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Bubbles and Collateral*

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Abstract

We construct a model of bubbles where an asset can be used as collateral primarily due to higher-order uncertainty: while both a lender and a borrower know that the intrinsic value of the asset is low, they may still believe that a "greater fool" exists who will purchase it at a much higher price. We show that such bubbles can lead to inefficient overinvestment under certain conditions. Using this framework, we also examine the impacts of macroprudential policies, as well as other regulatory measures such as interest rate hikes and the resolution of uncertainty.

Keywords: collateral; higher-order uncertainty; speculative bubbles

1 Introduction

While collateral and secured loans generally facilitate transactions, they also give rise to public concerns—namely, that they fuel *asset bubbles* or induce overinvestment. When lenders extend credit based on collateral values, borrowers may take excessive risks or inefficiently allocate substantial resources to risky projects. This can pose significant threats to economic stability and reduce social surplus. The prevailing explanation for these concerns is that the

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intrinsic value of collateral is often perceived to be too high relative to its true value. This typically arises when some agents, such as lenders, miscalculate an asset value or hold misguided beliefs.

In this paper, we argue that such first-order uncertainty regarding the value of collateral is not a necessary condition for bubbles and overinvestment to arise. In fact, collateral can induce socially inefficient overinvestment even in cases where all agents fully recognize that the collateralized asset lacks significant intrinsic value. This occurs when *higher-order uncertainty* plays a central role: even though both the lender and the borrower know that the asset has low intrinsic value to anyone in the economy, they may still believe it can be sold to a third party—potentially unaware of this fact (a "greater fool")—at a much higher price. This paper constructs a simple model of such *speculative bubbles* and examines the impact of macroprudential policies, as well as other regulatory measures such as interest rate hikes and the resolution of uncertainty.

Beyond the housing and stock markets, which naturally fit our framework, decentralized finance (DeFi) lending systems—particularly those relying heavily on crypto-collateral—also provide a relevant application.[1] MakerDAO, for example, issues the Dai stablecoin, which maintains a stable value pegged to 1 USD. Users can collateralize cryptocurrencies such as Bitcoin or Ethereum to borrow the stablecoin for investment purposes. These cryptocurrencies, arguably characterized by high volatility and speculative behavior, derive their value largely from market sentiment and higher-order uncertainty.[2] Nonetheless, as reported by Aramonte et al. (2022), DeFi lending platforms and crypto-backed lending have grown rapidly in recent years: "total value locked in DeFi lending protocols peaked at 50 billion USD in early 2022, up from nearly zero at the end of 2020."

More precisely, we consider an environment with three economic agents: a borrower, a lender, and a consumer. The borrower is an entrepreneur with an investment opportunity in a risky project but must borrow funds from the lender to pursue it. The lender, endowed with one unit of resources, allocates it between a safe project yielding a guaranteed but lower return, and the borrower's risky project, which offers a higher expected return. The borrower owns an asset that can be used as collateral. To sharpen the results, we assume this asset holds no intrinsic value for either the borrower or the lender but may have value to the consumer in certain states of the world. Importantly, (i) the borrower and the lender *commonly know*

[1] Alamsyah et al. (2024) provide an overview of the Defi ecosystem and related literature.

[2] Chiu et al. (2022) theoretically examine the role of cryptocurrency as collateral in the DeFi lending system, in which cryptocurrencies are used as a medium of exchange for consumption.

whether the collateral is valuable to the consumer, but (ii) they may not know whether the consumer *knows* this fact. In other words, there is higher-order uncertainty.

Within this framework, we compare two environments—one in which the asset can be used as collateral and one in which it cannot—and establish the following result:

Theorem. Collateral facilitates investment, even when all agents know it is worthless. Under certain conditions, it even induces overinvestment.

Policy Implications

The tractability of our model allows us to analyze several important policies associated with bubbles—*macroprudential policy*, *increase in the interest rate* and *resolving (higher-order) uncertainty*. As emphasized by Barlevy (2018), our analysis contributes to bridging the gaps between policymakers and economic theories of bubbles.

Macroprudential policy bans the trade of collateralized assets. This is a significant policy tool frequently used in practice. We ask: (i) can governments effectively control bubbles through such policies, and (ii) if so, to what extent should they intervene? We show that macroprudential policy influences both the size of bubbles and overall welfare. Further, the optimal strictness of such policy depends on the degree of overinvestment. If collateral only moderately facilitates investment, the optimal policy permits full asset trading. Conversely, when overinvestment is more severe, stricter policies that restrict certain trades are welfare-enhancing.

Interest rate policy serves as another crucial instrument for managing bubbles. In our model, an increase in the interest rate is reflected by a higher return on the safe project. However, it also has an indirect effect: it reduces equilibrium investment in the risky project. If there is underinvestment, the overall welfare impact is ambiguous, as these two effects offset each other. Conversely, if there is overinvestment, raising the interest rate unambiguously improves welfare by both increasing the return from the safe project and curbing excessive investment.

Finally, we analyze the effect of resolving higher-order uncertainty by publicly announcing the value of collateralized assets. Such a policy introduces greater volatility into the economy—it boosts investment when assets are revealed to be valuable and reduces it when they are not. We provide an example in which this announcement policy is welfare-reducing.

Related Literature

Our paper is related to several strands of the existing literature. First of all, it is most closely connected to the literature on greater fool bubbles (see, e.g., Allen, Morris, and Postlewaite (1993), Conlon (2004), Awaya, Iwasaki, and Watanabe (2022); for an excellent summary, see Barlevy (2018)). Like our work, this literature examines rational bubbles that arise due to information asymmetry and higher-order uncertainty among rational agents. Our main contribution is to show that such bubbles can emerge specifically in the context of *collateralized* assets–a topic not addressed in the existing literature. Further, to the best of our knowledge, many of the policy implications we derive are also novel.

The second relevant strand concerns bubbles under borrowing constraints (e.g. Martin and Ventura (2012) and Hirano and Yanagawa (2016)).[3] In these models, bubbles function as a *store of value* under symmetric information. In contrast, our model explores bubbles that arise from asymmetric information and serve as *collateral*. One might conjecture that these differences are superficial if collateral is isomorphic to bubble exchanges–that is, selling bubbly assets to finance investment would be equivalent to borrowing against them. However, we demonstrate that the two are not equivalent.

More fundamentally, our private information framework reveals that bubbles can reduce the dispersion of asset prices relative to fundamentals. When the asset has positive intrinsic value, its price remains bounded above by this value. Yet even when its intrinsic value is zero, a positive price can still be sustained through bubbles. This feature suggests that bubbles may play a socially useful role in contexts where reducing investment dispersion is welfare-enhancing–for example, in economies with both over- and under-investment relative to the efficient level. This mechanism, rooted in private information, differentiates our paper from the existing literature based on symmetric information models.

Finally, our paper also contributes to the literature on collateral and secured credit. In their seminal work, Kiyotaki and Moore (1997) show that collateral constraints are a central channel through which macroeconomic fluctuations are propagated and amplified. Building on this, Awaya, Fukai, and Watanabe (2021) endogenize the Kiyotaki-Moore constraint by demonstrating that collateral can facilitate trade even when the underlying asset lacks sufficient intrinsic value, because it functions as a monitoring device that prevents reneging. In contrast, our paper proposes a new mechanism to endogenize the collateral constraint: *bubbles*

[3] See also Kocherlakota (1992), Santos and Woodford (1997), Caballero and Krishnamurthy (2006), Farhi and Tirole (2012), and Guerron-Quintana et al. (2023), among others. Barlevy (2022) provides an excellent survey of this and other approaches.

in collateralized assets, which arise due to information asymmetry–even when all agents are fully rational.

Organization of the Paper

The paper is organized as follows. Section 2 introduces the model. Section 3 presents the analysis and main results. Section 4 discusses the policy implications. Section 5 briefly examines the payment puzzle.

2 Model

Environment The economy has three risk-neutral agents: a borrower, a lender, and a consumer. The borrower is an entrepreneur having an investment opportunity in a risky project, whose outcome is either success $y = G$ with probability $p \in (0, 1)$ or failure $y = B$ with remaining probability $1 - p$. If $x \geq 0$ is invested in this project, it generates a return $f(x)$ if $y = G$. If $y = B$, then the return is 0 regardless of x. To take advantage of its investment opportunity, the borrower needs to borrow funds from the lender, who has a total available resource of 1 and allocates it between the borrower's risky project and an alternative safe project, providing a per unit return $r > 0$ for certainty without cost.

Assumption 1. $f(x)$ is continuously differentiable with the following properties:

(i) $f(0) = 0$ and $f'(x) > 0 > f''(x)$ for all $x > 0$

(ii) $\lim_{x\to 0} f'(x) = \infty$ and $f'(1) < \frac{r}{p} < f(1)$.

The first item (i) simply means that the risky project's return is zero if no investment is made and increases in a strictly concave manner as more investments are made. The second item (ii) requires that the marginal return of the risky project approaches infinity when the investment level approaches zero; when all the resources are being invested, the average return (i.e., $f(1)/1 = f(1)$) remains high, but its marginal return falls below $\frac{r}{p}$. As one can easily check, this assumption ensures that the efficient investment level x^* that maximizes the joint surplus of the borrower and the lender

$$S(x) \equiv pf(x) + r(1 - x),$$

is characterized by the first order condition, $pf'(x^*) - r = 0$ with $x^* \in (0, 1)$.

Terms of Trade A key friction in this economy is a commitment problem in the relation: they cannot write down a contract on the division of the outcome upon success ex-ante, and they have to bargain about the division ex-post. To capture this environment, we assume that the borrower and the lender engage in Nash bargaining for a realized return $f(x)$, where outside option payoffs are zero, and the borrower's bargaining power is $\sigma \in (0,1)$. Then, the amount of the return given to the lender is $y = (1-\sigma) f(x)$, which solves

$$\max_{0 \leq y \leq f(x)} y^{1-\sigma} (f(x) - y)^{\sigma} .$$

Therefore, if x and $1-x$ are invested in the borrower's and the safe project, respectively, then the borrower obtains $p\sigma f(x)$, and the lender obtains

$$p(1-\sigma)f(x) + r(1-x).$$

Note that $\sigma \to 1$ corresponds to the case where the hold-up problem is extreme: the borrower has control rights over the return from the investment on the risky project and makes a take-it-or-leave-it offer y to the lender. This is an environment widely analyzed in the literature on hold-up problems, such as Gul (2001) and Pitchford and Snyder (2004). We also emphasize that the assumption of zero outside option payoffs can be relaxed as long as these values remain small enough, although it only complicates the analysis without offering additional insights.

Collateral The borrower has an asset that can be used as collateral, potentially serving as a safety device to the lender when the risky project does not work out. To make our results as stark as possible, this asset is assumed to be *worthless* for both the borrower and the lender, but it can be useful to the consumer.[4] More precisely, there are three states of the world with respect to the value of collateral, which are denoted by

$$\Omega \equiv \{\omega_v, \omega_0, \omega_\phi\} .$$

The collateral has a value $V = v > 0$ for the consumer if the state of the world is ω_v. Otherwise, in states $\{\omega_0, \omega_\phi\}$, it has no value to all agents, $V = 0$. The prior probability that the true state is ω_i is denoted by $q_i \in (0,1)$ for each $i = v, 0, \phi$.

[4] We emphasize that the assumption of "worthless" asset is only for clarifying our mechanism; our analysis and results continue to hold as long as the value of the asset is smaller for the borrower and the lender than for the consumer, and the consumer's valuation depends on the state of the world in a similar manner as above.

Information The information partition of the consumer is

$$\mathcal{P}^{\mathbf{c}} \equiv \{\{\omega_v, \omega_\phi\}, \{\omega_0\}\},$$

whereas the borrower and the lender's information partitions are equally

$$\mathcal{P} \equiv \{\{\omega_v\}, \{\omega_\phi, \omega_0\}\}.$$

Note that the second information set $\{\omega_0\}$ of the consumer corresponds to the case where the consumer knows that the collateral has no value, whereas the first information set $\{\omega_v, \omega_\phi\}$ indicates the case where the consumer does not know whether the collateral is worthless. Similarly, the first information set $\{\omega_v\}$ of the borrower and the lender corresponds to the case where they know that the collateral has a positive value to the consumer, whereas the second information set $\{\omega_\phi, \omega_0\}$ indicates that they know that the collateral is worthless but do not know whether the consumer knows this. In the following, we will often write as $\Omega_v^{\mathbf{c}} = \{\omega_v, \omega_\phi\}$ and $\Omega_0^{\mathbf{c}} = \{\omega_0\}$ for the information states of the consumer; $\Omega_v = \{\omega_v\}$ and $\Omega_0 = \{\omega_\phi, \omega_0\}$ for the information states of the borrower and the lender. Also, we will impose the following assumption later, which requires that the consumer's expected value of collateral is not too high.

Assumption 2. $E\left(V|\Omega_v^{\mathbf{c}}\right) = \frac{q_v}{q_v+q_\phi}v < \frac{r}{1-p}$.

There are several remarks. First, every agent in our model is fully rational. Second, the investment level x is not observable to the consumer, implying that no additional information is available for the consumer to update his belief. Third, each of the three states is a necessary ingredient of bubbles: State ω_v creates gains from the trade of the asset so that it can be used as collateral; State ω_0 establishes a situation where all agents know that the asset has no value; State ω_ϕ constructs a case where the asset value is zero, but the consumer does not know it—only the lender and the borrower know the fact that the value is zero.

Finally, while the setup is best illustrated by traditional markets such as housing and stocks, it can also be interpreted as the DeFi lending system. In this context, the borrower collateralizes cryptocurrency (e.g., Bitcoin or Ethereum) in order to borrow and invest an amount x of stablecoins (e.g., Dai stablecoin), which yields a return of $f(x)$ with probability p. In exchange, the lender is entitled to a repayment of $(1-\sigma)f(x)$ from the borrower, but if the borrower defaults, the lender retains the collateralized cryptocurrency. Importantly, it is common knowledge that the collateralized crypto asset has no intrinsic value, but there is uncertainty (asymmetric information) on consumers' knowledge about how much to appreciate it. Its potential value may come from capital gains (or losses) if resold further or liquidity premiums if used as a means of payment.

3 Analysis

3.1 Without Collateral

We first consider the case where there is no collateral, or equivalently, the collateralized contract is banned. Clearly, the information held by the borrower and the lender does not affect their investment incentives. The borrower's incentive compatibility condition is always satisfied since, for all $x \geq 0$,

$$p\sigma f(x) \geq 0,$$

whereas the lender's incentive compatibility condition is

$$p\,(1-\sigma)\,f(x) + r\,(1-x) \geq r \iff \mathcal{L}(x) \equiv p\,(1-\sigma)\,f(x) - rx \geq 0.$$

Note that $\mathcal{L}(0) = 0$; $\lim_{x\to 0} \mathcal{L}'(x) > 0$; and $\mathcal{L}(x)$ is strictly concave in x. We conclude that there exists a unique cutoff $x^N \in (0,1]$ such that the lender's incentive-compatibility condition is satisfied if and only if $x \leq x^N$. The following lemma summarizes this observation.

Lemma 1. Suppose Assumption 1 holds. Without collateral,

(i) The borrower is willing to borrow any $x \in [0,1]$.

(ii) There exists $x^N \in (0,1]$ such that the lender is willing to lend if and only if $x \leq x^N$.

Note that there is a continuum of investment levels x that satisfy both the borrower's and the lender's incentive-compatibility conditions. Among those, we will focus on the maximum investment level throughout the paper, implicitly assuming that the borrower has all the bargaining power regarding the choice of x. As explained later, this is not crucial for our results that asset bubbles can arise and facilitate investments in the presence of higher-order uncertainties. Henceforth, such a maximum investment level will be called *equilibrium investment level without collateral* and denoted by x^N.

The lender's opportunity cost for investing in the risky project increases as σ increases, which can be seen from the fact that $\mathcal{L}(x)$ monotonically decreases and converges to $-rx$ as σ increases to 1. Therefore, the equilibrium investment level becomes lower than x^* when σ is sufficiently high, resulting in insufficient investment level without collateral.

Lemma 2. Suppose Assumption 1 holds. Without collateral, $x^N < x^*$ for sufficiently large σ.

3.2 With Collateral

Now, we analyze the case where the borrower and the lender can trade the collateral. More specifically, we consider the contract of the form: "the borrower continues to hold the collateral if the project is successful, but otherwise, it is held by the lender."

The value of holding the collateral comes from the fact that it could be sold to the consumer. To derive such values, recall that the consumer perfectly knows that the collateral is worthless at information state $\Omega_0^{\mathbf{c}} = \{\omega_0\}$, in which case she would not buy it at any strictly positive price. Otherwise, the expected consumption value of the collateral to the consumer is given by

$$\Pr\left(\omega_v \middle| \, \Omega_v^{\mathbf{c}}\right) \times v = \Pr\left(\omega_v \middle| \, \{\omega_v, \omega_\phi\}\right) \times v = \left(\frac{q_v}{q_v + q_\phi}\right) v > 0.$$

Suppose now that the borrower and the lender's information state is Ω_0. If the holder of the collateral sets price $t = \left(\frac{q_v}{q_v+q_\phi}\right) v$, then the consumer will buy the good if and only if the true state of the world is ω_ϕ, whose likelihood conditional on $\Omega_0 = \{\omega_\phi, \omega_0\}$ is $\frac{q_\phi}{q_\phi+q_0}$. Therefore, the expected values of holding the collateral to the borrower and the lender are given as follows: at information state $\Omega_0 = \{\omega_0, \omega_\phi\}$,

$$W_0 \equiv \underbrace{\frac{q_\phi}{q_\phi + q_0}}_{\text{Probability that the good is sold}} \times \underbrace{\left(\frac{q_v}{q_v + q_\phi}\right) v}_{\text{Price paid by the consumer}} > 0,$$

and at information state $\Omega_v = \{\omega_v\}$,

$$W_v \equiv \underbrace{1}_{\text{Probability the good is sold}} \times \underbrace{\left(\frac{q_v}{q_v + q_\phi}\right) v}_{\text{Price paid by the consumer}} > W_0.$$

Given this, the borrower's incentive-compatibility condition is

$$p\left(W_k + \sigma f\left(x\right)\right) \geq W_k \iff f\left(x\right) \geq \frac{W_k\left(1-p\right)}{p\sigma},$$

which, by Assumption 1, is ensured to hold for some x when

$$f(1) > \frac{W_v(1-p)}{p\sigma}. \tag{1}$$

Compared to the case without collateral, the borrower now would not find it profitable to trade with the lender unless the level of investment is large enough. Also, the minimum investment level satisfying the borrower's incentive-compatibility condition, denoted by $\underline{x}_k$, depends on the state of the world.

Next, we consider the lender's incentive-compatibility condition. The lender would find it optimal to invest x in the borrower's project at Ω_k if and only if

$$\begin{aligned} & p\,(1-\sigma)\,f\,(x)+(1-p)\,W_k+r\,(1-x)\geq r \\ \Longleftrightarrow\; & \mathcal{L}_k\,(x)\equiv p\,(1-\sigma)\,f\,(x)+(1-p)\,W_k-rx\geq 0 \end{aligned}$$

Note that $\mathcal{L}_k\,(0)=(1-p)\,W_k>0$, and, by Assumption 1,

$$\lim_{x\to 0}\mathcal{L}_k'\,(x)=p\,(1-\sigma)\lim_{x\to 0}f'(x)-r>0.$$

Since $\mathcal{L}_k(x)$ is strictly concave in x, the lender's incentive condition imposes the upper bound on an investment level x. Also, compared to the case without collateral, the lender's incentive-compatibility condition is relaxed due to the positive value of holding the collateral, that is,

$$\mathcal{L}_k(x)=\mathcal{L}(x)+(1-p)W_k>\mathcal{L}(x)$$

for each x.

Assuming the condition (1), the necessary and sufficient condition for the existence of incentive-compatible investment is then, for each $k=v,0$,

$$\mathcal{L}_k\,(\underline{x}_k)=p\,(1-\sigma)\,f\,(\underline{x}_k)+(1-p)\,W_k-r\underline{x}_k\geq 0.$$

If this is the case, there is a nonempty interval $[\underline{x}_k,\bar{x}_k]\subset(0,1]$ such that both the borrower's and the lender's incentive constraints are satisfied for $x\in[\underline{x}_k,\bar{x}_k]$. By the definition of $\underline{x}_k$ and strictly increasing f, it can be checked that $\mathcal{L}_k\,(\underline{x}_k)\geq 0$ is equivalent to $\underline{x}_k\leq\frac{(1-p)W_k}{r\sigma}$, or, by taking f on both sides,

$$\frac{(1-p)W_k}{p\sigma}\leq f\left(\frac{(1-p)W_k}{r\sigma}\right) \tag{2}$$

To summarize,

Lemma 3. Suppose Assumption 1 and the conditions (1)–(2) hold. With collateral, there exists a nonempty interval $[\underline{x}_k,\bar{x}_k]\subset(0,1]$ for state Ω_k such that

(i) The borrower is willing to borrow if and only if $x\geq\underline{x}_k$.

(ii) The lender is willing to lend if and only if $x\leq\bar{x}_k$.

It is immediate that $\bar{x}_v\geq\bar{x}_0$ and $\underline{x}_v\geq\underline{x}_0$, where the inequalities are strict whenever one of the bounds in each inequality is strictly lower than 1. Also, when the borrower's bargaining power σ is sufficiently large, Assumption 2 that imposes an upper bound on W_k's turns out to ensure both the conditions (1) and (2), leading to the following corollary.

Corollary 1. Suppose Assumptions 1–2 hold. If σ is sufficiently large, the conditions (1)–(2) hold, so both parties are willing to trade $x \in [\underline{x}_k, \bar{x}_k]$ at state Ω_k.

As in Section 3.1, the maximum investment level satisfying both incentive constraints at Ω_k is called *equilibrium investment level with collateral at* Ω_k and simply denoted by x_k.

3.3 With vs. Without Collateral

3.3.1 Bubbles and Overinvestment

Now, we compare the equilibrium investment levels with and without collateral. Note first that the equilibrium investment levels with collateral are strictly lower than 1 if

$$\mathcal{L}_v(1) < 0 \iff W_v < \frac{r - p(1-\sigma)}{1-p}.$$

Note that under Assumption 2, we have $\frac{W_v(1-p)}{r} < 1$ since $\frac{r}{p} > f'(1)$, which implies that the above inequality holds for sufficiently large σ. Also, it can be checked that collateral induces overinvestment at Ω_k if and only if

$$\mathcal{L}_k(x^*) > 0 \iff p(1-\sigma) f(x^*) + (1-p) W_k - rx^* > 0.$$

For sufficiently large σ, this inequality holds when $W_k > \frac{r}{1-p} x^*$, while if $W_k < \frac{r}{1-p} x^*$, the reversed inequality holds, meaning that equilibrium investment level with collateral is still insufficient compared to the socially efficient one. This observation leads to the following main result of the paper.

Theorem 1. Suppose Assumptions 1–2 hold. For sufficiently large σ,

(i) If $W_0 > \frac{rx^*}{1-p}$, then $x^N < x^* < x_0 < x_v$.

(ii) If $W_v < \frac{rx^*}{1-p}$, then $x^N < x_0 < x_v < x^*$.

(iii) If $W_0 < \frac{rx^*}{1-p} < W_v$, then $x^N < x_0 < x^* < x_v$.

Theorem 1 (i) pertains to the case of primary interest, where the asset bubble, driven by collateral, becomes so extreme that it leads to excessive overinvestment in every possible state of the world. In particular, x_0 is the investment level when the state is ω_0, that is, when the collateralized asset has no value, and all agent knows this. The fact that $x_0 > x^N$ means that, despite that the collateralized asset has no value and all agent knows this, it still facilitates investment. Moreover, $x_0 > x^*$ means that it facilitates investment beyond the efficiency level.

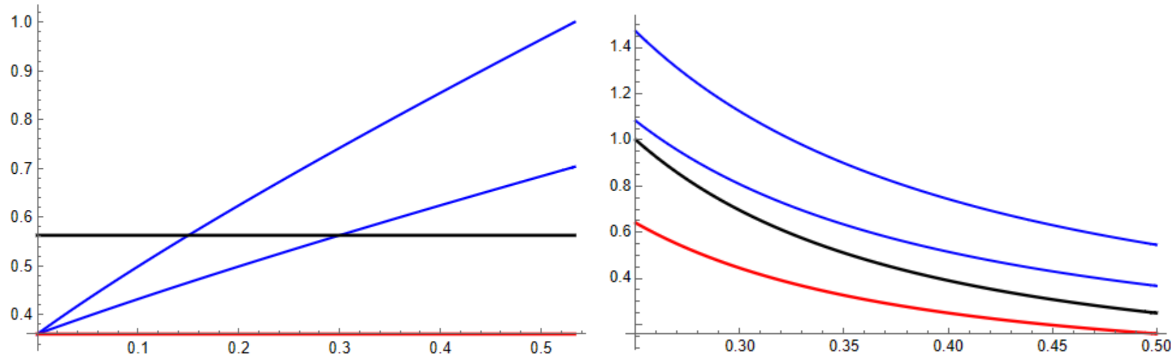

Figure 1: The efficient (black) and equilibrium investment levels without collateral (red) and with collateral (blue) as v (left) and r (right) vary when $f(x) = \sqrt{x}$, $p = \frac{1}{2}, \sigma = \frac{3}{5}$, $q = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, and $r = \frac{1}{3}$ (left) and $v = \frac{1}{2}$ (right).

It is worth emphasizing that, while the assumption that the borrower and the lender cannot commit to terms of trade before investment is crucial, our focus on the maximum incentive-compatible investment level is not essential for this result. For instance, we may alternatively assume either that the lender makes an offer to the borrower or that they agree on x that maximizes their joint surplus conditional on the incentive constraints. In either case, one can find similar conditions ensuring that the equilibrium investment levels are ordered as in Theorem 1 (i). To see this, note that without collateral, the maximum incentive-compatible investment level decreases in σ whereas the minimum incentive-compatible investment level remains at zero; with collateral, both the minimum and the maximum incentive-compatible investment levels increase in W_k. Thus, when σ is large enough, *any* incentive-compatible x without collateral must be smaller than x^*, and given such σ, *any* incentive-compatible x_k with collateral must exceed x^* when W_k is large enough.

Example. Suppose $f(x) = \sqrt{x}$, which is continuous, strictly increasing, and strictly concave with $f(0) = 0$. Since $f'(x) = \frac{1}{2\sqrt{x}}$, Assumption 1 holds so long as $\frac{1}{2} < \frac{r}{p} < 1$, and the socially efficient investment level is $x^* = \frac{p^2}{4r^2} \in (0, 1)$. Finally, Assumption 2 and the condition in Theorem 1 (i) respectively impose the upper and lower bounds on the expected values of holding the collateral: the former requires $W_v < \frac{r}{1-p}$, while the latter requires $W_0 > \frac{rx^*}{1-p} = \frac{p^2}{4r(1-p)}$. Letting $p = \frac{1}{2}$, for instance, it can be checked that all the conditions hold when $\frac{1}{4} < r < \frac{1}{2}$ and $\frac{1}{8r} < W_0 < W_v < 2r$. Similarly, underinvestment occurs even with collateral (Theorem 1 (ii)) if $W_v < \frac{rx^*}{1-p} = \frac{p^2}{4r(1-p)}$, which becomes $W_v < \frac{1}{8r}$ when $p = \frac{1}{2}$. Indeed, given the simplicity of this functional form, more thorough characterizations are possible; Figure 1 illustrates how the equilibrium investment levels depend on v and r for certain parameter values. $\diamond$

3.3.2 Welfare

Based on the above results, we analyze the impact of collateral on the joint surplus, assuming that the equilibrium investment levels are interior. The ex-ante expected joint surplus with collateral is then

$$S^C \equiv E\left(S\left(x_k\right)\right) = q_v S\left(x_v\right) + \left(1 - q_v\right) S\left(x_0\right),$$

whereas the expected joint surplus without collateral is

$$S^N \equiv S\left(x^N\right).$$

Clearly, as long as there is an underinvestment without collateral, the collateral improves social welfare when v is sufficiently small. Indeed, such a condition can be rewritten as

$$S^C \geq S^N \iff E\left(x_k\right) - x^N \geq \left(\frac{(1-p)q_v}{\sigma r}\right) v \tag{3}$$

because

$$S^N = \left(\frac{r\sigma}{1-\sigma}\right) x^N + r, \quad S^C = \left(\frac{r\sigma}{1-\sigma}\right) E\left(x_k\right) - \frac{1-p}{1-\sigma} E\left(W_k\right) + r$$

and

$$E(W_k) = q_v \left(\frac{q_v}{q_v + q_\phi}\right) v + \left(1 - q_v\right) \left(\frac{q_\phi}{1 - q_v}\right) \left(\frac{q_v}{q_v + q_\phi}\right) v = q_v v.$$

The condition (3) shows that, in order for collateral to improve social welfare, the increase of investments facilitated by collateral must be sufficiently large relative to the expected value of holding the collateral. If v is too large, for instance, the collateralized trade may potentially lead to excessive overinvestments, resulting in lower social welfare. We confirm this intuition with the functional form example of $f(x) = \sqrt{x}$.

Example. Suppose $f(x) = \sqrt{x}$ with $p = \frac{1}{2}, \sigma = \frac{3}{5}, q = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $r \in \left(\frac{1}{4}, \frac{1}{2}\right)$, and $v < 4r - \frac{4}{5}$. One can check that the equilibrium investment levels exist at interiors as follows:

$$x_v = \frac{2 + 25rv + 2\sqrt{1 + 25rv}}{100r^2} > x_0 = \frac{4 + 25rv + 2\sqrt{4 + 50rv}}{200r^2}$$

and $x^N = \frac{1}{25r^2}$. Since $E\left(W_k\right) = \frac{v}{3}$, we have

$$\frac{E\left(x_k\right) - x^N}{E\left(W_k\right)} = \frac{25rv - 3 + \sqrt{1 + 25rv} + \sqrt{4 + 50rv}}{50vr^2}.$$

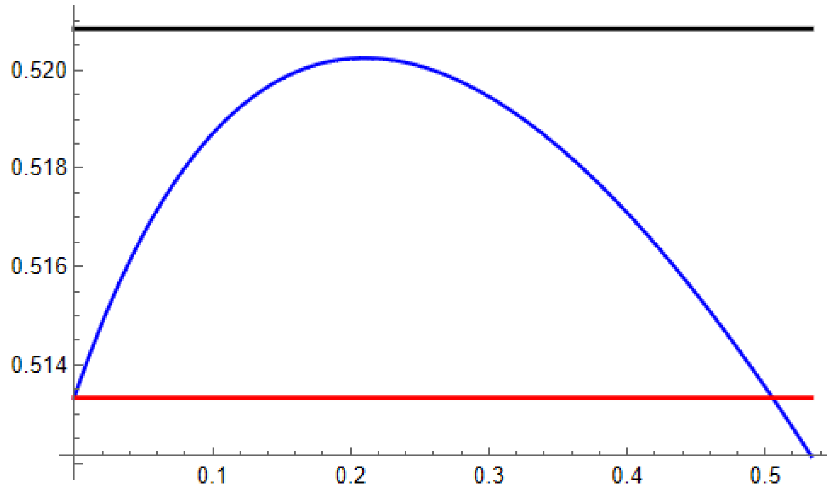

Figure 2: The ex-ante joint surplus under the efficient (black) and equilibrium investment levels without collateral (red) and with collateral (blue) as v varies when $f(x) = \sqrt{x}$, $p = \frac{1}{2}, \sigma = \frac{3}{5}, r = \frac{1}{3}$ and $q = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

The condition (3) is then equivalent to

$$\frac{25rv - 3 + \sqrt{1 + 25rv} + \sqrt{4 + 50rv}}{50vr} \geq \frac{5}{6}.$$

Clearly, the collateral improves social welfare when v is sufficiently small since

$$\lim_{v \to 0} \left(\frac{25rv - 3 + \sqrt{1 + 25rv} + \sqrt{4 + 50rv}}{50vr} \right) = 1 > \frac{5}{6}.$$

The opposite case may also arise: if $r = \frac{1}{3}$, for instance, then $4r - \frac{4}{5} = \frac{8}{15}$, and so

$$\begin{aligned} \lim_{v \to \frac{8}{15}} \left(\frac{25rv - 3 + \sqrt{1 + 25rv} + \sqrt{4 + 50rv}}{50vr} \right) \Big|_{r=\frac{1}{3}} &= \frac{1}{80} \left(34 + 6\sqrt{29}\right) \approx 0.82 \\ &< \frac{5}{6} \approx 0.83. \end{aligned}$$

Figure 2 depicts the expected social welfare under the efficient and the equilibrium investment levels when v varies. $\diamond$

Note that, although we focus only on the joint surplus of the borrower and the lender, the comparison between the two environments (with and without collateral) is unaffected when the consumer surplus is also considered. If we allow the trade of the good between the borrower and the consumer in the benchmark case without collateral, the borrower would offer the same price as that under the collateralized contract, and consumer welfare would remain the same under the two environments.

4 Policy Implications

The simplicity of our model enables the analysis of several important policies related to bubbles, as discussed below. For simplicity, we assume that equilibrium investment levels are interior, i.e., $x^N, x_v, x_0 \in (0, 1)$.

4.1 Macroprudential Policy

Suppose that the government bans the trade of the collateral with probability $\alpha \in [0, 1]$, which measures the level of macroprudential policy. Therefore, the expected value of holding the collateral with the borrower and the lender is reduced to $(1 - \alpha) W_k$.

Lemma 4. The equilibrium investment level x_k with collateral at Ω_k is strictly decreasing in α, σ, r, while strictly increasing in W_k.

Clearly, the lender's incentive constraint becomes tighter when α, σ, or r increases while being relaxed as W_k increases. Since the borrower has the bargaining power and the equilibrium investment levels are pinned down by the lender's incentive constraint, the comparative statics results in the above lemma hold.

Note that $S^C = S^N$ when $\alpha = 1$ as the collateral cannot play any role in this case. If the government slightly reduces the level of macroprudential policy α from 1, the social welfare will certainly be increased whenever the economy suffers underinvestment without collateral (i.e., $x^N < x^*$) because the collateralized trade will induce higher equilibrium investment levels. By the same argument, one can check that such a macroprudential policy is only detrimental if v is initially small enough and $x^N < x^*$. If this is not the case, for instance, when the condition (3) is violated, a positive degree of macroprudential policy can be welfare-improving.

Letting $\alpha^* \in [0, 1]$ denote the optimal level of macroprudential policy maximizing the ex-ante joint surplus, the above observations can be summarized as follows (Figure 3).

Proposition 1. The ex-ante joint surplus with collateral S^C is strictly concave in α and coincides with the surplus without collateral when $\alpha = 1$. Therefore,

(i) If $x^N < x^*$, then $\alpha^* < 1$. That is, a complete ban is not socially optimal.

(ii) If $x^N < x^*$ and v is sufficiently small, then $\alpha^* = 0$. That is, full allowance is socially optimal.

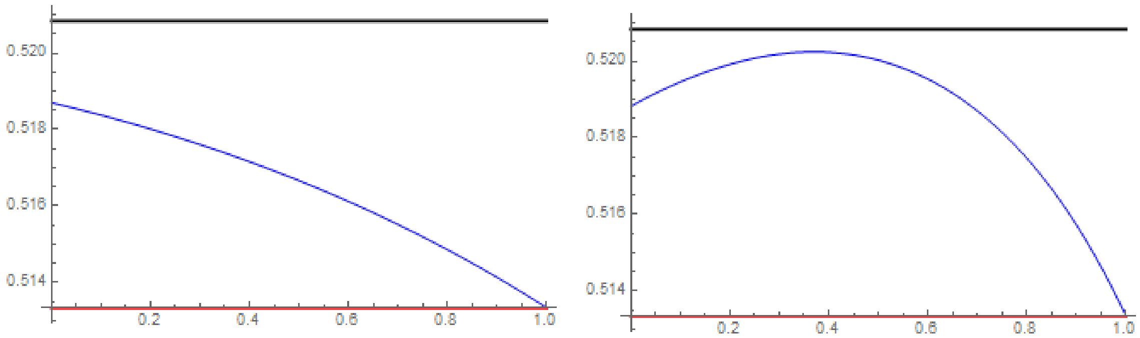

Figure 3: The ex-ante joint surplus under the efficient (black) and equilibrium investment levels without collateral (red) and with collateral (blue) as α varies when $f(x) = \sqrt{x}$, $p = \frac{1}{2}, \sigma = \frac{3}{5}, r = \frac{1}{3}, q = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, and $v = \frac{1}{10}$ (left) and $v = \frac{1}{3}$ (right).

Clearly, as v increases, overinvestment induced by the collateral becomes excessively inefficient, in which case the optimal level of macroprudential policy should be correspondingly higher. Similarly, the change of the prior from $q = (q_v, q_\phi, q_0)$ to $q' = (q_v + \epsilon, q_\phi, q_0 - \epsilon)$ induces increases in W_v and W_0. This means that the expected value of holding the collateral becomes higher as it is more likely that the asset has positive value v (i.e., $\omega = \omega_v$), while the event that every agent knows that the asset has no value (i.e., $\omega = \omega_0$) is less likely.[5] Finally, it must be the case that $x_v > x^* > x_0$ at the optimal level of macroprudential policy α^*: if both x_v and x_0 exceed x^*, then the government would rather increase α to lower these levels and improve social welfare, while in the opposite case, the government would decrease α.

To summarize,

Proposition 2. Assume $\alpha^* \in (0, 1)$. Then,

(i) $x_v > x^* > x_0$ at $\alpha = \alpha^*$.

(ii) α^* is strictly increasing in v.

(iii) α^* is strictly lower at $q = (q_v, q_\phi, q_0)$ than at $q' = (q_v + \epsilon, q_\phi, q_0 - \epsilon)$ for small $\epsilon > 0$.

Our analysis suggests that the impact of macroprudential policy crucially depends on the initial degree of the equilibrium investment levels with collateral. A stricter macroprudential policy may decrease social welfare if the collateral is facilitating investment moderately (as in the left panel of Figure 3), but may also improve social welfare if the collateral is inducing too much over-investment (as in the right panel of Figure 3).

[5] A similar conclusion cannot be made for the change to $q' = (q_v + \epsilon, q_\phi - \epsilon, q_0)$ because it reduces asset bubbles at Ω_0 through the decrease of $\Pr(\Omega_v^c | \Omega_0) = \Pr(\omega_\phi | \Omega_0)$.

Finally, note that similar analysis can also be applied to other policies that effectively limit the equilibrium investment levels. For instance, a *haircut* in our model can be defined as $h_k \equiv \frac{W_k - \left(x_k - x^N\right)}{W_k}$ for each state k, so a haircut floor is given by the lower bound $\underline{h}$ such that $\min_{k=0,v} h_k \geq \underline{h}$. That is, the haircut floor regulation imposes an *upper bound* on the investment level x, which brings about similar effects as increasing α.

4.2 Interest Rate Regulation

When there are inefficiently excessive or insufficient investments, governments may consider adjusting the interest rate, which is captured by r in our framework. A marginal increase in the interest rate impacts the joint surplus directly by improving the return of the safe project. However, there is also an indirect effect through the change of the equilibrium investment levels x_k's. More precisely,

$$\frac{dS\left(x_k\right)}{dr} = S'\left(x_k\right)\frac{\partial x_k}{\partial r} + 1 - x_k = \underbrace{\left(pf'\left(x_k\right) - r\right)\frac{\partial x_k}{\partial r}}_{\text{Indirect effect}} + \underbrace{1 - x_k}_{\text{Direct effect}}$$

Assuming the interior equilibrium, one can apply the implicit function theorem to $\mathcal{L}_k\left(x_k\right) = 0$ to show that

$$\frac{\partial x_k}{\partial r} = \frac{x_k}{\mathcal{L}_k'\left(x_k\right)} < 0.$$

That is, as r becomes higher, the investments for the risky project are reduced in equilibrium because the lender's opportunity cost is increased.

The indirect effect is thus negative if and only if $S'\left(x_k\right) > 0 \iff x_k < x^*$. This means that the direct and indirect effects are of opposite signs if there is underinvestment, $x_v, x_0 < x^*$, so it is unclear whether increasing the interest rate is welfare-improving. On the other hand, if there is overinvestment, $x_v, x_0 > x^*$, this policy unambiguously increases social welfare because not only is the return from the safe project improved, but also the equilibrium investment levels are reduced, mitigating overinvestment.

4.3 Resolving Uncertainty

Suppose that the central bank discloses the true state of the world by making public announcements. Since ω_i is common knowledge, the holder of the collateral knows that she can never sell the good to the consumer if $\omega \neq \omega_v$, meaning that the collateral does not affect the investment levels in such cases. That is,

$$x_0^F = x^N.$$

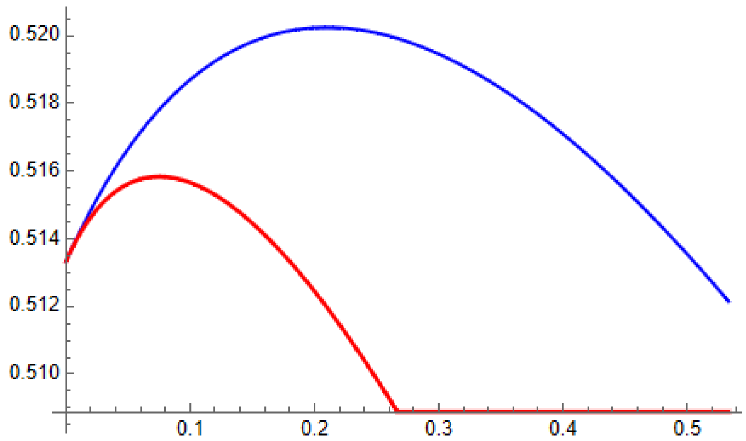

Figure 4: The ex-ante joint surplus under the equilibrium investment levels with uncertainty (blue) and without uncertainty (red) as v varies when $f(x) = \sqrt{x}$, $p = \frac{1}{2}, \sigma = \frac{3}{5}, r = \frac{1}{3}$ and $q = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

On the other hand, if $\omega = \omega_v$, the value of holding the collateral to the borrower and the lender becomes $W_v = v$ since the probability of selling it to the consumer is 1, implying

$$x_v^F = \min\{1, \bar{x}\},$$

where $\bar{x}$ is the unique solution to

$$p\,(1-\sigma)\,f\,(x) + (1-p)\,v - rx = 0.$$

Thus, the social welfare becomes higher by resolving uncertainty if and only if

$$q_v S\left(x_v^F\right) + (1-q_v)S\left(x^N\right) \geq q_v S(x_v) + (1-q_v)S(x_0)$$

Clearly, such a policy increases the equilibrium investment level at ω_v while lowering it at $\omega \neq \omega_v$, that is, $x_v^F > x_v$ and $x_0 > x^N$. In other words, the dispersion in the asset's price with respect to fundamentals is *increased* by resolving uncertainty in our setting, which is unlike the models with symmetric information (e.g., Martin and Ventura (2012) and Hirano and Yanagawa (2016)). An immediate implication is that the policy *strictly lowers* social welfare whenever overinvestment occurs only at Ω_v initially:

$$x_0 < x^* < x_v.$$

Indeed, our numerical exercise suggests that, even in other cases, resolving uncertainty does not improve social welfare, and that bubbles can play a socially beneficial role. (Figure 4). We leave further characterizations to future research.[6]

[6]The effect of information policies that aim to burst bubbles is well-examined in the literature. Examples

5 Payment Puzzle

Lagos (2011) raised the payment puzzle: "Why even to these days are assets used as collateral instead of simply as a means of payment?" His point is also suggested by Martin and Ventura (2016).[7] In many settings, these two are equivalent, as these papers have shown. But are there any positive reasons why collateral, rather than payment, is used in reality?

In our framework, payment and collateral are not equivalent. To see this, consider an alternative environment where the borrower *sells* the asset to the lender to get x, and the return from the project is solely consumed by the borrower. Notice that the lender would be willing to buy the asset at Ω_k if

$$r(1-x)+W_k \geq r \iff x \leq \frac{W_k}{r},$$

and the borrower would be willing to sell the asset if

$$pf(x) \geq W_k \iff x \geq f^{-1}\left(\frac{W_k}{p}\right).$$

Thus, there exists x at which both the borrower and the lender would trade if and only if

$$\begin{aligned} f^{-1}\left(\frac{W_k}{p}\right) \leq \frac{W_k}{r} \iff \frac{W_k}{p} \leq f\left(\frac{W_k}{r}\right) &\iff f\left(\frac{W_k}{r}\right) - \frac{W_k}{p} \geq 0 \\ &\iff \frac{f\left(W_k/r\right)}{W_k/r} \geq \frac{r}{p}. \end{aligned} \tag{4}$$

Compared to the equilibrium condition (2) with the collateralized contract,

$$\frac{(1-p)W_k}{p\sigma} \leq f\left(\frac{(1-p)W_k}{r\sigma}\right) \iff \frac{f\left((1-p)W_k/(r\sigma)\right)}{(1-p)W_k/(r\sigma)} \geq \frac{r}{p},$$

one can see that the two conditions are equivalent only in the knife-edge case, $1-p=\sigma$, whereas if $1-p<\sigma$, then the condition (2) is strictly implied by the condition (4). That is, an equilibrium may fail to exist with the selling contract while the collateralized trading mechanism supports an equilibrium. Further, even when trade is possible under both mechanisms, they predict different levels of investments and social welfare in general. For instance, if $\sigma \approx 1$, then the collateralized contract induces $x_k \approx \frac{(1-p)W_k}{r}$, which is strictly smaller than the maximum investment level $x = \frac{W_k}{r}$ that the lender would accept under the selling contract. In this case, the collateralized contract induces a higher joint surplus than the selling contract when the size of bubbles is too high.

include Asako and Ueda (2014) and Conlon (2015). Holt (2019) considers risk-averse agents and, like ours, identifies environments in which such policies are detrimental.

[7]See also Awaya et al. (2021) and Madison (2024).

The crucial difference between the two in our model is the following: Under the collateralized contract, the compensation made to the lender consists of its share of the investment return and holding the asset upon the failure of the project, but these are *dependent* on the project's success and bargaining power. If $\sigma = p = 1$, for instance, no investment return is given to the lender, and the asset always reverts to the borrower. The lender thus entirely loses an incentive to invest. This is obviously not the case under the payment contract where the borrower can compensate the lender only through the asset, but its value is *unconditional* on the project's success or its return.

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Appendix: Omitted Proofs

Proof of Corollary 1. Since $f(1) > \frac{r}{p}$, the condition (1) is satisfied for sufficiently large σ if $\frac{r}{p} > \frac{W_v(1-p)}{p}$, which is equivalent to Assumption 2. Further, the condition (2) is equivalent to

$$\frac{f\left((1-p)W_k/(r\sigma)\right)}{(1-p)W_k/(r\sigma)} \geq \frac{r}{p}.$$

Under Assumption 2, $\frac{(1-p)W_k}{r\sigma} < 1$ for sufficiently large σ. The strict concavity of $f(x)$ with $f(0) = 0$ then implies

$$\frac{f\left((1-p)W_k/(r\sigma)\right)}{(1-p)W_k/(r\sigma)} > f(1) > \frac{r}{p},$$

and therefore the condition (2) holds for large enough σ.

Proof of Lemma 4. Recall that x_k is the unique solution to

$$\mathcal{L}_k(x) = p(1-\sigma)f(x) + (1-p)(1-\alpha)W_k - rx = 0, \tag{5}$$

and that $\mathcal{L}'(x_k) = p(1-\sigma)f'(x_k) - r < 0$. By the implicit function theorem, we have

$$\frac{\partial x_k}{\partial \alpha} = \frac{(1-p)W_k}{p(1-\sigma)f'(x_k) - r} < 0.$$

Similarly,

$$\frac{\partial x_k}{\partial \sigma} = \frac{pf(x_k)}{p(1-\sigma)f'(x_k) - r} < 0,$$

$$\frac{\partial x_k}{\partial r} = \frac{x_k}{p(1-\sigma)f'(x_k) - r} < 0,$$

and

$$\frac{\partial x_k}{\partial W_k} = \frac{-(1-p)(1-\alpha)}{p(1-\sigma)f'(x_k) - r} > 0.$$

Proof of Proposition 1. It is clear that $S^C = S^N$ when $\alpha = 1$. To see strict concavity, note that, by the definition of x_k,

$$f(x_k) = \frac{rx_k - (1-p)(1-\alpha)W_k}{p(1-\sigma)}.$$

Therefore,

$$\begin{aligned} S^C &= E\left(pf\left(x_k\right) + r\left(1 - x_k\right)\right) \\ &= \frac{r\sigma}{1-\sigma} E\left(x_k\right) - \frac{(1-p)(1-\alpha)}{1-\sigma} E\left(W_k\right) + r, \end{aligned}$$

and so

$$\frac{\partial S^C}{\partial \alpha} = \frac{1-p}{1-\sigma}\left(r\sigma E\left(\frac{W_k}{p(1-\sigma)f'(x_k) - r}\right) + E\left(W_k\right)\right).$$

As α increases, x_k decreases by Lemma 4. In addition, the strict concavity of f implies that $f'(x_k)$ is increasing in α, and thus $\frac{\partial S^C}{\partial \alpha}$ decreases in α. This shows that S^C is strictly concave in α, which, in turn, implies that if v is sufficiently small so that $\frac{\partial S^C}{\partial \alpha} \leq 0$ at $\alpha = 0$, we have $\frac{\partial S^C}{\partial \alpha} \leq 0$ for all α. Clearly, the optimal macroprudential policy in this case is $\alpha^* = 0$.

Finally, observe that the derivative of S^C with respect to α can also be written as

$$\frac{\partial S^C}{\partial \alpha} = E\left(\left(pf'\left(x_k\right) - r\right)\frac{\partial x_k}{\partial \alpha}\right). \tag{6}$$

Since $x^N < x^*$ implies $f'\left(x^N\right) > \frac{r}{p}$, we have

$$\left.\frac{\partial S^C}{\partial \alpha}\right|_{\alpha=1} = E\left(\left(pf'\left(x^N\right) - r\right)\frac{\partial x_k}{\partial \alpha}\right)\Big|_{\alpha=1} < 0$$

when $x^N < x^*$.

Proof of Proposition 2. From the equation (6), the optimal $\alpha^* \in (0,1)$ should satisfy the following first-order condition:

$$E\left(\left(pf'\left(x_k\right) - r\right)\frac{\partial x_k}{\partial \alpha}\right) = 0 \text{ at } \alpha = \alpha^*.$$

Since $\frac{\partial x_k}{\partial \alpha} < 0$, it must be the case that

$$\left(pf'\left(x_v\right) - r\right)\left(pf'\left(x_0\right) - r\right) < 0.$$

Since $x_v > x_0$ and f is strictly concave, this implies

$$pf'\left(x_v\right) - r < 0 < pf'\left(x_0\right) - r,$$

which is equivalent to $x_v > x^* > x_0$.

To check (ii), note that

$$
\begin{aligned}
& E\left(\left(pf'\left(x_k\right)-r\right)\frac{\partial x_k}{\partial \alpha}\right)=0 \\
\Longleftrightarrow\ & q_v\left(1-p\right)W_v\left(\frac{pf'\left(x_v\right)-r}{p\left(1-\sigma\right)f'\left(x_v\right)-r}\right)+\left(1-q_v\right)\left(1-p\right)W_0\left(\frac{pf'\left(x_0\right)-r}{p\left(1-\sigma\right)f'\left(x_0\right)-r}\right)=0 \\
\Longleftrightarrow\ & q_v\left(\frac{1}{1-p\sigma\left(p-\frac{r}{f'(x_v)}\right)^{-1}}\right)+\left(1-q_v\right)\left(\frac{q_\phi}{q_\phi+q_0}\right)\left(\frac{1}{1-p\sigma\left(p-\frac{r}{f'(x_0)}\right)^{-1}}\right)=0 \\
\Longleftrightarrow\ & q_v\left(\frac{1}{1-p\sigma\left(p-\frac{r}{f'(x_v)}\right)^{-1}}\right)+q_\phi\left(\frac{1}{1-p\sigma\left(p-\frac{r}{f'(x_0)}\right)^{-1}}\right)=0. \qquad (7)
\end{aligned}
$$

As v increases, x_k also increases and so $f'\left(x_k\right)$ decreases. This shows that the left-hand side of the above equation will be increased as v increases without a change in α. Because x_k decreases in α, we conclude that the optimal α^* should correspondingly increase in α.

Finally, consider a change of prior from $q=(q_v,q_\phi,q_0)$ to $q'=(q_v+\epsilon,q_\phi,q_0-\epsilon)$. Since the expected values of holding collateral,

$$W_v=\left(\frac{q_v}{q_v+q_\phi}\right)v=\left(\frac{1}{1+\frac{q_\phi}{q_v}}\right)v,$$

and

$$W_0=\left(\frac{q_\phi}{q_\phi+q_0}\right)\left(\frac{q_v}{q_v+q_\phi}\right)v=\left(\frac{1}{1+\frac{q_0}{q_\phi}}\right)\left(\frac{1}{1+\frac{q_\phi}{q_v}}\right)v,$$

become correspondingly higher, the equilibrium investment level x_k also increases, while $f'\left(x_k\right)$ decreases for each $k=v,0$. As a result, each term in the bracket of the left-hand side of the equation (7) is increased. Furthermore, the first bracket is strictly positive, implying that the overall impact of such a change in the prior induces the left-hand side of the equation to strictly increase. Therefore, the optimal macropudential policy level α^* becomes higher since x_v and x_0 are strictly decreasing in α. This completes the proof.