, respectively, and add the Bellman equation that solves the value function V to the mapping F and the definition of welfare $\mathcal{W}$ to the mapping G, our framework computes welfare autmatically. To find the τ_Θ that maximizes ergodic welfare, we can use Lemma 1(SO) and take the expectation conditional on time 0 information to find

$$\mathbb{E}_0 [X_t] = \bar{X} + \frac{1}{2} \left(\sum_{s=0}^{t} \overline{X}_{ZZ,t-s,t-s} \sigma_{\mathcal{E}}^2 + \overline{X}_{\sigma\sigma,t} \right) + O\left(\underline{\mathcal{E}}^3\right),$$

where $\sigma_{\mathcal{E}}$ standard deviation of the exogenous shock $\mathcal{E}_t$. Taking the limit as $t \to \infty$ gives the long run ergodic means of the aggregate variables X as

$$\mathbb{E}[X] = \bar{X} + \frac{1}{2} \left(\sum_{s=0}^{\infty} \overline{X}_{ZZ,s,s} \sigma_{\mathcal{E}}^2 + \overline{X}_{\sigma\sigma,\infty} \right) + O\left(\underline{\mathcal{E}}^3\right)$$

where$\overline{X}_{\sigma\sigma,\infty} = \lim_{t\to\infty} \overline{X}_{\sigma\sigma,t}$. As mentioned before X contains a measure of welfare, this gives a quick and efficient algorithm for computing ergodic welfare as a function of a policy parameter since for given value of the policy parameter it is straightforward to compute the second-order derivatives $\bar{X}_{ZZ,t,t}$ and $\bar{X}_{\sigma\sigma,t}$.[15]

We illustrate this approach by computing optimal ergodic welfare as a function of the tax parameter τ_Θ and plotting ergodic welfare (computed as consumption equivalent relative to steady state) in Figure 4. For risk aversion set to 2, we see that relative to a laissez-faire policy, $\tau_\Theta = 0$, making the tax policy more countercyclical initially raises welfare with a distinct maximum achieved at $\tau_\Theta = -3.1$ which amounts to raising taxes by 3.1 percentage points for every percentage point decrease in TFP. In fact, roughly 22% of the welfare losses from business cycles in this model can be ameliorated by this tax policy. In Table 3, column τ^*_Θ, we report the optimal cyclicalty for other values of risk aversion. We find that higher the risk aversion, lower the cyclicality. Because gains costs of insurance increase with risk aversion, higher values of risk aversion are associated with lower values for tax cyclicality.

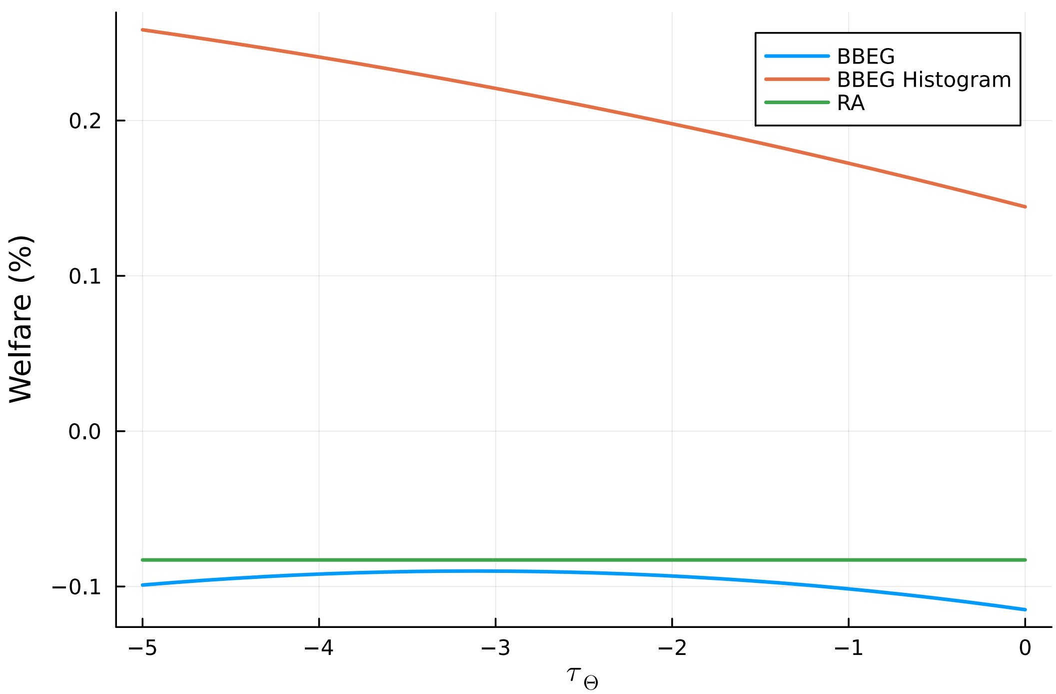

Figure 4: Welfare for various values of τ_Θ when risk aversion is set to 2.

This application also serves as a valuable tool for illustrating the shortcomings associated with employing the histogram technique. In Section 3.4, we emphasized the consequences of naively extending the histogram approach, which may overlook specific second-order terms.

[15]In principle ergodic welfare will also depend on the change in the steady state $\bar{X}$, in our example the steady state will not depend on the policy parameter.

These second-order terms become particularly crucial when calculating welfare derived from stabilization policy, which is inherently a second-order object. In columns $\frac{\mathcal{W}^{\text{hist}}\left(\tau_\Theta^*\right)}{\mathcal{W}\left(\tau_\Theta^*\right)}$ and $\frac{\tau_\Theta^{*,\text{hist}}}{\tau_\Theta^*}$ of Table 3, the ergodic welfare and optimal cyclicality under the histogram method are presented. It is evident that both the magnitude of welfare corresponding to a particular τ_Θ and the gradient of welfare in relation to τ_Θ are inaccurate when utilizing the histogram approach. As previously mentioned, the histogram method leads to discrepancies in the second derivatives that influence the welfare criterion. Furthermore, this analysis accentuates the significance of our method for accurately capturing the law of motion of the distribution.

Table 3: OPTIMAL CYLICALITY τ_Θ^*

risk aversion	τ_Θ^*	$\frac{\mathcal{W}^{\text{hist}}\left(\tau_\Theta^*\right)}{\mathcal{W}\left(\tau_\Theta^*\right)}$	$\frac{\tau_\Theta^{*,\text{hist}}}{\tau_\Theta^*}$
2	-3.10	-348%	161%
3	-1.90	-230%	209%
4	-1.03	-226%	167%
5	-0.69	-217%	125%
7	-0.52	-187%	67%

Notes: Optimal τ_Θ as we vary the risk aversion parameter. The $\mathcal{W}^{\text{hist}}\left(\tau_\Theta^*\right)$ uses the histogram method to compute the welfare and $\tau_\Theta^{*,\text{hist}}$ is the optimal policy using $\mathcal{W}^{\text{hist}}\left(\tau_\Theta\right)$ as the measure of welfare

5.3.2 Stochastic Volatility

Figure 5 plots the time-series for the CBOE Volatility Index (VIX) over the last twenty years. We observe large fluctuations, with rapid increases of about 4-5 times the average in 2008 and during the COVID pandemic, which take a couple of years to mean revert. Interpreting the VIX as a measure of macroeconomic risk, we are interested in understanding the aggregate and distributional consequences of an increase in volatility. We will apply the techniques described in Section 4.2, with the extension of the baseline model that uses equations (36)–(37) as the new process for aggregate shocks.

Motivated by the VIX data, we focus on the impulse response to a one-time, large but transitory shock to the uncertainty of the TFP process, similar to what we saw in the recent crisis. The shock increases the standard deviation by a factor of 5 and mean reverts with a persistence of 0.75. The impulse response to this shock is defined by

$$IRF_k^{\Upsilon}\left(\mathcal{E}_{\Upsilon,t}\right) = \mathbb{E}_t\left[X_{t+k}\middle|\mathcal{E}_{\Upsilon,t}\right] - \mathbb{E}\left[X_{t+k}|\mathcal{E}_{\Upsilon,t}=0\right].$$

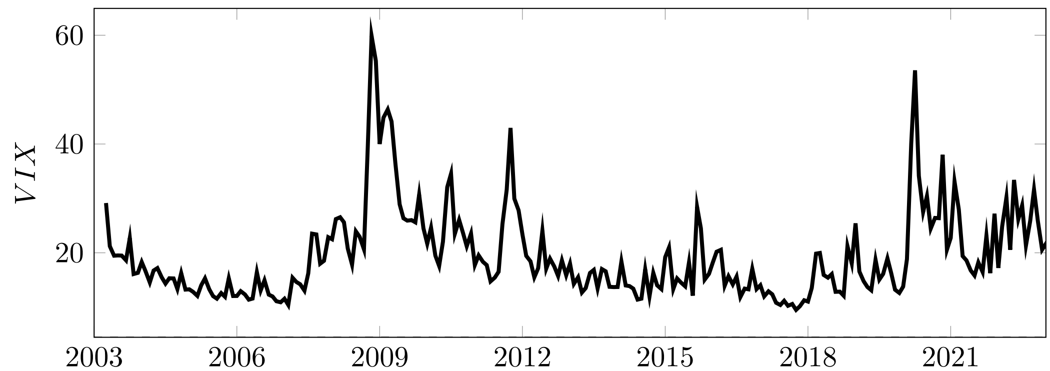

Figure 5: Time series for the CBOE Volatility Index

Applying Lemma 1(SV) we find that

$$IRF_k^{\Upsilon}\left(\mathcal{E}_{\Upsilon,t}\right) = \left(\sum_{j=0}^{k-1} \overline{X}_{ZZ,j,j} \rho_{\Upsilon}^{k-1-j} \sigma_{\Theta}^2 + \sum_{j=0}^{k} \bar{X}_{\Upsilon,j} \rho_{\Upsilon}^{k-j}\right) \mathcal{E}_{\Upsilon,t}. \tag{59}$$

In Figure 6, we illustrate the cumulative response of investment following the shock. We observe that the shock leads to a decrease in capital accumulation, resulting in a cumulative loss of approximately 6% in investment over a 5-year period. Interestingly, the response in the heterogeneous agent economy is about twice as large as that in the representative agent counterpart.

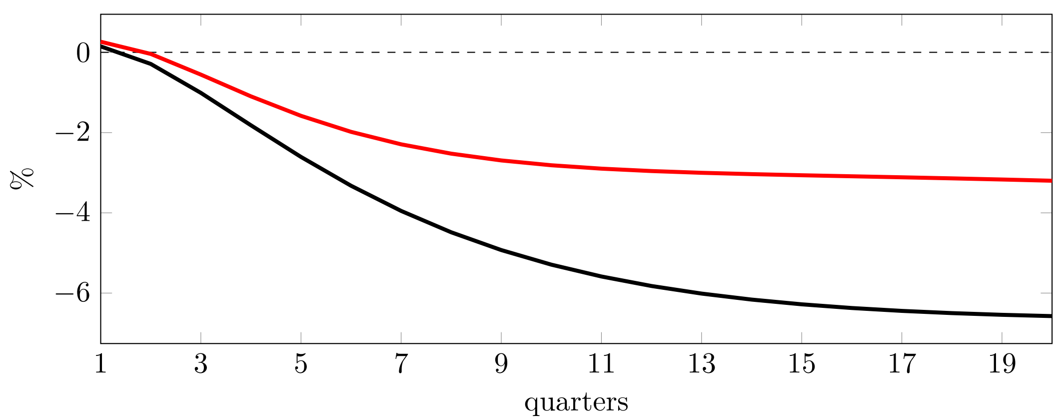

Figure 6: Cumulative change in investment after an uncertainty shock

In addition to the impact on aggregate variables, we investigate the effect of the shock on individual welfare. As in Section 5.3.1, we compute the effect on welfare using the response of individual value functions at the date of the shock and convert the magnitudes into certainty equivalents. These equivalents are measured in terms of the per-period consumption households would be willing to forgo to avoid the uncertainty shock's path. In Figure 7, we plot the

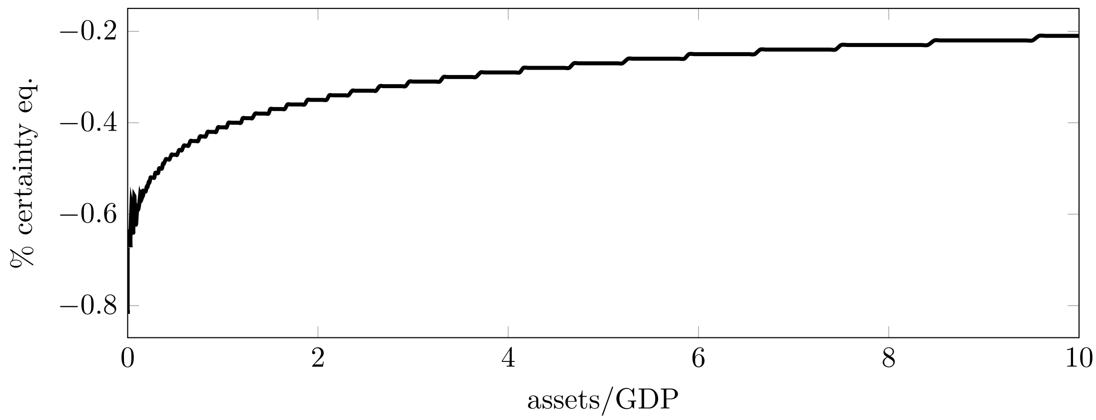

Figure 7: Distribution of per-period certainty equivalent that households forgo to avoid the one-time uncertainty shock

welfare losses by assets, normalized by per capita GDP. The average welfare loss amounts to approximately half a percentage point of per-period consumption, and these losses range from 0.81% to 0.20% across the asset distribution. The most significant welfare losses are experienced by asset-poor agents who are closer to the borrowing constraints.

5.3.3 Portfolio Choice

A prevalent empirical pattern emphasized in the literature studying household finance is the observation that the share of risky assets increases with wealth. See Yogo and Wachter (2011) who use data from the Survey of Consumer Finances. In this section, we explore the predictions of the basic Krusell-Smith framework for this particular moment. To do so, we extend the model, allowing agents to trade risk-free debt, b, which has a zero net supply, in addition to claims on capital. We impose a constraint that prevents households from short-selling capital. Subsequently, we apply the methods described in Section 4.3 to calculate the portfolio for all households.

In Figure 8, we depict the distribution of household portfolios by assets normalized by per capita GDP. The model qualitatively aligns with the observed pattern wherein poorer households hold more bonds and wealthier households hold more stocks. Households closest to the borrowing constraint are most exposed to aggregate shocks, and they optimally reduce their exposure by adjusting their portfolios towards risk-free bonds.

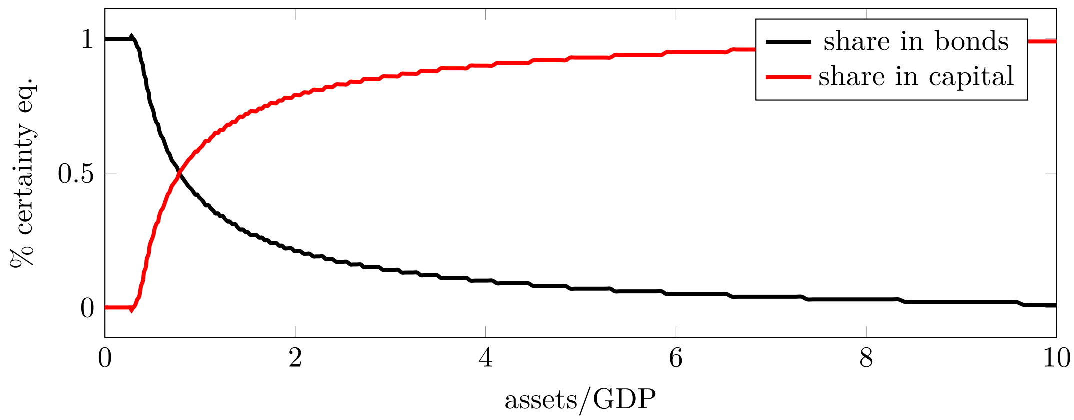

Figure 8: Distribution of household portfolios by assets

6 Conclusion

In this paper, we propose a novel perturbation technique to approximate a wide variety of stochastic heterogeneous-agent (HA) models. Our methods goes beyond the MIT shock approach found in existing literature by employing higher-order approximations. Utilizing Fréchet derivative techniques, we demonstrate that all-order approximations can be represented using analytically derived coefficients that are straightforward to implement numerically. Our approach broadens the range of research questions that can be addressed within these model classes. We showcase the practicality of our method by applying it to examine welfare implications of stabilization policies, portfolio choice, and time-varying uncertainty in a calibrated economy.

References

Achdou, Yves, Jiequn Han, Jean Michel Lasry, Pierre Louis Lions, and Ben Moll. 2020. "Income and Wealth Distribution in Macroeconomics: a continuous-time approach." *Review of Economic Studies*.

Ahn, SeHyoun, Greg Kaplan, Benjamin Moll, Thomas Winberry, and Christian Wolf. 2018. "When Inequality Matters for Macro and Macro Matters for Inequality." *NBER Macroeconomics Annual*, 32(1): 1–75.

Alvarez, Fernando, Andrea Ferrara, Erwan Gautier, Herve Le Bihan, and Francesco Lippi. 2023a. "Empirical Investigation of a Sufficient Statistic for Monetary Shocks."

Alvarez, Fernando, and Francesco Lippi. 2022. "The Analytic Theory of a Monetary Shock." *Econometrica*, 90(4): 1653–1682.

Alvarez, Fernando, Francesco Lippi, and Panagiotis Souganidis. 2023b. "Price Setting with Strategic Complementarities as a Mean Field Game."

Aparisi de Lannoy, Leo R, Anmol Bhandari, David Evans, Mikhail Golosov, and Thomas J Sargent. 2022. "Managing Public Portfolios." Working Paper 30501, National Bureau of Economic Research.

Auclert, Adrien, Bence Bardóczy, Matthew Rognlie, and Ludwig Straub. 2021. "Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models." *Econometrica*, 89(5): 2375–2408.

Bhandari, Anmol, David Evans, Mikhail Golosov, and Thomas J. Sargent. 2021. "Inequality, Business Cycles, and Monetary-Fiscal Policy." *Econometrica*, 89(6): 2559–2599.

Bigio, Saki, Galo Nuño, and Juan Passadore. 2023. "Debt-Maturity Management with Liquidity Costs."

Bilal, Adrien. 2023. "Solving Heterogeneous Agent Models with the Master Equation."

Blanchard, Olivier Jean, and Charles M Kahn. 1980. "The Solution of Linear Difference Models under Rational Expectations." *Econometrica* 1305–1311.

Boppart, Timo, Per Krusell, and Kurt Mitman. 2018. "Exploiting MIT Shocks in Heterogeneous-Agent Economies: The Impulse Response as a Numerical Derivative." *Journal of Economic Dynamics and Control*, 89 68–92.

Carroll, Christopher D. 2006. "The Method of Endogenous Gridpoints for Solving Dynamic Stochastic Optimization Problems." *Economics Letters*, 91(3): 312–320.

Childers, David, Jesús Fernández-Villaverde, Jesse Perla, Christopher Rackauckas, and Peifan Wu. 2022. "Differentiable State-Space Models and Hamiltonian Monte Carlo Estimation."Technical report, National Bureau of Economic Research.

Devereux, Michael B., and Alan Sutherland. 2011. "Country Portfolios in Open Economy Macro-Models." *Journal of the European Economic Association*, 9(2): 337–369.

Fernández-Villaverde, Jesús, Pablo Guerrón-Quintana, Juan F. Rubio-Ramírez, and Martin Uribe. 2011. "Risk Matters: The Real Effects of Volatility Shocks." *American Economic Review*, 101(6): 2530–61.

Han, Jiequn, Yucheng Yang et al. 2021. "Deepham: A global solution method for heterogeneous agent models with aggregate shocks." *arXiv preprint arXiv:2112.14377*.

Hayashi, Fumio. 1982. "Tobin's marginal Q and average Q: A Neoclassical Interpretation." *Econometrica: Journal of the Econometric Society* 213–224.

Judd, Kenneth L. 1998. *Numerical Methods in Economics.* MIT Press Books: The MIT Press.

Kahou, Mahdi Ebrahimi, Jesús Fernández-Villaverde, Jesse Perla, and Arnav Sood. 2021. "Exploiting symmetry in high-dimensional dynamic programming."Technical report, National Bureau of Economic Research.

Kaplan, Greg, Benjamin Moll, and Giovanni L Violante. 2018. "Monetary Policy according to HANK." *American Economic Review*, 108(3): 697–743.

Krusell, Per, and Anthony A Smith, Jr. 1998. "Income and Wealth Heterogeneity in the Macroeconomy." *Journal of Political Economy*, 106(5): 867–896.

Luenberger, David G. 1997. *Optimization by Vector Space Methods.* New York, NY, USA: John Wiley & Sons.

Maliar, Lilia, Serguei Maliar, and Pablo Winant. 2021. "Deep learning for solving dynamic economic models.." *Journal of Monetary Economics*, 122 76–101.

Reiter, Michael. 2009. "Solving Heterogeneous-Agent Models by Projection and Perturbation." *Journal of Economic Dynamics and Control*, 33(3): 649–665.

Schmitt-Grohé, Stephanie, and Martın Uribe. 2004. "Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function." *Journal of Economic Dynamics and Control*, 28(4): 755–775.

Viceira, Luis M. 2001. "Optimal Portfolio Choice for Long Horizon Investors with Nontradable Labor Income." *Journal of Finance*, 56(2): 433–470.

Yogo, Motohiro, and Jessica A. Wachter. 2011. "Why Do Household Portfolio Shares Rise in Wealth?" *Review of Financial Studies*, 24(11): 3929–3965.

Young, Eric R. 2010. "Solving the Incomplete Markets Model with Aggregate Uncertainty using the Krusell–Smith Algorithm and Non-Stochastic Simulations." *Journal of Economic Dynamics and Control*, 34(1): 36–41.

A Section 3.2 Proofs

A.1 Proof of Lemma 1(FO)

The path of aggregates, $X_t(\mathcal{E}^t;\sigma)$, as a history of aggregate shocks, $\mathcal{E}^t$, can be constructed from the recursive representation $\tilde{X}(Z;\sigma)$ and $\tilde{\Omega}(Z;\sigma)$. Begin by defining $Z_t(\mathcal{E}^t,\sigma) = \left[\Theta_t\left(\mathcal{E}^t;\sigma\right),\Omega_t\left(\mathcal{E}^t;\sigma\right)\right]^T$ recursively as follows: let

$$Z_t\left(\mathcal{E}^t;\sigma\right) = \left[\rho_\Theta\Theta_{t-1}\left(\mathcal{E}^{t-1};\sigma\right)+\sigma\mathcal{E}_t,\tilde{\Omega}\left(Z_{t-1}\left(\mathcal{E}^{t-1};\sigma\right);\sigma\right)\right], \tag{60}$$

where $Z_{-1} = Z^*$. The path of aggregates can then be constructed as

$$X_t(\mathcal{E}^t;\sigma) = \tilde{X}\left(Z_t\left(\mathcal{E}^t;\sigma\right);\sigma\right). \tag{61}$$

A first order expansion of equations (60) and (61) around the $\sigma = 0$ steady state yields the following recursive relationship

$$X_t\left(\mathcal{E}^t\right) = \overline{X} + \overline{X}_Z\left(Z_t\left(\mathcal{E}^t\right) - Z^*\right) + \overline{X}_\sigma + \mathcal{O}\left(\underline{\mathcal{E}}^2\right) \tag{62}$$

$$Z_t\left(\mathcal{E}^t\right) = Z^* + \overline{Z}_Z\left(Z_{t-1}\left(\mathcal{E}^{t-1}\right) - Z^*\right) + \hat{Z}_0\mathcal{E}_t + \overline{Z}_\sigma + \mathcal{O}\left(\underline{\mathcal{E}}^2\right). \tag{63}$$

with $\hat{Z}_0$ being the direction defined in the main text and $\overline{Z}_\sigma = \left[0,\overline{\Omega}_\sigma\right]$. Our first step is to show that $\overline{X}_\sigma$ and $\overline{Z}_\sigma$ are both 0 which we codify in the following lemma

Lemma 1. *The first derivatives with respect to σ, $\left(\overline{X}_\sigma, \overline{\Omega}_\sigma, \overline{x}_\sigma\right)$, are all* 0.

Proof. Differentiating the F, G, and LoM mappings with respect to σ yields the following system of equations[16]

$$
\begin{aligned}
0 &= \mathsf{F}_x(z,\theta)\overline{x}_\sigma(z,\theta) + \mathsf{F}_X(z,\theta)\overline{X}_\sigma + \mathsf{F}_{x'}(z,\theta)\left(\mathbb{E}\left[\overline{x}_z|z,\theta\right]\overline{z}_\sigma(z,\theta) + \mathbb{E}\left[\overline{x}_\sigma|z,\theta\right] + \mathbb{E}\left[\overline{x}_Z \cdot \overline{Z}_\sigma|z,\theta\right]\right) \\
0 &= \mathsf{G}_x \int \overline{x}_\sigma d\Omega^* + \mathsf{G}_X \overline{X}_\sigma \\
\overline{\Omega}_\sigma\langle z',\theta'\rangle &= -\iint \delta\left(\overline{z}(z,\theta) - z'\right)\iota(\rho_\theta\theta + \epsilon \leq \theta')\overline{z}_\sigma(z,\theta)\mu(\epsilon)d\epsilon d\Omega^*.
\end{aligned}
$$

This system of equations is homogeneous of degree 1 in $\left(\overline{X}_\sigma, \overline{\Omega}_\sigma, \overline{x}_\sigma\right)$ and, therefore, is solved by setting all terms to zero. $\square$

With the knowledge that $\overline{X}_\sigma$ and $\overline{Z}_\sigma$ are both zero and $Z_{-1} - Z^* = 0$ it's possible to roll forward equation (63) to find

$$
\begin{aligned}
Z_t\left(\mathcal{E}^t\right) - Z^* &= \sum_{s=0}^{t} \overline{Z}_Z^{t-s} \cdot \hat{Z}_0 \mathcal{E}_s + \mathcal{O}\left(\underline{\mathcal{E}}^2\right) \\
&= \sum_{s=0}^{t} \hat{Z}_{t-s}\mathcal{E}_s + \mathcal{O}\left(\underline{\mathcal{E}}^2\right), \qquad (64)
\end{aligned}
$$

where $\hat{Z}_t$ is defined in the main text. Plugging into equation (62) yields

$$
X_t\left(\mathcal{E}^t\right) = \overline{X} + \sum_{s=0}^{t} \overline{X}_Z \cdot \hat{Z}_{t-s}\mathcal{E}_s + \mathcal{O}\left(\underline{\mathcal{E}}^2\right) = \overline{X} + \sum_{s=0}^{t} \overline{X}_{Z,t-s}\mathcal{E}_s + \mathcal{O}\left(\underline{\mathcal{E}}^2\right)
$$

as desired.

A.2 Proof of Lemma 2(FO)

Differentiating the G mapping, equation (11), in direction $\hat{Z}_t = [\rho_\Theta^t, \hat{\Omega}_t]^\intercal$ is equivalent to differentiating

$$
G\left(\int \overline{x}\left(z,\theta,Z^* + \alpha\hat{Z}_t\right)d\left(\Omega^* + \alpha\hat{\Omega}_t\right), \overline{X}(Z^* + \alpha\hat{Z}_t), \rho_\Theta^t\alpha\right) = 0
$$

w.r.t. α. Doing so yields

$$
\mathsf{G}_x\left(\int \overline{x}_Z \cdot \hat{Z}_t d\Omega^* + \int \overline{x} d\hat{\Omega}_t\right) + \mathsf{G}_X \overline{X}_Z \cdot \hat{Z}_t + \mathsf{G}_\Theta \rho_\Theta^t = 0
$$

where these integrals are well defined from Assumption 1 (e). Replacing $\overline{x}_Z \cdot \hat{Z}_t = \overline{x}_{Z,t}$ and $\overline{X}_Z \cdot \hat{Z}_t = \overline{X}_{Z,t}$ yields equation (14) in the text.

[16] Here we have exploited the knowledge that $\mathbb{E}\left[\overline{x}_\Theta(z',\theta')\mathcal{E}'\right] = 0$

A.3 Proof of Lemma 3(FO)

We begin by differentiating the F mapping, equation (10), in direction $\hat{Z}_t$. Doing so yields

$$\mathsf{F}_x(z,\theta)\overline{x}_Z(z,\theta)\cdot\hat{Z}_t+\mathsf{F}_X(z,\theta)\overline{X}_Z\cdot\hat{Z}_t+\mathsf{F}_{x^e}(z,\theta)\left(\mathbb{E}\left[\overline{x}_z|z,\theta\right]\mathsf{P}\overline{x}_Z(z,\theta)\cdot\hat{Z}_t+\mathbb{E}\left[\overline{x}_Z\cdot\overline{Z}_Z\cdot\hat{Z}_t|z,\theta\right]\right)=0$$

Replacing $\overline{x}_Z\cdot\hat{Z}_t=\overline{x}_{Z,t}$, $\overline{X}_Z\cdot\hat{Z}_t=\overline{X}_{Z,t}$ and $\hat{Z}_{t+1}=\overline{Z}_Z\cdot\hat{Z}_t$ we get the difference equation

$$\mathsf{F}_x(z,\theta)\overline{x}_{Z,t}(z,\theta)+\mathsf{F}_X(z,\theta)\overline{X}_{Z,t}+\mathsf{F}_{x^e}(z,\theta)\left(\mathbb{E}\left[\overline{x}_z|z,\theta\right]\mathsf{P}\overline{x}_{Z,t}(z,\theta)+\mathbb{E}\left[\overline{x}_{Z,t+1}|z,\theta\right]\right)=0. \quad (65)$$

Our claim is that $\overline{x}_{Z,t}=\sum_{s=0}^{\infty}\mathsf{x}_s\overline{X}_{Z,t+s}$ solves this equation where x_s are defined via (16) and (17). To see this, note that

$$\begin{aligned}\mathsf{F}_{x^e}(z,\theta)\mathbb{E}\left[\overline{x}_{Z,t+1}|z,\theta\right]&=\sum_{s=0}^{\infty}\mathsf{F}_{x^e}(z,\theta)\mathbb{E}\left[\mathsf{x}_s|z,\theta\right]\overline{X}_{Z,t+1+s}\\&=-\left(\mathsf{F}_x(z,\theta)+\mathsf{F}_{x^e}(z,\theta)\mathbb{E}\left[\overline{x}_z|z,\theta\right]\mathsf{P}\right)\sum_{s=0}^{\infty}\mathsf{x}_{s+1}(z,\theta)\overline{X}_{Z,t+1+s}\\&=-\left(\mathsf{F}_x(z,\theta)+\mathsf{F}_{x^e}(z,\theta)\mathbb{E}\left[\overline{x}_z|z,\theta\right]\mathsf{P}\right)\sum_{s=1}^{\infty}\mathsf{x}_{s}(z,\theta)\overline{X}_{Z,t+s}\end{aligned}$$

where the second line comes from applying equation (17). Combined with equation (16) we have

$$\mathsf{F}_X(z,\theta)+\mathsf{F}_{x^e}(z,\theta)\mathbb{E}\left[\overline{x}_{Z,t+1}|z,\theta\right]=-\left(\mathsf{F}_x(z,\theta)+\mathsf{F}_{x^e}(z,\theta)\mathbb{E}\left[\overline{x}_z|z,\theta\right]\mathsf{P}\right)\overline{x}_{Z,t}(z,\theta)$$

which guarantees (65) and completes the proof.

A.4 Assumptions

Before proceeding let's layout the two assumptions I'll be using throughout this. First we assume that $\tilde{x}(z,\theta,Z)$ is continuous and piecewise smooth with kinks defined by the functions $z_j^{\vee}(\theta,Z)$.

Next we assume that the steady state density has a finite number of mass points

$$d\Omega^*\langle z,\theta\rangle=\left(\omega^*(z,\theta)+\sum_n\omega^*_{\delta,n}(\theta)\delta(z-z^*_n)\right)dzd\theta \quad (66)$$

with $\omega^*(z,\theta)$ being smooth. We assume that the set of all kinks is measure 0 under this density.

A.5 Proof of Lemma 4(FO)

Differentiating the LoM, equation 12, in direction $\hat{Z}_t$ is equivalent to differentiating

$$\overline{\Omega}(Z^*+\alpha\hat{Z}_t)\langle z',\theta'\rangle=\iint\iota\left(\bar{z}(z,\theta,Z^*+\alpha\hat{Z}_t)\leq z'\right)\iota\left(\rho_\theta\theta+\epsilon\leq\theta'\right)\mu(\epsilon)d\epsilon d\left(\Omega^*+\alpha\hat{\Omega}_t\right)\langle z,\theta\rangle$$

with respect to α. This yields

$$\overline{\Omega}_Z \cdot \hat{Z}_t \langle z', \theta' \rangle = -\iint \delta\left(\overline{z}(z,\theta) - z'\right) \iota\left(\rho_\theta \theta + \epsilon \leq \theta'\right) \mu(\epsilon) d\epsilon \overline{z}_Z(z,\theta) \cdot \hat{Z}_t d\Omega^* \langle z, \theta \rangle$$
$$+ \iint \iota\left(\overline{z}(z,\theta) \leq z'\right) \iota\left(\rho_\theta \theta + \epsilon \leq \theta'\right) \mu(\epsilon) d\epsilon d\hat{\Omega}_t \langle z, \theta \rangle.$$

Applying $\frac{d}{d\theta'}$ to both sides and substituting for $\overline{z}_{Z,t}$ yields

$$\begin{aligned}
\frac{d}{d\theta'} \hat{\Omega}_{t+1} \langle z', \theta' \rangle &= -\sum_{s=0}^{\infty} \int \overbrace{\delta\left(\overline{z}(z,\theta) - z'\right) \mu(\theta' - \rho_\theta \theta)}^{\overline{\Lambda}(z',\theta',z,\theta)} \mathsf{z}_s(z,\theta) d\Omega^* \langle z, \theta \rangle \overline{X}_{Z,t+s} \\
&\quad + \int \iota\left(\bar{z}(z,\theta) \leq z'\right) \mu(\theta' - \rho_\theta \theta) \mu(\epsilon) d\epsilon d\hat{\Omega}_t \langle z, \theta \rangle. \\
&= -\sum_{s=0}^{\infty} \left(\mathcal{M} \cdot \mathsf{z}_s\right) \langle z', \theta' \rangle \overline{X}_{Z,t+s} \\
&\quad + \int \overbrace{\delta\left(\overline{z}(z,\theta) - z'\right) \mu(\theta' - \rho_\theta \theta)}^{\overline{\Lambda}(z',\theta',z,\theta)} \bar{z}_z(z,\theta) \frac{d}{d\theta} \hat{\Omega}_t \langle z, \theta \rangle dz d\theta \\
&= -\sum_{s=0}^{\infty} \left(\mathcal{M} \cdot \mathsf{z}_s\right) \langle z', \theta' \rangle \overline{X}_{Z,t+s} + \left(\mathcal{L} \cdot \frac{d}{d\theta} \hat{\Omega}_t\right) \langle z', \theta' \rangle
\end{aligned}$$

Where the second and third lines are achieved via integration by parts. To conclude, we need to show that all the integrals are well defined. We start with the following Claim

Claim 1. If y is a piecewise smooth with kinks at $\overline{z}_j^{\vee}(\theta)$ then $\mathcal{M} \cdot \mathsf{y}$ is a density with a finite number of mass points z_n^*.

Proof. We will show that $\left(\mathcal{M} \cdot \mathsf{y}\right) \langle z', \theta' \rangle$ is of the form

$$\mathsf{m}(z', \theta') + \sum_n \mathsf{m}_{\delta,n}(\theta') \delta(z - z_n^*).$$

Define the function $\overline{\theta}(z, z')$ as the implicit function

$$\begin{aligned}
\left(\mathcal{M} \cdot \mathsf{y}\right) \langle z', \theta' \rangle &= \iint \overline{\Lambda}(z', \theta', z, \theta) \mathsf{y}(z,\theta) \left(\omega^*(z,\theta) + \sum_n \omega_{\delta,n}^*(\theta) \delta(z - z_n^*)\right) dz d\theta \\
&= \iint \overline{\Lambda}(z', \theta', z, \theta) \mathsf{y}(z,\theta) \omega^*(z,\theta) dz d\theta \\
&\quad + \sum_n \int \overline{\Lambda}(z', \theta', z_n^*, \theta) \mathsf{y}(z_n^*, \theta) \omega_{\delta,n}^*(\theta) d\theta
\end{aligned}$$

For points $z' \neq z_n^*$ we have

$$\omega^*(z', \theta') = \iint \overline{\Lambda}(z', \theta', z, \theta) \omega^*(z,\theta) dz d\theta + \sum_n \int \overline{\Lambda}(z', \theta', z_n^*, \theta) \omega_{\delta,n}^*(\theta) d\theta,$$

and since y is piecewise smooth with discontinuities that don't align with the mass-points z_n^* we conclude that

$$\begin{aligned}\mathsf{m}(z',\theta') = & \iint \overline{\Lambda}(z',\theta',z,\theta)\mathsf{y}(z,\theta)\omega^*(z,\theta)dzd\theta \\ & + \sum_n \int \overline{\Lambda}(z',\theta',z_n^*,\theta)\mathsf{y}(z_n^*,\theta)\omega_{\delta,n}^*(\theta)d\theta\end{aligned}$$

exists and is continuous for all $z' \neq z_n^*$ At the mass points we have

$$\iint \overline{\Lambda}(z_n^*,\theta',z,\theta)\omega^*(z,\theta)dzd\theta + \sum_m \int \overline{\Lambda}(z_n^*,\theta',z_m^*,\theta)\omega_{\delta,m}^*(\theta)d\theta = \omega_n^*(\theta')\delta(z-z_n^*)$$

where

$$\omega_n^*(\theta') = \int\int_{\overline{\theta}(z,z_n^*)} \mu\left(\theta' - \rho_\theta\theta\right)\omega^*(z,\theta)d\theta dz + \sum_m \int_{\overline{\theta}(z_m^*,z_n^*)} \mu(\theta'-\rho_\theta\theta)\omega_{\delta,m}^*(\theta)d\theta$$

with $\overline{\theta}(z,z') = \{\theta : \overline{z}(z,\theta) = z'\}$. As y is piecewise smooth with discontinuities that don't align with the mass-points z_n^*

$$\begin{aligned}\mathsf{m}_{\delta,n}(\theta') = & \int\int_{\overline{\theta}(z,z_n^*)} \mu\left(\theta'-\rho_\theta\theta\right)\omega^*(z,\overline{\theta}(z,z_n^*))\mathsf{y}\left(z,\theta\right)d\theta dz \\ & + \sum_m \int_{\overline{\theta}(z_m^*,z_n^*)} \mu(\theta'-\rho_\theta\theta)\omega_{\delta,m}^*(\theta)\mathsf{y}\left(z_m^*,\theta\right)d\theta.\end{aligned}$$

□

This claim implies that $\mathcal{M}\cdot\mathsf{z}_s$ is a density with a finite number of mass-points as z_n^*. As $\hat{\Omega}_0 = 0$ we conclude that

$$\frac{d}{d\theta}\hat{\Omega}_1 = -\sum_{s=0}^{\infty}\left(\mathcal{M}\cdot\mathsf{z}_s\right)\overline{X}_{Z,s}$$

is a density with a finite number of mass-points. Our next claim allows us to extend this to all $\frac{d}{d\theta}\hat{\Omega}_t$ via induction

Claim 2. If $\frac{d}{d\theta}\hat{\Omega}$ is a density with a finite number of mass points z_n^* then $\mathcal{L}\cdot\frac{\mathsf{d}}{\mathsf{d}\theta}\hat{\Omega}$ is a density with a finite number of mass points z_n^*.

Proof. Repeat the steps of the Claim 1 replacing y with $\overline{z}_z$ and

$$\omega^*(z,\theta) + \sum_n \omega_{\delta,n}^*(\theta)\delta(z-z_n^*)$$

with $\frac{d}{d\theta}\hat{\Omega}$. □

A.6 Proof of Corollary 2(FO)

We start with our first claim

Claim 3. $\frac{d}{d\theta}\hat{\Omega}_t$ is given by

$$\frac{d}{d\theta}\hat{\Omega}_t = -\sum_{s=0} \mathsf{A}_{t,s}\overline{X}_{Z,s}$$

where $\mathsf{A}_{t,s}$ is as defined in Corollary 2(FO).

Proof. We proceed by induction. It's trivially true from $t = 0$ as $\mathsf{A}_{0,s} = 0$ and $\frac{d}{d\theta}\hat{\Omega}_0$. We then proceed by induction

$$\begin{aligned}
\frac{d}{d\theta}\hat{\Omega}_{t+1} &= \mathcal{L} \cdot \frac{d}{d\theta}\hat{\Omega}_t - \sum_{j=0}^{\infty} \mathsf{a}_j \overline{X}_{Z,t+j} \\
&= \mathcal{L} \cdot \left(-\sum_{s=0}^{\infty} \mathsf{A}_{t,s}\overline{X}_{Z,s} \right) - \sum_{s=0}^{\infty} \mathsf{a}_{s-t}\overline{X}_{Z,s} \\
&= \sum_{s=0}^{\infty} - \left(\mathcal{L} \cdot \mathsf{A}_{t,s} + \mathsf{a}_{s-t}\right) \overline{X}_{Z,s} \\
&\equiv \sum_{s=0}^{\infty} \mathsf{A}_{t+1,s}\overline{X}_{Z,s}
\end{aligned}$$

where the second equality is achieved by letting $s = t + j$ and WLOG setting $\mathsf{a}_k = 0$ for $k < 0$. $\square$

Applying integration by parts we have

$$\int \overline{x} d\hat{\Omega}_t = -\int \overline{x}_z \frac{d}{d\theta}\hat{\Omega}_t dz d\theta := -\mathcal{I} \cdot \frac{d}{d\theta}\hat{\Omega}_t.$$

From the proof of Lemma 4(FO) we know that $\frac{d}{d\theta}\hat{\Omega}_t$ is a density with mass points at a finite number of points z_n^* which implies that the set of points where $\overline{x}_z$ is not defined is measure zero under $\frac{d}{d\theta}\hat{\Omega}_t dz d\theta$ so $\mathcal{I} \cdot \frac{d}{d\theta}\hat{\Omega}_t$ is well defined. Therefore

$$\int \overline{x} d\hat{\Omega}_t = -\mathcal{I} \cdot \left(-\sum_{s=0} \mathsf{A}_{t,s}\overline{X}_{Z,s} \right) = \sum_s \left(\mathcal{I} \cdot \mathsf{A}_{t,s}\right) \overline{X}_{Z,s}$$

as desired.

A.7 Proof Of Proposition 1(FO)

Combining 1(FO) and 2(FO) we have

$$\int \overline{x}_{Z,t} d\Omega^* + \int \overline{x} d\hat{\Omega}_t = \sum_{j=0}^{\infty} \left(\int \mathsf{x}_j d\Omega^* \right) \overline{X}_{Z,t+j} + \sum_s \left(\mathcal{I} \cdot \mathsf{A}_{t,s} \right) \overline{X}_{Z,s}$$
$$= \sum_{s=0}^{\infty} \left(\underbrace{\int \mathsf{x}_{t-s} d\Omega^* + \mathcal{I} \cdot \mathsf{A}_{t,s}}_{\mathsf{J}_{t,s}} \right) \overline{X}_{Z,s}.$$

Combined with Lemma 2(FO) immediately yields equation (19).

A.8 Proof of Lemma 1(SO)

We proceed by taking a second order expansion of (60) and (61) to find[17]

$$X_t\left(\mathcal{E}^t\right) = \overline{X} + \overline{X}_Z\left(Z_t\left(\mathcal{E}^t\right) - Z^*\right) + \frac{1}{2}\left(\overline{X}_{ZZ} \cdot \left(Z_{t-1}\left(\mathcal{E}^{t-1}\right) - Z^*, Z_{t-1}\left(\mathcal{E}^{t-1}\right) - Z^*\right) + \overline{X}_{\sigma\sigma}\right) + \mathcal{O}\left(\underline{\mathcal{E}}^3\right) \tag{67}$$

$$Z_t\left(\mathcal{E}^t\right) = Z^* + \overline{Z}_Z\left(Z_{t-1}\left(\mathcal{E}^{t-1}\right) - Z^*\right) + \hat{Z}_0\mathcal{E}_t + \frac{1}{2}\left(\overline{Z}_{ZZ} \cdot \left(Z_{t-1}\left(\mathcal{E}^{t-1}\right) - Z^*, Z_{t-1}\left(\mathcal{E}^{t-1}\right) - Z^*\right) + \overline{Z}_{\sigma\sigma}\right) + \mathcal{O}\left(\underline{\mathcal{E}}^3\right), \tag{68}$$

where $\overline{Z}_{ZZ}$ is defined in the main text and $\overline{Z}_{\sigma\sigma} = \left[0, \overline{\Omega}_{\sigma\sigma}\right]^T$. As $Z_{t-1}\left(\mathcal{E}^{t-1}\right) - Z^*$ satisfies (64) and both $\overline{X}_{ZZ}$ and $\overline{Z}_{ZZ}$ are quadratic forms we can conclude that

$$\overline{X}_{ZZ} \cdot \left(Z_t\left(\mathcal{E}^t\right) - Z^*, Z_t\left(\mathcal{E}^t\right) - Z^*\right) = \sum_{s=0}^{t} \sum_{m=0}^{t} \overline{X}_{ZZ} \cdot \left(\hat{Z}_{t-s}, \hat{Z}_{t-m}\right) \mathcal{E}_s \mathcal{E}_m + \mathcal{O}\left(\underline{\mathcal{E}}^3\right)$$
$$\overline{Z}_{ZZ} \cdot \left(Z_t\left(\mathcal{E}^t\right) - Z^*, Z_t\left(\mathcal{E}^t\right) - Z^*\right) = \sum_{s=0}^{t} \sum_{m=0}^{t} \overline{Z}_{ZZ} \cdot \left(\hat{Z}_{t-s}, \hat{Z}_{t-m}\right) \mathcal{E}_s \mathcal{E}_m + \mathcal{O}\left(\underline{\mathcal{E}}^3\right).$$

Therefore if we define the direction $\hat{Z}_t^{(2)}\left(\mathcal{E}^t\right)$ recursively via $\hat{Z}_0^{(2)}\left(\mathcal{E}^0\right) = \mathbf{0}$ and

$$\hat{Z}_t^{(2)}\left(\mathcal{E}^t\right) = \overline{Z}_Z \hat{Z}_{t-1}^{(2)}\left(\mathcal{E}^{t-1}\right) + \sum_{s=0}^{t-1} \sum_{m=0}^{t-1} \overline{Z}_{ZZ} \cdot \left(\hat{Z}_{t-1-s}, \hat{Z}_{t-1-m}\right) \mathcal{E}_s \mathcal{E}_m + \overline{Z}_{\sigma\sigma} \tag{69}$$

then we can conclude that

$$Z_t\left(\mathcal{E}^t\right) - Z^* = \sum_{s=0}^{t} \hat{Z}_{t-s}\mathcal{E}_s + \frac{1}{2}\hat{Z}_t^{(2)}\left(\mathcal{E}^t\right) + \mathcal{O}\left(\underline{\mathcal{E}}^3\right)$$

and

$$X_t\left(\mathcal{E}^t\right) - \overline{X} = \sum_{s=0}^{t} \overline{X}_{Z,t-s}\mathcal{E}_s + \frac{1}{2}\left(\sum_{s=0}^{t} \sum_{m=0}^{t} \overline{X}_{ZZ} \cdot \left(\hat{Z}_{t-s}, \hat{Z}_{t-m}\right) \mathcal{E}_s \mathcal{E}_m + \overline{X}_Z \cdot \hat{Z}_t^{(2)}\left(\mathcal{E}^t\right)\right) + \mathcal{O}\left(\underline{\mathcal{E}}^2\right). \tag{70}$$

[17] There are also $\overline{X}_{\sigma Z}$ and $\overline{Z}_{\sigma Z}$ terms but they are 0 following the same logic as $\overline{X}_\sigma$ and $\overline{Z}_\sigma$ being 0 in the proof of Lemma 1

By construction it is straightforward to use show that (69) implies that

$$\hat{Z}_t^{(2)}\left(\mathcal{E}^t\right)=\hat{Z}_{\sigma\sigma,t}+\sum_{s=0}^{t}\sum_{m=0}^{t}\hat{Z}_{t-s,t-m}\mathcal{E}_s\mathcal{E}_m$$

where both $\hat{Z}_{t,s}$ and $\hat{Z}_{\sigma\sigma,t}$ are as defined in the main text. Substituting for $\hat{Z}_t^{(2)}\left(\mathcal{E}^t\right)$ in (70) and applying the definitions of $\overline{X}_{\sigma\sigma,t}$ and $\overline{X}_{ZZ,t,s}$ immediately yields

$$X_t\left(\mathcal{E}^t\right)=...+\frac{1}{2}\left(\sum_{s=0}^{t}\sum_{m=0}^{t}\overline{X}_{ZZ,t-s,t-m}\mathcal{E}_s\mathcal{E}_m+\overline{X}_{\sigma\sigma,t}\right)+\mathcal{O}\left(\underline{\mathcal{E}}^2\right)$$

where ... are the first-order terms, as desired.

A.9 Proof of Lemma 3(SO)

Assumption 1 states that the functions $\tilde{x}(z,\theta,Z;\sigma)$ are continuous and piece-wise smooth in some neighborhood Z^* and $\sigma=0$. We'll start by assuming a single point where the functions are joined defined by $z^*(\theta,Z;\sigma)$. We'll let $\tilde{x}^p$ and $\tilde{x}^m$ denote the functions on either side of that point thus

$$\tilde{x}(z,\theta,Z;\sigma)=\begin{cases}\tilde{x}^1(z,\theta,Z;\sigma) & z\geq z^*(\theta,Z;\sigma)\\ \tilde{x}^0(z,\theta,Z;\sigma) & z\leq z^*(\theta,Z;\sigma)\end{cases},$$

or

$$\tilde{x}(z,\theta,Z;\sigma)=\tilde{x}^0(z,\theta,Z;\sigma)\iota\left(z\leq z^*(\theta,Z;\sigma)\right)+\tilde{x}^1(z,\theta,Z;\sigma)\iota\left(z\geq z^*(\theta,Z;\sigma)\right),$$

where $z^*(\theta,Z;\sigma)$ is determined by the agent being on the budget constraint while unconstrained, i.e.

$$\tilde{z}^1\left(z^*(\theta,Z;\sigma),\theta,Z;\sigma\right)=\underline{\mathrm{z}}$$

If we differentiate $\tilde{x}(z,\theta,Z;\sigma)$ with respect to Z in direction $\hat{Z}_i$ we get

$$\begin{aligned}\overline{x}_Z(z,\theta)\cdot\hat{Z}_i&=\overline{x}_Z^0(z,\theta)\cdot\hat{Z}_i\iota\left(z\leq\overline{z}^*(\theta)\right)+\overline{x}_Z^1(z,\theta)\cdot\hat{Z}_i\iota\left(z\geq\overline{z}^*(\theta)\right)+\left(\overline{x}^1(z,\theta)-\overline{x}^0(z,\theta)\right)\overline{z}_Z^*(\theta)\cdot\hat{Z}_i\delta(z-\overline{z}^*(\theta))\\&=\overline{x}_Z^\circ(z,\theta)\cdot\hat{Z}_i+\left(\overline{x}^1(z,\theta)-\overline{x}^0(z,\theta)\right)\overline{z}_Z^*(\theta)\cdot\hat{Z}_i\delta(z-\overline{z}^*(\theta))\\&=\overline{x}_Z^\circ(z,\theta)\cdot\hat{Z}_i\end{aligned}$$

where $\overline{x}_Z^\circ(z,\theta)\cdot\hat{Z}_i$ is the piece-wise smooth component of $\overline{x}_Z(z,\theta)\cdot\hat{Z}_i$. Continuity of $\tilde{x}(z,\theta,Z;\sigma)$ implies that $\left(\overline{x}^1(\overline{z}^*(\theta),\theta)-\overline{x}^0(\overline{z}^*(\theta),\theta)\right)=0$ which give the third equality. Therefore, all the proofs in the main text can proceed as they do.

Differentiating twice with respect to Z implies that

$$
\begin{aligned}
\overline{x}_{ZZ}(z,\theta)\cdot\left(\hat{Z}_i,\hat{Z}_j\right) =& \overline{x}^{\circ}_{ZZ}(z,\theta)\cdot\left(\hat{Z}_i,\hat{Z}_j\right)+\left(\overline{x}^1_Z(z,\theta)\cdot\hat{Z}_j-\overline{x}^0_Z(z,\theta)\cdot\hat{Z}_j\right)\overline{z}^*_Z(\theta)\cdot\hat{Z}_i\delta(z-\overline{z}^*(\theta))\\
&+\left(\overline{x}^1_Z(z,\theta)\cdot\hat{Z}_j-\overline{x}^0_Z(z,\theta)\cdot\hat{Z}_j\right)\overline{z}^*_Z(\theta)\cdot\hat{Z}_j\delta(z-\overline{z}^*(\theta))\\
&-\left(\overline{x}^1(z,\theta)-\overline{x}^0(z,\theta)\right)\left(\overline{z}^*_Z(\theta)\cdot\hat{Z}_i\right)\left(\overline{z}^*_Z(\theta)\cdot\hat{Z}_j\right)\delta'(z-\overline{z}^*(\theta))\\
=& \overline{x}^{\circ}_{ZZ}(z,\theta)\cdot\left(\hat{Z}_i,\hat{Z}_j\right)+\overline{x}^{\delta}_{ZZ,i,j}(\theta)\delta(z-\overline{z}^*(\theta))
\end{aligned}
$$

where $\overline{x}^{\circ}_{ZZ}(z,\theta)\cdot\left(\hat{Z}_i,\hat{Z}_j\right)$ is the piece-wise smooth component of $\overline{x}_{ZZ}(z,\theta)\cdot\left(\hat{Z}_i,\hat{Z}_j\right)$ and $\overline{x}^{\delta}_{ZZ,i,j}(\theta)$ is given by

$$
\overline{x}^{\delta}_{ZZ,i,j}(\theta)=\overline{x}^{\Delta}_Z(\theta)\cdot\hat{Z}_j\overline{z}^*_Z(\theta)\cdot\hat{Z}_i+\overline{x}^{\Delta}_Z(\theta)\cdot\hat{Z}_i\overline{z}^*_Z(\theta)\cdot\hat{Z}_j+\overline{x}^{\Delta}_z(\theta)\left(\overline{z}^*_Z(\theta)\cdot\hat{Z}_i\right)\left(\overline{z}^*_Z(\theta)\cdot\hat{Z}_j\right)
$$

with $\overline{x}^{\Delta}_Z(\theta)\cdot\hat{Z}_i\equiv\overline{x}^1_Z(\overline{z}^*(\theta),\theta)\cdot\hat{Z}_j-\overline{x}^0_Z(\overline{z}^*(\theta),\theta)\cdot\hat{Z}_j$ and $\overline{x}^{\Delta}_z(\theta)\equiv\left(\overline{x}^1_z(\overline{z}^*(\theta),\theta)-\overline{x}^0(\overline{z}^*(\theta),\theta)\right)$. Finally, we note that $\overline{z}^*_Z(\theta)\cdot\hat{Z}_i$ is determined by

$$
\overline{z}^*_Z(\theta)\cdot\hat{Z}_i=-\overline{z}^1_z\left(\overline{z}^*(\theta),\theta\right)^{-1}\overline{z}^1_Z\left(\overline{z}^*(\theta),\theta\right)\cdot\hat{Z}_i.
$$

Note that all of the terms of $\overline{x}^{\delta}_{ZZ}(\theta)$ are known to first order. Combining all of these facts together we have that

$$
\overline{x}_{ZZ,i,j}(z,\theta)=\overline{x}_Z(z,\theta)\cdot\hat{Z}_{i,j}+\overline{x}_{ZZ}(z,\theta)\cdot\left(\hat{Z}_i,\hat{Z}_j\right)=\overline{x}^{\circ}_{ZZ,i,j}(z,\theta)+\overline{x}^{\delta}_{ZZ,i,j}(\theta)\delta(z-\overline{z}^*(\theta))
$$

where $\overline{x}^{\circ}_{ZZ,i,j}(z,\theta)$ is the piece-wise smooth component of $\overline{x}_{ZZ,i,j}(z,\theta)$.

To determine $\overline{x}^{\circ}_{ZZ,i,j}(z,\theta)$ we differentiate the F mapping twice in direction $\hat{Z}_i$ and $\hat{Z}_j$ at any point $(z,\theta)\neq(z^*(\theta),\theta)$ and add to it the derivative of F in direction $\hat{Z}_{i,j}$ to get

$$
0=\mathsf{F}_x(z,\theta)\overline{x}^{\circ}_{ZZ,i,j}(z,\theta)+\mathsf{F}_X(z,\theta)\overline{X}_{ZZ,ij}+\mathsf{F}_{x'}(z,\theta)\mathbb{E}\left[\overline{x}^{\circ}_{ZZ,i+1,j+1}(,)+\overline{x}_z(,)\mathsf{p}\overline{x}^{\circ}_{ZZ,i,j}(z,\theta)\right]+\mathsf{F}_{i,j}(z,\theta) \tag{71}
$$

where $\mathsf{F}_{i,j}(z,\theta)$ contains all the interaction terms known to first order

$$
\begin{aligned}
\mathsf{F}_{i,j}(z,\theta) =& \mathsf{F}_{x'}(z,\theta)\mathbb{E}\left[\overline{x}^{\circ}_{zz}()\overline{z}_{Z,i}(z,\theta)\overline{z}_{Z,j}(z,\theta)+\overline{x}^{\circ}_{zZ,j+1}()\overline{z}_{Z,i}(z,\theta)+\overline{x}^{\circ}_{zZ,i+1}()\overline{z}_{Z,j}(z,\theta)\right]\\
&+\mathsf{F}_{xx}(z,\theta)\cdot(\overline{x}_{Z,i}(z,\theta),\overline{x}_{Z,j}(z,\theta))+\mathsf{F}_{xX}(z,\theta)\cdot\left(\overline{x}_{Z,i}(z,\theta),\overline{X}_{Z,j}\right)+\mathsf{F}_{xx'}(z,\theta)\cdot\left(\overline{x}_{Z,i}(z,\theta),\overline{x}^+_{Z,j}(z,\theta)\right)\\
&+\mathsf{F}_{Xx}(z,\theta)\cdot\left(\overline{X}_{Z,i},\overline{x}_{Z,j}(z,\theta)\right)+\mathsf{F}_{XX}(z,\theta)\cdot\left(\overline{X}_{Z,i},\overline{X}_{Z,j}\right)+\mathsf{F}_{Xx'}(z,\theta)\cdot\left(\overline{X}_{Z,i},\overline{x}^+_{Z,j}(z,\theta)\right)\\
&+\mathsf{F}_{x'x}(z,\theta)\cdot\left(\overline{x}^+_{Z,i}(z,\theta),\overline{x}_{Z,j}(z,\theta)\right)+\mathsf{F}_{x'X}(z,\theta)\cdot\left(\overline{x}^+_{Z,i}(z,\theta),\overline{X}_{Z,j}\right)+\mathsf{F}_{x'x'}(z,\theta)\cdot\left(\overline{x}^+_{Z,i}(z,\theta),\overline{x}^+_{Z,j}(z,\theta)\right)\\
&+\mathsf{F}_{x'}(z,\theta)\frac{\mu(\overline{\theta}^*(\overline{z}(z,\theta))-\rho_\theta\theta)}{\overline{z}^*_\theta(\theta)}\Bigg(\overline{x}^{\delta}_{zz}(\overline{\theta}^*(\overline{z}(z,\theta)),\overline{z}(z,\theta))\overline{z}_{Z,i}(z,\theta)\overline{z}_{Z,j}(z,\theta)+\overline{x}^{\delta}_{zZ,j+1}(\overline{\theta}^*(\overline{z}(z,\theta)),\overline{z}(z,\theta))\overline{z}_{Z,i}(z,\theta\\
&\quad+\overline{x}^{\delta}_{zZ,i+1}(\overline{\theta}^*(\overline{z}(z,\theta)),\overline{z}(z,\theta))\overline{z}_{Z,j}(z,\theta)+\overline{x}^{\delta}_{ZZ,i+1,j+1}(\overline{\theta}^*(\overline{z}(z,\theta)),\overline{z}(z,\theta))\Bigg).
\end{aligned}
$$

where $\overline{x}^{+}_{Z,i}(z,\theta) = \mathbb{E}\left[\overline{x}_z(,)\overline{x}_{Z,i}(z,\theta) + \overline{x}_{Z,i+1}(,)|z,\theta\right]$. It is straightforward to verify that (71) is solved by

$$\overline{x}^{\circ}_{ZZ,i,j}(z,\theta) = \sum_{s=0}^{\infty} \mathsf{x}_s(z,\theta)\overline{X}_{ZZ,i+s,j+s} + \mathsf{y}^{\circ}_{i,j}(z,\theta)$$

where $\mathsf{x}^{\circ}_{i,j}(z,\theta)$ solves the recursive equation

$$\mathsf{x}^{\circ}_{i,j}(z,\theta) = \left(\mathsf{F}_x(z,\theta) + \mathsf{F}_{x'}(z,\theta)\mathbb{E}\left[\overline{x}_z(,)|z,\theta\right]\mathsf{p}\right)^{-1}\left(\mathsf{F}_{i,j}(z,\theta) + \mathsf{F}_{x'}(z,\theta)\mathbb{E}\left[\mathsf{y}^{\circ}_{i+1,j+1}(,)|z,\theta\right]\right).$$

The first equation of the Lemma is therefore satisfied by

$$\mathsf{x}_{t,k}(z,\theta) = \mathsf{x}^{\circ}_{t,k}(z,\theta) + \overline{x}^{\delta}_{ZZ,t,k}(\theta)\delta(z - \overline{z}^{*}(\theta))$$

with all terms known to first order or lower.

Finally, we move on to the second half of Lemma. Assuming knowledge of $\overline{X}_{ZZ,t,t}$ we can construct $\overline{x}_{\Theta\Theta}(z,\theta) = \overline{x}_{ZZ,0,0}(z,\theta)$. To find $\overline{x}_{\sigma\sigma}(z,\theta)$ differentiate the F mapping twice with respect to σ and add to it the derivative of F in direction $\hat{Z}_{\sigma\sigma,t}$

$$0 = \mathsf{F}_x(z,\theta)\overline{x}_{\sigma\sigma,t}(z,\theta) + \mathsf{F}_X(z,\theta)\overline{X}_{\sigma\sigma,t} + \mathsf{F}_{x'}(z,\theta)\mathbb{E}\left[\overline{x}_{\Theta\Theta}(,)\sigma^2_{\mathcal{E}} + \overline{x}_{\sigma\sigma,t+1}(,) + \overline{x}_z(,)\mathsf{p}\overline{x}_{\sigma\sigma,t}(z,\theta)\right].$$

We begin by letting $\mathsf{x}_{\sigma\sigma}(z,\theta)$ be the function that solves the following linear functional equation

$$0 = \mathsf{F}_x(z,\theta)\mathsf{x}_{\sigma\sigma}(z,\theta) + \mathsf{F}_{x'}(z,\theta)\mathbb{E}\left[\overline{x}_{\Theta\Theta}(,)\sigma^2_{\mathcal{E}} + \mathsf{x}_{\sigma\sigma}(z,\theta)(,) + \overline{x}_z(,)\mathsf{p}\mathsf{z}(z,\theta)\right].$$

Subtracting these two equations and defining $\hat{x}_{\sigma\sigma,t}(z,\theta) = \overline{x}_{\sigma\sigma,t}(z,\theta) - \mathsf{x}_{\sigma\sigma}(z,\theta)$ we see that

$$0 = \mathsf{F}_x(z,\theta)\hat{x}_{\sigma\sigma,t}(z,\theta) + \mathsf{F}_X(z,\theta)\overline{X}_{\sigma\sigma,t} + \mathsf{F}_{x'}(z,\theta)\mathbb{E}\left[\hat{x}_{\sigma\sigma,t+1}(,) + \overline{x}_z(,)\mathsf{p}\hat{x}_{\sigma\sigma,t}(z,\theta)\right].$$

This is identical to system of equations solved by $\overline{x}_{Z,t}$ which allows us to conclude that

$$\hat{x}_{\sigma\sigma,t}(z,\theta) = \sum_{s=0}^{\infty} \mathsf{x}_s(z,\theta)\overline{X}_{\sigma\sigma,t+s}$$

and hence

$$\overline{x}_{\sigma\sigma,t}(z,\theta) = \sum_{s=0}^{\infty} \mathsf{x}_s(z,\theta)\overline{X}_{\sigma\sigma,t+s} + \mathsf{x}_{\sigma\sigma}(z,\theta).$$

A.10 Proof of Lemma 4(SO)

Differentiating the law of motion of Ω twice in direction $\hat{Z}_t$ and $\hat{Z}_k$ and adding to it the derivative in direction $\hat{Z}_{t,k}$ gives

$$\begin{aligned}
&\mathcal{D}_\theta \cdot \hat{\Omega}_{t+1,k+1}\langle z', \theta' \rangle = \\
&-\int \overline{\Lambda}(z',\theta',z,\theta)\overline{z}_{ZZ,t,k}(z,\theta)d\Omega^* + \int \overline{\Lambda}_{z'}(z',\theta',z,\theta)\overline{z}_{Z,t}(z,\theta)\overline{z}_{Z,k}(z,\theta)d\Omega^* \\
&+\int \overline{\Lambda}(z',\theta',z,\theta)\overline{z}_{zZ,k}(z,\theta)\mathcal{D}_\theta \cdot \hat{\Omega}_t\langle z,\theta\rangle d\theta dz + \int \overline{\Lambda}(z',\theta',z,\theta)\overline{z}_{zZ,t}(z,\theta)\mathcal{D}_\theta \cdot \hat{\Omega}_k\langle z,\theta\rangle d\theta dz \\
&-\int \overline{\Lambda}_{z'}(z',\theta',z,\theta)\overline{z}_{Z,k}(z,\theta)\overline{z}_z(z,\theta)\mathcal{D}_\theta \cdot \hat{\Omega}_t\langle z,\theta\rangle d\theta dz - \int \overline{\Lambda}_{z'}(z',\theta',z,\theta)\overline{z}_{Z,k}(z,\theta)\overline{z}_z(z,\theta)\mathcal{D}_\theta \cdot \hat{\Omega}_t\langle z,\theta\rangle d\theta dz \\
&+\int \overline{\Lambda}(z',\theta',z,\theta)\overline{z}_z(z,\theta)\mathcal{D}_\theta \cdot \hat{\Omega}_{t,k}\langle z,\theta\rangle d\theta dz
\end{aligned}$$

where $\overline{\Lambda}_{z'}(z',\theta',z,\theta) = -\int \delta'(\overline{z}(z,\theta) - z')\delta(\rho_\theta\theta + \epsilon - \theta')\mu(\epsilon)d\epsilon$. Substituting for $\overline{z}_{ZZ,t,k}(z,\theta)$ using Lemma 3(SO) we have the first equation in Corollary 2(SO) with

$$\begin{aligned}
\mathsf{b}_{t+1,k+1}\langle z',\theta'\rangle = &\int \overline{\Lambda}(z',\theta',z,\theta)\mathsf{y}_{t,k}(z,\theta)d\Omega^* - \int \overline{\Lambda}(z',\theta',z,\theta)\overline{z}_{zZ,k}(z,\theta)\mathcal{D}_\theta \cdot \hat{\Omega}_t\langle z,\theta\rangle d\theta dz \\
&- \int \overline{\Lambda}(z',\theta',z,\theta)\overline{z}_{zZ,t}(z,\theta)\mathcal{D}_\theta \cdot \hat{\Omega}_k\langle z,\theta\rangle d\theta dz
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{c}_{t+1,k+1}\langle z',\theta'\rangle = &\int \overline{\Lambda}(z',\theta',z,\theta)\overline{z}_{Z,t}(z,\theta)\overline{z}_{Z,k}(z,\theta)d\Omega^* - \int \overline{\Lambda}(z',\theta',z,\theta)\overline{z}_{Z,k}(z,\theta)\overline{z}_z(z,\theta)\mathcal{D}_\theta \cdot \hat{\Omega}_t\langle z,\theta\rangle d\theta dz \\
&- \int \overline{\Lambda}(z',\theta',z,\theta)\overline{z}_{Z,k}(z,\theta)\overline{z}_z(z,\theta)\mathcal{D}_\theta \cdot \hat{\Omega}_t\langle z,\theta\rangle d\theta dz.
\end{aligned}$$

Next, differentiating the LOM of Ω twice with respect to σ and adding to it the derivative of the LOM in direction $\hat{Z}_{\sigma\sigma,t}$ yields

$$\mathcal{D}_\theta \cdot \hat{\Omega}_{\sigma\sigma,t+1}\langle z',\theta'\rangle = -\int \overline{\Lambda}(z',\theta',z,\theta)\overline{z}_{\sigma\sigma,t}(z,\theta)d\Omega^* + \int \overline{\Lambda}(z',\theta',z,\theta)\overline{z}_z(z,\theta)\mathcal{D}_\theta \cdot \hat{\Omega}_{\sigma\sigma,t}\langle z,\theta\rangle d\theta dz.$$

Substituting for $\overline{z}_{\sigma\sigma,t}$ using Lemma 3(SO) immediately obtains the second equation of the Lemma with

$$\mathsf{a}_{\sigma\sigma}\langle z',\theta'\rangle = \int \overline{\Lambda}(z',\theta',z,\theta)\mathsf{x}_{\sigma\sigma}(z,\theta)d\Omega^*.$$

A.11 Proof of Corollary 2(SO)

We begin with a expression for the operator $\mathcal{L}\cdot\mathcal{D}_z$. We note that

$$\begin{aligned}
\mathcal{L}\cdot\mathcal{D}_z\cdot\hat{\Omega}\langle z',\theta'\rangle &= \int \overline{\Lambda}(z',\theta',z,\theta)\overline{z}_z(z,\theta)\mathcal{D}_z\cdot\hat{\Omega}\langle z,\theta\rangle dzd\theta \\
&= -\int \overline{\Lambda}_z(z',\theta',z,\theta)\overline{z}_z(z,\theta)\hat{\Omega}\langle z,\theta\rangle dzd\theta - \int \overline{\Lambda}(z',\theta',z,\theta)\overline{z}_{zz}(z,\theta)\hat{\Omega}\langle z,\theta\rangle dzd\theta \\
&= \int \overline{\Lambda}_{z'}(z',\theta',z,\theta)\overline{z}_z(z,\theta)\overline{z}_z(z,\theta)\hat{\Omega}\langle z,\theta\rangle dzd\theta - \int \overline{\Lambda}(z',\theta',z,\theta)\overline{z}_{zz}(z,\theta)\hat{\Omega}\langle z,\theta\rangle dzd\theta \\
&= \mathcal{D}_z\cdot\mathcal{L}^{(z,z)}\cdot\hat{\Omega}\langle z',\theta'\rangle - \mathcal{L}^{(zz)}\cdot\hat{\Omega}\langle z',\theta'\rangle
\end{aligned}$$

where $\mathcal{L}\cdot\hat{\Omega}\langle z',\theta'\rangle \equiv \int\overline{\Lambda}(z',\theta',z,\theta)\overline{z}_z(z,\theta)\overline{z}_z(z,\theta)\hat{\Omega}\langle z,\theta\rangle dzd\theta$ and $\mathcal{L}^{(zz)}\cdot\hat{\Omega}\langle z',\theta'\rangle \equiv \int\overline{\Lambda}(z',\theta',z,\theta)\overline{z}_{zz}(z,\theta)\hat{\Omega}\langle z,\theta\rangle d$

From the definition of $\left(\overline{\int xd\Omega}\right)_{ZZ,t,k}$ we have

$$\begin{aligned}
\left(\overline{\int xd\Omega}\right)_{ZZ,t,k} &= \int \overline{x}_{ZZ,t,k}d\Omega^* + \int \overline{x}_{Z,t}d\hat{\Omega}_k + \int \overline{x}_{Z,k}d\hat{\Omega}_t + \int \overline{x}d\hat{\Omega}_{t,k}. \\
&= \int \overline{x}_{ZZ,t,k}d\Omega^* - \int \overline{x}_{zZ,t}\mathcal{D}_\theta\cdot\hat{\Omega}_k dzd\theta - \int \overline{x}_{zZ,k}\mathcal{D}_\theta\cdot\hat{\Omega}_t dzd\theta - \int \overline{x}_z\mathcal{D}_\theta\cdot\hat{\Omega}_{t,k}dzd\theta
\end{aligned} \tag{72}$$

We'll proceed under the assumption that $k \geq t$, the case where $t > k$ is symmetric. Our first claim is that

$$\mathcal{D}_\theta\cdot\hat{\Omega}_{t,k} = \sum_{j=0}^{t-1}\sum_{s=0}^{\infty}\mathcal{L}^{t-1-j}\cdot\mathsf{a}_{s-j}\overline{X}_{ZZ,s,k-t+s} - \mathsf{B}_{t,k} + \mathcal{D}_z\cdot\mathsf{C}_{t,k}$$

Proof is via induction starting with our knowledge that $\mathcal{D}_\theta\cdot\hat{\Omega}_{0,k-t} = 0$. Applying the LOM for $\mathcal{D}_\theta\cdot\hat{\Omega}_{t,k}$ we have

$$\begin{aligned}
\mathcal{D}_\theta\cdot\hat{\Omega}_{t+1,k+1} =& \mathcal{L}\cdot\left(\sum_{j=0}^{t-1}\sum_{s=0}^{\infty}\mathcal{L}^{t-1-j}\cdot\mathsf{a}_{s-j}\overline{X}_{ZZ,s,k-t+s} - \mathsf{B}_{t,k} + \mathcal{D}_z\cdot\mathsf{C}_{t,k}\right) - \sum_{s=0}^{\infty}\mathsf{a}_{s-t}\overline{X}_{ZZ,s,k-t+s} \\
&- \mathsf{b}_{t+1,k+1} + \mathcal{D}_z\mathsf{c}_{t+1,k+1} \\
=& \sum_{j=0}^{t}\sum_{s=0}^{\infty}\mathcal{L}^{t-j}\cdot\mathsf{a}_{s-j}\overline{X}_{ZZ,s,k-t+s} - \mathcal{L}\cdot\mathsf{B}_{t,k} - \mathsf{b}_{t+1,k+1} + \mathcal{L}\cdot\mathcal{D}_z\cdot\mathsf{C}_{t,k} + \mathcal{D}_z\cdot\mathsf{c}_{t+1,k+1} \\
=& \sum_{j=0}^{t}\sum_{s=0}^{\infty}\mathcal{L}^{t-j}\cdot\mathsf{a}_{s-j}\overline{X}_{ZZ,s,k-t+s} - \mathcal{L}\cdot\mathsf{B}_{t,k} - \mathsf{b}_{t+1,k+1} - \mathcal{L}^{(zz)}\cdot\mathsf{C}_{t,k} + \mathcal{D}_z\cdot\mathcal{L}^{(z,z)}\cdot\mathsf{C}_{t,k} + \mathcal{D}_z\cdot\mathsf{c}_{t+1,k+1}
\end{aligned}$$

We conclude that

$$\mathsf{B}_{t+1,k+1} = \mathcal{L}\cdot\mathsf{B}_{t,k} + \mathsf{b}_{t+1,k+1} + \mathcal{L}^{(zz)}\cdot\mathsf{C}_{t,k}$$

and

$$\mathsf{C}_{t+1,k+1} = \mathcal{L}^{(z,z)}\cdot\mathsf{C}_{t,k} + \mathsf{c}_{t+1,k+1}.$$

with the initial conditions $\mathsf{B}_{0,k-t} = \mathsf{C}_{0,k-t} = 0$. Substituting for $\mathcal{D}_\theta \cdot \hat{\Omega}_{t,k}$ and $\overline{x}_{ZZ,t,k}$ in (72) yields the first equation in Corollary 1(SO) with

$$\mathsf{H}_{t,k} = \int \mathsf{y}_{t,k} d\Omega^* - \int \overline{x}_{zZ,t}\mathcal{D}_\theta \cdot \hat{\Omega}_k dz d\theta - \int \overline{x}_{zZ,k}\mathcal{D}_\theta \cdot \hat{\Omega}_t dz d\theta + \int \overline{x}_z \mathsf{B}_{t,k} dz d\theta + \int \overline{x}_{zz}\mathsf{C}_{t,k} dz d\theta.$$

For the second half of the corollary we note that we can iterate the LOM for $\mathcal{D}_\theta \cdot \hat{\Omega}_{\sigma\sigma,t}$ forward starting from the initial condition $\mathcal{D}_\theta \cdot \hat{\Omega}_{\sigma\sigma,0} = 0$ to get

$$\mathcal{D}_\theta \cdot \hat{\Omega}_{\sigma\sigma,t} = -\sum_{k=0}^{t-1}\sum_{s=0}^{\infty} \mathcal{L}^{t-1-k} \cdot \mathsf{a}_{s-k}\overline{X}_{\sigma\sigma,s} - \sum_{k=0}^{t-1} \mathcal{L}^{t-1-k} \cdot \mathsf{a}_{\sigma\sigma}.$$

By construction and applying integration by parts yields

$$\left(\overline{\int x d\Omega}\right)_{\sigma\sigma,t} = \int \overline{x}_{\sigma\sigma} d\Omega^* - \int \overline{x}_z \mathcal{D}_\theta \cdot \hat{\Omega}_{\sigma\sigma,t} dz d\theta$$

substituting for $\mathcal{D}_\theta \cdot \hat{\Omega}_{\sigma\sigma,t}$ and $\overline{x}_{\sigma\sigma}$ using Lemma 3(SO) we have equation the second equation in Corollary 1(SO) with

$$\mathsf{J}_{\sigma\sigma,t} = \int \mathsf{x}_{\sigma\sigma} d\Omega^* + \sum_{k=0}^{t-1} \mathcal{I} \cdot \mathcal{L}^{t-1-k} \cdot \mathsf{a}_{\sigma\sigma}$$

which conveniently satisfies the recursion $\mathsf{J}_{\sigma\sigma,t} = \mathsf{J}_{\sigma\sigma,t-1} + \mathcal{I} \cdot \mathcal{L}^{t-1} \cdot \mathsf{a}_{\sigma\sigma}$.

A.12 Proof of Proposition 1(SO)

To show equation (32) we differentiate the G mapping twice with respect to $\hat{Z}_t$ and $\hat{Z}_k$ and add to it the derivative of the G mapping in direction $\hat{Z}_{t,k}$ to get

$$\mathsf{G}_x \left(\overline{\int x d\Omega}\right)_{ZZ,t,k} + \mathsf{G}_X \overline{X}_{ZZ,t,k} + \mathsf{G}_{\Theta,t,k} = 0$$

where

$$\begin{aligned}
\mathsf{G}_{\Theta,t,k} =& \mathsf{G}_{xx} \cdot \left(\left(\overline{\int x d\Omega}\right)_{Z,t}, \left(\overline{\int x d\Omega}\right)_{Z,k}\right) + \mathsf{G}_{xX} \cdot \left(\left(\overline{\int x d\Omega}\right)_{Z,t}, \overline{X}_{Z,k}\right) + \mathsf{G}_{x\Theta} \cdot \left(\left(\overline{\int x d\Omega}\right)_{Z,t}, \rho_\Theta^k\right) \\
&+ \mathsf{G}_{Xx} \cdot \left(\overline{X}_{Z,t}, \left(\overline{\int x d\Omega}\right)_{Z,k}\right) + \mathsf{G}_{xX} \cdot \left(\overline{X}_{Z,t}, \overline{X}_{Z,k}\right) + \mathsf{G}_{x\Theta} \cdot \left(\overline{X}_{Z,t}, \rho_\Theta^k\right) \\
&+ \mathsf{G}_{\Theta x} \cdot \left(\rho_\Theta^t, \left(\overline{\int x d\Omega}\right)_{Z,k}\right) + \mathsf{G}_{\Theta X} \cdot \left(\rho_\Theta^t, \overline{X}_{Z,k}\right) + \mathsf{G}_{\Theta\Theta} \cdot \left(\rho_\Theta^t, \rho_\Theta^k\right).
\end{aligned}$$

Substituting for $\left(\overline{\int x d\Omega}\right)_{ZZ,t,k}$ using Corollary 1(SO) yields the desired expression.

To show equation (32) we differentiate the G mapping twice with respect to σ and add to it the derivative of the G mapping in direction $\hat{Z}_{\sigma\sigma,t}$ to find

$$\mathsf{G}_x \left(\overline{\int x d\Omega}\right)_{\sigma\sigma,t} + \mathsf{G}_X \overline{X}_{\sigma\sigma,t} = 0.$$

The desired expression is then obtained by substituting for $\left(\overline{\int x d\Omega}\right)_{\sigma\sigma,t}$ again by using Corollary 1(SO).