A Perturbational Approach for Approximating Heterogeneous-Agent Models

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Motivation

- Canonical framework to study aggregate fluctuations
  - aggregate shocks + incomplete markets + het. agents (HA)
- Challenge: equilibria are difficult to compute
  - distribution of individual characteristics is a state variable
  - distribution follows complicated LoM with agg shocks
- Existing methods often rely on 1st order appr. and MIT shocks
  - cannot study stabilization policies, risk, asset prices, portfolio choice
- This paper: proposes a novel method to approx HA economies
  - fast, efficient, and easy to implement
  - scalable to higher-order approximations

What is novel?

- Standard approach (Reiter, Mitman, Auclert...)
  - discretize distribution and its LoM (e.g., "histogram method")
  - obtain 1st order approx via Taylor expansions (MIT shocks)
- Our approach
  - derive exact theoretical responses for any given order of appr.
  - compute those expressions numerically via discretization
- 1st order:
  - two approaches agree as grid size $\rightarrow 0$
  - ours is faster since we can utilize exact analytical expressions
- higher orders:
  - naive extensions of existing methods to higher order miss terms
  - MIT shocks do not recover effects of risk

Canonical HA representation

Eqm condititions in HA models:

$$F\left(z_{i,t-1}, x_{i,t}, \mathbb{E}_{i,t}x_{i,t+1}, X_t, \theta_{i,t}\right) = 0 \text{ for all } i, t \tag{1}$$

$$G\left(\int x_{i,t}di, X_t, \Theta_t\right) = 0 \text{ for all } t \tag{2}$$

where

- $\theta_{i,t}$, Θ_t: indiv and agg exogenous shocks, AR(1)

$$\theta_{i,t} = \rho_\theta \theta_{i,t-1} + \varepsilon_{i,t}$$
$$\Theta_t = \rho_\Theta \Theta_{t-1} + \mathcal{E}_t$$

- $x_{i,t}$, X_t: are indiv and agg endogenous variables
    - $z_{i,t-1} \in x_{i,t-1}$ predetermined in $t-1$
- Initial conditions: Θ_{-1} and distribution Ω_{-1} over $\left(z_{i-1}, \theta_{i,-1}\right)$
- Eqm given initial conditions is given by: $\left\{X_t\left(\mathcal{E}^t\right), x_t\left(\varepsilon_i^t, \mathcal{E}^t\right)\right\}$

Recursive representation

Let $Z = [\Theta, \Omega]^T$: aggregate state

- $\widetilde{x}\,(z, \theta, Z)$, $\widetilde{X}\,(Z)$, $\widetilde{\Omega}\,(Z)$ are indiv and agg policy functions
- $\widetilde{z}\,(z, \theta, Z) = \mathrm{P}\widetilde{x}\,(z, \theta, Z)$

Recursive representation

$$F\left(z, \widetilde{x}, \mathbb{E}\widetilde{x}, \widetilde{X}, \theta\right) = 0 \text{ for all } z, \theta, Z$$

$$G\left(\int \widetilde{x} d\Omega, \widetilde{X}, \Theta\right) = 0 \;\text{ for all } Z$$

$$\widetilde{\Omega}\,\langle z', \theta' \rangle = \iint \iota\left(\widetilde{z}(z, \theta, Z) \leq z'\right) \iota(\rho_\theta \theta + \epsilon \leq \theta') d \Pr\left(\epsilon\right) d\Omega \text{ for all } Z$$

Example: Krusell-Smith

- Households

$$\max_{\{c_{i,t},k_{i,t}\}_{t\geq 0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_{i,t})$$

$$c_{i,t} + k_{i,t} = R_t k_{i,t-1} + W_t \exp(\theta_{i,t})$$

$$k_{i,t} \geq 0$$

- Firms

$$\max_{N_t,K_t} \exp(\Theta_t) K_t^{\alpha} N_t^{1-\alpha} + (1-\delta)K_t - W_t N_t - R_t K_t$$

- Market clearings

$$K_t = \int k_{i,t} di, \qquad N_t = \int \exp(\theta_{i,t}) \, di$$

Mapping of KS economy

- Variables:

$$X_t = (K_t, W_t, R_t),\ x_{i,t} = (k_{i,t}, c_{i,t}, \lambda_{i,t}, \zeta_{i,t}),\ z_{i,t} = k_{i,t}$$

- Mapping F:

$$\begin{aligned} c_{i,t} + k_{i,t} - R_t k_{i,t-1} - W_t \exp(\theta_{i,t}) &= 0 \\ \lambda_{i,t} - R_t u_c(c_{i,t}) &= 0 \\ u_c(c_{i,t}) + \zeta_{i,t} - \beta \mathbb{E}_t \lambda_{i,t+1} &= 0 \\ k_{i,t}\zeta_{i,t} &= 0 \end{aligned}$$

- Mapping G:

$$\begin{aligned} K_t - \int k_{i,t-1} di &= 0 \\ R_t + \delta - 1 - \alpha \exp\left(\Theta_t\right) K_t^{\alpha-1} &= 0 \\ W_t - (1-\alpha) \exp\left(\Theta_t\right) K_t^{\alpha} &= 0 \end{aligned}$$

Standard perturbational approach

1. Scale aggregate shocks by $\sigma \geq 0$
    - shock process: $\Theta_t = \rho_\Theta \Theta_{t-1} + \sigma \mathcal{E}_t$
    - policy functions: $\widetilde{X}(Z;\sigma)$, $\widetilde{\Omega}(Z;\sigma)$, ...
2. Find steady state (SS) for $\sigma = 0$ economy
3. Use Taylor expansions w.r.t. σ to approximate stochastic economy around that SS

- Quick, standard way to solve RA-DSGE models
    - runs into trouble when Z is high-dimensional

0th order economy

0th Order

- Notation for $\sigma = 0$ economy
  - $\overline{X}(Z) := \widetilde{X}(Z;0)$, $\overline{x}(z,\theta,Z) := \widetilde{x}(z,\theta,Z;0)$, etc
  - $\overline{Z}(Z) := \left[\rho_\Theta \Theta, \quad \overline{\Omega}(Z)\right]$
- Steady state: $Z^* = [0, \quad \Omega^*]$
  - Ω^* : invariant distribution without agg. shocks
  - $\Lambda(z', \theta', z, \theta)$: transition probability density
  - $\overline{X} := \overline{X}(Z^*)$, $\overline{x}(z,\theta) := \overline{x}(z,\theta,Z^*)$, etc
  - by definition, $\overline{Z} = Z^*$

Solving 0th Order

- Ω^*, $\overline{X}$, $\overline{x}(z,\theta)$ can be found with standard methods
  - appr. policy rules with quadratic splines (basis functions)
  - solve for optimal policy with endog. grid method
- Basis functions also give $\overline{x}_z(z,\theta)$, $\overline{x}_{zz}(z,\theta)$, etc
- Automatic differentiation gives all derivatives of $\overline{F}$ and $\overline{G}$
  - denote $\mathbf{G}_x$, $\mathbf{G}_X$, $\mathbf{G}_\Theta$, etc.
- Treat all of these objects as known

Assumptions

- Stability and smoothness assumptions:
  1. $\lim_{t\to\infty} \overline{Z}_t(Z_0) = Z^*$ for all Z_0 in a neighborhood of Z^*;
  2. $\widetilde{X}(Z;\sigma)$ is sufficiently differentiable at $(Z,\sigma)=(Z^*,0)$
  3. $\widetilde{x}(z,\theta,Z;\sigma)$ is continuous and piecewise sufficiently differentiable at $(Z,\sigma)=(Z^*,0)$ for all (z,θ)
  4. Ω^* has a finite number of mass-points $\{z_n^*\}_n$
- Remarks
  - 1. and 2. are standard (Blanchard-Kahn)
  - 3 is analogue of 2 for individual policy functions with kinks.
  - 4. allow for mass-points in Ω and kinks in $\tilde{x}$

Notation

- $\overline{X}_Z$ is the Frechet derivative of $\tilde{X}$ evaluated at $(Z^*, 0)$ review
- $\overline{X}_Z \cdot \hat{Z}$ is the value of derivative in direction $\hat{Z}$
  - i.e. how much X changes if state changes to $Z^* + \hat{Z}$
- Similarly for $\bar{x}_Z(z, \theta)$ and $\bar{\Omega}_Z$
- Extends to higher orders, i.e. $\overline{X}_{ZZ} \cdot \left(\hat{Z}_1, \hat{Z}_2\right)$

Computing Taylor Expansions

- Solving brute force (Dynare) is impractical
  - $\bar{\Omega}_Z$ is approximately $N \times N$
  - $\bar{\Omega}_{ZZ}$ is approximately $N \times N \times N$
- Idea: only evaluate in direction needed for expansion
  - $\overline{X}_Z$ is large
  - $\overline{X}_Z \cdot \hat{Z}$ is not
- Use analytical expressions
  - constructed with Frechet derivatives and linear operators
  - extends to higher order Taylor expansion

1st order expansions

Directions of interest

- Define sequence of directions $\{\hat{Z}_t\}_t$ recursively

$$\hat{Z}_0 := [1, \quad \mathbf{0}]^T,$$

$$\hat{Z}_1 := \overline{Z}_Z \cdot \hat{Z}_0 = \left[\rho_\Theta, \quad \overline{\Omega}_Z \cdot \hat{Z}_0\right]^T$$

$$\hat{Z}_t := \overline{Z}_Z \cdot \hat{Z}_{t-1} = \left[\rho_\Theta^t, \quad \overline{\Omega}_Z \cdot \hat{Z}_{t-1}\right]^T$$

- Let

$$\overline{X}_{Z,t} := \overline{X}_Z \cdot \hat{Z}_t$$

- Intuition:
  - $\{\hat{Z}_t\}_t$ traces changes of agg state due to shock to Θ in pd 0
  - $\{\overline{X}_{Z,t}\}_t$ is the IR to an "MIT shock"

1st Order Approximation

Lemma
To the first order approximation X_t satisfies

$$X_t\left(\mathcal{E}^t\right) = \overline{X} + \sum_{s=0}^{t} \overline{X}_{Z,t-s}\mathcal{E}_s + O\left(\|\mathcal{E}\|^2\right).$$

- Solving 1st order approximation = finding response to MIT shock (Boppart et al, 2018)
- Same information contained in impulse responses

$$\mathbb{E}\left[X_t|\mathcal{E}_0\right] - \mathbb{E}\left[X_t|\mathcal{E}_0 = 0\right] = \overline{X}_{Z,t}\mathcal{E}_0 + O\left(\underline{\mathcal{E}}^2\right)$$

- Need to find $\{\overline{X}_{Z,t}\}_t$

Finding $\left\{\overline{X}_{Z,t}\right\}$

- Recall
$$G\left(\int \overline{x} d\Omega, \overline{X}, \Theta\right) = 0 \text{ for all } Z$$

Finding $\{\overline{X}_{Z,t}\}$

- Recall

$$G\left(\int \overline{x} d\Omega, \overline{X}, \Theta\right) = 0 \text{ for all } Z$$

- Differentiate at $Z = Z^*$ in direction $\hat{Z}_t$:

$$\mathrm{G}_x\left[\int \overline{x}_{Z,t} d\Omega^* + \int \overline{x} d\hat{\Omega}_t\right] + \mathrm{G}_X \overline{X}_{Z,t} + \mathrm{G}_\Theta \rho_\Theta^t = 0$$

Finding $\{\overline{X}_{Z,t}\}$

- Recall
$$G\left(\int \overline{x} d\Omega, \overline{X}, \Theta\right) = 0 \text{ for all } Z$$

- Differentiate at $Z = Z^*$ in direction $\hat{Z}_t$:
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- Step 1: characterize $\overline{x}_{Z,t}$ and then $\int \overline{x}_{Z,t} d\Omega^*$
- Step 2: characterize $d\hat{\Omega}_t$ and then $\int \overline{x} d\hat{\Omega}_t$
- Step 3: plug in the eqn above to find $\{\overline{X}_{Z,t}\}_t$

Step 1

Lemma

$$\bar{x}_{Z,t}\left(z,\theta\right) = \sum_{s=0}^{\infty} \underbrace{\mathrm{x}_s\left(z,\theta\right)}_{=\partial x_t / \partial X_{t+s}} \bar{X}_{Z,t+s}$$

where $\mathrm{x}_s\left(z,\theta\right)$ *are known from zeroth order*

Step 1

Lemma

$$\bar{x}_{Z,t}(z,\theta) = \sum_{s=0}^{\infty} \underbrace{\mathrm{x}_s(z,\theta)}_{=\partial x_t/\partial X_{t+s}} \bar{X}_{Z,t+s}$$

where $\mathrm{x}_s(z,\theta)$*are known from zeroth order*

$$\mathrm{x}_0(z,\theta) = -\left(\mathrm{F}_x(z,\theta) + \mathrm{F}_{x'}(z,\theta)\overline{x}_z^+(z,\theta)\,\mathrm{P}\right)^{-1}\mathrm{F}_X(z,\theta)$$

$$\mathrm{x}_{s+1}(z,\theta) = -\left(\mathrm{F}_x(z,\theta) + \mathrm{F}_{x'}(z,\theta)\overline{x}_z^+(z,\theta)\,\mathrm{P}\right)^{-1}\mathrm{F}_{x'}(z,\theta)\mathrm{x}_s^+(z,\theta)$$

where $\mathrm{x}_s^+(z,\theta) = \mathbb{E}\left[\mathrm{x}_s(,)|z,\theta\right]$ *and* $\overline{x}_z^+(z,\theta) = \mathbb{E}\left[\bar{x}_z(\cdot,\cdot)|z,\theta\right].$

Step 1

Lemma

$$\bar{x}_{Z,t}(z,\theta)=\sum_{s=0}^{\infty}\underbrace{\mathrm{x}_s(z,\theta)}_{=\partial x_t/\partial X_{t+s}}\bar{X}_{Z,t+s}$$

where $\mathrm{x}_s(z,\theta)$*are known from zeroth orde*

- Intuition: individuals only care about effect on prices $\left\{\bar{X}_{Z,s}\right\}_s$
- We now can replace $\{\overline{x}_{Z,t}(z,\theta)\}_{(z,\theta)}$ with $\left\{\overline{X}_{Z,s}\right\}_s$:

$$\int\overline{x}_{Z,t}d\Omega^*=\sum_{s=0}^{\infty}\left(\int\mathrm{x}_s d\Omega^*\right)\overline{X}_{Z,t+s}$$

Step 2: $\mathcal{M}$ and $\mathcal{L}$

- Now we want to characterize

$$\int \overline{x} d\hat{\Omega}_t$$

- Linear operators $\mathcal{M}$ and $\mathcal{L}$ help to characterize $\hat{\Omega}_t$
- For any $y : (z, \theta) \to \mathbb{R}$ they return

$$(\mathcal{M} \cdot y) \langle z', \theta' \rangle := \int \overline{\Lambda}(z', \theta', z, \theta) y(z, \theta) \, d\Omega^* (z, \theta)$$

$$(\mathcal{L} \cdot y) \langle z', \theta' \rangle := \int \overline{\Lambda}(z', \theta', z, \theta) \overline{z}_z(z, \theta) y(z, \theta) \, dz d\theta$$

- Intuition: suppose indiv. policy functions are perturbed by $\hat{z}_0(z, \theta)$
  - effect on agg. distribution in pd 1: $\frac{d}{d\theta}\hat{\Omega}_1 = \mathcal{M} \cdot \hat{z}_0$
  - effect on agg. distribution in pd 2: $\frac{d}{d\theta}\hat{\Omega}_2 = \mathcal{L} \cdot \frac{d}{d\theta}\hat{\Omega}_1$

Step 2: recursive LoM

Lemma
$\frac{d}{d\theta}\hat{\Omega}_t$ satisfies a recursion

$$\frac{d}{d\theta}\hat{\Omega}_t = \mathcal{L} \cdot \frac{d}{d\theta}\hat{\Omega}_{t-1} - \sum_{s=0}^{\infty} \mathcal{M} \cdot \mathrm{z}_s \overline{X}_{Z,t+s},$$

where $\frac{d}{d\theta}\hat{\Omega}_0 = \mathbf{0}$.

- Intuition:
  - operator $\mathcal{L}$ captures first-order effect of past changes agg dist
  - $\mathcal{M} \cdot \mathrm{z}_s$ captures first-order effect of ind. policy functions to change in aggregates s periods ahead

Step 2: characterize $\int \overline{x} d\hat{\Omega}_t$

- We have

$$\int \overline{x} d\hat{\Omega}_t = -\int \overline{x}_z \frac{d}{d\theta} \hat{\Omega}_t dz d\theta := -\mathcal{I} \cdot \frac{d}{d\theta} \hat{\Omega}_t$$

- Together with previous results this implies that

$$\int \overline{x} d\hat{\Omega}_t = \sum_{s=0}^{\infty} (\mathcal{I} \cdot \mathsf{A}_{t,s}) \overline{X}_{Z,s},$$

where $\{\mathsf{A}_{t,s}\}_{t,s}$ follow a recursion $\mathsf{A}_{0,s} = 0$ and

$$\mathsf{A}_{t,s} = \mathcal{L} \cdot \mathsf{A}_{t-1,s} + \mathcal{M} \cdot \mathsf{z}_{s-t-1}$$

Step 3: solve 1st order appr

Proposition

$\left\{\overline{X}_{Z,t}\right\}_t$ *is the solution to*

$$\mathsf{G}_x \sum_{s=0}^{\infty} \mathsf{J}_{t,s}\overline{X}_{Z,s} + \mathsf{G}_X\overline{X}_{Z,t} + \mathsf{G}_\Theta \rho_\Theta^t = 0,$$

where $\{\mathsf{J}_{t,s}\}_{t,s}$ *satisfies*

$$\mathsf{J}_{t,s} = \int \mathsf{x}_{s-t} d\Omega^* + \mathcal{I} \cdot \mathsf{A}_{t,s}$$

- Linear system of equations that determines $\{\overline{X}_{Z,t}\}_t$

2nd order expansions

Higher-order approximations

- Same approach extends with **minimal** changes to higher orders
  - exactly the same steps to derive approx terms
  - almost the same mathematical form of equations
  - many 1st order terms get recycled for higher-order computations
- I will illustrate intuition for this using a simple example

Super simple example

- In RCE policy functions depend on other policy functions, e.g.

$$f\left(g\left(a\right)\right)$$

- First order expansion:

$$f_a = f_g g_a$$

- Second order expansion:

$$f_{aa} = f_g g_{aa} + f_{gg} g_a g_a$$

- Note the general structure of second order terms
    - will be useful to think about directions

Super simple example

- Our procedure for 1st order approximation:
  - we know f_g from 0th order
  - we developed a way to find f_a and g_a with

$$f_a = f_g g_a \tag{3}$$

Super simple example

- Our procedure for 1st order approximation:
  - we know f_g from 0th order
  - we developed a way to find f_a and g_a with

$$f_a = f_g g_a \tag{3}$$

- Our procedure for 2nd order approximation:
  - know f_g, f_{gg} from 0th order, g_a from 1st order
  - need to develop a way to find f_{aa} and g_{aa} with

$$f_{aa} = f_g g_{aa} + c \tag{4}$$

where $c = f_{gg} g_a g_a$ is known

Super simple example

- Our procedure for 1st order approximation:
  - we know f_g from 0th order
  - we developed a way to find f_a and g_a with

$$f_a = f_g g_a \tag{3}$$

- Our procedure for 2nd order approximation:
  - know f_g, f_{gg} from 0th order, g_a from 1st order
  - need to develop a way to find f_{aa} and g_{aa} with

$$f_{aa} = f_g g_{aa} + c \tag{4}$$

  where $c = f_{gg} g_a g_a$ is known

- (1) and (2) have almost identical structure!

2nd order directions

- Non-linearities from shocks

$$\hat{Z}_{t,s} = \overline{Z}_Z \cdot \hat{Z}_{t-1,s-1} + \overline{Z}_{ZZ} \cdot \left(\hat{Z}_{t-1}, \hat{Z}_{s-1}\right)$$

$$\overline{X}_{ZZ,t,k} := \overline{X}_Z \cdot \hat{Z}_{t,k} + \overline{X}_{ZZ} \cdot \left(\hat{Z}_t, \hat{Z}_k\right)$$

- Precautionary motives:

$$\hat{Z}_{\sigma\sigma,t} = \left[0, \overline{\Omega}_{\sigma\sigma}\right]^T + \overline{Z}_Z \cdot \hat{Z}_{\sigma\sigma,t-1}$$

$$\overline{X}_{\sigma\sigma,t} := \overline{X}_{\sigma\sigma} + \overline{X}_Z \cdot \hat{Z}_{\sigma\sigma,t}$$

where $\overline{X}_{\sigma\sigma} := \frac{\partial^2}{\partial\sigma^2}\widetilde{X}\left(Z^*;\sigma\right)\Big|_{\sigma=0}$, etc

Recycle 1st order for 2nd order

- $\{\overline{X}_{ZZ,t,k}\}_{t,k}$ and $\{\overline{X}_{\sigma\sigma,t}\}_t$ recover second-order approximation:

$$X_t\left(\mathcal{E}^t\right) = \ldots + \frac{1}{2}\left(\sum_{s=0}^{t}\sum_{m=0}^{t}\overline{X}_{ZZ,t-s,t-m}\mathcal{E}_s\mathcal{E}_m + \overline{X}_{\sigma\sigma,t}\right) + O\left(\|\mathcal{E}\|^3\right)$$

- Finding components of $\overline{X}_{ZZ,t,k}$ and $\overline{X}_{\sigma\sigma,t}$
  - $\overline{X}_{ZZ}\cdot\left(\hat{Z}_t,\hat{Z}_k\right)$: explicit formula in terms of 1st and 0th order
  - $\overline{X}_Z\cdot\hat{Z}_{t,k}$ and $\overline{X}_Z\cdot\hat{Z}_{\sigma\sigma,t}$: determined almost identically to $\overline{X}_Z\cdot\hat{Z}_t$
  - Impulse responses are insufficient

$$\mathbb{E}\left[X_t|\mathcal{E}_0\right] - \mathbb{E}\left[X_t|\mathcal{E}_0=0\right] = \ldots + \overline{X}_{ZZ,t,t}\mathcal{E}_0^2 + O\left(\underline{\mathcal{E}}^3\right),$$

Linear system for $\left\{\bar{X}_{ZZ,t,k}\right\}_{t,k}$ and $\left\{\bar{X}_{\sigma\sigma,t}\right\}_{t}$

$$\mathsf{G}_x \sum_{s=0}^{\infty} \mathsf{J}_{t,s}\overline{X}_{\sigma\sigma,s} + \mathsf{G}_x\mathsf{H}_{\sigma\sigma,t} + \mathsf{G}_X\overline{X}_{\sigma\sigma,t} = 0, \tag{5}$$

and

$$\mathsf{G}_x \sum_{s=0}^{\infty} \mathsf{J}_{t,s}\overline{X}_{ZZ,t-k+s,s} + \mathsf{G}_x\mathsf{H}_{t,k} + \mathsf{G}_X\overline{X}_{ZZ,t,k} + \mathsf{G}_{\Theta,t,k} = 0. \tag{6}$$

the expressions for $\mathsf{G}_{\Theta,t,k}$and $\mathsf{H}_{t,k}, \mathsf{H}_{\sigma\sigma,t}$ are in the paper

Comparison to existing approaches

- State of the art: Auclert et al. (ABRS 2021)
  - first order expansions of similar class of economies
  - MIT shocks, histogram method, numerical derivatives
- 1st order: we are theoretically equivalent to ABRS
  - their computations converge to our formulas as grid size $\to 0$
  - our method faster because we can use exact formulas
- 2nd and higher orders: ABRS doesn't work
  - MIT shocks do not capture effects of risk
  - histogram method fails (misses $f_{gg} g_a g_a$ terms)

$$\lim_{\text{num grid points}\to\infty} \overline{X}^{HIST}_{ZZ,t,s} \neq \overline{X}_{ZZ,t,s}$$

Histogram method (Review)

- Histogram (bins,mass points) to approximate Ω
  - grid $\{\mathrm{z}_i\}_{i=0}^N$ represent midpoints of bins
  - $\{\omega_i^z\}$ mass at points $\{\mathrm{z}_i\}_{i=0}^N$
- Functions $\left\{\mathcal{P}^i\left(\cdot\right)\right\}$ so for $z \in [\mathrm{z}_i, \mathrm{z}_{i+1}]$ only non-zero values

$$\mathcal{P}^i\left(z\right) = \frac{\mathrm{z}_{i+1} - z}{\mathrm{z}_{i+1} - \mathrm{z}_i}, \quad \mathcal{P}^{i+1}\left(z\right) = \frac{z - \mathrm{z}_i}{\mathrm{z}_{i+1} - \mathrm{z}_i}.$$

  - $\mathcal{P}^i\left(z\right)$: the probability z is assigned to bin with midpoint z_i.
- Applications: Linear approximates for aggregates and LOM
  - $\int x\left(z,\theta\right) d\Omega \approx \int \sum_i x\left(\mathrm{z}_i,\theta\right) \omega_i^z dF\left(\theta\right)$
  - $\tilde{\omega}_j^z(\Theta,\omega) \approx \sum_i \omega_i^z \int \mathcal{P}^j\left(\tilde{z}\left(\mathrm{z}_i,\theta,\Theta,\omega^z\right)\right) dF\left(\theta\right)$
- Standard approach: Differentiate after applying discretizing using histogram method

Why does Histogram method fail? Simple Example

Histogram method approximates $f(z) \approx \sum_{i=0}^{N} \mathcal{P}^i(z) f(\mathrm{z}_i)$. Now...

- Expand LHS $f(z+\hat{z})$

$$f(z) + f'(z)\hat{z} + \frac{1}{2} f''(z)\hat{z}^2 + o\left(\hat{z}^2\right)$$

- Expand RHS $\sum_{i=0}^{N} \mathcal{P}^i(z+\hat{z}) f(\mathrm{z}_i)$

$$\sum_{i=0}^{N} \mathcal{P}^i(z) f(\mathrm{z}_i) + \sum_{i=0}^{N} \mathcal{P}_z^i(z) f(\mathrm{z}_i)\hat{z} + \frac{1}{2}\sum_{i=0}^{N} \mathcal{P}_{zz}^i(z) f(\mathrm{z}_i)\hat{z}^2 + o\left(\hat{z}^2\right)$$

- Now take limits as $N \to \infty$
  - zeroth order $\sum_{i=0}^{N} \mathcal{P}^i(z) f(\mathrm{z}_i) \to f(z)$

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- Expand RHS $\sum_{i=0}^{N} \mathcal{P}^i(z+\hat{z}) f(\mathrm{z}_i)$

$$\sum_{i=0}^{N} \mathcal{P}^i(z) f(\mathrm{z}_i) + \sum_{i=0}^{N} \mathcal{P}^i_z(z) f(\mathrm{z}_i)\hat{z} + \frac{1}{2}\sum_{i=0}^{N} \mathcal{P}^i_{zz}(z) f(\mathrm{z}_i)\hat{z}^2 + o\left(\hat{z}^2\right)$$

- Now take limits as $N \to \infty$
    - first order $\sum_{i=0}^{N} \mathcal{P}^i_z(z) f(\mathrm{z}_i)\hat{z} = \frac{f\left(\mathrm{z}_{i+1}\right) - f(\mathrm{z}_i)}{\mathrm{z}_{i+1} - \mathrm{z}_i} \to f'(z)\hat{z}$

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Histogram method approximates $f(z) \approx \sum_{i=0}^{N} \mathcal{P}^i(z) f(\mathrm{z}_i)$. Now...

- Expand LHS $f(z+\hat{z})$

$$f(z) + f'(z)\hat{z} + \frac{1}{2}f''(z)\hat{z}^2 + o\left(\hat{z}^2\right)$$

- Expand RHS $\sum_{i=0}^{N} \mathcal{P}^i(z+\hat{z}) f(\mathrm{z}_i)$

$$\sum_{i=0}^{N} \mathcal{P}^i(z) f(\mathrm{z}_i) + \sum_{i=0}^{N} \mathcal{P}_z^i(z) f(\mathrm{z}_i)\hat{z} + \frac{1}{2}\sum_{i=0}^{N} \mathcal{P}_{zz}^i(z) f(\mathrm{z}_i)\hat{z}^2 + o\left(\hat{z}^2\right)$$

- Now take limits as $N \to \infty$
    - second order $\sum_{i=0}^{N} \mathcal{P}_{zz}^i(z) f(\mathrm{z}_i)\hat{z}^2 = 0 \nrightarrow f''(z)\hat{z}^2$

Why does Histogram method fail?

- Tractability of histogram methods come from “uniform” lotteries
    - preserves mass and conditional means

$$\sum_i \mathcal{P}^i(z) = 1$$

$$\sum_i \mathcal{P}^i(z)\,\mathrm{z}_i = z$$

    - which works for first-order but not higher in presence of curvature
- Our approach discretizes after differentiating
    - approximates $f''(z)\,\hat{z}$ instead of $\sum_{i=0}^{N} \mathcal{P}^i_{zz}(z)\, f(\mathrm{z}_i)\,\hat{z}^2$
    - works for all orders
- Show later in the application than the missing terms can affect conclusions

Applications

Goals

- Use a calibrated version of the basic model to assess the method
    - speed, accuracy comparisons,role of nonlinearities
- Applications to illustrate usefulness
    1. welfare from stabilization policies
    2. impact of uncertainty
    3. household portfolios
    4. transitions

Comparisons

First Order		Second Order		
Step	Time	Step	Time (ZZ)	Time($\sigma\sigma$)
		Additional 1st order terms	0.70s	
Compute $\{x_s\}$	0.07s	Compute $\{x_{t,k}\}$ and $\{x_{\sigma\sigma}\}$	0.64s	0.05s
Compute $\mathcal{L}$ and $\{a_t\}_t$	0.13s	Compute $\{b_{t,k}, c_{t,k}\}$ and $\{b_{\sigma\sigma}\}$	0.21s	0.45s
Compute $\{J_{t,s}\}_{t,s}$	0.17s	Compute $H_{t,k}$ and $H_{\sigma\sigma,t}$	0.07s	0.05s
Compute $\{\overline{X}_{Z,t}\}_t$	0.13s	Compute $\{\overline{X}_{ZZ,t,k}\}_{t,k}$, $\{\overline{X}_{\sigma\sigma,t}\}_t$	0.19s	0.28s
Total	0.5s		1.81s	0.83s
ABRS	1.51s			

Stabilization Policy

- Simple model of stabilization policy: choose optimal τ_Θ in

$$\tau_t = \overline{\tau} + \tau_\Theta \Theta_t$$

- Stabilization policy is a second order question
    - τ_Θ has no effect on welfare to the first order
- Add extra equation $\mathcal{W}\left(\Omega, \Theta; \tau_\theta\right) = \int V\left(k, \theta; \tau_\theta\right) d\Omega$ to G and use

$$\mathbb{E}\left[\mathcal{W}\right] = \bar{\mathcal{W}} + \frac{1}{2}\left(\sum_{s=0}^{\infty} \overline{\mathcal{W}}_{ZZ,s,s}\sigma_{\mathcal{E}}^2 + \overline{\mathcal{W}}_{\sigma\sigma,\infty}\right) + O\left(\underline{\mathcal{E}}^3\right)$$

- Compare answers if we tried to track distribution using the histogram method

Stabilization Policy: Results

- Optimal policy: Countercyclical fiscal policy
  - Raise taxes by 300 basis points for a 1% fall in TFP

risk aversion	τ_{Θ}^{*}	$\frac{\mathcal{W}^{\text{hist}}(\tau_{\Theta}^{*})}{\mathcal{W}\left(\tau_{\Theta}^{*}\right)}$	$\frac{\tau_{\Theta}^{*,\text{hist}}}{\tau_{\Theta}^{*}}$
2	-3.10	-348%	161%
3	-1.90	-230%	209%
4	-1.03	-226%	167%
5	-0.69	-217%	125%
7	-0.52	-187%	67%

The $\mathcal{W}^{\text{hist}}(\tau_{\Theta}^{*})$ uses the histogram method to compute the welfare and $\tau_{\Theta}^{*,\text{hist}}$ is the optimal policy using $\mathcal{W}^{\text{hist}}(\tau_{\Theta})$ as the measure of welfare

Effects on Uncertainty

- Large empirical literature about macroeconomic uncertainty
- What are the aggregate and distributional effects of uncertainty?
- Calibrate uncertainty shock to capture changes in VIX during Covid

Effects on Uncertainty: Methodology

- Conventional wisdom: requires 3rd or higher order expansion

- In paper: slight modification to second order expansion is sufficient

    - Extend shock process to allow for time varying volatility:

$$\mathcal{E}_t = \sqrt{1 + \Upsilon_{t-1}}\mathcal{E}_{\Theta,t}, \tag{7}$$

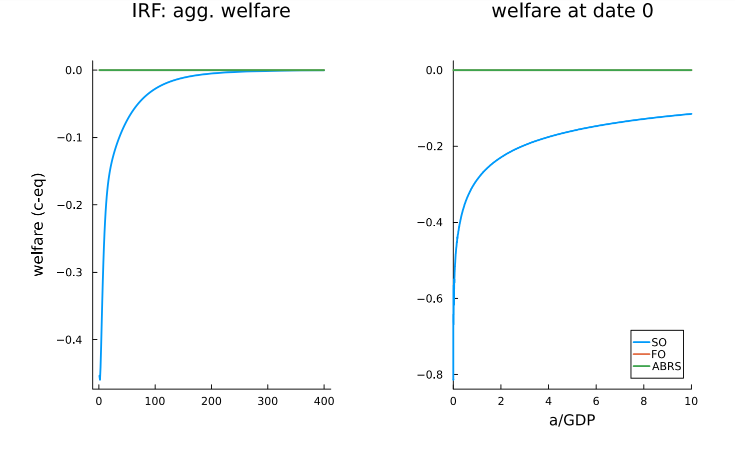
$$\Upsilon_t = \rho_\Upsilon \Upsilon_{t-1} + \mathcal{E}_{\Upsilon,t}, \tag{8}$$

    - Construct a few new terms

$$\overline{X}_{\sigma\sigma,t}\left(\mathcal{E}^t_\Upsilon\right) = \overline{X}_{\sigma\sigma,t} + \sum_{s=0}^{t} \overline{X}_{\Upsilon,t-s}\mathcal{E}_{\Upsilon,s},$$

Effects on Uncertainty: Results

- Average $\approx \frac{1}{2}\%$ of per-period consumption over their life
    - larger losses for low net worth



Household Portfolios

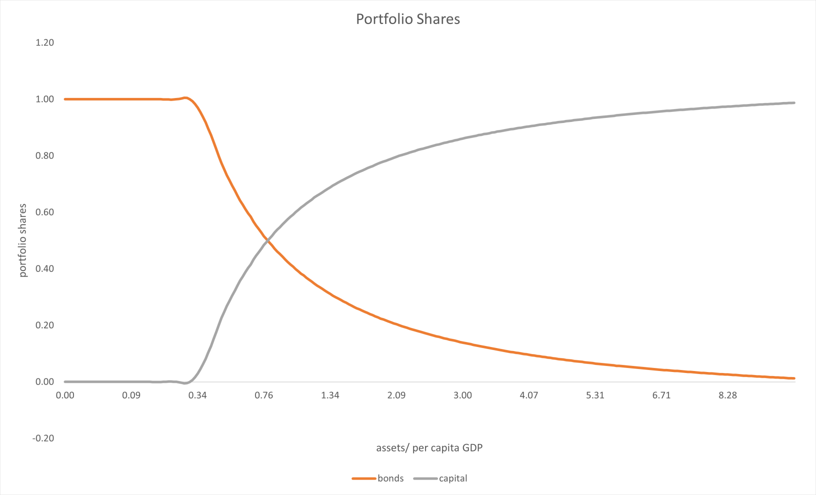
- Large empirical literature on household portfolios (Viceria, Campbell, Yogo etc)
- What are the predictions from standard macro model?

Household Portfolios: Methodology

- Key problem: HH portfolios not pinned down at $\sigma = 0$
- Use second order expansions to solve the limiting ($\sigma \to 0$) portfolio
  - HH portfolios linear equation given exposures of excess agg. returns
  - boils down to one nonlinear equation in exposure of excess returns
  - extends the Devereux Sutherland insight to HA economies
- Correct portfolio matters even for first order expansion

Household Portfolios: Results

- Standard model predicts that stock share is increasing in wealth



Transitions

- Often interested in the entire transition path after reforms (permanent changes)
  - economy transitions to a new steady state
  - welfare gains on transitions are large
- Same objects can be recycled to get approximations to transition paths

Transitions: Methodology

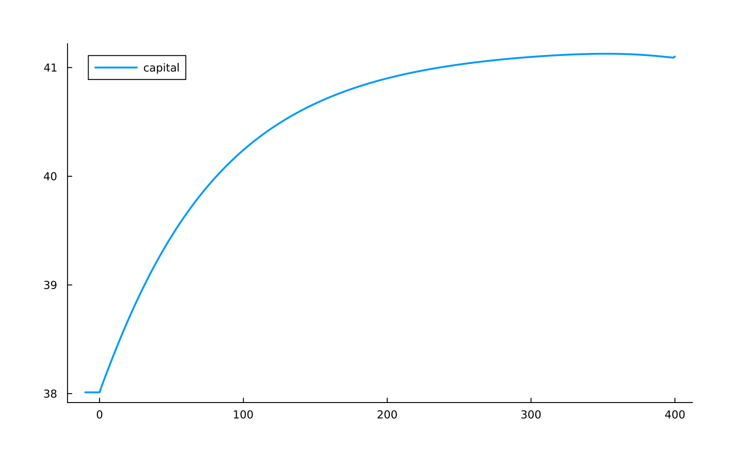
- Relax the assumption that the initial distribution $\Omega_0 = \Omega^*$
- Directions
  - Date 0 direction $\hat{Z}_{\Omega,0} = [0, \hat{\Omega}_0]^{\mathrm{T}}$ where $\hat{\Omega}_0 = \Omega^* - \Omega_0$
  - Date t directions as before $\hat{Z}_{\Omega,t} = \overline{Z}_Z \cdot \hat{Z}_{\Omega,t-1}$
  - To first-order $\overline{X}_{\Omega,t} := \overline{X}_Z \cdot \hat{Z}_{\Omega,t}$
- $\left\{\overline{X}_{\Omega,t}\right\}_t$ is the solution to

$$\underbrace{\mathsf{G}_x \sum_{s=0}^{\infty} \mathsf{J}_{t,s} \overline{X}_{\Omega,s} + \mathsf{G}_X \overline{X}_{\Omega,t}}_{\text{same as before}} + \mathsf{G}_x \mathsf{J}_{\Omega,t} = 0,$$

where $\mathsf{J}_{\Omega,t} = \mathcal{I} \cdot \mathcal{L}^t \cdot \frac{d}{d\theta} \hat{\Omega}_0$.

Transition: Results

Path of capital stock after one-time permanent 5% change in agg. TFP



Conclusions

- Tool for higher order approximations of heterogeneous agent models
    - with occasionally binding constraints
- Extends to
    - time varying volatility
    - portfolio problems
    - transitions

Frechet derivatives (Review)

- Consider a function $f : O \to Y$. The change of f from $\Omega \to \Omega + \hat{\Omega}$ is approximated as

$$f\left(\Omega + \hat{\Omega}\right) = f(\Omega) + f_{\Omega}(\Omega) \cdot \hat{\Omega} + o\left(\hat{\Omega}\right)$$

  - $f_{\Omega}(\Omega)$ is a linear operator ("huge matrix") on directions $\hat{\Omega}$
  - Eg: Jacobian matrix $\left[f^{j}_{\Omega_i}\right]_{i,j}$ or a kernel

- Easily computed using Gateaux $f_{\Omega}(\Omega) \cdot \hat{\Omega} = \lim_{\alpha \to 0} \frac{\|f\left(\Omega + \alpha\hat{\Omega}\right) - f(\Omega)\|}{\alpha}$

- Extends naturally to higher orders

$$f_{\Omega}\left(\Omega + \hat{\Omega}_2\right) \cdot \hat{\Omega}_1 = f_{\Omega}(\Omega) \cdot \hat{\Omega}_1 + f_{\Omega\Omega}(\Omega) \cdot \left(\hat{\Omega}_1, \hat{\Omega}_2\right) + o\left(\hat{\Omega}_2\right)$$

  - $f_{\Omega\Omega}(\Omega)$ is a bilinear operator ("3d tensor") on directions $\left(\hat{\Omega}_1, \hat{\Omega}_2\right)$
  - Eg, collection of Hessians $\left\{\left[f^{j}_{\Omega_i\Omega_k}\right]_{i,k}\right\}_{j}$

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Appendix: Baseline Calibration

Parameter	Description	Value
α	Capital share	0.36
β	Discount factor	0.983
σ	Risk aversion	2
δ	Depreciation rate of capital	1.77%
ϕ	Adjustment cost of capital	125
ρ_θ	Idiosyncratic mean reversion	0.966
$\sigma_\theta/\sqrt{1-\rho_\theta^2}$	Cross-sectional std of log earnings	0.503
ρ_Θ	Persistence of TFP shock	0.80
σ_Θ	Std of Aggregate TFP growth rate	0.014
N_ϵ	Points in Markov chain for ϵ	7
N_z	Grid points for the policy rule $\bar{x}^i(z)$	60
I_z	Grid points for the distribution $\bar{\omega}_i$	1000
T	Time horizon (in quarters) for IRF	400

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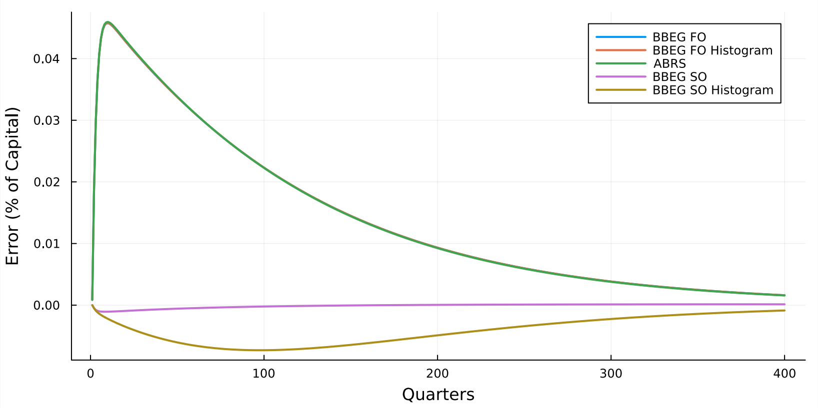
Appendix: Accuracy

- Check accuracy in approximated perfect foresight equilibrium
- Let $\hat{X}_t$ path of aggregates from approximation
- Given $\hat{X}$ solve for agent behavior and resulting path of distribution
- Aggregate to compute resulting path of aggregates $\tilde{X}$
    - In equilibrium we should have $\tilde{X} = \hat{X}$
- Measure accuracy by difference $\tilde{X} - \hat{X}$

Appendix: Accuracy Comparison



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