On the Dynamics of a Big Push *

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Abstract

We study a dynamic model of economic development that stems from the adoption of complementary technologies, and where government policies can affect development paths. We consider a growing technology frontier, which is consistent with multiple distributions of technology gaps and different degrees of technology adoption rates. We characterize equilibrium and optimal dynamic paths; surprisingly, a simple policy that corrects for the static markup distortion equalizes these paths. A particular case with a fixed technological frontier can be analyzed as a version of the Neoclassical Growth Model with a s-shaped production function. We focus on the effect of markups on the long run adoption of technologies, and on the effect of policies aimed at correcting the inefficiencies that spur from markups. A Big Push—sustained growth towards a stable steady state—occurs when the optimal static policy eliminates (bad) steady states. Otherwise, in spite of the existence of better steady states, the optimal policy does not generate a transition away from the bad steady state, in spite of being feasible. Finally, we do not find any role for policy as an equilibrium selection device.

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1 Introduction

In modern economies, the flow of new production techniques is accompanied by a continuous process where firms replace their older, less productive, techniques with those at the technological frontier. When frontier technologies are produced with goods, as firms adopt modern techniques the cost of adoption falls for all firms, fostering more adoption. These complementary actions can result in the coexistence of poor and rich economies: the poor one featuring infrequent adoption and large dispersion of production techniques, and the the rich economy featuring frequent adoption and low dispersion of techniques. A dynamic Big Push can result when a policy intervention, aimed at reducing the distortions arising from markups or at coordinating actions, embarks the poor economy in a sustained development path. Understanding the optimal development policy in this framework is the main object of study in this paper.

We provide a dynamic environment where the technological frontier advances exogenously through time, and were firms produce intermediate goods under monopolistic competition. Intermediate goods are aggregated to produce final consumption and modern technologies. At any point in time, firms can receive a free opportunity to adopt the technology at the frontier. Otherwise, the firm must acquire the frontier technique. As other firms adopt frontier techniques, a firm's older technique must be replaced as the firm's effective productivity declines. Because investing in techniques is lumpy, firms adjust their technology in discrete intervals, doing so whenever their technology gap to the frontier, g, is larger than $G(t)$. A dynamic equilibrium is characterized by a forward-looking optimality condition for $G(t)$ at each t, and a backward-looking distribution of gaps at t, $m(g,t)$. Using tools from the Mean Field Games theory we characterize a dynamic equilibrium and the planner's problem. We show that the solutions do not coincide. First, markup distortions generate a static and dynamic inefficiencies by raising the price of the intermediate aggregate.[1] The static inefficiency stems from a under-utilization of intermediate inputs, which lowers aggregate output. The dynamic inefficiency stems from a sub-optimal technology adoption policy that follows from the fact that markups also affect the price of adoption. Surprisingly, a revenue subsidy that corrects for the markup distortion fixes both margins of the inefficiency, while an adoption subsidy can correct the dynamic inefficiency margin. Second, there can be multiple equilibrium dynamic paths that stem from strategic complementarities among firms' adoption decisions. Strategic complementarities manifest in the aggregate production function: in spite a firm's production function being concave, when complementarities are strong, $\zeta > 1$, the aggregate production function exhibits a convex region as a function of $m(g,t)$. Multiple equilibrium dynamic paths result because when ζ is large there can be three equilibrium balanced-growth paths: one with no costly adoption (a poor economy), one with infrequent costly adoption, and one with frequent costly adoption (a rich economy). We show that the poor and rich equilibrium balanced-growth paths are stable, while the intermediate is generically unstable.[2] We interpret stability of the poor and rich as paths that

[1] Consistent with the empirical evidence in Hsieh and Klenow (2009) and Ayerst et al. (2023), more distorted 'less developed' economies will feature not only lower aggregate productivity, but also higher productivity dispersion across firms.

[2] We are yet to prove part of this statement in this general formulation of the problem, but we conjecture that this will be the case.

should be observed in the long run: when hit by small shocks, the equilibrium allocation should return to any of these two paths. On the contrary, the instability of the intermediate balanced-growth path provides that we should not observe it the long run. Furthermore, because of the nature of multiple equilibria, the rich balanced-growth path is 'discretely' away from the poor one. Because of their stability and discrete differences, we consider a Big Push policy as one that is able to transition the economy from the a poor to a rich balanced-growth path.

The multi-dimensional feature of the distribution $m(g,t)$ constrains our ability to make further progress within the general formulation of the model described above. As a result, we then study a particular case of the economy where the technology frontier is fixed. Firms are born with a gap g to the fixed frontier, and can pay the fixed cost to adopt and have a gap equal to zero. This economy maintains all the relevant features of the more general environment, but it is simpler to analyze. In particular, rather than keeping track of the distribution $m(g,t)$, the relevant *unidimensional* state variable is the fraction of firms that adopted the frontier technology, $K(t)$. We interpret $K(t)$ as the capital stock of the economy, and we show that this economy can be analyzed as a version of the Neoclassical Growth Model, but with a convex-concave shaped production function as in Skiba (1978) and Brock and Dochert (1983). As in the general model, there is the potential for multiple dynamic equilibrium paths when ζ is large, that manifests by the presence of three steady states: a stable 'poor' one with no adoption, an unstable one with infrequent adoption, and a stable 'rich' one with frequent adoption. Furthermore, markup distortions generate the same inefficiencies as in the general formulation of the model. Can policy generate a Big Push? Will the planner implement a Big Push? As described above, the presence of strategic complementarities, i.e. $\zeta > 1$, is necessary as it provides the grounds for multiple steady states. Moreover, it is also necessary that there is a unique steady state under the optimal policy. The next two cases study this.

The first case we study is one where there are multiple equilibrium steady states, but only the rich one—corrected for markup distortions—surviving under the optimal static policy. In this case, the planner can implement a Big Push transition from the poor to the rich steady state by setting a revenue subsidy equal to the markup distortion. Unlike in Matsuyama (1991) where the role of policy is aimed at equilibrium selection towards industrialization, here the optimal policy generates a Big Push by correcting the static and dynamic inefficiencies that stem from markup distortions.[3] We also show that a Big Push can be generated with an adoption subsidy. This one is particularly interesting as the subsidy only corrects for the dynamic inefficiency that stems from markups, and thus the policy has no measurable effect on aggregate production on impact; rather, it only manifests through time through its effect on $G(t)$.

The second case we study is one where there are multiple equilibrium steady states, even when corrected for markup distortions. When the intertemporal elasticity of substitution $\frac{1}{\theta}$ is low, we show that there exists a value $\hat{K}$ such that if the initial mass of adopters $K(0)$ is below $\hat{K}$ then the planner's allocation converges uniquely to the poor steady state with no adoption, and when $K(0) \geq \hat{K}$ then the planners' allocation

[3]This also relates to Matsuyama (1995), where it is noted that a potential role for policy is to eliminate complete the bad equilibrium.

converges uniquely to the rich steady state. This implies that the planner would not find it optimal to implement a Big Push if the economy starts in the poor steady state. As a result, in spite the presence of multiple steady states, there is no role for policy as an equilibrium selection tool. When the intertemporal elasticity of substitution $\frac{1}{\theta}$ is high, $\hat{K}$ is a Skiba-point. A Skiba-point is one where, if $K(0) = \hat{K}$, there are two dynamic solutions consistent with the planning problem, one converging to the poor steady state, and one to the rich one. Here, when $K(0) = \hat{K}$, the role of policy is two-fold: it corrects distortions, and it is an equilibrium selection tool. However, as in the previous case, if $K(0)$ is such that the economy starts in the poor steady state, then the planner finds it optimal to correct the markup distortion and remain in the poor steady state. As a result, the optimal policy does not achieve a Big Push, nor equilibrium selection is a consideration. Constraining the planner to use an adoption subsidy achieves the same results.

1.1 Related Literature

Understanding the role of complementarities as the underlying mechanism for Big Push has been explored by important contributions in the Macroeconomics Development literature. Murphy et al. (1989) advanced a static setting where multiple equilibria results from coordination failures. In settings with multiple steady states, Krugman (1991), Matsuyama (1991), Matsuyama (1995), Chen and Shimomura (1998) and Adserà and Ray (1998) argue that expectations play an important role in generating multiple dynamic equilbria and that the planner can instrument a Big Push by selecting among them. We contribute to these studies by showcasing a model where expectations and equilibrium selection, in spite of being present, are not the underlying cause for a Big Push. Rather, a Big Push occurs when the planner is able to correct markup distortions and, by such, also able to eliminate undesirable long run outcomes.

Another literature within Macroeconomics Development argues that distortions are important for understanding the lack of development of an economy (Restuccia and Rogerson, 2008; Hsieh and Klenow, 2009; Bento and Restuccia, 2017), and how distortions can be amplified through the production structure of the economy to generate amplification in static settings (Jones, 2011; Liu, 2019; Baqaee and Farhi, 2020; Buera et al., 2021; Buera and Trachter, 2024). In our paper we analyze a fully dynamic economy subject to roundabout production, characterize dynamic equilibria and the planning problem, and show the way dynamics are a key consideration for a planner aiming to instrument a Big Push and correct distortions.[4]

Our work also relates to studies that advanced tools to study optimal control problems under the presence of non-convexities in the production function (Clark, 1971; Skiba, 1978; Brock and Dochert, 1983; Dechert and Nishimura, 1983). We apply these tools to study the optimal control of technology adoption by heterogeneous firms where the aggregate production function is non-convex endogenously. Likewise, we relate to studies pointing out to the potential for multiple equilibria arising as a result of these non-convexities, see Buera (2009).

[4]Another related paper is Aghion et al. (2024) that studies the transition dynamics to a Green economy in an input-output economy. More broadly, our paper also relates to studies linking macroeconomic fluctuations with coordination failures. See (Schaal and Taschereau-Dumouchel, 2023; Taschereau-Dumouchel and Schaal, 2015), and citation therein.

2 Setup

We consider an economy with a measure 1 of households and firms. The household demands a composite of varieties and supplies labor to firms. Firms produce differentiated varieties using a constant returns to scale technology. Firms can upgrade their production techniques to the technological frontier at any point in time. The technological frontier is continuously evolving. We now describe the preferences and technologies for this economy.

Preferences and Taxes. Household utility is given by

$$\int_0^\infty e^{-\rho t} \frac{C(t)^{1-\theta} - 1}{1-\theta} \, dt \ ,$$

where $C(t)$ denotes aggregate consumption at time t. The household also supplies an inelastic amount of labor N in every period t.

The government uses lump sum taxes $T(t)$ to households to cover revenue and adoption subsidies/taxes $s_r(t)$ and $s_a(t)$.

Technology adoption. The technological frontier of the economy grows at an exogenous rate γ at t. Any firm can use κ_t aggregate final goods at time t and adopt the frontier technology, i.e. their neutral TFP becomes $e^{\gamma t}$. We refer to this as a costly adoption event. Furthermore, with probability q per unit of time, the firm can adopt the frontier technology without paying the fixed cost κ_t. We refer to this event as a free adoption opportunity. The firm's productivity stays constant until the firm either pays a fixed cost again or a free adoption opportunity arrives.

We assume that the adoption cost scales with the level of the technological frontier measured in goods, that is $\kappa(t) = e^{\frac{\gamma}{1-\nu} t} \kappa_0$ where κ_0 is a positive parameter.

Dynamics of distribution. Let $m(g,t)$ be the density at time t of the firms indexed by the technology gap g. Let $G(t)$ be the threshold at time t at which firms will adopt the frontier technology, and therefore their gap will be zero. Let $m_0(g)$ be the initial density at time $t = 0$. We derive the p.d.e. for the density m starting with the following discrete-time discrete state approximation

$$m(g + dt, t + dt) = m(g,t)(1 - q dt) \text{ for } 0 < g < G(t) \ .$$

Using a Taylor expansion on the right hand side, and taking limits as $dt \downarrow 0$ we obtain:

$$m_t(g,t) + m_g(g,t) + q m(g,t) = 0 \ , \text{ for } 0 \le g \le G(t) \ . \tag{1}$$

We use that m is a density, i.e. $1 = \int_0^{G(t)} m(g,t)dg$, for all $t > 0$. Differentiating this equation with respect to time we get the so called mass preservation condition,

$$\begin{aligned} 0 &= \int_0^{G(t)} m_t(g,t)dg + m(G(t),t)G'(t) \text{ for all } t > 0 \ , \\ &= m(0,t) - m(G(t),t) - q\int_0^{G(t)} m(g,t)dg + m(G(t),t)G'(t) \text{ for all } t > 0 \ , \end{aligned} \tag{2}$$

where the second equation follows from replacing the p.d.e.. In what follows we use $G'(t)$ for the time derivative of $G(t)$. Note that, because $G(t)$ is the upper bound of the support of gaps g and because the gap of each firm that did not adjust moves dt with time, it follows from the definition of $G(t)$ that $G'(t) \leq 1$.

In a period of length dt, firms that are located at $g = 0$ are the ones that just adopted. In other words, the density of firms at $g = 0$ at time t can only come from adoption at $t - dt$. If we use $m(0,t)$ for the density, then they are $m(0,t)dg$ firms that have just adopted. Since we measure gaps in units of time we can also write $m(0,t)dt$. Thus, $m(0,t)dt$ is the number of firms that adopt in an interval of length dt. Using $a(t)$ for the adoption rate we have:

$$a(t) = m(0,t) = m(G(t),t) + q\int_0^{G(t)} m(g,t)dg - m(G(t),t)G'(t) \ . \tag{3}$$

The boundary condition in (2) and hence the adoption rate in (3) states that the density of firms that adopt at time t, denoted by $m(0,t)$, is given by the firms that reach the threshold $G(t)$ at time t, plus all the firms with a free adoption opportunity, plus the adoption that occurs if the threshold $G(t)$ has decreased.

Feasibility. Aggregate consumption $C(t)$ is net final aggregate output $Y(t)$ minus adoption costs,

$$C(t) = Y(t) - \kappa(t)\left(m(0,t) - q\int_0^{G(t)} m(g,t)dg\right) , \tag{4}$$

where $\kappa(t)$ accounts for the number of final goods used by a firm to adopt by paying the cost, and where $m(0,t)$ is the rate per unit of time at which firms adopt the frontier technology. We subtract $q\int_0^{G(t)} m(g,t)dg$ from the adoption rate, since adoption was free from these firms.

The feasibility condition 4 implies that the distribution of gaps cannot have any mass points, i.e. that $G(t)$ is differentiable. This follows as, given that $C(t)$, $Y(t)$ and $\kappa(t)$ are flows, there are not sufficient resources in an instant to pay for the stock of adoption costs implied by a mass point.

There are two types of firms. The final good is produced by a constant return CES technology, with elasticity of substitution η using each of the goods produced by the firms producing differentiated goods. The firms producing differentiated goods use labor and the aggregate good, with a constant returns to scale technology—where ν is the elasticity of output to the intermediate aggregate, to produce their differentiated goods.

Each firm produces a variety of goods using labor and intermediate aggregate goods. The production function for a firm with gap g to the frontier at time t, hiring $n(g,t)$ workers and using $x(g,t)$ aggregate intermediate inputs is,

$$e^{(t-g)\gamma}\, b\, x(g,t)^{\nu}\, n(g,t)^{1-\nu} \ , \tag{5}$$

where ν is the share of intermediate input, and $1-\nu$ is the share of labor in production for each firm, and where $b \equiv (1-\nu)^{\nu-1}\nu^{-\nu}$ is defined to simplify the notation. At time t, the TFP of a firm that adopted the frontier technology g time ago is $e^{(t-g)\gamma}$. Net final aggregate output $Y(t)$ is given by gross output across firms $Q(t)$ minus aggregate intermediate inputs $X(t)$:

$$Q(t) \equiv Y(t) + X(t) = \left[\int_0^{G(t)} \left(e^{(t-g)\gamma}\, b\, x(g,t)^{\nu} n(g,t)^{1-\nu}\right)^{1-\frac{1}{\eta}} m(g,t)\, dg\right]^{\frac{1}{1-1/\eta}} \ , \tag{6}$$

$$X(t) = \int_0^{G(t)} x(g,t)\, m(g,t)\, dg \ . \tag{7}$$

Aggregate (exogenous) labor supply N equals labor demand across firms:

$$N = \int_0^{G(t)} n(g,t)\, m(g,t)\, dg \ . \tag{8}$$

3 Equilibrium

We describe the equilibrium of this economy in steps.

Household Problem. The problem of the household is simple. They receive the profit from the aggregate portfolio of firms and wages, and can borrow and lend at rate $r(t)$ at t. Their Arrow-Debreu budget constraint is:

$$0 = \int_0^{\infty} e^{-\int_0^t r(s)ds} \left[P(t)C(t) - \Pi(t) - Nw(t) + T(t)\right] dt \ ,$$

where $r(t)$ is the interest rate (in terms of the numeraire), $w(t)$ is the time t wage rate, $P(t)$ is the time t price of the aggregate final good, and $\Pi(t)$ are the time t profits of the the portfolio of firms, and $T(t)$ are the time t lump sum taxes/tranfers received by the household. This implies the following Euler first order condition

$$e^{-\rho t} C(t)^{-\theta} = \lambda_H e^{-\int_0^t r(s)ds} P(t) \ ,$$

where λ_H is the Lagrange multiplier of the Arrow-Debreu budget constraint. Taking logs, and differentiating with respect to time,

$$r(t) = \rho + \theta \frac{\dot{C}(t)}{C(t)} + \frac{\dot{P}(t)}{P(t)} \ . \tag{9}$$

Firms producing final goods. Firms buy differentiated goods indexed by their gap g in the amount $y(g,t)$ of each of the density of firms $m(g,t)$ with gap g at t, and produce the final good that can be used for consumption, as the adoption good, or as an intermediate good in the production of the differentiated goods. Let $Q(t)$ denote the gross aggregate output,

$$Q(t) \equiv \left[\int_0^{G(t)} y(g,t)^{1-\frac{1}{\eta}} \, m(g,t) dg \right]^{\frac{1}{1-\frac{1}{\eta}}} \quad , \tag{10}$$

providing the standard input demand equation,

$$y(g,t) = \left(\frac{p(g,t)}{P(t)} \right)^{-\eta} Q(t) \ , \tag{11}$$

where the price of the aggregate final good is given by

$$P(t) \equiv \left[\int_0^{G(t)} p(g,t)^{1-\eta} m(g,t) dg \right]^{\frac{1}{1-\eta}} \quad . \tag{12}$$

The static problem of a differentiated good's firm We assume that firms engage in monopolistic competition. In period t, firms set their price to maximize static profits. They take as given the period price for aggregate output, denoted by $P(t)$, and the period wages, denoted by $w(t)$. Their demand function is given by (11), and the cost minimization problem is given by

$$\min_{x,n} Px + nw + \lambda_c \left(y - e^{\gamma(t-g)} b x^{\nu} n^{1-\nu} \right) \ ,$$

with fist order conditions:

$$Px = \lambda_c \nu e^{\gamma(t-g)} \, b \, x^{\nu} n^{1-\nu} \ , \text{ and} \tag{13}$$

$$wn = \lambda_c (1-\nu) e^{\gamma(t-g)} \, b \, x^{\nu} n^{1-\nu} \ . \tag{14}$$

We can combine these expressions to obtain that the marginal cost is given by $\lambda_c = P^{\nu} w^{1-\nu} e^{(g-t)\gamma}$. The larger the firm's gap to the frontier, the higher the marginal cost of the firm. Further, $\frac{\partial}{\partial y}\mathcal{C}(w,P,y) = \lambda_c = e^{\gamma(g-t)} w(t)^{1-\nu} P(t)^{\nu}$. Then, we can write the firm's static pricing problem as:

$$\pi(g,t) \equiv \max_p \left(\frac{p}{P(t)} \right)^{-\eta} Q(t) \left[s_r(t) p - e^{\gamma(g-t)} w(t)^{1-\nu} P(t)^{\nu} \right] \ ,$$

where $s_r(t)$ is a revenue subsidy/tax to the firm. If $s_r(t) > 1$ is a subsidy proportional to revenue, and if $s_r(t) < 1$ it is a tax proportional to revenue. It is immediate to obtain that the optimal price of a firm satisfies the Lerner condition,

$$p(g,t) = \frac{\eta}{\eta-1}\frac{1}{s_r(t)}e^{\gamma(g-t)}w(t)^{1-\nu}P(t)^{\nu} \ . \tag{15}$$

Then, the maximized profits are given by

$$\pi(g,t) = s_r(t)^{\eta}\, P(t)Q(t)\, \left(\frac{P(t)}{w(t)}\right)^{(1-\nu)(\eta-1)} e^{\gamma(t-g)(\eta-1)}\left(\frac{1}{\eta-1}\right)\left(\frac{\eta}{\eta-1}\right)^{-\eta} \ . \tag{16}$$

The technology adoption problem of a differentiated good's firm Taking the profit function as given, the adoption problem for the firm can be characterized using standard dynamic programming methods. The state of the firm problem is the pair (g,t), and we let $V(g,t)$ denote the firm's value function for the given state. The value function satisfies the following Hamilton-Jacobi-Bellman (HJB) equation,

$$r(t)V(g,t) = \max\left\{r(t)\left(V(0,t) - s_a(t)\kappa(t)P(t)\right)\ ,\ \pi(g,t) + V_g(g,t) + V_t(g,t) + q\left(V(0,t) - V(g,t)\right)\right\}\ , \tag{17}$$

where $r(t)$ is the real interest rate in (9), and $\pi(g,t)$ is the profit of firm at time t with gap g in (16). The first term in the right hand side gives the value of a costly adoption. The second term gives the continuation in the case of at which there is no costly adoption. The value function changes with both time t and gap g. The gap increases one by one with time. For a costly adoption the firm must pay $\kappa(t)$ final goods, where each final good costing $P(t)$, and were we assume that the firm is subject to a gross tax/subsidy to adoption $s_a(t)$. In particular, if $s_a(t) < 1$, this is a subsidy, in the sense that it reduces the cost paid by the firm. Time is part of the state of the firm's problem because $r(t), P(t), s_a(t), \kappa(t)$ and $\pi(g,t)$ are all functions of time. The term $q\left(V(0,t) - V(g,t)\right)$ is the expected value of a free adoption opportunity.

We can also cast the dynamic problem of the firm as an optimal stopping time problem,

$$V(0,t) = \max_{\tau \geq t} \int_t^{\tau} e^{-\int_t^s q + r(u)\,du}\left[\pi(s-t,s) + qV(0,s)\right] ds + e^{-\int_t^{\tau} q+r(s)\,ds}\left[V(0,\tau) - s_a(\tau)\kappa(\tau)P(\tau)\right]\ . \tag{18}$$

and where the stopping time τ can be characterized using the first order conditions of problem in (18),

$$0 = \pi(\tau - t,\tau) + qV(0,\tau) - (q + r(\tau))\left[V(0,\tau) - \kappa(\tau)P(\tau)\right] + V_t(0,\tau) - \frac{d}{dt}s_a(\tau)\kappa(\tau)P(\tau)\ . \tag{19}$$

Given that for all t the flow profit $\pi(g,t)$ is decreasing in the gap g, it can be shown that $V(g,t)$ is also decreasing in g. This implies that the optimal firm adoption policy is of a threshold type $G(t)$ for each t, such that for all $0 \leq g \leq G(t)$ we have that

$$r(t)V(g,t) = \pi(g,t) + V_g(g,t) + V_t(g,t) + q\left[V(0,t) - V(g,t)\right]\ , \tag{20}$$

and for all $g > G(t)$ we have that

$$V(g,t) = V(0,t) - s_a(t)\kappa(t)P(t) \ , \text{ and thus } 0 = V_g(g,t) \ .$$

Since the value function is continuous, we can evaluate the previous expression at $g = G(t)$ obtaining the value matching condition,

$$V(G(t),t) = V(0,t) - \kappa(t)P(t) \text{ for all } t > 0 \ . \tag{21}$$

If, furthermore, the value function is also differentiable at $g = G(t)$, we obtain the smooth pasting condition,

$$0 = V_g(G(t),t) \text{ for all } t > 0 \ . \tag{22}$$

We can use the expression characterizing τ (19), and rewrite τ it in terms of $G(t)$. We use a change of variables as $t \to t + \tau$ and $\tau \to G(t)$ to write:

$$0 = \pi(G(t),t) + qV(0,t) - (r(t)+q)\left[V(0,t) - s_a(t)\kappa(t)P(t)\right] + V_t(0,t) - \frac{d}{dt}(s_a(t)\kappa(t)P(t)) \ .$$

We assume value matching, i.e. that $V(G(t),t) = V(0,t) - s_a(t)\kappa(t)P(t)$ and, if V is differentiable at that point, we have $V_g(G(t),t)G'(t) + V_t(G(t),t) = V_t(0,t) - \frac{d}{dt}(s_a(t)\kappa(t)P(t))$,

$$r(t)V(G(t),t) = \pi(G(t),t) + q(V(0,t) - V(G(t),t)) + V_g(G(t),t)G'(t) + V_t(G(t),t) \ .$$

Let's subtract the p.d.e. in (20) evaluated at $\{t, G(t)\}$ to obtain,

$$0 = V_g(G(t),t)(G'(t) - 1) \ , \tag{23}$$

which gives smooth pasting for $G'(t) < 1$.

3.1 Temporary Equilibrium

We fix $m(\cdot,t)$ in period t and we characterize the equilibrium conditions only involving static decisions. We refer to this as a temporary equilibrium.

Definition 1 *Fix $m(\cdot,t)$ and a subsidy $s_r(t)$. The triplet $\{Y(t), Q(t), w(t), P(t)\}$ satisfies the temporary equilibrium conditions, if there exist $\{n(\cdot,t), x(\cdot,t), p(g,t)\}$ such that:*

1. *Given $P(t), w(t), s_r(t), Q(t)$ the price $p(g,t)$ maximizes the profit of firm indexed by g, i.e. $p(g,t)$ satisfies (15).*

2. *Firm g choice of inputs minimizes cost, i.e. (14) and (13) hold.*

3. *Given $\{p(\cdot,t)\}$, the aggregate price satisfies the zero profit condition in (12) maximizes the profit of firm g.*

4. *Labor market clearing holds, i.e. (8) holds.*

5. *Intermediate good market clearing holds, i.e. (7) holds.*

6. *$Y(t)$ is given the by the production function, i.e. (6) holds.*

The next proposition characterizes the temporary equilibrium conditions.

Proposition 1 *Fix a time $t > 0$, and consider the static equilibrium corresponding to $m(\cdot,t)$. Then, gross output $Q(t)$, GDP $Y(t)$, employment $n(g,t)$, intermediate input demand $x(g,t)$, and real wages $w(t)/P(t)$ are given by:*

$$Q(t) = \frac{N}{1-\nu}\frac{1}{s_r(t)}\left(\frac{\eta}{\eta-1}\right)\frac{w(t)}{P(t)} \ , \tag{24}$$

$$Y(t) = N\left(\frac{\frac{1}{s_r(t)}\left(\frac{\eta}{\eta-1}\right)-\nu}{1-\nu}\right)\frac{w(t)}{P(t)} \ , \tag{25}$$

$$n(g,t) = N\frac{e^{-g\gamma(\eta-1)}}{\int_0^{G(t)} e^{-g\gamma(\eta-1)}m(g,t)dg} \tag{26}$$

$$x(g,t) = \frac{\nu}{1-\nu}n(g,t)\frac{w(t)}{P(t)} \tag{27}$$

$$\frac{w(t)}{P(t)} = \left[s_r(t)\left(\frac{\eta-1}{\eta}\right)\right]^{\frac{1}{1-\nu}} e^{\frac{\gamma}{1-\nu}t}\left[\int_0^{G(t)} e^{-\gamma g(\eta-1)}m(g,t)dg\right]^{\frac{1}{(\eta-1)(1-\nu)}} \tag{28}$$

The proof of this proposition is available in Appendix B.1. Combining (25) with (28) provides that aggregate output satisfies

$$Y(t) = e^{\frac{\gamma}{1-\nu}t}A(s_r(t))F(m(g,t)) \ , \tag{29}$$

where

$$A(s_r(t)) \equiv \frac{N}{1-\nu}\left[\frac{1}{s_r(t)}\frac{\eta}{\eta-1}-\nu\right]\left[s_r(t)\frac{\eta-1}{\eta}\right]^{\frac{1}{1-\nu}} \ , \tag{30}$$

and where

$$F(m(g,t)) \equiv \left[\int_0^{G(t)} e^{-\gamma g(\eta-1)}m(g,t)dg\right]^{\frac{1}{(\eta-1)(1-\nu)}} \tag{31}$$

can be interpreted as the aggregate production function in the economy. Aggregate output at time t expands with the technology frontier γt, and depends on the degree of static misallocation, given by $A(\cdot)$, and on aggregate production $F(m(g,t))$. Dynamic inefficiencies, through their effect on the optimal choice of the threshold $G(t)$, will affect the distribution of $m(g,t)$ and thus aggregate production in a temporary equilibrium at t.

Proposition 2 further characterizes the cost of adoption $P(t)\kappa(t)$, aggregate demand $P(t)Q(t)$, and a firm's profit $\pi(g,t)$ in any temporary equilibrium.

Proposition 2 *Using Proposition 1, normalizing wages, i.e. $w(t) = 1 \; \forall \; t$, and setting the subsidy constant through time, i.e. $s_r(t) = s_r \; \forall \; t$, we obtain the following in a temporary equilibrium,*

1. $P(t)\kappa(t) = \kappa_0 \left(\frac{1}{s_r} \; \frac{\eta}{\eta-1}\right)^{\frac{1}{1-\nu}} \left[\int_0^{G(t)} e^{-\gamma g(\eta-1)} m(g,t)dg\right]^{-\frac{1}{(\eta-1)(1-\nu)}}$,
2. $P(t)Q(t) = \frac{N}{1-\nu} \; \frac{1}{s_r} \; \frac{\eta}{\eta-1}$, *and*
3. $\pi(g,t) = \frac{N}{1-\nu} \; \frac{1}{\eta-1} \; \frac{e^{-\gamma g(\eta-1)}}{\int_0^{G(t)} e^{-\gamma(\eta-1)g'} m(g',t)dg'}$.

The proposition states that several important objects do not depend depend on time and, if they do, they do so only indirectly through $m(\cdot,t)$. It is interesting to note that the revenue subsidy s_r does not affect the firm's profit $\pi(g,t)$. From (16), we see that the subsidy affects a firm's profit in three ways. First, there is the direct effect of the subsidy, which raises the profit. Second, there is an indirect effect of s that depresses the price of other firms, decreasing a given firm's profit. Third, the negative effect on the aggregate price $P(t)$ is stronger than the positive effect on gross output $Q(t)$, so that a subsidy depresses aggregate demand $P(t)Q(t)$. These three forces exactly counterbalance each other on a firm's profit, and thus a revenue subsidy may only affect $\pi(g,t)$ through its effect on $m(\cdot,t)$.

Definition 2 *Let $m_0(\cdot)$ be the initial distribution of technology gaps at time $t = 0$. An* ***equilibrium*** *is given by a flow of densities $\{m(\cdot,t)\}$, a path of thresholds $\{G(t)\}$, and a value function $\{V(\cdot,t)\}$, such that*

1. *The allocation $\{Y(t), Q(t), P(t), C(t)\}$ is a static equilibrium given $m(\cdot,t)$ for every t.*
2. *The path of distribution $\{m(\cdot,t\}$ and thresholds $\{G(t)\}$ solves the p.e.d and boundary conditions in (1) and (2).*
3. *The path of threshold $G(t)$ solves the value function $V(t,g)$ given the path of $\{m(\cdot,t)\}$ and $\{r(t)\}$ and path of subsidies $\{s_a(t)\}$.*
4. *The path of interest rates and prices $\{r(t), P(t)\}$ solve the Euler equation (9) of the households for $C(t) = Y(t) - a(t)$, where $a(t)$ is given in (3).*

The next Lemma further characterizes the equilibrium interest rate at any $t > 0$. The lemma shows that $r(t)$ depends on time t only through the distribution $m(\cdot,t)$.

Lemma 1 *In equilibrium, for $t > 0$, the interest rate $r(t)$ satisfies*

$$r(t) = \rho + \theta \frac{\dot{C}(t)}{C(t)} + \frac{\dot{P}(t)}{P(t)} = \bar{\rho} + \theta \frac{d}{dt} \log c(t) - \frac{1}{1-\nu} \frac{d}{dt} \ln Z(t) \ ,$$

where

$$\bar{\rho} \equiv \rho + (\theta - 1) \frac{\gamma}{1-\nu} \ , \tag{32}$$

$$Z(t) \equiv \left[\int_0^{G(t)} e^{-\gamma g(\eta-1)} m(g,t) dg \right]^{\frac{1}{\eta-1}} \quad \textit{for } t > 0 \ , \tag{33}$$

$$C(t) = c(t) e^{\frac{\gamma}{1-\nu} t} \ , \tag{34}$$

where $Z(t)$ denotes the intermediate input productivity in the economy. The intermediate input productivity depends on calendar time t solely through the effect of time on the distribution of gaps g, $m(\cdot, t)$. A proof of the lemma is straightforward, and follows from replacing $C(t)$ and $P(t)$ into (9).

Mean Field Game Formulation We can use the some of the equilibrium relationship, and obtain two p.d.e.'s that characterize an equilibrium. We use the normalization that $w(t) = 1$ and assume that the subsidy is constant $s_r(t) = s_r$.

The equilibrium of the MFG is given by two functions $\{V(g,t), m(g,t)\}$ for all $t > 0$ and $g \in [0, G(t)]$, and paths $\{G(t), C(t), r(t)\}$ for all $t > 0$ such that:

$$\begin{aligned}
r(t)V(g,t) &= \pi(g,t) + V_g(g,t) + V_t(g,t) + q\left(V(0,t) - V(g,t)\right), g \in [0, G(t)],\ t \geq 0 \ , \\
V(0,t) &= V(G(t),t) + s_a \kappa_0 \left(\frac{1}{s_r} \frac{\eta}{\eta - 1} \right)^{\frac{1}{1-\nu}} Z(t)^{-\frac{1}{1-\nu}} \ , \\
V_g(G(t),t) &= 0 \ , \\
m_t(g,t) &= -m_g(g,t) - q\ m(g,t), g \in [0, G(t)] \ , \\
1 &= \int_0^{G(t)} m(g,t) dg \ , \\
r(t) &= \bar{\rho} + \theta \frac{d}{dt} \log c(t) - \frac{1}{1-\nu} \frac{d}{dt} \log Z(t) \ , \\
c(t) &= A(s_r) Z(t)^{\frac{1}{1-\nu}} - \kappa_0 \left(m(0,t) - q \right) \ ,
\end{aligned}$$

where $Z(t)$ and $\pi(g,t)$ are defined in (33) and item 3 in Proposition 2. We remind the reader that $c(t), Z(t)$ and $\pi(g,t)$ are functions of time solely through the effect on $m(\cdot, t)$.

4 Planner's Problem

We first describe the static efficient conditions for the planner problem. Then we describe the full planning problem.

Temporary Planner's Problem. Fix the density $\{m(g,t)\}$ and maximize aggregate output $Y(t)$ by choosing $\{x(g,t), n(g,t)\}$ subject to (6), (7) and (8). We use a Lagrange multiplier $\mathcal{W}$ for (8) and replace $X(t)$ from (7) into (6).

$$\mathcal{Y}(m,t) \equiv \max_{y,n} \left[\int_0^{G(t)} \left(e^{(t-g)\gamma t}\, b\, x(g,t)^{\nu} n(g,t)^{1-\nu} \right)^{1-\frac{1}{\eta}} m(g,t)\, dg \right]^{\frac{1}{1-1/\eta}} - \int_0^{G(t)} x(g,t)\, m(g,t)\, dg + \mathcal{W} \left[N - \int_0^{G(t)} n(g,t)\, m(g,t)\, dg \right] . \tag{35}$$

Proposition 3 *The solution of the problem defined in (35) is given by:*

$$\mathcal{W}(m,t) = e^{\frac{\gamma}{1-\nu}t} \left[\int_0^{G(t)} e^{-g\gamma(\eta-1)}\, m(g,t)\, dg \right]^{\frac{1}{(\eta-1)(1-\nu)}} = e^{\frac{\gamma}{1-\nu}t} Z(t)^{\frac{1}{1-\nu}} .$$

$$\mathcal{Q}(t) = \frac{1}{1-\nu} N \mathcal{W}(m,t,1) ,$$

$$\mathcal{Y}(m,t) = N\, \mathcal{W}(m,t) = N\, e^{\frac{\gamma}{1-\nu}t} Z(t)^{\frac{1}{1-\nu}} ,$$

$$n(g,t) = N \frac{e^{-g\gamma(\eta-1)}}{\int_0^{G(t)} e^{-g\gamma(\eta-1)} m(g,t) dg} ,$$

$$x(g,t) = \frac{\nu}{1-\nu} n(g,t) \mathcal{W}(m,t) .$$

A proof of the proposition is available in Appendix B.2.

Comparing the characterization of the temporary equilibrium in Proposition 1 and the characterization of the static planner's problem in Proposition 3, we obtain the following corollary:

Corollary 1 *Fix any $m(\cdot,t)$. The static equilibrium conditions coincide with the conditions for the static planner's problem if and only if*

$$\text{If } 0 < \nu \leq 1, \text{ then } s_r(t) = \frac{\eta}{\eta - 1} , \tag{36}$$

$$\text{If } 0 = \nu, \qquad \text{then } s_r(t) > 0 . \tag{37}$$

The economics behind this corollary is as follows. As it can be seen from Proposition 1 and Proposition 3 the labor allocation is undistorted for any $s_r(t)$. This is because constant markup an isoelastic demand do not distort relative labor allocations, and labor supply is inelastic. Thus, if $\nu = 0$, so that there is no intermediate inputs, for any s the static equilibrium is (statically) efficient. On the other hand, when $\nu > 0$, since markups decrease the price of labor relative to intermediate inputs, i.e. $w(t)/P(t)$, then markups uniformly decrease $x(t,t)/n(g,t)$. Since the intermediate input is "elastically" produced, this leads to a lower than efficient use of it. This can be seen in the expression for $Y(t)$ in (25) in the temporary equilibrium. A subsidy that matches the markup, as in the corollary, corrects the equilibrium value of $w(t)/P(t)$, and hence reestablishes the efficient use of intermediate inputs.

An alternative way to state this corollary is to write net aggregate output as a function of the gross corrected markup, defined as

$$\mu \equiv \frac{1}{s_r}\left(\frac{\eta}{\eta-1}\right) .$$

Using this definition, the combined the expressions above, and the expression for aggregate output in (29), we can write:

$$Y(\mu,t) = e^{\frac{\gamma}{1-\nu}t}F(m(g,t))\tilde{A}(\mu) = e^{\frac{\gamma}{1-\nu}t}Z(t)^{\frac{1}{1-\nu}}\tilde{A}(\mu) \ , \text{ where } \tilde{A}(\mu) \equiv \frac{N}{1-\nu}(\mu-\nu)\left(\frac{1}{\mu}\right)^{\frac{1}{1-\nu}} .$$

Note that $Y(1,t) = \mathcal{Y}(m,t)$, so that output is equal to the optimal when the gross corrected markup is one. It is easy to see, for $\nu > 0$, the derivative of Y with respect to μ is decreasing, and crosses to zero at $\mu = 1$ since $\tilde{A}'(\mu) = \frac{N}{1-\nu}\frac{\nu}{1-\nu}\mu^{-\frac{1}{1-\nu}-1}(1-\mu)$.

The next proposition explores several properties of the solution of the static social planner problem. The first result shows the semi-elasticity of the efficient net output with respect to reshuffling firms with respect to their gaps to the technological frontier. The second results states that the efficient static output is concave on the density m only if $1/((\eta-1)(1-\nu)) < 1$. The third result shows that the monopolist profit aligns with the consumer surplus accrued through adoption under the optimal revenue subsidy. Informally, this alignment is because with a CES consumer surplus on a good equals revenue times $1/(\eta-1)$, and so does the static profit. Nevertheless, without any revenue subsidy, the equilibrium revenues of the monopolist are depressed, since the monopolist markup depresses demand.

For the next proposition, start with a density m which is strictly positive in two gaps, g_1 and g_2. Take two positive numbers α and ϵ, and define $m^{\epsilon,\alpha}$ as a perturbation on the original density m. In particular, we add a uniform density of height ϵ and width α, centered around gap g_1, and subtract a uniform density of the same height and width centered around gap g_2. For small α and ϵ it defines the following density:

$$m^{\epsilon,\alpha}(g,t) = m(g,t) + \epsilon\left(1_{\{|g-g_1|<\alpha/2\}} - 1_{\{|g-g_2|<\alpha/2\}}\right) \text{ for all } g \ . \tag{38}$$

In words, relative to m, the density $m^{\alpha,\epsilon}$ reshuffles probability density from a neighborhood of g_2 to one around g_1. This experiment will turn of interest for the study of technology adoption, since we model that activity as one where a number of firms discretely eliminate their technology gap with respect to the frontier.

Proposition 4 *Fix a density m and two points g_1, g_2 with strictly positive density. Let $m^{\epsilon,\alpha}$ be defined as in (38). Then,*

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\frac{d}{d\epsilon}\log\mathcal{Y}(m^{\epsilon,\alpha})|_{\epsilon=0} = \frac{1}{(\eta-1)(1-\nu)}\frac{e^{-\gamma(\eta-1)g_1}-e^{-\gamma(\eta-1)g_2}}{\int_0^{G(t)} e^{-\gamma(\eta-1)g'}m(g',t)dg'}\ , \tag{39}$$

$$\mathcal{Y}(m^{\epsilon,\alpha}) \text{ is concave in } \epsilon \iff \frac{1}{(\eta-1)(1-\nu)} \leq 1\ , \tag{40}$$

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\frac{d\mathcal{Y}(m^{\epsilon,\alpha})}{d\epsilon} = \frac{\pi(g_1,t)-\pi(g_2,t)}{P(t)}\left[\frac{1}{s_r(t)}\left(\frac{\eta}{\eta-1}\right)\right]^{\frac{1}{1-\nu}}\ . \tag{41}$$

A proof of the proposition is available in Appendix B.3. We discuss the proposition using the following combination of parameters:

$$\zeta \equiv \frac{1}{(\eta-1)(1-\nu)}\ . \tag{42}$$

The expression in (39) measures the increase in the social planner's final output due to the reshuffling of probability density. This reshuffling in density changes $\mathcal{Y}$ by an amount that on the difference in productivity gaps g_1 and g_2, normalized by the average productivity gaps, multiplied by ζ. Whether this change is less than linear (i.e. concave in ϵ), linear, or more than linear (i.e. convex in ϵ) depends on whether $\zeta \geq 1$ or not, as also stated in (40). The convexity of this mapping is controlled by ζ through two mechanisms. The first mechanism is the roundabout nature of production, captured by ν. The strength of this mechanism increases as ν increases towards one. The second mechanism is that the economy features complementarities in the production of intermediate goods. These complementarities are controlled by η, and they increase as η decreases towards one. While both mechanisms operate through ζ in similar ways, they do have relevant differences. In particular, while the multiplier effect of ν as $\nu \uparrow 1$ is unbounded, the effect as $\eta \downarrow 1$ is bounded. In particular, as $\eta \downarrow 1$, we have $d\mathcal{Y}/d\epsilon = (g_2 - g_1)/(1-\nu)$.

Finally, (41) connects the increase in the social planner's final output due to the reshuffling of probability density with static profits, measured in units of final output. We note on three important implications of this expression. First, the relative valuation (i.e across different g's) in the planning problem and in the equilibrium with monopolistic competition are the same. Second, because the equilibrium profits are depressed relative to the social planner's valuation of the consumption flow produced by a firm with g (as long as $s_r(t) < \eta/(\eta-1)$), the increase in output through reshuffling is larger in the planning problem. Third, because the planner's valuation and the monopolist profits are identical when the revenue subsidy is equal to the markup, when $r_s(t) = \eta/(\eta-1)$ the gains accrued through reshuffling are the same in the planning and equilibrium.

Now we are ready to write the "dynamic" planner's problem.

Definition 3 *Fix an initial condition $m_0(\cdot)$. The planner chooses the change in time of the threshold path $\{u(t)\}$, where $u(t) = G'(t)$. The objective is to maximize*

$$\int_0^\infty e^{-\rho t} \frac{C(t)^{1-\theta} - 1}{1-\theta} dt$$

subject to:

$$C(t) = \mathcal{Y}(m(\cdot,t),t) - \kappa(t) \left(m(0,t) - q \int_0^{G(t)} m(g,t) dg \right) \tag{43}$$

and the law of motion of m,

$$m_t(g,t) = -m_g(g,t) - q\, m(g,t) \text{ for } g \in [0, G(t)] \ , \tag{44}$$

$$1 = \int_0^{G(t)} m(g,t) dg, \text{ for all } t, \tag{45}$$

$$G'(t) = u(t), \text{ for all } t \tag{46}$$

The next proposition characterizes the necessary first order conditions for the planner's problem.

Proposition 5 *Let m_0 be the the density with the initial conditions for the planner. Let $e^{\bar{\rho}t}\lambda(g,t)$ be the Lagrange multiplier of the p.d.e. for the evolution of the density, i.e. (44) and $e^{\bar{\rho}t}\omega(t)$ the Lagrange multiplier of the mass preservation (45). Let $\{C(t)\}$ the optimal path of consumption, let $\{G(t)\}$ the optimal path of for the adoption threshold, and let $\{m(g,t)\}$ the density of gaps in the optimal path. Define $\bar{\rho} \equiv \rho - (1-\theta)\frac{\gamma}{1-\nu}$. Then, the Lagrange multipliers λ and ω satisfy the following p.d.e. and boundary conditions:*

$$\begin{aligned} \bar{\rho}\lambda(g,t) &= c(t)^{-\theta} Z(t)^{\frac{1}{1-\nu}} \pi(g,t) - \omega(t) \\ &\quad + \lambda_t(g,t) + \lambda_g(g,t) + q\left(\lambda(0,t) - \lambda(g,t)\right) \text{ for } t \geq 0 \text{ and } g \in [0, G(t)] \end{aligned} \tag{47}$$

$$\lambda(0,t) = c(t)^{-\theta}\kappa_0, \text{ for all } t > 0 \tag{48}$$

$$\lambda(G(t),t) = 0, \text{ for all } t > 0 \tag{49}$$

$$\lambda_g(G(t),t) = 0, \text{ for all } t > 0 \tag{50}$$

$$0 = \lim_{T\to\infty} e^{-\bar{\rho}T} \lambda(g,T) m(g,T) \text{ for all } 0 \leq g < \lim_{T\to\infty} G(T) \tag{51}$$

where

$$c(t) = N\, Z(t)^{\frac{1}{1-\nu}} - \kappa_0 \left(m(0,t) - q \int_0^{G(t)} m(g,t) dg \right) \text{ so that} \tag{52}$$

$$C(t) = c(t)\, e^{\frac{\gamma}{1-\nu} t} \text{ for } t > 0 \tag{53}$$

The proof is available in Appendix B.4.

The next proposition states that an optimal allocation can be decentralized as an equilibrium by correcting the static markup with a constant revenue subsidy. In other words, once the static markup, and associated real wages (and it reciprocal the price of the adoption good), are corrected, the market determined technology adoption becomes optimal. The market equilibrium without any tax/subsidy is inefficient due to two market failures. One is the inefficient use of intermediate inputs, as highlighted in Proposition 3 and Corollary 1. This makes the temporary equilibrium inefficient, which is completely static in its nature, and unrelated to technology adoption by definition. The second market failure is the effect of the distorted relative price of the adoption good, i.e. the inverse of the real wage. Again, the revenue subsidy s_r corrects this market failure. Interestingly, an adoption subsidy s_a can correct the effect on the relative price on the incentive to adoption, but it does not corrects the first static inefficiency, unless, of course, $\nu = 0$ in which case intermediate inputs are not used at all.

There are other potential differences in the planner vs market allocation. For instance, inefficiencies in the investment of production across different goods. Or differences in the valuation of the monopolist and the planer. But in our case, as long as the optimal revenue subsidy is in place, they do not materialize.

Proposition 6 *Let $\{m, G, C, \lambda, \omega\}$ be the allocation and multipliers that solved the necessary firsr order conditions for the planner problem as in Proposition 5. Then, for a constant revenue subsidy $s_r = \eta/(\eta-1)$, or in the case of $\nu = 0$ with an adoption subsidy $s_a = (\eta-1)/\eta$, then $\{m, G, C, V, r\}$ is an equilibrium where the firm's value functions are:*

$$V(g,t) = \frac{\lambda(g,t)}{c(t)^{-\theta} Z(t)^{\frac{1}{1-\nu}}} + \int_0^\infty e^{-\bar{\rho}s} \frac{\omega(t+s)}{c(t+s)^{-\theta} Z(t+s)^{\frac{1}{1-\nu}}} ds \text{ all } g \in [0, G(t)],\ t \geq 0 \tag{54}$$

where $Z(t), c(t), r(t)$ as defined in (33), (52), and in Lemma 1 respectively depend on t solely through $m(\cdot, t)$, and where the constant $\bar{\rho}$ is defined in (32).

A proof of the proposition is available in Appendix B.5.

The next proposition proves a kind of converse of this. It shows that, given an equilibrium, we can find the Lagrange multipliers solving the first order condition for an interior solution of a planner's problem.

Proposition 7 *Let $\{m, G, C, V, r\}$ be the allocation, value function, and interest rate in an equilibrium with a constant revenue subsidy $s_r = \eta/(\eta-1)$, or in the case of $\nu = 0$ and adoption subsidy of $s_a = (\eta-1)/\eta$, then, for the same allocation, we can construct multipliers $\{\lambda, \omega\}$ that solve the necessary first order conditions for the planner's problem of Proposition 5.*

$$\lambda(g,t) = c(t)^{-\theta} Z(t)^{\frac{1}{1-\nu}} \left(V(g,t) - V(G(t),t)\right) \text{ all } g \in [0, G(t)],\ t \geq 0 \tag{55}$$

$$\omega(t) = r(t) V(G(t), t) \text{ all } t \geq 0 \tag{56}$$

A proof of the proposition is available in Appendix B.6.

5 Balanced Growth Path(s)

In this section we analyze the equilibrium balanced growth paths. In a balanced growth path, the density $m(g,t)$ and the value function $V(g,t)$ stay constant through time, so we define $\bar{m}(g) = m(g,t)$ and $\bar{V}(g) = V(g,t)$. Likewise, the threshold $G(t)$ is also time-invariant, so with a slight abuse of notation we denote it by $G = G(t)$. The price $P(t)$ decays at the rate $\gamma/(1-\nu)$ so we represent it by $\bar{P}(Z)$ where $P(t) = \bar{P}(Z)e^{\frac{-\gamma}{1-\nu}t}$. The de-trended consumption $c(t)$ is constant, denoted by $\bar{c}$. The interest rate $r(t)$ is constant through time and equal to $\bar{\rho}$. The intermediate input productivity, given by the integral $Z(t)$, is also constant through time, so again, with a slight abuse of notion, we denote it simply by Z. We denote profits by $\bar{\pi}(g;Z)$ as function g and Z. Profits are given by,

$$\bar{\pi}(g;Z) = \frac{N}{1-\nu}\,\frac{1}{\eta-1}\frac{e^{-g\gamma(\eta-1)}}{Z^{\eta-1}} = N\zeta\frac{e^{-g\gamma(\eta-1)}}{Z^{\eta-1}}\ . \tag{57}$$

Given Z, the value function V and G solves the following o.d.e. and boundary conditions:

$$(\bar{\rho}+q)\bar{V}(g) = \bar{\pi}(g;Z) + \bar{V}_g(g) + q\bar{V}(0) \text{ for } g \in [0,G] \tag{58}$$

$$\bar{V}(G) = \bar{V}(0) - s_a\kappa_0\bar{P}(Z) \tag{59}$$

$$\bar{V}_g(G) = 0 \tag{60}$$

Given G, the distribution satisfies the following o.d.e. and boundary condition:

$$\bar{m}_g(g) + q\bar{m}(g) = 0 \text{ for } g \in [0,G]\ , \tag{61}$$

$$\bar{m}(0) = \bar{m}(G) + q\int_0^G \bar{m}(g)dg\ . \tag{62}$$

Finally, the intermediate input productivity Z, the price index $\bar{P}(Z)$, and consumption $\bar{c}(Z)$ are given by

$$Z = \left[\int_0^G e^{-\gamma g(\eta-1)}\bar{m}(g)dg\right]^{\frac{1}{\eta-1}}\ , \tag{63}$$

$$\bar{P}(Z) = \left[\frac{1}{s_r}\left(\frac{\eta}{\eta-1}\right)\right]^{\frac{1}{1-\nu}} Z^{-\frac{1}{1-\nu}}\ , \tag{64}$$

$$\bar{c}(Z) = A(s_r)\,Z^{\frac{1}{1-\nu}} - \kappa_0\,(\bar{m}(0)-q)\ . \tag{65}$$

We can now define an equilibrium balanced growth path.

Definition 4 *A balanced growth path with costly adoption is given by the value function of $\bar{V}(g)$, the density $\bar{m}(g)$, a threshold G^*, and intermediate input productivity Z^* such that (58), (59), (60), (61), (62), (63) and (64) hold.*

Solving for a balanced growth path requires to solve for the equilibrium, time-invariant, threshold rule G^*. This object is the convolution of how the optimal G of a firm depends on the equilibrium, aggregate, threshold rule in the economy, and how the equilibirum threshold rule is consistent with the optimal G of a firm (i.e. a fixed point). For a firm, this dependence plays solely through the intermediate input productivity Z. Then, through aggregation, we build the fixed point and study it.

For a given Z, we are going to analyze the optimal choice of G. We denote this optimal choice as $\bar{G}(Z)$. We define the function $R(G)$ as an intermediate step, and notice some of its properties.

Lemma 2 *Define*

$$R(G) \equiv \frac{1}{q+\bar{\rho}} \left\{ 1 - e^{-\gamma(\eta-1)G} - \frac{\gamma(\eta-1)}{q+\bar{\rho}+\gamma(\eta-1)} \left[1 - e^{-(q+\bar{\rho}+\gamma(\eta-1))G} \right] \right\} . \tag{66}$$

The function R satisfies: (i) $R(0) = 0$, (ii) $R(G) > 0$ for $G > 0$, (iii) R is strictly increasing in G with $0 < R'(G) < \gamma(\eta-1)/(q+\bar{\rho})e^{-\gamma(\eta-1)G}$, (iv) $R(G) = \frac{\gamma(\eta-1)}{2}G^2 + o(G)$, and (v) R has an asymptote to $1/(q+\bar{\rho}+\gamma(\eta-1))$. Given these properties the function $R(G)$ has an inverse $R^{-1} : [0, \frac{1}{q+\bar{\rho}+\gamma(\eta-1)}] \to [0, \infty]$.

A proof of the lemma is available in Appendix B.7. For a firm with gap G, $R(G)\zeta Z^{-(\eta-1)}N$ accounts for the net gain of adopting the frontier technology, discounted at rate $q+\bar{\rho}$: The term $1 - e^{-\gamma(\eta-1)G}$ is the productivity gain accrued from adoption, the term $\frac{\gamma(\eta-1)}{q+\bar{\rho}+\gamma(\eta-1)} \left[1 - e^{-(q+\bar{\rho}+\gamma(\eta-1))G}\right]$ accounts for the economic depreciation of the technology that follows from this technology adoption policy, and $N\zeta Z^{-(\eta-1)}$ translates this gain into profit. The optimal threshold G solves

$$\overbrace{R(G)\zeta Z^{-(\eta-1)}N}^{\text{discounted net gain}} = \overbrace{\kappa_0 s_a \left(\frac{1}{s_r} \left(\frac{\eta}{\eta-1} \right) \right)^{\frac{1}{1-\nu}} Z^{-\frac{1}{1-\nu}}}^{\text{net cost}} .$$

At the optimal threshold, the discounted net gains from adoption are equal to the net cost. The optimal firm's threshold G depends on aggregate behavior through intermediate input productivity Z. Let $\bar{G}(Z)$ denote this optimal threshold. The next proposition uses the function $R(G)$ to define and characterize $\bar{G}(Z)$.

Lemma 3 *The optimal threshold G as a function of Z, denoted by $\bar{G}(Z)$, is the solution of:*

$$\bar{G}(Z) = R^{-1} \left(\frac{\kappa_0}{N} s_a \left(\frac{1}{s_r} \left(\frac{\eta}{\eta-1} \right) \right)^{\frac{1}{1-\nu}} \left(\frac{1}{\zeta} \right) Z^{(\eta-1)(1-\zeta)} \right) . \tag{67}$$

If the adoption cost is large, i.e.,

$$\frac{\kappa_0}{N} s_a \left(\frac{1}{s_r} \left(\frac{\eta}{\eta-1} \right) \right)^{\frac{1}{1-\nu}} \left(\frac{1}{\zeta} \right) Z^{(\eta-1)(1-\zeta)} > \frac{1}{q+\bar{\rho}+\gamma(\eta-1)} , \tag{68}$$

then the value $\bar{G}(Z) = +\infty$. Otherwise, if the cost is low, i.e., (68) is not satisfied, then $\bar{G}(Z)$ is finite. Furthermore, in this case: (i) the function $\bar{G}$ is decreasing in Z if $\zeta > 1$, and increasing if $\zeta < 1$, and (ii) the value $\bar{G}(Z)$ is strictly increasing in κ_0 and s_a, and decreasing in s_r. For κ_0 sufficiently small, the solution is approximately given by:

$$\bar{G}(Z) \approx \sqrt{\frac{2}{\gamma}\frac{\kappa_0}{N}(1-\nu)s_a\left[\frac{1}{s_r}\left(\frac{\eta}{\eta-1}\right)\right]^{\frac{1}{1-\nu}} Z^{(\eta-1)(1-\zeta)}}\ . \tag{69}$$

The proof is available in Appendix B.8. Close-form expressions for the value function V as a function of Z are presented in the body of the proof.

The next lemma uses aggregation to find the relationship between the intermediate input productivity Z and the threshold G, defining a function $\bar{Z}(G)$, and provides a close-form expression for the stationary distribution of technology gaps.

Lemma 4 *Given a threshold G, the density $\bar{m}(g)$ is given by*

$$\bar{m}(g) = \frac{qe^{-qg}}{1-e^{-qG}} \text{ for } g \in [0, G]\ ,$$

and the value of Z as a function of G, $\bar{Z}(G)$, is obtained by computing the integral is:

$$\bar{Z}(G) = \left(\frac{q}{q+\gamma(\eta-1)}\frac{1-e^{-(q+\gamma(\eta-1))G}}{1-e^{-qG}}\right)^{\frac{1}{\eta-1}}\ .$$

The function $\bar{Z}$ is decreasing in G.

The proof is a direct computation so it is omitted. Given Lemmas 3 and 4, a balance growth path with costly adoption is given by the solution of the following fixed point,

$$G^* = \bar{G}(\bar{Z}(G^*)) \text{ and } Z^* = \bar{Z}(G^*)\ .$$

We rewrite the fixed point as the zero of an equation to analyze the existence and uniqueness of balanced growth path. Direct use of the previous results provide the following proposition. In particular, using the functions $R(G)$ and $\bar{Z}(G)$ we define a new function $\bar{H}(G)$, so that a balance growth path is given by the solution of $\bar{\kappa} = \bar{H}(G^*)$, where $\bar{\kappa}$ is an adjusted fixed cost. The proposition characterizes the cases whether there is a unique or multiple balance growth paths. Notably, the necessary condition for multiple balance growth paths is the presence of complementarities, i.e., $\zeta > 1$.

Proposition 8 *We define the adjusted fixed cost $\bar{\kappa}$, and using the previously defined functions R and $\bar{Z}$,* *we define the constant $\bar{\kappa}$ and the function $\bar{H}$ as:*

$$\bar{\kappa} \equiv \frac{\kappa_0}{N}\, s_a \left(\frac{1}{s_r}\left(\frac{\eta}{\eta-1}\right)\right)^{\frac{1}{1-\nu}}, \quad \text{and} \quad \bar{H}(G) \equiv R(G)\, \zeta\, \bar{Z}(G)^{(\eta-1)(\zeta-1)} . \tag{70}$$

A balanced growth path gap with costly adoption G^* is a solution of the equation: $\bar{\kappa} = \bar{H}(G^*)$. We have:

1. *The function $\bar{H}$ satisfies: $\bar{H}(0) = 0$, $\bar{H}'(0) > 0$, and has an asymptote:*

$$\bar{H}(\infty) \equiv \lim_{G\to\infty} \bar{H}(G) = \left(\frac{q}{q+\gamma(\eta-1)}\right)^{\zeta-1} \frac{\zeta}{q+\bar{\rho}+\gamma(\eta-1)} .$$

2. *If $q > 0$ and $\bar{\kappa} > \bar{H}(\infty)$, then there is a balanced growth path without costly adoption, i.e. $G^* = \infty$.*

3. *For small $\bar{\kappa}$ there is always a balanced growth path with costly adoption, which can be approximated by:*

$$G^* \approx \sqrt{\frac{2\,(1-\nu)}{\gamma}\frac{s_a\,\kappa_0}{N}} \left(\frac{1}{s_r}\left(\frac{\eta}{\eta-1}\right)\right)^{\frac{1}{2(1-\nu)}} . \tag{71}$$

4. *If $\zeta \leq 1$, the function $\bar{H}(G)$ is monotone increasing in G. Then there is at most one balanced growth path. If $\bar{\kappa} \leq \bar{H}(\infty)$ then there is unique $G^* < \infty$, which is increasing in the adjusted fixed cost $\bar{\kappa}$.*

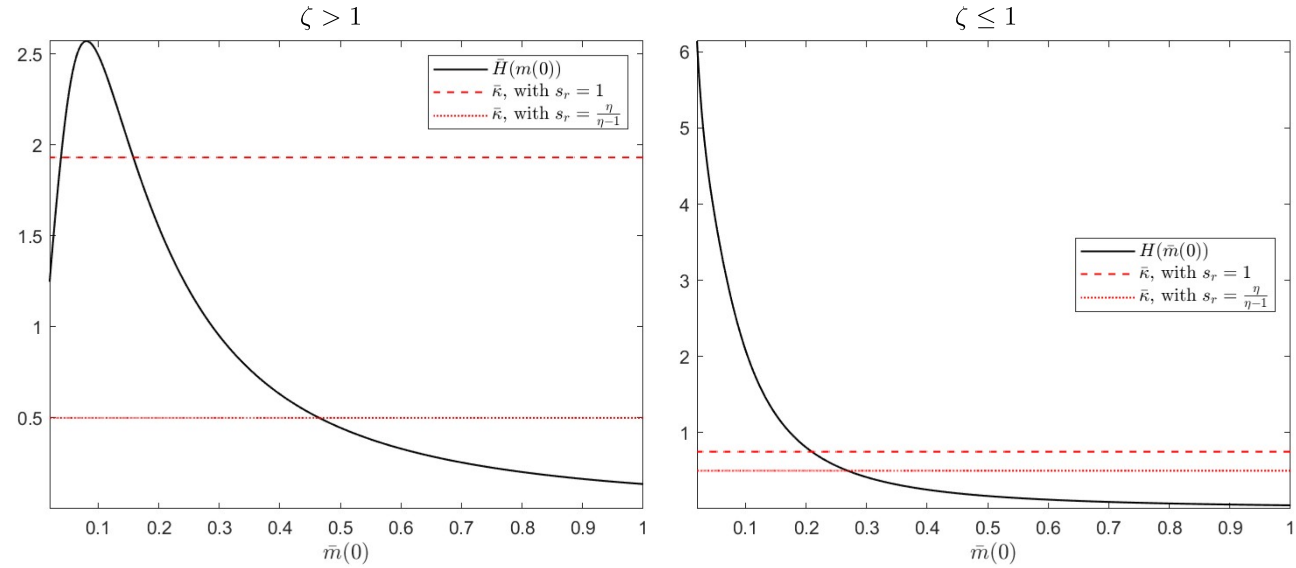
5. *If $\zeta > 1$ and ν is large enough, $\bar{H}(G)$ is not monotone in G. Then if $\bar{\kappa}$ is not too large, there are multiple balanced growth paths.*

The proposition follows directly from the previous lemmas, and thus the proof is omitted. The proposition shows that if there are no strategic complementarities, i.e. when $\zeta < 1$ either there is a unique balanced growth path with costly adoption, or if $\bar{\kappa}$ is large enough the balanced growth path has no costly adoption. The threshold for interior balanced growth path G^* is increasing in the adjusted adoption cost, $\bar{\kappa}$. Moreover, note that the adjusted adoption cost $\bar{\kappa}$ decreases with subsidy s_r, i.e. as s_r increases then $\bar{\kappa}$ decreases and thus G^* decreases, so that the balanced growth path has more adoption.

Instead, with strategic complementarities, i.e. when $\zeta > 1$, while there is always a balanced growth path with costly adoption for low $\bar{\kappa}$, but there can be multiple balanced growth path with costly adoption if $\bar{\kappa}$ is in an intermediate range and ν is large. We stress that this is a sufficient condition for the existence of multiple balanced growth paths. It is easy to construct examples where $\zeta > 1$ and ν is small, and where multiple balanced growth paths occur.[5] We next provide a numerical exploration of Proposition 8. We do so by exemplifying how the functions $\bar{H}$ and $\bar{\kappa}$, economic outcomes, and the set of equilibria, differ between a parameterization of the economy with complementarities and one without complementarities.

[5] For example, choosing η close to one.

Figure 1: Balanced-growth path equilibria: $\bar{H}(\cdot)$ and $\bar{\kappa}$



Notes. Parameters : $q = 0.02$, $\gamma = 0.10$, $\eta = 3$, $\theta = 1$, $\rho = 0.05$, $N = 1$, and $\kappa_0 = 0.5$. Left plot: $\nu = 0.7$. Right plot: $\nu = 0$. The left plot presents, for $\zeta > 1$, $\bar{H}(\cdot)$ as a function of $\bar{m}(0)$, and considers the case where there are two interior balanced-growth path equilibria (and one equilibrium with no adoption, i.e. $\bar{m}(0) \to 0$), when $s_r = s_a = 1$, but where there is one interior balanced-growth path equilibrium (and no equilibrium with no adoption) when $s_r = \frac{\eta}{\eta-1}$ and $s_a = 1$. The right plot presents, for $\zeta \leq 1$, $\bar{H}(\cdot)$ as a function of $\bar{m}(0)$, and considers the case where there is one interior balanced-growth path equilibrium, when $s_r = s_a = 1$, and where there is one interior balanced-growth path equilibrium when $s_r = \frac{\eta}{\eta-1}$ and $s_a = 1$.

Figure 1 displays the functions $\bar{H}$ and $\bar{\kappa}$, as a function of $\bar{m}(0)$, i.e. the density of firms with $g = 0$. From Lemma 4, we obtain that $\bar{m}(0)$ is monotone decreasing in $\bar{G}$. The density of firms with $g = 0$ provides a clear notion of the amount of adoption at every instant. The left panel in the figure presents the case where there are strategic complementarities, i.e. when $\zeta > 1$, while the right panel presents the case when there are not. It can be seen in the left panel that $\bar{H}$ is hump shaped. The two horizontal lines, corresponding to the same value for κ_0, present different values for the adjusted $\bar{\kappa}$. These two values differ in the revenue subsidy. The highest $\bar{\kappa}$ corresponds to the laissez faire, i.e. no revenue subsidy (i.e. $s_r = 1$), while the lowest $\bar{\kappa}$ corresponds to the case with the optimal subsidy described in Corollary 1, (i.e. $s_r = \eta/(\eta - 1)$). In the laissez faire case, there are two intersections, which corresponds to two balanced growth path with costly adoption, and one balanced growth path with no costly adoption (i.e. $G^* = +\infty$ or $\bar{m}(0) = q$). In the case with the optimal subsidy, there is a unique balanced-growth path equilibrium, and this equilibrium requires costly adoption. In the case with no strategic complementarities presented in the right panel, the function $\bar{H}(\cdot)$ is monotone decreasing towards its asymptote. For both the laissez faire and optimal subsidy cases, provided that $\bar{\kappa}$ is low we obtain a unique balanced growth path with costly adoption.

Table I presents some economic outcomes for the different equilibria observed in Figure 1. In particular, it presents the equilibrium gaps G^*, the expected time until next adoption for a firm that just adjusted, output $\bar{y}$, intermediate productivity $\bar{Z}$, and the density of firms at $g = 0$, for the different equilibria in the laissez faire ($s_r = 1$) and optimum ($s_r = \frac{\eta}{\eta-1}$) balanced-growth paths.[6] With strategic complementarities, the laissez faire case exhibits three equilibrium gaps G^*: there is one equilibrium where costly adoption is frequent ($G^* = 7$), and thus there is a significant density of firms at zero gap at every moment ($\bar{m}(0) = 0.16$) where these firms are expected to adopt again in 6 years, and output is 'high' ($\bar{y} = 0.37$ relative to optimum). There is another equilibrium were costly adoption is less frequent ($G^* = 103$), the density of firms at zero gap is lower ($\bar{m}(0) = 0.04$) where these firms at zero gap are expected to adopt again in 26 years, and output is low ($\bar{y} = 0.06$ relative to optimum). There is also one equilibrium with no costly adoption ($G^* = \infty$), where the density of firms at zero gap is the lowest possible ($\bar{m}(0) = q = 0.02$) and where these firms are expected to again in $1/q = 50$ years, and where output is the lowest among the equilibria ($\bar{y} = 0.01$ relative to optimum). The optimal equilibrium features costly adoption. Relative to the laissez faire equilibria, costly adoption is substantially more frequent in the optimal balanced-growth path ($G^* = 2$), there is a larger density of firms at zero gap ($\bar{m}(0) = 0.16$) where a firm that just adjusted is expected to adopt again in 2 years. Naturally, a higher frequency of adoption results in higher output too ($\bar{y} = 1.00$). The case with no strategic complementarities is substantially different, exhibiting little variation in outcomes from the single laissez faire equilibrium and optimum.

To conclude the numerical exploration of Proposition 8, we study the effect of distortions, interpreted as variations in s_r, in shaping up the set of equilibrium balanced growth paths. From the definition of $\bar{\kappa}$ in the proposition, we have that $d\bar{\kappa}/ds_r < 0$. This implies that economies with high distortions ($s_r \leq \underline{s_r}$) will feature an unique equilibrium with no costly adoption, and economies with low distortions ($s_r > \bar{s}_r$) a unique equilibrium, and this equilibrium is with costly adoption. Furthermore, depending of the extent of strategic complementarities, an intermediate intermediate level of distotions implies either one equilibrium or three equilibria. Figure 2 presents the set of equilibria for both the cases with and without strategic complementarities. Naturally, in the case with strategic complementarities, there is a region in the space of s_r where a small change in the distortion has a discrete effect in equilibrium outcomes. For example, variation in distortions around $\bar{s_r}$ can have massive effects in aggregate outcomes, if the economy 'jumps' from an equilibrium without costly adoption (or the interior equilibrium with low costly adoption) below $\bar{s}_r$ to the unique equilibrium above $\bar{s}_r$. This behavior cannot happen in the economy with $\zeta \leq 1$. If we were to interpret s_r as a policy instrument instead than a distortion, a small change in the policy change can have large aggregate effects. Of course, this 'jump' feature cannot really occur in a fully dynamic economy, something that we tackle more carefully later on.

The next proposition shows that the balanced growth path without costly adoption is locally stable. To interpret the proposition recall that the distribution of gaps in an equilibrium without costly adoption is given by the density qe^{-qg}.

Proposition 9 *Assume $q > 0$ and consider a balanced growth path without costly adoption for which the asymptote is strictly smaller than the adjusted cost, i.e: $\bar{H}(\infty) < \bar{\kappa}$. Then, this balanced growth path is*

[6] The expected time until next adjustment for a firm with gap g and maximum gap G is equal to $\frac{1}{q}\left[1 - e^{q(g-G)}\right]$.

Table I: Statistics for different balanced-growth path equilibria

		Laissez faire, $s_r = 1$			Optimum, $s_r^* = \frac{\eta}{\eta-1}$
		no costly adoption	low interior	high interior	
$\zeta > 1$	Maximum gap, G^* (years)	∞	36	7	2
	Exp. Duration (years)	50	26	6	2
	Output, $\bar{y}$	0.01	0.06	0.37	1.00
	Interm. Prod., Z	0.30	0.42	0.75	0.90
	Density at $g = 0$, $\bar{m}(0)$	0.02	0.04	0.16	0.47
		Laissez faire, $s_r = 1$			Optimum, $s_r^* = \frac{\eta}{\eta-1}$
$\zeta \leq 1$	Maximum gap, G^* (years)	5			4
	Exp. Duration (years)	5			4
	Output, $\bar{y}$	0.95			1.00
	Interm. Prod., Z	0.80			0.84
	Density at $g = 0$, $\bar{m}(0)$	0.21			0.27

Notes. Parameters : $q = 0.02$, $\gamma = 0.10$, $\eta = 3$, $\theta = 1$, $\rho = 0.05$, $N = 1$, and $\kappa_0 = 0.5$. For $\zeta > 1$: $\nu = 0.7$. For $\zeta \leq 1$: $\nu = 0$. Output is normalized by the Output at the optimum.

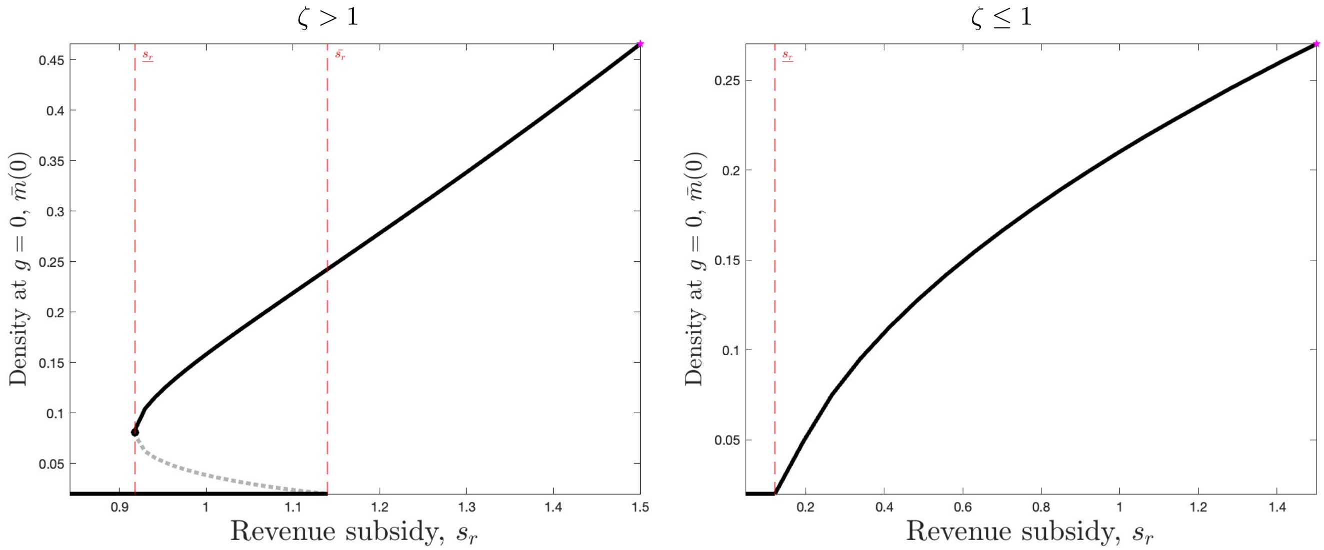
locally stable, i.e. there exist an $\epsilon > 0$ such that for all m_0 with $m_0(0) = q$ for which $|\Delta(m_0)| < \epsilon$ where

$$\Delta(m_0) \equiv \int_0^\infty e^{-g\gamma(\eta-1)} \left(m_0(g) - qe^{-qg}\right) dg \tag{72}$$

there is an equilibrium that converges to the balanced growth path without costly adoption.

A proof of the proposition is available in Appendix B.9.

Figure 2: The set of balanced growth paths as a function of s_r



Notes. The figure presents the set of balanced growth paths with and without strategic complementarities. Common parameters: $q = 0.02$, $\gamma = 0.10$, $\theta = 1$, $\rho = 0.05$, $\kappa_0 = 0.5$, $N = 1.0$, $s_a = 1$. Left plot: $\eta = 3$, and $\nu = 0.7$. Right plot: $\eta = 3$, $\nu = 0$. The pink star presents the balanced growth path at the optimum.

6 Model with a Static Frontier

We consider a version of the model with no growth of the frontier—normalized to be equal to 1, and no free adoption opportunities, i.e. $q = 0$. We say that a firm has a gap z if its productivity equals $e^{-z} < 1$, where $z \geq 0$. A firm in the frontier has a productivity gap $z = 0$, i.e. productivity equal to one. Relative to the version of the model with growth, here gaps are expressed in terms of log productivity units instead of time. We consider equilibria where the adoption decision follows a threshold rule. In particular, at time t all firms with productivity gaps $z \leq G(t)$ are yet to adopt the frontier technology. Given the threshold rule, the state of the economy is characterized by a density $m_0(z)$ of firms with productivity gaps $z \in (0, G(t)]$ and a mass of forms that already adopted $K(t)$. Further, we allow firms to revert their adoption decisions. That is, at any point in time, a firm that adopted can give up the acquired technique and return to its gap z. Finally, we assume that the economy starts with a distribution of productivity gaps $m_0(z)$ with support $z \in (0, G_0]$ and an initial mass point at $z = 0$ which we denote $K(0)$.

We being the analysis by noting that there is a one-to-one relationship between the marginal adopter $G(t)$ and the mass of firms that already adopted by time t— denoted by $K(t) \in [0, 1]$. This relationship satisfies:

$$K(t) = 1 - \int_0^{G(t)} m_0(z)\, dz \ . \tag{73}$$

Taking time derivatives, we obtain the evolution of the distribution of productivity z, which is characterized by the accumulation of mass at $z = 0$ and the evolution of the gap, $G(t)$,

$$\dot{K}(t) = -m_0(G(t))\, G'(t)\,, \text{ for } G(t) > 0 \;. \tag{74}$$

For $K(t) = 1$, or equivalently, $G(t) = 0$, $\dot{K}(t) \leq 0$. This follows from $G(t) \geq 0$.

As in the general model with growth, firms that adopt the modern technology incur in a cost κ, measured in units of output. When a firm adopts the modern technology, it joins the mass of firms that adopted. Combining this with the aggregate feasibility constraint we obtain the counterpart to (4) in the general formulation of the model,

$$\kappa\dot{K}(t) + C(t) = Y(t) \equiv A(s_r)F(K(t))\,, \text{ for } K(t) < 1 \;, \tag{75}$$

where, in a static equilibrium, the analogous of the aggregate production function in the general formulation of the model in (31) is given by

$$F(K) = \left[\int_0^{\hat{G}(K)} e^{-z(\eta-1)} m_0(z)\, dz + K\right]^{\zeta} \text{ with} \tag{76}$$

$$K = 1 - \int_0^{\hat{G}(K)} m_0(z) dz \text{ or } \hat{G}(K) = M_0^{-1}(1-K) \;,$$

where $M_0(\cdot)$ is the CDF of m_0, G is defined implicitly as a function of K in (73), and where $A(s_r)$ is defined in (30). When $K(t) = 1$, feasibility requires $C(t) \leq A(s_r)F(1)$.

In this simpler economy, the mass of adopters K can be interpreted as the productive capital stock, conspicuously labeled K. Indeed, the law of motion of this stock, and the intertemporal preferences, are akin to those in the Neoclassical Growth Model. A crucial difference with the standard version, is that the production function is only guaranteed to be concave if $\zeta \leq 1$. Given the relevance of the shape of the production function, for future reference we provide expressions for the marginal product of capital,

$$\frac{\partial}{\partial K}F(K) = \zeta\left[\int_0^{\hat{G}(K)} e^{-z(\eta-1)} m_0(z)\, dz + K\right]^{\zeta-1}\left(1 - e^{-\hat{G}(K)(\eta-1)}\right) \geq 0 \;, \tag{77}$$

x‘which for $K = 0$ gives

$$\frac{\partial}{\partial K}F(0) = \zeta\left[\int_0^{G_0} e^{-z(\eta-1)} m_0(z)\, dz\right]^{\zeta-1}\left(1 - e^{-G_0(\eta-1)}\right) = F(0)\frac{\zeta\left(1 - e^{-G_0(\eta-1)}\right)}{\int_0^{G_0} e^{-z(\eta-1)} m_0(z)\, dz} \;, \tag{78}$$

where G_0 is the upper bound of the support of m_0. To better understand the shape of F we compute its curvature:

$$\frac{\partial^2/\partial K^2 F(K)}{\partial/\partial K F(K)} = (\zeta-1)\frac{1-e^{-\hat{G}(K)(\eta-1)}}{\int_0^{\hat{G}(K)} e^{-z(\eta-1)} m_0\left(z\right) dz + K} - (\eta-1)\frac{e^{-\hat{G}(K)(\eta-1)}}{1-e^{-\hat{G}(K)(\eta-1)}}\frac{1}{m_0(\hat{G}(K))}\ . \tag{79}$$

The curvature at the two extremes values of K is equal to:

$$\frac{F''(1)}{F'(1)} = -\infty, \text{ and } \frac{F''(0)}{F'(0)} = (\zeta-1)\frac{1-e^{-G_0(\eta-1)}}{\int_0^{G_0} e^{-z(\eta-1)} m_0\left(z\right) dz} - (\eta-1)\frac{e^{-G_0(\eta-1)}}{1-e^{-G_0(\eta-1)}}\frac{1}{m_0(G_0)}\ . \tag{80}$$

To analyze the dynamics of this system, it is therefore useful to characterize the properties of the function $F(K)$. It is straightforward to see that $F(K) > 0$, $F'(K) > 0$, and that $F(K)$ is bounded. As mentioned above, the concavity or convexity on K depends on the parameter ζ. In our setup, the production function could be globally concave or s-shaped. With this in mind, we define an inflection point K^i as a value such that $F''(K) > 0$ if $K < K^i$, $F''(K) < 0$ if $K > K^i$, and $F''(K) = 0$ if $K = K^i$. The following proposition gives a characterization of the shape of the aggregate production function $F(\cdot)$.

Proposition 10 *Assume that $m_0(z) > 0$ in its domain. Then*

1. *If $\zeta \leq 1$ then $F(\cdot)$ is globally concave.*
2. *Regardless of ζ, then $F(\cdot)$ is concave near $K = 1$.*
3. *A necessary condition for $F(\cdot)$ to be convex around $K = 0$ is that $\zeta > 1$.*
4. *If $\zeta > 1$ and $e^{-z(\eta-1)}/m_0(z)$ is decreasing in z, then $F(\cdot)$ has at most one inflection point. Furthermore, if $\frac{F''(0)}{F'(0)} > 0$ then then there is an interior inflection point.*
5. *If in addition to $\zeta > 1$, either ν or G_0 are large enough, then $\frac{F''(0)}{F'(0)} > 0$.*

A proof of the proposition is available in Appendix B.10. The proposition shows that there is one inflection point provided enough complementarities. That is, when ζ is sufficiently large.

Proposition 10 gives a quite complete characterization of the shape of F. This is illustrated below with some figures for the case of interest. Before that we give an intuitive explanation of the forces in the model for concavity and convexity of F as a function of K. The force for concavity is simple, given the threshold nature of an equilibrium, in particular that $\hat{G}(K)$ is decreasing. For low values of K, i.e. when adoption is low, an increase in one unit of K moves firms with low productivity z to the frontier. Instead, for high K, an increase in one unit of K moves firms with high productivity to the frontier. So, this force pushes the marginal productivity to be decreasing. The force for convexity, in the case of $\zeta > 1$ has been analyzed and explained in Proposition 4. Recall that there we fixed two gaps, and consider an arbitrary movement of density from a one productivity to another. The concavity/convexity analyzed there was fixing the value of the gaps, and changing the amount of density transferred.

The case of truncated exponential m_0. In this case the density m_0 is given by $m_0(z) = \frac{\chi e^{-\chi z}}{1-e^{-\chi G_0}}$, with CDF given by $M_0(z) = \frac{1-e^{-\chi z}}{1-e^{-\chi G_0}}$. This example is particularly relevant as it preserves the same shape of the stationary density of gaps in the balanced growth path in the model with a growing technological frontier. In this case

$$\hat{G}(K) = -\frac{\log\left[e^{-\chi G_0} + K(1-e^{-\chi G_0})\right]}{\chi} \ , \ F(K)^{1/\zeta} = \frac{\chi}{\eta-1+\chi} \frac{\left\{1 - \left[e^{-\chi G_0} + K(1-e^{-\chi G_0})\right]^{\frac{\eta-1+\chi}{\chi}}\right\}}{1-e^{-\chi G_0}} + K \ .$$

Figures 3 and 4 display the production function $F(K)$ and the marginal product of capital $\partial F(K)/\partial K$ as we vary ζ for the case in which m_0 is given by a truncated exponential. In Figure 3 we keep ν fixed (in particular, $\nu = 0$), and we vary ζ by varying η. In Figure 4 we keep η fixed (in particular, $\eta = 3$), and we vary ζ by varying ν. In each plot, the black thick line depicts the case for $\zeta = 1$. In each plot, each different line corresponds to a different value of ζ. At $K = 0$, lines with higher values correspond to lower values of ζ. As it is clear from the figures, $\zeta \leq 1$ correspond to concave production functions. While this can be observed in the left panel of both figures, it can be easily seen in the right panels, where for the $\zeta \leq 1$ cases, the marginal productivity of capital is decreasing in the capital level. For $\zeta > 1$, the production function is convex for low values of capital and concave for high values of capital. In terms of marginal product of capital, this corresponds to having an increasing marginal product of capital for low levels of the capital stock, and decreasing marginal product of capital for high values of the capital stock. In summary, for large ζ, either due to high ν or low η, the production function is S-shaped.

Equilibrium for Static Frontier Case. In this section we specialize the expressions for the different problems in the static frontier case. Profits, measured as a function of productivity gaps, are given by

$$\pi(z,t) = \frac{N}{1-\nu}\frac{1}{\eta-1}\frac{e^{-z(\eta-1)}}{\int_0^{G(t)} e^{-\tilde{z}(\eta-1)} m_0(\tilde{z})\, d\tilde{z} + K(t)} = N\zeta \frac{e^{-z(\eta-1)}}{\int_0^{G(t)} e^{-\tilde{z}(\eta-1)} m_0(\tilde{z})\, d\tilde{z} + K(t)} \ , \tag{81}$$

and the price of the final good is

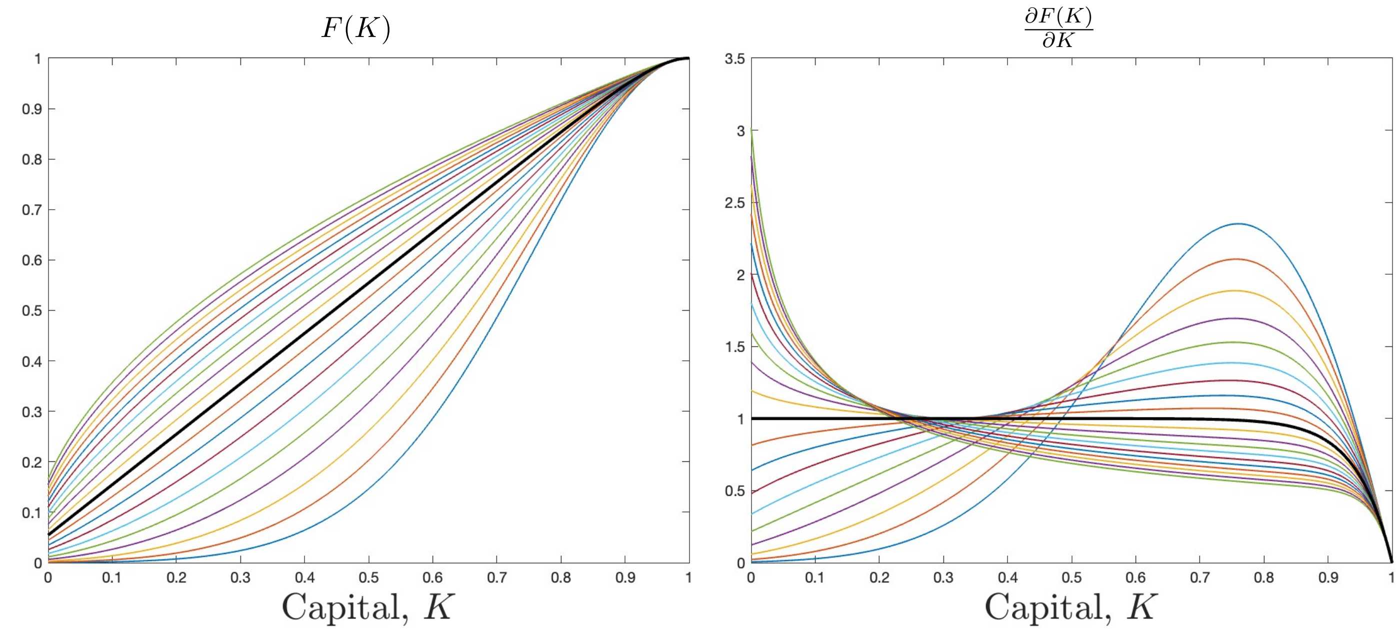
$$P(t) = \left(s_r \frac{\eta-1}{\eta}\right)^{-\frac{1}{1-\nu}} \left[\int_0^{G(t)} e^{-z(\eta-1)} m_0(z)\, dz + K(t)\right]^{-\zeta} \ . \tag{82}$$

The value at t of a firm with gap z is given by

$$V(z,t) = \max_{\tau \geq t} \int_t^{\tau} e^{-\int_t^s r(\tilde{s})d\tilde{s}} \pi(z,s)\, ds + e^{-\int_t^{\tau} r(\tilde{s})d\tilde{s}} \left[V^0(z,\tau) - s_a P(\tau)\kappa\right] \ . \tag{83}$$

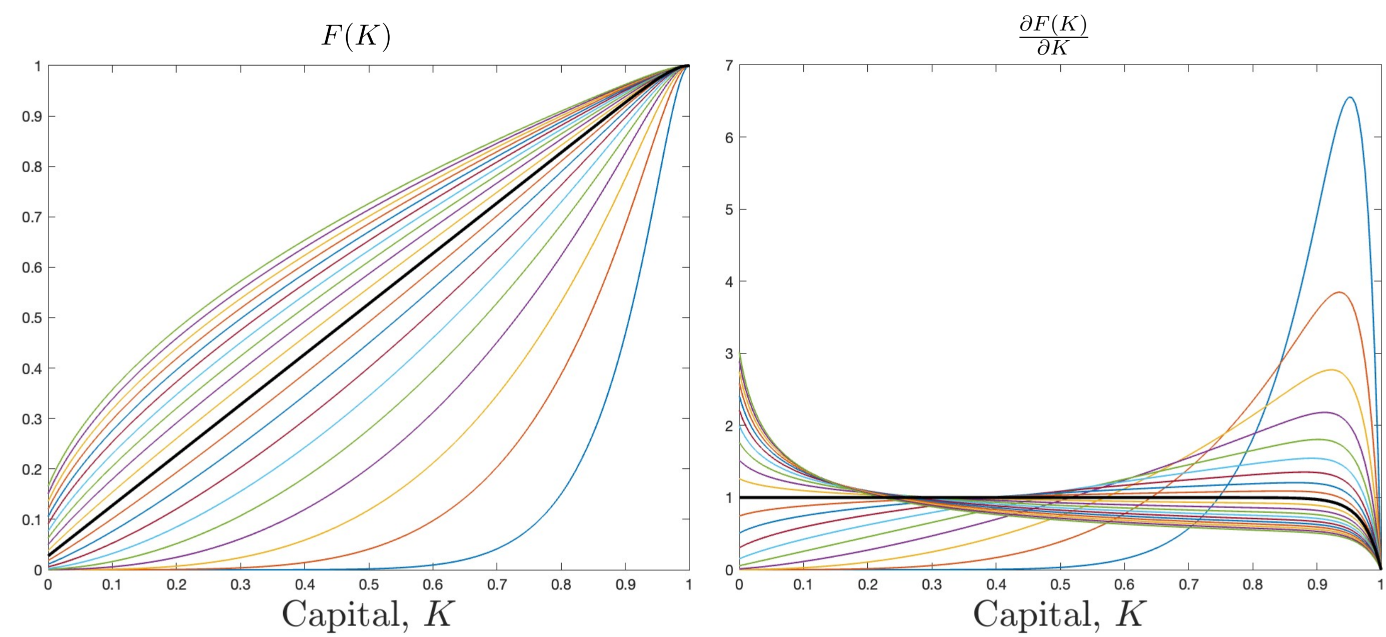
Although the value function is indexed by (z,t), in this version of the model the first index is fixed unless the firm adopts. We let $V^0(z,t)$ the value of a z firm that has adopted the frontier technology, and can produce with it. The technology index of the firm stays with it forever. In particular we allow a z firm that

Figure 3: $F(K)$ and $\frac{\partial F(K)}{\partial K}$, for various η



Notes: m_0 is a truncated exponential $m_0(z) = \chi \exp(-\chi z)/(1 - \exp(-\chi G_0))$ with $\chi = 0.01$ and $G_0 = 20$. Other parameters: $\nu = 0$, $N = 1$, and η varying from 1.1 to 3. The thick black line depicts the $\zeta = 1$ case.

Figure 4: $F(K)$ and $\frac{\partial F(K)}{\partial K}$, for various ν



Notes. m_0 is a truncated exponential $m_0(z) = \chi \exp(-\chi z)/(1 - \exp(-\chi G_0))$ with parameters: $\chi = 0.01$ and $G_0 = 20$. Other parameters: $\eta = 3$, $N = 1$, and ν varies from 0 to 0.95. The thick black line depicts the $\zeta = 1$ case.

has adopted the technology to disinvest, recovering the adoption cost, and use its original technology again. Then,

$$V^0(z,t) = \max_{\{\tau \geq t\}} \int_t^\tau e^{-\int_t^s r(\tilde{s})d\tilde{s}} \pi(0,s)\, ds + e^{-\int_t^\tau r(\tilde{s})d\tilde{s}} \left[V(\tau,z) + \kappa s_a P(\tau)\right] \ . \tag{84}$$

The remaining equilibrium objects and definitions are the complete analog to the general case. An equilibrium, given the density m_0, subsidies s_r, s_a, consists of paths of $\{K(t), C(t), G(t), r(t), P(t), V(z,t)\}$ for $t \geq 0$ and $z \in [0,1]$ such that: (i) given $\{r(t), P(t), \pi(t,z)\}$ optimal adoption is given by $\{G(t)\}$, (ii) given $\{K(t), P(t)\}$, then $\pi(z,t)$ is given by (81), (iii) given $\{G(t), K(t)\}$ prices are given by (82), (iv) $\{C(t), r(t), P(t)\}$ satisfy the Euler equation (9), and (v) the allocation $\{C(t), K(t), \dot{K}(t)\}$ is a static equilibrium.

The next proposition gives a very simple characterization of equilibrium as the solution of a system of two o.d.e.'s for $\{K(t), C(t)\}$ with an initial condition for K and a terminal boundary condition.

Proposition 11 *Fix s_r, s_a and the initial capital $K(0)$. A necessary and sufficient condition for an interior equilibrium is that $\{C(t), K(t)\}$ solve the following system of o.d.e.'s:*

$$\kappa \dot{K}(t) = A(s_r)F(K(t)) - C(t), \ \text{for } 0 \leq K(t) \leq 1 \ , \tag{85}$$

where $\dot{K}(t) \leq 0$ if $K(t) = 1$ and $\dot{K}(t) \geq 0$ if $K(t) = 0$,

$$\theta \frac{\dot{C}(t)}{C(t)} = B(s_r, s_a) A(s_r) \frac{1}{\kappa} \frac{\partial}{\partial K} F(K(t)) - \rho \ , \text{where} \tag{86}$$

$$B(s_r, s_a) \equiv \left(\frac{1-\nu}{\frac{1}{s_r}\frac{\eta}{\eta-1} - \nu} \right) \frac{1}{s_a} \tag{87}$$

for all $t \geq 0$, with terminal condition given by

$$0 = \lim_{T \to \infty} e^{-\rho T} C(T)^{-\theta} A(s_r) F(K(T)) \ . \tag{88}$$

A proof of the proposition is available in Appendix B.11.

Steady States For a given m_0 and subsidies s_a, s_r, an interior steady state equilibrium of the model is given by a capital level $K^* \geq 0$ such that:

$$\frac{\rho \kappa}{A(s_r) B(s_r, s_a)} = \frac{\partial}{\partial K} F(K^*) \ . \tag{89}$$

An important corollary of Proposition 10 is the following partial characterization of the steady states.

Corollary 2 ***Steady States.*** *Consider two cases:*

1. *If $\zeta \leq 1$, so that $F(\cdot)$ is strictly concave, then there is at most one interior steady state. If in addition, $F'(0) > \frac{\rho\kappa}{A(s_r)B(s_r,s_a)}$, there is an interior steady state.*

2. *If $\zeta > 1$, $\frac{F''(0)}{F'(0)} > 0$, and $e^{-z(\eta-1)}/m_0(z)$ is decreasing in z, so that $F(\cdot)$ is S-shaped, then there are at most two interior steady states. If, in addition $F'(0) > \frac{\rho\kappa}{A(s_r)B(s_r,s_a)}$ then there is exactly one interior steady state. Alternatively if $F'(0) < \frac{\rho\kappa}{A(s_r)B(s_r,s_a)}$, and there is $0 < \hat{K} < 1$ for which $F'(\hat{K}) > \frac{\rho\kappa}{A(s_r)B(s_r,s_a)}$ there are exactly two interior steady states.*

It is useful to analyze the local dynamics around an interior steady sate (K^*, C^*). Linearizing (85) and (86) around an interior steady state we get:

$$\begin{bmatrix} \dot{K}(t) \\ \dot{C}(t) \end{bmatrix} = M \begin{bmatrix} K(t) - K^* \\ C(t) - C^* \end{bmatrix} \equiv \begin{bmatrix} \frac{\rho}{B(s_r,s_a)} & -\frac{1}{\kappa} \\ \rho\frac{C^*}{\theta}\frac{F''(K^*)}{F'(K^*)} & 0 \end{bmatrix} \begin{bmatrix} K(t) - K^* \\ C(t) - C^* \end{bmatrix} . \tag{90}$$

We denote the eigenvalues of M by σ_1, σ_2. Then, we have,

$$\sigma_1 + \sigma_2 = Tr(M) = \frac{\rho}{B(s_r, s_a)} \text{ and } \sigma_1\sigma_2 = Det(M) = \frac{\rho}{\kappa}\frac{C^*}{\theta}\frac{F''(K^*)}{F'(K^*)} ,$$

$$\sigma_{1,2} = \frac{\rho/B(s_r, s_a) \pm \sqrt{(\rho/B(s_r, s_a))^2 - 4\frac{\rho}{\kappa}\frac{C^*}{\theta}\frac{F''(K^*)}{F'(K^*)}}}{2} .$$

From here we immediately obtain the following result.

Proposition 12 *Assume that the conditions in Corollary 2 so that there are two interior steady states holds and denote them by $0 < K_L^* < K_H^* < 1$, with corresponding consumption $0 < C_L^* < C_H^*$. Then:*

1. *The high adoption interior steady state (K_H^*, C_H^*) is a saddle, both eigenvalues of M are real with $\sigma_1 < 0 < \rho < \sigma_2$.*

2. *The low adoption interior steady state (K_L^*, C_L^*) is a source, both eigenvalues of M have positive real part. In particular, there is a threshold θ^* given by*

$$\theta^* \equiv 4\, C_L^* \frac{B(s_r, s_a)^2}{\rho\kappa} \frac{\partial^2/\partial K^2 F(K_L^*)}{\partial/\partial K F(K_L^*)} > 0 , \tag{91}$$

such that:

   (a) *If $\theta \geq \theta^*$ both eigenvalues of M are real and strictly positive, $0 < \sigma_1 < \frac{\rho}{B(s_r,s_a)} < \sigma_2$.*

   (b) *If $\theta < \theta^*$ the eigenvalues of M are complex conjugates, with strictly positive real part equal to to $\frac{\rho}{2B(s_r,s_a)}$.*

Proposition 12 has important consequences. First, the high adoption interior steady state is saddle-path stable, so for $K(0)$ close to K_H^* one can construct an equilibrium path that converges to it. Second, the smaller interior steady state K_L^* is not stable. This lack of stability provides that this equilibrium should not be observed, since any perturbation from the steady state will make the equilibrium to move away from it. In particular, the condition in (91) can be written as a critical value for the curvature of $U(\cdot)$ relative to the curvature on $F(\cdot)$. As it is well known in the case of concave $F(\cdot)$, the speed of convergence of the neoclassical growth model depends on this ratio, as well as on the discount factor. Less well known is that in the S-shaped case, this quantity plays a crucial role on determining the nature of the unstable dynamics of the lower steady state, i.e. whether the lower interior steady state is a nodal source or a spiral source. Third, there can be multiple equilibrium paths starting from the same initial condition, as a consequence of coordinated expectations about different future paths for prices. For instance, some of these equilibria can have oscillatory initial behavior as in the case where $\theta < \theta^*$ (a spiral source). Another possibility is that for a given initial condition one can construct an equilibrium where $K(t) \to 0$ or one where $K(t) \to K_H^*$.

The next proposition analyzes the no-adoption steady state:

Proposition 13 *Assume that*

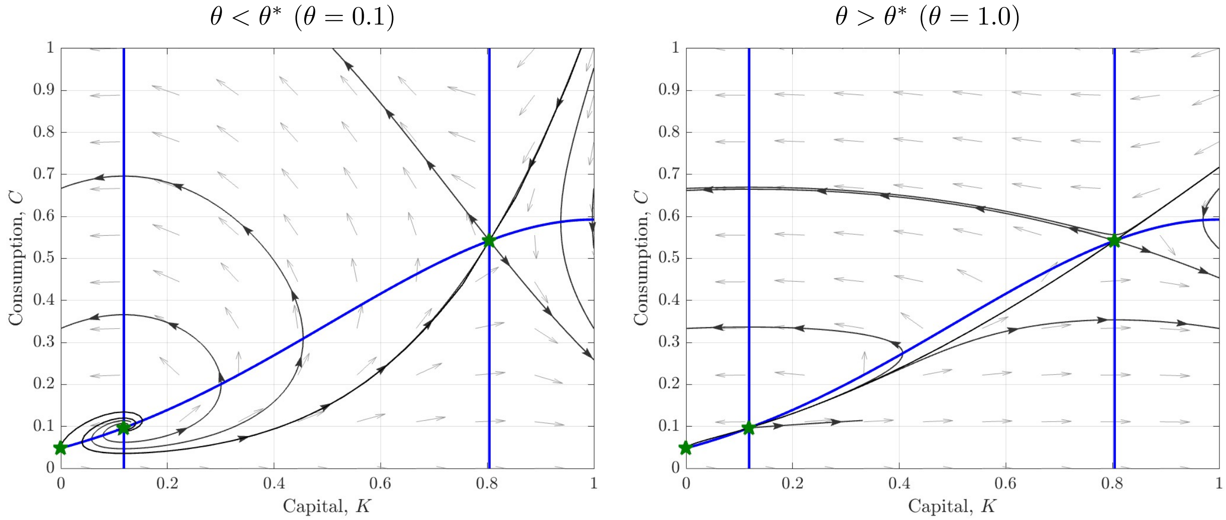
$$\frac{\partial}{\partial K} F(0) < \frac{\rho\kappa}{A(s_r)B(s_r, s_a)} \ . \tag{92}$$

Then $K^* = 0$ *and* $C^* = A(s_r)F(0)$ *is a stationary equilibrium with no adoption (i.e. non-interior). Furthermore, the no adoption stationary equilibrium is locally stable, i.e. there exists an* $\epsilon > 0$ *such that for all* $0 < K(0) \leq \epsilon$*, there is an equilibrium path* $\{C(t), K(t)\}_{t\in[0,T]}$ *for which* $C(T) = F(0)$ *and* $K(T) = 0$*, i.e. the convergence is in finite time.*

A proof is available in Appendix B.12.

Figure 5 shows the phase diagrams illustrating the equilibrium dynamics of an economy featuring strong complementarities, i.e., $\zeta > 1$. In each plot, the blue line depicts aggregate output $A(1)F(K)$, the green filled stars account for steady-state equilibria, the gray arrows provides the direction that the system $\{K, C\}$ will move to at a given point, and the black lines account for different examples of dynamic equilibrium paths. As the figure shows, there are three steady state equilibria: (i) a steady state with no adoption, $K = 0$, (ii) an interior steady steady state with low adoption, $K = K_L^*$, and (iii) an interior steady state with high adoption, $K = K_H^*$. The left panel considers the case where the elasticity of intertemporal substitution is low ($\theta = 0.1$), and below the threshold θ^*, while the right panel considers the case where the elasticity of intertemporal substitution is high ($\theta = 1.0$), and above the threshold θ^*. Both cases share the same steady states, which depends only on production parameters and the discount factor ρ, as evident in equation (89). As shown in the figures, the no-adoption steady state and the interior steady state with high adoption are stable. Furthermore, the interior high adoption steady state is a saddle: locally, there is one equilibrium path that converges to it. While in both figures the low adoption interior steady state is unstable—evident

Figure 5: Phase diagrams: laissez faire equilibrium dynamics, $s_r = 1$



Notes. m_0 is a truncated exponential $m_0(z) = \chi \exp(-\chi z)/(1 - \exp(-\chi G_0))$ with parameters: $\chi = 0.8$ and $G_0 = 10$. Other parameters: $\eta = 3$, $N = 1$, $s_r = s_a = 1$, and $\nu = 0.75$.

by the fact that the arrows push the equilibrium away from it, the case with low θ presented in the left panel exhibits a spiral source, while the case with high θ does not. While we do not care per se about the low adoption interior steady state, the differential equilibrium dynamics sourcing from it are relevant when studying policies with the potential to transition the economy from a steady state with no adoption to a interior high adoption steady state. Understanding the potential for these policies to achieve outcomes of this sort is the objective of the next section.

Planner's Problem. The sequence problem is:

$$\max_{C(\cdot)} \int_0^\infty e^{-\rho t} U(C(t))dt \text{ subject to } \kappa \dot{K}(t) = A(s_r^*)F(K(t)) - C(t) \ ,$$

with $K(0)$ given. There are three necessary conditions for an optimal path that are immediate: (i) that the optimal static subsidy is required $s_r = s_r^* \equiv \eta/(\eta - 1)$, (ii) that the Euler equation has to hold, i.e. the o.d.e. (86) for all $t > 0$, and (iii) that the Transversality condition has to hold, i.e. the boundary condition (51). As it is standard, a necessary condition for K^* to be an optimal steady state is that $A(s_r^*)F(K^*) = NF(K^*) = C^*$ and that

$$\frac{\rho\kappa}{N} = \frac{\partial F(K^*)}{\partial K} \ .$$

If $\zeta \leq 1$, so that $F(\cdot)$ is concave, these conditions are sufficient, and together with feasibility characterize a solution. In this case, there is a unique saddle path, so that from any $K(0)$ there is monotone convergence of C and K to the unique stationary equilibrium.

In the case of $\zeta > 1$, the three conditions above are only necessary, since the problem is not globally concave. This case has been studied rigorously at least since Skiba (1978), who proposed the study of the neoclassical growth model with an S-shaped production function. Further results, which gives a complete characterization are in Dechert and Nishimura (1983) and in Brock and Dochert (1983).[7] We summarize here the relevant ones:

1. Suppose that $\zeta > 1$ and consider the case in which there are two interior steady states $K_L^* < K_H^*$. In this case, K_L^* cannot be stable. This follows from Proposition 12 by setting $s_a = 1$ and $s_r = s_r^*$.

2. Suppose that $\zeta > 1$, the function $F(\cdot)$ is S-shaped, and there are two interior steady states $K_L^* < K_H^*$. If $\theta < \theta^*$, as defined in (91), then the solution K_L^* is not efficient, i.e. if $K(0) = K_L^*$ the efficient allocation moves away from K_L^*. This is shown by considering the continuous time limit of Lemma 4 in Dechert and Nishimura (1983).

3. If $F'(0) > \frac{\rho\kappa}{N}$, then the efficient allocation converges to the unique steady state K_H^* from any $K(0)$.

6.1 Big Push with Revenue Subsidies

In this section we analyze a case where the introduction of a subsidy in an economy that is in a "trap", an inefficient steady state with out adoption, moves the economy to an equilibrium that is efficient, and whose steady state is far away.

We start with a proposition about how the equilibrium steady state move as a function of the subsidy. We consider the case without complementarities, i.e. with $\zeta \leq 1$, and the one with complementarities,i.e. with $\zeta > 1$ separately.

The next proposition gives conditions under which without subsidy there exists a stationary equilibrium without adoption, and if the optimal subsidy is implemented, the only equilibrium is one converging to the highest interior steady state. Furthermore, the resulting equilibrium path is optimal. We view this intervention as describing what the literature refers to a big push, since the interior efficient steady state can be very far away from the equilibrium with no adoption.

Proposition 14 *Assume that $F(\cdot)$ is S-shaped and that*

$$\left(\frac{\eta - 1}{\eta}\right)^{\frac{1}{1-\nu}} \frac{\partial F(0)}{\partial K} < \frac{\rho\kappa}{N} < \frac{\partial F(0)}{\partial K} \ .$$

Let the economy start with no technology adoption, i.e. with $K(0) = 0$.

[7] The setup in these papers is almost identical to the one for the efficient allocation in the model of this section. There are minor differences, for instance, these papers typically have $F(0) = 0$ and Dechert and Nishimura (1983) uses a discrete time version.

1. *If we set the subsidy to zero, i.e. if* $s_r = s_a = 1$*, there are two interior steady states* $0 < K^*_{L,\{1,1\}} < K^*_{H,\{1,1\}}$ *and one no-adoption steady state. Then there is an equilibrium where there is no adoption at any time, i.e.* $K(t) = K(0) = 0$ *for all* $t \geq 0$*.*

2. *If we set the revenue subsidy to its static optimal level, i.e.* $s_r = s^*_r \equiv \eta/(\eta-1)$*, then there is a* unique *steady state, and the* unique *equilibrium converges to it, i.e.* $K(t) \to K^*_{H,\{s^*_r,1\}}$ *for any* $K(0) \geq 0$*.*

3. *In the case where* $s_r = s^*_r \equiv \eta/(\eta-1)$*, the equilibrium path solves the planner's problem, i.e. the allocation is Pareto Optimal.*

Proposition 14 has the interpretation that correcting the markup distortion can cause a big push, i.e. it can move an economy from a stable trap of no adoption, through a process of technology adoption, with a path that implements a Pareto Optimal allocation. In particular, under the stated circumstances, an economy can be in a stationary equilibrium without adoption in a laissez faire equilibrium, i.e. $K(0) = 0$ with $s_r = 1$. Furthermore, this equilibrium is locally stable. Then, consider a permanent unexpected changed in policy that corrects the markup distortion, setting $s_r = s^*_r = \eta/(\eta-1)$. This policy has the effect of an immediate permanent static gain in output, equal to a factor $A(s^*_r)/A(1)$ where

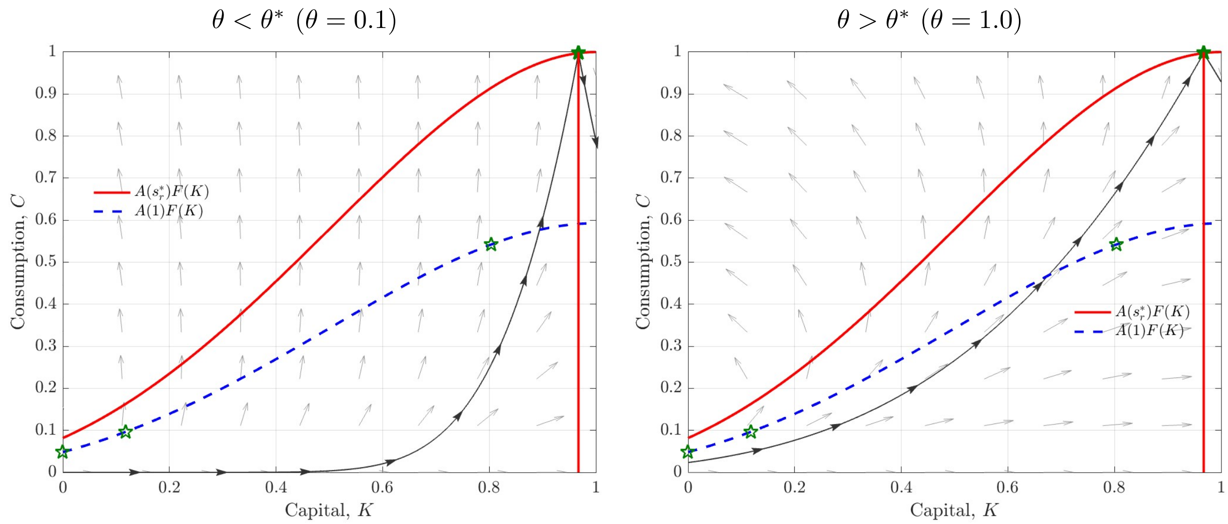
$$\frac{A(s^*_r)}{A(1)} = \frac{1-\nu}{\frac{\eta}{\eta-1} - \nu}\left(\frac{\eta}{\eta-1}\right)^{\frac{1}{1-\nu}} .$$

Note that, for a given markup, this static effect is increasing in the share ν. In particular, the static effect vanishes if $\nu = 0$, i.e. $\frac{A(s^*_r)}{A(1)} = 1$. Importantly, under the stated assumptions, the economy converges monotonically from its non-adoption "trap" to its unique steady state with adoption. This equilibrium is also unique, so it requires no coordination. Furthermore, the resulting equilibrium is Pareto Optimal.

Figure 6 mimics the analysis performed in Figure 5, but for the planning problem, i.e. setting $s_r = s^*_r$. The red curved line in both is the locus obtained from setting $\dot{K}(t) = 0$, which corresponds to aggregate consumption $C(t)$ being equal to aggregate output $A(s^*_r)F(K)$. The vertical red line is the locus corresponding to $\dot{C}(t) = 0$. As discussed, there is a unique steady state under the optimal subsidy s^*_r, depicted by a filled green star. The steady state is interior, and it is stable. The blue dashed line in both plots corresponds to aggregate output in the laissez faire economy, i.e. $A(1)F(K)$, and the unfilled green stars correspond to the equilibrium steady states in the laissez faire economy.

Consider the following exercise. The economy is in laissez faire ($s_r = 1$) equilibrium steady state with no adoption. As previously discussed, this steady state equilibrium is stable. Now, unexpectedly, the optimal policy is implemented i.e. the new subsidy is $s_r = s^*_r$. Either with high or low θ, there is path towards the optimal steady state: on impact, consumption discretely falls and the equilibrium, under the optimal policy, transitions towards the optimal steady state through the saddle path.

Figure 6: Phase Diagrams with the Optimal Static Subsidy



Notes. m_0 is a truncated exponential $m_0(z) = \chi \exp(-\chi z)/(1 - \exp(-\chi G_0))$ with parameters: $\chi = 0.8$ and $G_0 = 10$. Other parameters: $\eta = 3$, $N = 1$, $s_r = s_r^*$, and $\nu = 0.75$.

6.2 Big Push with Adoption Subsidies

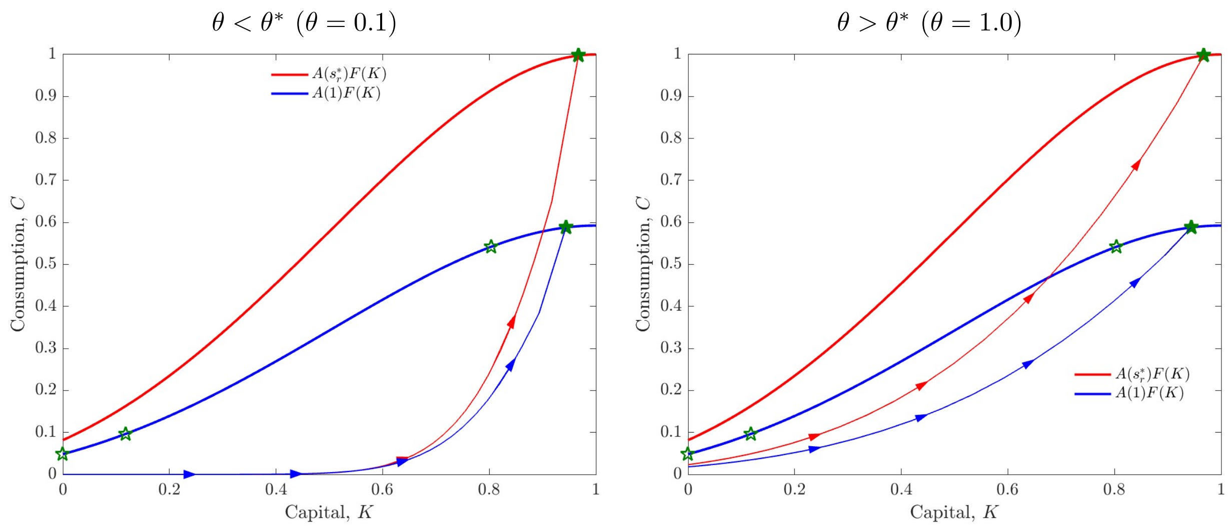
In this section we describe the constrained efficient allocation, and how to decentralized it. We define a constrained efficient allocation as the one that maximizes the utility of the representative agent, subject to being able to being able to decentralize it as an equilibrium. The constrained is that the planner can use a time varying adoption subsidy, i.e. a path of $\{s_a(t)\}$ but it cannot use revenue tax/subsidy, i.e. it must set $s_r(t) = 1$ for all $t \geq 0$. There are two type of inefficiencies in the laissez-faire equilibrium of our model. The first inefficiency is present even in a static set up if $\nu > 0$, as described in Corollary 1, which implies that the output level is lower than what is feasible given the technology in place. We refer to this as the static level effect. The second inefficiency is dynamic in nature: the path of technology adoption differs from the what would be optimal. Our definition of constrained efficiency aims at focusing on the second, and to ignore the static level effect. In particular, our interest in this definition is that the adoption subsidy is often viewed as direct tool to affect technology adoption, while the revenue tax/subsidy if often viewed as a tool to correct the effect of monopoly power, which is static in nature.

Proposition 15 *Fix the initial capital $K(0)$. The constrained efficient allocation has a constant subsidy that solves $1 = B(1, s_a^*)$ or,*

$$s_a(t) = s_a^* \equiv \frac{1-\nu}{\frac{\eta}{\eta-1} - \nu} \; < \; 1 \; . \tag{93}$$

The constrained efficient allocation satisfies the same properties as the efficient allocation with a aggregate

Figure 7: Big Push with Revenue vs. Adoption Subsidies



Notes. m_0 is a truncated exponential $m_0(z) = \chi \exp(-\chi z)/(1 - \exp(-\chi G_0))$ with parameters: $\chi = 0.8$ and $G_0 = 10$. Other parameters: $\eta = 3$, $N = 1$, and $\nu = 0.75$.

output $Y(K) = A(1)F(K)$.

Few comments are in order. First, the interpretation of $s_a^* < 1$ is that this is the fraction of the price of the adoption good that the firm pays, so that a smaller value of s_a^* is a larger subsidy. Second, note that $(\eta - 1)/\eta < s_a^* < 1$, so the optimal subsidy is positive, but bounded by the markup. Third, the subsidy is increasing in ν, i.e s_a^* is decreasing in ν. Forth, by design, notice that this policy has *no static effect* on output. In particular, at time $t = 0$ the output is the same as in the laissez faire equilibrium. Importantly, this lack of static level effect holds for any value of ν. Fifth, in this case if the initial condition is

$$\frac{\partial F(0)}{\partial K} < \frac{\rho\kappa}{A(1)} < \frac{\partial F(0)}{\partial K}\frac{1}{s_a^*} ,$$

and if $F(\cdot)$ is S-shaped, then we will obtain a similar conclusion than in Proposition 14. The proof is available in Appendix B.13. In particular there will be a "big push" as the economy moves from the no adoption steady state trap to the constrained efficient allocation with high adoption. The main difference is that in this case in the "big push" there is no static level effect. It is purely an effect due to the change in adoption incentives. Because the adoption subsidy does not correct for the static inefficiency, the constrained planner steady state will have lower adoption, i.e. lower K, and lower consumption than in the case with the planner's steady state with revenue subsidies. This is shown in Figure 7, for both low and high values of θ.

6.3 No Big Push with Revenue Subsidies

ADD TEXT

Proposition 16 *Assume that $F(\cdot)$ is S-shaped and that*

$$\frac{\partial F(0)}{\partial K} < \frac{\rho\kappa}{N} .$$

1. *If we set the subsidy to zero, i.e. if $s_r = s_a = 1$, there are two interior steady states $0 < K^*_{L,\{1,1\}} < K^*_{H,\{1,1\}}$ and one no-adoption steady state. Then there is an equilibrium where there is no adoption at any time, i.e. $K(t) = K(0) = 0$ for all $t \geq 0$.*

2. *If we set the revenue subsidy to its static optimal level, i.e. $s_r = s^*_r \equiv \eta/(\eta - 1)$, there are two interior steady states $K^*_{L,\{s^*_r,1\}} < K^*_{H,\{s^*_r,1\}}$ and one no-adoption steady state. Then, there is an equilibrium where there is no adoption at any time, i.e. $K(t) = K(0) = 0$ for all $t \geq 0$.*

3. *Let $\theta \geq \theta^*$ and $s_r(t) = s^*_r$ for all t. Then, if $K(0) < K^*_{L,\{s^*_r,1\}}$ the equilibrium converges to the equilibrium with no adoption, i.e. $K(t) \to K(0) = 0$, and if $K(0) > K^*_{L,\{s^*_r,1\}}$ the equilibrium converges to the best interior steady state with adoption, i.e. $K(t) \to K^*_{H,\{s^*_r,1\}}$. Further, if $K(0) = K^*_{L,\{s^*_r,1\}}$, then $K(0) > K^*_{L,\{s^*_r,1\}}$ for all t.*

4. *Let $\theta < \theta^*$ and $s_r(t) = s^*_r$ for all t. Then, there exists a $\hat{K}$ such that if $K(0) < \hat{K}$ then the equilibrium converges to the equilibrium with no adoption, i.e. $K(t) \to K(0) = 0$, if $K(0) > \hat{K}$ then the equilibrium converges to the best interior steady state equilibrium with adoption, i.e. $K(t) \to K^*_{H,\{s^*_r,1\}}$, and if $K(0) = \hat{K}$ there is one path converging to the steady state equilibrium with no adoption and one path converging to $K^*_{H,\{s^*_r,1\}}$.*

When $\rho\kappa/N$ is large enough, the equilibrium with no adoption survives even under the optimal static policy. When ρ is high, the gains of transitioning away from the equilibrium with no adoption are heavily discounted, and thus little is gained by following suit. Similarly, when κ is high, the cost of this transition is high. Following Dechert and Nishimura (1983), the proposition shows that if the economy begins with low $K(0)$ the optimal path converges to the equilibrium with no adoption but corrected markup distortion with the optimal subsidy, and if $K(0)$ is high the optimal path converges to the best interior steady state with corrected markup distortion, $K^*_{H,\{s^*_r,1\}}$. Furthermore, if θ is low, if $K(0) = \hat{K}$ there are two optimal paths, one converging to the equilibrium with no adoption, and the other one converging to the best interior steady state; in both cases the markup distortion is fully corrected. If the economy were to start with $K(0) = \hat{K}$, there would be an equilibrium selection role for the planner. Interestingly, even in the presence of multiple steady states, if the economy is initially trapped in the equilibrium with no adoption, i.e. $K(0) = 0$, the planner finds optimal to correct the markup distortion by setting $s_r = s^*_r$ *but* the economy remains in the no adoption equilibrium. The next corollary makes this point.

Corollary 3 *Suppose that the conditions in Proposition 16 hold so that there are multiple equilibrium steady states in the laissez faire and in the optimal subsidy s_r^* economies. Then, if the laissez faire economy is initially 'trapped' in the equilibrium steady state with no adoption, i.e. $K(0) = 0$, then the economy will remain with no adoption under the optimal subsidy s_r^*. In other words, a Big Push is not optimal.*

A similar result follows for the constrained planner allocation, where the planner is only allowed to use adoption subsidies. Here, if $\frac{\partial F(0)}{\partial K}\frac{1}{s_a^*} < \frac{\rho\kappa}{A(1)}$, a Big Push is not optimal if the economy starts with $K(0) = 0$. Interestingly, because firms do not adopt, no firm uses the adoption subsidy, and thus the constrained-optimal allocation for the planner can be decentralized with $s_a = 1$.

References

Adserà, A. and Ray, D. (1998). History and coordination failure. *Journal of Economic Growth*, 3(3):267–276.

Aghion, P., Barrage, L., Hemous, D., and Liu, E. (2024). Transition to green technology along the supply chain. CEP Discussion Papers dp2017, Centre for Economic Performance, LSE.

Ayerst, S., Brandt, L., and Restuccia, D. (2023). Distortions, producer dynamics, and aggregate productivity: A general equilibrium analysis. Working Paper 30985, National Bureau of Economic Research.

Baqaee, D. R. and Farhi, E. (2020). Productivity and Misallocation in General Equilibrium. *The Quarterly Journal of Economics*, 135(1):105–163.

Bento, P. and Restuccia, D. (2017). Misallocation, Establishment Size, and Productivity. *American Economic Journal: Macroeconomics*, 9(3):267–303.

Brock, W. A. and Dochert, W. D. (1983). The Generalized Maximum Principle. SSRI Workshop Series 292592, University of Wisconsin-Madison, Social Systems Research Institute.

Buera, F. (2009). A dynamic model of entrepreneurship with borrowing constraints: theory and evidence. *Annals of Finance*, 5(3):443–464.

Buera, F. J., Hopenhayn, H., Shin, Y., and Trachter, N. (2021). Big Push in Distorted Economies. Nber working papers, National Bureau of Economic Research, Inc.

Buera, F. J. and Trachter, N. (2024). Sectoral development multipliers. Working Paper 32230, National Bureau of Economic Research.

Chen, B.-L. and Shimomura, K. (1998). Self-fulfilling expectations and economic growth: A model of technology adoption and industrialization. *International Economic Review*, 39(1):151–170.

Clark, C. W. (1971). Economically optimal policies for the utilization of biologically renewable resources. *Mathematical Biosciences*, 12(3):245–260.

Dechert, W. D. and Nishimura, K. (1983). A complete characterization of optimal growth paths in an aggregated model with a non-concave production function. *Journal of Economic Theory*, 31(2):332–354.

Hsieh, C.-T. and Klenow, P. J. (2009). Misallocation and Manufacturing TFP in China and India. *Quarterly Journal of Economics*, 124(4):1403–1448.

Jones, C. I. (2011). Intermediate Goods and Weak Links in the Theory of Economic Development. *American Economic Journal: Macroeconomics*, 3(2):1–28.

Krugman, P. (1991). History versus expectations. *The Quarterly Journal of Economics*, 106(2):651–667.

Liu, E. (2019). Industrial Policies in Production Networks. *The Quarterly Journal of Economics*, 134(4):1883–1948.

Matsuyama, K. (1991). Increasing Returns, Industrialization, and Indeterminacy of Equilibrium. *The Quarterly Journal of Economics*, 106(2):617–650.

Matsuyama, K. (1995). Complementarities and Cumulative Processes in Models of Monopolistic Competition. *Journal of Economic Literature*, 33(2):701–729.

Murphy, K. M., Shleifer, A., and Vishny, R. W. (1989). Industrialization and the Big Push. *Journal of Political Economy*, 97(5):1003–1026.

Restuccia, D. and Rogerson, R. (2008). Policy Distortions and Aggregate Productivity with Heterogeneous Plants. *Review of Economic Dynamics*, 11(4):707–720.

Schaal, E. and Taschereau-Dumouchel, M. (2023). Herding through booms and busts. *Journal of Economic Theory*, 210(C).

Skiba, A. K. (1978). Optimal Growth with a Convex-Concave Production Function. *Econometrica*, 46(3):527–539.

Taschereau-Dumouchel, M. and Schaal, E. (2015). Coordinating Business Cycles. Technical report.

Appendix

A Discrete time, discrete state version

In this section we discretize time and gaps as follows. We let $i = 1, 2, 3, \ldots$ index time, and $j = 1, 2, 3, \ldots$ index gap g. Both have a step size Δ. In particular, we let $t = (i-1)\Delta$ and $g = (j-1)\Delta$ for all i and j. We let $m_{i,j}\Delta$ be fraction of firms with gap $g = (j-1)\Delta$ at time $t = (i-1)\Delta$, so that $m_{i,j}$ has the units of a density. The law of motion of $m_{i,j}$ is given by

$$m_{j+1,i+1} = (1 - q\Delta)\left(m_{j,i} - a_{j,i}\right) \text{ for } i, j = 1, 2, \ldots \ , \tag{94}$$

where $a_{j,i}\Delta$ is the number of firms of type j at time i that are going through costly adoption, so $a_{j,i}$ has the units of a density. Moreover,

$$0 \leq a_{j,i} \leq m_{j,i} \text{ for } i, j = 1, 2, \ldots \ , \tag{95}$$

which states that the density of adopters for a given gap at a given time cannot be larger that the density of firms with that given gap and time.[8] Finally, it must also be the case that, at any time $i = 1, 2, \ldots$, adding up all fraction of firms with different gaps $j = 1, 2, \ldots$ equals one,

$$\sum_j m_{j,i}\Delta = 1 \text{ for all } i \ . \tag{96}$$

Using that this condition provides that $m_{1,i+1}\Delta = 1 - \sum_{j\geq 2} m_{j,i+1}\Delta$, we can use (94) to get

$$\begin{aligned} m_{1,i+1}\Delta &= 1 - \sum_{j\geq 1} (1 - q\Delta)\left(m_{j,i} - a_{j,i}\right)\Delta \ , \\ &= 1 + \sum_{j\geq 1} (1 - q\Delta)\, a_{j,i}\Delta - (1 - q\Delta)\sum_{j\geq 1} m_{j,i}\Delta \ , \\ &= \sum_{j\geq 1} (1 - q\Delta)\, a_{j,i}\Delta + q\Delta \sum_{j\geq 1} m_{j,i}\Delta \ , \end{aligned}$$

[8] Notice that we can rewrite the difference equation in (94), by subtracting $m_{i,j+1}$ from both sides and dividing by Δ,

$$\frac{m_{j+1,i+1} - m_{j+1,i}}{\Delta} = \frac{m_{j,i} - m_{j+1,i}}{\Delta} - q m_{j,i} - a_{j,i}(1 - q\Delta)/\Delta \ ,$$

and if we let $\Delta \downarrow 0$ we obtain that

$$m_t(g,t) = \begin{cases} -m_g(g,t) - q m(g,t) & \text{if } a(g,t) = 0 \ , \\ -\infty \text{ or not defined} & \text{if } a(g,t) > 0 \ . \end{cases}$$

where in the last step we used (96). Finally, dividing by Δ,

$$m_{1,i+1} = (1 - q\Delta) \sum_{j \geq 1} a_{j,i} + q \sum_{j \geq 1} m_{j,i} \Delta \ . \tag{97}$$

Note that $m_{1,i+1}\Delta$ is the number of firms that adopt the technology in a period of length Δ between times indexed by i and $i+1$. The fraction of firms that adopt the technology in a costly manner during this period is thus given by,

$$\left(m_{1,i+1} - q \sum_{j \geq 1} m_{j,i} \right) \Delta = (1 - q\Delta) \sum_{j \geq 1} a_{j,i} \Delta \ .$$

Feasibility. Consumption during a period of length Δ is given by:

$$c_i \Delta = N \left(\sum_{j \geq 1} e^{-\gamma(\eta-1)(j-1)\Delta} \, m_{j,i} \, \Delta \right)^{\zeta} \Delta - \kappa_0 \left(m_{1,i+1} - q \sum_{j \geq 1} m_{j,i} \Delta \right) \Delta$$

where $N \left(\sum_{j \geq 1} e^{-\gamma(\eta-1)(j-1)\Delta} \, m_{j,i} \, \Delta \right)^{\zeta} \Delta$ is the output in a period of length Δ, and where $\sum_{j \geq 1} a_{j,i}\Delta$ is the fraction of firms going over a costly adoption of the technology in a period of length Δ. It is immediate to obtain that

$$c_i = N \left(\sum_{j=1,2\ldots} e^{-\gamma(\eta-1)(j-1)\Delta} \, m_{j,i} \, \Delta \right)^{\zeta} - \kappa_0 \left(m_{1,i+1} - q \sum_{j=1}^{\infty} m_{j,i} \Delta \right) \ . \tag{98}$$

A.1 Planner's problem

The planner's problem is given by

$$\max_{\{a_{j,i}\}} \sum_{i \geq 1} \beta^i \, \frac{c_i^{1-\theta}}{1-\theta} \, \Delta \ , \text{ where } \beta = \frac{1}{1 + \bar{\rho}\Delta} \ ,$$

given an initial $\{m_{j,1}\}$ and subject to (94), (95), (97), (96), and (98). Note that in the objective function we adjust the discount factor β to be consistent with the discount rate $\bar{\rho}$, and use Δ so that the discounted sum of utilities converges to an integral of the discounted utilities for the continuum case.

The next proposition shows that the solution of the planner problem is of the threshold type.

Proposition 17 *Let $\{m_{j,i}, a_{j,i}\}$ be the solution of the planner's problem. If $a_{j,i} > 0$, then for all integers $k > 0$ we have that $a_{j+k,i} = m_{j+k,i}$.*

A proof is available in Appendix B.14. As in the continuum case, if there is a productivity gap for which the planner finds it optimal to do costly adoption, every productivity gap that is larger should also adopt.

We can write the Lagrangian of the planner's problem as follows,

$$\mathcal{L}(a,m,\omega,\lambda) = \sum_{i=1}^{\infty} \beta^i \frac{\left(N\left(\sum_{j=1}^{\infty} e^{-\gamma(\eta-1)(j-1)\Delta}\, m_{j,i}\,\Delta\right)^{\zeta} - \kappa_0\left(m_{1,i+1} - q\sum_{j=1}^{\infty} m_{j,i}\Delta\right)\right)^{1-\theta}}{1-\theta}\Delta$$
$$+ \sum_{i=1}^{\infty}\beta^i\omega_i\left[1-\sum_{j=1}^{\infty} m_{j,i}\Delta\right]\Delta + \sum_{i=1}^{\infty}\beta^i\left(\sum_{j=1}^{\infty}\frac{\lambda_{j,i}\left[(1-\Delta q)\left(m_{j,i}-a_{j,i}\right)-m_{j+1,i+1}\right]}{\Delta}\Delta\right)\Delta\,.$$

Here we use $\beta^i\omega\Delta$ as the multiplier of (96) at time i and use $\beta^i\lambda_{j,i}\Delta^2$ as the multiplier of (94). Note that in general we have added the strictly positive multiplicative constant $\Delta > 0$ within the sums. Relative to the continuum case, the sum correspond to integrals, and Δ takes the place of both dt and dg. Further, we have written the law of motion for m dividing it by Δ because this is the object that converges to a non-trivial expression as Δ shrinks. Canceling Δ's,

$$\mathcal{L}(a,m,\omega,\lambda) = \sum_{i=1}^{\infty} \beta^i \frac{\left(N\left(\sum_{j=1}^{\infty} e^{-\gamma(\eta-1)(j-1)\Delta}\, m_{j,i}\,\Delta\right)^{\zeta} - \kappa_0\left(m_{1,i+1} - q\sum_{j=1}^{\infty} m_{j,i}\Delta\right)\right)^{1-\theta}}{1-\theta}\Delta$$
$$+ \sum_{i=1}^{\infty}\beta^i\omega_i\left[1-\sum_{j=1}^{\infty} m_{j,i}\Delta\right]\Delta + \sum_{i=1}^{\infty}\beta^i\left(\sum_{j=1}^{\infty}\lambda_{j,i}\left[(1-\Delta q)\left(m_{j,i}-a_{j,i}\right)-m_{j+1,i+1}\right]\right)\Delta\,.$$

The first order condition of $m_{j,i}$ for $j = 2, 3, \ldots$ and $i = 1, 2, \ldots$, in the case where $m_{j,i} > 0$ is:

$$0 = \frac{\partial\mathcal{L}(a,m,\omega,\lambda)}{\partial m_{j,i}} = \beta^i c_i^{-\theta}\frac{\partial Y_i}{m_{i,j}}\Delta + \beta^i c_i^{-\theta}\kappa_0 q\Delta^2 - \beta^i\omega_i\Delta^2$$
$$+ \beta^i\lambda_{j,i}(1-q\Delta)\Delta - \beta^{i-1}\lambda_{j-1,i-1}\Delta$$

where

$$\frac{\partial Y_i}{m_{i,j}} = \frac{Y_i}{(\eta-1)(1-\nu)}\frac{e^{-\gamma(\eta-1)(j-1)\Delta}}{\sum_{j'=1}^{\infty} e^{-\gamma(\eta-1)(j'-1)\Delta}\, m_{j',i}\,\Delta}\Delta$$

or dividing by β^i

$$0 = c_i^{-\theta}\frac{Y_i}{(\eta-1)(1-\nu)}\frac{e^{-\gamma(\eta-1)(j-1)\Delta}}{\sum_{j'=1}^{\infty} e^{-\gamma(\eta-1)(j'-1)\Delta}\, m_{j',i}\,\Delta}\Delta^2$$
$$+ c_i^{-\theta}\kappa_0 q\Delta^2 - \omega_i + \lambda_{j,i}(1-q\Delta) - \lambda_{j-1,i-1}/\beta$$

Dividing by Δ:

$$0 = c_i^{-\theta} \frac{Y_i}{(\eta-1)(1-\nu)} \frac{e^{-\gamma(\eta-1)(j-1)\Delta}}{\sum_{j'=1}^{\infty} e^{-\gamma(\eta-1)(j'-1)\Delta} m_{j',i}\,\Delta} \Delta$$
$$+ c_i^{-\theta}\kappa_0 q\Delta - \omega_i\Delta + \lambda_{j,i}(1-q\Delta) - \lambda_{j-1,i-1}/\beta$$

Replacing by $1/\beta = 1 + \Delta\bar{\rho}$

$$0 = c_i^{-\theta} \frac{Y_i}{(\eta-1)(1-\nu)} \frac{e^{-\gamma(\eta-1)(j-1)\Delta}}{\sum_{j'=1}^{\infty} e^{-\gamma(\eta-1)(j'-1)\Delta} m_{j',i}\,\Delta} \Delta$$
$$+ c_i^{-\theta}\kappa_0 q\Delta - \omega_i\Delta + \lambda_{j,i}(1-q\Delta) - \lambda_{j-1,i-1}(1+\Delta\bar{\rho})$$

Rearranging it:

$$\Delta\left(\lambda_{j-1,i-1}\bar{\rho} + \lambda_{j,i}\bar{\rho}\right)$$
$$= c_i^{-\theta}\left(\frac{Y_i}{(\eta-1)(1-\nu)} \frac{e^{-\gamma(\eta-1)(j-1)\Delta}}{\sum_{j'=1}^{\infty} e^{-\gamma(\eta-1)(j'-1)\Delta} m_{j',i}\,\Delta} + \kappa_0 q\right)\Delta - \omega_i\Delta + \lambda_{j,i} - \lambda_{j-1,i-1}$$

Adding and subtracting $\lambda_{j,i-1}$ and dividing by Δ:

$$\lambda_{j-1,i-1}\bar{\rho} + \lambda_{j,i}\bar{\rho}$$
$$= c_i^{-\theta}\left(\frac{Y_i}{(\eta-1)(1-\nu)} \frac{e^{-\gamma(\eta-1)(j-1)\Delta}}{\sum_{j'=1}^{\infty} e^{-\gamma(\eta-1)(j'-1)\Delta} m_{j',i}\,\Delta} + \kappa_0 q\right) - \omega_i + \frac{\lambda_{j,i} - \lambda_{j,i-1}}{\Delta} + \frac{\lambda_{j,i-1} - \lambda_{j-1,i-1}}{\Delta}$$

As we let $\Delta \downarrow 0$:

$$(\bar{\rho} + q)\lambda(g,t) = c(t)^{-\theta}\left(\frac{\partial Y(t)}{\partial m(g,t)} + \kappa q\right) - \omega(t) + \lambda_g(g,t) + \lambda_t(g,t)$$

The first order condition for $m_{1,i+1}$ for $i = 1, 2, \dots$ is

$$0 = \frac{\partial \mathcal{L}(a,m,\omega,\lambda)}{\partial m_{1,i+1}}$$
$$= -\beta^i c_i^{-\theta}\kappa_0\Delta + \beta^{i+1}\lambda_{1,i+1}(1-\Delta q)\Delta$$
$$- \beta^{i+1}\omega_{i+1}\Delta^2 + \beta^{i+1}c_{i+1}^{-\theta}\frac{\partial Y_{i+1}}{\partial m_{1,i+1}}\Delta + \beta^{i+1}c_{i+1}^{-\theta}q\kappa_0\Delta^2$$

Dividing it by $\Delta\beta^{i+1}$ and replacing the expression for $\frac{\partial Y_{i+1}}{\partial m_{1,i+1}}$, we get:

$$
\begin{aligned}
0 = &-(1+\Delta\bar{\rho})c_i^{-\theta}\kappa_0 + \lambda_{1,i+1}(1-\Delta q) \\
&+ c_{i+1}^{-\theta}\left(\frac{Y_{i+1}}{(\eta-1)(1-\nu)}\frac{1}{\sum_{j'=1}^{\infty} e^{-\gamma(\eta-1)(j'-1)\Delta}\, m_{j',i}\,\Delta} + \kappa_0 q\right)\Delta - \omega_{i+1}\Delta
\end{aligned}
$$

If we let $\Delta \downarrow 0$, then the second line converges to zero, and we get:

$$\lambda(0,t) = c(t)^{-\theta}\kappa_0 \tag{99}$$

Combining the two equations we can write:

$$(\bar{\rho}+q)\lambda(g,t) = c(t)^{-\theta}\frac{\partial Y(t)}{\partial m(g,t)} + q\lambda(0,t) - \omega(t) + \lambda_g(g,t) + \lambda_t(g,t) \tag{100}$$

The first order condition for $a_{j,i}$, if for that (j,i) then $0 < a_{j,i} \leq m_{j,i}$ is interior, is:

$$0 \leq \frac{\partial \mathcal{L}(a,m,\omega,\lambda)}{\partial a_{j,i}} = -\beta^i \lambda_{j,i}(1-q\Delta)\Delta$$

or

$$\lambda_{j,i} \leq 0$$

If $a_{j,i} < m_{i,j}$ then this has to hold with equality, i.e.. if $0 < a_{j,i} < m_{j,i}$, then:

$$\lambda_{j,i} = 0$$

For those $0 = a_{j,i} < m_{j,i}$ then

$$0 \geq \frac{\partial \mathcal{L}(a,m,\omega,\lambda)}{\partial a_{j,i}} = -\beta^i \lambda_{j,i}(1-q\Delta)\Delta$$

or

$$0 \geq \frac{\partial \mathcal{L}(a,m,\omega,\lambda)}{\partial a_{j,i}} = -\beta^i \lambda_{j,i}(1-q\Delta)\Delta$$

or

$$\lambda_{j,i} \geq 0$$

These three cases for the first order condition for a imply that if $a(G,t) > 0$ then:

$$\lambda(G(t), t) = 0 \tag{101}$$

Concavity of the planner's period return function Let $F(m, m_1')$ be defined as the period return function of the planner problem, i.e.:

$$F(m, m_1') = \frac{\left(N\left(\sum_{j=1}^{\infty} y e^{-\gamma(\eta-1)(j-1)\Delta}\, m_j\, \Delta\right)^{\zeta} - \kappa_0 \left(m_1' - q\sum_{j=1}^{\infty} m_{j,i}\Delta\right)\right)^{1-\theta}}{1-\theta}$$

Note that, the constraints of the planner defined a convex set, so the planner problem is a convex if F is concave. We study this below.

We can write the recursive equation for the planner using a distribution $m = \{m_1, m_2, \ldots\}$ as the current state state and m' as the next period state, and u as the value function, as follows:

$$\begin{gathered} u\,(m) = \max_{m' \in \Gamma(m)} \Delta\, F(m, m_1') + \beta\, u\left(m'\right) \\ \text{where } \Gamma(m) = \{\, m' \,: 0 \leq m_{j+1}' \leq (1-\Delta q) m_j \text{ for } j = 1, 2, 3, \ldots \text{ and} \\ 0 \leq m_1'\,\Delta = 1 - \sum_{j=1}^{\infty} m_j'\,\Delta \,\} \end{gathered}$$

Note that m is, in principle, infinite dimensional. We claim that u is bounded. F is bounded.

Proposition 18 *F is concave if and only if $\zeta \leq 1$.*

A proof of the proposition is available in Appendix 18.

B Proofs and derivations

B.1 Proof of Proposition 1

We start establishing (28). We replace the optimal price given in (15) into the ideal price $P(t)$ given by (12) to obtain:

$$P(t) = \left[\int_0^{G(t)} \left(\frac{\eta}{\eta-1}\frac{1}{s_r(t)}e^{\gamma(g-t)}w(t)^{1-\nu}P(t)^{\nu}\right)^{1-\eta} m(g,t)dg\right]^{\frac{1}{1-\eta}}$$
$$= e^{-\gamma t}\frac{\eta}{\eta-1}\frac{1}{s_r(t)}w(t)^{1-\nu}P(t)^{\nu}\left[\int_0^{G(t)} e^{\gamma(1-\eta)g}\, m(g,t)dg\right]^{\frac{1}{1-\eta}}$$

Solving for $P(t)$ we get:

$$P(t) = e^{-\frac{\gamma}{1-\nu}t}\left(\frac{\eta}{\eta-1}\frac{1}{s_r(t)}\right)^{\frac{1}{1-\nu}} w(t)\left[\int_0^{G(t)} e^{\gamma(1-\eta)g}\, m(g,t)dg\right]^{\frac{1}{(1-\eta)(1-\nu)}}$$

which gives the expression for (28).

The first order conditions of the cost minimization problem (13) and (14) give:

$$\frac{w(t)}{P(t)}n(g,t)\frac{\nu}{1-\nu} = x(g,t) \tag{102}$$

We can replace the optimal x into the production function to obtain a function linear in n as follows:

$$y(g,t) = e^{\gamma(t-g)}\;\; \nu^{-\nu}(1-\nu)^{-(1-\nu)}\, n(g,t)^{1-\nu}\left(\frac{\nu}{1-\nu}\frac{w(t)}{P(t)}n(g,t)\right)^{\nu}$$
$$= e^{\gamma(t-g)}\, n(g,t)\frac{1}{1-\nu}\left(\frac{w(t)}{P(t)}\right)^{\nu}.$$

Taking a ratio for $g_1 = g$ and $g_2 = 0$,

$$\frac{y(g,t)}{y(0,t)} = e^{-\gamma g}\,\frac{n(g,t)}{n(0,t)}$$

Using (15), we obtain $y(g,t) = \left(\frac{p}{P(t)}\right)^{-\eta} Q(t) = \left(\frac{\eta}{\eta-1}\frac{1}{s_r(t)}e^{\gamma(g-t)}\left(\frac{w(t)}{P(t)}\right)^{1-\nu}\right)^{-\eta} Q(t)$. For $g_1 = g$ and $g_2 = 0$,

$$\frac{y(g,t)}{y(0,t)} = e^{-\eta\gamma g}\;.$$

We can combine these last two expressions to obtain

$$n(g,t) = n(0,t)e^{-(\eta-1)\gamma g} \ .$$

Using market clearing,

$$N = \int_0^{G(t)} n(g,t)m(g,t)dg = n(0,t)\int_0^{G(t)} e^{-(\eta-1)\gamma g}m(g,t)dg \ ,$$

and using this result in the expression for $n(g,t)$,

$$n(g,t) = N\frac{e^{-(\eta-1)\gamma g}}{\int_0^{G(t)} e^{-(\eta-1)\gamma g}m(g,t)dg} \ ,$$

which is (26). Also, (27) follows by combining this last expression with (102).

Using the expression for $Q(t)$ in (10) together with the definition of $y(g,t)$,

$$Q(t) = e^{\gamma t}\left[\int_0^{G(t)} \left(e^{-g\gamma}\, b\, x(g,t)^{\nu} n(g,t)^{1-\nu}\right)^{1-\frac{1}{\eta}} \ m(g,t)\, dg\right]^{\frac{1}{1-1/\eta}}$$

using (102),

$$Q(t) = \frac{1}{1-\nu}e^{\gamma t}\left(\frac{w(t)}{P(t)}\right)^{\nu}\left[\int_0^{G(t)} \left(e^{-g\gamma}\, n(g,t)\right)^{1-\frac{1}{\eta}} \ m(g,t)\, dg\right]^{\frac{1}{1-1/\eta}} \ ,$$

and using (26),

$$\begin{aligned}
Q(t) =& \frac{N\frac{1}{1-\nu}e^{\gamma t}}{\int_0^{G(t)} e^{-(\eta-1)\gamma g}m(g,t)dg}\left(\frac{w(t)}{P(t)}\right)^{\nu}\left[\int_0^{G(t)} \left(e^{-(\eta-1)\gamma g}\right) \ m(g,t)\, dg\right]^{\frac{1}{1-1/\eta}} \ , \\
=& N\frac{1}{1-\nu}e^{\gamma t}\left(\frac{w(t)}{P(t)}\right)^{\nu}\left[\int_0^{G(t)} \left(e^{-(\eta-1)\gamma g}\right) \ m(g,t)\, dg\right]^{\frac{1}{\eta-1}} \ , \\
=& N\frac{1}{1-\nu}e^{\gamma t}\left(\frac{w(t)}{P(t)}\right)^{\nu}\left[\int_0^{G(t)} \left(e^{-(\eta-1)\gamma g}\right) \ m(g,t)\, dg\right]^{\frac{1}{\eta-1}} \ .
\end{aligned}$$

Using (28),

$$\left(\frac{w(t)}{P(t)}\right)^{1-\nu}\left(s_r(t)\left(\frac{\eta-1}{\eta}\right)\right)^{-1} = e^{\gamma t}\left[\int_0^{G(t)} e^{\gamma g(1-\eta)}m(g,t)dg\right]^{\frac{1}{(\eta-1)}} \ .$$

Replacing this expression back into $Q(t)$,

$$Q(t) = \frac{N}{1-\nu}\frac{1}{s_r(t)}\left(\frac{\eta}{\eta-1}\right)\frac{w(t)}{P(t)} \ .$$

This is the expression in (24). Aggregate intermediate inputs are given by (7),

$$X(t) = \int_0^{G(t)} x(g,t)\, m(g,t)\, dg = \frac{w(t)}{P(t)}\frac{\nu}{1-\nu}\int_0^{G(t)} n(g,t)\, m(g,t)\, dg \ = \frac{w(t)}{P(t)}\frac{\nu}{1-\nu}N \ .$$

B.2 Proof of Proposition 3

The first order conditions are:

$$0 = \left[\int_0^{G(t)} \left(e^{(t-g)\gamma t}\, b\, x(g,t)^{\nu} n(g,t)^{1-\nu}\right)^{1-\frac{1}{\eta}} \ m(g,t)\, dg\right]^{\frac{1}{1/\eta}} \left(e^{(t-g)\gamma t}\, b\, x(g,t)^{\nu} n(g,t)^{1-\nu}\right)^{-\frac{1}{\eta}} m(g,t)\times$$
$$e^{(t-g)\gamma t}\, b\nu\, x(g,t)^{\nu-1} n(g,t)^{1-\nu} - m(g,t)$$
$$0 = \left[\int_0^{G(t)} \left(e^{(t-g)\gamma t}\, b\, x(g,t)^{\nu} n(g,t)^{1-\nu}\right)^{1-\frac{1}{\eta}} \ m(g,t)\, dg\right]^{\frac{1}{1/\eta}} \left(e^{(t-g)\gamma t}\, b\, x(g,t)^{\nu} n(g,t)^{1-\nu}\right)^{-\frac{1}{\eta}} m(g,t)\times$$
$$e^{(t-g)\gamma t}\, b(1-\nu)\, x(g,t)^{\nu} n(g,t)^{-\nu} - \mathcal{W}\, m(g,t)$$

We can rewrite the foc as:

$$x(g,t) = B(m,y,n,t) \ \ b\nu$$
$$n(g,t)\mathcal{W} = B(m,y,n,t)\, b(1-\nu)$$

or

$$x(g,t) = \frac{\nu}{1-\nu} n(g,t)\mathcal{W}$$

Thus

$$\mathcal{Y}(m,t) = \max_n e^{\gamma t} b\left(\frac{\nu}{1-\nu}\right)^{\nu} \mathcal{W}^{\nu}\left[\int_0^{G(t)} \left(e^{-g\gamma}\, n(g,t)\right)^{1-\frac{1}{\eta}} \ m(g,t)\, dg\right]^{\frac{1}{1-1/\eta}}$$
$$+ \mathcal{W}N - \mathcal{W}\left(1+\frac{\nu}{1-\nu}\right)\int_0^{G(t)} n(g,t)\, m(g,t)\, dg$$

We use that

$$b\left(\frac{\nu}{1-\nu}\right)^{\nu} = (1-\nu)^{\nu-1}\nu^{-\nu}\left(\frac{\nu}{1-\nu}\right)^{\nu} = \frac{1}{1-\nu}$$

This problem is clearly homogeneous of degree one in N i.e.:

$$\mathcal{Y}(m,t,N) = N\,\mathcal{Y}(m,t)$$

We note that the Lagrange multiplier $\mathcal{W}$ is independent of N.

Setting $L=1$ and taking the first order condition:

$$\mathcal{W}\left(1+\frac{\nu}{1-\nu}\right) = e^{\gamma t} b \frac{\nu}{1-\nu}\left[\int_0^{G(t)} \left(e^{-g'\gamma}\, n(g',t)\right)^{1-\frac{1}{\eta}}\, m(g',t)\, dg'\right]^{\frac{1}{1-1/\eta}-1} \left(e^{-g\gamma}\, n(g,t)\right)^{-\frac{1}{\eta}} e^{-g\gamma}$$

Take any two $0 \leq g_1 < g_2 \leq G(t)$:

$$n(g_1,t)^{-\frac{1}{\eta}} e^{-g_1\gamma\left(1-\frac{1}{\eta}\right)} = n(g_2,t)^{-\frac{1}{\eta}} e^{-g_2\gamma\left(1-\frac{1}{\eta}\right)}$$

or

$$n(g_1,t) = n(g_2,t)e^{(g_2-g_1)\gamma(\eta-1)} \text{ or } n(g,t) = n(0,t)e^{-g\gamma(\eta-1)}$$

Using **??** we get

$$N = \int_0^{G(t)} n(g,t)m(g,t)dg = n(0,t)\int_0^{G(t)} m(g,t)e^{-g\gamma(\eta-1)}dg$$

or

$$n(g,t) = N\,\frac{e^{-g\gamma(\eta-1)}}{\int_0^{G(t)} e^{-g\gamma(\eta-1)}m(g,t)dg}$$

Replacing into the objective function:

$$
\begin{aligned}
\mathcal{Y}(m,t) =& \frac{1}{1-\nu} N \frac{e^{\gamma t}\mathcal{W}(m,t,1)^{\nu}}{\int_0^{G(t)} e^{-g'\gamma(\eta-1)} m(g',t) dg'} \left[\int_0^{G(t)} \left(e^{-g\gamma}\, e^{-g\gamma(\eta-1)}\right)^{1-\frac{1}{\eta}} \; m(g,t)\, dg\right]^{\frac{1}{1-1/\eta}} \\
& - \frac{\nu}{1-\nu} N \mathcal{W}(m,t,1) \\
&= \frac{1}{1-\nu} N \left(e^{\gamma t}\mathcal{W}(m,t,1)^{\nu} \frac{\left[\int_0^{G(t)} \left(e^{-g\gamma}\, e^{-g\gamma(\eta-1)}\right)^{1-\frac{1}{\eta}} \; m(g,t)\, dg\right]^{\frac{1}{1-1/\eta}}}{\int_0^{G(t)} e^{-g'\gamma(\eta-1)} m(g',t) dg'} - \nu \mathcal{W}(m,t,1) \right) \\
&= \frac{1}{1-\nu} N \left(e^{\gamma t}\mathcal{W}(m,t,1)^{\nu} \frac{\left[\int_0^{G(t)} \left(e^{-g\gamma\eta}\right)^{1-\frac{1}{\eta}} \; m(g,t)\, dg\right]^{\frac{1}{1-1/\eta}}}{\int_0^{G(t)} e^{-g'\gamma(\eta-1)} m(g',t) dg'} - \nu \mathcal{W}(m,t,1) \right) \\
&= \frac{1}{1-\nu} N \left(e^{\gamma t}\mathcal{W}(m,t,1)^{\nu} \frac{\left[\int_0^{G(t)} e^{-g\gamma(\eta-1)} \; m(g,t)\, dg\right]^{\frac{1}{1-1/\eta}}}{\int_0^{G(t)} e^{-g'\gamma(\eta-1)} m(g',t) dg'} - \nu \mathcal{W}(m,t,1) \right) \\
&= \frac{1}{1-\nu} N \left(e^{\gamma t}\mathcal{W}(m,t,1)^{\nu} \left[\int_0^{G(t)} e^{-g\gamma(\eta-1)} \; m(g,t)\, dg\right]^{\frac{1}{1-1/\eta}-1} - \nu \mathcal{W}(m,t,1) \right)
\end{aligned}
$$

or

$$
\mathcal{Y}(m,t) = \frac{1}{1-\nu} N \left(e^{\gamma t}\mathcal{W}(m,t,1)^{\nu} \left[\int_0^{G(t)} e^{-g\gamma(\eta-1)} \; m(g,t)\, dg\right]^{\frac{1}{\eta-1}} - \nu \mathcal{W}(m,t,1) \right) \tag{103}
$$

Next we solve for $\mathcal{W}(m,0,1)$. On the one hand, we use the envelope in the definition of $\mathcal{Y}$ in (35) we get

$$
\frac{d}{dN}\mathcal{Y}(m,t) = \mathcal{W}(m,t,1)
$$

On the other hand, we can differentiate the expression (103)

$$
\frac{d}{dN}\mathcal{Y}(m,t) = \frac{1}{1-\nu} \left(e^{\gamma t}\mathcal{W}(m,t,1)^{\nu} \left[\int_0^{G(t)} e^{-g\gamma(\eta-1)} \; m(g,t)\, dg\right]^{\frac{1}{\eta-1}} - \nu \mathcal{W}(m,t,1) \right)
$$

Equating both expressions we get:

$$\mathcal{W}(m,t)=\frac{1}{1-\nu}\left(e^{\gamma t}\mathcal{W}(m,t,1)^{\nu}\left[\int_0^{G(t)} e^{-g\gamma(\eta-1)}\,m(g,t)\,dg\right]^{\frac{1}{\eta-1}}-\nu\mathcal{W}(m,t,1)\right)$$

$$\mathcal{W}(m,t)=e^{\gamma t}\mathcal{W}(m,t,1)^{\nu}\left[\int_0^{G(t)} e^{-g\gamma(\eta-1)}\,m(g,t)\,dg\right]^{\frac{1}{\eta-1}}$$

or

$$\mathcal{W}(m,t)=e^{\frac{\gamma}{1-\nu}t}\left[\int_0^{G(t)} e^{-g\gamma(\eta-1)}\,m(g,t)\,dg\right]^{\frac{1}{(\eta-1)(1-\nu)}}$$

Thus replacing $\mathcal{W}$ into

$$\mathcal{Y}(m,t)=\frac{1}{1-\nu}N\left(e^{\gamma t}\mathcal{W}(m,t,1)^{\nu}\left[\int_0^{G(t)} e^{-g\gamma(\eta-1)}\,m(g,t)\,dg\right]^{\frac{1}{\eta-1}}-\nu\mathcal{W}(m,t)\right)$$

we obtain:

$$\mathcal{Y}(m,t)=N\,\mathcal{W}(m,t,1)=N\,e^{\frac{\gamma}{1-\nu}t}\left[\int_0^{G(t)} e^{-g\gamma(\eta-1)}\,m(g,t)\,dg\right]^{\frac{1}{(\eta-1)(1-\nu)}}$$

Also, using the expressions above, we can obtain

$$\mathcal{Q}(t)=\frac{1}{1-\nu}N\mathcal{W}(m,t,1)\ .$$

B.3 Proof of Proposition 4

From the definition of $\mathcal{Y}(m,t)$ in Proposition 3,

$$\log\mathcal{Y}(m^{\epsilon,\alpha})=\ln\left(N\,e^{\frac{\gamma}{1-\nu}t}\right)+\frac{\log\left[\int_0^G e^{-\gamma(\eta-1)g)}m(g)dg+\epsilon\left(\int_{g_1-\alpha/2}^{g_1+\alpha/2} e^{-\gamma(\eta-1)g}dg-\int_{g_2-\alpha/2}^{g_2+\alpha/2} e^{-\gamma(\eta-1)g}dg\right)\right]}{(\eta-1)(1-\nu)}\ .$$

Differentiating with respect to ϵ and evaluating at $\epsilon=0$,

$$\frac{d}{d\epsilon}\log\mathcal{Y}(m^{\epsilon,\alpha})\bigg|_{\epsilon=0}=\frac{1}{(\eta-1)(1-\nu)}\frac{\int_{g_1-\alpha/2}^{g_1+\alpha/2} e^{-\gamma(\eta-1)g}dg-\int_{g_2-\alpha/2}^{g_2+\alpha/2} e^{-\gamma(\eta-1)g}dg}{\int_0^G e^{-\gamma(\eta-1)g)}m(g)dg}\ ,$$

dividing both sides by α, and taking the limit as α approaches zero provides (39).

For (40) we begin by computing the second derivative of $\mathcal{Y}$ with respect to ϵ and evaluate it $\epsilon = 0$,

$$\frac{d^2}{d\epsilon^2}\mathcal{Y}(m^{\epsilon,\alpha})\Big|_{\epsilon=0} = N\, e^{\frac{\gamma}{1-\nu}t}\left[\frac{1}{(\eta-1)(1-\nu)} - 1\right]\frac{\left[\int_0^G e^{-\gamma(\eta-1)g}m(g)dg\right]^{\frac{1}{(\eta-1)(1-\nu)}-2}}{(\eta-1)(1-\nu)}$$
$$\times\left(\int_{g_1-\alpha/2}^{g_1+\alpha/2} e^{-\gamma(\eta-1)g}dg - \int_{g_2-\alpha/2}^{g_2+\alpha/2} e^{-\gamma(\eta-1)g}dg\right)^2 .$$

It is now immediate to see that $\mathcal{Y}$ is concave in ϵ only if $\frac{1}{(\eta-1)(1-\nu)} \leq 1$, which is the condition presented in (40).

Following similar steps as above

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\frac{d}{d\epsilon}\mathcal{Y}(m^{\epsilon,\alpha})\Big|_{\epsilon=0} = \frac{N\, e^{\frac{\gamma}{1-\nu}t}}{(\eta-1)(1-\nu)}\frac{e^{-\gamma(\eta-1)g_1} - e^{-\gamma(\eta-1)g_2}}{\left[\int_0^G e^{-\gamma(\eta-1)g}m(g)dg\right]^{\frac{1}{1-(\eta-1)(1-\nu)}}} .$$

or,

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\frac{d}{d\epsilon}\mathcal{Y}(m^{\epsilon,\alpha})\Big|_{\epsilon=0} = \frac{N\, e^{\frac{\gamma}{1-\nu}t}}{(\eta-1)(1-\nu)}\left\{e^{\frac{\gamma}{1-\nu}t}\frac{N}{1-\nu}\frac{1}{\eta-1}\left[s_r(t)\left(\frac{\eta-1}{\eta}\right)\right]^{\frac{1}{1-\nu}}\right\}^{-1}\frac{\pi(g_1,t)-\pi(g_2,t)}{P(t)} ,$$
$$= \frac{\pi(g_1,t)-\pi(g_2,t)}{P(t)}\left[\frac{1}{s_r(t)}\left(\frac{\eta}{\eta-1}\right)\right]^{\frac{1}{1-\nu}} ,$$

where we used the expression for the profit in Proposition 2, and the expression for $w(t)/P(t)$ in (28). The expression that we obtained is that one in (41).

B.4 Proof of Proposition 5

As a preliminary step we write:

$$\int_0^\infty e^{-\rho t}\frac{\left(\mathcal{Y}(m(\cdot,t),t) - \kappa(t)(m(0,t) - \int_0^{G(t)} m(g,t)dg)\right)^{1-\theta} - 1}{1-\theta}dt$$
$$= \int_0^\infty e^{\left(-\rho+(1-\theta)\frac{\gamma}{1-\nu}\right)t}\frac{\left(\mathcal{Y}(m(\cdot,t)) - \kappa_0(m(0,t) - \int_0^{G(t)} m(g,t)dg)\right)^{1-\theta} - 1}{1-\theta}dt$$

where

$$\mathcal{Y}(m(\cdot,t),0) \equiv N\left[\int_0^{G(t)} e^{-g\gamma(\eta-1)}\, m(g,t)\, dg\right]^{\frac{1}{(\eta-1)(1-\nu)}} \quad \text{so that}$$
$$\frac{\mathcal{Y}(m(\cdot,t),0)}{N} = Z(t)^{\frac{1}{1-\nu}}$$

We attach a Lagrange multiplier $e^{-\bar{\rho}t}\lambda(g,t)$ to the constraint (44) for each (g,t), a Lagrange multiplier $e^{-\bar{\rho}t}\omega(t)$ to the constraint (45) for each t, and a Lagrange multiplier $e^{-\bar{\rho}t}\xi(t)$ to the constraint that $G(t) = u(t)$.

The solution of the planner problem is attained by computing $\max_{u,G,m} \min_{\lambda,\omega,zeta} \mathcal{L}(m, G, u\lambda, \omega, \xi)$ where the Lagrangian is defined as:

$$
\begin{aligned}
\mathcal{L}(m,G,u,\lambda,\omega,\xi) = \lim_{T\to\infty} \int_0^T e^{-\bar{\rho}t} \frac{\left(\mathcal{Y}(m(\cdot,t),0) - \kappa_0 \left(m(0,t) - q\int_0^{G(t)} m(g,t)dg\right)\right)^{1-\theta} - 1}{1-\theta} dt \\
\lim_{T\to\infty} \int_0^T \int_0^{G(t)} e^{-\bar{\rho}t}\lambda(g,t) \left[m_t(g,t) + m_g(g,t) + q\, m(g,t)\right] dgdt \\
+ \lim_{T\to\infty} \int_0^T e^{-\bar{\rho}t}\omega(t) \left[1 - \int_0^{G(t)} m(g,t)dg\right] dt \\
+ \lim_{T\to\infty} \int_0^T e^{-\bar{\rho}t}\xi(t) \left[u(t) - G'(t)\right] dt
\end{aligned}
$$

We will use integration by parts to rewrite the the Lagrangian. In particular

$$
\begin{aligned}
&\lim_{T\to\infty} \int_0^T \int_0^{G(t)} e^{-\bar{\rho}t}\lambda(g,t) \left[m_t(g,t) + m_g(g,t)\right] dgdt \\
&= \lim_{T\to\infty} \int_0^{G(t)} \int_0^T e^{-\bar{\rho}t}\lambda(g,t) m_t(g,t) dtdg + \int_0^\infty \int_0^{G(t)} e^{-\bar{\rho}t}\lambda(g,t) m_g(g,t) dgdt \\
&= -\lim_{T\to\infty} \int_0^{G(t)} \int_0^T e^{-\bar{\rho}t} \left[-\bar{\rho}\lambda(g,t)m(g,t) + \lambda_t(g,t)m(g,t)\right] dtdg + \lim_{T\to\infty} \int_0^{G(t)} e^{-\bar{\rho}t}\lambda(g,t)m(g,t)|_{t=0}^{t=T} dg \\
&- \lim_{T\to\infty} \int_0^T \int_0^{G(t)} e^{-\bar{\rho}t}\lambda(g,t)m(g,t)dgdt - \lim_{T\to\infty} \int_0^T e^{-\bar{\rho}t}\lambda(g,t)m(g,t)|_{g=0}^{g=G(t)} dt
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}(m,G,u,\lambda,\omega,\xi) = \lim_{T\to\infty} \int_0^T e^{-\bar{\rho}t} \frac{\left(\mathcal{Y}(m(\cdot,t),0) - \kappa_0 (m(0,t) - q\int_0^{G(t)} m(g,t)dg)\right)^{1-\theta} - 1}{1-\theta} dt \\
- \lim_{T\to\infty} \int_0^T \int_0^{G(t)} e^{-\bar{\rho}t} m(g,t) \left[q\lambda(g,t) + \bar{\rho}\lambda(g,t) - \lambda_t(g,t) - \lambda_g(g,t)\right] dgdt \\
- \lim_{T\to\infty} \int_0^{G(t)} e^{-\bar{\rho}t}\lambda(g,t)m(g,t)|_{t=0}^{t=T} dg - \lim_{T\to\infty} \int_0^T e^{-\bar{\rho}t}\lambda(g,t)m(g,t)|_{g=0}^{g=G(t)} dt \\
+ \lim_{T\to\infty} \int_0^T e^{-\bar{\rho}t}\omega(t) \left[1 - \int_0^{G(t)} m(g,t)dg\right] dt \\
+ \lim_{T\to\infty} \int_0^T e^{-\bar{\rho}t} \left[\left(\xi_t(t) - \bar{\rho}\xi(t)\right) G(t) + \xi(t)u(t)\right] dt - \lim_{T\to\infty} e^{-\bar{\rho}t}\xi(t)G(t)|_{t=0}^{t=T}
\end{aligned}
$$

The first order conditions with respect to $m(g,t)$ for $t > 0$ and $g \in (0, G)$ are:

$$\begin{aligned}
\frac{\partial}{\partial m(g,t)} \mathcal{L}(m, G, u, \lambda, \omega, \xi) &= e^{-\bar{\rho} t} \left(C(t)\right)^{-\theta} \frac{\partial}{\partial m(g,t)} \mathcal{Y}(m(\cdot,t), 0)\, dt + q\kappa_0 C(t)^{-\theta} dt \\
&\quad - e^{-\bar{\rho} t} \left[q\lambda(g.t) - \omega(t) + \bar{\rho}\lambda(g,t) - \lambda_t(g,t) - \lambda_g(g,t)\right] dg\, dt \leq 0 \\
\frac{\partial}{\partial m(g,t)} \mathcal{L}(m, \lambda, \mu, G) &= 0 \text{ if } m(g,t) > 0 \\
.\text{where } c(t) &= \mathcal{Y}(m(\cdot,t), 0) - \kappa_0 (m(0,t) - q) \\
\frac{\partial}{\partial m(g,t)} \mathcal{Y}(m(\cdot,t)) &= \frac{N}{(\eta-1)(1-\nu)} \left[\int_0^{G(t)} e^{-g'\gamma(\eta-1)}\, m(g',t)\, dg'\right]^{\frac{1}{(\eta-1)(1-\nu)}-1} e^{-g\gamma(\eta-1)}\, dg \\
&= Z(t)^{\frac{1}{1-\nu}} \frac{N}{(\eta-1)(1-\nu)} \frac{e^{-g\gamma(\eta-1)}\, dg}{\int_0^{G(t)} e^{-g'\gamma(\eta-1)}\, m(g',t)\, dg'} \\
&= Z(t)^{\frac{1}{1-\nu}}\, \pi(g,t)
\end{aligned}$$

for $t > 0$ and $g \in (0, G(t))$.

We can thus rewrite this first order condition when $m(g,t) > 0$ as:

$$\bar{\rho}\lambda(g,t) = c(t)^{-\theta} Z(t)^{\frac{1}{1-\nu}}\, \pi(g,t) + \lambda_g(g,t) + \lambda_t(g,t) + q\kappa_0 c(t)^{-\theta} - \omega(t) - q\lambda(g,t)$$

Now we take the first order conditions with respect to the boundary terms

$$\begin{aligned}
0 &= \frac{\partial}{\partial m(0,t)} \mathcal{L}(m, G, u, \lambda, \omega, \xi) = e^{-\bar{\rho} t} \left[\lambda(0,t) - \kappa_0 c(t)^{-\theta}\right] dt \\
0 &= \frac{\partial}{\partial m(G(t),t)} \mathcal{L}(m, G, u, \lambda, \omega, \xi) = -e^{-\bar{\rho} t} \lambda(G(t),t) dt
\end{aligned}$$

From here we conclude that:

$$\lambda(0,t) = \kappa_0 c(t)^{-\theta}, \text{ and } \lambda(G(t),t) = 0$$

The first order conditions with respect to $u(t)$

$$\frac{\partial}{\partial u(t)} \mathcal{L}(m, G, u, \lambda, \omega, \xi) = e^{-\bar{\rho} t} \xi(t) dt = 0$$

So we conclude that :

$$\xi(t) = 0$$

The first order condition with respect to $G(t)$:

$$
\begin{aligned}
0 &= \frac{\partial}{\partial G(t)}\mathcal{L}(m,G,u,\lambda,\omega,\xi) \\
&= e^{-\bar{\rho}t}c(t)^{-\theta}\frac{\partial}{\partial G(t)}\mathcal{Y}(m(\cdot,t),0)dt + e^{-\bar{\rho}t}c(t)^{-\theta}\kappa_0\, q\, m(G(t),t)dt \\
&- e^{-\bar{\rho}t}m(G(t),t)\left[q\lambda(G(t),t)+\omega(t)+\bar{\rho}\lambda(G(t),t)-\lambda_t(G(t),t)-\lambda_g(G(t),t)\right] dt \\
&- e^{-\bar{\rho}t}\left[\lambda_g(G(t),t)m(G(t),t)+\lambda(G(t),t)m_g(G(t),t)\right] dt \\
&+ e^{-\bar{\rho}t}\left[\xi_t(t)-\bar{\rho}\xi(t)\right] dt
\end{aligned}
$$

Note that

$$
\begin{aligned}
\frac{\partial}{\partial G(t)}\mathcal{Y}(m(t)) &= \left[\int_0^{G(t)} e^{-g\gamma(\eta-1)}\, m(g,t)\, dg\right]^{\frac{1}{(\eta-1)(1-\nu)}} N\, \frac{e^{-G(t)\gamma(\eta-1)}\, m(G(t),t)}{\int_0^{G(t)} e^{-g\gamma(\eta-1)}\, m(g,t)\, dg} \\
&= m(G(t),t)Z(t)^{\frac{1}{1-\nu}}\, \frac{N}{(\eta-1)(1-\nu)}\, \frac{e^{-G(t)\gamma(\eta-1)}}{\int_0^{G(t)} e^{-g'\gamma(\eta-1)}\, m(g',t)\, dg'} \\
&= m(G(t),t)Z(t)^{\frac{1}{1-\nu}}\, \pi(G(t),t)
\end{aligned}
$$

Replacing this expression and using the first order conditions with respect to $m(g,t)$ we have:

$$
\begin{aligned}
0 = &- e^{-\bar{\rho}t}\left[\lambda_g(G(t),t)m(G(t),t)+\lambda(G(t),t)m_g(G(t),t)\right] dt \\
&+ e^{-\bar{\rho}t}\left[\xi_t(t)-\bar{\rho}\xi(t)\right] dt
\end{aligned}
$$

using that $\lambda(G(t),t)=\xi(t)=0$ all $t>0$

$$
0 = \lambda_g(G(t),t)m(G(t),t)
$$

B.5 Proof of Proposition 6

Using Proposition 5 we start with the Lagrange multipliers λ and ω satisfying:

$$
\begin{aligned}
\bar{\rho}\lambda(g,t) &= c(t)^{-\theta}Z(t)^{\frac{1}{1-\nu}}\, \pi(g,t)+\lambda_g(g,t)+\lambda_t(g,t)-\omega(t)+q(\lambda(0,t)-\lambda(g,t)) \\
\lambda(0,t) &= \kappa_0 c(t)^{-\theta} \\
\lambda(G(t),t) &= 0 \\
\lambda_g(G(t),t) &= 0
\end{aligned}
$$

We define Λ as follows:

$$\Lambda(g,t) = \frac{\lambda(g,t)}{c(t)^{-\theta} Z(t)^{\frac{1}{1-\nu}}} + \chi(t) \text{ and } \Omega(t) = \frac{\omega(t)}{c(t)^{-\theta} Z(t)^{\frac{1}{1-\nu}}}$$

The boundary condition from λ above implies:

$$\Lambda(G(t),t) - \Lambda(0,t) = \kappa_0 Z(t)^{-\frac{1}{1-\nu}}$$
$$\Lambda_g(G(t),t) = 0$$

Dividing the p.d.e. for λ by $c(t)^{-\theta} Z(t)^{\frac{1}{1-\nu}}$,

$$\bar{\rho}(\Lambda(g,t) - \chi(t)) = \pi(g,t) + \Lambda_g(g,t) + \frac{\lambda_t(g,t)}{c(t)^{-\theta} Z(t)^{\frac{1}{1-\nu}}} - \Omega(t) + q(\Lambda(0,t) - \Lambda(g,t))$$

The time derivative of Λ is:

$$\begin{aligned}
\Lambda_t(g,t) &= \frac{\lambda_t(g,t)}{c(t)^{-\theta} Z(t)^{\frac{1}{1-\nu}}} + \lambda(g,t) \frac{d}{dt} \frac{1}{c(t)^{-\theta} Z(t)^{\frac{1}{1-\nu}}} + \chi_t(t) \\
&= \frac{\lambda_t(g,t)}{c(t)^{-\theta} Z(t)^{\frac{1}{1-\nu}}} \\
&\quad + \Lambda(g,t) \left(c(t)^{-\theta} Z(t)^{\frac{1}{1-\nu}} \right) \frac{d}{dt} \frac{1}{c(t)^{-\theta} Z(t)^{\frac{1}{1-\nu}}} + \chi_t(t) \\
&= \frac{\lambda_t(g,t)}{c(t)^{-\theta} Z(t)^{\frac{1}{1-\nu}}} - \Lambda(g,t) \frac{d}{dt} \log \left(c(t)^{-\theta} Z(t)^{\frac{1}{1-\nu}} \right) + \chi_t(t)
\end{aligned}$$

Replacing back into the p.d.e:

$$\begin{aligned}
\left(\bar{\rho} - \frac{d}{dt} \log \left(c(t)^{-\theta} Z(t)^{\frac{1}{1-\nu}} \right) \right) \Lambda(g,t) &= \pi(g,t) + \Lambda_g(g,t) + \Lambda_t(g,t) \\
&\quad + \bar{\rho}\chi(t) - \chi_t(t) - \Omega(t) + q(\Lambda(0,t) - \Lambda(g,t))
\end{aligned}$$

Letting $\chi(t)$ be:

$$\chi_t(t) = \bar{\rho}\chi(t) - \Omega(t) \text{ or } \chi(t) = \int_0^\infty e^{-\bar{\rho}s} \Omega(t+s) ds$$

Then we get:

$$\begin{aligned}
&\left(\bar{\rho} - \frac{d}{dt} \log \left(c(t)^{-\theta} Z(t)^{\frac{1}{1-\nu}} \right) \right) \Lambda(g,t) \\
&= \pi(g,t) + \Lambda_g(g,t) + \Lambda_t(g,t) + q(\Lambda(0,t) - \Lambda(g,t))
\end{aligned}$$

Note that by Lemma 1 we have:

$$\begin{aligned}\bar{\rho} - \frac{d}{dt}\log\left(c(t)^{-\theta}Z(t)^{\frac{1}{1-\nu}}\right) &= \bar{\rho} + \theta\frac{d}{dt}\log C(t) - \frac{1}{1-\nu}\frac{d}{dt}\log Z(t) \\ &= r(t)\end{aligned}$$

Thus letting $V(g,t) = \Lambda(g,t)$ we have found that this function solves the p.d.e for the HJB equation for $g \in [0, G(t)]$ in (20) as well as value matching and smooth pasting in (21) and (22).

B.6 Proof of Proposition 7

We start with the p..d.e. and boundary conditions for the firm's value function in an equilibrium, which is given by:

$$\begin{aligned} r(t)V(t) &= \pi(g,t) + V_g(g,t) + V_t(g,t) + q\left(V(0,t) - V(g,t)\right) \text{ for all } g \in [0, G(t)],\, t \geq 0 \\ V(G(t),t) - V(0,t) &= \kappa_0 Z(t)^{-\frac{1}{1-\nu}} \\ V_g(G(t),t) &= 0 \end{aligned}$$

and where r satisfies

$$r(t) = \bar{\rho} + \theta\frac{d}{dt}\ln c(t) - \frac{1}{1-\nu}\frac{d}{dt}\ln Z(t)\ .$$

We will construct λ and ω satisfying the p.d.e. and boundary conditions described in Proposition 5. We guess, and proceed to verify, that the expression (55) and (56) solve the relevant p.d.e. and boundary conditions.

Using value matching and smooth pasting for V given by (21) and (22), the expression in (55) for λ immediately implies (49) and (50) holds. It remains to be shown that the p.d.e. for V in (20) implies that λ and ω defined in (55) and (56) solve the p.d.e. in (47). We turn that next.

Multiplying both side of the p.d.e. for V above by $c(t)^{-\theta}Z(t)^{\frac{1}{1-\nu}}$ we obtain:

$$\begin{aligned} r(t)\lambda(t) + r(t)V(G(t),t) = {}& c(t)^{-\theta}Z(t)^{\frac{1}{1-\nu}}\pi(g,t) + \lambda_g(g,t) \\ & + c(t)^{-\theta}Z(t)^{\frac{1}{1-\nu}}V_t(g,t) + q\left(\lambda(0,t) - \lambda(g,t)\right) \text{ for all } g \in [0, G(t)],\, t \geq 0 \end{aligned}$$

We differentiate with respect to time the expression (55). We obtain:

$$\lambda_t(g,t) = \left(V(g,t) - V(G(t),t)\right)\frac{d}{dt}c(t)^{-\theta}Z(t)^{\frac{1}{1-\nu}} + c(t)^{-\theta}Z(t)^{\frac{1}{1-\nu}}\left(V_t(g,t) - V_t(G(t),t)\right)$$

or

$$
\begin{aligned}
c(t)^{-\theta}Z(t)^{\frac{1}{1-\nu}}\left(V_t(g,t)-V_t(G(t),t)\right) &= \lambda_t(g,t)-\left(V(g,t)-V(G(t),t)\right)\frac{d}{dt}c(t)^{-\theta}Z(t)^{\frac{1}{1-\nu}} \\
&= \lambda_t(g,t)-\left(\lambda(g,t)-\lambda(G(t),t)\right)\frac{d}{dt}\ln\left(c(t)^{-\theta}Z(t)^{\frac{1}{1-\nu}}\right) \\
&= \lambda_t(g,t)-\left(\lambda(g,t)-\lambda(G(t),t)\right)\left(\bar{\rho}-r(t)\right)
\end{aligned}
$$

where we use the guess for λ and the form of the equilibrium interest rate. We replace this term back into the p.d.e.:

$$
\begin{aligned}
r(t)\lambda(t)+r(t)V(G(t),t) = {}& c(t)^{-\theta}Z(t)^{\frac{1}{1-\nu}}\pi(g,t)+\lambda_g(g,t) \\
&+\lambda_t(g,t)+(r(t)-\bar{\rho})(\lambda(g,t)-\lambda(G(t),t))+q\left(\lambda(0,t)-\lambda(g,t)\right)
\end{aligned}
$$

Simplifying the term $r(t)\lambda(g,t)$ in both sides:

$$
\begin{aligned}
\bar{\rho}\lambda(t) = {}& c(t)^{-\theta}Z(t)^{\frac{1}{1-\nu}}\pi(g,t)+\lambda_g(g,t)+\lambda_t(g,t)+q\left(\lambda(0,t)-\lambda(g,t)\right) \\
&+(r(t)-\bar{\rho})\lambda(G(t),t)-r(t)V(G(t),t)
\end{aligned}
$$

We use that in our guess $\lambda(G(t),t)=0$, so we define $\omega(t)$ as in (56). Thus we get:

$$
\bar{\rho}\lambda(t)=c(t)^{-\theta}Z(t)^{\frac{1}{1-\nu}}\pi(g,t)+\lambda_g(g,t)+\lambda_t(g,t)+q\left(\lambda(0,t)-\lambda(g,t)\right)-\omega(t)
$$

B.7 Proof of Lemma 2

(i) and (ii) follow immediately. For (iii) and (iv) let's write R as:

$$
\begin{aligned}
R(G) = {}& \frac{1}{q+\bar{\rho}+\gamma(\eta-1)}\left(1-e^{-\gamma(\eta-1)G}+\frac{\gamma(\eta-1)}{q+\bar{\rho}}e^{-\gamma(\eta-1)G}\left(e^{-(q+\bar{\rho})G}-1\right)\right)\ , \\
= {}& \frac{1}{a+b}\left(\frac{a}{b}e^{-aG}\left(e^{-bG}-1\right)+\left(1-e^{-aG}\right)\right)\ ,
\end{aligned}
$$

with the obvious definition of a and b. We have:

$$
\begin{aligned}
\frac{d}{dG}\frac{1}{(a+b)}\left(\frac{a}{b}e^{-aG}\left(e^{-bG}-1\right)+\left(1-e^{-aG}\right)\right) &= \frac{a}{b}\left(e^{-aG}-e^{-G(a+b)}\right)>0 \\
\frac{1}{a+b}\left(\frac{a}{b}e^{-aG}\left(e^{-bG}-1\right)+\left(1-e^{-aG}\right)\right) &= \frac{1}{2}aG^2+o(G^2)
\end{aligned}
$$

Thus

$$0 < R'(G) < \frac{\gamma(\eta-1)}{q+\bar{\rho}} e^{-\gamma(\eta-1)G}$$

(v) follows from a straight computation.

B.8 Proof of Lemma 3

The solution of (61) and (62), for a given G is:

$$m(g) = \frac{qe^{-qg}}{1-e^{-qG}}$$

Given m, we can solve for Z as a function of G, which defines the following function:

Note that $\bar{Z}$ is a decreasing function of G, since $\gamma > 0$ and $\eta > 1$.

Now we turn to the solution of the function $V(\cdot)$ and the scalar G as function of Z. (58) is a constant coefficient first order o.d.e. Its solution is the sum of the particular solution plus the solution of the homogeneous equation, i.e.

$$V(p) = V^p(g) + Ae^{(\bar{\rho}+q)g}$$

for an arbitrary constant A. We guess a particular solution of the form:

$$V^p(g) = \beta_0 e^{-\gamma(\eta-1)g} + \beta_1$$

for two constants β_0, β_1. Replacing this into the o.d.e.:

$$(q+\bar{\rho})\beta_0 e^{-\gamma(\eta-1)g} + (\bar{\rho}+q)\beta_1 = \frac{N}{(1-\nu)(\eta-1)Z^{\eta-1}} e^{-g\gamma(\eta-1)} - \gamma(\eta-1)\beta_0 e^{-\gamma(\eta-1)g} + qV(0)$$

From here we can solve for β_0 and β_1 as:

$$\beta_0 = \frac{N}{(q+\bar{\rho}+\gamma(\eta-1))(1-\nu)(\eta-1)Z^{\eta-1}} \text{ and } \beta_1 = \frac{q}{q+\bar{\rho}} V(0)$$

Now we use the o.d.e. at $g = 0$, i.e.:

$$\bar{\rho}V(0) = \frac{N}{(1-\nu)(\eta-1)Z^{\eta-1}} + V_g(0) \quad (104)$$

$$\bar{\rho}(A+\beta_0+\beta_1) = \frac{N}{(1-\nu)(\eta-1)Z^{\eta-1}} + A(\bar{\rho}+q) - \beta_0(\eta-1)\gamma \quad (105)$$

We can write the o.d.e. at $g = 0$

$$\bar{\rho}V(0) = \frac{N}{(1-\nu)(\eta-1)Z^{\eta-1}} + V_g(0)$$

We eliminate $V(0)$ in terms of β_1 from above to get:

$$\bar{\rho}\beta_1\frac{q+\bar{\rho}}{q} = \frac{N}{(1-\nu)(\eta-1)Z^{\eta-1}} + A(\bar{\rho}+q) - \beta_0(\eta-1)\gamma$$

Next we manipulate this equation to solve of β_1 as follows:

$$\begin{aligned}
\bar{\rho}\beta_1\frac{q+\bar{\rho}}{q} &= \frac{N}{(1-\nu)(\eta-1)Z^{\eta-1}} + A(\bar{\rho}+q) - \frac{N}{(q+\bar{\rho}+\gamma(\eta-1))(1-\nu)(\eta-1)Z^{\eta-1}}(\eta-1)\gamma \\
\bar{\rho}\beta_1\frac{q+\bar{\rho}}{q} &= \frac{N}{(1-\nu)(\eta-1)Z^{\eta-1}} + A(\bar{\rho}+q) - \frac{N}{(1-\nu)(\eta-1)Z^{\eta-1}}\frac{(\eta-1)\gamma}{(q+\bar{\rho}+\gamma(\eta-1))} \\
\bar{\rho}\beta_1\frac{q+\bar{\rho}}{q} &= \frac{N}{(1-\nu)(\eta-1)Z^{\eta-1}}\left(1 - \frac{(\eta-1)\gamma}{(q+\bar{\rho}+\gamma(\eta-1))}\right) + A(\bar{\rho}+q) \\
\bar{\rho}\beta_1\frac{q+\bar{\rho}}{q} &= \frac{N}{(1-\nu)(\eta-1)Z^{\eta-1}}\frac{q+\bar{\rho}}{q+\bar{\rho}+\gamma(\eta-1)} + A(\bar{\rho}+q) \\
\beta_1\frac{\bar{\rho}}{q} &= \frac{N}{(1-\nu)(\eta-1)Z^{\eta-1}}\frac{1}{q+\bar{\rho}+\gamma(\eta-1)} + A \\
\beta_1 &= \frac{q}{\bar{\rho}}\frac{N}{(1-\nu)(\eta-1)Z^{\eta-1}}\frac{1}{q+\bar{\rho}+\gamma(\eta-1)} + \frac{q}{\bar{\rho}}A
\end{aligned}$$

Now we use value matching: $V(0) - V(G) = s_a\kappa_0\bar{P}(Z)$

$$V(0) - V(G) = \beta_0\left(1 - e^{-\gamma(\eta-1)G}\right) + A\left(1 - e^{(\bar{\rho}+q)G}\right) \tag{106}$$

Thus, A is equal to:

$$A = \frac{s_a\kappa_0\bar{P}(Z) - \beta_0\left(1 - e^{-\gamma(\eta-1)G}\right)}{1 - e^{(\bar{\rho}+q)G}} \tag{107}$$

Finally, smooth pasting $V_g(G) = 0$ gives us:

$$0 = (q+\bar{\rho})Ae^{(q+\bar{\rho})G} - (\gamma(\eta-1))\beta_0 e^{-\gamma(\eta-1)G}$$

Notice that this implies that $A > 0$. We can replace A to obtain one equation that links G with Z.

$$(q+\bar{\rho})e^{(q+\bar{\rho})G}\left(\frac{s_a\kappa_0\bar{P}(Z) - \beta_0\left(1 - e^{-\gamma(\eta-1)G}\right)}{1 - e^{(\bar{\rho}+q)G}}\right) = \gamma(\eta-1)\beta_0 e^{-\gamma(\eta-1)G}$$

Thus:

$$(q+\bar{\rho})e^{(q+\bar{\rho})G}\left(s_a\kappa_0\bar{P}(Z)-\beta_0\left(1-e^{-\gamma(\eta-1)G}\right)\right)=\gamma(\eta-1)\beta_0 e^{-\gamma(\eta-1)G}\left(1-e^{(\bar{\rho}+q)G}\right)$$

or

$$\begin{aligned}&(q+\bar{\rho})e^{(q+\bar{\rho})G}s_a\kappa_0\bar{P}(Z)\\&=(\gamma(\eta-1))\beta_0 e^{-\gamma(\eta-1)G}\left(1-e^{(\bar{\rho}+q)G}\right)+\beta_0\left(1-e^{-\gamma(\eta-1)G}\right)(q+\bar{\rho})e^{(q+\bar{\rho})G}\end{aligned}$$

or

$$\begin{aligned}&(q+\bar{\rho})e^{(q+\bar{\rho})G}s_a\kappa_0\bar{P}(Z)\\&=\beta_0\left[(\gamma(\eta-1))e^{-\gamma(\eta-1)G}\left(1-e^{(\bar{\rho}+q)G}\right)+\left(1-e^{-\gamma(\eta-1)G}\right)(q+\bar{\rho})e^{(q+\bar{\rho})G}\right]\end{aligned}$$

Now we can replace β_0 to obtain:

$$\begin{aligned}&(q+\bar{\rho})e^{(q+\bar{\rho})G}s_a\kappa_0\bar{P}(Z)\\&=\frac{N}{(q+\bar{\rho}+\gamma(\eta-1))(1-\nu)(\eta-1)Z^{\eta-1}}\times\\&\left[(\gamma(\eta-1))e^{-\gamma(\eta-1)G}\left(1-e^{(\bar{\rho}+q)G}\right)+\left(1-e^{-\gamma(\eta-1)G}\right)(q+\bar{\rho})e^{(q+\bar{\rho})G}\right]\end{aligned}$$

or

$$\begin{aligned}&(q+\bar{\rho})e^{(q+\bar{\rho})G}s_a\kappa_0(q+\bar{\rho}+\gamma(\eta-1))(1-\nu)(\eta-1)\bar{P}(Z)Z^{\eta-1}\\&=N\left[\gamma(\eta-1)e^{-\gamma(\eta-1)G}\left(1-e^{(\bar{\rho}+q)G}\right)+\left(1-e^{-\gamma(\eta-1)G}\right)(q+\bar{\rho})e^{(q+\bar{\rho})G}\right]\end{aligned}$$

or

$$\begin{aligned}&(q+\bar{\rho})s_a\kappa_0(q+\bar{\rho}+\gamma(\eta-1))(1-\nu)(\eta-1)\bar{P}(Z)Z^{\eta-1}\\&=N\left[\gamma(\eta-1)e^{-\gamma(\eta-1)G}\left(e^{-(q+\bar{\rho})G}-1\right)+\left(1-e^{-\gamma(\eta-1)G}\right)(q+\bar{\rho})\right]\end{aligned}$$

or

$$\begin{aligned}&s_a\kappa_0(q+\bar{\rho}+\gamma(\eta-1))(1-\nu)(\eta-1)\bar{P}(Z)Z^{\eta-1}\\&=N\left[\frac{\gamma(\eta-1)}{q+\bar{\rho}}e^{-\gamma(\eta-1)G}\left(e^{-(q+\bar{\rho})G}-1\right)+\left(1-e^{-\gamma(\eta-1)G}\right)\right]\end{aligned}$$

Replacing $\bar{P}(Z)$ we get:

$$s_a \frac{\kappa_0}{N}(q+\bar{\rho}+\gamma(\eta-1))(1-\nu)(\eta-1)\left[\frac{1}{s_r}\left(\frac{\eta}{\eta-1}\right)\right]^{\frac{1}{1-\nu}} Z^{(\eta-1)\left(1-\frac{1}{(1-\nu)(\eta-1)}\right)}$$
$$= \frac{\gamma(\eta-1)}{q+\bar{\rho}} e^{-\gamma(\eta-1)G}\left(e^{-(q+\bar{\rho})G}-1\right)+\left(1-e^{-\gamma(\eta-1)G}\right)$$

This equation can be rewritten as:

$$R(G) = \frac{\kappa_0}{N} s_a \left[\frac{1}{s_r}\left(\frac{\eta}{\eta-1}\right)\right]^{\frac{1}{1-\nu}} Z^{(\eta-1)(1-\zeta)}\,\frac{1}{\zeta} \tag{108}$$

where R is defined in (66) The properties of the solution $\bar{G}(Z)$ follow immediately from Lemma 2.

B.9 Proof of Proposition 9

The proof of this proposition proceed in steps.

1. We solve for the path of the density m from an arbitrary initial condition, assuming that $G(t) = \infty$ for all t.

2. We compute bounds at each t for the difference on $Z(t)$ and $\bar{Z}$ as a function of $\Delta(m_0)$.

3. We compute bounds at each t for the difference on $c(t)$, $P(t)\kappa(t)$, $\pi(g,t)$, and $e^{\int_0^t r(s)ds}$ and their balanced growth values as a function of $\Delta(m_0)$, under the assumption that $G(t) = \infty$ for all $t \geq 0$.

4. The stability result then follows directly by a continuity argument, since all the elements that define the value function are arbitrary close to their balanced growth values. That is, we can use these bounds to bound the gain on the deviation of the policy.

The first steps are developed as separate lemmas.

The first lemma construct the path of density m. In particular, consider the case where $G(t) = +\infty$ but there is an arbitrary initial distribution. We want to solve

$$m_t(g,t) + m_g(g,t) + qm(g,t) = 0 \text{ for all } g, t \geq 0 \tag{109}$$

for $g \in [0,\infty)$ with initial condition $m_0(g)$. Assume that $m_0(0) = q$, that $\int_0^\infty m_0(g)dg = 1$, and that $m_0(g) \to 0$ as $g \to \infty$ at a exponential rate. For mass preservation, we must check that for all t:

$$0 = \frac{d}{dt}\int_0^\infty m(g,t)dg = \int_0^\infty m_t(g,t)dg = -\int_0^\infty m_g(g,t)dg - q\int_0^\infty m(g,t)dg$$

Provided that $m(\cdot,t)$ has right and left derivatives, and that $m(\cdot,t)$ is continuous, mass preservation requires that: $m(0,t) = q$.

Lemma 5 *Take an arbitrary initial condition that satisfies $m_0(0) = q$. The continuous solution for m of the transport equation (109) with initial condition, $m(g,0) = m_0(g)$, satisfying mass preservation is:*

$$m(g,t) = \begin{cases} q\, e^{-qg} & \text{if } g < t \\ m_0(g-t)\, e^{-qt} & \text{if } g \geq t \end{cases}$$

From this lemma it is clear that for each g, the density $m(g,t)$ converges to its steady state value $q\exp(-qg)$ in finite time,i.e. after $t \geq g$.

Lemma 6 *Let $Z(t)$ the path of productivity in an equilibrium without costly adoption, and let $\bar{Z}$ be its steady state value. Then:*

$$Z(t)^{\eta-1} - \bar{Z}^{\eta-1} = e^{-a\,t}\Delta(m_0) \text{ where } a \equiv \gamma(\eta-1) + q$$

We collect here the objects that enter into the definition of the objective function for the firm described in (17) and (18):

$$\begin{aligned} c(t) &= N\left(\frac{\frac{1}{s_r}\left(\frac{\eta}{\eta-1}\right) - \nu}{1-\nu}\right)\left[s_r\left(\frac{\eta-1}{\eta}\right)\right]^{\frac{1}{1-\nu}} Z(t)^{\frac{1}{1-\nu}} \\ \int_0^t r(s)ds - \bar{\rho}t &= \theta\log\frac{c(t)}{c(0)} - \frac{1}{1-\nu}\log\frac{Z(t)}{Z(0)} \\ P(t)\kappa(t) &= \kappa_0\left(\frac{1}{s_r}\,\frac{\eta}{\eta-1}\right)^{\frac{1}{1-\nu}} Z(t)^{-\frac{1}{1-\nu}} \\ \pi(g,t) &= \frac{N}{1-\nu}\,\frac{1}{\eta-1}\,\frac{e^{-\gamma g(\eta-1)}}{Z(t)^{\eta-1}} \end{aligned}$$

All the expressions above are functions of $Z(t)$. The first expression is consumption when there is no costly adoption, which we only include to bound the second expression, which is the discount function. The latter enters directly in the firm objective function. The third expression is the cost of adoption. The last expression is the flow profits of the firm.

Proof. (of Lemma 5) The solution

$$m(g,t) = \begin{cases} I(t-g)e^{-q(2g-t)} & \text{if } g < t \\ J(g-t)e^{-qt} & \text{if } g \geq t \end{cases}$$

First we check that the transport equation holds in the interior of each of the cases. For $g < t$ then:

$$m_t(g,t) = I'(t-g)e^{-q(2g-t)} + qI(t-g)e^{-q(2g-t)} \text{ and}$$
$$m_g(g,t) = -I'(t-g)e^{-q(2g-t)} - 2qI(t-g)e^{-q(2g-t)}$$

thus the p.d.e. holds in this domain. Also notice that for. For $g \geq t$

$$m_t(g,t) = -J'(g-t)e^{-qt} - qJ(g-t)e^{-qt} \text{ and}$$
$$m_g(g,t) = J'(g-t)e^{-qt}$$

which also satisfies the p.d.e in this domain. Now we use the boundary conditions to determine I and J. One of the boundary conditions is that $m(g,0) = m_0(g)$ for all $g \geq 0$. For this we use:

$$m_0(g) = J(g) \text{ for all } g \geq 0$$

Now we use that the second boundary condition, namely that $m_(0,t) = q$ for all $t \geq 0$. For this we use:

$$q = I(t)e^{qt} \text{ for all } t \geq 0 \text{ or}$$
$$I(t) = qe^{-qt} \text{ for all } t \geq 0$$

Thus, replacing the expression for I and J our proposed solution is:

$$m(g,t) = \begin{cases} q\,e^{-qg} & \text{if } g < t \\ m_0(g-t)\,e^{-qt} & \text{if } g \geq t \end{cases}$$

Finally, fix t and consider the two branches of the proposed solution for $g > t$ and $g < t$:

$$\lim_{g \to t} m(g,t) = \begin{cases} q\,e^{-qt} & \text{if } g < t \\ m_0(0)\,e^{-qt} & \text{if } g > t \end{cases}$$

Note that these two expressions are the same, i.e. $m(g,t)$ is continuous, provided that $m_0(0) = q$, which is a property that we assumed. ■

Proof. (of Lemma 6) In an equilibrium without costly adoption we can then compute the path of Z:

$$
\begin{aligned}
Z(t)^{\eta-1} &= \int_0^\infty e^{-\gamma g(\eta-1)} m(g,t) dg \\
&= \int_0^t q e^{-g(\gamma(\eta-1)+q)} dg + e^{-qt} \int_t^\infty e^{-g\gamma(\eta-1)} m_0(g-t) dg
\end{aligned}
$$

Thus, letting $barZ$ the value in the steady state without costly adoption:

$$
\begin{aligned}
Z(t)^{\eta-1} - \bar{Z}^{\eta-1} &= e^{-qt} \int_t^\infty e^{-g\gamma(\eta-1)} \left(m_0(g-t) - q e^{-q(g-t)} \right) dg \\
&= e^{-t(\gamma(\eta-1)+q)} \int_t^\infty e^{-(g-t)\gamma(\eta-1)} \left(m_0(g-t) - q e^{-q(g-t)} \right) dg \\
&= e^{-t(\gamma(\eta-1)+q)} \int_0^\infty e^{-\tilde{g}\gamma(\eta-1)} \left(m_0(\tilde{g}) - q e^{-q\tilde{g}} \right) d\tilde{g}
\end{aligned}
$$

■

B.10 Proof of Proposition 10

The proof of this proposition follows quite directly from utilizing the expressions (79) and (80). Item 1 follows directly from (79). Item 2 follows directly from (80). Item 3 follows directly from (79). Item 4 it uses the stated assumptions right hand side of (79) is monotone increasing in K, and hence $F''(\cdot; s_r)$ can cross zero only once. Item 5, using (80), gives sufficient conditions for $F''(0)/F'(0) > 0$, and using 4 gives its existence.

B.11 Proof of Proposition 11

The o.d.e. (85) is feasibility. Next we turn to the necessity of (86) and of the terminal condition (88). We will return to sufficiency below.

Necessity of (86). We obtain the o.d.e. (86) from the first order condition for adoption of the marginal firm at each time, replacing the expressions for the profit functions $\pi(z,t)$ for the marginal firm and $\pi(0,t)$, and using the Euler equation to replace the interest rate $r(t)$. The boundary condition (88) is necessary for the finite value of $V(0,t)$.

We expand into the details to obtain (86). First we consider the case of adoption. The first order condition with respect τ of problem (83) is:

$$
\begin{aligned}
& e^{\int_t^{t+\tau} r(\tilde{s}) d\tilde{s}} \pi(z, t+\tau) - r(t+\tau) e^{-\int_t^{t+\tau} r(\tilde{s}) d\tilde{s}} \left[V^0(z, t+\tau) - P(t) s_a \kappa \right] \\
& + e^{-\int_t^{t+\tau} r(\tilde{s}) d\tilde{s}} \left[V_t^0(z, t+\tau) - \dot{P}(t+\tau) s_a \kappa \right] = 0
\end{aligned}
$$

which simplifies to:

$$\pi\left(z,t+\tau\right)-r\left(t+\tau\right)\left[V^{0}\left(z,t+\tau\right)-P\left(t\right)s_{a}\kappa\right]+\left[V_{t}^{0}\left(z,t+\tau\right)-\dot{P}\left(t+\tau\right)s_{a}\kappa\right]=0 \tag{110}$$

A firm with this z at time t finds it optimal to adopt at time $t+\tau$. Defining the value of z for the marginal adopter at time t as $G\left(t\right)$ gives:

$$\pi\left(G\left(t\right),t\right)-r\left(t\right)\left[V^{0}\left(z,t\right)-P\left(t\right)s_{a}\kappa\right]+V_{t}^{0}\left(z,t\right)-\dot{P}\left(t\right)s_{a}\kappa=0 \tag{111}$$

Using that the time derivatives of the value function is:

$$V_{t}^{0}\left(z,t\right)=-\pi\left(0,t\right)+r\left(t\right)V^{0}\left(z,t\right)\ \text{ for all } z \text{ and } t$$

evaluated at $z=0$ we get

$$\pi\left(G\left(t\right),t\right)-r\left(t\right)\left[V^{0}\left(z,t\right)-P\left(t\right)s_{a}\kappa\right]-\pi\left(0,t\right)+r\left(t\right)V^{0}\left(z,t\right)-\dot{P}\left(t\right)s_{a}\kappa=0$$

cancelling terms

$$\pi\left(0,t\right)-\pi\left(G\left(t\right),t\right)=r\left(t\right)P\left(t\right)s_{a}\kappa-\dot{P}\left(t\right)s_{a}\kappa$$

dividing by $P(t)s_a\kappa$ and rearranging

$$r\left(t\right)=\frac{1}{s_{a}\kappa}\frac{\pi\left(0,t\right)-\pi\left(G\left(t\right),t\right)}{P\left(t\right)}+\frac{\dot{P}\left(t\right)}{P\left(t\right)}$$

equating $r(t)$ to the expression given by Euler equation (9)

$$r(t)=\rho+\theta\frac{\dot{C}(t)}{C(t)}+\frac{\dot{P}(t)}{P(t)}$$

we obtain:

$$\rho+\theta\frac{\dot{C}(t)}{C(t)}+\frac{\dot{P}(t)}{P(t)}=\frac{1}{s_{a}\kappa}\frac{\pi\left(0,t\right)-\pi\left(G\left(t\right),t\right)}{P\left(t\right)}+\frac{\dot{P}\left(t\right)}{P\left(t\right)}$$

simplifying:

$$\theta\frac{\dot{C}(t)}{C(t)}=\frac{1}{s_{a}\kappa}\frac{\pi\left(0,t\right)-\pi\left(G\left(t\right),t\right)}{P\left(t\right)}-\rho$$

Using the expression for $\pi(z,t)$ for $z=G(t)$ and $z=0$ and the one for $P(t)$ to obtain:

$$\frac{\pi\left(0,t\right)-\pi\left(G\left(t\right),t\right)}{P\left(t\right)}=\frac{N}{\left(1-\nu\right)}\frac{1}{\left(\eta-1\right)}\left(s_{r}\frac{\eta-1}{\eta}\right)^{\frac{1}{1-\nu}}\frac{1-e^{-G(t)(\eta-1)}}{\left[\int_{0}^{G(t)}e^{-z(\eta-1)}m_{0}\left(z\right)dz+K\left(t\right)\right]^{1-\zeta}}$$

and replacing into the previous expression we obtain

$$\theta\frac{\dot{C}\left(t\right)}{C\left(t\right)}=\frac{1}{s_{a}\kappa}\frac{N}{(1-\nu)(\eta-1)}\left(s_{r}\frac{\eta-1}{\eta}\right)^{\frac{1}{1-\nu}}\frac{1-e^{-G(t)(\eta-1)}}{\left[\int_{0}^{G(t)}e^{-z(\eta-1)}m_{0}\left(z\right)dz+K\left(t\right)\right]^{1-\zeta}}-\rho.$$

Finally, the derivative of $A(s_r)F(K)$ with respect to K computed in (77) is

$$A(s_{r})F'\left(K;s_{r}\right)=\left(\frac{\frac{1}{s_{r}}\frac{\eta}{\eta-1}-\nu}{1-\nu}\right)\frac{N}{(1-\nu)(\eta-1)}\left(s_{r}\frac{\eta-1}{\eta}\right)^{\frac{1}{1-\nu}}\frac{\left(1-e^{-G(\eta-1)}\right)}{\left[\int_{0}^{G}e^{-z(\eta-1)}m_{0}\left(z\right)dz+K\right]^{1-\zeta}}$$

Thus

$$\theta\frac{\dot{C}\left(t\right)}{C\left(t\right)}=\left(\frac{1-\nu}{\frac{1}{s_{r}}\frac{\eta}{\eta-1}-\nu}\right)\frac{1}{s_{a}\kappa}A(s_{r})\frac{\partial F(K)}{\partial K}-\rho.$$

Now we sketch the analogous case of dis-investment on technology corresponding to problem (84). the proof follows the same steps as the adoption case, so we only sketch them. the first order condition for the optimal time to dis-invest gives, after a simple cancellation:

$$\pi\left(0,t\right)-r\left(t\right)\left[V\left(z,t\right)+P\left(t\right)s_{a}\kappa\right]+V_{t}(z,t)+s_{a}\kappa\dot{P}(t)$$

We also have $r(t)V\left(z,t\right)=\pi(z,t)+V_{t}(z,t)$, which replacing it back into the f.o.c yields:

$$\pi\left(0,t\right)-r\left(t\right)\left[V\left(z,t\right)+P\left(t\right)s_{a}\kappa\right]-\pi\left(z,t\right)+r\left(t\right)V\left(z,t\right)+\dot{P}\left(t\right)s_{a}\kappa=0$$

Denoting the z which is indifferent as $G(t)$ and cancelling terms we get:

$$\pi\left(0,t\right)-r\left(t\right)P\left(t\right)s_{a}\kappa-\pi\left(G(t),t\right)+\dot{P}\left(t\right)s_{a}\kappa=0$$

which is the same expression as for the adoption case. The remaining steps to get the o.d.e. 86 are identical, so we omit them.

Now we turn to show that the boundary condition (88). For $V(0,t)$ to be finite we require the tail of the integral to go to zero, i.e.

$$0 = \lim_{T\to\infty} e^{-\int_t^T r(s)ds}\pi\left(0,T\right)$$

Since $\pi(0,T)$ is bounded below and above, we require:

$$0 = \lim_{T\to\infty} e^{-\int_t^T r(s)ds}$$

Replacing $r(s)$ from the Euler equation we have:

$$-\int_t^T r\left(s\right)ds = -\int_t^T \left(\rho + \theta\frac{\dot{C}(s)}{C(s)} + \frac{\dot{P}(s)}{P(s)}\right)ds = -\rho(T-t) - \theta\log\frac{C(T)}{C(t)} - \log\frac{P(T)}{P(t)}$$

Thus,

$$e^{-\int_t^T r(s)ds} = e^{-\rho(T-t)-\theta\log\frac{C(T)}{C(t)}-\log\frac{P(T)}{P(t)}} = e^{-\rho(T-t)}\left(\frac{C(T)}{C(t)}\right)^{-\theta}\frac{P(t)}{P(T)}$$

Using the for expression for $P(T)$ and $F(K(T);s_r)$ get:

$$\frac{1}{P\left(T\right)} = \left(s_r\frac{\eta-1}{\eta}\right)^{\frac{1}{1-\nu}}\left[\int_0^{G(T)} e^{-z(\eta-1)}m_0\left(z\right)dz + K\left(T\right)\right]^{\zeta}$$

$$F\left(K(T);s_r\right) = N\frac{\left(\frac{1}{s_r}\frac{\eta}{\eta-1}-\nu\right)}{1-\nu}\frac{1}{P(T)}$$

Thus

$$\begin{aligned}
0 &= \lim_{T\to\infty} e^{-\int_t^T r(s)ds}\pi\left(0,T\right) \text{ which is equivalent to} \\
0 &= \lim_{T\to\infty} e^{-\rho T}\left(C(T)\right)^{-\theta}\frac{1}{P(T)} \text{ which is equivalent to} \\
0 &= \lim_{T\to\infty} e^{-\rho T}\left(C(T)\right)^{-\theta}A(s_r)F(K(T))
\end{aligned}$$

Sufficiency . Now we turn to shock that these conditions are sufficient for an equilibrium. In particular, we take path that solves the o.d.e. system and relevant boundary conditions and recover the path of prices $P(t)$ and interest rates $r(t)$ using (82) and the Euler equation in (9), respectively. By construction the path solves the first order conditions for the household and firms problem. The first order condition for the household problem, together with the boundary condition (88) are sufficient for the path to be optimal. The first order condition for the adoption (or dis-investment) are also satisfied by construction. The next lemma shows that, when $\{P(t), r(t)\}$ are constructed as indicated above, the corresponding times solve the firm's problem.

Lemma 7 *Let $\{K(t), C(t)\}$ solve the o.d.e's (85) and (86). Let $\{P(t), r(t)\}$ be defined using (82) and (9). Then the threshold $G(t)$ that solves the first order condition of the firms adoption or dis-investment problem achieves the optima.*

Proof. (of Lemma 7)

We will consider first the objective function for the adoption problem $f(z,t)$, and then the analogous dis-investment problem. We will let $t^*(z)$ a time for which $f_t(z, t^*(z)) = 0$, i.e. $t^*(z)$ satisfies the first order condition of the firms' problem. We will restrict ourselves to objective functions evaluated at path of prices and interest rates which comes from solution to the o.d.e.'s. We will show that such objective function is strictly locally concave at $t^*(z)$. Because this property holds at any solution of the first order condition, the firms' objective function is single-peaked, and hence the time that satisfies the fist order condition is an optimum.

A solution of the o.d.e.'s that is continuous in time can either start at a steady state (and stay there), or it cannot reach a steady state in finite time. Because of this we will treat two cases separately, first the case where $\dot{K}(t) \neq 0$ for almost all times, and then the case where the economy starts at steady state. For the first case we will show that

$$f_{tt}(G(t), t) = -(\eta - 1)\pi(G(t), t)\frac{|\dot{K}(t)|}{m_0(G(t)}$$

We will treat the stationary solutions separately.

Adoption case. Let $f(t,z)$ be the objective of a firm for the adoption problem is:

$$f(t,z) = \int_0^t e^{-\int_0^s r(\tilde{s})d\tilde{s}} \pi(z,s)\, ds + e^{-\int_0^t r(\tilde{s})d\tilde{s}} \left[V^0(z,t) - s_a P(t)\kappa\right]$$

Taking the derivative w.r.t. t we get:

$$\begin{aligned} f_t(t,z) &= e^{-\int_0^t r(\tilde{s})d\tilde{s}} \\ &\quad \times \left[\pi(z,t) - r(t)\left[V^0(z,t) - P(t)\, s_a\kappa\right] + \left[V_t^0(z,t) - \dot{P}(t)\, s_a\kappa\right]\right] \end{aligned}$$

Using that the time derivatives of the value function is:

$$V_t^0(z,t) = -\pi(z,t) + r(t)\, V^0(z,t) \;\text{ for all } z \text{ and } t$$

so we get:

$$
\begin{aligned}
f_t(t,z) &= e^{-\int_0^t r(\tilde{s})d\tilde{s}} \\
&\quad \times \left[\pi\left(z,t\right) - r\left(t\right)\left[V^0\left(z,t\right) - P\left(t\right)s_a\kappa\right] + \left[V_t^0\left(z,t\right) - \dot{P}\left(t\right)s_a\kappa\right]\right] \\
&= e^{\int_0^t r(\tilde{s})d\tilde{s}}\left[\pi\left(z,t\right) - \pi\left(0,t\right) + r(t)P\left(t\right)s_a\kappa - \dot{P}\left(t\right)s_a\kappa\right]
\end{aligned}
$$

Using the Euler equation:

$$
r(t)P(t)s_a\kappa = P(t)s_a\kappa\rho + \theta\frac{\dot{C}(t)}{C(t)}P(t)s_a\kappa + \dot{P}(t)s_a\kappa
$$

Thus

$$
f_t(t,z) = e^{-\int_0^t r(\tilde{s})d\tilde{s}}\left[\pi\left(z,t\right) - \pi\left(0,t\right) + P(t)s_a\kappa\left(\rho + \theta\frac{\dot{C}(t)}{C(t)}\right)\right]
$$

The o.d.e. which is necessary condition in equilibrium gives:

$$
\left(\theta\frac{\dot{C}(t)}{C(t)} + \rho\right)P(t)s_a\kappa = \pi\left(0,t\right) - \pi\left(G\left(t\right),t\right)
$$

or

$$
-\left(\theta\frac{\dot{C}(t)}{C(t)} + \rho\right)P(t)s_a\kappa - \pi\left(G\left(t\right),t\right) + \pi\left(z,t\right) = \pi\left(z,t\right) - \pi\left(0,t\right)
$$

Replacing it back:

$$
\begin{aligned}
f_t(t,z) &= e^{-\int_0^t r(\tilde{s})d\tilde{s}} \\
&\quad \times \left[-\left(\theta\frac{\dot{C}(t)}{C(t)} + \rho\right)P(t)s_a\kappa - \pi\left(G\left(t\right),t\right) + \pi\left(z,t\right) + P(t)s_a\kappa\left(\rho + \theta\frac{\dot{C}(t)}{C(t)}\right)\right] \\
&= e^{-\int_0^t r(\tilde{s})d\tilde{s}}\left[\pi\left(z,t\right) - \pi\left(G\left(t\right),t\right)\right]
\end{aligned}
$$

We can differentiate again with respect to time to get:

$$
\begin{aligned}
f_{tt}(t,z) &= -r(t)e^{-\int_0^t r(\tilde{s})d\tilde{s}}\left[\pi\left(z,t\right) - \pi\left(G\left(t\right),t\right)\right] \\
&\quad + e^{-\int_0^t r(\tilde{s})d\tilde{s}}\left[\pi_t\left(z,t\right) - \pi_z\left(G\left(t\right),t\right)\dot{G}(t) - \pi_t\left(G\left(t\right),t\right)\right]
\end{aligned}
$$

If we evaluate it at $z = G(t)$ then

$$f_{tt}(t, G(t)) = -e^{-\int_0^t r(\tilde{s})d\tilde{s}} \pi_z (G(t), t) \dot{G}(t))$$

Recall that $\dot{G}(t) = -\frac{\dot{K}(t)}{m_0(G(t))}$ so

$$f_{tt}(t, G(t)) = e^{-\int_0^t r(\tilde{s})d\tilde{s}} \pi_z (G(t), t) \frac{\dot{K}(t)}{m_0(G(t))}$$

and using the expression for $\pi(z, t)$

$$\pi_z (z, t) = -(\eta - 1)\pi(z, t)$$

so

$$f_{tt}(t, G(t)) = -(\eta - 1)e^{-\int_0^t r(\tilde{s})d\tilde{s}} \pi (G(t), t) \frac{\dot{K}(t)}{m_0(G(t))},$$

where $\dot{K}(t) > 0$, as we are considering the case of firms adopting the frontier technology.

Now we consider the case where $K(0) = K^*$ and $C(t) = C^*$ are a stationary interior solution of the o.d.e.'s. In this case the derivative of the objective function gives:

$$f_t(t, z) = e^{-\rho t} [\pi^* (z) - \pi^* (0) + \rho P^* s_a \kappa]$$

where $P^*, \pi^*(z)$ and $\pi^*(0)$ are the steady state versions of the price and profit functions. It is immediate that $f(z, t)$ is either strictly increasing in time (for high z), strictly decreasing in time (for low z), and constant for a particular z.

Dis-investment case Now we consider the problem of firms that wants to dis-invest. Again, let $f(t, z)$ be the objective of a firm:

$$f(t, z) = \int_0^t e^{-\int_0^s r(\tilde{s})d\tilde{s}} \pi (0, s) \, ds + e^{-\int_0^t r(\tilde{s})d\tilde{s}} [V (z, t) + s_a P (t) \kappa]$$

Taking the derivative w.r.t. t we get:

$$\begin{aligned} f_t(t, z) = {} & e^{-\int_0^t r(\tilde{s})d\tilde{s}} \\ & \times \left[\pi (0, t) - r (t) [V (z, t) + P (t) s_a \kappa] + \left[V_t (z, t) + \dot{P} (t) s_a \kappa\right]\right] \end{aligned}$$

Using that the time derivatives of the value function is:

$$V_t(z,t) = -\pi(z,t) + r(t)\,V(z,t) \text{ for all } z \text{ and } t$$

we get

$$f_t(t,z) = e^{-\int_0^t r(\tilde{s})d\tilde{s}} \times \left[\pi(0,t) - \pi(z,t) - \left(r(t)\,P(t)\,s_a\kappa - \dot{P}(t)\,s_a\kappa\right)\right]$$

Using the Euler equation:

$$r(t)P(t)s_a\kappa - \dot{P}(t)s_a\kappa = P(t)s_a\kappa\rho + \theta\frac{\dot{C}(t)}{C(t)}P(t)s_a\kappa$$

thus,

$$f_t(t,z) = e^{-\int_0^t r(\tilde{s})d\tilde{s}} \times \left[\pi(0,t) - \pi(z,t) - \left(\rho + \theta\frac{\dot{C}(t)}{C(t)}\right)P(t)s_a\kappa\right]$$

Using the o.d.e. which is necessary condition in equilibrium gives:

$$\left(\theta\frac{\dot{C}(t)}{C(t)} + \rho\right)P(t)s_a\kappa = \pi(0,t) - \pi(G(t),t)$$

we obtain

$$f_t(t,z) = e^{-\int_0^t r(\tilde{s})d\tilde{s}} \times \left[\pi(G(t),t) - \pi(z,t)\right].$$

Using the same steps as before, we obtain

$$f_{tt}(t,G(t)) = (\eta-1)e^{-\int_0^t r(\tilde{s})d\tilde{s}}\pi(G(t),t)\,\frac{\dot{K}(t)}{m_0(G(t))},$$

where $\dot{K}(t) < 0$, as we are considering the case of firms that are dis-investing.

Now we consider the case where $K(0) = K^*$ and $C(t) = C^*$ are a stationary interior solution of the o.d.e.'s. In this case the derivative of the objective function gives:

$$f_t(t,z) = e^{-\rho t}\left[\pi^*(0) - \pi^*(z) - \rho P^* s_a\kappa\right]$$

where $P^*, \pi^*(z)$ and $\pi^*(0)$ are the steady state versions of the price and profit functions. It is immediate that $f(z,t)$ is either strictly increasing in time (for low z), strictly decreasing in time (for high z), and constant for a particular z.

■

B.12 Proof of Proposition 13

We begin by showing that the no-adoption stationary equilibrium, $C(t) = A(s_r)F(0)$ satisfies all the relevant conditions if (92) holds.

For this we need to go back to the proof of Proposition 11 and modify it suitably. The first order condition for τ in (110) holds with inequality, so that (111) becomes,

$$\pi\left(1,t\right) - r\left(t\right)\left[V\left(0,t\right) - P\left(t\right)s_a\kappa\right] + V_t\left(0,t\right) - \dot{P}\left(t\right)s_a\kappa \le 0 \ ,$$

since $G(t) = 1$. The Euler equation for $\dot{C}$ and the equations for profits and $\partial F/\partial K$ still hold with equality, so this means that replacing them we obtain:

$$\theta\frac{\dot{C}\left(t\right)}{C\left(t\right)} \le \left(\frac{1-\nu}{\frac{1}{s_r}\frac{\eta}{\eta-1}-\nu}\right)\frac{1}{s_a\kappa}A(s_r)\frac{\partial F(0)}{\partial K} - \rho \ .$$

Since by feasibility at $K = 0$, then $\dot{K} \ge 0$, if

$$\left(\frac{1-\nu}{\frac{1}{s_r}\frac{\eta}{\eta-1}-\nu}\right)\frac{1}{s_a\kappa}A(s_r)\frac{\partial F(0)}{\partial K} \le \rho$$

then $\dot{C}(t) = 0$ satisfies all the sufficient conditions.

On the stability, the proof follows from examining the phase diagram. In particular, given the assumptions, we can show that there exists a $\epsilon > 0$ such that for all $0 < K(0) \le \epsilon$, there is a path $\{C(t), K(t)\}$ for $t \in [0,T]$ with $T < \infty$ which satisfy the o.d.e. given by (86) and (85) for $0 \le t < T$ for which $C(T) = F(0,s_r)$ and $K(T) = 0$. To see this, we reverse time defining $\tau = t_0 - t$ or $t = t_0 - \tau$ and define $c(\tau) = C(t_0 - \tau)$ and $k(\tau) = K(t_0 - \tau)$. The o.d.e. for $\{c,k\}$ are obtained by multiplying by minus one the expression for the time derivatives of C, K. We then run the system for $\{c,k\}$ with initial conditions $c(0) = F(0;s_r)$ and $k(0) = 0$. At $\tau = 0$, in the c,k plane, the system starts vertically up. Since the $\dot{k} = 0$ locus is upward slopping with finite slope, for any $\tau > 0$ the system is in the quadrant for which $\frac{d}{d\tau}c(\tau) > 0$ and $\frac{d}{d\tau}k(\tau) > 0$. By reversing time again, so that we go back to t instead of τ we found the desired path.

B.13 Proof of Proposition 15

Let's consider the planner problem where aggregate output is $A(1)F(K)$ but the planner can directly set the path $\{C(t)\}$ subject to feasibility $C(t) + \kappa\dot{K}(t) = A(1)F(K(t))$ for all $t \geq 0$. This is a relaxed problem with a larger feasible set, since we are not checking that the allocation can be obtained as an equilibrium. The final step is to notice that the first order necessary conditions of this problem coincide with the ones for an equilibrium where $B(1, s_a(t)) = 1$, or $s_a(t) = s_a^*$.

B.14 Proof of Proposition 17

The proof proceeds by a contradiction argument. Suppose that $a_{i,j} > 0$ and there is an integer $k > 0$ for which $a_{j+k,i} < m_{j+k,i}$. We will find an alternative policy with the same measure of adoption at all times but with higher consumption in period $i+1$. We will denote the new policy by $\{\tilde{a}_{j,i}\}$. To set this policy, let the strictly positive scalar ϵ be defined as:

$$\epsilon = \frac{1}{2} \min \{a_{j,i}\, ,\, m_{j+k,i} - a_{j+k,i}\} > 0$$

The policy $\tilde{a}$ is identical to a at all times and gaps, except for $(j,i), (j+1, i+1), (j+k, i)$ and $(j+k+1, i+1)$. In particular we let:

$$\begin{aligned} \tilde{a}_{j,i} &= a_{j,i} - \epsilon \\ \tilde{a}_{j+1,i+1} &= a_{j+1,i+1} + \epsilon(1 - \Delta q) \\ \tilde{a}_{j+k,i} &= a_{j+k,i} + \epsilon \\ \tilde{a}_{j+k+1,i+1} &= a_{j+k+1,i+1} - \epsilon(1 - \Delta q) \end{aligned}$$

Letting $\tilde{m}$ the new process for the fraction of firms, we have that:

$$\begin{aligned} \tilde{m}_{j+1,i+1} &= m_{j+1,i+1} + (1 - \Delta q)\epsilon \\ \tilde{m}_{j+k+1,i+1} &= m_{j+k+1,i+1} - (1 - \Delta q)\epsilon \end{aligned}$$

but for all other j', i' we have $\tilde{m}_{j',i'} = m_{j',i'}$. Clearly $Y_{i+1}(\hat{m}_{i+1}) > Y_{i+1}(m_{i+1})$, and thus $\hat{c}_{i+1} > c_{i+1}$.

B.15 Proof of Proposition 18

If $\zeta \leq 1$, and $\theta \geq 0$, then F is the composition of concave functions, and hence concave.

if $\zeta > 1$, then we the F is not concave, regardless of θ. In particular, we can show that the function is not quasi-concave. Recall that F is quasi-concave if the upper contour set is convex. To show that if $\zeta > 1$

the function is not quasi-concave, define

$$Q(M,x)=\frac{\left(NM^{\zeta}-\kappa_0x+\kappa_0q\right)^{1-\theta}}{1-\theta}$$

To find the upper contour set fix $\bar{c}>0$, define

$$Q^{\bar{c}}\equiv\{M,x:\frac{\left(NM^{\zeta}-\kappa_0(x-q)\right)^{1-\theta}}{1-\theta}\geq\frac{\bar{c}^{1-\theta}}{1-\theta}\}$$

or equivalently:

$$\begin{aligned}Q^{\bar{c}}&=\left\{M,x:NM^{\zeta}-\kappa_0(x-q)\geq\bar{c}\right\}\\&=\left\{M,x:M\geq\psi(x;\bar{c})\equiv\left(\frac{\bar{c}+\kappa(x-q)}{N}\right)^{1/\zeta}\right\}\end{aligned}$$

Since $\psi(\cdot;\bar{c})$ is strictly concave for $\psi>1$, then $Q^{\bar{c}}$ is not convex.

C Linearization of the KFE

In this section we give an expression for a perturbation of the density m with respect to an initial distribution and a path of adoption.

The perturbation is done around a the steady state density $\bar{m}:[0,\bar{G}]$ corresponding to a balance growth path. Consider a perturbed initial condition $m_0(g,\epsilon)=\bar{m}(g)+\epsilon\omega(g)$ where ω is a bounded function. Let also define $G(t,\epsilon)$ be a path G differentiable on time for each ϵ. In particular consider a path of $G(\cdot,\epsilon)$ parametrized by the function $\hat{G}_t$ and the number ϵ that satisfies:

$$G(t,\epsilon)=\bar{G}+\epsilon\,\hat{G}(s)\text{ for all }s\geq0\text{ and }\epsilon$$

For any ϵ and t, the value of $G(t,\epsilon)$ is the upper bound of the support of $m(\cdot,t,\epsilon)$.

Fixing a number ϵ, a function $\omega(g)$ and a differentiable path $\hat{G}(t)$, we are looking for a solution of the p.d.e.:

$$m_t(g,t,\epsilon)+m_g(g,t,\epsilon)+qm(g,t,\epsilon)=0\text{ for all }t\geq0,\ g\in[0,G(t,\epsilon)]\tag{112}$$

with initial condition:

$$m(g,0,\epsilon)=m_0(g,\epsilon)\equiv\bar{m}(g)+\epsilon\omega(g)\text{ for all }g\in[0,G(0,\epsilon)]\tag{113}$$

and for each t we have mass preservation:

$$\int_0^{G(t,\epsilon)} m(g,t,\epsilon)dg = 1 \text{ for all } t \geq 0. \tag{114}$$

For future reference we note that if $\epsilon = 0$, then $m(g,t,0) = \bar{m}(g)$ for all $g \in [0, \bar{G}]$. This holds because if $\epsilon = 0$, then $m_0(g) = \bar{m}(g)$ and $G(t,0) = \bar{G}$. Note that this also implies $m_g(g,t,0) = \bar{m}_g(g)$ for all g.

We define the derivative with respect to ϵ of the system the system given by (112), (113) and (114). We use the notation:

$$\hat{m}(g,t) \equiv \frac{\partial}{\partial \epsilon} m(g,t,\epsilon)|_{\epsilon=0}.$$

The following lemma describes the p.d.e. and boundary conditions that $\hat{m}$ must satisfy.

Lemma 8 *Fix a differentiable path $\hat{G}(t)$ for $t \in [0,T]$ and initial condition $\omega(g)$ for $g \in [0, \bar{G}]$, which satisfies*

$$0 = \bar{m}(\bar{G})\hat{G}(0) + \int_0^{\bar{G}} \omega(g)dg \text{ and} \tag{115}$$

$$\omega(\bar{G}) - \hat{m}(0,0) = \ell(0) \equiv \bar{m}(\bar{G}) \left[\hat{G}'(0) + q\hat{G}(0)\right]. \tag{116}$$

Then $\hat{m}(g,t)$ must solve the p.d.e.:

$$\hat{m}_t(g,t) + \hat{m}_g(g,t) + q\hat{m}(g,t) = 0 \text{ for all } t \geq 0,\ g \in [0, \bar{G}] \tag{117}$$

with initial conditions given by:

$$\hat{m}(g,0) = \omega(g) \text{ for } g \in (0, \bar{G}] \tag{118}$$

and lateral boundary conditions:

$$\hat{m}(\bar{G},t) - \hat{m}(0,t) = \ell(t) \equiv \bar{m}(\bar{G}) \left[\hat{G}'(t) + q\hat{G}(t)\right] \text{ for all } t \geq 0 \tag{119}$$

The initial condition $\omega(\cdot)$ and $\hat{G}(0)$ has to satisfy the condition (115) because $\hat{G}$ is the derivative of $G(0,\epsilon)$, the upper bound of the support of $m(\cdot,0,\epsilon)$, with respect to ϵ. Condition (119) is obtained by using the time derivative of an analogous condition for any $t > 0$. Taking the limit as $t \downarrow 0$ we obtained (116). Note that, in principle, we allow $\hat{m}(g,0)$ to differ from $\omega(g)$ only at $g = 0$, which is reflected in the domain of condition (118). We can regard that, given $\hat{G}(0)$ which is determined in (115), the value of $\hat{G}'(0)$ determines the density $\hat{m}(0,0)$.

Proof. (of Lemma 8) (117) follows directly from differentiating (112) with respect to ϵ and evaluating at $\epsilon = 0$. Differentiating (114) with respect to ϵ and evaluating at $\epsilon = 0$ we obtain:

$$0 = \bar{m}(\bar{G})\hat{G}(t) + \int_0^{\bar{G}} \hat{m}(g,t)dg = 0 \tag{120}$$

Evaluating this equation at $t = 0$ we obtain (115). Differentiating this equation with respect to time we obtain

$$0 = \bar{m}(\bar{G})\hat{G}'(t) + \int_0^{\bar{G}} \hat{m}_t(g,t)dg = 0$$

Replacing $\hat{m}_t$ from (117) we obtain

$$0 = \bar{m}(\bar{G})\hat{G}'(t) - \int_0^{\bar{G}} \hat{m}_g(g,t)dg - q\int_0^{G} \hat{m}(g,t)dg$$

Using (120) to replace $\int_0^G \hat{m}(g,t)dg$ we get

$$\begin{aligned} 0 &= \bar{m}(\bar{G})\hat{G}'(t) - \int_0^{\bar{G}} \hat{m}_g(g,t)dg + q\bar{m}(\bar{G})\hat{G}(t) \\ &= \bar{m}(\bar{G})\hat{G}'(t) - \hat{m}(\bar{G},t) + \hat{m}(0,t) + q\bar{m}(\bar{G})\hat{G}(t) \end{aligned}$$

which gives (119). ■

Given this lemma, we can solve for $\hat{m}$ as in the next proposition.

Proposition 19 *Fix two differentiable functions $\omega : [0,\bar{G}] \to \mathbb{R}$ and $\hat{G} : [0,T] \to \mathbb{R}$ satisfying (115) and (116). The solution for $\hat{m} : [0,\bar{G}] \times [0,T] \to \mathbb{R}$, differentiable on $g \in (0,\bar{G}), t \in (0,T)$, and $t \neq g + (k-1)\bar{G}\}$ for integer k, whose p.d.e. and boundary conditions are described in Lemma 8 is given by:*

$$\hat{m}(g,t) = \begin{cases} \omega(g-t)e^{-qt} & \text{if } t < g \leq \bar{G} \\ M(t-g)e^{-qt} & \text{if } 0 \leq g \leq t \end{cases} \tag{121}$$

where

$$M(t) = -\sum_{j=0}^{\lfloor \frac{t}{\bar{G}} \rfloor} e^{q(t-j\bar{G})}\bar{m}(\bar{G})\left[\hat{G}'(t-j\bar{G}) + q\hat{G}(t-j\bar{G})\right] + \omega\left(\bar{G} + \lfloor \tfrac{t}{\bar{G}} \rfloor \bar{G} - t\right) \text{ for } t \geq 0.$$

and thus $M(t-g)$ is given by:

$$M(t-g)e^{-qt} = -\sum_{j=0}^{\lfloor \frac{t-g}{\bar{G}} \rfloor} e^{-q(g+j\bar{G})}\bar{m}(\bar{G})\left[\hat{G}'(t-g-j\bar{G}) + q\hat{G}(t-g-j\bar{G})\right] + e^{-qt}\omega\left(\bar{G} + \lfloor \tfrac{t-g}{\bar{G}} \rfloor \bar{G} - t + g\right) \tag{122}$$

for $t \geq g$, $g \in [0, \bar{G}]$. If the boundary condition in (116) holds with $\omega(0) = \hat{m}(0,0)$, then the function $\hat{m} : [0, \bar{G}] \times [0, T] \to \mathbb{R}$ is continuous and solves the p.d.e. in a weak sense.

Few comments are in order. First, if $\omega(0) \neq \hat{m}(0,0)$, the solution of $\hat{m}(0,t)$ is discontinuous at $t = \bar{G}k$ for any integer $k \geq 1$. Second, if $\omega(0) = \hat{m}(0,0)$, then the initial conditions (115) and (116) jointly determine $\hat{G}(0)$ and $\hat{G}'(0)$.

Proof. (of Proposition 19) Since the p.d.e. for $\hat{m}$ is a constant coefficient, homogeneous transport equation, its solution is given by:

$$\hat{m}(g,t) = \begin{cases} \omega(g-t)e^{-qt} & \text{if } t < g \leq \bar{G} \\ M(t-g)e^{-qt} & \text{if } 0 \leq g \leq t \end{cases} \tag{123}$$

for some function M, which will be chosen to satisfy the lateral boundary condition. By construction, the initial condition is satisfied, i.e. $\hat{m}(g,0) = \omega(g)$ for all g. We will check that given the properties of M the lateral boundary condition is satisfied for all $t \geq 0$. We verify the lateral boundary conditions recursively. First we solve for $t \in [0, \bar{G}]$:

$$\hat{m}(\bar{G},t) - \hat{m}(0,t) = \left[\omega(\bar{G}-t) - M(t)\right]e^{-qt} = \ell(t) \text{ or } M(t) = \omega(\bar{G}-t) - e^{qt}\ell(t),$$

where $\ell(t) \equiv \bar{m}(\bar{G})\left[\hat{G}'(t) + q\hat{G}(t)\right]$.

For all $t > g$ we define M recursively:

$$\ell(t) = e^{-qt}\left[M(t-\bar{G}) - M(t)\right] \text{ or } M(t) = M(t-\bar{G}) - e^{qt}\ell(t).$$

Replacing this expression for large t

$$M(t) = -e^{qt}\ell(t) - e^{q(t-\bar{G})}\ell(t-\bar{G}) + M(t-2\bar{G})$$

continuing this substitution gives:

$$M(t) = -\sum_{j=0}^{\lfloor \frac{t}{\bar{G}} \rfloor} e^{q(t-j\bar{G})}\ell(t-j\bar{G}) + \omega\left(\bar{G} + \lfloor \tfrac{t}{\bar{G}} \rfloor \bar{G} - t\right) \text{ for } t \geq 0.$$

Evaluating this expression on $t-g$, gives the solution for $t \geq g$ and $g \in [0, \bar{G}]$:

$$M(t-g)e^{-qt} = -\sum_{j=0}^{\lfloor \frac{t-g}{\bar{G}} \rfloor} e^{-q(g+j\bar{G})}\ell(t-g-j\bar{G}) + e^{-qt}\omega\left(\bar{G} + \lfloor \tfrac{t-g}{\bar{G}} \rfloor \bar{G} - t + g\right) .$$

Next we show that $\hat{m}$ is continuous everywhere. We do this in two steps.

Step 1. Given that $\hat{m}(g,t)$ is defined by two different functions in different regions, we check that it is continuous at $g = t$ for $g \in [0, \bar{G}]$. For these combinations we compute:

$$\lim_{t \downarrow g} \hat{m}(t,g) = e^{-qg}\omega(0) \text{ and } \lim_{t \uparrow g} \hat{m}(t,g) = e^{-qg}M(0)$$

Thus continuity requires that $\omega(0) = M(0)$. Direct computation evaluating the expression for $M(t)$ gives $M(0) = -\ell(0) + \omega\left(\bar{G}\right)$ Then, if the boundary condition (116) holds with $\omega(0) = \hat{m}(0,0)$ then $\hat{m}(g,t)$ is continuous at $(t,g) = (g,g)$.

Step 2. Given the use of the floor function in the definition of M, we check that if the boundary condition (116) holds with $\omega(0) = \hat{m}(0,0)$, then function M is indeed continuous around the points of discontinuity of the floor function. In particular fix $g \in (0, \bar{G})$, fix k as a strictly positive integer, and let $\delta \geq 0$ as a small positive real. Let $t(\delta)$ be parametrized by δ as follows: $t(\delta) = g + (k-\delta)\bar{G}$. Thus

$$\lfloor \tfrac{t(\delta)-g}{\bar{G}} \rfloor = \begin{cases} k-1 & \text{if } \delta > 0 \\ k & \text{if } \delta = 0 \end{cases}$$

In this we can compute $M\left(t(\delta) - t\right)$ as:

$$\begin{aligned} &M(t(0)-g)e^{-qt(0)} \\ &= -\sum_{j=0}^{k} e^{-q(g+j\bar{G})}\ell(k\bar{G} - j\bar{G}) + e^{-qt(0)}\omega\left(\bar{G} + k\bar{G} - k\bar{G}\right) \\ &= -\sum_{j=0}^{k} e^{-q(g+j\bar{G})}\ell((k-j)\bar{G}) + e^{-q(g+k\bar{G})}\omega\left(\bar{G}\right) \end{aligned}$$

and for $\delta > 0$:

$$\begin{aligned}
&M(t(\delta) - g)e^{-qt(\delta)} \\
&= -\sum_{j=0}^{k-1} e^{-q(g+j\bar{G})}\ell((k-\delta)\bar{G} - j\bar{G}) + e^{-qt(\delta)}\omega\left(\bar{G} + (k-1)\bar{G} - (k-\delta)\bar{G}\right) \\
&= -\sum_{j=0}^{k-1} e^{-q(g+j\bar{G})}\ell((k-j)\bar{G} - \delta\bar{G}) + e^{-q(g+(k-\delta)\bar{G})}\omega\left(\delta\bar{G}\right)
\end{aligned}$$

Now we can take the limit:

$$\lim_{\delta\downarrow 0} M(t(\delta) - g)e^{-qt(\delta)} = -\sum_{j=0}^{k-1} e^{-q(g+j\bar{G})}\ell((k-j)\bar{G}) + e^{-q(g+k\bar{G})}\omega(0)$$

Finally, we can compute the difference:

$$\begin{aligned}
&M(t(0) - g)e^{-qt(0)} - \lim_{\delta\downarrow} M(t(\delta) - g)e^{-qt(\delta)} \\
&= -\sum_{j=0}^{k} e^{-q(g+j\bar{G})}\ell((k-j)\bar{G}) + e^{-q(g+k\bar{G})}\omega\left(\bar{G}\right) \\
&+ \sum_{j=0}^{k-1} e^{-q(g+j\bar{G})}\ell((k-j)\bar{G}) - e^{-q(g+k\bar{G})}\omega(0) \\
&= e^{-q(g+k\bar{G})}\left[-\ell(0) + \omega\left(\bar{G}\right) - \omega(0)\right] = 0
\end{aligned}$$

which is zero, given the boundary condition.

Verifying that $\hat{m}$ is a weak solution. Since, by construction, $\hat{m}$ is continuous, but not differentiable for $t = g + (k-1)\bar{G}$ for integer k, we check that the proposed solution solves the p.d.e. in weak sense. The function $\hat{m}$ solves p.d.e. in a weak sense if

$$0 = \int_0^T \int_0^{\bar{G}} \hat{m}(g,t)\left[-h_t(g,t) - h_g(g,t) + qh(g,t)\right] dgdt - \int_0^{\bar{G}} h(g,0)\omega(g)dg$$

for all $h \in C^1([0,\bar{G}] \times [0,T])$ and such that $h(0,t) = h(\bar{G},t) = 0$ for all $t > 0$, and $h(g,T) = 0$ for all $g \in [0,\bar{G}]$. Alternatively, this is the set of functions h which are zero outside a compact support which is included in $(0,\bar{G}) \times [0,T)$, and differentiable.

We will show that:

$$0 = \sum_{k=1}^{K} \int_{(k-1)\bar{G}}^{k\bar{G}} \int_{0}^{\bar{G}} \hat{m}(g,t)\left[-h_t(g,t) - h_g(g,t) + qh(g,t)\right] dgdt - \int_{0}^{\bar{G}} h(g,0)\omega(g)dg$$

where we are assuming, to simplify the computations, that $T = K\bar{G}$ for an integer K. Take an integer k with $1 \leq K$, then define

$$I_k \equiv \int_{(k-1)\bar{G}}^{k\bar{G}} \int_{0}^{\bar{G}} \hat{m}(g,t)\left[-h_t(g,t) - h_g(g,t) + qh(g,t)\right] dgdt$$

where the proposed solution found for $\hat{m}(g,t)$ can be written as:

$$\hat{m}(g,t) = \begin{cases} u_k(t-g)e^{-qt} & \text{if } t \geq g + \bar{G}(k-1) \\ u_{k-1}(t-g)e^{-qt} & \text{if } t < g + \bar{G}(k-1) \end{cases} \tag{124}$$

where, as shown above since $\hat{m}(g,t)$ is continuous, then $u_k(\bar{G}(k-1)) = u_{k-1}((\bar{G}(k-1)))$ and where u_k and u_{k-1} are differentiable functions defined in the interior of each of the domains. The functions u_k's are related to our proposed solution as follows. Since $\hat{m}(g,t) = e^{-qt}\omega(g-t)$ for $t \leq g$, then $u_0(t-g) = \omega(g-t)$, or just $u_0(x) = \omega(-x)$ for all $x \in [-\bar{G}, 0]$. For $k \geq 1$ we have $M(t-g) = u_k(t-g)$ for all $g + \bar{G}(k-1) \leq t \leq g + \bar{G}k$ and $g \in [0, \bar{G}]$.

Taking an arbitrary k, we can rewrite the integral I_k as:

$$\begin{aligned} I_k =& \int_{(k-1)\bar{G}}^{k\bar{G}} \int_{0}^{\bar{G}} \hat{m}(g,t)\left[-h_t(g,t) - h_g(g,t) + qh(g,t)\right] dgdt \\ =& -\int_{0}^{\bar{G}} \left[\int_{(k-1)\bar{G}}^{(k-1)\bar{G}+g} \hat{m}(g,t)h_t(g,t)dt + \int_{(k-1)\bar{G}+g}^{k\bar{G}} \hat{m}(g,t)h_t(g,t)dt\right] dg \\ & - \int_{(k-1)\bar{G}}^{k\bar{G}} \left[\int_{0}^{t-(k-1)\bar{G}} \hat{m}(g,t)h_g(g,t)dg + \int_{t-(k-1)\bar{G}}^{\bar{G}} \hat{m}(g,t)h_g(g,t)dg\right] dt \\ & + \int_{(k-1)\bar{G}}^{k\bar{G}} \int_{0}^{\bar{G}} q\hat{m}(g,t)h(g,t)dgdt \end{aligned}$$

Now we use the form of the proposed solution for $\hat{m}$ in terms of q and the the functions u_k and u_{k-1}:

$$\begin{aligned}I_k = &-\int_0^{\bar{G}}\left[\int_{(k-1)\bar{G}}^{(k-1)\bar{G}+g} u_{k-1}(t-g)e^{-qt}h_t(g,t)dt + \int_{(k-1)\bar{G}+g}^{k\bar{G}} u_k(t-g)e^{-qt}h_t(g,t)dt\right]dg\\ &-\int_{(k-1)\bar{G}}^{k\bar{G}}\left[\int_0^{t-(k-1)\bar{G}} u_k(t-g)e^{-qt}h_g(g,t)dg + \int_{t-(k-1)\bar{G}}^{\bar{G}} u_{k-1}(t-g)e^{-qt}h_g(g,t)dg\right]dt\\ &+\int_{(k-1)\bar{G}}^{k\bar{G}}\int_0^{\bar{G}} q\hat{m}(g,t)h(g,t)dgdt\end{aligned}$$

We can integrate by parts the first four terms of I_k:

$$\begin{aligned}&I_k =\\ &\int_0^{\bar{G}}\int_{(k-1)\bar{G}}^{(k-1)\bar{G}+g}\left(u'_{k-1}(t-g) - qu_{k-1}(t-g)\right)e^{-qt}h(g,t)\,dt\,dg\\ &+\int_0^{\bar{G}}\int_{(k-1)\bar{G}+g}^{k\bar{G}}\left(u'_k(t-g) - qu_k(t-g)\right)e^{-qt}h(g,t)\,dt\,dg\\ &-\int_0^{\bar{G}}\left[u_{k-1}(t-g)e^{-qt}h(g,t)|_{t=(k-1)\bar{G}}^{t=(k-1)\bar{G}+g} + u_k(t-g)e^{-qt}h(g,t)|_{t=(k-1)\bar{G}+g}^{t=k\bar{G}}\right]dg\\ &-\int_{(k-1)\bar{G}}^{k\bar{G}}\left[\int_0^{t-(k-1)\bar{G}} u'_k(t-g)e^{-qt}h(g,t)dg + \int_{t-(k-1)\bar{G}}^{\bar{G}} u'_{k-1}(t-g)e^{-qt}h(g,t)dg\right]dt\\ &-\int_{(k-1)\bar{G}}^{k\bar{G}}\left[u_k(t-g)e^{-qt}h(g,t)|_{g=0}^{g=t-(k-1)\bar{G}} + u_{k-1}(t-g)e^{-qt}h(g,t)|_{g=g=t-(k-1)\bar{G}}^{g=\bar{G}}\right]dt\\ &+\int_{(k-1)\bar{G}}^{k\bar{G}}\int_0^{\bar{G}} q\hat{m}(g,t)h(g,t)dgdt\end{aligned}$$

Using the definition of $\hat{m}$ we can cancel the terms proportional to q obtaining:

$$\begin{aligned}&I_k =\\ &\int_0^{\bar{G}}\left[\int_{(k-1)\bar{G}}^{(k-1)\bar{G}+g} u'_{k-1}(t-g)e^{-qt}h(g,t)dt + \int_{(k-1)\bar{G}+g}^{k\bar{G}} u'_k(t-g)e^{-qt}h(g,t)dt\right]dg\\ &-\int_0^{\bar{G}}\left[u_{k-1}(t-g)e^{-qt}h(g,t)|_{t=(k-1)\bar{G}}^{t=(k-1)\bar{G}+g} + u_k(t-g)e^{-qt}h(g,t)|_{t=(k-1)\bar{G}+g}^{t=k\bar{G}}\right]dg\\ &-\int_{(k-1)\bar{G}}^{k\bar{G}}\left[\int_0^{t-(k-1)\bar{G}} u'_k(t-g)e^{-qt}h(g,t)dg + \int_{t-(k-1)\bar{G}}^{\bar{G}} u'_{k-1}(t-g)e^{-qt}h(g,t)dg\right]dt\\ &-\int_{(k-1)\bar{G}}^{k\bar{G}}\left[u_k(t-g)e^{-qt}h(g,t)|_{g=0}^{g=t-(k-1)\bar{G}} + u_{k-1}(t-g)e^{-qt}h(g,t)|_{g=t-(k-1)\bar{G}}^{g=\bar{G}}\right]dt\end{aligned}$$

Using that $h(0,t) = h(\bar{G},t) = 0$ and that $u_k((k-1)\bar{G}) = u_{k-1}((k-1)\bar{G})$ then the last term is zero, and thus:

$$
\begin{aligned}
&I_k = \\
&\int_0^{\bar{G}} \left[\int_{(k-1)\bar{G}}^{(k-1)\bar{G}+g} u_{k-1}'(t-g) e^{-qt} h(g,t) dt + \int_{(k-1)\bar{G}+g}^{k\bar{G}} u_k'(t-g) e^{-qt} h(g,t) dt \right] dg \\
&- \int_0^{\bar{G}} \left[u_{k-1}(t-g) e^{-qt} h(g,t) |_{t=(k-1)\bar{G}}^{t=(k-1)\bar{G}+g} + u_k(t-g) e^{-qt} h(g,t) |_{t=(k-1)\bar{G}+g}^{t=k\bar{G}} \right] dg \\
&- \int_{(k-1)\bar{G}}^{k\bar{G}} \left[\int_0^{t-(k-1)\bar{G}} u_k'(t-g) e^{-qt} h(g,t) dg + \int_{t-(k-1)\bar{G}}^{\bar{G}} u_{k-1}'(t-g) e^{-qt} h(g,t) dg \right] dt
\end{aligned}
$$

The first and the third terms of the previous equation are equal and of opposite sign, so we have:

$$
I_k = - \int_0^{\bar{G}} \left[u_{k-1}(t-g) e^{-qt} h(g,t) |_{t=(k-1)\bar{G}}^{t=(k-1)\bar{G}+g} + u_k(t-g) e^{-qt} h(g,t) |_{t=(k-1)\bar{G}+g}^{t=k\bar{G}} \right] dg
$$

Using that $u_{k-1}((k-1)\bar{G}) = u_k((k-1)\bar{G})$ we have:

$$
I_k = \int_0^{\bar{G}} u_{k-1}((k-1)\bar{G} - g) e^{-q(k-1)\bar{G}} h(g,(k-1)\bar{G}) dg - \int_0^{\bar{G}} u_k(k\bar{G} - g) e^{-qk\bar{G}} h(g,k\bar{G}) dg
$$

Thus we have:

$$
\begin{aligned}
&\sum_{k=1}^{K} \int_{(k-1)\bar{G}}^{k\bar{G}} \int_0^{\bar{G}} \hat{m}(g,t) \left[-h_t(g,t) - h_g(g,t) + q h(g,t) \right] dg dt = \sum_{k=1}^{K} I_k \\
&= \sum_{k=1}^{K} \int_0^{\bar{G}} u_{k-1}((k-1)\bar{G} - g) e^{-q(k-1)\bar{G}} h(g,(k-1)\bar{G}) dg \\
&- \sum_{k=1}^{K} \int_0^{\bar{G}} u_k(k\bar{G} - g) e^{-qk\bar{G}} h(g,k\bar{G}) dg \\
&= \int_0^{\bar{G}} u_0(-g) h(g,0) dg - \int_0^{\bar{G}} u_K(K\bar{G} - g) e^{-qK\bar{G}} h(g,K\bar{G}) dg
\end{aligned}
$$

where the last line uses that $K-1$ terms enter with opposite sides of the two sums. Using the boundary condition on the h's, namely that $h(g,\bar{K}) = 0$ for all $g \in [0,\bar{G}]$ we have:

$$
\begin{aligned}
&\sum_{k=1}^{K} \int_{(k-1)\bar{G}}^{k\bar{G}} \int_0^{\bar{G}} \hat{m}(g,t) \left[-h_t(g,t) - h_g(g,t) + q h(g,t) \right] dg dt \\
&= \int_0^{\bar{G}} u_0(-g) h(g,0) dg = \int_0^{\bar{G}} \omega(g) h(g,0) dg
\end{aligned}
$$

where the last line uses the definition of u_0 in our proposed solution. Thus. we have verified that our proposed solution is weak solution of the p.d.e.

■

D Linearization of the HJB equation

In this section we give an expression for a perturbation of the optimal threshold as a function of the future path of productivity, prices and interest rates.

We start defining the perturbations from a steady state for the path that affect the firm problem. In particular we define the following paths for $t \in [0, T]$:

$$Z(t, \epsilon) = \bar{Z} + \epsilon \hat{Z}(t)\,,\; r(t, \epsilon) = \bar{\rho} + \epsilon \hat{r}(t) \;\text{ and }\; P(t, \epsilon) = \bar{P} + \epsilon \hat{P}(t),\; \text{ for } t \in [0, T] \tag{125}$$

where ϵ is a scalar, and $\hat{Z}, \hat{r}$ and $\hat{P}$ are perturbations, i.e. arbitrary paths. For technical reasons, the notation allow for the perturbation to stop at finite T, but we can, if needed, take $T \to \infty$.

We consider the value function $V(g, t, \epsilon)$ and optimal threshold $G(t, \epsilon)$ which satisfy:

$$\begin{aligned} r(t, \epsilon) V(g, t, \epsilon) &= \bar{\pi}(g, Z(t, \epsilon)) + V_t(g, t, \epsilon) + V_g(g, t, \epsilon) \\ &\quad + q\left(V(0, t, \epsilon) - V(g, t, \epsilon)\right) \;\text{ for } g \in [0, G(t, \epsilon)] \text{ and } t \in [0, T], \end{aligned} \tag{126}$$

satisfying value matching and smooth pasting:

$$V(G(t, \epsilon), t, \epsilon) = V(0, t, \epsilon) - \kappa P(t, \epsilon), \;\text{ for } t \in [0, T] \tag{127}$$

$$V_g(G(t, \epsilon), t, \epsilon) = 0, \;\text{ for } t \in [0, T], \tag{128}$$

and where we force that at $t = T$ the continuation of the firms' value is the steady state one:

$$V(g, T, \epsilon) = \bar{V}(g) \text{ for } g \geq 0. \tag{129}$$

We note that for $\epsilon = 0$ the paths are constant equal to the steady state values, and hence the value function and policy are the ones that correspond to that steady state, namely $\bar{V}(g), \bar{G}$.

We define $\hat{V}$ and $\hat{G}$ as follows:

$$\begin{aligned} \hat{V}(g, t) &\equiv \frac{d}{d\epsilon} V(g, t, \epsilon)|_{\epsilon=0} \text{ for } g \in [0, \bar{G}] \text{ and } t \in [0, T] \\ \hat{G}(t) &\equiv \frac{d}{d\epsilon} G(t, \epsilon)|_{\epsilon=0} \text{ for } t \in [0, T]. \end{aligned}$$

Proposition 20 *Consider the problem of a firm facing arbitrary paths as described in (125). The derivative of the optimal threshold at each time t is given by:*

$$\hat{G}(t) = \frac{\hat{V}_g(\bar{G},t)}{-\bar{V}_{gg}(\bar{G})} \text{ for all } t \in [0,T]$$

Solving for these two derivatives we get:

$$\hat{G}(t) = \frac{a(t)}{\bar{\pi}_g(\bar{G},\bar{Z})} + e^{-(q+\bar{\rho})\bar{G}}\,\hat{G}(t+\bar{G}) \tag{130}$$

with $\hat{G}(\tau) = 0$ *for* $\tau \geq T$*, and where*

$$\begin{aligned} a(t) \equiv & \int_0^{\bar{G}} e^{-(q+\bar{\rho})\tau}\bar{\pi}_{gZ}(\tau,\bar{Z})\hat{Z}(t+\tau)d\tau - \int_0^{\bar{G}} e^{-(q+\bar{\rho})\tau}\bar{\pi}_g(\tau,\bar{Z})\left(\int_t^{t+\tau}\hat{r}(s)ds\right)d\tau \\ & + \left[\bar{\pi}_Z(0,\bar{Z}) - \bar{\pi}_Z(\bar{G},\bar{Z})\right]\tilde{Z}(t) + \kappa\hat{P}'(t) - (\bar{\rho}+q)\kappa\hat{P}(t) - \kappa\bar{P}\hat{r}(t) \end{aligned} \tag{131}$$

Proof. (of Proposition 20) Differentiating the HJB in (126) with respect to ϵ we obtain:

$$\begin{aligned} (\bar{\rho}+q)\hat{V}(g,t) &= \hat{V}_t(g,t) + \hat{V}_g(g,t) + \hat{S}(g,t) \\ \hat{S}(g,t) &\equiv \bar{\pi}_Z(g,\bar{Z})\hat{Z}(t) - \hat{r}(t)\bar{V}(g) + q\hat{V}(0,t) \end{aligned}$$

for $g \in [0,\bar{G}]$ and $t \in [0,T]$, and differentiating the boundary conditions with respect to ϵ:

$$\hat{V}(\bar{G},t) = \hat{V}(0,t) - \kappa\hat{P}(t), \text{ for } t \in [0,T] \tag{132}$$

$$\bar{V}_{gg}(\bar{G})\hat{G}(t) = -\hat{V}_g(\bar{G},t), \text{ for } t \in [0,T], \tag{133}$$

where we use smooth pasting, i.e. that $\bar{V}_g(\bar{G},t) = 0$ in (132). The corresponding terminal condition is:

$$\hat{V}(g,T) = 0 \text{ for } g \geq 0. \tag{134}$$

We are interesting in solving for the path $\hat{G}$ as a function of the paths $\hat{Z}, \hat{P}, \hat{r}$, the number $\bar{V}_{gg}(\bar{Z})$ and the path for derivative $\hat{V}_g(\bar{G},\cdot)$. To obtain this last path we will solve for the level of $\hat{V}(g,t)$ and then differentiate with respect to g. The solution, for a given path $\{\hat{V}(0,t)\}$, is given by:

$$\hat{V}(g,t) = \begin{cases} \int_t^{t-g+\bar{G}} e^{-(\bar{\rho}+q)(\tau-t)} \hat{S}\left(g-t+\tau,\tau\right) d\tau \\ +e^{-(\bar{\rho}+q)(\bar{G}-g)} \left[\hat{V}(0,\bar{G}-g+t) - \kappa \hat{P}\left(\bar{G}-g+t\right)\right] & \text{if } 0 \leq t \leq T-\bar{G} \\ \\ \int_t^T e^{-(\bar{\rho}+q)(\tau-t)} \hat{S}\left(g-t+\tau,\tau\right) d\tau & \text{if } T-\bar{G} \leq t \leq T \end{cases}$$

To verify that this is the solution, we can compute the derivatives with respect to g and t in each of the two cases. For $0 \leq t \leq T - \bar{G}$ we have:

$$\begin{aligned} \hat{V}_t(g,t) &= -\hat{S}(g,t) + \hat{S}(\bar{G}, t-g+\bar{G}) - \int_t^{t-g+\bar{G}} e^{-(\bar{\rho}+q)(\tau-t)} \hat{S}_1\left(g-t+\tau,\tau\right) d\tau \\ &+ (\bar{\rho}+q) \int_t^{t-g+\bar{G}} e^{-(\bar{\rho}+q)(\tau-t)} \hat{S}\left(g-t+\tau,\tau\right) d\tau + e^{-(\bar{\rho}+q)(\bar{G}-g)} \left[\hat{V}_t(0,\bar{G}-g+t) - \kappa \hat{P}'\left(\bar{G}-g+t\right)\right] \\ \hat{V}_g(g,t) &= -\hat{S}(\bar{G}, t-g+\bar{G}) + \int_t^{t-g+\bar{G}} e^{-(\bar{\rho}+q)(\tau-t)} \hat{S}_1\left(g-t+\tau,\tau\right) d\tau \\ &- e^{-(\bar{\rho}+q)(\bar{G}-g)} \left[\hat{V}_t\left(0,\bar{G}-g+t\right) - \kappa \hat{P}'\left(\bar{G}-g+t\right)\right] \\ &+ (\bar{\rho}+q) e^{-(\bar{\rho}+q)(\bar{G}-g)} \left[\hat{V}\left(0,\bar{G}-g+t\right) - \kappa \hat{P}\left(\bar{G}-g+t\right)\right] \end{aligned}$$

Thus adding them:

$$\begin{aligned} &\hat{V}_t(g,t) + \hat{V}_g(g,t) = -\hat{S}(g,t) \\ &+ (\bar{\rho}+q) \left(\int_t^{t-g+\bar{G}} e^{-(\bar{\rho}+q)(\tau-t)} \hat{S}\left(g-t+\tau,\tau\right) d\tau + e^{-(\bar{\rho}+q)(\bar{G}-g)} \left[\hat{V}\left(0,\bar{G}-g+t\right) - \kappa \hat{P}\left(\bar{G}-g+t\right)\right] \right) \end{aligned}$$

so that the p.d.e. holds in this case. In this case for $g = \bar{G}$ we have that $\hat{V}(\bar{G},t) = \hat{V}(0,t) + \kappa P(t)$ so it satisfies the lateral boundary condition. A similar –and simpler– calculation holds for the case of $T = \bar{G} \leq t \leq T$.

Differentiating the solution for $\hat{V}$ with respect g and evaluating it at $\bar{G}$ we have:

$$\begin{aligned} \hat{V}_g(\bar{G},t) &= -\hat{S}(\bar{G},t) + \kappa \hat{P}'\left(t\right) - \hat{V}_t(0,t) + (\bar{\rho}+q) \left[\hat{V}\left(0,t\right) - \kappa \hat{P}\left(t\right)\right] \\ &= -\bar{\pi}_Z(\bar{G},\bar{Z}) \hat{Z}(t) + \hat{r}(t) \bar{V}(\bar{G}) - q \hat{V}(0,t) \\ &+ \kappa \hat{P}'\left(t\right) - \hat{V}_t(0,t) + (\bar{\rho}+q) \left[\hat{V}\left(0,t\right) - \kappa \hat{P}\left(t\right)\right] \\ &= -\bar{\pi}_Z(\bar{G},\bar{Z}) \hat{Z}(t) + \hat{r}(t) \bar{V}(\bar{G}) + \kappa \hat{P}'\left(t\right) - (\bar{\rho}+q) \kappa \hat{P}\left(t\right) \\ &+ \bar{\rho} \hat{V}\left(0,t\right) - \hat{V}_t(0,t) \end{aligned}$$

Thus, to evaluate $\hat{V}_g(\bar{G},t)$ we need to solve for $\hat{V}(0,t)$ and $\hat{V}_t(0,t)$. For this we evaluate the solution to $\hat{V}(g,t)$ in $g=0$ to obtain the following integral equation:

$$\hat{V}(0,t) = \begin{cases} s(t) + q\int_t^{t+\bar{G}} e^{-(\bar{\rho}+q)(\tau-t)}\hat{V}(0,\tau)\,d\tau + e^{-(\bar{\rho}+q)\bar{G}}\hat{V}(0,\bar{G}+t) & \text{if } 0 \le t \le T-\bar{G} \\ \\ s(t) + q\int_t^{T} e^{-(\bar{\rho}+q)(\tau-t)}\hat{V}(0,\tau)\,d\tau & \text{if } T-\bar{G} \le t \le T \end{cases}$$

where $s(\cdot)$ is given by

$$s(t) \equiv \begin{cases} \int_0^{\bar{G}} e^{-(\bar{\rho}+q)\tau} S(\tau, t+\tau)\,d\tau - e^{-(\bar{\rho}+q)\bar{G}}\kappa\hat{P}\left(\bar{G}+t\right) & \text{if } 0 \le t \le T-\bar{G} \\ \\ \int_t^{T} e^{-(\bar{\rho}+q)(\tau-t)} S(\tau-t,\tau)\,d\tau & \text{if } T-\bar{G} \le t \le T \end{cases}$$

and where $S(\cdot)$ is given by

$$S(g,t) \equiv \bar{\pi}_Z(g,\bar{Z})\hat{Z}(t) - \bar{V}(g)\hat{r}(t) \tag{135}$$

Now we can compute the derivative of $\hat{V}(0,t)$ with respect to t obtaining:

$$\hat{V}_t(0,t) = \begin{cases} s_t(t) + qe^{-(\bar{\rho}+q)\bar{G}}\hat{V}\left(0,t+\bar{G}\right) - q\hat{V}(0,t) \\ +(q+\bar{\rho})\left[\hat{V}(0,t) - s(t) - e^{-(\bar{\rho}+q)\bar{G}}\hat{V}(0,\bar{G}+t)\right] \\ +e^{-(\bar{\rho}+q)\bar{G}}\hat{V}_t(0,\bar{G}+t) & \text{if } 0 \le t \le T-\bar{G} \\ \\ s_t(t) + (q+\bar{\rho})\left[\hat{V}(0,t) - s(t)\right] - q\hat{V}(0,t) & \text{if } T-\bar{G} \le t \le T \end{cases}$$

It is convenient to define $\tilde{v}$ as:

$$\tilde{v}(t) \equiv \bar{\rho}\hat{V}(0,t) - \hat{V}_t(0,t)$$

and using the expression for $\hat{V}_t(0,t)$ we obtain:

$$\tilde{v}(t) = \begin{cases} -s_t(t) + (q+\bar{\rho})s(t) + e^{-(\bar{\rho}+q)\bar{G}}\tilde{v}(t+\bar{G}) & \text{if } 0 \le t \le T-\bar{G} \\ \\ -s_t(t) + (q+\bar{\rho})s(t) & \text{if } T-\bar{G} \le t \le T \end{cases} \tag{136}$$

This gives $\tilde{v}(t)$ for each t as the solution to a converging difference equation with known terminal condition.

Now we can replace back $\tilde{v}(t) = \bar{\rho}\hat{V}(0,t) - \hat{V}_t(0,t)$ into the expression for $\hat{V}_g(\bar{G},t)$ to obtain:

$$\hat{V}_g(\bar{G},t) = -\bar{\pi}_Z(\bar{G},\bar{Z})\hat{Z}(t) + \hat{r}(t)\bar{V}(\bar{G}) + \kappa\hat{P}'(t) - (\bar{\rho}+q)\kappa\hat{P}(t) + \tilde{v}(t)$$

Now we express the term in $\tilde{v}(t)$ given by

$$-s_t(t) + (q+\bar{\rho})s(t) = \tilde{a}_1(t) + \tilde{a}_2(t) + \tilde{a}_3(t)$$

The first term contains the expressions involving $\bar{P}$ and equals

$$\tilde{a}_1(t) = e^{-(q+\bar{\rho})G}\left[\kappa\hat{P}'(t+\bar{G}) - (\bar{\rho}+q)\kappa\hat{P}(t+\bar{G})\right]$$

The other two terms are the integrals of S. Given the expression fore $S(g,t)$ as a product of a function of g, denoted by f times a function of t, denoted by y we can write them as:

$$\tilde{a}_i(t) = (q+\bar{\rho})\int_0^{\bar{G}} e^{-(q+\bar{\rho})\tau} f(\tau)y(t+\tau)d\tau - \int_0^{\bar{G}} e^{-(q+\bar{\rho})\tau} f(\tau)y'(t+\tau)d\tau$$

for $i = 2, 3$. Using integration by parts we obtain:

$$\tilde{a}_i(t) = \int_0^{\bar{G}} e^{-(q+\bar{\rho})\tau} f'(\tau)y(t+\tau)d\tau + f(0)y(t) - e^{-(q+\bar{\rho})\bar{G}} f(\bar{G})y(\bar{G})$$

If we take $f(\tau)y(t+\tau) = \bar{\pi}_Z(\tau,\bar{Z})\hat{Z}(t+\tau)$ thus:

$$\tilde{a}_2(t) = \int_0^{\bar{G}} e^{-(q+\bar{\rho})\tau}\bar{\pi}_{gZ}(\tau,\bar{Z})\hat{Z}(t+\tau)d\tau + \bar{\pi}_Z(0,\bar{Z})\hat{Z}(t) - e^{-(q+\bar{\rho})\bar{G}}\bar{\pi}_Z(\bar{G},\bar{Z})\hat{Z}(t+\bar{G})$$

Finally if we take $f(\tau)y(t+\tau) = -\bar{V}(\tau)\hat{r}(t+\tau)$ we obtain:

$$\tilde{a}_3(t) = -\int_0^{\bar{G}} e^{-(q+\bar{\rho})\tau}\bar{V}_g(\tau)\hat{r}(t+\tau)d\tau - \bar{V}(0)\hat{r}(t) + e^{-(q+\bar{\rho})\bar{G}}\bar{V}(\bar{G})\hat{r}(t+\bar{G})$$

We can integrate by parts again, to obtain:

$$\begin{aligned}\tilde{a}_3(t) = &\int_0^{\bar{G}} e^{-(q+\bar{\rho})\tau}\left[\bar{V}_{gg}(\tau) - (q+\bar{\rho})\bar{V}_g(\tau)\right]\left(\int_t^{t+\tau}\hat{r}(s)ds\right)d\tau \\ &- e^{-(q+\bar{\rho})\bar{G}}\bar{V}_g(\bar{G})\left(\int_t^{t+\bar{G}}\hat{r}(t+s)ds\right) - \bar{V}(0)\hat{r}(t) + e^{-(q+\bar{\rho})\bar{G}}\bar{V}(\bar{G})\hat{r}(t+\bar{G})\end{aligned}$$

Note that $(q+\bar{\rho})\bar{V}_g(g) = \pi_g(g, \bar{Z}) + \bar{V}_{gg}(g)$ that $\bar{V}(\bar{G}) = \bar{V}(0) - \kappa\bar{P}$ and that $\bar{V}_g(\bar{G}) = 0$, so we can write:

$$\begin{aligned}\tilde{a}_3(t) = &- \int_0^{\bar{G}} e^{-(q+\bar{\rho})\tau}\bar{\pi}_g(\tau, \bar{Z}) \left(\int_t^{t+\tau} \hat{r}(s)ds\right) d\tau \\ &- \bar{V}(\bar{G})\hat{r}(t) - \kappa\bar{P}\hat{r}(t) + e^{-(q+\bar{\rho})\bar{G}}\bar{V}(\bar{G})\hat{r}(t+\bar{G})\end{aligned}$$

Thus we have:

$$\begin{aligned}-s_t(t) + (q+\bar{\rho})s(t) \equiv \tilde{a}(t) &= \tilde{a}_1(t) + \tilde{a}_2(t) + \tilde{a}_3(t) \\ &= \bar{\pi}_Z(0, \bar{Z})\hat{Z}(t) - \bar{V}(\bar{G})\hat{r}(t) - \kappa\bar{P}\hat{r}(t) \\ &+ \int_0^{\bar{G}} e^{-(q+\bar{\rho})\tau}\bar{\pi}_{gZ}(\tau, \bar{Z})\hat{Z}(t+\tau)d\tau - \int_0^{\bar{G}} e^{-(q+\bar{\rho})\tau}\bar{\pi}_g(\tau, \bar{Z}) \left(\int_t^{t+\tau} \hat{r}(s)ds\right) d\tau \\ &+ e^{-(q+\bar{\rho})\bar{G}} \left[\kappa\hat{P}'(t+\bar{G}) - (\bar{\rho}+q)\kappa\hat{P}(t+\bar{G}) + \bar{V}(\bar{G})\hat{r}(t+\bar{G}) - \bar{\pi}_Z(\bar{G}, \bar{Z})\hat{Z}(t+\bar{G})\right]\end{aligned}$$

We can then solve for $\tilde{v}(t)$ as:

$$\tilde{v}(t) = \tilde{a}(t) + e^{-(q+\bar{\rho})\bar{G}}\tilde{v}(t+\bar{G}) \text{ or } \tilde{v}(t) = \sum_{j=0}^{\lfloor \frac{T-t}{\bar{G}} \rfloor + 1} e^{-(q+\bar{\rho})j\bar{G}}\, \tilde{a}(t+j\bar{G})$$

where we use that $\hat{P}(t) = \hat{Z}(t) = \hat{r}(t) = 0$ for $t \geq T$.

Defining $a(t)$ as:

$$\begin{aligned}a(t) \equiv &\int_0^{\bar{G}} e^{-(q+\bar{\rho})\tau}\bar{\pi}_{gZ}(\tau, \bar{Z})\hat{Z}(t+\tau)d\tau - \int_0^{\bar{G}} e^{-(q+\bar{\rho})\tau}\bar{\pi}_g(\tau, \bar{Z}) \left(\int_t^{t+\tau} \hat{r}(s)ds\right) d\tau \\ &+ \left[\bar{\pi}_Z(0, \bar{Z}) - \bar{\pi}_Z(\bar{G}, \bar{Z})\right] \tilde{Z}(t) + \kappa\hat{P}'(t) - (\bar{\rho}+q)\kappa\hat{P}(t) - \kappa\bar{P}\hat{r}(t)\end{aligned}$$

Note that:

$$\begin{aligned}\tilde{a}(t) = &a(t) \\ &+ e^{-(q+\bar{\rho})\bar{G}}\bar{V}(\bar{G})\hat{r}(t+\bar{G}) - \bar{V}(\bar{G})\hat{r}(t) \\ &- \left[e^{-(q+\bar{\rho})\bar{G}}\bar{V}(\bar{G})\hat{r}(t+\bar{G})\bar{\pi}_Z(\bar{G}, \bar{Z}) - \bar{\pi}_Z(\bar{G}, \bar{Z})\hat{Z}(t)\right] \\ &+ e^{-(q+\bar{\rho})\bar{G}} \left[\kappa\hat{P}'(t+\bar{G}) - (\bar{\rho}+q)\kappa\hat{P}(t+\bar{G})\right] - \left[\kappa\hat{P}'(t) - (\bar{\rho}+q)\kappa\hat{P}(t)\right]\end{aligned}$$

Thus, for any integer $J \geq 1$

$$\sum_{j=0}^{J-1} e^{-(q+\bar{\rho})j\bar{G}} \tilde{a}(t+j\bar{G}) = \sum_{j=0}^{J-1} e^{-(q+\bar{\rho})j\bar{G}} a(t+j\bar{G})$$
$$-\bar{V}(\bar{G})\hat{r}(t) + \bar{\pi}_Z(\bar{G},\bar{Z})\hat{Z}(t) - \left[\kappa\hat{P}'(t) - (\bar{\rho}+q)\kappa\hat{P}(t)\right]$$
$$+\bar{V}(\bar{G})\hat{r}(t+J\bar{G}) - \bar{\pi}_Z(\bar{G},\bar{Z})\hat{Z}(t+J\bar{G}) + \left[\kappa\hat{P}'(t+\bar{J}) - (\bar{\rho}+q)\kappa\hat{P}(t+\bar{J})\right]$$

Then

$$\tilde{v}(t) = \bar{\pi}_Z(\bar{G},\bar{Z})\hat{Z}(t) - \bar{V}(\bar{G})\hat{r}(t) - \kappa\hat{P}'(t) + (\bar{\rho}+q)\kappa\hat{P}(t) + \sum_{j=0}^{J-1} e^{-(q+\bar{\rho})j\bar{G}} \bar{a}(t+j\bar{G})$$
$$+\bar{V}(\bar{G})\hat{r}(t+J\bar{G}) - \bar{\pi}_Z(\bar{G},\bar{Z})\hat{Z}(t+J\bar{G}) + \left[\kappa\hat{P}'(t+\bar{J}) - (\bar{\rho}+q)\kappa\hat{P}(t+\bar{J})\right]$$

Replacing $\tilde{v}(t)$ into the expression for $\hat{V}_g(\bar{G},t)$ we get:

$$\begin{aligned}\hat{V}_g(\bar{G},t) &= -\bar{\pi}_Z(\bar{G},\bar{Z})\hat{Z}(t) + \hat{r}(t)\bar{V}(\bar{G}) + \kappa\hat{P}'(t) - (\bar{\rho}+q)\kappa\hat{P}(t) + \tilde{v}(t) \\ &= \sum_{j=0}^{J-1} e^{-(q+\bar{\rho})j\bar{G}}\, a(t+j\bar{G}) \\ &\quad + \bar{V}(\bar{G})\hat{r}(t+J\bar{G}) - \bar{\pi}_Z(\bar{G},\bar{Z})\hat{Z}(t+J\bar{G}) + \left[\kappa\hat{P}'(t+\bar{J}) - (\bar{\rho}+q)\kappa\hat{P}(t+\bar{J})\right]\end{aligned}$$

Setting $J = \lfloor \frac{T-t}{\bar{G}} \rfloor + 2$

$$\hat{V}_g(\bar{G},t) = \sum_{j=0}^{\lfloor \frac{T-t}{\bar{G}} \rfloor + 1} e^{-(q+\bar{\rho})j\bar{G}}\, a(t+j\bar{G})$$

Finally, differentiating the o.d.e. for $\bar{V}$ and using smooth pasting, we get $-\bar{V}_{gg}(\bar{G}) = \bar{\pi}_g(\bar{G},\bar{Z})$, so that

$$\hat{G}(t) = \frac{1}{\bar{\pi}_g(\bar{G},\bar{z}Z)} \sum_{j=0}^{\lfloor \frac{T-t}{\bar{G}} \rfloor + 1} e^{-(q+\bar{\rho})j\bar{G}}\, a(t+j\bar{G})$$

or

$$\hat{G}(t) = \frac{a(t)}{\bar{\pi}_g(\bar{G},\bar{Z})} + e^{-(q+\bar{\rho})\bar{G}}\, \hat{G}(t+\bar{G})$$

with $\hat{G}(t) = 0$ for $t \geq T$.

This finishes the solution of $\hat{G}(t)$. ■

E Alternative Derivation of $\hat{G}(t)$

In this section we derive a integral equation in $\hat{G}(t)$. We start first with a representation of the first order condition for $G(t)$, and then we differentiate with respect to a path of $\hat{r}, \hat{P}, \hat{Z}$. The Bellman equation for a firm is:

$$\begin{aligned} V(g_0, t_0) = & \max_{\tau \geq t_0} \int_{t_0}^{\tau} e^{-\int_{t_0}^{s}[q+r(u)]du} \left[\pi(g_0 + s - t_0, s) + qV(0, s)\right] ds \\ & + e^{-\int_{t_0}^{\tau}[q+r(s)]ds} \left[V(0, \tau) - \kappa(\tau) P(\tau)\right]. \end{aligned} \tag{137}$$

for all g_0, t_0. Using the envelope theorem, the derivative of the value function with respect to time is

$$\begin{aligned} V_t(g_0, t_0) = & - \left[\pi(g_0, t_0) + qV(0, t_0)\right] \\ & - \int_{t_0}^{\tau^*(g_0, t_0)} e^{-\int_{t_0}^{s}[q+r(u)]du} \pi_g(g_0 + s - t_0, s) ds \\ & + \left[q + r(t_0)\right] V(g_0, t_0). \end{aligned} \tag{138}$$

for all g_0, t_0, where $\tau^*(t_0, g_0)$ is first time that a firm that at (g_0, t_0) decides to adopt the technology. The first order condition of the problem in (137) with respect to τ and evaluating it at $\tau = \tau^*(g_0, t_0)$ is given by

$$\begin{aligned} 0 = & \pi(g_0 + \tau - t_0, \tau) + qV(0, \tau) - (q + r(\tau)) \left[V(0, \tau) - \kappa(\tau) P(\tau)\right] \\ & + V_t(0, \tau) - \frac{d}{dt}\left[\kappa(\tau) P(\tau)\right]. \end{aligned}$$

Evaluating the first order condition at $g_0 + \tau^*(g_0, t_0) - t_0 = G(t)$ and $\tau^*(g_0, t_0) = t$

$$\begin{aligned} 0 = & \pi(G(t), t) + qV(0, t) - (q + r(t)) \left[V(0, t) - \kappa(t) P(t)\right] \\ & + V_t(0, t) - \frac{d}{dt}\left[\kappa(t) P(t)\right]. \end{aligned}$$

Evaluating $V_t(0, t_0)$ at $t = t_0$ and $g_0 = 0$ from (138), we obtain:

$$\begin{aligned} V_t(0, t) = & - \left[\pi(0, t) + qV(0, t)\right] - \int_{t}^{\tau^*(0, t)} e^{-\int_{t}^{s}[q+r(u)]du} \pi_g(s - t, s) ds \\ & + \left[q + r(t)\right] V(0, t). \end{aligned}$$

for all $t \geq 0$. In words, $G(t)$ is the value of the gap for the firm that adjust at time t. Substituting this expression into the first order condition we have

$$\begin{aligned} 0 =& \pi\left(G\left(t\right),t\right)+qV\left(0,t\right)-\left(q+r\left(t\right)\right)\left[V\left(0,t\right)-\kappa\left(t\right)P\left(t\right)\right] \\ & -\left[\pi\left(0,t\right)+qV\left(0,t\right)\right] \\ & -\int_{t}^{\tau^{*}\left(0,t\right)}e^{-\int_{t}^{s}\left[q+r\left(u\right)\right]du}\pi_{g}\left(s-t,s\right)ds \\ & +\left(q+r\left(t\right)\right)V\left(0,t\right)-\frac{d}{dt}\left[\kappa\left(t\right)P\left(t\right)\right]. \end{aligned}$$

Canceling terms

$$\begin{aligned} 0 =& \pi\left(G\left(t\right),t\right)-\pi\left(0,t\right)+\left(q+r\left(t\right)\right)\kappa\left(t\right)P\left(t\right) \\ & -\int_{t}^{\tau^{*}\left(0,t\right)}e^{-\int_{t}^{s}\left[q+r\left(u\right)\right]du}\pi_{g}\left(s-t,s\right)ds-\frac{d}{dt}\left[\kappa\left(t\right)P\left(t\right)\right], \end{aligned}$$

Including ϵ explicitly in the notation, $G(t,\epsilon)$ satisfies for all t:

$$\tau^{*}\left(0,t;\epsilon\right):G\left(\tau^{*}\left(0,t;\epsilon\right),\epsilon\right)=\tau^{*}\left(0,t;\epsilon\right)-t.$$

for given paths $Z(t,\varepsilon), P(t,\varepsilon)$ and $r(t,\varepsilon)$. Consider the optimal threshold $G\left(t,\varepsilon\right)$, and using $\bar{\pi}$ as the profits as function of g and Z, we can write

$$\begin{aligned} 0 =& \bar{\pi}\left(G\left(t,\varepsilon\right),Z\left(t,\varepsilon\right)\right)-\bar{\pi}\left(0,Z\left(t,\varepsilon\right)\right)+\left(q+r\left(t,\varepsilon\right)\right)\bar{\kappa}P\left(t,\varepsilon\right) \\ & -\int_{t}^{\tau^{*}\left(t,\varepsilon\right)}e^{-\int_{t}^{s}\left[q+r\left(u,\varepsilon\right)\right]du}\bar{\pi}_{g}\left(s-t,Z\left(s,\varepsilon\right)\right)ds \\ & -\bar{\kappa}\frac{d}{dt}P\left(t,\varepsilon\right) \end{aligned} \tag{139}$$

Differentiating the last equation with respect to ε

$$G_{t}\left(\tau^{*}\left(0,t,\varepsilon\right),\varepsilon\right)\tau_{\varepsilon}^{*}\left(0,t,\varepsilon\right)+G_{\varepsilon}\left(\tau^{*}\left(0,t,\varepsilon\right),\varepsilon\right)=\tau_{\varepsilon}^{*}\left(0,t,\varepsilon\right)$$

Evaluating at $\varepsilon=0$, using that $G_t(\tau,0)=0$, and using the $\hat{f}\equiv\frac{d}{d\epsilon}f(\epsilon)|_{\epsilon=0}$ notation, then

$$\hat{G}\left(t+\bar{G}\right)=\hat{\tau}^{*}\left(0,t\right).$$

Differentiating (139) with respect to ε and evaluating at $\varepsilon = 0$,

$$\begin{aligned}0 &= \bar{\pi}_g\left(\bar{G},\bar{Z}\right)\hat{G}\left(t\right)+\bar{\pi}_Z\left(\bar{G},\bar{Z}\right)\hat{Z}\left(t\right)-\bar{\pi}_Z\left(0,\bar{Z}\right)\hat{Z}\left(t\right)+\left(q+\bar{\rho}\right)\bar{\kappa}\hat{P}\left(t\right)+\bar{\kappa}\bar{P}\hat{r}\left(t\right)\\&+\int_t^{t+\bar{G}}\left[\left(\int_t^s\hat{r}\left(u\right)du\right)e^{-\int_t^s[q+\bar{\rho}]du}\bar{\pi}_g\left(s-t,\bar{Z}\right)-e^{-\int_t^s[q+\bar{\rho}]du}\pi_{gZ}\left(s-t,\bar{Z}\right)\hat{Z}\left(s\right)\right]ds\\&-e^{-[q+\bar{\rho}]\bar{G}}\bar{\pi}_g\left(\bar{G},\bar{Z}\right)\hat{G}\left(t+\bar{G}\right)-\bar{\kappa}\hat{P}_t\left(t\right)\end{aligned}$$

or

$$\hat{G}\left(t\right)=\frac{a\left(t\right)}{\bar{\pi}_g\left(\bar{G},\bar{Z}\right)}+e^{-[q+\bar{\rho}]\bar{G}}\hat{G}\left(t+\bar{G}\right)$$

where

$$\begin{aligned}\frac{a\left(t\right)}{\bar{\pi}_g\left(\bar{G},\bar{Z}\right)}=\frac{1}{\bar{\pi}_g\left(\bar{G},\bar{Z}\right)}\Big\{&\left[\bar{\pi}_Z\left(0,\bar{Z}\right)-\bar{\pi}_Z\left(\bar{G},\bar{Z}\right)\right]\hat{Z}\left(t\right)\\&+\int_t^{t+\bar{G}}e^{-[q+\bar{\rho}](s-t)}\left[\bar{\pi}_{gZ}\left(s-t,\bar{Z}\right)\,\hat{Z}\left(s\right)-\bar{\pi}_g\left(s-t,\bar{Z}\right)\,\left(\int_t^s\hat{r}\left(u\right)du\right)\right]ds\\&-\left(q+\bar{\rho}\right)\bar{\kappa}\hat{P}\left(t\right)-\bar{\kappa}\bar{P}\hat{r}\left(t\right)+\bar{\kappa}\hat{P}_t\left(t\right)\Big\}\end{aligned}$$

This completes the alternative derivation.

F Structure of the Linear System

Proposition 21 *Given a differentiable path $\hat{G}\equiv\{\hat{G}(t)\}_t\geq 0$, and the initial condition $\omega\equiv\{\omega(g)\}_{g\in[0,\bar{G}]}$, the linearization of feasibility and the law of motion of m is summarized by two linear functions $\hat{\mathcal{C}}$ and $\hat{\mathcal{Z}}$. For each $t\geq 0$, the linear function $\hat{\mathcal{C}}_t$ maps $\left(\{\hat{G}(s)\}_{s=0}^t,\{\hat{G}'(s)\}_{s=0}^t\right)$ and ω into $\hat{c}_t$, and the linear function $\hat{\mathcal{Z}}_t$ maps $\{\hat{G}(s)\}_{s=0}^t$ and $\omega(g)$ into $\hat{Z}(t)$, i.e. $\hat{c}(t)=\hat{\mathcal{C}}_t(\hat{G},\hat{G}',\omega)$ and $\hat{Z}(t)=\hat{\mathcal{Z}}_t(\hat{G},\omega)$ given by*

$$\frac{\hat{Z}(t)}{\bar{Z}}=\frac{1}{\eta-1}\frac{\hat{I}(t)}{\bar{I}}\;\text{ and }\;\hat{c}(t)=A\,\zeta Z^{\frac{1}{1-\nu}}\,\frac{\hat{I}(t)}{\bar{I}}-\kappa_0\hat{m}(0,t),\;\text{ for all }t\geq 0$$

where

$$\hat{m}(0,s)=-\,\bar{m}(\bar{G})\sum_{j=0}^{\lfloor\frac{s}{\bar{G}}\rfloor}e^{-qj\bar{G}}\left[\hat{G}'(s-j\bar{G})+q\hat{G}(s-j\bar{G})\right]+e^{-qs}\omega\left(\bar{G}+\lfloor\tfrac{s}{\bar{G}}\rfloor\bar{G}-s\right),\;\text{ for all }s\geq 0$$

and where

$$\hat{I}(t) = e^{-(q+\gamma(\eta-1))t}\left(\int_0^{\bar{G}} e^{-\gamma(\eta-1)g}\omega(g)dg\right) + \left(1 - e^{-\gamma(\eta-1)\bar{G}}\right)\int_0^t e^{-(q+\gamma(\eta-1))(t-s)}\hat{m}(0,s)ds$$
$$- qe^{-\gamma(\eta-1)\bar{G}}\bar{m}(\bar{G})\int_0^t e^{-(q+\gamma(\eta-1))(t-s)}\hat{G}(s)ds \text{ for all } t \geq 0$$

The expressions above define the coefficients of the linear functions. In the case of $\hat{\mathcal{Z}}_t$ it is given by the sum of two integrals of the paths of $\hat{G}$ and $\hat{G}'$ between 0 and t. Furthermore, in the case of $\hat{\mathcal{C}}_t$, the linear function also includes the sum of finitely many values of $\hat{G}$ and $\hat{G}'$, evaluated at points between 0 and t. The backward-looking nature of these two functions comes from the forward propagation of the law of motion of the density m.

Proposition 22 *Using the expression for Proposition 21, one can write an explicit expression for $\hat{Z}(t)$ as function of the path $\hat{G}(s)$ for $s \in [0,t]$ as:*

$$\begin{aligned}\frac{\hat{Z}(t)}{\bar{Z}} &= \frac{e^{-(q+\gamma(\eta-1))t}}{\bar{I}(\eta-1)}\left[\int_0^{\bar{G}} e^{-\gamma(\eta-1)g}\omega(g)dg + \left(1-e^{-\gamma(\eta-1)\bar{G}}\right)\int_0^t e^{\gamma(\eta-1)s}\omega\left(\bar{G}+\lfloor\tfrac{s}{\bar{G}}\rfloor\bar{G}-s\right)ds\right] \\ &- \frac{1}{\bar{I}(\eta-1)}\bar{m}(\bar{G})e^{-\gamma(\eta-1)\bar{G}}\int_0^t e^{-(q+\gamma(\eta-1))(t-s)}\, q\,\hat{G}(s)ds \\ &- \frac{\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)}{\bar{I}(\eta-1)}e^{-(q+\gamma(\eta-1))t}\,\bar{m}(\bar{G})\times \\ &\left[\sum_{j=0}^{\lfloor\frac{t}{\bar{G}}\rfloor-1}\int_{j\bar{G}}^{(j+1)\bar{G}}\sum_{k=0}^{j} e^{(q+\gamma(\eta-1))s-q\bar{G}k}\left[\hat{G}'(s-k\bar{G})+q\hat{G}(s-k\bar{G})\right]ds\right. \\ &\left.+\int_{\lfloor\frac{t}{\bar{G}}\rfloor\bar{G}}^{t}\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{(q+\gamma(\eta-1))s-q\bar{G}k}\left[\hat{G}'(s-k\bar{G})+q\hat{G}(s-k\bar{G})\right]ds\right]\end{aligned} \tag{140}$$

Solving it completely we obtain

$$\frac{\hat{Z}(t)}{\bar{Z}} = \frac{e^{-(q+\gamma(\eta-1))t}}{\bar{I}(\eta-1)} \left[\int_0^{\bar{G}} e^{-\gamma(\eta-1)g} \omega(g) dg + \left(1 - e^{-\gamma(\eta-1)\bar{G}}\right) \int_0^t e^{\gamma(\eta-1)s} \omega\left(\bar{G} + \lfloor \tfrac{s}{\bar{G}} \rfloor \bar{G} - s\right) ds \right]$$
$$- \frac{1}{\bar{I}(\eta-1)} \bar{m}(\bar{G}) q e^{-\gamma(\eta-1)\bar{G}} \int_0^t e^{-(q+\gamma(\eta-1))(t-s)} \hat{G}(s) ds$$
$$+ \frac{\gamma}{\bar{I}} \bar{m}(\bar{G}) \left(1 - e^{-\gamma(\eta-1)\bar{G}}\right) \sum_{k=0}^{\lfloor \frac{t}{\bar{G}} \rfloor} e^{-q\bar{G}k} \int_0^{t-k\bar{G}} e^{-(q+\gamma(\eta-1))(t-k\bar{G}-s)} \hat{G}(s) ds \tag{141}$$
$$+ \frac{1}{\bar{I}(\eta-1)} \bar{m}(\bar{G}) \left(1 - e^{-\gamma(\eta-1)\bar{G}}\right) \left(\sum_{k=0}^{\lfloor \frac{t}{\bar{G}} \rfloor} e^{-q\bar{G}k} e^{-(q+\gamma(\eta-1))(t-k\bar{G})} \hat{G}(0) - \sum_{k=0}^{\lfloor \frac{t}{\bar{G}} \rfloor} e^{-q\bar{G}k} \hat{G}\left(t - k\bar{G}\right) \right)$$

As a corollary of this proposition, using the expression (140) and fixing $\hat{G}(0)$ we obtain the following comparative statics on the path of Z. Let $\{\hat{G}_r'(t)\}_{t=0}^T$ for $r = a, b$ be two paths with $\hat{G}_a'(t) \geq \hat{G}_b'(t)$ and $\hat{G}_a(0) = \hat{G}_b(0)$, then the corresponding paths for Z satisfy $Z_a(t) \geq \hat{Z}_b(t)$ for all $t \in [0, T]$.

If we consider the case of $q = 0$, the expression (140) gives $Z(t)$ as a function of $\hat{G}(0)$ and integrals of the path of $\hat{G}$'s. Regardless of the value of q, the expression in (141), is obtained by explicitly solving these integrals. The expression in (141) is useful for setting the integral equation whose solution is the equilibrium fixed point.

Proof. (of Proposition 22) We can replace the expression for $\hat{m}(0, s)$ into the one for $\hat{Z}(t)$ to obtain:

$$\hat{Z}(t) = \frac{\bar{Z}}{\bar{I}} \frac{1}{\eta-1} \left[e^{-(q+\gamma(\eta-1))t} \left(\int_0^{\bar{G}} e^{-\gamma(\eta-1)g} \omega(g) dg \right) \right.$$
$$\left. + \left(1 - e^{-\gamma(\eta-1)\bar{G}}\right) \int_0^t e^{-(q+\gamma(\eta-1))(t-s)} e^{-qs} \omega\left(\bar{G} + \lfloor \tfrac{s}{\bar{G}} \rfloor \bar{G} - s\right) ds \right]$$
$$- \frac{\bar{Z}}{\bar{I}} \frac{1}{\eta-1} q e^{-\gamma(\eta-1)\bar{G}} \bar{m}(\bar{G}) \int_0^t e^{-(q+\gamma(\eta-1))(t-s)} \hat{G}(s) ds$$
$$- \frac{\bar{Z}}{\bar{I}} \frac{1}{\eta-1} \left(1 - e^{-\gamma(\eta-1)\bar{G}}\right) e^{-(q+\gamma(\eta-1))t} \bar{m}(\bar{G}) \times \mathcal{H}(t; \hat{G})$$

$$\frac{\hat{Z}(t)}{\bar{Z}} = \frac{1}{\bar{I}}\frac{e^{-(q+\gamma(\eta-1))t}}{\eta-1}\left[\left(\left(\int_0^{\bar{G}} e^{-\gamma(\eta-1)g}\omega(g)dg\right)\right.\right.$$

$$+\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)\int_0^t e^{\gamma(\eta-1)s}\omega\left(\bar{G}+\lfloor\tfrac{s}{\bar{G}}\rfloor\bar{G}-s\right)ds$$

$$-\frac{1}{\eta-1}qe^{-\gamma(\eta-1)\bar{G}}\bar{m}(\bar{G})\int_0^t e^{-(q+\gamma(\eta-1))(t-s)}\hat{G}(s)ds$$

$$\left.-\frac{1}{\eta-1}\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)e^{-(q+\gamma(\eta-1))t}\,\bar{m}(\bar{G})\times\mathcal{H}(t;\hat{G})\right]$$

where

$$\mathcal{H}(t;\hat{G})$$

$$=\int_0^t e^{(q+\gamma(\eta-1))s}\sum_{k=0}^{\lfloor\frac{s}{\bar{G}}\rfloor}e^{-qk\bar{G}}\left[\hat{G}'(s-k\bar{G})+q\hat{G}(s-k\bar{G})\right]ds$$

$$=\int_0^t \sum_{k=0}^{\lfloor\frac{s}{\bar{G}}\rfloor}e^{(q+\gamma(\eta-1))s}e^{-qk\bar{G}}\left[\hat{G}'(s-k\bar{G})+q\hat{G}(s-k\bar{G})\right]ds$$

$$=\sum_{j=0}^{\lfloor\frac{t}{\bar{G}}\rfloor-1}\int_{j\bar{G}}^{(j+1)\bar{G}}\sum_{k=0}^{\lfloor\frac{s}{\bar{G}}\rfloor}e^{(q+\gamma(\eta-1))s}e^{-qk\bar{G}}\left[\hat{G}'(s-k\bar{G})+q\hat{G}(s-k\bar{G})\right]ds$$

$$+\int_{\lfloor\frac{t}{\bar{G}}\rfloor\bar{G}}^{t}\sum_{k=0}^{\lfloor\frac{s}{\bar{G}}\rfloor}e^{(q+\gamma(\eta-1))s}e^{-qk\bar{G}}\left[\hat{G}'(s-k\bar{G})+q\hat{G}(s-k\bar{G})\right]ds$$

$$=\sum_{j=0}^{\lfloor\frac{t}{\bar{G}}\rfloor-1}\int_{j\bar{G}}^{(j+1)\bar{G}}\sum_{k=0}^{j}e^{(q+\gamma(\eta-1))s-q\bar{G}k}\left[\hat{G}'(s-k\bar{G})+q\hat{G}(s-k\bar{G})\right]ds$$

$$+\int_{\lfloor\frac{t}{\bar{G}}\rfloor\bar{G}}^{t}\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor}e^{(q+\gamma(\eta-1))s-q\bar{G}k}\left[\hat{G}'(s-k\bar{G})+q\hat{G}(s-k\bar{G})\right]ds$$

Replacing this expression we obtain (140). Integrating by parts

$$\mathcal{H}(t;\hat{G})$$

$$=\sum_{j=0}^{\lfloor\frac{t}{\bar{G}}\rfloor-1}\sum_{k=0}^{j}e^{-q\bar{G}k}\left[\int_{j\bar{G}}^{(j+1)\bar{G}}e^{(q+\gamma(\eta-1))s}\hat{G}'(s-k\bar{G})ds+q\int_{j\bar{G}}^{(j+1)\bar{G}}e^{q+\gamma(\eta-1))s}\hat{G}(s-k\bar{G})ds\right]$$

$$+\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor}e^{-q\bar{G}k}\left[\int_{\lfloor\frac{t}{\bar{G}}\rfloor\bar{G}}^{t}e^{(q+\gamma(\eta-1))s}\hat{G}'(s-k\bar{G})ds+q\int_{\lfloor\frac{t}{\bar{G}}\rfloor\bar{G}}^{t}e^{(q+\gamma(\eta-1))s}\hat{G}(s-k\bar{G})ds\right]$$

$$
\begin{aligned}
&\mathcal{H}(t;\hat{G})\\
&= \sum_{j=0}^{\lfloor\frac{t}{\bar{G}}\rfloor-1}\sum_{k=0}^{j} e^{-q\bar{G}k}\left[-(q+\gamma(\eta-1))\int_{j\bar{G}}^{(j+1)\bar{G}} e^{(q+\gamma(\eta-1))s}\hat{G}(s-k\bar{G})ds + q\int_{j\bar{G}}^{(j+1)\bar{G}} e^{q+\gamma(\eta-1))s}\hat{G}(s-k\bar{G})ds\right]\\
&+ \sum_{j=0}^{\lfloor\frac{t}{\bar{G}}\rfloor-1}\sum_{k=0}^{j} e^{-q\bar{G}k}\left[e^{(q+\gamma(\eta-1))(j+1)\bar{G}}\,\hat{G}\left((j+1-k)\bar{G}\right) - e^{(q+\gamma(\eta-1))j\bar{G}}\,\hat{G}\left((j-k)\bar{G}\right)\right]\\
&+ \sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}\left[-(q+\gamma(\eta-1)\int_{\lfloor\frac{t}{\bar{G}}\rfloor\bar{G}}^{t} e^{q+\gamma(\eta-1))s}\hat{G}(s-k\bar{G})ds + q\int_{\lfloor\frac{t}{\bar{G}}\rfloor\bar{G}}^{t} e^{q+\gamma(\eta-1))s}\hat{G}(s-k\bar{G})ds\right]\\
&+ \sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}\left[e^{(q+\gamma(\eta-1))t}\,\hat{G}\left(t-k\bar{G}\right) - e^{(q+\gamma(\eta-1))\lfloor\frac{t}{\bar{G}}\rfloor\bar{G}}\,\hat{G}\left(\lfloor\tfrac{t}{\bar{G}}\rfloor\bar{G}-k\bar{G}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{H}(t;\hat{G})\\
&= -\gamma(\eta-1)\sum_{j=0}^{\lfloor\frac{t}{\bar{G}}\rfloor-1}\sum_{k=0}^{j} e^{-q\bar{G}k}\int_{j\bar{G}}^{(j+1)\bar{G}} e^{(q+\gamma(\eta-1))s}\hat{G}(s-k\bar{G})ds\\
&+ \sum_{j=0}^{\lfloor\frac{t}{\bar{G}}\rfloor-1}\sum_{k=0}^{j} e^{-q\bar{G}k}\left[e^{(q+\gamma(\eta-1))(j+1)\bar{G}}\,\hat{G}\left((j+1-k)\bar{G}\right) - e^{(q+\gamma(\eta-1))j\bar{G}}\,\hat{G}\left((j-k)\bar{G}\right)\right]\\
&- \gamma(\eta-1)\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}\int_{\lfloor\frac{t}{\bar{G}}\rfloor\bar{G}}^{t} e^{q+\gamma(\eta-1))s}\hat{G}(s-k\bar{G})ds\\
&+ \sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}\left[e^{(q+\gamma(\eta-1))t}\,\hat{G}\left(t-k\bar{G}\right) - e^{(q+\gamma(\eta-1))\lfloor\frac{t}{\bar{G}}\rfloor\bar{G}}\,\hat{G}\left(\lfloor\tfrac{t}{\bar{G}}\rfloor\bar{G}-k\bar{G}\right)\right]
\end{aligned}
$$

or

$$
\begin{aligned}
&\mathcal{H}(t;\hat{G}) \\
&= -\gamma(\eta-1)\sum_{j=0}^{\lfloor \frac{t}{\bar{G}}\rfloor -1}\sum_{k=0}^{j} e^{-q\bar{G}k}\int_{j\bar{G}}^{(j+1)\bar{G}} e^{(q+\gamma(\eta-1))s}\hat{G}(s-k\bar{G})ds \\
&+ \sum_{k=0}^{\lfloor \frac{t}{\bar{G}}\rfloor -1}\sum_{j=k}^{\lfloor \frac{t}{\bar{G}}\rfloor -1} e^{-q\bar{G}k}\left[e^{(q+\gamma(\eta-1))(j+1)\bar{G}}\,\hat{G}\left((j+1-k)\bar{G}\right) - e^{(q+\gamma(\eta-1))j\bar{G}}\,\hat{G}\left((j-k)\bar{G}\right)\right] \\
&- \gamma(\eta-1)\sum_{k=0}^{\lfloor \frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}\int_{\lfloor \frac{t}{\bar{G}}\rfloor\bar{G}}^{t} e^{q+\gamma(\eta-1))s}\hat{G}(s-k\bar{G})ds \\
&+ \sum_{k=0}^{\lfloor \frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}\left[e^{(q+\gamma(\eta-1))t}\,\hat{G}\left(t-k\bar{G}\right) - e^{(q+\gamma(\eta-1))\lfloor \frac{t}{\bar{G}}\rfloor\bar{G}}\,\hat{G}\left(\lfloor \tfrac{t}{\bar{G}}\rfloor\bar{G}-k\bar{G}\right)\right]
\end{aligned}
$$

or

$$
\begin{aligned}
&\mathcal{H}(t;\hat{G}) \\
&= -\gamma(\eta-1)\sum_{j=0}^{\lfloor \frac{t}{\bar{G}}\rfloor -1}\sum_{k=0}^{j} e^{-q\bar{G}k}\int_{j\bar{G}}^{(j+1)\bar{G}} e^{(q+\gamma(\eta-1))s}\hat{G}(s-k\bar{G})ds \\
&+ \sum_{k=0}^{\lfloor \frac{t}{\bar{G}}\rfloor -1} e^{-q\bar{G}k}\left[e^{(q+\gamma(\eta-1))\lfloor \frac{t}{\bar{G}}\rfloor\bar{G}}\,\hat{G}\left((\lfloor \tfrac{t}{\bar{G}}\rfloor-k)\bar{G}\right) - e^{(q+\gamma(\eta-1))k\bar{G}}\,\hat{G}\left(0\right)\right] \\
&- \gamma(\eta-1)\sum_{k=0}^{\lfloor \frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}\int_{\lfloor \frac{t}{\bar{G}}\rfloor\bar{G}}^{t} e^{q+\gamma(\eta-1))s}\hat{G}(s-k\bar{G})ds \\
&+ \sum_{k=0}^{\lfloor \frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}\left[e^{(q+\gamma(\eta-1))t}\,\hat{G}\left(t-k\bar{G}\right) - e^{(q+\gamma(\eta-1))\lfloor \frac{t}{\bar{G}}\rfloor\bar{G}}\,\hat{G}\left(\lfloor \tfrac{t}{\bar{G}}\rfloor\bar{G}-k\bar{G}\right)\right]
\end{aligned}
$$

pr

$$
\begin{aligned}
&\mathcal{H}(t;\hat{G})\\
&= -\gamma(\eta-1)\sum_{j=0}^{\lfloor\frac{t}{\bar{G}}\rfloor-1}\sum_{k=0}^{j} e^{-q\bar{G}k}\int_{j\bar{G}}^{(j+1)\bar{G}} e^{(q+\gamma(\eta-1))s}\hat{G}(s-k\bar{G})ds\\
&-\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}e^{(q+\gamma(\eta-1))k\bar{G}}\,\hat{G}\left(0\right)\\
&-\gamma(\eta-1)\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}\int_{\lfloor\frac{t}{\bar{G}}\rfloor\bar{G}}^{t} e^{q+\gamma(\eta-1))s}\hat{G}(s-k\bar{G})ds\\
&+\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}e^{(q+\gamma(\eta-1))t}\,\hat{G}\left(t-k\bar{G}\right)
\end{aligned}
$$

or

$$
\begin{aligned}
&\mathcal{H}(t;\hat{G})\\
&= -\gamma(\eta-1)\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor-1} e^{-q\bar{G}k}\sum_{j=k}^{\lfloor\frac{t}{\bar{G}}\rfloor-1}\int_{j\bar{G}}^{(j+1)\bar{G}} e^{(q+\gamma(\eta-1))s}\hat{G}(s-k\bar{G})ds\\
&-\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}e^{(q+\gamma(\eta-1))k\bar{G}}\,\hat{G}\left(0\right)\\
&-\gamma(\eta-1)\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}\int_{\lfloor\frac{t}{\bar{G}}\rfloor\bar{G}}^{t} e^{q+\gamma(\eta-1))s}\hat{G}(s-k\bar{G})ds\\
&+\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}e^{(q+\gamma(\eta-1))t}\,\hat{G}\left(t-k\bar{G}\right)
\end{aligned}
$$

or

$$\mathcal{H}(t;\hat{G}) = -\gamma(\eta-1)\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor-1} e^{-q\bar{G}k}\int_{k\bar{G}}^{\lfloor\frac{t}{\bar{G}}\rfloor\bar{G}} e^{(q+\gamma(\eta-1))s}\hat{G}(s-k\bar{G})ds$$
$$-\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}e^{(q+\gamma(\eta-1))k\bar{G}}\,\hat{G}\,(0)$$
$$-\gamma(\eta-1)\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}\int_{\lfloor\frac{t}{\bar{G}}\rfloor\bar{G}}^{t} e^{q+\gamma(\eta-1))s}\hat{G}(s-k\bar{G})ds$$
$$+\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}e^{(q+\gamma(\eta-1))t}\,\hat{G}\left(t-k\bar{G}\right)$$

or

$$\mathcal{H}(t;\hat{G}) = -\gamma(\eta-1)\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor-1} e^{-q\bar{G}k}\int_{k\bar{G}}^{t} e^{(q+\gamma(\eta-1))s}\hat{G}(s-k\bar{G})ds$$
$$-\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}e^{(q+\gamma(\eta-1))k\bar{G}}\,\hat{G}\,(0)$$
$$-\gamma(\eta-1)e^{-q\bar{G}\lfloor\frac{t}{\bar{G}}\rfloor}\int_{\lfloor\frac{t}{\bar{G}}\rfloor\bar{G}}^{t} e^{q+\gamma(\eta-1))s}\hat{G}(s-\lfloor\tfrac{t}{\bar{G}}\rfloor\bar{G})ds$$
$$+\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}e^{(q+\gamma(\eta-1))t}\,\hat{G}\left(t-k\bar{G}\right)$$

or

$$\mathcal{H}(t;\hat{G}) = -\gamma(\eta-1)\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}\int_{k\bar{G}}^{t} e^{(q+\gamma(\eta-1))s}\hat{G}(s-k\bar{G})ds$$
$$-\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}e^{(q+\gamma(\eta-1))k\bar{G}}\,\hat{G}\,(0)+\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}e^{(q+\gamma(\eta-1))t}\,\hat{G}\left(t-k\bar{G}\right)$$

Replacing $\mathcal{H}$ into the expression for $\hat{Z}(t)$ we get:

$$\frac{\hat{Z}(t)}{\bar{Z}} = \frac{e^{-(q+\gamma(\eta-1))t}}{\bar{I}(\eta-1)}\left[\int_0^{\bar{G}} e^{-\gamma(\eta-1)g}\omega(g)dg + \left(1-e^{-\gamma(\eta-1)\bar{G}}\right)\int_0^t e^{\gamma(\eta-1)s}\omega\left(\bar{G}+\lfloor\tfrac{s}{\bar{G}}\rfloor\bar{G}-s\right)ds\right]$$
$$-\frac{1}{\bar{I}(\eta-1)}\bar{m}(\bar{G})qe^{-\gamma(\eta-1)\bar{G}}\int_0^t e^{-(q+\gamma(\eta-1))(t-s)}\hat{G}(s)ds$$
$$+\frac{\gamma}{\bar{I}}\bar{m}(\bar{G})\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}\int_{k\bar{G}}^t e^{-(q+\gamma(\eta-1))(t-s)}\hat{G}(s-k\bar{G})ds$$
$$+\frac{1}{\bar{I}(\eta-1)}\bar{m}(\bar{G})\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)\left(\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}e^{-(q+\gamma(\eta-1))(t-k\bar{G})}\hat{G}(0)-\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}\hat{G}\left(t-k\bar{G}\right)\right)$$

We use change of variables for each k. $s' = s - k\bar{G}$. Using this change of variables, the original lower bound $s = k\bar{G}$, becomes $s' = 0$, and the original upper bound $s = t$ becomes $s' = t - k\bar{G}$. Thus

$$\int_{k\bar{G}}^t e^{-(q+\gamma(\eta-1))(t-s)}\hat{G}(s-k\bar{G})ds$$
$$=\int_0^{t-k\bar{G}} e^{-(q+\gamma(\eta-1))(t-k\bar{G}-s')}\hat{G}(s')ds'$$

We can thus write:

$$\frac{\hat{Z}(t)}{\bar{Z}} = \frac{e^{-(q+\gamma(\eta-1))t}}{\bar{I}(\eta-1)}\left[\int_0^{\bar{G}} e^{-\gamma(\eta-1)g}\omega(g)dg + \left(1-e^{-\gamma(\eta-1)\bar{G}}\right)\int_0^t e^{\gamma(\eta-1)s}\omega\left(\bar{G}+\lfloor\tfrac{s}{\bar{G}}\rfloor\bar{G}-s\right)ds\right]$$
$$-\frac{1}{\bar{I}(\eta-1)}\bar{m}(\bar{G})qe^{-\gamma(\eta-1)\bar{G}}\int_0^t e^{-(q+\gamma(\eta-1))(t-s)}\hat{G}(s)ds$$
$$+\frac{\gamma}{\bar{I}}\bar{m}(\bar{G})\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}\int_0^{t-k\bar{G}} e^{-(q+\gamma(\eta-1))(t-k\bar{G}-s')}\hat{G}(s')ds'$$
$$+\frac{1}{\bar{I}(\eta-1)}\bar{m}(\bar{G})\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)\left(\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}e^{-(q+\gamma(\eta-1))(t-k\bar{G})}\hat{G}(0)-\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}\hat{G}\left(t-k\bar{G}\right)\right)$$

■

Proof. (of Proposition 21). Define

$$I(t,\epsilon) \equiv \int_0^{G(t,\epsilon)} e^{-\gamma(\eta-1)g}m(g,t,\epsilon)dg$$

Then

$$\frac{d}{d\epsilon}I(t,\epsilon)|_{\epsilon=0} \equiv \hat{I}(t) = \int_0^{\bar{G}} e^{-\gamma(\eta-1)g}\hat{m}(g,t)dg + e^{-\gamma(\eta-1)\bar{G}}\bar{m}(\bar{G})\hat{G}(t)$$

We can now differentiate this with respect to time:

$$\begin{aligned}
\hat{I}'(t) &= \int_0^{\bar{G}} e^{-\gamma(\eta-1)g}\hat{m}_t(g,t)dg + e^{-\gamma(\eta-1)\bar{G}}\bar{m}(\bar{G})\hat{G}'(t) \\
&= \int_0^{\bar{G}} e^{-\gamma(\eta-1)g}\left(-q\hat{m}(g,t) - \hat{m}_g(g,t)\right)dg + e^{-\gamma(\eta-1)\bar{G}}\bar{m}(\bar{G})\hat{G}'(t) \\
&= -q\int_0^{\bar{G}} e^{-\gamma(\eta-1)g}\hat{m}(g,t)dg - \gamma(\eta-1)\int_0^{\bar{G}} e^{-\gamma(\eta-1)g}\hat{m}(g,t)dg \\
&- e^{-\gamma(\eta-1)\bar{G}}\hat{m}(\bar{G},t) + \hat{m}(0,t) + e^{-\gamma(\eta-1)\bar{G}}\bar{m}(\bar{G})\hat{G}'(t) \\
&= -\left[q+\gamma(\eta-1)\right]\int_0^{\bar{G}} e^{-\gamma(\eta-1)g}\hat{m}(g,t)dg + \hat{m}(0,t) - e^{-\gamma(\eta-1)\bar{G}}\hat{m}(\bar{G},t) + e^{-\gamma(\eta-1)\bar{G}}\bar{m}(\bar{G})\hat{G}'(t) \\
&= -\left[q+\gamma(\eta-1)\right]\left(\hat{I}(t) - e^{-\gamma(\eta-1)\bar{G}}\bar{m}(\bar{G})\hat{G}(t)\right) + \\
&+ \hat{m}(0,t) - e^{-\gamma(\eta-1)\bar{G}}\hat{m}(\bar{G},t) + e^{-\gamma(\eta-1)\bar{G}}\bar{m}(\bar{G})\hat{G}'(t)
\end{aligned}$$

We can use the boundary conditions for $\hat{m}$:

$$\hat{m}(\bar{G},t) = \hat{m}(0,t) + \bar{m}(\bar{G})\hat{G}'(t) + q\bar{m}(\bar{G})\hat{G}(t)$$

So we get:

$$\hat{I}'(t) = -\left[q+\gamma(\eta-1)\right]\hat{I}(t) + \hat{m}(0,t)\left(1 - e^{-\gamma(\eta-1)\bar{G}}\right) - qe^{-\gamma(\eta-1)\bar{G}}\bar{m}(\bar{G})\hat{G}(t)$$

We can write this simply as:

$$\hat{I}'(t) = -a\hat{I}(t) + b(t)$$

So its solution is:

$$\begin{aligned}
&\hat{I}(t) = e^{-at}\hat{I}(0) + \int_0^t e^{-a(t-s)}b(s)ds \text{ where } a \equiv q + \gamma(\eta-1) \text{ and} \\
&b(t) \equiv \hat{m}(0,t)\left(1 - e^{-\gamma(\eta-1)\bar{G}}\right) - qe^{-\gamma(\eta-1)\bar{G}}\bar{m}(\bar{G})\hat{G}(t)
\end{aligned}$$

Define $\hat{Z}, \hat{P}$ and $\hat{c}$ in terms of $\hat{I}(t)$ and $\hat{m}(0,t)$

$$Z(t,\epsilon) = \bar{Z}\left(\frac{I(t,\epsilon)}{\bar{I}}\right)^{1/(\eta-1)}, \text{ thus } \hat{Z}(t) = \bar{Z}\frac{1}{\eta-1}\frac{\hat{I}(t)}{\bar{I}},$$

$$c(t,\epsilon) = AZ(t,\epsilon))^{\frac{1}{1-\nu}} - \kappa_0(m(0,t;\epsilon)-q) \implies \hat{c}(t) = A\frac{1}{1-\nu}\bar{Z}^{\frac{1}{1-\nu}}\frac{\hat{Z}(t)}{\bar{Z}} - \kappa_0\hat{m}(0,t), \text{ thus}$$

$$\hat{c}(t) = A\zeta\bar{Z}^{\frac{1}{1-\nu}}\frac{\hat{I}(t)}{\bar{I}} - \kappa_0\hat{m}(0,t)$$

Finally, the solution for $\hat{m}(0,t)$ is given in Proposition 19. ■

Proposition 23 *Given a differentiable path $\hat{c} \equiv \{\hat{c}(t)\}_{t\geq 0}$, and one for $\hat{Z} \equiv \{\hat{Z}(t)\}_{t\geq 0}$, the linearization of the optimality conditions for the firms and for the households are summarized by a linear functions $\hat{\mathcal{G}}$. For each $t \geq 0$, the linear function $\hat{\mathcal{G}}_t$ maps $\left(\{\hat{Z}(s)\}_{s=t}^{\infty}, \{\hat{c}(s)\}_{s=t}^{\infty}, \{\hat{c}'(s)\}_{s=t}^{\infty},\right)$ into $\hat{G}_t$. The linear function $\hat{G}(t) = \hat{\mathcal{G}}_t\left(\hat{Z}, \hat{c}, \hat{c}'\right)$ is given by*

$$\hat{G}\left(t\right) = \sum_{j=0}^{\infty} e^{-j\,[q+\bar{\rho}]\bar{G}}\,\frac{a\left(t+j\bar{G}\right)}{\bar{\pi}_g\left(\bar{G},\bar{Z}\right)}$$

where:

$$\begin{aligned}\frac{a\left(t\right)}{\bar{\pi}_g\left(\bar{G},\bar{Z}\right)} = \left(\eta-1\right)\left(1-\zeta\right)&\left\{\left(\frac{e^{\gamma(\eta-1)\bar{G}}-1}{\gamma(\eta-1)}\right)\frac{\hat{Z}\left(t\right)}{\bar{Z}} - e^{\gamma(\eta-1)\bar{G}}\int_t^{t+\bar{G}} e^{-[q+\bar{\rho}+\gamma(\eta-1)](s-t)}\,\frac{\hat{Z}\left(s\right)}{\bar{Z}}ds\right\}\\ &+\theta\left\{\frac{e^{\gamma(\eta-1)\bar{G}}-e^{-[\bar{\rho}+q]\bar{G}}}{q+\bar{\rho}+\gamma(\eta-1)}\frac{\hat{c}(t)}{\bar{c}} - e^{\gamma(\eta-1)\bar{G}}\int_t^{t+\bar{G}} e^{-[q+\bar{\rho}+\gamma(\eta-1)](s-t)}\,\frac{\hat{c}(s)}{\bar{c}}ds\right.\\ &\left.\quad - \frac{\bar{\kappa}\bar{P}}{\bar{\pi}_g\left(\bar{G},\bar{Z}\right)}\frac{\hat{c}'(t)}{\bar{c}}\right\}\end{aligned}$$

Proof. (of Proposition 23). The proof takes the expression for $\hat{G}$ given in (130) from Proposition 20, and iterates forward to obtain the first equation. The remaining part of the proof rewrites the expression for $a(t)$ so that it only depends on $\hat{Z}, \hat{c}$ and $\hat{c}'$.

Using that $\hat{r}(t) = \frac{\hat{P}'(t)}{\bar{P}} + \theta \frac{\hat{c}'(t)}{\bar{c}}$ to rewrite the expression for $a(t)$ we get:

$$
\begin{aligned}
\frac{a(t)}{\bar{\pi}_g\left(\bar{G},\bar{Z}\right)} = \frac{1}{\bar{\pi}_g\left(\bar{G},\bar{Z}\right)} \Big\{ & \left[\bar{\pi}_Z\left(0,\bar{Z}\right) - \bar{\pi}_Z\left(\bar{G},\bar{Z}\right)\right]\hat{Z}(t) \\
& + \int_t^{t+\bar{G}} e^{-[q+\bar{\rho}](s-t)} \left[\bar{\pi}_{gZ}\left(s-t,\bar{Z}\right)\, \hat{Z}(s) - \bar{\pi}_g\left(s-t,\bar{Z}\right)\, \left(\frac{\hat{P}(s)-\hat{P}(t)}{\bar{P}}\right)\right] ds \\
& -\theta \int_t^{t+\bar{G}} e^{-[q+\bar{\rho}](s-t)} \bar{\pi}_g\left(s-t,\bar{Z}\right)\, \left(\frac{\hat{c}(s)-\hat{c}(t)}{\bar{c}}\right) ds \\
& - (q+\bar{\rho})\,\bar{\kappa}\hat{P}(t) - \theta\bar{\kappa}\bar{P}\frac{\hat{c}'(t)}{\bar{c}} \Big\}
\end{aligned}
$$

Using that :

$$
P(t,\epsilon) = \frac{\bar{P}}{\bar{Z}^{-\frac{1}{1-\nu}}} Z(t,\epsilon)^{-\frac{1}{1-\nu}} \ , \ \hat{P}(t) = -\frac{\bar{P}}{1-\nu}\frac{\hat{Z}(t)}{\bar{Z}}, \text{ and } \hat{P}'(t) = -\frac{\bar{P}}{1-\nu}\frac{\hat{Z}'(t)}{\bar{Z}}
$$

we obtain:

$$
\begin{aligned}
\frac{a(t)}{\bar{\pi}_g\left(\bar{G},\bar{Z}\right)} = \frac{1}{\bar{\pi}_g\left(\bar{G},\bar{Z}\right)} \Big\{ & \left[\bar{\pi}_Z\left(0,\bar{Z}\right) - \bar{\pi}_Z\left(\bar{G},\bar{Z}\right)\right]\hat{Z}(t) \\
& + \int_t^{t+\bar{G}} e^{-[q+\bar{\rho}](s-t)} \left[\bar{\pi}_{gZ}\left(s-t,\bar{Z}\right)\, \hat{Z}(s) + \bar{\pi}_g\left(s-t,\bar{Z}\right)\, \frac{1}{1-\nu} \left(\frac{\hat{Z}(s)-\hat{Z}(t)}{\bar{Z}}\right)\right] ds \\
& -\theta \int_t^{t+\bar{G}} e^{-[q+\bar{\rho}](s-t)} \bar{\pi}_g\left(s-t,\bar{Z}\right)\, \left(\frac{\hat{c}(s)-\hat{c}(t)}{\bar{c}}\right) ds \\
& + (q+\bar{\rho})\,\bar{\kappa}\bar{P}\frac{1}{1-\nu}\frac{\hat{Z}(t)}{\bar{Z}} - \theta\bar{\kappa}\bar{P}\frac{\hat{c}'(t)}{\bar{c}} \Big\}
\end{aligned}
$$

Now we use that $\bar{\pi}$ satisfies:

$$
\begin{aligned}
\bar{\pi}(g,\bar{Z}) &= \frac{N}{1-\nu}\ \frac{1}{\eta-1}\ \frac{e^{-\gamma g(\eta-1)}}{\bar{Z}^{\eta-1}} \ , \ \bar{\pi}_g(g,Z) = -\gamma(\eta-1)\bar{\pi}(g,\bar{Z}) \\
\bar{\pi}_{gZ}(g,\bar{Z}) &= \gamma(\eta-1)^2\frac{\bar{\pi}(g,\bar{Z})}{\bar{Z}} = -\frac{(\eta-1)}{\bar{Z}}\bar{\pi}_g(g,\bar{Z}) \\
\bar{\pi}(0,\bar{Z}) - \bar{\pi}(\bar{G},\bar{Z}) &= \frac{N}{1-\nu}\ \frac{1}{\eta-1}\ \frac{1-e^{-\gamma\bar{G}(\eta-1)}}{\bar{Z}^{\eta-1}} \\
\bar{\pi}_Z(0,\bar{Z}) - \bar{\pi}_Z(\bar{G},\bar{Z}) &= -\frac{(\eta-1)}{\bar{Z}}\left(\bar{\pi}(0,\bar{Z}) - \bar{\pi}(\bar{G},\bar{Z})\right) = \frac{1}{\gamma\bar{Z}}\left(\bar{\pi}_g(0,\bar{Z}) - \bar{\pi}_g(\bar{G},\bar{Z})\right)
\end{aligned}
$$

So that:

$$\frac{a(t)}{\bar{\pi}_g(\bar{G},\bar{Z})}$$

$$= \frac{1}{\bar{\pi}_g(\bar{G},\bar{Z})}\left\{\left[\bar{\pi}_Z(0,\bar{Z}) - \bar{\pi}_Z(\bar{G},\bar{Z})\right]\hat{Z}(t)\right.$$
$$+(\eta-1)\int_t^{t+\bar{G}} e^{-[q+\bar{\rho}](s-t)}\left[-\bar{\pi}_g(s-t,\bar{Z})\,\frac{\hat{Z}(s)}{\bar{Z}} + \bar{\pi}_g(s-t,\bar{Z})\,\frac{1}{(1-\nu)(\eta-1)}\left(\frac{\hat{Z}(s)-\hat{Z}(t)}{\bar{Z}}\right)\right]ds$$
$$-\theta\int_t^{t+\bar{G}} e^{-[q+\bar{\rho}](s-t)}\bar{\pi}_g(s-t,\bar{Z})\left(\frac{\hat{c}(s)-\hat{c}(t)}{\bar{c}}\right)ds$$
$$\left.+(q+\bar{\rho})\,\bar{\kappa}\bar{P}\frac{1}{1-\nu}\frac{\hat{Z}(t)}{\bar{Z}} - \theta\bar{\kappa}\bar{P}\frac{\hat{c}'(t)}{\bar{c}}\right\}$$

or

$$\frac{a(t)}{\bar{\pi}_g(\bar{G},\bar{Z})} = \frac{1}{\bar{\pi}_g(\bar{G},\bar{Z})}\left\{-(\eta-1)\left(\bar{\pi}(0,\bar{Z}) - \bar{\pi}(\bar{G},Z)\right)\frac{\hat{Z}(t)}{\bar{Z}} + (q+\bar{\rho})\,\bar{\kappa}\bar{P}\frac{1}{1-\nu}\frac{\hat{Z}(t)}{\bar{Z}}\right.$$
$$+(\eta-1)\int_t^{t+\bar{G}} e^{-[q+\bar{\rho}](s-t)}\left[-\bar{\pi}_g(s-t,\bar{Z})\,[1-\zeta]\,\frac{\hat{Z}(s)}{\bar{Z}} - \bar{\pi}_g(s-t,\bar{Z})\,\zeta\,\frac{\hat{Z}(t)}{\bar{Z}}\right]ds$$
$$\left.-\theta\int_t^{t+\bar{G}} e^{-[q+\bar{\rho}](s-t)}\bar{\pi}_g(s-t,\bar{Z})\left(\frac{\hat{c}(s)-\hat{c}(t)}{\bar{c}}\right)ds - \theta\bar{\kappa}\bar{P}\frac{\hat{c}'(t)}{\bar{c}}\right\}$$

or

$$\frac{a(t)}{\bar{\pi}_g(\bar{G},\bar{Z})} = \frac{1}{\bar{\pi}_g(\bar{G},\bar{Z})}\left\{-(\eta-1)\left(\bar{\pi}(0,\bar{Z}) - \bar{\pi}(\bar{G},Z)\right)\frac{\hat{Z}(t)}{\bar{Z}} + (\eta-1)\zeta\,(q+\bar{\rho})\,\bar{\kappa}\bar{P}\frac{\hat{Z}(t)}{\bar{Z}}\right.$$
$$-\left[(\eta-1)\zeta\int_t^{t+\bar{G}} e^{-[q+\bar{\rho}](s-t)}\bar{\pi}_g(s-t,\bar{Z})ds\right]\frac{\hat{Z}(t)}{\bar{Z}}$$
$$-(\eta-1)(1-\zeta)\int_t^{t+\bar{G}} e^{-[q+\bar{\rho}](s-t)}\bar{\pi}_g(s-t,\bar{Z})\,\frac{\hat{Z}(s)}{\bar{Z}}ds$$
$$\left.-\theta\int_t^{t+\bar{G}} e^{-[q+\bar{\rho}](s-t)}\bar{\pi}_g(s-t,\bar{Z})\left(\frac{\hat{c}(s)-\hat{c}(t)}{\bar{c}}\right)ds - \theta\bar{\kappa}\bar{P}\frac{\hat{c}'(t)}{\bar{c}}\right\}$$

Note that the first order condition (139) at a steady state gives:

$$\bar{\pi}(0,\bar{Z}) - \bar{\pi}(\bar{G},\bar{Z}) = (q+\bar{\rho})\,\bar{\kappa}\bar{P} - \int_t^{t+\bar{G}} e^{-[q+\bar{\rho}](s-t)}\bar{\pi}_g(s-t,\bar{Z})\,ds$$

Replacing the steady state into the expression for a :

$$\frac{a(t)}{\bar{\pi}_g(\bar{G},\bar{Z})} = -\frac{1}{\bar{\pi}_g(\bar{G},\bar{Z})}\Bigg\{(\eta-1)(1-\zeta)\left(\bar{\pi}(0,\bar{Z})-\bar{\pi}(\bar{G},Z)\right)\frac{\hat{Z}(t)}{\bar{Z}}$$
$$+(\eta-1)(1-\zeta)\int_t^{t+\bar{G}} e^{-[q+\bar{\rho}](s-t)}\bar{\pi}_g\left(s-t,\bar{Z}\right)\frac{\hat{Z}(s)}{\bar{Z}}ds$$
$$+\theta\int_t^{t+\bar{G}} e^{-[q+\bar{\rho}](s-t)}\bar{\pi}_g\left(s-t,\bar{Z}\right)\left(\frac{\hat{c}(s)-\hat{c}(t)}{\bar{c}}\right)ds+\theta\bar{\kappa}\bar{P}\frac{\hat{c}'(t)}{\bar{c}}\Bigg\}$$

We can rewrite this as:

$$\frac{a(t)}{\bar{\pi}_g(\bar{G},\bar{Z})} = -(\eta-1)(1-\zeta)\left\{\frac{\left(\bar{\pi}(0,\bar{Z})-\bar{\pi}(\bar{G},Z)\right)}{\bar{\pi}_g(\bar{G},\bar{Z})}\frac{\hat{Z}(t)}{\bar{Z}}+\int_t^{t+\bar{G}} e^{-[q+\bar{\rho}](s-t)}\frac{\bar{\pi}_g\left(s-t,\bar{Z}\right)}{\bar{\pi}_g(\bar{G},\bar{Z})}\frac{\hat{Z}(s)}{\bar{Z}}ds\right\}$$
$$-\frac{\theta}{\bar{\pi}_g(\bar{G},\bar{Z})}\left\{\int_t^{t+\bar{G}} e^{-[q+\bar{\rho}](s-t)}\bar{\pi}_g\left(s-t,\bar{Z}\right)\left(\frac{\hat{c}(s)-\hat{c}(t)}{\bar{c}}\right)ds+\bar{\kappa}\bar{P}\frac{\hat{c}'(t)}{\bar{c}}\right\}$$

From the previous expressions:

$$\frac{\bar{\pi}(0,\bar{Z})-\bar{\pi}(\bar{G},\bar{Z})}{\bar{\pi}_g(\bar{G},\bar{Z})} = \frac{1}{\gamma(\eta-1)}\frac{\bar{\pi}_g(\bar{G},\bar{Z})-\bar{\pi}_g(0,\bar{Z})}{\bar{\pi}_g(\bar{G},\bar{Z})} = \frac{1-e^{\gamma(\eta-1)\bar{G}}}{\gamma(\eta-1)}$$
$$\frac{\bar{\pi}_g\left(s-t,\bar{Z}\right)}{\bar{\pi}_g(\bar{G},\bar{Z})} = e^{\gamma(\eta-1)(\bar{G}-(s-t))}$$

replacing them back into $a(t)$:

$$\frac{a(t)}{\bar{\pi}_g(\bar{G},\bar{Z})} = -(\eta-1)(1-\zeta)\left\{\left(\frac{1-e^{\gamma(\eta-1)\bar{G}}}{\gamma(\eta-1)}\right)\frac{\hat{Z}(t)}{\bar{Z}}+\int_t^{t+\bar{G}} e^{-[q+\bar{\rho}+\gamma(\eta-1)](s-t)+\bar{G}\gamma(\eta-1)}\frac{\hat{Z}(s)}{\bar{Z}}ds\right\}$$
$$-\theta\left\{\int_t^{t+\bar{G}} e^{-[q+\bar{\rho}+\gamma(\eta-1)](s-t)+\bar{G}\gamma(\eta-1)}\left(\frac{\hat{c}(s)-\hat{c}(t)}{\bar{c}}\right)ds+\frac{\bar{\kappa}\bar{P}}{\bar{\pi}_g(\bar{G},\bar{Z})}\frac{\hat{c}'(t)}{\bar{c}}\right\}$$

We have:

$$\int_t^{t+\bar{G}} e^{-[q+\bar{\rho}+\gamma(\eta-1)](s-t)+\bar{G}\gamma(\eta-1)}ds = e^{\gamma(\eta-1)\bar{G}}\frac{\left(1-e^{-[\bar{\rho}+q+\gamma(\eta-1)]\bar{G}}\right)}{q+\bar{\rho}+\gamma(\eta-1)}$$
$$= \frac{e^{\gamma(\eta-1)\bar{G}}-e^{-[\bar{\rho}+q]\bar{G}}}{q+\bar{\rho}+\gamma(\eta-1)}$$

Then

$$\frac{a(t)}{\bar{\pi}_g(\bar{G},\bar{Z})} = (\eta-1)(1-\zeta)\left\{\left(\frac{e^{\gamma(\eta-1)\bar{G}}-1}{\gamma(\eta-1)}\right)\frac{\hat{Z}(t)}{\bar{Z}} - e^{\gamma(\eta-1)\bar{G}}\int_t^{t+\bar{G}} e^{-[q+\bar{\rho}+\gamma(\eta-1)](s-t)}\,\frac{\hat{Z}(s)}{\bar{Z}}ds\right\}$$
$$+\theta\left\{\frac{e^{\gamma(\eta-1)\bar{G}}-e^{-[\bar{\rho}+q]\bar{G}}}{q+\bar{\rho}+\gamma(\eta-1)}\frac{\hat{c}(t)}{\bar{c}} - e^{\gamma(\eta-1)\bar{G}}\int_t^{t+\bar{G}} e^{-[q+\bar{\rho}+\gamma(\eta-1)](s-t)}\,\frac{\hat{c}(s)}{\bar{c}}ds\right.$$
$$\left.-\ \frac{\bar{\kappa}\bar{P}}{\bar{\pi}_g(\bar{G},\bar{Z})}\frac{\hat{c}'(t)}{\bar{c}}\right\}$$

■

The case of $\theta = 0$. *Step 1.* We write the coefficients for the linear equation fo $\hat{G}$ as a function of future values of $\hat{Z}(s)$ for $s \geq t$. This summarizes the optimal choices of firms and households as obtained in Proposition 23.

We can write $\hat{G}(t)$ as follows:

$$\hat{G}(t) = \int_t^T b(s-t)\frac{\hat{Z}(s)}{\bar{Z}}ds\ +\sum_{j=0}^{\lfloor\frac{T-t}{\bar{G}}\rfloor} B(j\bar{G})\frac{\hat{Z}(t+j\bar{G})}{\bar{Z}}$$

where

$$b(s-t) = -(\eta-1)(1-\zeta)e^{\gamma(\eta-1)}\,e^{-(\bar{\rho}+q)(s-t)}\,e^{-\gamma(\eta-1)\left(s-t-\lfloor\frac{s-t}{\bar{G}}\rfloor\bar{G}\right)}\ \text{and}$$
$$B(s-t) = \begin{cases}(\eta-1)(1-\zeta)\frac{e^{\gamma(\eta-1)\bar{G}}-1}{\gamma(\eta-1)}\,e^{-\lfloor\frac{s-t}{\bar{G}}\rfloor\bar{G}\,(\bar{\rho}+q)} & \text{if } t-s=\bar{G}\lfloor\frac{s-t}{\bar{G}}\rfloor\\ 0 & \text{if } t-s\neq\bar{G}\lfloor\frac{s-t}{\bar{G}}\rfloor\end{cases}$$
$$b(s-t) = B(s-t) = 0 \text{ if } s<t$$

Without loss of generality we let $T = K\bar{G}$, where K is a positive integer. Then we can write, more compactly:

$$\hat{G}(t) = \int_0^{K\bar{G}-t} b(v)\,\frac{\hat{Z}(t+v)}{\bar{Z}}dv\ +\sum_{j=0}^{K-\lfloor\frac{t}{\bar{G}}\rfloor} B(j\bar{G})\,\frac{\hat{Z}(t+j\bar{G})}{\bar{Z}}$$

where

$$
\begin{aligned}
b(v) &= -(\eta-1)(1-\zeta)e^{\gamma(\eta-1)}\, e^{-(\bar{\rho}+q)v}\, e^{-\gamma(\eta-1)\left(v-\lfloor\frac{v}{\bar{G}}\rfloor\bar{G}\right)} \quad \text{and} \\
B(v) &= \begin{cases} (\eta-1)(1-\zeta)\frac{e^{\gamma(\eta-1)\bar{G}}-1}{\gamma(\eta-1)}\; e^{-\lfloor\frac{v}{\bar{G}}\rfloor\bar{G}\,(\bar{\rho}+q)} & \text{if } v=\bar{G}\lfloor\frac{v}{\bar{G}}\rfloor \\ 0 & \text{if } v\neq\bar{G}\lfloor\frac{v}{\bar{G}}\rfloor \end{cases} \\
b(v) &= B(v) = 0 \text{ if } v<0
\end{aligned}
$$

Step 2. We write the coefficients for the linear equation fo $\hat{Z}$ as a function of past values of $\hat{G}(s)$ for $s \leq t$ and also ω's. This summarizes the solution of the Kolmogorov forward equation as obtained in Proposition 21.

$$
\begin{aligned}
\frac{\hat{Z}(t)}{\bar{Z}} &= \frac{e^{-(q+\gamma(\eta-1))t}}{\bar{I}(\eta-1)}\left[\int_0^{\bar{G}} e^{-\gamma(\eta-1)g}\omega(g)dg + \left(1-e^{-\gamma(\eta-1)\bar{G}}\right)\int_0^t e^{\gamma(\eta-1)s}\omega\left(\bar{G}+\lfloor\tfrac{s}{\bar{G}}\rfloor\bar{G}-s\right)ds\right] \\
&- \frac{1}{\bar{I}(\eta-1)}\bar{m}(\bar{G})qe^{-\gamma(\eta-1)\bar{G}}\int_0^t e^{-(q+\gamma(\eta-1))(t-s)}\hat{G}(s)ds \\
&+ \frac{\gamma}{\bar{I}}\bar{m}(\bar{G})\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}\int_0^{t-k\bar{G}} e^{-(q+\gamma(\eta-1))(t-k\bar{G}-s)}\hat{G}(s)ds \\
&+ \frac{1}{\bar{I}(\eta-1)}\bar{m}(\bar{G})\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)\left(\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}e^{-(q+\gamma(\eta-1))(t-k\bar{G})}\,\hat{G}(0) - \sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}\,\hat{G}\left(t-k\bar{G}\right)\right)
\end{aligned}
$$

We can write:

$$
\frac{\hat{Z}(t)}{\bar{Z}} = \int_0^{\bar{G}} \check{h}(t,g)\omega(g)dg + \int_0^t n(t-s)\hat{G}(s)ds + \sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} N(k\bar{G})\hat{G}(t-k\bar{G}) + M(t)\hat{G}(0)
$$

where

$$
\begin{aligned}
\int_0^{\bar{G}} \check{h}(t,g)\omega(g)dg &= \frac{1}{\bar{\bar{I}}}\frac{1}{\eta-1}\left[e^{-(q+\gamma(\eta-1))t}\left(\int_0^{\bar{G}} e^{-\gamma(\eta-1)g}\omega(g)dg\right)\right.\\
&\left.+\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)\int_0^t e^{-(q+\gamma(\eta-1))(t-s)}e^{-qs}\omega\left(\bar{G}+\lfloor\tfrac{s}{\bar{G}}\rfloor\bar{G}-s\right)ds\right]\\
&= \frac{1}{\bar{\bar{I}}}\frac{1}{\eta-1}e^{-(q+\gamma(\eta-1))t}\left[\left(\int_0^{\bar{G}} e^{-\gamma(\eta-1)g}\omega(g)dg\right)\right.\\
&\left.+\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)\int_0^t e^{\gamma(\eta-1)s}\omega\left(\bar{G}+\lfloor\tfrac{s}{\bar{G}}\rfloor\bar{G}-s\right)ds\right]\\
&= \frac{1}{\bar{\bar{I}}}\frac{1}{\eta-1}e^{-(q+\gamma(\eta-1))t}\left[\left(\int_0^{\bar{G}} e^{-\gamma(\eta-1)g}\omega(g)dg\right)\right.\\
&+\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)\sum_{i=0}^{\left(\lfloor\frac{t}{\bar{G}}\rfloor-1\right)\bar{G}}\int_{i\bar{G}}^{(i+1)\bar{G}} e^{\gamma(\eta-1)s}\omega\left(\bar{G}+\lfloor\tfrac{s}{\bar{G}}\rfloor\bar{G}-s\right)ds\\
&\left.+\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)\int_{\lfloor\frac{t}{\bar{G}}\rfloor\bar{G}}^t e^{\gamma(\eta-1)s}\omega\left(\bar{G}+\lfloor\tfrac{s}{\bar{G}}\rfloor\bar{G}-s\right)ds\right]\\
&= \frac{1}{\bar{\bar{I}}}\frac{1}{\eta-1}e^{-(q+\gamma(\eta-1))t}\left[\left(\int_0^{\bar{G}} e^{-\gamma(\eta-1)g}\omega(g)dg\right)\right.\\
&+\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)\sum_{i=0}^{\lfloor\frac{t}{\bar{G}}-1\rfloor\bar{G}} e^{\gamma(\eta-1)(\bar{G}i+\bar{G})}\int_{i\bar{G}}^{(i+1)\bar{G}} e^{\gamma(\eta-1)(s-\bar{G}i-\bar{G})}\omega\left(\bar{G}+\lfloor\tfrac{s}{\bar{G}}\rfloor\bar{G}-s\right)ds\\
&\left.+\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)e^{\gamma(\eta-1)(\lfloor\frac{t}{\bar{G}}\rfloor\bar{G}+\bar{G})}\int_{\lfloor\frac{t}{\bar{G}}\rfloor\bar{G}}^t e^{\gamma(\eta-1)(s-\lfloor\frac{t}{\bar{G}}\rfloor\bar{G}-\bar{G})}\omega\left(\bar{G}+\lfloor\tfrac{s}{\bar{G}}\rfloor\bar{G}-s\right)ds\right]
\end{aligned}
$$

Fixing an interval $s \in (i\bar{G}, (i+1)\bar{G})$, doing a chain of variables $g = \bar{G} + i\bar{G} - s$,

$$
\begin{aligned}
&= \frac{1}{\bar{\bar{I}}} \frac{1}{\eta - 1} e^{-(q+\gamma(\eta-1))t} \left[\left(\int_0^{\bar{G}} e^{-\gamma(\eta-1)g} \omega(g) dg \right) \right. \\
&+ \left(1 - e^{-\gamma(\eta-1)\bar{G}}\right) \sum_{i=0}^{\lfloor \frac{t}{\bar{G}} - 1 \rfloor \bar{G}} e^{\gamma(\eta-1)(\bar{G}i + \bar{G})} \int_{\bar{G}}^{0} e^{-\gamma(\eta-1)g} \omega\left(g\right)\left(-dg\right) \\
&+ \left. \left(1 - e^{-\gamma(\eta-1)\bar{G}}\right) e^{\gamma(\eta-1)(\lfloor \frac{t}{\bar{G}} \rfloor \bar{G} + \bar{G})} \int_{\bar{G}}^{\bar{G} + \lfloor \frac{t}{\bar{G}} \rfloor \bar{G} - t} e^{-\gamma(\eta-1)g} \omega\left(g\right)\left(-dg\right) \right] \\
&= \frac{1}{\bar{\bar{I}}} \frac{1}{\eta - 1} e^{-(q+\gamma(\eta-1))t} \left[\left(\int_0^{\bar{G}} e^{-\gamma(\eta-1)g} \omega(g) dg \right) \right. \\
&+ \left(1 - e^{-\gamma(\eta-1)\bar{G}}\right) \sum_{i=0}^{\lfloor \frac{t}{\bar{G}} - 1 \rfloor \bar{G}} e^{\gamma(\eta-1)\bar{G}i + \bar{G}} \int_0^{\bar{G}} e^{-\gamma(\eta-1)g} \omega\left(g\right) dg \\
&+ \left. \left(1 - e^{-\gamma(\eta-1)\bar{G}}\right) e^{\gamma(\eta-1)\bar{G}} e^{\gamma(\eta-1)\bar{G}\lfloor \frac{t}{\bar{G}} \rfloor} \int_{\bar{G} + \lfloor \frac{t}{\bar{G}} \rfloor \bar{G} - t}^{\bar{G}} e^{-\gamma(\eta-1)g} \omega\left(g\right) dg \right] \\
&= \frac{1}{\bar{\bar{I}}} \frac{1}{\eta - 1} e^{-(q+\gamma(\eta-1))t} \left[\left(\int_0^{\bar{G}} e^{-\gamma(\eta-1)g} \omega(g) dg \right) \right. \\
&+ \left(1 - e^{-\gamma(\eta-1)\bar{G}}\right) e^{\gamma(\eta-1)\bar{G}} \left(\int_0^{\bar{G}} e^{-\gamma(\eta-1)g} \omega(g) dg \right) \sum_{i=0}^{\lfloor \frac{t}{\bar{G}} - 1 \rfloor} e^{\gamma(\eta-1)\bar{G}i} \\
&+ \left. \left(1 - e^{-\gamma(\eta-1)\bar{G}}\right) e^{\gamma(\eta-1)\bar{G}} e^{\gamma(\eta-1)\bar{G}\lfloor \frac{t}{\bar{G}} \rfloor} \int_{\bar{G} + \lfloor \frac{t}{\bar{G}} \rfloor \bar{G} - t}^{\bar{G}} e^{-\gamma(\eta-1)g} \omega\left(g\right) dg \right]
\end{aligned}
$$

Using that $\sum_{i=0}^{\lfloor \frac{t}{\bar{G}} - 1 \rfloor} e^{\gamma(\eta-1)\bar{G}i} = \left(1 - e^{\gamma(\eta-1)\lfloor \frac{t}{\bar{G}} \rfloor \bar{G}}\right) / \left(1 - e^{\gamma(\eta-1)\bar{G}}\right)$ if $t > \bar{G}$

$$
\left(1 - e^{-\gamma(\eta-1)\bar{G}}\right) e^{\gamma(\eta-1)\bar{G}} \frac{\left(1 - e^{\gamma(\eta-1)\lfloor \frac{t}{\bar{G}} \rfloor \bar{G}}\right)}{\left(1 - e^{\gamma(\eta-1)\bar{G}}\right)} = \left(e^{\gamma(\eta-1)\lfloor \frac{t}{\bar{G}} \rfloor \bar{G}} - 1\right)
$$

$$
\begin{aligned}
\int_0^{\bar{G}} \check{h}(t,g)\omega(g) dg = \frac{1}{\bar{\bar{I}}} \frac{1}{\eta - 1} e^{-(q+\gamma(\eta-1))t} &\left[\left(\int_0^{\bar{G}} e^{-\gamma(\eta-1)g} \omega(g) dg \right) \right. \\
&+ \left(e^{\gamma(\eta-1)\lfloor \frac{t}{\bar{G}} \rfloor \bar{G}} - 1\right) \left(\int_0^{\bar{G}} e^{-\gamma(\eta-1)g} \omega(g) dg \right) 1_{\{t > \bar{G}\}} \\
&+ \left. \left(1 - e^{-\gamma(\eta-1)\bar{G}}\right) e^{\gamma(\eta-1)\bar{G}} e^{\gamma(\eta-1)\bar{G}\lfloor \frac{t}{\bar{G}} \rfloor} \int_{\bar{G} + \lfloor \frac{t}{\bar{G}} \rfloor \bar{G} - t}^{\bar{G}} e^{-\gamma(\eta-1)g} \omega\left(g\right) dg \right]
\end{aligned}
$$

and Thus we have:

We also have (NEW PROPOSED VERSION)

$$
\begin{aligned}
n(t-s) =& -\frac{1}{\bar{\bar{I}}}\frac{1}{\eta-1} q e^{-\gamma(\eta-1)\bar{G}} \bar{m}(\bar{G}) e^{-(q+\gamma(\eta-1))(t-s)} \\
&+ \frac{\gamma}{\bar{\bar{I}}}\bar{m}(\bar{G})\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)\sum_{k=0}^{\lfloor \frac{t}{\bar{G}} \rfloor} 1_{\{s\leq t-k\bar{G}\}} e^{-q\bar{G}k} e^{-(q+\gamma(\eta-1))(t-k\bar{G}-s)} \\
=& -\frac{1}{\bar{\bar{I}}}\frac{1}{\eta-1} q e^{-\gamma(\eta-1)\bar{G}} \bar{m}(\bar{G}) e^{-(q+\gamma(\eta-1))(t-s)} \\
&+ \frac{\gamma}{\bar{\bar{I}}}\bar{m}(\bar{G})\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)\sum_{k=0}^{\lfloor \frac{t}{\bar{G}} \rfloor} 1_{\{k\leq \frac{t-s}{\bar{G}}\}} e^{-q\bar{G}k} e^{-(q+\gamma(\eta-1))(t-k\bar{G}-s)} \\
=& -\frac{1}{\bar{\bar{I}}}\frac{1}{\eta-1} q e^{-\gamma(\eta-1)\bar{G}} \bar{m}(\bar{G}) e^{-(q+\gamma(\eta-1))(t-s)} \\
&+ \frac{\gamma}{\bar{\bar{I}}}\bar{m}(\bar{G})\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)\sum_{k=0}^{\lfloor \frac{t-s}{\bar{G}} \rfloor} e^{-q\bar{G}k} e^{-(q+\gamma(\eta-1))(t-k\bar{G}-s)}
\end{aligned}
$$

TO BE REVISED (OLD VERSION) We also have:

$$
\begin{aligned}
n(t-s) = & -\frac{1}{\bar{I}}\frac{1}{\eta-1}qe^{-\gamma(\eta-1)\bar{G}}\bar{m}(\bar{G})e^{-(q+\gamma(\eta-1))(t-s)} \\
& + \frac{\gamma}{\bar{I}}\bar{m}(\bar{G})\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)\sum_{k=0}^{\lfloor\frac{t-s}{\bar{G}}\rfloor} e^{-q\bar{G}k}e^{-(q+\gamma(\eta-1))(t-k\bar{G}-s)} \\
= & -\frac{1}{\bar{I}}\frac{1}{\eta-1}qe^{-\gamma(\eta-1)\bar{G}}\bar{m}(\bar{G})e^{-(q+\gamma(\eta-1))(t-s)} \\
& + \frac{\gamma}{\bar{I}}\bar{m}(\bar{G})\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)e^{-(q+\gamma(\eta-1))(t-s)}\sum_{k=0}^{\lfloor\frac{t-s}{\bar{G}}\rfloor} e^{-q\bar{G}k}e^{(q+\gamma(\eta-1))k\bar{G}} \\
= & -\frac{1}{\bar{I}}\frac{1}{\eta-1}qe^{-\gamma(\eta-1)\bar{G}}\bar{m}(\bar{G})e^{-(q+\gamma(\eta-1))(t-s)} \\
& + \frac{\gamma}{\bar{I}}\bar{m}(\bar{G})\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)e^{-(q+\gamma(\eta-1))(t-s)}\sum_{k=0}^{\lfloor\frac{t-s}{\bar{G}}\rfloor} e^{\gamma(\eta-1)k\bar{G}} \\
= & -\frac{1}{\bar{I}}\frac{1}{\eta-1}qe^{-\gamma(\eta-1)\bar{G}}\bar{m}(\bar{G})e^{-(q+\gamma(\eta-1))(t-s)} \\
& + \frac{\gamma}{\bar{I}}\bar{m}(\bar{G})\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)e^{-(q+\gamma(\eta-1))(t-s)}\frac{e^{\gamma(\eta-1)\lfloor\frac{t-s}{\bar{G}}+1\rfloor\bar{G}}-1}{e^{\gamma(\eta-1)\bar{G}}-1} \\
= & -\frac{1}{\bar{I}}\frac{1}{\eta-1}\bar{m}(\bar{G})qe^{-\gamma(\eta-1)\bar{G}}e^{-(q+\gamma(\eta-1))(t-s)} \\
& + \frac{\gamma}{\bar{I}}\bar{m}(\bar{G})e^{-\gamma(\eta-1)\bar{G}}e^{-(q+\gamma(\eta-1))(t-s)}\left(e^{\gamma(\eta-1)\lfloor\frac{t-s}{\bar{G}}+1\rfloor\bar{G}}-1\right)
\end{aligned}
$$

$$
N(k\bar{G}) = -\frac{1}{\bar{I}(\eta-1)}\bar{m}(\bar{G})\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)e^{-q\bar{G}k}
$$

$$
\begin{aligned}
M(t) &= \frac{1}{\bar{I}(\eta-1)}\bar{m}(\bar{G})\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}e^{-(q+\gamma(\eta-1))(t-k\bar{G})} \\
&= \frac{1}{\bar{I}(\eta-1)}\bar{m}(\bar{G})\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)e^{-(q+\gamma(\eta-1))t}\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{-q\bar{G}k}e^{(q+\gamma(\eta-1))k\bar{G}} \\
&= \frac{1}{\bar{I}(\eta-1)}\bar{m}(\bar{G})\left(1-e^{-\gamma(\eta-1)\bar{G}}\right)e^{-(q+\gamma(\eta-1))t}\sum_{k=0}^{\lfloor\frac{t}{\bar{G}}\rfloor} e^{\gamma(\eta-1)k\bar{G}}
\end{aligned}
$$

Using that $\sum_{k=0}^{\lfloor \frac{t}{\bar{G}} \rfloor} e^{\gamma(\eta-1)\bar{G}k} = \left(1 - e^{\gamma(\eta-1)\bar{G}\left(\lfloor \frac{t}{\bar{G}} \rfloor + 1\right)}\right) / \left(1 - e^{\gamma(\eta-1)\bar{G}}\right)$

$$
\begin{aligned}
M(t) &= \frac{1}{\bar{I}(\eta-1)} \bar{m}(\bar{G}) \left(1 - e^{-\gamma(\eta-1)\bar{G}}\right) e^{-(q+\gamma(\eta-1))t} \frac{1 - e^{\gamma(\eta-1)\bar{G}\left(\lfloor \frac{t}{\bar{G}} \rfloor + 1\right)}}{1 - e^{\gamma(\eta-1)\bar{G}}} \\
&= \frac{1}{\bar{I}(\eta-1)} \bar{m}(\bar{G}) \left(e^{\gamma(\eta-1)\bar{G}} - 1\right) e^{-\gamma(\eta-1)\bar{G}} e^{-(q+\gamma(\eta-1))t} \frac{1 - e^{\gamma(\eta-1)\bar{G}\left(\lfloor \frac{t}{\bar{G}} \rfloor + 1\right)}}{1 - e^{\gamma(\eta-1)\bar{G}}} \\
&= \frac{1}{\bar{I}(\eta-1)} \bar{m}(\bar{G}) \left(e^{\gamma(\eta-1)\bar{G}\lfloor \frac{t}{\bar{G}} + 1 \rfloor} - 1\right) e^{-\gamma(\eta-1)\bar{G}} e^{-(q+\gamma(\eta-1))t} \\
&= \frac{1}{\bar{I}(\eta-1)} \bar{m}(\bar{G}) \left(e^{\gamma(\eta-1)\bar{G}\lfloor \frac{t}{\bar{G}} \rfloor} - e^{-\gamma(\eta-1)\bar{G}}\right) e^{-(q+\gamma(\eta-1))t}
\end{aligned}
$$

Using (115)

$$
M(t)\hat{G}(0) = -\frac{1}{\bar{I}(\eta-1)} \left(e^{\gamma(\eta-1)\bar{G}\lfloor \frac{t}{\bar{G}} \rfloor} - e^{-\gamma(\eta-1)\bar{G}}\right) e^{-(q+\gamma(\eta-1))t} \int_0^{\bar{G}} \omega(g) dg
$$

We can write:

$$
\frac{\hat{Z}(t)}{\bar{Z}} = \int_0^{\bar{G}} h(t,g)\omega(g) dg + \int_0^t n(r)\hat{G}(t-r) dr + \sum_{k=0}^{\lfloor \frac{t}{\bar{G}} \rfloor} N(k\bar{G})\hat{G}(t - k\bar{G})
$$

where

$$
\begin{aligned}
h(t,g) &= \check{h}(t,g) - \frac{1}{\bar{I}(\eta-1)} \left(e^{\gamma(\eta-1)\bar{G}\lfloor \frac{t}{\bar{G}} \rfloor} - e^{-\gamma(\eta-1)\bar{G}}\right) e^{-(q+\gamma(\eta-1))t} \\
&= \frac{1}{\bar{I}} \frac{1}{\eta-1} e^{-(q+\gamma(\eta-1))t} e^{\gamma(\eta-1)\bar{G}\lfloor \frac{t}{\bar{G}} \rfloor} \left[1 + \left(e^{\gamma(\eta-1)\bar{G}} - 1\right) 1_{\{g \geq \bar{G} + \lfloor \frac{t}{\bar{G}} \rfloor \bar{G} - t\}}\right] e^{-\gamma(\eta-1)g} \\
&- \frac{1}{\bar{I}(\eta-1)} \left(e^{\gamma(\eta-1)\bar{G}\lfloor \frac{t}{\bar{G}} \rfloor} - e^{-\gamma(\eta-1)\bar{G}}\right) e^{-(q+\gamma(\eta-1))t} \\
&= \frac{1}{\bar{I}} \frac{1}{\eta-1} e^{-qt} \left\{ e^{\gamma(\eta-1)\left(\bar{G}\lfloor \frac{t}{\bar{G}} \rfloor - t\right)} \left[1 + \left(e^{\gamma(\eta-1)\bar{G}} - 1\right) 1_{\{g \geq \bar{G} + \lfloor \frac{t}{\bar{G}} \rfloor \bar{G} - t\}}\right] e^{-\gamma(\eta-1)g} \right. \\
&\left. - e^{\gamma(\eta-1)\left(\bar{G}\lfloor \frac{t}{\bar{G}} \rfloor - t\right)} + e^{-\gamma(\eta-1)(t+\bar{G})} \right\}
\end{aligned}
$$

Replacing the expression for $\hat{G}$

$$
\hat{G}(t-r) = \int_0^{K\bar{G}-t+r} b(v) \frac{\hat{Z}(t-r+v)}{\bar{Z}} dv + \sum_{j=0}^{K - \lfloor \frac{t-r}{\bar{G}} \rfloor} B(j\bar{G}) \frac{\hat{Z}(t-r+j\bar{G})}{\bar{Z}}
$$

into the one for $\hat{Z}$ we get: We can write:

$$
\begin{aligned}
\frac{\hat{Z}(t)}{\bar{Z}} &= \int_0^{\bar{G}} h(t,g)\omega(g)dg + M(t)\hat{G}(0) \\
&+ \int_0^t n(r) \left[\int_0^{K\,\bar{G}-t+r} b(v)\, \frac{\hat{Z}(t-r+v)}{\bar{Z}} dv + \sum_{j=0}^{K-\lfloor \frac{t-r}{\bar{G}} \rfloor} B(j\bar{G})\, \frac{\hat{Z}(t-r+j\bar{G})}{\bar{Z}} \right] dr \\
&+ \sum_{k=0}^{\lfloor \frac{t}{\bar{G}} \rfloor} N(k\bar{G}) \left[\int_0^{K\,\bar{G}-t+k\bar{G}} b(v)\, \frac{\hat{Z}(t-k\bar{G}+v)}{\bar{Z}} dv + \sum_{j=0}^{K-\lfloor \frac{t-k\bar{G}}{\bar{G}} \rfloor} B(j\bar{G})\, \frac{\hat{Z}(t-k\bar{G}+j\bar{G})}{\bar{Z}} \right]
\end{aligned}
$$

Discretized version. Consider a discrete grid for time $\mathbb{T} = \{0, \Delta, 2\Delta, \ldots, T-\Delta, T\}$.

$$
\Delta = \bar{G}/\bar{k} \text{ and } T = \bar{K}\,\bar{G}
$$

where $\bar{k}$ and $\bar{K}$ are integers. So that each grid point is separated in times units by Δ. The time horizon T is chosen to be an integer multiple of $\bar{G}$. We can write $\hat{G}$ as a vector of $J \equiv \bar{k}\bar{K}$ dimensional vector. Likewise we can consider $\hat{Z}$ a J dimensional vector. We can write:

$$
\begin{aligned}
\hat{G}_i &= \sum_{j=0}^{J-i} \tilde{b}_j\, \frac{\hat{Z}_{i+j}}{\bar{Z}} \text{ for } i = 1, \ldots, J \text{ where} \\
\tilde{b}_j &\equiv \begin{cases} b(j\Delta)\Delta + B(j\Delta) & \text{for } j = 0, 1, \ldots, J \text{ and } j = k\bar{G} \text{ for integer } k \\ b(j\Delta)\Delta & \text{for other } j = 0, 1, \ldots, J \end{cases} \\
\frac{\hat{Z}_j}{\bar{Z}} &= H_j(\omega) + (M_j + \tilde{n}_j)\hat{G}(0) + \sum_{i=0}^{j-1} \tilde{n}_i\, \hat{G}_{j-i} \text{ for } j = 1, \ldots, J \text{ where} \\
\tilde{n}_i &\equiv \begin{cases} n(i\Delta)\Delta + N(i\Delta) & \text{for } i = 0, 1, \ldots, J \text{ and } i = k\bar{G} \text{ for integer } k \\ n(i\Delta)\Delta & \text{for other } i = 0, 1, \ldots, J \end{cases} \\
\tilde{M}_j &= M(j\Delta) + \tilde{n}_j \text{ for } j = 1, 2, \ldots, J
\end{aligned}
$$

Where $\hat{G}(0)$ is given by the initial conditions, and where $\hat{Z}(0) = 0$. We can collect the $\tilde{b}$'s into a $J \times J$ upper triangular matrix denoted by $\tilde{B}$, and likewise we can collect the entries of $\tilde{n}$'s into a $J \times J$ matrix denoted by $\tilde{N}$. Thus, the linear system becomes:

$$\hat{Z} = \bar{Z}\hat{H}(\omega) + \bar{Z}\tilde{M}\hat{G}(0) + \tilde{N}\tilde{B}\,\hat{Z}$$
$$\hat{Z} = \left(I_J - \tilde{N}\tilde{B}\right)^{-1}\bar{Z}\left[\hat{H}(\omega) + \hat{M}\,\hat{G}(0)\right]$$