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Growth and Fluctuations Economies with Land Speculation

Tomohiro Hirano and Joseph E. Stiglitz[1]

Abstract

This paper considers growth and fluctuations in a standard Overlapping Generations (OLG) model with rational expectations, with land (a non-produced asset), credit frictions, and endogenous growth. Under plausible conditions, there can be multiple momentary equilibria, with the multiplicity itself depending on capital and land prices; this can give rise in turn to an infinity of rational expectations trajectories, all operating within bounds that can be calculated. Improvements in technology, while in the short run increasing GDP, may result in the equilibrium being unstable and fragile—and in the long run lead to a stagnation trap with lower GDP. The introduction of land increases the scope for fluctuations; the only rational expectations trajectories may entail fluctuations, with episodic unemployment and dynamic inefficiencies. With credit frictions, expansionary credit and financial policies may lead to lower growth, with the additional funds unevenly going to land speculation, diverting savings from productive investments, results consistent with empirical evidence. The analysis resolves several theoretical puzzles, such as how can land prices be finite with an interest rate less than the growth rate. It shows that even with two state variables, a tractable OLG model can be constructed providing a global analysis of complex dynamics.

Keywords: Multiplicity of momentary equilibria, Wobbly dynamics, Land speculation, Phase Transitions, Two-sector growth economies, Credit expansions, Low interest rates,

JEL Classification: C61 (Dynamic Analysis), E32 (Business Fluctuations, Cycles), O11 (Macroeconomic aspects of economic development)

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1. Introduction

Financial liberalization and expansionary monetary policy have been widely supported as leading to enhanced economic growth. The experience has been otherwise. Verner (2019) showed that rapid expansion in credit systematically predicts growth slowdowns.[2] Müller and Verner (2023) confirm that this is typical: using a novel database on the sectoral distribution of private credit for 117 countries starting in 1940, they show that credit expansions to the construction and real estate industries systematically predict subsequent productivity and economic slowdowns, while credit expansions to the manufacturing sector are associated with higher productivity and stronger macroeconomic performance. Banerjee, Mehrotra, and Zampolli (2024), who study emerging market economies, also show empirically that the larger the credit expansion to construction and real estate, the bigger the drop in labour productivity growth and TFP growth because productivity gains are generally smaller in these sectors, suggesting that credit expansion to the real estate sector reduces growth. On the other hand, productivity growth tends to be higher in countries with a greater share of loans to the manufacturing sector.

There are two distinct reasons for these adverse effects. The credit expansion of the early part of this century is consistent with both: (a) credit went disproportionately into real estate—so much so that the associated real estate boom crowded out more productive investments in other sectors[3]; (b) real estate speculation can give rise to destabilizing macroeconomic activities.

In the currently fashionable standard macroeconomic models based on individuals living infinitely long with rational expectations,[4] convergence to steady state is smooth and growth in the long run is determined by the rate of technological change (assumed exogenous) and population. Looser monetary or credit policy leads to more investment in the only asset, “k”, which is productive.

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2 Mian, Sufi, and Verner (2017) also found empirically that an increase in the household debt-to-GDP ratio predicts lower GDP growth.

3 Chakraborty, Goldstein, and MacKinlay (2018) empirically studied the effect of housing prices on bank commercial lending and firm investment in the U.S. in the period between 1988 and 2006. They found housing price booms led to crowding-out of commercial investment due to decreased lending by banks to credit-constrained firms. They conclude that housing price booms have negative spillovers to the real economy.

4 In standard models, it is also assumed that there are no new economic agents coming in, who earn wages.

We frame here a set of models not only more consistent with observed macroeconomic phenomena (with these facts) and underlying microeconomics, but simultaneously far richer in the set of feasible dynamics. For instance, we assume that individuals are finitely lived (a fact that can be established without extensive econometrics), that many do not have children (again, a fact that can be established without extensive econometrics), and that those that do, on average, do not act perfectly altruistic, the import of which is that the dynastic model (typically put up to justify the claim that the economy acts as if it consisted of individuals living infinitely long maximizing their utility) provides a poor description of the microeconomics underlying savings. The overlapping generations model provides a more robust foundation.[5][6]

The difficulty with our approach is that the analysis quickly gets quite complex: (a) in the simplest of models, even without land, there are multiple momentary equilibria—the model captures the spirit of Keynes' animal spirits, but because of the extreme (but standard) assumptions of wage and price flexibility, there is no Keynesian unemployment (and accordingly, we think of this paper as a *prologue* to a Keynesian theory of unemployment dynamics);[7] whether multiple momentary equilibria arise or not depends on a state variable such as capital stock; and the multiple momentary equilibria give rise to economic fluctuations—there are an infinity of rational expectations paths, but still, we can say a lot about the economy: the fluctuations lie within well-defined bounds. Importantly, the fluctuations are *endogenous*, i.e., arise from the internal workings of the economy, not from exogenous shocks of macroeconomic significance arbitrarily assumed in the standard model to hit the economy. (b) Adding land to the model adds a further complexity, giving rise to a richer set of possible endogenous fluctuations, with land speculation diverting resources from productive investments, and with the bursting of rational land booms contributing to an even richer set of fluctuations.[8] (c) With endogenous growth, with the sources of

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[5] Even if it is assumed that individuals care *some* for their children, the dynamics are little changed from that presented here.

[6] We employ the standard two-period overlapping generations model to describe the essence. Our analysis can be extended into the model proposed by Blanchard (1985), where individuals may live longer and new economic agents with earning wages come in in every period.

[7] In Part I and II, we employ the Leontief production function for analytical tractability. In this case, involuntary unemployment occurs when capital shortage occurs. If one considers a situation in which the substitutability between capital and labor may be difficult in the short run, employing the Leontief production function may not be implausible.

[8] Land is also important because it provides a non-produced store of value, so that Say's Law no longer need apply. In this paper, where we do not explore Keynesian unemployment, we do not pursue this implication; nonetheless, as we show in Part II, the existence of an alternative store of value has profound implications for the dynamics of the economy.

growth arising in only one sector (the manufacturing sector or productive sector), land speculation has the further effect of diverting investment away from the productive sector and therefore lowering the growth rate. (d) With credit frictions, there is the further possibility that looser monetary policy or financial market deregulation results in more land speculation, rather than more productive investment, decreasing long run productivity and growth. Analysing these effects requires adding still two more complexities to the analysis—one has to move out of the standard one sector aggregate model into a two-sector model, with a learning or scale effect in one sector, with a possible contagion to the other; and one has to introduce, in a tractable way, credit frictions.

Part of the achievement of this paper and the series of which it is a part (Hirano and Stiglitz 2022a, 2022b, 2024) is that we have found tractable formulations that enable us to address the issues surrounding endogenous fluctuations, land speculation, and endogenous growth. This paper is meant to provide an overview of the research program, providing the simplest models illustrating each of the issues addressed, and is not intended to provide a complete analysis of any of the models presented. More comprehensive analyses are provided in the cited papers. While in some cases, we illustrate with a parametric model entailing strong assumptions, the cited papers also establish that the key results are more robust.

Several of the results run contrary to much received wisdom. For instance: (a) Standard overlapping generating models, focusing on steady states, show that with land there cannot be over saving, because were that the case, the value of land would be infinite; by contrast, while in our model, there cannot be *permanent* over saving, there can be episodic over savings, and under some parameter values, there cannot exist a steady state with a positive price of land: the only rational expectations dynamics entail persistent and endogenous fluctuations, some of which may entail over saving; (b) in the standard models, by the same token, the rate of interest cannot be below the rate of growth, while the evidence[9] is that that has long been the case. The variant of our model with credit frictions shows that that is indeed normally the case; (c) We show that under plausible conditions, there may be an infinity of rational expectations trajectories, while in the standard model, there is a unique rational expectations trajectory; (d) Among the rational expectations trajectories are those marked by persistent fluctuations, even in the absence of external shocks, while in the standard model, in the absence of

[9] See, e.g. Blanchard (2019)

external shocks, there is smooth convergence; (e) In a standard model with a representative agent maximizing intertemporal utility, given the initial capital stock, there is a unique value of land consistent with rational expectations, while in our model that may or may not be true; in some cases, there can be a whole range of initial prices of land consistent with rational expectations trajectories.

The paper is divided into four parts. In Part I, we present the basic overlapping generating model, which is similar to that of Diamond (1967), and explore the possible patterns of global dynamics. We present a markedly different picture of the dynamics of capitalism from that associated with standard models with individuals living infinitely long and no new economic agents with earning wages coming in. In Part II, we extend the model further by incorporating land. In Parts III and IV we introduce endogenous growth, without and with financial frictions.

Part I: Wobbly Economies

2. The Basic Model and The Basic Analytical Results[10]

We develop a simple overlapping generations model in which everyone in each generation is identical.[11] In each period young agents are born and live for two periods. There is no population growth rate (Hirano and Stiglitz 2022a study the case with population growth). Each person is endowed with one unit of labor when young, and supplies it inelastically, receiving wage income, w_t. For simplicity, we assume the labor supply is fixed and normalized at unity. Each young person also has e units of consumption goods as an endowment,[12] and saves a fraction s_t of the total income $(w_t + e)$, generating the first and the second period consumption of

(2.1) $c_{1t} = (1 - s_t)(w_t + e)$ and $c_{2t} = (1 + r_{t+1})s_t(w_t + e)$,

where $1 + r_{t+1}$ is the interest rate between period t and $t + 1$. The holdings of capital by the young at time t becomes the capital stock at $t + 1$. This generates the dynamic equation of aggregate capital stock, i.e.,

(2.2) $k_{t+1} = s_t(w_t + e)$.

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[10] The model we present here is based on Hirano and Stiglitz (2022a).
[11] The two-period overlapping generations model is the simplest model with heterogeneous agents, and heterogeneity is crucial for multiplicity of momentary equilibria to arise.
[12] This can be thought of as other fixed income, or inheritance from parents.

The savings rate is chosen to maximize utility $u_t = u(c_{1t}, c_{2t})$ subject to their budget constraint, yielding the first order condition:

$$\frac{\partial u(c_{1t}, c_{2t})/\partial c_{1t}}{\partial u(c_{1t}, c_{2t})/\partial c_{2t}} = 1 + r_{t+1}.$$

Competitive firms produce output by using capital and labor. Each firm has a standard neoclassical constant return to scale production function. Output per capita, y_t, is a function of capital per capita k_t,

(2.3) $\quad y_t = f(k_t) = f(K_t/L_t),$

where K_t and $L_t = 1$ are aggregate capital and labor inputs, and given our normalization, $k_t = K_t$. We assume a constant rate of depreciation of capital, $\delta \in [0,1]$. Rental and wage rates, R_t and w_t, satisfy

(2.4a) $\quad R_t = f'(k_t),$

and

(2.4b) $\quad w_t = f(k_t) - f'(k_t)k_t = w(k_t),$

with $f'(k_t) < 0$ and $w'(k_t) > 0$. The interest rate equals the (net) return to holding capital.

(2.4c) $\quad 1 + r_{t+1} = f'(k_t) + 1 - \delta.$

The dynamic equation for k_t can be written as

(2.5a) $\quad k_{t+1} = s_t(w(k_t) + e).$

If s were a constant, there is a unique momentary equilibrium, i.e., for any value of k_t, there is a unique value of k_{t+1}. We focus on the more interesting case where s is a function of the return on capital, which in turn depends on k_{t+1}. We assume in particular that $s_t = s(r_{t+1}) = s(r(k_{t+1}))$.[13] Define $\Omega(k_{t+1}) \equiv \frac{k_{t+1}}{s_t}$ and $W(k_t) \equiv k_t + e$. Then the economy's evolution is governed by the equation:

[13] A still more general savings function would have the savings rate a function of the wage and interest rate. Extending the model to incorporate this is straightforward. What is crucial for our analysis is the dependence of s on k_{t+1} (in our analysis, through the effect on the rationally expected return to capital).

(2.5b) $\Omega(k_{t+1}) = W(k_t)$

Central to this paper is the result that under quite general and plausible conditions, Ω is not monotonic, so there may be, at least for some values of k_t, multiple values of k_{t+1} satisfying (2.5b). Figure 1 illustrates what happens if there are multiple values of k_{t+1} corresponding to any Ω. Given k_t, there is a particular value of $W(k_t)$, but for a wide range of $W(k_t)$, there will be multiple values of k_{t+1}. $\Psi(k_t)$ is defined as the set of k_{t+1} for any k_t.

Differentiating $\Omega(k_{t+1})$ with respect to k_{t+1} yields

(2.6) $$\Omega'(k_{t+1}) = \frac{1}{s_t}\left(1 - \frac{dlog(s_t)}{dlog(1+r_{t+1})}\frac{dlog(1+r_{t+1})}{dlog(k_{t+1})}\right)$$

where $\frac{dlog(s_t)}{dlog(1+r_{t+1})}$ is the interest rate elasticity of savings. $\frac{dlog(1+r_{t+1})}{dlog(k_{t+1})} < 0$ is the elasticity of the interest rate with respect to the capital stock. These elasticities depend on the intertemporal elasticity of substitution in consumption (IES) and the elasticity of substitution between capital and labor (ES), respectively. For instance, if individuals have a separable utility function with a constant elasticity of utility with respect to consumption, θ, then

(2.7) $$\frac{dlog(s_t)}{dlog(1+r_{t+1})} = (1 - s_t)(\theta - 1),$$

$\frac{dlog(s_t)}{dlog(1+r_{t+1})}$ is negative (positive) if $\theta < 1(\theta > 1)$. The borderline case is the logarithmic utility function $(\theta = 1)$, for which $\frac{dlog(s_t)}{dlog(1+r_{t+1})} = 0$.

Similarly,

(2.8) $$\frac{dlog(1+r_{t+1})}{dlog(k_{t+1})} = \frac{k_{t+1}f''(k_{t+1})}{f'(k_{t+1})}\frac{f'(k_{t+1})}{f'(k_{t+1})+1-\delta} = -\frac{h(k_{t+1})s_L(k_{t+1})}{\sigma}$$

where σ is the elasticity of substitution, h is the ratio of the rental rate to the return to holding capital, $h(k_{t+1}) \equiv \frac{f'(k_{t+1})}{f'(k_{t+1})+1-\delta} < 1$, and $s_L(k_{t+1}) < 1$ is the share of labor.

Thus, if δ is large, σ is small, and S_L is large, $\frac{dlog(1+r_{t+1})}{dlog(k_{t+1})}$ is more negative.

From (2.6), a sufficient condition for $\Omega'(k_{t+1}) > 0$ is that $\frac{dlog(s_t)}{dlog(1+r_{t+1})} \geq 0$. That is, if the saving rate is a monotonically increasing function of the interest rate, there is a unique momentary equilibrium. If, however, for some values of k_{t+1}, $\frac{dlog(s_t)}{dlog(1+r_{t+1})} < 0$, Ω may not be invertible, i.e., for some values of k_t, there may be multiple values of k_{t+1}, all consistent with rational expectations. Figure 1 illustrates this. Intuitively, if everyone believes that the interest rate is low (investment is expected to be high), they will save a great deal, and the interest rate will be low (investment will be high). This is the case for a separable utility function if θ is sufficiently less than 1, so that marginal utility decreases strongly with consumption and the elasticity of substitution is sufficiently small enough (sufficiently less than unity), so that the increases in savings/investment lead to large decreases in the return to capital.

Define $\underline{\Omega}$ as the minimum value of Ω for which there are multiple values of k solving $\underline{\Omega} = \Omega(k)$; and similarly, $\overline{\Omega}$ as the maximum value of Ω for which there are multiple value of k. Then so long as for some value of k_t,

$$\underline{\Omega} < W(k_t) < \overline{\Omega} \ ,$$

there will be indeterminacy in the dynamic trajectory of the economy. Since $W'(k_t) > 0$ under standard assumptions on production functions, and $W(k_t = 0) = e,$ there exists values of $W(k_t)$ for which, for some values of e and for some value of k_t, there exist multiple values of k_{t+1} which satisfies (2.5b). Define $\underline{k}$ as the solution to $w(k_t) + e = \underline{\Omega}$ and similarly for $\overline{k}$. There are multiple momentary equilibrium when k is between $\underline{k}$ and $\overline{k}$.

Steady states

A steady state is defined by

(2.9) $\Omega(k^*) = W(k^*)$.

If Ω is monotonic, there is a unique k_{t+1} for any k_t, i.e., a unique momentary equilibrium. Even if $\Omega'(k_{t+1}) >\ 0$, so there is a unique momentary equilibrium, there may be multiple steady-states, i.e., multiple values of k such that $k^* = \Omega^{-1}(W(k^*))$. Obviously, in the more general case, explored here, there may be multiple steady states.

The right panel of Figure 2 illustrated that when there are multiple momentary equilibria, there can easily be three steady states, two of which are locally stable in a normal sense, that is, if at those steady states, there are multiple momentary equilibria, and if at the upper steady state, the economy "selects" the upper value of the correspondence, and at the lower one it selects the lower one; then with those selections, the economy converges to the given steady state for a small perturbation from the equilibrium.

3. Equilibrium aggregate dynamics

The competitive equilibrium is then defined as a set of prices $\{R_t, w_t\}_{t=0}^{\infty}$ and quantities $\{c_{1t}, c_{2t}, y_t, k_t\}_{t=0}^{\infty}$, given initial k_0 and $Y = y_0$, such that (i) each young agent chooses consumption and capital investment to maximize the expected utility under the budget constraints and the non-negative constraints, and (ii) the market clearing condition for goods, capital and labor are all satisfied.

3.1 Phase Transitions

Multiplicity of momentary equilibria translates into an infinity of dynamic paths, all consistent with rational expectations, though as we shall show, there are bounds within which the economy must oscillate. The economy may wobble, neither settling down to a steady state equilibrium nor diverging in an explosive manner. The economy can suddenly switch from one momentary equilibrium (say with a low savings rate) to another (with a high savings rate), showing that a laissez-faire market economy can be, in this sense, unstable. In each state where there are multiple momentary equilibria, the outcome depends on beliefs, a world of Keynesian animal spirits. With bullish expectations, interest rates are low, so savings and investment are high, sustaining that equilibrium; and similarly for bearish expectations. In the right figure in Figure 2, we trace out one possible "wobbly" trajectory, where the economy neither converges to a steady state or even a limit cycle. By contrast, the left figure in Figure 2 illustrates the standard dynamic process showing convergence to a steady state in an economy with a unique momentary equilibrium. Given k_t, there is a unique value of k_{t+1}, and that determines, in turn, k_{t+2}, etc.

We focus on the case where the correspondence ψ defined by (2.5b) can take three values of k_{t+1} for a given k_t over an interval $\underline{k} < k < \overline{k}$. Since k is *endogenous,* changing over time, the economy can move from a situation where $\underline{k} < k_t < \overline{k}$ to one in which $k_{t+1} < \underline{k}$ or $k_{t+1} > \overline{k}$, i.e., from a situation where there are multiple momentary equilibria to one in which there is a unique equilibrium or vice versa. We refer to this as a phase transition.

The dynamics of the economy depends crucially on the relationships between $\underline{k}, \overline{k}, k^L$ and k^H. Figure 3 presents possible relationships.

3.2 Persistent wobbles

Perhaps the most interesting situation is State (c), where there are three steady states, all unstable**.** This arises when $0 < \underline{k} < k^L < k^H < \overline{k}$. The wobbles are bound by k^L and k^H for large t. Even when the economy is at say k^H, the economy may suddenly jump in a fully rational expectations equilibrium to a smaller value of k. Nothing in the theory ensures that it will remain at k^H. The economy can bounce around infinitely without converging. A phase transition from a state with a unique momentary equilibrium to a state with multiple momentary equilibria occurs when the economy initially starts from the outer region of $\underline{k}$ or $\overline{k}$.

The other cases are: Three steady states, two stable, with unstable wobbly dynamics' three steady states, with the higher k stable and others unstable and unstable wobbly dynamic; and three steady states, two stable but unstable wobbly dynamics.

3.3 Local stability with global instability

As States (a), (b), and (d) show, when there is a stable steady state, the presence of multiple momentary equilibria may not be observed so long as there are only small perturbations. When the size of the shocks is sufficiently large, however, its hidden presence in the global system is suddenly revealed, and exhibited through large and persistent macroeconomic instabilities.[14]

3.4 Welfare decreasing innovations

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[14] One of the criticisms of multiple equilibria is that economic variables are not as volatile as models with multiple equilibria suggest. This criticism may not necessarily apply to our model because the existence of multiple momentary equilibria depends on the endogenous state variable, i.e. capital stock. This means that once the economy settles down into one of the stable steady-states, the macroeconomy exhibits only small changes in economic variables. It is only when the economy is sufficiently away from a stable steady-state that macroeconomic instabilities emerge.

Which of the configurations describes the economy depends on the parameters of the production and utility functions as well as the other parameters of the model and e. That means, of course, that changes in those parameters will change the economy's regime.[15] That raises one interesting possibility: An economy can initially be in a stable boom, with $k^H > \overline{k}$, i.e., the initial regime of the economy corresponds to State (a), but a seeming productivity improvement, which increases k^H increases $\overline{k}$ even more, moving the economy into a new state where the equilibrium is unstable, so that while the boom is strengthened—so long as it lasts—it becomes fragile, and eventually breaks. In other words, the regime of the economy changes to State (b) from State (a) and the economy may then converge to the (new) k^L, with per capita income and consumption sustained at a level markedly below the level prior to the innovation.[16]

Hysteresis effects of temporary shocks

With the productivity improvement, there are three different scenarios for a temporary technology change. Figure 4 illustrates this.

The first is that the economic boom might persist until the productivity reverts to the previous level. If this is the case, the economy converges back to the original steady-state, k^H, and the economic boom ends with a mild decline (from k^{HH} to k^H). This case can be interpreted as a normal business fluctuation.

The second is that even if the economy experiences a large-scale-collapse, if the decline is not deep enough, the economy can produce self-recovery and eventually converges back to the original steady-state, even if aggregate output is lower than the trend level temporarily. The dotted line in Figure 4 illustrates this scenario.

The third scenario is that if the decline gets sufficiently deep following the large-scale collapse, the economy can no longer generate self-recovery and ends up in the stagnation trap. The temporary boom has resulted in the economy moving from the upper stable equilibrium to the lower stable equilibrium.

4. A Parametric Model: CES utility and production functions

Assume that the representative individual's utility function entails constant elasticity of substitution between consumption in the two periods:

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[15] Hirano and Stiglitz (2022) refer to this as a state transition.
[16] See Section 6 of Hirano and Stiglitz (2022) for the detailed analysis of this case.

(4.1) $$u_t = \left((a_1)^{\frac{1}{\theta}}(c_{1t})^{\frac{\theta-1}{\theta}} + (a_2)^{\frac{1}{\theta}}(c_{2t})^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}},$$

where θ is intertemporal elasticity of substitution in consumption. c_{1t} and c_{2t} are gross complements (gross substitutes) if $\theta < 1$ ($\theta > 1$). a_1 and a_2 are weights on consumption in working and retirement periods, respectively, and affect the optimal consumption ratio between c_{1t} and c_{2t}.

Assume, moreover, a constant elasticity of substitution production function.

(4.2) $$Y_t = A\left(\alpha(\frac{K_t}{\omega_1})^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(\frac{L_t}{\omega_2})^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}},$$

where σ is the elasticity of substitution (ES). K_t and L_t are gross complements (gross substitutes) if $\sigma < 1$ ($\sigma > 1$). A is a productivity parameter, and $\frac{1}{\omega_1}$ and $\frac{1}{\omega_2}$ are capital productivity and labor productivity, respectively. $\alpha \in (0,1)$ reflects capital intensity in production.

It is straightforward to show

Proposition 1: If $\sigma < 1 - \theta$ and A is large enough (or given large enough A, if a_1/a_2 is large enough), for any given k_t, there are multiple values of k_{t+1} for some e.

Comparative statics

In our parametric model, it is easy to calculate how changing the parameters of the utility and/or production function or e changes the shape of $\Omega(k)$ and the value of $W(k)$. For instance, with greater e, $\underline{k}$ becomes smaller. This is because with greater e, aggregate savings get larger even for small k_t, so that expectations of high investments associated with a low interest rate can be consistent with rational expectations even in the region with small k_t. Likewise, with greater a_1/a_2, $\overline{k}$ becomes larger. This is because each person is more impatient, so that the saving rate gets lower and aggregate savings become small even for large k_t. Expectations of low investments associated with a high interest rate can be self-fulfilling even for large k_t.[17]

[17] Hirano and Stiglitz (2022) provides a numerical analysis showing how different parameter configuration in the (θ, σ)-plane give rise to each of the four cases, given other parameters.

5. A complete analytical characterization of a wobbly economy with involuntary unemployment -- The Leontief case

By focusing on a specific case where both utility and production functions are of Leontief forms, we can provide the necessary and sufficient condition for wobbly dynamics, and we can also provide a complete characterization analytically for all possible patterns of wobbly dynamics. The Leontief case corresponds to the limiting case of $\sigma \to 0$ and $\theta \to 0$ in (4.1) and (4.2). Moreover, unlike the previous analysis showing wobbly dynamics with full employment, in this case, wobbly dynamics with involuntary unemployment can arise if k is small enough.

The utility function and the aggregate production function are $u_t = min\left(\frac{c_{1t}}{a_1}, \frac{c_{2t}}{a_2}\right)$ and $Y_t = A\, min\, (\frac{k_t}{\omega_1}, \frac{L_t}{\omega_2})$. $k_t = \frac{\omega_1}{\omega_2} \equiv k^f$ is per capita capital level required to have full employment. If $k_t < \frac{\omega_1}{\omega_2}$, involuntary unemployment occurs, while if $k_t > \frac{\omega_1}{\omega_2}$, full employment is achieved, but not all capital is utilized. As k becomes lower compared with k^f, there is more involuntary unemployment. The optimal consumption between the working period and the retirement period satisfies $\frac{c_{1t}}{a_1} = \frac{c_{2t}}{a_2}$. Moreover, the saving rate is given by $s_t = \frac{1}{1+\frac{a_1}{a_2}(1+r_{t+1})}$ and it is a decreasing function of the interest rate.

K_{t+1} in general depends on r_{t+1} and w_t, and factor payments clearly depend on whether k_t is greater or less than k_f, with an inderminacy arising when $k_t = k^f$, though there may be a unique distribution consistent with the prior period's rational expectations.

Accordingly, the function $\Omega(k_{t+1})$ and $W(k_t)$ are written as follows:

(5.1a) $\Omega(k_{t+1}) = \left(1 + \frac{a_1}{a_2}(1 + r_{t+1})\right)k_{t+1} =$

$$\begin{cases} \left(1 + \frac{a_1}{a_2}\left(\frac{A}{\omega_1} + 1 - \delta\right)\right)k_{t+1} \quad if\ k_{t+1} < \frac{\omega_1}{\omega_2} \\ \left(1 + \frac{a_1}{a_2}(1 - \delta)\right)k_{t+1} \quad if\ k_{t+1} > \frac{\omega_1}{\omega_2} \end{cases}$$

and

(5.1b) $$W(k_t) = \begin{cases} e \quad if\ k_{t+1} < \frac{\omega_1}{\omega_2} \\ \frac{A}{\omega_2} + e \quad if\ k_{t+1} > \frac{\omega_1}{\omega_2} \end{cases}$$

Figure 5 illustrates the relationship between $\Omega(k_{t+1})$ and $W(k_t)$. $\Omega(k_{t+1})$ increases linearly with k_{t+1}, with slope $\left(1 + \frac{a_1}{a_2}\left(\frac{A}{\omega_1} + 1 - \delta\right)\right)$ until k^f is reached, then jumps down, and then increases again linearly but now at a lower slope, $1 + \frac{a_1}{a_2}(1 - \delta)$. As we can see, the relationship doesn't change much compared to the general case.

The maximum value of Ω in the capital shortage regime, i.e., $\overline{\Omega} = \left(1 + \frac{a_1}{a_2}\left(\frac{A}{\omega_1} + 1 - \delta\right)\right) \frac{\omega_1}{\omega_2}$ and the minimum value of Ω in the capital surplus regime, i.e., $\underline{\Omega} = \left(1 + \frac{a_1}{a_2}(1 - \delta)\right) \frac{\omega_1}{\omega_2}$. Note that neither depends on e. On the other hand, W clearly depends on e. There is a critical value of e at which $\underline{\Omega}$ just equals e, i.e., for low k_t there exists a unique momentary equilibrium, and another critical value of e at which $\overline{\Omega}$ just equals $\frac{A}{\omega_2} + e$, i.e., for high k_t there exists a unique momentary equilibrium.

Then the necessary and sufficient condition for stable wobbly dynamics, i.e., for reverse switching to be possible in both the capital shortage and capital surplus regimes, is that

(5.2a) $$\overline{\Omega} = \left(1 + \frac{a_1}{a_2}\left(\frac{A}{\omega_1} + 1 - \delta\right)\right)\frac{\omega_1}{\omega_2} > \frac{A}{\omega_2} + e.$$

i.e., when there is a capital surplus, the economy can switch to the capital shortage regime; and

(5.2b) $$\underline{\Omega} = \left(1 + \frac{a_1}{a_2}(1 - \delta)\right)\frac{\omega_1}{\omega_2} < e$$

i.e., when there is a capital shortage, the economy can switch to the capital surplus regime.

Proposition 2: The necessary and sufficient condition for stable wobbly dynamics in the Leontief case is given by

$$(5.3) \qquad e_2 \equiv \left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2} < e < \left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2} + \frac{A}{\omega_2}\left(\frac{a_1}{a_2}-1\right) \equiv e_3.$$

It is easy to see that there exists sets of parameter values for which (5.3) can be satisfied if $\frac{a_1}{a_2} > 1$.

When there are stable wobbly dynamics, there are also three steady states, $k^H = \frac{\frac{A}{\omega_2}+e}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)}$, $k^L = \frac{e}{\left(1+\frac{a_1}{a_2}\left(\frac{A}{\omega_1}+1-\delta\right)\right)}$ and k^f, where the latter is supported by a particular distribution of income, $\{w, r\}$. Each of the steady states is unstable, i.e., the economy can be in k^L, but bullish expectations lead to the belief that there will be high levels of investment and low interest rates, and individuals will save more, supporting those beliefs in a rational expectations trajectory. The economy can wobble infinitely.

We can indeed identify the parameter space where multiplicity of momentary equilibria can occur. We require

$$(5.4) \qquad \left[1+\frac{a_1}{a_2}(1-\delta)\right]\frac{\omega_1}{\omega_2} < \frac{A}{\omega_2} + e$$

and

$$(5.5) \qquad \left[1+\frac{a_1}{a_2}\left(\frac{A}{\omega_1}+1-\delta\right)\right]\frac{\omega_1}{\omega_2} > e.$$

By rearranging (5.4) and (5.5), we can show that there are four states, corresponding to the four states identified earlier in the more general case. We can also provide a complete characterization of all possible patterns of wobbly dynamics in the Leontief case. For the complete characterization, see Hirano and Stiglitz (2022a), who clarify the necessary and sufficient conditions under which each pattern arises. Furthermore, by using the analytically tractable Leontief-case, Hirano and Stiglitz (2022a) provide a complete analytic representation of all the possible state transitions, i.e., how an exogenous change in a parameter induces a regime shift of the macroeconomy from one state to another and then another.

Part II. Wobbly Economies with Land[18]

6. Introducing land

6.1. Basic Model

Introducing land into the model of Part I is easy—even if doing so adds considerable complexity to the dynamics. The production function of this economy is now $Y_t = F(K_t, L_t, T_t)$, where Y_t, K_t, L_t and T_t are, respectively, output, aggregate capital stock, labor force and land at date t. The aggregate supply of land is fixed $T_t = T$. Output per capita can be written as $y_t = f(\frac{K_t}{L_t}, \frac{T_t}{L_t})$. We normalize L and T at unity, and take produced output (which can be used either for consumption or investment) as our numeraire. Without loss of generality, we write $y_t = f(k_t)$ where $k_t \equiv \frac{K_t}{L_t}$ and $f_k(k_t)$ is the rental rate of capital, $w_t = f_L(k_t)$ is the wage rate, and $f_T(k_t)$ is the rental rate of land.

Total returns to owning land have to equal the return to capital, the rental rate minus depreciation, which equals the interest rate. That is, the capital arbitrage equation is

$$(6.1) \quad \frac{f_T(k_{t+1})}{P_t} + \frac{P_{t+1}}{P_t} = f_k(k_{t+1}) + 1 - \delta = 1 + r_{t+1},$$

where P_t is the market price of land at date t, δ is the rate of depreciation of capital, and, as before, $1 + r_{t+1}$ is the interest rate between period t and $t + 1$.

The competitive equilibrium is defined as a set of prices $\{1 + r_t, P_t, w_t\}_{t=0}^{\infty}$ and quantities, $\{c_{1t}, c_{2t}, k_t, y_t\}_{t=0}^{\infty}$, given initial k_0, such that (i) each young agent chooses consumption, land holdings, and capital investment to maximize expected utility under the budget constraints, and (ii) the competitive market clearing condition for goods, land, capital and labor are all satisfied.

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[18] The model we present here is based on Hirano and Stiglitz (2022b).

The big difference from the previous model is that savings can be used to hold land as well as capital, so the savings/capital accumulation equation becomes

(6.2) $k_{t+1} + P_t = s_t(w_t + e)$.

Other things being equal, higher land prices reduce capital investment. This is the obvious sense in which land holdings crowd out real capital accumulation.

Equilibrium paths consistent with rational expectations have to satisfy (6.1) and (6.2) for all dates. Given P_t and k_t, if s_t were fixed, we could easily solve (6.1) and (6.2) for P_{t+1} and k_{t+1}. But in general, the saving rate is a function of the interest rate, which in turn depends on k_{t+1}. Then equation (6.2) can be rewritten as specifying k_{t+1} as a function (correspondence) of k_t and P_t.

(6.3) $\Omega(k_{t+1}, P_t) \equiv \frac{k_{t+1}}{s_t(k_{t+1})} + \frac{P_t}{s_t(k_{t+1})} = w(k_t) + e \equiv W(k_t, e)$.

Here, we ask, how does the presence of land change wobbly dynamics? Multiplicity of momentary equilibria can still occur under the same general conditions. We can see in equation (6.3) that an increase in P_t shifts the function Ω up but (at least for small P_t), under the conditions in which there were a multiplicity of momentary equilibria without land, there still exists a multiplicity of momentary equilibria for some values of k_t. The dashed line in Figure 6-1 illustrates the situation. The reasoning is changed only slightly: given P_t, if individuals believe that the interest rates will be low, they save a lot, financing a high level of capital accumulation *beyond their land purchases*, thereby leading to low interest rates. Conversely, if they believe that the interest rates will be high, they save less and less savings finances less capital accumulation *beyond their land holdings*, leading to high interest rates.

As before, we can translate these results into a relationship between k_t and k_{t+1}, and see how the correspondence between k_t and k_{t+1} is affected by an increase in P_t. Going back to Figure 6-1, the increase in P_t doesn't shift Ω up uniformly: the amount by which it shifts up is proportional to $\frac{1}{s_t}$. If the savings rate increases with k_{t+1}, it means that an increase in P_t increases $\Omega(k_{t+1})$ more for low values of k_{t+1} than for high values. That means that as the price of land increases, we might go from a situation where corresponding to a particular k_t there were three values of k_{t+1}, to one where there is a single value of k_{t+1} (that is, initially $\underline{\Omega} \leq W(k_t, e) \leq \overline{\Omega}$ but as P_t increases, $\underline{\Omega} > W(k_t, e)$; or alternatively, we might go from a situation where there was a single value of k_{t+1}, to one in which there is now multiple values of k_{t+1}, i.e., initially, $\overline{\Omega} < W(k_t, e)$, with the increase in P_t, $\overline{\Omega} > W(k_t, e)$: *There is an endogenous transition between regimes in which there is a unique momentary equilibrium and multiple momentary equilibria*. Within a certain range of P_t, there are still multiple values of k_{t+1}, given k_t but once prices reach critical values, there is a single value of k_{t+1}.

This in turn means that if the land price gets too high, for a low value of $W(k_t, e)$ there is a unique equilibrium, entailing low k_{t+1}, high returns, and thus (as we will later show) explosive price dynamics. Thus, before P_t reaches such levels, the economy must switch to the low return equilibrium, which in turn leads to the crash of real estate prices. Moreover, because Ω is shifted up more for low values of k_{t+1} than for high values, at least for some production and utility functions, including the Leontief utility function upon which we focus, the range of k_t for which there are multiple equilibria is increased as P_t rises. This implies an increased range of potential variability in economic activity.

6.2. Land price dynamics

So far, we have focused on the dynamics of k_t. From (6.1) we can see the dynamics of P_t, which is interlinked with that of k_t:

(6.4) $$\frac{P_{t+1}}{P_t} = 1 + r_{t+1}(k_{t+1}) - \frac{D(k_{t+1})}{P_t} > \text{ or } < 1 \text{ as } P_t > \text{or} < \frac{D(k_{t+1})}{r_{t+1}(k_{t+1})},$$

where land rents, $D(k_{t+1}) \equiv f_T(k_{t+1})$: land prices go up or down depending on whether P_t is greater or less than $\frac{D(k_{t+1})}{r_{t+1}(k_{t+1})}$.

(6.4) and (6.2) define (together with the standard boundary value conditions, $0 \leq \{k_{t+1}, P_t\} < \infty$) the set of rational expectations dynamic trajectories. The basic insight of wobbly dynamics is that because of the multiplicity of momentary equilibria, k_t can suddenly increase dramatically, causing the interest rate to fall, leading prices of land to start declining: previously, it may have looked as if the economy was on a trajectory with an explosive real estate boom, but with the fall in r, the real estate boom collapses. Of course, we have to check *simultaneously* movements in k and P, showing that they are consistent with a rational expectations equilibrium going forward. The following analysis does this.

Figure 7 illustrates wobbly dynamics with endogenous fluctuations in land prices. The curve giving k_{t+1} as a function of k_t constantly shifts down and up as the price of land increases and decreases. Land price booms in the real estate sector crowd out productive capital in the capital-intensive sector (manufacturing).

At the same time, we can trace out the dynamics of P, from equation (6.4), noting that if the economy selects a high return (a low k) momentary equilibrium, the interest rate r will be high, so the curve giving P_{t+1} as a function of P_t will be steep—land prices will look like they are exploding. As land prices increase, land speculation crowds out capital accumulation, so the rate of interest increases even more. Moreover, wages decrease, lowering capital accumulation further. The explosion in land prices accelerates. But, of course, along a rational expectations equilibrium this can't continue forever, and the market knows this. Thus, at some time, the economy *must* select a low return equilibrium, leading land prices to collapse, and return to a sustainable level. This has to happen before land prices rise so high that there is a unique momentary equilibrium—providing the upper bound on land prices within wobbly dynamics.

This provides the heuristics of the two-dimensional dynamics. In Part I (and more extensively in Hirano and Stiglitz (2022a)), we analysed dynamics with only capital, we showed the essence of the analysis can be illustrated with the Leontief production and utility functions, in which all possible trajectories can easily be traced out. The same is true here. Accordingly, we focus on that case.

7. An analytically tractable Case -- The Leontief case

The utility function is the same as in section 5, but the production function is now

(7.1) $$Y_t = A \min\left[\frac{K_t}{\omega_1}, \frac{L_t}{\omega_2}\right] + DT_t,$$

where the return to land is fixed and does not depend on the amount of labor or capital. One interpretation of this aggregate production function is that there are two sectors. In one (real estate), land is the primary input and we simplify by assuming it is the only input. In the other (capital-intensive sector), capital and labor are the main inputs for production. With this interpretation, our analysis shows how fluctuations in the real estate sector affect production and employment in the productive, capital-intensive sector.[19][20]

With land, the savings/capital accumulation equation is written as

$$(7.2)\quad k_{t+1}+P_t=\frac{w_t+e}{1+\frac{a_1}{a_2}(1+r_{t+1})}.$$

The capital arbitrage equation becomes simplified to

$$(7.3)\quad \frac{D}{P_t}+\frac{P_{t+1}}{P_t}=1+r_{t+1}=\begin{cases}\frac{A}{\omega_1}+1-\delta & if\ k_{t+1}<\frac{\omega_1}{\omega_2}\\[2ex] 1-\delta & if\ k_{t+1}>\frac{\omega_1}{\omega_2}\end{cases}$$

(7.2) and (7.3), together with k_0 and P_0, define the dynamics system for which we will provide a complete global analysis. The key property of the price dynamics which simplifies the analysis is that the dynamics simply depend on whether $k_{t+1}>or<\frac{\omega_1}{\omega_2}$, i.e., on whether there is unused capital or labor.

The model entails five key production and technology parameters, $\frac{A}{\omega_1}, \frac{A}{\omega_2}, D, e, \delta$, and one key taste parameter $\frac{a_1}{a_2}$. We solve for six endogenous variables at each date t, $\{y,\ k$, employment, $w,\ r$, and $P\}$. In the following sections, we will demonstrate the remarkable richness of dynamics that can be generated by such a simple model.

7.1 The Implications of the savings-investment equation

Key to preventing prices of land from exploding along a trajectory with a real estate boom (or imploding after the crash) is the existence of multiple momentary equilibria, which allows the economy to switch from a high return equilibrium to a low or vice versa. Earlier, we noted that whether there were multiple momentary equilibria depends

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[19] As before, capital is assumed to depreciate at the fixed rate δ, and the net return to investment when capital is scarce, $\frac{A}{\omega_1}-\delta$, is assumed positive, while the net return when capital is abundant is just $-\delta$.

[20] See Hirano and Stiglitz (2022b) for the case where D endogenously changes depending on the amount of capital or labor.

on the value of P. We now investigate the ranges of values in our specific model for which there are multiple momentary equilibria.

The functions Ω and W are written as

(7.4) $$\Omega(k_{t+1}, P_t) \equiv (k_{t+1} + P_t)\left(1 + \frac{a_1}{a_2}(1 + r_{t+1})\right) = w_t + e \equiv W(w_t, e).$$

where r_{t+1} depends on whether $k_{t+1} > or < \frac{\omega_1}{\omega_2}$. Figure 6-2 (which redraws Figure 6-1 for the specific preferences and technology assumed here) illustrates.

$\Omega(k_{t+1}, P_t)$ increases linearly with k_{t+1}, with slope $1 + \frac{a_1}{a_2}\left(1 + \frac{A}{\omega_1} - \delta\right)$, until k^f is reached, then jumps down, and then increases again linearly but now at a lower slope, $1 + \frac{a_1}{a_2}(1 - \delta)$. Moreover, $W(k_t, e) = e$ or $\frac{A}{\omega_2} + e$ depending on whether $k_t < or > \frac{\omega_1}{\omega_2}$. As we can see, the relationship doesn't change much compared to the general case.

The model without land is just the special case where $P_t = 0$. Note that $\Omega(0,0) = 0$. By continuity, if (5.2a) and (5.2b) are strictly satisfied, for small P_t there still exists multiple momentary equilibria. We now investigate more precisely the conditions under which multiple equilibria occur. To do this, we derive several critical values of land prices. For most of the analysis, we focus only on trajectories where there is strictly a capital shortage or surplus. Even so restricting ourselves, we show that there can be multiple dynamic paths (indeed, in some cases an infinity of such paths) all consistent with rational expectations in which land prices can endogenously fluctuate without converging.[21]

Wobbly dynamics requires only that when prices are seemingly exploding and becoming high, with the economy in a high return regime, it can switch into a low return regime; and when prices are imploding, and becoming too low, with the economy in a low return regime, it can switch into a high return regime.

In Figure 6-2, the dashed line shows how an increase in P_t shifts Ω up. We define the point B where the "low return" line of Ω hits the full employment line $\frac{\omega_1}{\omega_2}$. As P_t

[21] Hirano and Stiglitz (2022b) show that when the economy sometimes selects the momentary equilibrium with just full employment, there is an even richer set of wobbly dynamics.

increases, B moves up, and eventually, there does not exist a low return (high k) equilibrium when $W_t = e$. There either exists no equilibrium, or only the high return (low k) equilibrium. The maximum value of P before the high k equilibrium disappears is called P_2. It is the solution to

(7.5a) $\left(\frac{\omega_1}{\omega_2} + P_2\right)\left(1 + \frac{a_1}{a_2}(1-\delta)\right) = e$; or $P_2 \equiv \frac{e - \left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)}$.

Clearly, for there to exist wobbly dynamics, $P_2 > 0$, i.e.,

(7.6a) $e > \left(1 + \frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2} = e_2$.

Otherwise, once in the capital shortage (high return) equilibrium, the economy could never switch out, and the price of land would increase without bound.

Similarly, define the value of P_t, P_3, where land holdings just crowd out capital accumulation enough that at the high wage there is a capital shortage. P_3 satisfies

(7.5b) $(\frac{\omega_1}{\omega_2} + P_3)\left(1 + \frac{a_1}{a_2}(\frac{A}{\omega_1} + 1 - \delta)\right) = \frac{A}{\omega_2} + e$; or $P_3 = \frac{e - \left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2} - \left(\frac{a_1}{a_2} - 1\right)\frac{A}{\omega_2}}{1+\frac{a_1}{a_2}(\frac{A}{\omega_1}+1-\delta)}$.

$P_2 > P_3$ if

(7.6b) $e > \frac{\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}}{\frac{a_1}{a_2}} = \frac{e_2}{\frac{a_1}{a_2}}$.

So long as $\frac{a_1}{a_2} > 1$, i.e., so long as individuals put more weight on consumption when young than when old (which is equivalent to discounting future consumption), (7.6b) is satisfied if (7.6a) holds. Note that if $P_3 < 0$, if the economy is in the low return equilibrium, it can always switch into the high return equilibrium. Moreover,

(7.6c) $P_3 < or > 0$ as $e < or > \left(1 + \frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2} + \left(\frac{a_1}{a_2} - 1\right)\frac{A}{\omega_2} = e_2 + \left(\frac{a_1}{a_2} - 1\right)\frac{A}{\omega_2} = e_3$.

Assume that the economy is in a high return equilibrium, with land prices increasing. Once land prices get near P_2 from below, the only rational expectations equilibrium trajectory is one entailing switching to a low return regime—otherwise, the price the following period will exceed P_2, and it will not be possible to switch to a low return regime, so that prices would have to explode. Once it switches (and the switch can occur well before reaching P_2), land prices start to fall. But they can't fall too far, for we know if they fall below P_3, the unique momentary equilibrium is the low return equilibrium and prices would fall forever, eventually becoming zero or negative (if the capital arbitrage equation is to be satisfied). Hence, so long as the economy switches back to the high return regime before P reaches P_3, prices won't implode.

Tightening the bounds

We can put somewhat tighter bounds on fluctuations. There is a critical value of P, P_1, such for any P_t higher than P_1, when wages are low, there is (at most) a single low-return momentary equilibrium. P_1 is given by the solution to

(7.5c) $$P_1\left(1+\frac{a_1}{a_2}\left(\frac{A}{\omega_1}+1-\delta\right)\right)=e;\text{ or }P_1\equiv\frac{e}{\left(1+\frac{a_1}{a_2}\left(\frac{A}{\omega_1}+1-\delta\right)\right)}.$$

P_1 is the value of P where $\Omega(0,P)=e$, i.e. Ω intersects the vertical axis at e (where wages are zero) and is labelled A in Figure 6-2. Depending on parameter values, $P_1>P_2$ or $P_2>P_1$. Also, $P_1>P_3$ if

(7.6d) $$\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}+\left(\frac{a_1}{a_2}-1\right)\frac{A}{\omega_2}>0.$$

If $\frac{a_1}{a_2}>1$, as we have assumed, (7.6d) is always satisfied.

Thus, all that is required for switching to be possible (looking just as the savings-investment equation) is that $P_3<P_2$, a sufficient conditions for which (under are maintained hypothesis that $\frac{a_1}{a_2}>1$) is that (7.5a) be satisfied.

A full dynamic analysis, however, has to look *simultaneously* at the capital arbitrage and the savings-investment equations and has to consider the possibility of a "just full employment" equilibrium. In the general case with $D>0$, that turns out to be conceptually straightforward, but notationally complex, so we focus on a special case of interest in its own right, where $D=0$. The key results extend to the case where $D>0$, as shown in Hirano and Stiglitz (2022b).

8. Dynamics Unproductive land (pure land bubbles)

When $D = 0$, land is just a store of value. Land may have value today because it can be sold tomorrow for a positive price. Such situations have come to be called "bubbles". From (6.1), when $D = 0$, land prices grow at the rate of the return to capital. Since in both the capital shortage and capital abundance regimes, the return to capital is fixed, this means that in the former, land prices increase exponentially, and in the latter they fall exponentially. This greatly simplifies the analysis.

8.1 The existence and non-existence of a steady state

We first derive conditions under which a steady-state with positive values of land bubbles exists. If a steady state exists, the net return on capital must be zero—if it is positive, land prices must be ever increasing; if negative, ever decreasing. But this means that the steady state must entail just full employment. The savings rate is then just $\frac{1}{1+\frac{a_1}{a_2}}$. Moreover, the exhaustion of product equation gives us

(8.1) $\omega_1(r^* + \delta) + \omega_2 w^* = A,$

which determines $w^* = \frac{A}{\omega_2} - \frac{\omega_1}{\omega_2}\delta$ when $r^* = 0$. In steady state P^* is constant. If savings are just sufficient to sustain full employment,

(8.2) $P^* = \frac{\frac{A}{\omega_2} - \frac{\omega_1}{\omega_2}\delta + e}{1+\frac{a_1}{a_2}} - \frac{\omega_1}{\omega_2}.$[22]

The existence of a steady state with land having a positive value requires $P^* > 0$, i.e., $\frac{\frac{A}{\omega_2} - \frac{\omega_1}{\omega_2}\delta + e}{1+\frac{a_1}{a_2}} > \frac{\omega_1}{\omega_2}.$

Proposition 3. There exists a steady state with positive land prices in a bubble economy if and only if

[22] This is derived directly from the steady-state investment equals savings equation: $k^f + P^* = s^*(w^* + e)$, where $s^* = \frac{1}{1+\frac{a_1}{a_2}}$ and $P^* \geq 0$.

(8.3) $\quad e > \frac{\omega_1}{\omega_2}\left[1 + \delta + \frac{a_1}{a_2} - \frac{A}{\omega_1}\right] = e_2 + \left(\frac{a_1}{a_2}\delta + \delta - \frac{A}{\omega_1}\right)\frac{\omega_1}{\omega_2} \equiv \hat{e},$

where, it will be recalled, $e_2 \equiv \left(1 + \frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}$. (8.3) is always satisfied if $\frac{A}{\omega_1} > 1 + \delta + \frac{a_1}{a_2}$, i.e., if the productivity of capital is high enough. If $\frac{A}{\omega_1} < 1 + \delta + \frac{a_1}{a_2}$, for $P^* > 0$, e has to be sufficiently large. If $e \leq \hat{e}$, no steady state with land bubbles exists. The intuition is that to sustain just full-employment k^f and a strictly positive land price, the productivity of the economy, $\frac{A}{\omega_1}$, and/or e, has to be large enough so that the economy can generate enough savings.

When there exists a steady state full employment equilibrium, at that equilibrium, with $w = w^*$, there are multiple momentary equilibria, and so that equilibrium is not stable in the sense defined earlier. Of course, we have to check that any such deviation from the steady state is consistent with a rational expectations trajectory going off infinitely far into the future. We now show that that is in general the case, by exploring in greater detail wobbly dynamics.

8.2 Wobbly dynamics in the case with $D = 0$

The conditions for wobbly dynamics can now be easily ascertained and compared to those for the existence of a steady state. The price dynamics are given by, for the high and low return regimes respectively:

(8.4) $\quad P_{t+1} = \left(\frac{A}{\omega_1} + 1 - \delta\right)P_t$ and $P_{t+1} = (1-\delta)P_t,$

implying that in the high return regime, prices rise at the rate $\frac{A}{\omega_1} + 1 - \delta$ and in the low return regime, they fall at the rate $1 - \delta$. Price dynamics with $D = 0$ are illustrated in Figure 8, showing the price initially rising exponentially, then falling.

When $D = 0$, for wobbly dynamics to occur, the lowest possible land price satisfying both the capital arbitrage and the saving-investment equations along a rational expectations trajectory is $P_{min} \equiv max\{0, P_3\}$.

Consider first the case where $P_3 < 0$. Assume for the moment that $P_1 > P_2$. Consider a trajectory which begins with P_t small ($< P_2$). There are multiple momentary equilibria. Assume the economy chooses the high return equilibrium. The price of land starts to rise. So long as the economy switches back to a low return equilibrium before $P_t = P_2$, it can be on a rational expectations trajectory. If it switches, then the price of land falls exponentially. It can then switch back, at any time, to the high return equilibrium.

As P_t increases, Ω shifts up, to the point where eventually point B in Figure 6-2 is about to rise above the $W_t = e$ line. Just prior to hitting that point, the only equilibrium is that with low returns, for if the economy were to remain in the high return regime, the price the following period would exceed P_2, and would have to increase thereafter. Once the economy switches to the low return equilibrium, land prices start falling.

Within the bounds of land prices where there are multiple momentary equilibria, the economy with land can go from one momentary equilibrium to another, with prices of land rising and falling by the arbitrage equation, endogenously fluctuating within a range of $(0, P_2]$, neither converging nor diverging. The case of $P_3 < 0$ but $P_1 < P_2$, is similar except the moment the price exceeds P_1, the economy switches into the low return regime.

The dynamics for the case of $P_3 > 0$ is the same, with the analysis only slightly more complex. Assume $P_1 > P_2$ and that initially P_t lies between P_3 and P_2. Assume the economy chooses the high return regime. Then P_t increases exponentially. So long as it switches back to the low return equilibrium before P_t exceeds P_2, it can be on a rational expectations trajectory. The case where $P_1 < P_2$ can be handled similarly, except now, when P exceeds P_1, the only equilibrium is the low return equilibrium, and the economy immediately switches to falling prices.

Critical Points and Endogenous phase transitions

Note that there may be a unique momentary equilibrium *consistent with a rational expectations trajectory,* even if at that land price, there is still multiple momentary equilibria according to the savings-investment equation. For if the economy doesn't select the "right" equilibrium, the capital arbitrage equations result in land prices moving to values where there is a unique momentary equilibrium, such that going forward prices either implode or explode. We now derive those critical points, which help refine the boundaries of the economy's fluctuations.

If land prices get near P_2 from below, there exists a land price level

$$\hat{P} \equiv min\left(\frac{P_2}{\frac{A}{\omega_1}+1-\delta}, P_1\right)$$

such that if $\hat{P} < P_t$, the low return equilibrium is the unique momentary equilibrium. We have already explained the reason: if land prices were to continue to rise (in the high return equilibrium), price the next period would exceed P_2, which would imply that they have to increase forever, inconsistent with a rational expectations trajectory. Hence, after P_t exceeds $\hat{P}$, land prices will surely start to fall, that is, a rational expectations trajectory *must* endogenously become bearish, leading to an endogenous crash. The "bearish" momentary equilibrium with collapsing land prices is the *unique* momentary equilibrium.

From another angle, as land prices are rising explosively, the crowding out effect gets strong, so the resource allocation deteriorates over time. Once the deterioration exceeds a certain threshold, the endogenous crash in land prices occurs.

A similar logic applies as prices fall. If $P_3 > 0$, there exists a price, P_{min}, below which the price implodes. If the economy remains in a low return regime, with prices falling exponentially, eventually $P_t < P_3$, after which the economy cannot return to the high return regime, and P would continue to fall, and would asymptotically converge to zero. From (8.4), $P_{t+1} = (1-\delta)P_t \geq P_3$ or $P_t \geq \frac{P_3}{1-\delta}$. This means that when $P_3 \leq P_t < \frac{P_3}{1-\delta}$, for wobbly dynamics to arise, the economy needs to switch to the high return regime: there is a unique momentary equilibrium consistent with rational expectations wobbles.

If $P_3 \leq 0$, the economy can always switch to the high return regime. Hence there is no lower limit to the price, and correspondingly no lower limit of price at which there must be a unique momentary equilibrium, with the economy switching to "bullish."

We can summarize: for wobbly dynamics to arise, the lowest level that price can achieve on a rational expectations trajectory is $P_{min} = min\{0, P_3\}$ and in a downward movement of prices (a "crash") the lowest level that price can attain along a rational expectations trajectory, without a switch to a bullish equilibrium is $\underline{P} \equiv min\left\{0, \frac{P_3}{1-\delta}\right\}$.

Proposition 4. Sufficient conditions for wobbly dynamics with $D = 0$ to exist are

(8.5) $P_3 \leq min\left(\frac{P_2(1-\delta)}{\frac{A}{\omega_1}+1-\delta}, P_1(1-\delta)\right),$

that is,

(a) $P_2 > 0$ and $P_3 \leq 0$.

(b) $P_2 > 0, P_3 > 0, \frac{P_2}{\frac{A}{\omega_1}+1-\delta} < P_1,$ and $\frac{P_2}{\frac{A}{\omega_1}+1-\delta}(1-\delta) \geq P_3$.

(c) $P_2 > 0, P_3 > 0, P_1 < \frac{P_2}{\frac{A}{\omega_1}+1-\delta},$ and $P_1(1-\delta) \geq P_3$.

Because P_i is a function of e, for each of these cases, we can define critical values of e, say, as a function of the other parameters, for which wobbly dynamics exists. We can define e_{12} as the values of e for which $P_1 = \frac{P_2}{\frac{A}{\omega_1}+1-\delta}$; e_{23} as the values of e for which $\frac{P_2}{\frac{A}{\omega_1}+1-\delta}(1-\delta) = P_3$; and e_{13} as the values of e for which $P_1(1-\delta) = P_3$. e_2 and e_3, defined earlier are just the values of e at which $P_2 = 0$ and $P_3 = 0$, respectively: $e_3 = e_2 + \left(\frac{a_1}{a_2} - 1\right)\frac{A}{\omega_2} > e_2$ if $\frac{a_1}{a_2} > 1$ as we have assumed. Using this notation, a straightforward translation of the conditions for cases (a) to (c) establishes

Proposition 5. Sufficient conditions for wobbly dynamics with $D = 0$, corresponding to the three cases identified in Proposition 4, are that

(a) $e_2 < e \leq e_3$

(b1) $\delta < \frac{a_2}{a_1}, e_3 < e < e_{23}\ and\ \widehat{\left(\frac{A}{\omega_1}\right)} \leq \frac{A}{\omega_1},$

(b2) $\delta < \frac{a_2}{a_1}, e_3 < e < min\{e_{23}, e_{12}\}\ and\ \delta < \frac{A}{\omega_1} < \widehat{\left(\frac{A}{\omega_1}\right)}$

(b3) $\frac{a_2}{a_1} \leq \delta < \frac{A}{\omega_1}, e_3 < e < min\{e_{23}, e_{12}\}$

(c) $\frac{a_2}{a_1} \leq \delta < \frac{A}{\omega_1}, max\{e_3, e_{12}\} < e \leq e_{13}.$

where $\widehat{\left(\frac{A}{\omega_1}\right)} \equiv \frac{\delta\left(1+\frac{a_1}{a_2}(1-\delta)\right)}{1-\frac{a_1}{a_2}\delta}$.

The Appendix of Hirano and Stiglitz (2022b) shows that the set of values satisfying these restrictions, in each of the cases, is non-empty. While there is a rich set of parameters for which wobbly dynamics exists, wobbly dynamics cannot occur if, given $\frac{A}{\omega_1}$, e is too small or too large. In Part I, we established that that wobbly dynamics exists if and only if condition (a) of Proposition 4 is satisfied. *With land, wobbly dynamics can arise in a richer set of parameters.*

Wobbly dynamics and steady states

The parameter space can now be divided into four regions, depending on whether there exists both a rational expectations steady state and wobbly dynamics, neither, or only one or the other.

Consider, for instance, case (a). Recall that a necessary and sufficient condition for a rational expectations steady state is that $e > \hat{e}$. From (8.3), $\hat{e}$ is a negatively sloped line with the slope -1 (in the case where $\frac{\omega_1}{\omega_2} = 1$), depicted in Figure 9. *Hence, between e_3 and e_2, and below the line $\hat{e}$, the only rational expectations equilibrium with $P_t > 0$ is a wobbly economy.*

A similar analysis holds for case (b1) except now, whenever there exists wobbly dynamics, there also exists a steady state with just full employment. There also exist parameter values for which there exists a rational expectations just full employment steady state but no wobbly dynamics, and still others where there exists no rational expectations trajectory with positive land prices.[23]

Land price boom-bust cycles

Figure 10 illustrates endogenous land price boom-bust cycles for the case where $P_3 > 0$. Within the bounds $\underline{P} \leq P_t \leq \hat{P}$, a boom can always crash, and a downturn can always be reversed. Once prices go above $\hat{P}$, there has to be a crash, and once prices go below $\underline{P}$, there can be a boom, so long as they remain above P_{min}.

With rising and falling land prices and capital accumulation, aggregate wealth defined as $(k_t + P_t)$ is also rising and falling. Aggregate consumption also shows large swings, i.e., with the savings rate low in the high return regime but wages low, and

[23] Proposition 2d in Hirano and Stiglitz (2022b) provides a complete characterization of the relationship of wobbly dynamics with and without land and the existence of a steady state, with the parameters giving rise to each situation illustrated in Figure 4-2 of that paper.

conversely in the low return regime. Moreover, the macroeconomy can be plagued by repeated periods of unemployment.

Part III. Endogenous Growth without financial frictions

9. The Basic Model: Endogenous Growth with Land

9.1 The Environment

We continue with a variant of the standard two-period overlapping generations model where there is a competitive economy with productive capital and labor. For simplicity in this long run analysis, we employ a special parametric utility function, the same for each person:

(9.1) $u_t = \log(c_t) + \beta\log(c_{t+1})$,

where u_t is the utility of agent, and c_t and c_{t+1} are consumption levels when young and old, respectively.

As in the previous section, there are two sectors, a productive sector (in our simple version, the only sector using capital; more generally, the capital-intensive sector) and a real estate sector (in our simple version, it only uses land; more generally, land is the primary input for production).

The budget constraint of each agent is unchanged, but for convenience, we rewrite as

(9.2) $k_{t+1} + P_t x_t = w_t$ and $c_{t+1} = R_t^c k_{t+1} + R_t^x P_t x_t$,

where k_{t+1} and x_t are the capital investment and land holdings of each young person at date t. $R_t^c = R_{t+1} + 1 - \delta$ is the total return per unit of capital investment made at date t, where R_{t+1} is the rental rate of capital at date $t+1$ and δ is the depreciation rate of capital. P_t is, as before, the price of land at date t in terms of consumption goods. $R_t^x = \frac{D_{t+1}}{P_t} + \frac{P_{t+1}}{P_t}$ is the return to land between date t and $t+1$, where D_{t+1} is land rents at date $t+1$.

Since both capital investment and land holdings are perfect substitutes, the rates of returns must be the same. Hence, the no-arbitrage equation is, as before

(9.3a) $R_{t+1} + 1 - \delta = \frac{D_{t+1}}{P_t} + \frac{P_{t+1}}{P_t}$.

Each young person maximizes (9.1) subject to (9.2) by considering (9.3a). The solutions are given by $k_{t+1} + P_t x_t = \frac{\beta}{1+\beta} w_t \equiv s w_t$ and $c_t = \frac{1}{1+\beta} w_t$, and $c_{t+1} = (R^c_{t+1} + 1 - \delta) \frac{\beta}{1+\beta} w_t$, where s is the saving rate.

9.2 Production side of the economy

We simplify the production function by assuming a Cobb Douglus production function, but now, we assume the production (capital-intensive) sector exhibits standard Marshallian external increasing returns to scale in capital (Aoki 1970, 1971; Frankel 1962; Romer 1986).[24] The production function of each firm j is thus given by

(9.4a) $y_{tj} = \left(k_{tj}\right)^{\alpha} \left(\chi(K_t) l_{tj}\right)^{1-\alpha}$,

where k_{tj} and l_{tj} are capital and labor inputs of firm j. $\chi(K_t)$ is labor productivity, with $\chi'(K_t) > 0$, where K_t is aggregate capital stock at date t. When we aggregate over all firms, we get at the aggregate level returns to scale. We assume $\chi(K_t)$ takes on a particular functional form.

(9.5) $\chi(K_t) = a K_t$.

This is a key (though conventional) simplifying assumption to generate endogenous growth.

In the real estate sector, one unit of land produces D_t units of consumption goods. To ensure the existence of the balanced growth path, we focus on a case where there is a full spillover from the productive sector to the real estate sector, so that land productivity grows at the economy's growth rate,

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[24] There is a large literature justifying the presence of these Marshallian increasing returns externalities, which we will not repeat; the existence of these effects is at the center of much of the endogenous growth literature. Stiglitz and Greenwald (2014), explain why innovation is likely to be more centered in the capital-intensive (industrial) sector rather than the agricultural or craft sector, and why the capital-intensive sector generates more externalities to the rest of the economy.

(9.6) $D_t = \epsilon\chi(K_t) = \epsilon a K_t,$

where $\epsilon \geq 0$ is a parameter that captures the level of land productivity relative to labor productivity in the productive sector. ϵ can also be interpreted as the parameter that captures the size of the spillover to the real estate sector from the productive sector; when ϵ is larger, with an increase in K_t, the productivity increase in the real estate sector is larger. When $\epsilon = 0$, there is no spillover.

Apart from the introduction of land, this is the standard AK model. Aggregate output Y_t is then

(9.7a) $Y_t = (K_t)^{\alpha}(\chi(K_t)L_t)^{1-\alpha} + D_t X_t,$

The aggregate supply of land and labor are fixed and we again normalize L and X at unity.[25]

9.3 The behaviour of individual firms

The individual firm ignores its tiny influence on the aggregate capital stock and thus on the productivity of its own worker. Thus, each firm employs capital and labor up to the point where its private marginal product equals the rental rate of capital and the wage rate, respectively (see Appendix O1 of Hirano and Stiglitz (2024) for proof).

(9.8a) $R_t = \alpha A,$

(9.8b) $w_t = (1-\alpha)AK_t,$

where $A \equiv a^{1-\alpha}$. The wage rate grows at the same rate of aggregate capital stock, so $\frac{w_{t+1}}{w_t} = \frac{K_{t+1}}{K_t}$ holds, and the rental rate of capital is constant. It follows directly from (9.8a) that

(9.8c) $R^c \equiv R + 1 - \delta = \alpha A + 1 - \delta.$

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[25] As another interpretation, this production function corresponds to the limiting case ($\sigma \rightarrow \infty$) of the following CES production function, $Y_t = \left(\gamma_1((K_t)^{\alpha}(\chi(K_t)N_t)^{1-\alpha})^{\frac{\sigma-1}{\sigma}} + \gamma_2(D_t X_t)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$, where σ is the elasticity of substitution between the manufacturing and real estate sectors, i.e., between $(K_t)^{\alpha}(\chi(K_t)L_t)^{1-\alpha}$ and $D_t X_t$, and γ_1 and γ_2 are parameters and are set to be unity.

The aggregate production function (9.7a) can be written as

(9.7b) $Y_t = AK_t + D_t = AK_t + \epsilon a K_t = (1 + \epsilon a^{\alpha}) AK_t,$

where AK_t is aggregate production in the productive sector in equilibrium. Aggregate output grows at the same rate of aggregate capital. (9.8a) holds at every moment of time, implying that R and R^c are fixed and do not change over time.

9.4 The competitive equilibrium

The competitive equilibrium is defined as a set of prices $\{P_t, w_t, R_t\}_{t=0}^{\infty}$ and quantities $\{c_t, x_t, k_{t+1}, D_t, K_t, Y_t\}_{t=0}^{\infty}$, given an initial K_0, such that (i) each young person chooses consumption when young and old, land holdings, and capital investment to maximize utility (9.1) under the budget constraint (9.2), and (ii) the market clearing conditions for capital, labor, land, and goods are all satisfied. In the initial period $t = 0$, all land is held by the initial old agents.

9.5 Aggregate Dynamics

To analyze dynamics, we introduce a variable $\phi_t = \frac{P_t}{AK_t}$, i.e., the size of land speculation relative to production in the capital-intensive sector (for shorthand, we will refer to this as the relative size of land speculation).[26] From the definition of ϕ_t, we can derive the evolution of ϕ_t.

(9.9a) $\phi_{t+1} = \left(\frac{P_{t+1}/P_t}{K_{t+1}/K_t}\right) \phi_t = \left(\frac{P_{t+1}/P_t}{1+g_t}\right) \phi_t,$

where $\frac{K_{t+1}}{K_t} \equiv 1 + g_t$. The evolution of ϕ_t depends on the growth rate of aggregate capital and that of land prices.

Aggregating $k_{t+1} = sw_t - P_t x_t$ across young persons, we obtain

[26] It should be clear that the results would be unchanged if we measured land speculation by the ratio of the value of land to the value of the capital stock.

(9.10a) $K_{t+1} = sw_t - P_t = sA(1-\alpha)K_t - P_t = [s(1-\alpha) - \phi_t]AK_t.$

Hence, we have the growth rate of aggregate capital stock.

(9.10b) $1 + g_t \equiv \frac{K_{t+1}}{K_t} = [s(1-\alpha) - \phi_t]A.$

A rise in ϕ_t (land speculation) crowds savings away from productive capital, and therefore reduces the growth rate of aggregate capital, which in turn leads to a decreased growth rate of labor and land productivities.

The growth of land prices can be derived from the no-arbitrage condition, which can be rewritten (using ϕ_t) as

(9.11) $\frac{P_{t+1}}{P_t} = R^c - \frac{D_{t+1}}{P_t} = R^c - \frac{\epsilon a K_{t+1}}{P_t} = R^c - \frac{\epsilon a^\alpha A K_t}{P_t}\frac{K_{t+1}}{K_t} = R^c - \frac{\epsilon a^\alpha (1+g_t)}{\phi_t}.$

Substituting (9.10b) and (9.11) into (9.9a) yields

(9.9b) $\phi_{t+1} = \frac{R^c \phi_t}{[s(1-\alpha) - \phi_t]A} - \epsilon a^\alpha,$

which, given any initial value of ϕ_0, can be solved for ϕ_t for all t. From (9.10b), g_t is then defined for all t, which in turn means that given an initial value of K_0, K_t can be solved for all t, thereby fully characterizing the dynamics of this economy.

Since ϕ_{t+1} is a convex function of ϕ_t, with the negative intercept when $\epsilon > 0$ and $\phi_{t+1} \to \infty$, as $\phi_t \to s(1-\alpha)$, there exists a unique and positive value of ϕ, where $\phi_t = \phi_{t+1} \equiv \phi^* < s(1-\alpha)$, i.e., there is a unique steady state. We impose an assumption concerning parameter values to ensure $\phi^* > 0$, even when $\epsilon = 0$.[27]

Assumption 1. $s(1-\alpha)A > R^c = \alpha A + 1 - \delta > 1$

Under Assumption 1, the net return from capital investment is strictly greater than one

[27] It follows directly from (9.9b) that when $\epsilon = 0$, $\phi^* = s(1-\alpha)A - R^c$. If Assumption 1 holds, equilibria with pure land bubbles can arise. This also implies the following. Suppose $s(1-\alpha) > \alpha$. Then, for sufficiently small A, Assumption 1 is not satisfied and hence, there are no equilibria with pure land bubbles with $P_t > 0$. However, once A gets large enough and Assumption 1 becomes satisfied, then equilibria with pure land bubbles can emerge. In other words, an improvement in technological progress may create the formation of pure land bubbles and introduce macroeconomic fragility. Also, it is straightforward from (9.10b) that the existence of pure land bubbles reduces the growth rate of aggregate capital.

and the slope of (9.9b) evaluated at $\phi_t = 0$ is strictly less than one.

9.6 Steady state

From (9.9b), we can derive the quadratic equation determining ϕ^*.

(9.9c) $A(\phi^*)^2 + [-s(1-\alpha)A + R^c + \epsilon a^\alpha]\phi^* - s(1-\alpha)A\epsilon a^\alpha = 0.$

Solving (9.9c) for ϕ^* gives two solutions, only one of which is positive (when $\epsilon = 0$, the other solution is zero). The positive solution is given by

(9.9d) $$\phi^* = \frac{-(-s(1-\alpha)A+R^c+\epsilon a^\alpha)+\sqrt{[-s(1-\alpha)A+R^c+\epsilon a^\alpha]^2+4A^2s(1-\alpha)\epsilon a^\alpha}}{2A}.$$

By substituting (9.9d) into (9.10b), we have the steady-state growth rate of aggregate capital.

(9.12) $$1 + g^* = (1-\alpha)A - \frac{-(-s(1-\alpha)A+R^c+\epsilon a^\alpha)A+A\sqrt{[-s(1-\alpha)A+R^c+\epsilon a^\alpha]^2+4A^2s(1-\alpha)\epsilon a^\alpha}}{2}.$$

(9.9d) and (9.12) allow us to conduct comparative statics assessing the effects of changes in parameter values.

9.7 Dynamics

The dynamics is straightforward. It is apparent that the only value of ϕ_0 that is consistent with rational expectations (i.e. the only value for which ϕ_t remains within the economically meaningful bounds of $0 < \phi_t < \infty$) is $\phi_0 = \phi^*$. Given an initial state variable K_0, there exists a unique P_0 that immediately achieves the steady-state value of ϕ^*, that is, $P_0 = \phi^* A K_0$. There is no transitional dynamics. Once the economy jumps to the steady state, the growth rate of aggregate capital also becomes constant. The absence of transitional dynamics means that the impact of land speculation on long-run economic growth is captured by the impact on steady-state growth.[28]

[28] Note that the dynamics without transitional dynamics is one of the features in standard AK models

We summarize this in the following Proposition.

Proposition 6. Existence of balanced growth path and uniqueness of rational expectations trajectory Under Assumption 1, for any $\epsilon \geq 0$, there exists a unique steady-state growth path where the size of land speculation ϕ^*, the growth rate of aggregate capital $1 + g^*$, which equals the growth rate of aggregate output, and the growth rate of asset prices $\frac{P_{t+1}}{P_t}$ are constant over time, with ϕ^* and g^* given by (9.9d) and (9.12). This is the unique dynamic path consistent with rational expectations: The economy immediately converges to that path, through the setting of the initial land price, $P_0 = AK_0\phi^*$.

9.8 Industrial policies

There are two market failures that have not been well investigated within OLG models: within the manufacturing sector, firms fail to take into account the externality that their investments have for the productivity of other firms in the sector (the returns to scale effect, the within sector spillover); and firms within that sector fail to take into account the productivity effects of their investments on the other sector (the cross sector spillover.) There is an additional problem that arises in all overlapping generations models: this generation, in thinking about how much to save, doesn't take into account the benefits to future generations as a result of the higher wage they receive.

The implication is clear: any government attempting to maximize intergenerational social welfare would encourage investment in the productive sector and discourage land speculation, i.e., it would strive to shift savings from land speculation to productive investment, and it would encourage overall savings,[29] even inducing this generation to save more than would maximize its income, because of the benefit to future generations.

In this simple model, we can introduce several simple instruments to do this—a tax on land, providing a subsidy to the return to capital, or even a lump sum transfer to workers. Consider the latter case, with a 100% tax on real estate, so the price of land

(see, e.g., Chapter 11 of Acemoglu (2009)).

[29] So long as there is not "oversaving." But oversaving cannot occur within our model, at least in steady state (and as we have proven, the only rational expectations paths are steady states), because that would entail a return on capital lower than the growth rate of land rents, which would in turn imply an infinite price of land.

would be zero. Then

$$K_{t+1} = sw_t + s\epsilon a K_t = s(1-\alpha)AK_t + s\epsilon a K_t,$$

Aggregate capital would then grow at a steady rate of

(9.13) $$\frac{K_{t+1}}{K_t} = s(1-\alpha)A + s\epsilon a > s(1-\alpha)A - A\phi^*.$$

It is straightforward that this policy leads to increased welfare, except for the initial old generation who initially holds land.[30][31]

Part IV: Land Speculation with Credit Frictions

10. An Extended Model: Endogenous Growth with Land and Credit

10.1 The environment[32]

To study credit frictions, we need to construct a model with some heterogeneity, i.e., with some young people wanting to invest more than they can save on their own, and others providing those savings. We assume within any generation a fraction η of the young are entrepreneurs who have investment opportunities. The remaining fraction

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[30] In this special logarithmic model, the return to capital doesn't affect the savings rate; more generally, of course, it does. At a low enough social discount rate, an intertemporal social welfare function would always impose a 100% tax on the return to land. At higher social discount rates, there will be an optimal tax on land, balancing the marginal benefits to later generations with the marginal costs to the initially old individuals.

[31] We can also consider a tax on land rents. Then, the no-arbitrage equation is changed to
$R_{t+1} + 1 - \delta = \frac{(1-\tau)D_{t+1}}{P_t} + \frac{P_{t+1}}{P_t},$
where τ is a tax rate. Using this, (9.9b) becomes
$\phi_{t+1} = \frac{R^c \phi_t}{[s(1-\alpha)-\phi_t]A} - (1-\tau)\epsilon a^{\alpha}.$
It is straightforward that for any value of $\phi_t (< s(1-\alpha))$, ϕ_{t+1} increases as τ increases, thereby leading to the decreased ϕ^*. Then,
$K_{t+1} = sw_t + s\tau\epsilon a K_t - P_t = s(1-\alpha)AK_t + s\tau\epsilon a K_t - P_t,$
Therefore, aggregate capital grows at a steady rate of
$\frac{K_{t+1}}{K_t} = s(1-\alpha)A + s\tau\epsilon a - A\phi^*(\tau > 0) > s(1-\alpha)A - A\phi^*(\tau = 0).$
That is, this tax policy leads to higher economic growth.

[32] This model is a variant of the model proposed by Hirano and Stiglitz (2024), who provide a parsimonious and tractable framework for examining the impact of credit expansions arising from increases in collateral values or lower interest rate policies on long-run productivity and economic growth in a two-sector endogenous growth economy with credit frictions, with the driver of growth lying in one sector (manufacturing) but not in the other (real estate).

$1-\eta$ are savers who don't have investment opportunities. We consider a small open economy where the interest rate r is exogenously given,[33] and, for simplicity, that only entrepreneurs participate in the real estate market.[34] The utility function of each young person is given by (9.1).

The budget constraint of an entrepreneur i is given by

(10.1) $c_t^i + k_{t+1}^i + P_t x_t^i = w_t + b_t^i$ and $c_{t+1}^i = (R_{t+1} + 1 - \delta)k_{t+1}^i + R_t^x P_t x_t^i - (1+r)b_t^i$,

where k_{t+1}^i and x_t^i are the entrepreneur's capital investment and land holdings at date t. b_t^i is the amount of borrowing at date t if $b_t^i > 0$, and lending if $b_t^i < 0$. $R_t^c \equiv R_{t+1} + 1 - \delta$ is the total return per unit of capital investment made at date t. P_t and D_{t+1} are the price of land at date t and land rents at date $t+1$ in terms of consumption goods. R_t^x is the unleveraged return to land,

(10.2) $$R_t^x = \frac{D_{t+1}}{P_t} + \frac{P_{t+1}}{P_t},$$

i.e., the return per dollar spent on land holdings without using borrowing between date t and $t+1$.

Since savers don't have investment opportunities and don't participate in the real estate market, their behavior is simple, i.e., they lend all savings either to entrepreneurs and/or to abroad at the exogenously given interest rate r.

We assume credit frictions. The entrepreneurs cannot borrow unless they have collateral. The borrowing constraint is

(10.3) $$b_t^i \le \theta k_{t+1}^i + \theta^x P_t x_t^i,$$

where the first and second terms of the right-hand side in (10.3) reflect that only a

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[33] See Hirano and Stiglitz (2024), who study a closed monetary economy where there are three stores of values, capital, money, and land. They show that macroeconomic consequences are markedly different between a two-asset model (capital and money) of Tobin (1965) and a three-asset model of Hirano and Stiglitz (2024). As Hirano and Stiglitz (2024) show, the very existence of land changes results substantially.

[34] One interpretation of this assumption is that there is considerable land heterogeneity and workers are less informed about real estate (with high costs associated with obtaining the relevant information), making real estate unattractive for them. There are several other justifications. See Hirano and Stiglitz (2024) for detail.

fraction $\theta \in [0,1]$ of the value of capital and only a fraction $\theta^x \in [0,1]$ of the value of land can be used as collateral.[35]

The production side of the economy is the same as the one in the basic model of Section 9.[36]

10.2 The behavior of entrepreneurs and competitive equilibrium

We focus on the case where the borrowing constraint (10.3) binds for entrepreneurs in equilibrium at each date, so that

$$(10.4) \quad R^c = R + 1 - \delta = \alpha A + 1 - \delta > 1 + r,$$

where R^c is the same as (9.8c). (10.4) represents that the return on capital net of depreciation is greater than the safe rate of interest, implying that entrepreneurs would want to borrow as much as they could.

Optimal portfolio allocations

Entrepreneurs allocate their portfolio between capital and land, both using borrowing. Since they are perfect substitutes, in equilibrium, the leveraged rates of returns on each must be the same:

$$(10.5) \qquad \underbrace{\frac{R^c-(1+r)\theta}{1-\theta}}_{\substack{\text{leveraged rate of return} \\ \text{on capital investment}}} = \underbrace{\frac{R_t^x-(1+r)\theta^x}{1-\theta^x}}_{\substack{\text{leveraged rate of return} \\ \text{on land speculation}}}$$

(10.5) determines the equilibrium value of R_t^x, which we can solve for explicitly as

$$(10.6) \qquad R_t^x = \left[\frac{R-(1+r)\theta}{1-\theta}\right](1-\theta^x) + (1+r)\theta^x \equiv R^x,$$

R^x is fixed, but depend on technology parameters and policy variables.

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[35] Hirano and Stiglitz (2024) consider a different borrowing constraint that a fraction θ of future returns from capital investments and a fraction θ^x of future returns from land holdings (land rents plus the value of land in the next period) can be used as collateral. The results are parallel to those here, but their formulation introduces significant complexities.

[36] The Appendix O2 of Hirano and Stiglitz (2024) also study an extended case where the real estate sector uses both labor and land as inputs for production. We can apply their analysis here.

Recalling the definition of R_t^x, we obtain the first basic difference equation.

(10.7a) $P_{t+1} = P_t R_t^x - D_{t+1} = P_t R_t^x - \epsilon\chi(K_{t+1}) = P_t R^x - \epsilon a K_{t+1}$.

The capital-investment function

By substituting (10.3) into (10.1) and solving for k_{t+1}^i, we can derive the capital investment function of entrepreneurs when the borrowing constraint binds.[37]

(10.8) $$k_{t+1}^i = \underbrace{\frac{1}{1-\theta}}_{\text{leverage}} \left[\underbrace{sw_t}_{\text{saving}} - \underbrace{(1-\theta^x)P_t x_t^i}_{\substack{\text{total down-payment} \\ \text{in buying land}}} \right],$$

i.e., we can calculate an entrepreneur's capital holdings by taking his saving, subtracting what he has to pay to buy the land he holds (which depends on land leverage); and leveraging up that amount up (through borrowing). $1-\theta^x$ is the down-payment of a unit of land purchase.

10.3 The competitive equilibrium

The competitive equilibrium is defined as a set of prices $\{P_t, w_t, R_t, R_t^x\}_{t=0}^{\infty}$ and quantities $\left\{c_t^i, b_t^i, x_t^i, k_{t+1}^i, \int c_t^i di, \int b_t^i di, \int x_t^i di, \int k_t^i di, Y_t\right\}_{t=0}^{\infty}$, given an initial K_0, such that (i) each entrepreneur chooses land holdings, capital investment, and borrowing to maximize utility under the budget and the borrowing constraints, and (ii) each lender (saver) lends all savings either to entrepreneurs and/or to abroad, and (iii) the market clearing conditions for land, capital, labor and goods are all satisfied. In the initial period $t = 0$, all land is held by the initial old entrepreneurs.

10.4 Aggregate Dynamics

We are now ready to derive aggregate dynamics. When we aggregate (10.8) across

[37] Since we employ log-utility, each young person saves a constant fraction s of wages. For derivations, see Appendix O1 in Hirano and Stiglitz (2024). Note that although Hirano and Stiglitz (2024) employ a different borrowing constraint, the derivation method is the same.

young entrepreneurs, we have

(10.9) $K_{t+1} = \frac{1}{1-\theta}[\eta s w_t - (1-\theta^x)P_t] = \frac{1}{1-\theta}[\eta s(1-\alpha)AK_t - (1-\theta^x)P_t]$.

Substituting into (10.7a) yields

(10.7b) $P_{t+1} = P_t R^x - \frac{\epsilon a}{1-\theta}[\eta s(1-\alpha)AK_t - (1-\theta^x)P_t] = P_t\left\{R^x + \frac{\epsilon a(1-\theta^x)}{1-\theta}\right\} - \frac{\epsilon a \eta s(1-\alpha)A}{1-\theta}K_t$.

The dynamics of this economy is characterized by (10.7b) and (10.9), which define $\{K_{t+1}, P_{t+1}\}$ as simple linear functions of $\{K_t, P_t\}$.

To solve the dynamics, we again solve for the dynamics $\phi_t \equiv \frac{P_t}{AK_t}$: From the definition of R_t^x,

(10.6b) $\frac{P_{t+1}}{P_t} = R_t^x - \frac{D_{t+1}}{P_t} = R_t^x - \frac{\epsilon a K_{t+1}}{P_t} = R_t^x - \frac{\epsilon a}{A}\frac{AK_t}{P_t}\frac{K_{t+1}}{K_t} = R^x - \epsilon a^\alpha \frac{1+g_t}{\phi_t}$.

From (10.9), we can calculate the growth rate of aggregate capital as a function of ϕ_t and the various parameters of the problem:

(10.10a) $1 + g_t = \frac{A}{1-\theta}[\eta s(1-\alpha) - (1-\theta^x)\phi_t]$.

Other things being constant, a rise in ϕ_t (land speculation) crowds out savings away from capital investment, reducing the growth rate of the economy. On the other hand, an increase in θ increases leverage in capital investment. With a decline in θ^x, the down-payment of a unit of land purchase decreases, generating more capital investment. We call these “crowding-in” effects on capital investment. The question is what are the circumstances under which each effect dominates? Substituting (10.6b) and (10.10a) into (9.9a) yields

(10.11) $\phi_{t+1} = \frac{\phi_t R^x}{1+g_t} - \epsilon a^\alpha = \phi_t R^x / \left\{\frac{A}{1-\theta}[\eta s(1-\alpha) - (1-\theta^x)\phi_t]\right\} - \epsilon a^\alpha$.

ϕ_t, given initial conditions, can be solved on its own, with the dynamics depending just on the technological parameters A and ϵ, on the share of the population that are entrepreneurs, η, and the market/policy parameters $\{r, \theta, \theta^x\}$. Much of the discussion below focuses on how changes in these parameters affect the economy's trajectory and long-run productivity and economic growth through both crowding-in and -out effects.

10.5 Special case of unproductive land

Consider the special case of $\epsilon = 0$, i.e., land is unproductive. Even so, land can still be used as collateral and as a store of value. It follows directly from (10.11) that if there is a steady state, $1 + g^* = R^x$. Moreover, (10.11) becomes

(10.12) $$\phi_{t+1} = \phi_t R^x / \left\{\frac{A}{1-\theta}[\eta s(1-\alpha) - (1-\theta^x)\phi_t]\right\}.$$

The RHS is convex in ϕ_t and equals zero when $\phi_t = 0$, so there is a unique steady state with land having a positive price under Assumption 2 below. The effects of changes in r and collateral requirements on growth can easily be determined: they just depend on how those variables affect R^x. Because as $\epsilon \to 0$, $1 + g^* \to R^x$, these results extend to small (but non zero) ϵ.

In this case, we can also solve explicitly for the steady-state equilibrium level of land speculation.

(10.13) $$\phi^* = \{\eta s(1-\alpha)A - R^x(1-\theta)\}/\ A(1-\theta^x).$$

We impose an assumption concerning parameter values to ensure $\phi^* > 0$ even when $\epsilon = 0$.

Assumption 2. $\frac{A\eta s(1-\alpha)}{1-\theta} > R^x = \left[\frac{R^c-(1+r)\theta}{1-\theta}\right](1-\theta^x) + (1+r)\theta^x \leftrightarrow A\eta s(1-\alpha) > R^c(1-\theta^x) + (1+r)(\theta^x - \theta)$

Under this assumption, the slope of (10.11) evaluated at $\phi_t = 0$ is strictly less than one.[38] Assumption 2 is more likely to be satisfied when the values of θ and θ^x are

[38] If this condition is not satisfied, the only steady state entails ϕ^*= 0.

larger.[39]

10.6 General case and dynamics

From (10.11), the steady-state value of ϕ, when $\epsilon \neq 0$, satisfies the following quadratic equation.

$$(10.14) \qquad A\frac{1-\theta^x}{1-\theta}(\phi^*)^2 + \left(-\frac{A\eta s(1-\alpha)}{1-\theta} + R^x + \epsilon a^\alpha A\frac{1-\theta^x}{1-\theta}\right)\phi^* - \frac{A\eta s(1-\alpha)}{1-\theta}\epsilon a^\alpha = 0.$$

Under Assumption 2, solving (10.14) for ϕ gives two solutions, i.e., one with the positive value and the other with the negative value (when $\epsilon = 0$, the other solution is zero). The positive solution is given by

$$(10.15) \qquad \phi^* = \frac{-\left(-\frac{A\eta s(1-\alpha)}{1-\theta}+R^x+\epsilon a^\alpha A\frac{1-\theta^x}{1-\theta}\right)+\sqrt{\left(-\frac{A\eta s(1-\alpha)}{1-\theta}+R^x+\epsilon a^\alpha A\frac{1-\theta^x}{1-\theta}\right)^2+\frac{4(1-\theta^x)\eta s(1-\alpha)\epsilon a^\alpha A^2}{(1-\theta)^2}}}{2A(1-\theta^x)/(1-\theta)},$$

This gives ϕ^* as a function of the parameters of the model. As $\epsilon \to 0$, (10.15)→(10.13).

Since ϕ_{t+1} is a convex function of ϕ_t, with the negative intercept when $\epsilon > 0$ and $\phi_{t+1} \to \infty$, as $\phi_t \to \bar{\phi} \equiv \eta s(1-\alpha)/(1-\theta^x)$, there exists a unique and positive value of ϕ where $\phi_t = \phi_{t+1} \equiv \phi^* < \bar{\phi}$. Figure 11 illustrates this. Note that as $\phi_t \to \bar{\phi}$, $1 + g_t \to 0$ because the crowding-out effect becomes so large.

It is clear that unless ϕ_0 is set at ϕ^*, the economy does not converge. Given the initial value of K_0, there is a unique value of P_0 such that $\phi_0 = \phi^*$. ϕ_t remains at that value forever: there are no transitional dynamics for ϕ. Given this, g_t is fixed:

$$(10.16) \qquad 1 + g_t = \frac{A}{1-\theta}[\eta s(1-\alpha) - (1-\theta^x)\phi^*] \equiv 1 + g^*.$$

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[39] When Assumption 2 is satisfied, pure land bubbles can occur. Under some conditions, if θ and θ^x are sufficiently small, there are no equilibria with $P_t > 0$. But, if they get large enough, equilibria with $P_t > 0$ can arise. Thus, financial market deregulations might enhance the possibility of pure land bubbles and lower economic growth simultaneously. See also Hirano and Yanagawa (2017), who study the relationship between financial development and the possibility of asset bubbles, and Hirano and Toda (2025), who prove the inevitability of asset bubbles within modern macroeconomic models.

Capital grows at a constant rate, and given the constancy of ϕ, so does the price of land. Note that the extent of spillover (ϵ) affects g^* only through its effects on the equilibrium level of land speculation ϕ^*. We can substitute (10.15) into (10.16) to obtain g^* as a function of all the parameters and policy variables.

We have thus established

Proposition 7 Existence and Uniqueness of Rational Expectations Trajectory[40]

Under Assumption 2, for any $\epsilon \geq 0$, there exists a unique balanced growth path where the value of land relative to production in the productive sector ϕ^*, the economic growth rate $1 + g^*$, and the growth rate of land prices $\frac{P_{t+1}}{P_t}$ are constant over time, with ϕ^* and g^* given by (10.15) and (10.16). The economy immediately converges to that path, through the setting of the initial price of land, $P_0 = AK_0\phi^*$. This is the unique rational expectations trajectory with positive land prices.

Moreover, as $\epsilon \to 0$, $1 + g^* \to R^x$ and $\phi^* \to \{\eta s(1-\alpha)A - R^x(1-\theta)\}/\ A(1-\theta^x)$.

10.7 Comparative dynamics

The variable of greatest interest is the growth rate. Using (10.15) and (10.16), we can conduct various comparative dynamic exercises with changes in the market/policy parameters $\{r, \theta, \theta^x\}$. Straightforward differentiation enables us to show the following Proposition. We provide proofs in the Appendix.

Proposition 8 Impact of changes in the collateral values on long-run economic growth

(i) If ϵ is small, greater collateral values of land encourage land speculation with leverage, retarding long-run economic growth. That is, $\frac{d(1+g^*)}{d\theta^x} < 0$: The crowding out

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[40] For completeness, we note that when $\epsilon = 0$, there is another steady state equilibrium with $\phi^* = 0$, where land is valued at zero, and with growth higher than in the case where land has a positive price; this equilibrium is stable; if $\phi_t < \phi^*$, the rational expectations trajectory converges to the landless economy.

effect of land speculation overwhelms the crowding in effect.

(ii) $\frac{d(1+g^*)}{d\theta} > 0$. With an increase in θ, the crowding-in effects associated with greater collateral values of capital investment dominate the crowding-out effect, so the increase in θ leads to higher economic growth. This is true, even though the equilibrium ϕ^* rises.

As Proposition 8 shows, the impact on long-run productivity and economic growth of an increase in liquidity (a relaxation of the collateral constraints) is markedly different depending on how it arises. If it is a result of relaxation in real estate financing, it will be productivity-and growth-retarding in the long run. On the other hand, if it rises mainly with the relaxation in capital investment financing, it will be productivity-and growth-enhancing.

Proposition 9 Impact of changes in interest rates on long-run economic growth

$$sign\frac{d(1+g^*)}{dr} = sign(\theta^x - \theta).$$

That is to say, the long-run impact of low interest rates on economic growth depends on the relative size of θ^x and θ. When $\theta^x > \theta$, the crowding-out effect caused by low interest rates (with land prices increasing) dominates the crowding-in effects; low interest rates encourage land speculation with leverage more than capital investment, reducing long-run productivity and economic growth. On the other hand, when $\theta^x < \theta$, just the opposite is true.[41]

Intuitively, in economies where the collateral value of land is greater than that of capital investment, lower interest rates mean more funds unevenly flow into the real estate market, encouraging land speculation with leverage rather than increasing productive capital. Therefore, they can be harmful to long-run productivity growth and economic growth.[42]

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[41] Proposition 9 and 8 (ii) are true regardless of the size of ϵ.

[42] We can obtain further insight into how growth is affected by changes in the interest rate or collateral requirements by decomposing the effects of changes into a direct (partial equilibrium) effect and a general equilibrium effect, where the latter represents the impact on the equilibrium level of speculation. See Hirano and Stiglitz (2024) for a full decomposition.

10.8 Welfare analysis and government policy

Given an initial capital stock K_0, and therefore the initial wage, the first generation's welfare, whether that of the worker or the entrepreneur, depends simply on $1 + r$ and the leveraged return, respectively, and increasing the capital investment leverage increases the latter and leaves the former unchanged. Since that leads to a higher growth of K, and higher wages, *in every subsequent period,* workers are better off. But so are entrepreneurs. The older generation at time 0 are also better off: the policy change leads to a sudden increase in the value of land (given K_0, $P_0 = \phi^{**} A K_0$ rises, where ϕ^{**} is the value of land speculation *after* the increase in capital investment leverage), and the entrepreneurs of that generation holding land receive a large capital gain. Hence, increasing capital investment leverage in this model is a Pareto improvement.[43]

But increasing land leverage has more ambiguous effects. In the 0th period, the price of land increases so the leveraged return to land remains equal to the (unchanged) leveraged return to capital. Neither workers nor entrepreneurs of the 0th generation are affected. If it results in lower growth (the conditions for which we have already identified), then wages *in every subsequent period* will be lower, so workers in every subsequent period are worse off. But the leveraged return on capital investment is unchanged, so that in spite of the higher land leverage, given the general equilibrium adjustments, entrepreneurs in every subsequent period are also worse off with the decrease in wages. *Increasing land leverage in these circumstances is almost Pareto inferior—everyone except the elderly at time 0 are worse off.* The elderly entrepreneurs, who hold the land, experience an unanticipated capital gain from a higher land price.

Intervention can, of course, take many other forms. For instance, tightening θ^x induces a portfolio shift from land speculation to capital investment, which leads to reduced ϕ^*, thereby increasing growth. Or a tax on land rents with its proceeds transferred to young workers increases aggregate savings of the young and leads to reduced ϕ^*, therefore financing more capital investments and increases growth. These policies increase the welfare of all generations except for the old generation who holds land when this policy change is introduced.

11. Literature Review

Parts I and II of this paper focus on endogenous economic fluctuations—quite different

[43] With output first period fixed, one might ask how can consumption of the old increase and investment by the young? The answer is that workers/savers lend less to foreigners.

from those induced by exogenous shocks as in the DSEG models. The earlier literature exploring endogenous non-linear business cycles, such as that associated with Samuelson and Goodwin, rely on more irrationality in expectations than was acceptable in an era in which rational expectations dominated. This is even more true of the work of Minksy (1986) where there is a critical point (a "Minsky moment") when financial speculation reaches an extreme and an unsustainable bull market suddenly comes to an end. But the booms that precede Minsky moments are based on "irrational exuberance" (like Keynes' animal spirits). By contrast with this earlier literature, we present models of economic fluctuations with "rational exuberance" (rational expectations) and an endogenous collapse of land prices occurs.[44]

This paper draws on several disparate and large literatures in macroeconomic dynamics and finance, exploring steady states, and local and global dynamics. See Hirano and Stiglitz (2022a, 2022b), and (2024) for the extensive literature review of each Part. Most importantly, it draws upon the Samuelson (1958) pure consumption OLG, and more relevantly, Diamond's 1965 paper, where he introduced capital. But that paper focused on steady states, while Part I shows the complex dynamics that arise.[45]

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[44] There is a small literature trying to reconcile asset booms that endogenously crash with a modicum of rationality. Abreu and Brunnermeier (2003) do so in a model with dispersed opinions about the timing of collapse. Abreu and Brunnermeier (2003) is a partial equilibrium model without investment and production, while key to our dynamic general equilibrium analysis is the strong interaction between land prices and capital accumulation, with a critical price at which the land price boom *must* break. The standard reasoning that knowing that an explosive increase in land price must break at t means that it must break at t-1, which means that it must break at t-2—making impossible any unsustainable dynamics in land prices from the beginning—does not apply in our model precisely because of the large general equilibrium effects that we identify and analyze, which entail, amongst other things, large endogenous changes in the real interest rate.

[45] There were two foundational literatures that could have given rise to wobbly dynamics had there been further exploration. One set are the literatures on multiple momentary equilibria. A recent strand of macroeconomics (Vines and Wills 2020) puts multiple equilibria at the center of macroeconomic analysis.

There has, for instance, been a large literature in macroeconomic models showing the existence of multiple equilibria in static or two-or-three-period models (see, e.g., Diamond 1982 and Cooper and John 1988 for static models, and see Neary and Stiglitz 1983 and Stiglitz 1994 and Lamont 1995 for two-period models, and Diamond and Dybvig 1983 for a three-period model).

Even earlier, Uzawa (1961, 1963) noted the possibility of multiplicity of momentary equilibria, related to general equilibrium distributional effects. It is likely that had any of these works been extended into a formal dynamic model, they would naturally exhibit wobbly dynamics, suggesting that wobbly dynamics may arise in a wide variety of models. Stiglitz (1967) had shown that in such models simple savings behavior, consistent with OLG models, could give rise to cyclical dynamics.

The possibility of multiplicity of momentary equilibria in the standard life-cycle model developed by Diamond (1965) has been recognized for a long time, but seems little explored. (Stiglitz (1973), Azariadis (1993), De La Croix and Michel (2002), Evans and Honkapohja (2012) and Romer (2019).) These papers, while mentioning the possibility of multiple momentary equilibria, do not provide either necessary or sufficient conditions within a broad class of utility and production functions. These papers do not investigate what patterns of global macro-dynamics may emerge, and how changes in the underlying parameter values will alter the phase of global dynamics.

Still another strand of work recognizing the multiplicity of multiple equilbira are the pure

We show that even a small change in assumptions in the Diamond model can produce a markedly different picture of capitalism, unlike the standard representative agent dynastic utility model, where typically there is smooth convergence to a unique equilibrium. We present the possible patterns of global dynamics.

Introducing a non-produced asset (land or money), as we do in Part II, has long been known to change key results, such as *in steady state* there cannot be oversaving. (See Calvo 1978; Scheinkman 1980; Woodford 1984; Tirole 1985; McCallum 1987; Muller and Woodford 1988; Rhee 1991; and Mountford 2004). While Tirole (1985) developed an OLG model with capital and pure bubble assets, he imposed a condition ensuring a unique momentary equilibrium. In that setting, he showed that there is a unique saddle path converging to a steady state with positive bubbles, and the presence of bubbles restores dynamic efficiency. It is clear that our results are markedly different in both respects. That is to say, land prices and other key macro variables endogenously "wobble" within well-identified ranges with repeated boom-bust cycles, without converging to a steady state.

Part III draws on the enormous literature on endogenous growth, too extensive to cite, but expanding the analysis to a two sector model with endogenous growth effects originating in one but spilling over to the other, with results overturning long established results about the effects of lower interest rates and financial market liberalization. Part IV then brings into that analysis credit frictions, which reinforce the earlier conclusion. That Part is related to the vast modern literatures on credit and financial frictions[46], land, monetary economics, OLG, and endogenous growth. It brings together in a parsimonious model key insights from each. Most interestingly, it incorporates two separate literatures: In the standard overlapping generations model with land and exogenous growth, land holdings crowd out capital accumulation (Deaton and Laroque 2001; Mountford 2004) but increases welfare if the landless economy is

consumption/endowment sunspot models-- Cass and Karl (1983), Grandmont (1985), Matsuyama (1991), and Golosov and Menzio (2020). Also, most of the sunspot literature focuses on multiplicity of momentary equilibria in *monetary* economies without capital investment within a two-period overlapping generations framework (see also Azariadis 1981; Azariadis and Guesnerie 1986). By contrast, we introduce productive capital, and in Part II, we have both a store of value (land) and productive capital. The existence of productive capital linking one period with the other is of critical importance in our model, and we believe, in the world. Moreover, the focus of our paper is wobbly fluctuations, while Grandmont (1985) focuses on deterministic cycles or chaotic dynamics. A subsequent paper by Reichlin (1986) proves the existence of limit cycles in a productive economy with a Leontief production function. The literature on chaos with the application of Li and Yorke (1975)'s theorem focusses on a unique momentary equilibrium. There are many further differences between our paper and this earlier work, such as the endogeneity of whether there are or are not multiple momentary equilibrium, depending on the values of the state variables of the economic system.

[46] Including seminal papers by Stiglitz and Weiss (1981), Bernanke (1983), and Diamond (1984)

dynamically inefficient. By contrast, in our models with endogenous growth, welfare implications are markedly different, i.e., land speculation encouraged by lower interest rates and financial market liberalization reduces long-run economic growth and welfare, and whether the economy is dynamically efficient or not is not relevant. In a recent strand of literature on credit frictions and macroeconomic fluctuations, including Kiyotaki and Moore (1997) and many of subsequent papers,[47] land plays an important role as collateral. An increase in the value of land relaxes borrowing constraints of entrepreneurs who have productive investments, allowing them to make more of their productive investments, increasing the efficiency of macroeconomy. But it is assumed by model construction that the only thing productive entrepreneurs can do with borrowed funds is to engage in entrepreneurial activity with high returns. In contrast, in our model, entrepreneurs may use borrowed funds for land speculation as well as for productive investment. An increase in the collateral value of land produces a "crowding in" effect, as in their model, but through a quite different mechanism, i.e., in our model, entrepreneurs engage in both productive capital investments and less productive land investments, both using borrowing. Entrepreneurs decide an optimal portfolio allocation between those two assets. If the pledgeability of land rises, it leads to a lower downpayment to buy real estate, which in turn leaves more funds available for capital investment. More importantly, unlike their model, the increase in the collateral value of land produces a general equilibrium "crowding out" effect, as entrepreneurs' portfolios shift toward land speculation, rather than productive investments, which leads to increased land speculation, crowding out more productive capital investments.

12. Concluding remarks

Macro dynamics, both in the short run and the long, are markedly different in OLG models than in the standard dynastic models with individuals maximizing utility over an infinite horizon. Though rational expectations puts strong bounds on the set of dynamics, they are nothing like the stringent conditions implied by the Euler equations and transversality conditions. Even without adding intra sectoral and cross sectoral spillovers, there is no presumption of efficiency—and we have shown that indeed, there exist rational expectations trajectories that are not dynamically efficient. Taking into

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[47] A comprehensive literature review is too large for this paper; for a more complete discussion, see the 2022 Nobel Memorial Prize in Economics https://www.nobelprize.org/uploads/2022/10/advanced-economicsciencesprize2022-2.pdf

account the existence of a non-produced store of value (land) expands the range of instabilities of the economy—the economy can wobble, though within bounds. Indeed, under certain circumstances (parameter values) the only rational expectations trajectory with positive land prices entails wobbly dynamics.

While it has been almost 70 years since Samuelson first introduced the overlapping generations model, the complex dynamics to which it can give rise--especially when there are a multiplicity of stores of value (capital, money, and land) giving rise a multiplicity of state variables—has made it largely intractable, and accordingly undermined its use both for analysis and policy; it seemed as if only steady state analysis and a local analysis in the neighbourhood of the steady state, one entailing with smooth convergence to a steady state were analytically tractable, especially because even when individuals lived only two periods, they had to contemplate the prices at which they could sell their assets, and that in turn would depend on what succeeding generations could sell their assets for. In other words, individuals need to formulate expectations infinitely far into the future of whether asset price dynamics is sustainable or not.

Part of the analytical achievement of this paper and other related papers is to show how, at least under certain circumstances, one can nonetheless provide a *global* analysis (not limited to small deviations from a steady state), and such a global analysis could provide strong strictures on the set of possible rational expectations trajectories—even if it couldn't specify which trajectory the economy would choose. Animal spirits still can reign supreme in a world of rational expectations. Moreover, we have explained why such a global analysis is so important: a steady state may be locally stable, but globally unstable—not returning to the initial equilibrium when confronted with a large enough shock, precisely because of the multiplicity of equilibria. Furthermore, multiple momentary equilibria can not only arise but also whether they arise or not depends on endogenous state variables. Hence, the economy may endogenously go back and forth between regions where there are multiple momentary equilibria and regions where there is a unique momentary equilibrium.

As Vines and Willis (2021) have argued, multiplicity of equilibria lies at the heart of understanding Keynesian economics. The standard models center on cases where there is a unique equilibrium. In the models explored here, not only may multiple momentary equilibria easily arise, there are an infinity of rational expectations trajectories, and under plausible conditions, there is a presumption that there are multiple equilibria.

Whether there are may, in addition, vary with the state of the economy—there are what we have identified as endogenous phase transitions. Moreover, improvements in technology, while in the short run increasing GDP, may result in the equilibrium being unstable and fragile—and in the long run lead to a shift in the equilibrium and a lowering of GDP.

Importantly, policy in these models may have markedly different effects. In this overview, we have been able to touch on only a few of these. Most importantly, looser monetary and financial policy may lead not to more productive investment and higher growth, but simply to more speculation---so much so that growth is actually harmed.

We believe models like those put forward here provide a fertile area for future macroeconomic research, including by incorporating wage and price rigidities in the models of parts I and II, to give rise to fluctuations in involuntary unemployment (even when labor and capital are substitutable), by incorporating credit frictions in the models of parts I and II, to give rise to endogenous land price booms and their crashes with credit expansions and contractions, and by incorporating learning by doing in the models of Parts III and IV. There are also important interactions between fluctuations and growth that should be explored: the real estate booms that we note have adverse effects on endogenous growth, and more broadly, the uncertainty created by fluctuations dampens incentives to invest, including in R&D. Thus, models we presented here can be thought of as prototypes of how to analyse global dynamics and forward dynamics when the existence of multiplicity of momentary equilibrium depends on endogenous state variables.

Appendix for Part IV

Proof of Proposition 8(i)

As $\epsilon \to 0$,

$$1 + g^* \to R^x = \left[\frac{R^c-(1+r)}{1-\theta} + (1+r)\right](1-\theta^x) + (1+r)\theta^x = \frac{R^c(1-\theta^x)+(1+r)(\theta^x-\theta)}{1-\theta}.$$

Then, we obtain

$$\frac{dR^x}{d\theta^x} = \frac{-[R^c-(1+r)]}{1-\theta} < 0.$$

Proof of Proposition 8(ii)

(10.11) can be written as

$$\phi_{t+1} = \frac{R^x \phi_t}{\frac{A}{1-\theta}[\eta s(1-\alpha)-(1-\theta^x)\phi_t]} - \epsilon a^\alpha = \frac{[R^c(1-\theta^x)+(1+r)(\theta^x-\theta)]\phi_t}{[\eta s(1-\alpha)-(1-\theta^x)\phi_t]A} - \epsilon a^\alpha.$$

For any value of $\phi_t < \bar{\phi}$, ϕ_{t+1} decreases as θ increases. That is, we obtain $\frac{d\phi^*}{d\theta} > 0$.

(10.11) can also be written as $1 + g^* = \frac{R^x}{1+(\epsilon a^\alpha/\phi^*)}$. Differentiating $1 + g^*$ with respect to θ yields

$$\frac{d(1+g^*)}{d\theta} = \frac{\frac{dR^x}{d\theta}[1+(\epsilon a^\alpha/\phi^*)]}{[1+(\epsilon a^\alpha/\phi^*)]^2} + \frac{\frac{\epsilon a^\alpha}{(\phi^*)^2}\frac{d\phi^*}{d\theta}}{[1+(\epsilon a^\alpha/\phi^*)]^2} > 0,$$

because $\frac{dR^x}{d\theta} = \frac{[R^c-(1+r)](1-\theta^x)}{(1-\theta)^2} > 0$ and $\frac{d\phi^*}{d\theta} > 0$.

Proof of Proposition 9

From (10.11), we learn that for any value of $\phi_t < \bar{\phi}$, ϕ_{t+1} decreases as $1 + r$ falls if and only if $\theta^x > \theta$. That is, we obtain $sign\frac{d\phi^*}{dr} = sign(\theta - \theta^x)$. From (10.16), we know $sign\frac{d(1+g^*)}{dr} = sign(-\frac{d\phi^*}{dr})$. Therefore, we obtain Proposition 8.

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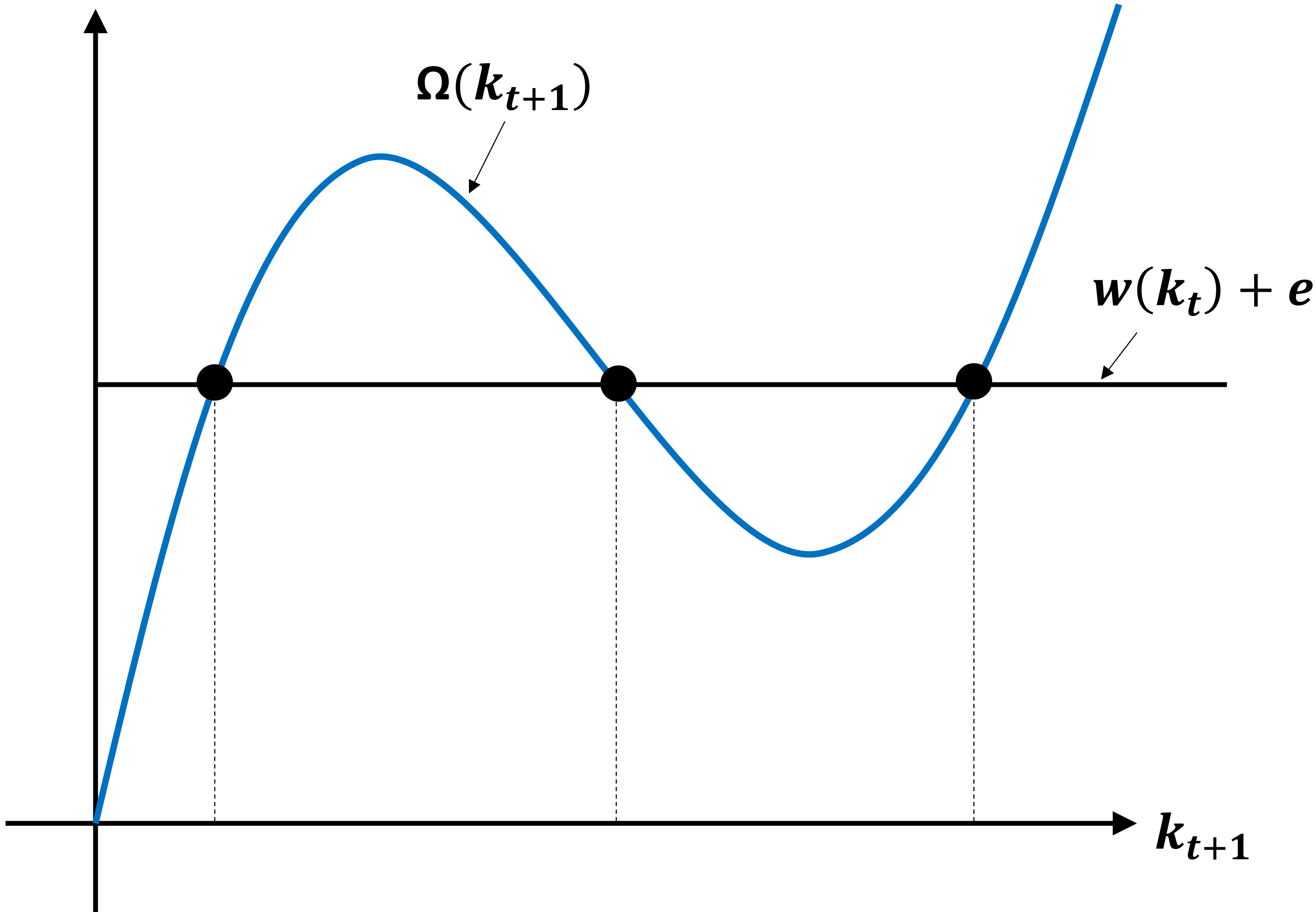

Figure 1: Existence of multiplicity of momentary equilibria

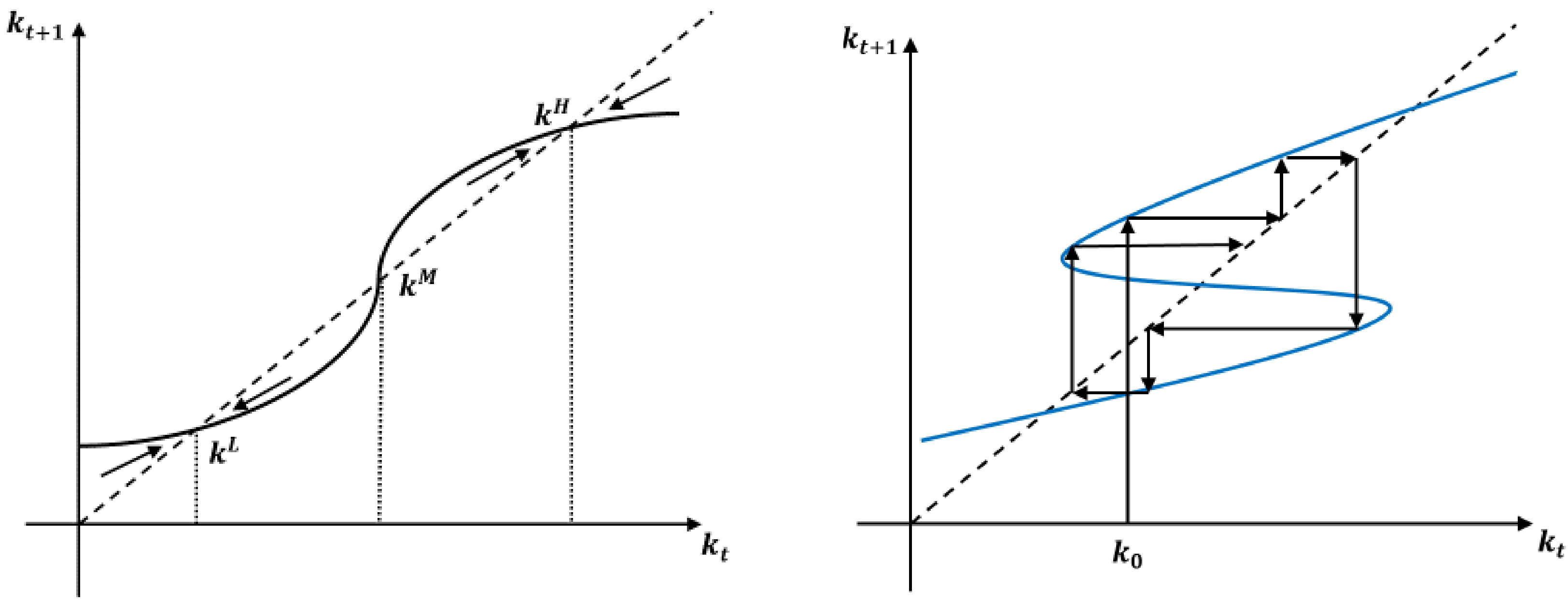

Figure 2: An example of a wobbly trajectory vs the typical dynamics with a unique equilibrium

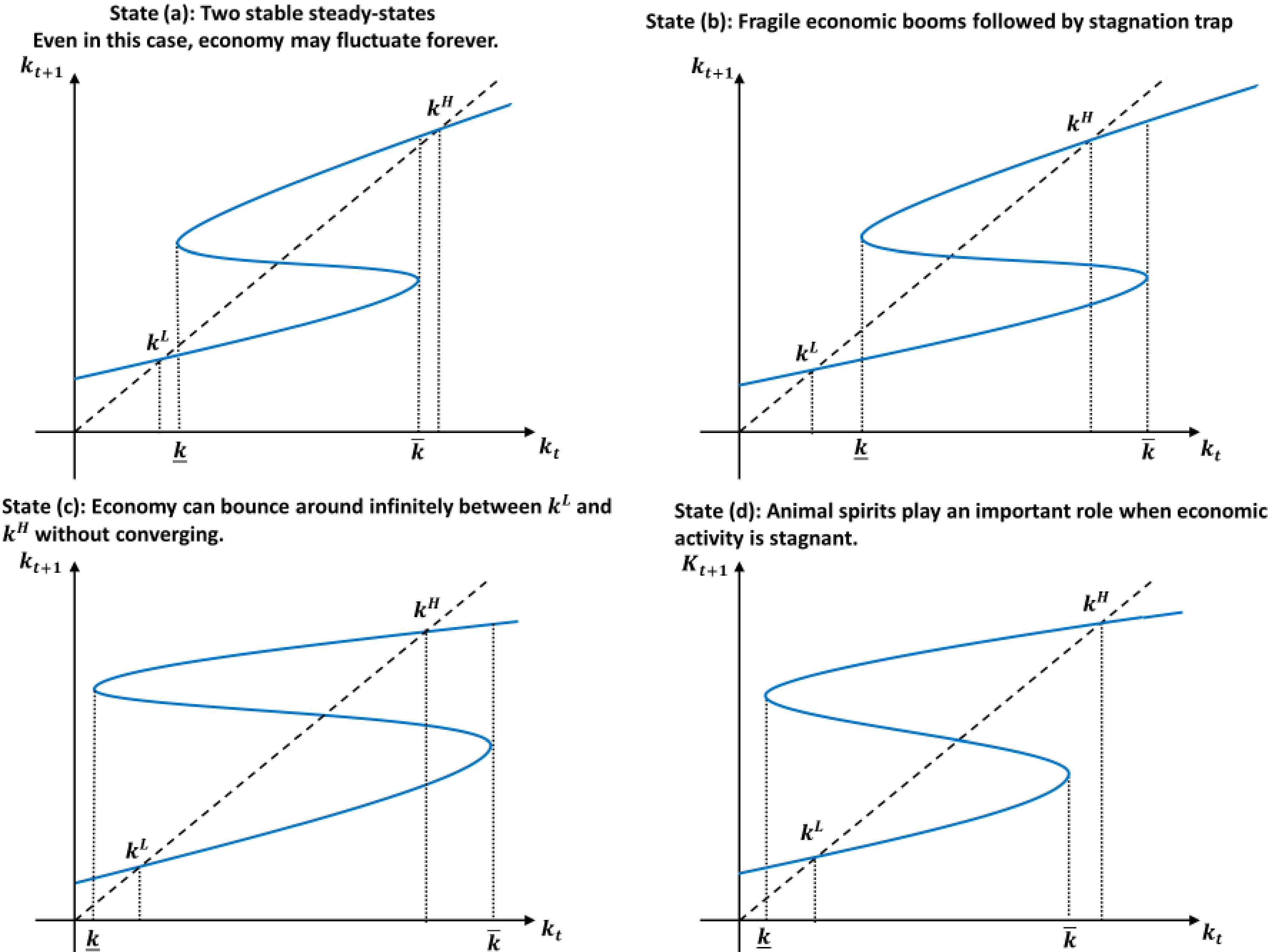

Figure 3: four typical patterns of the wobbly macro-dynamics

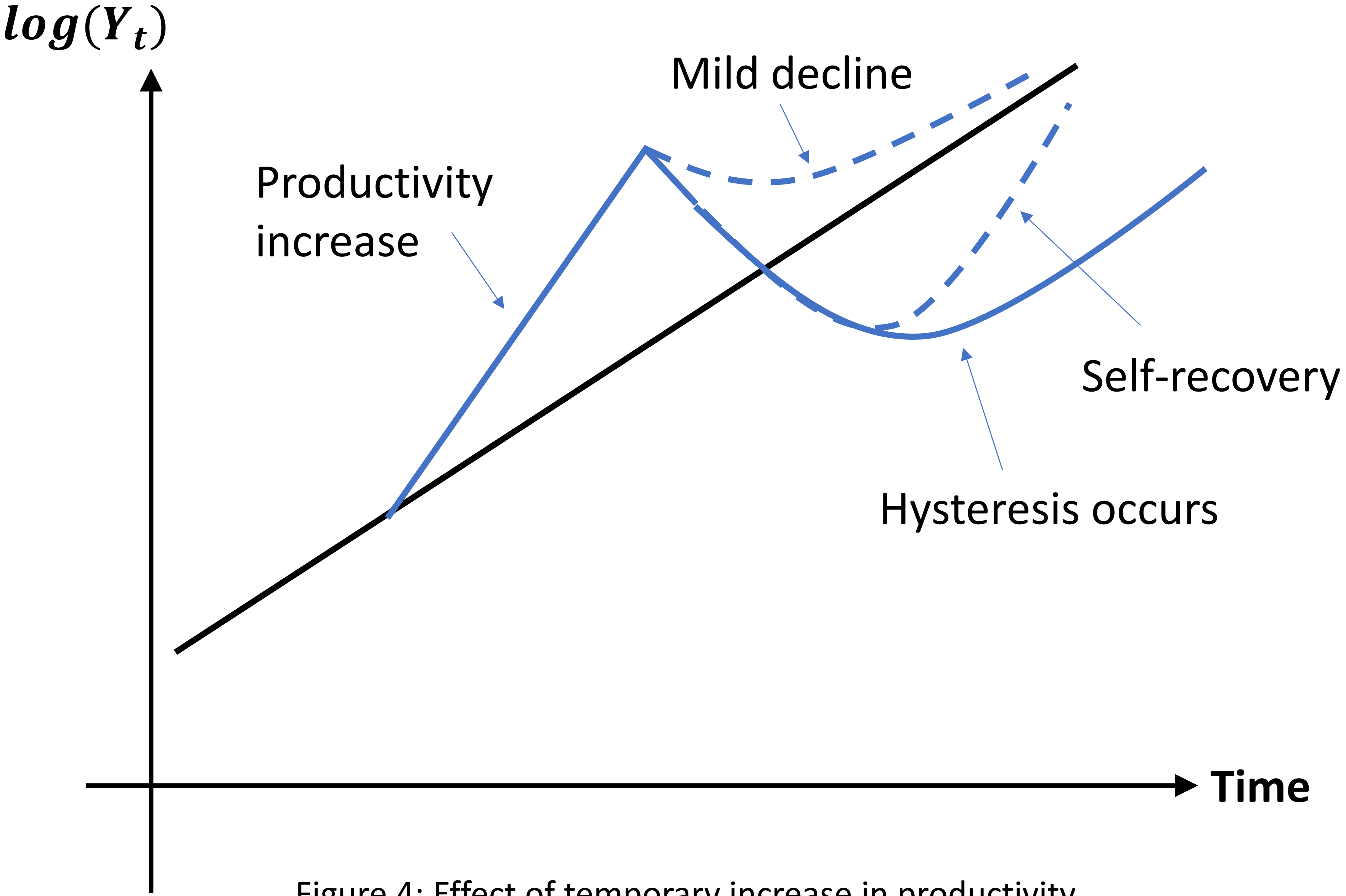

Figure 4: Effect of temporary increase in productivity

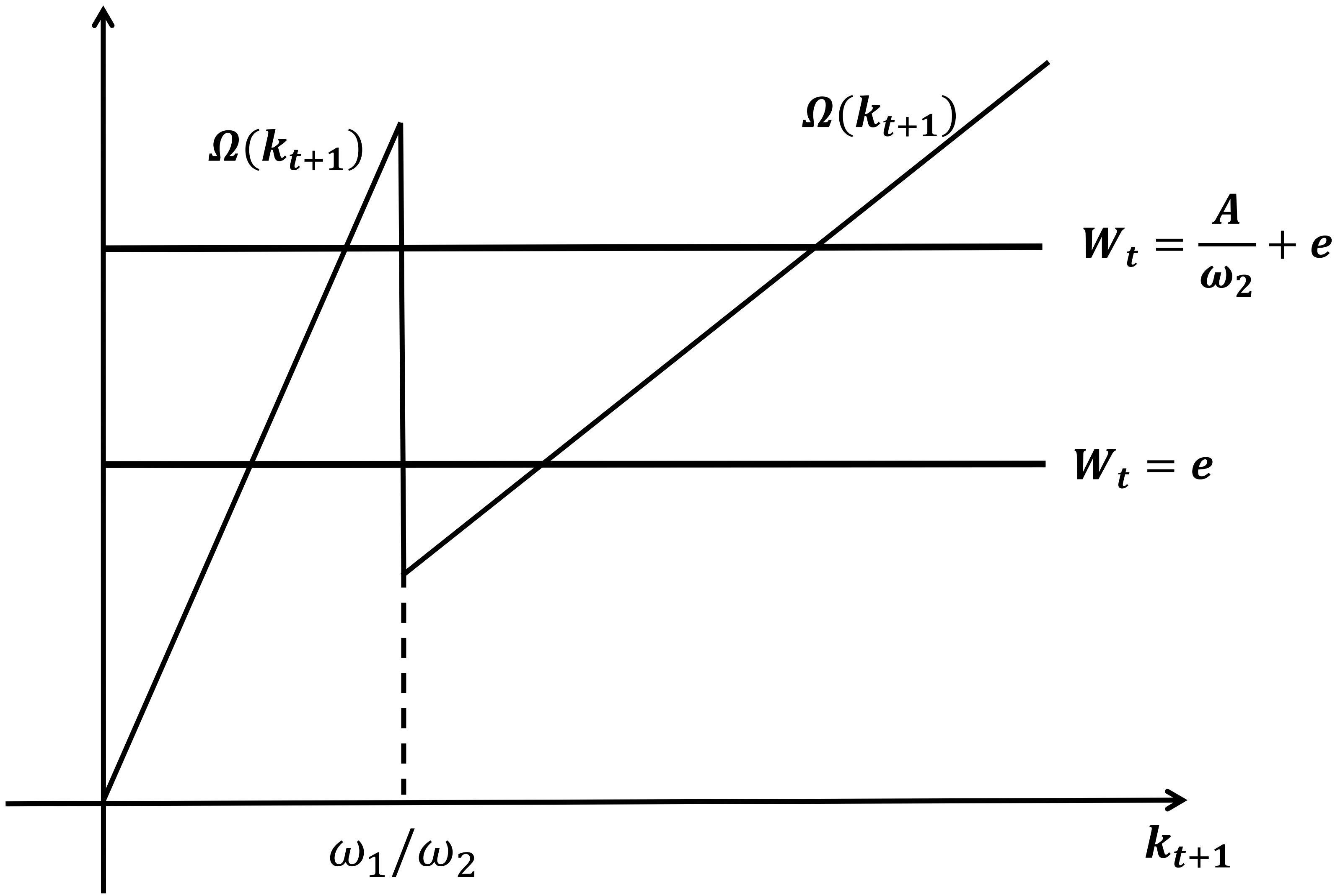

Figure 5: Existence of multiplicity of equilibria in the Leontief case

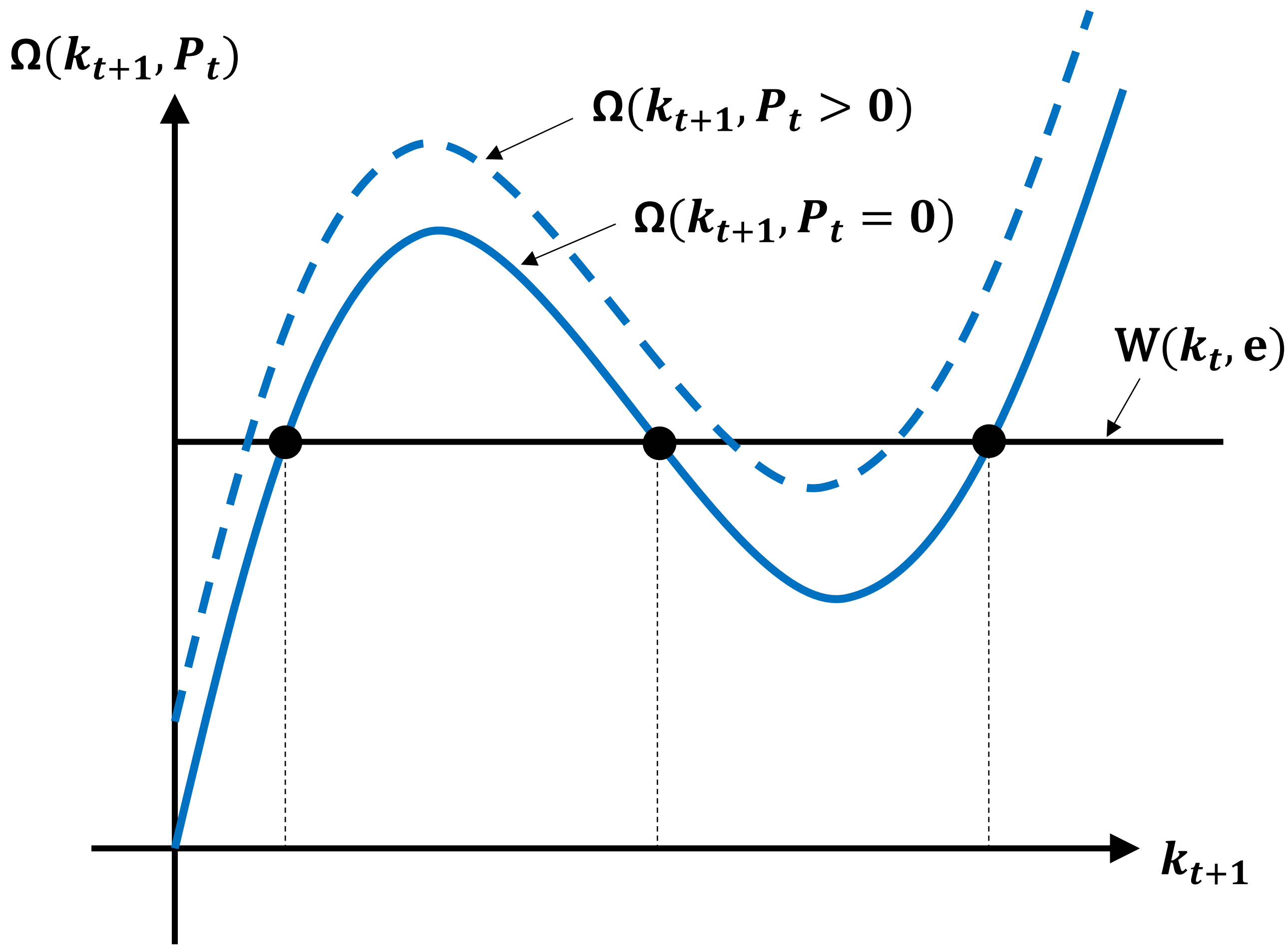

Figure 6-1: The existence of multiplicity of equilibria in a general case

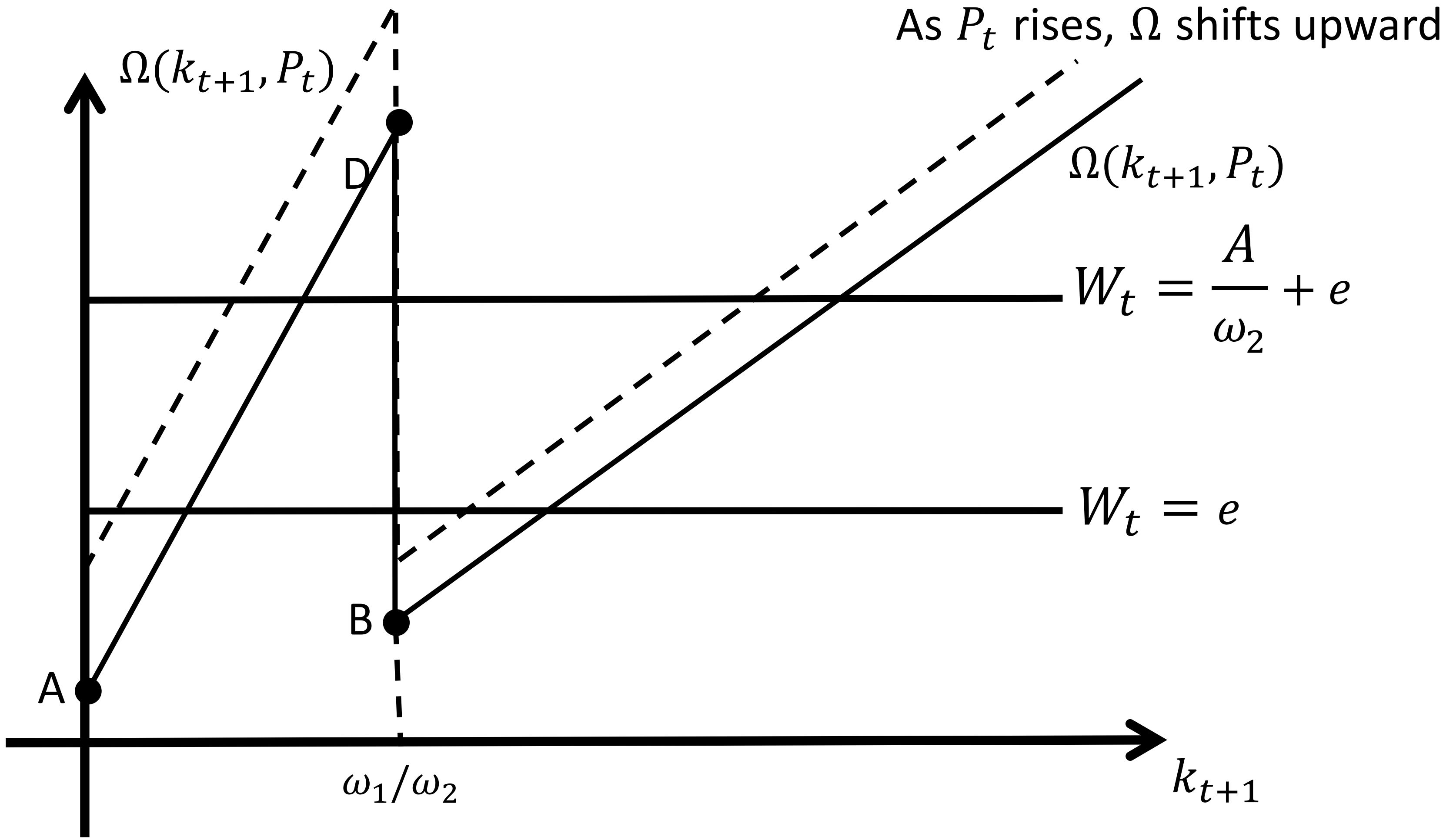

Figure 6-2: The existence of multiplicity of equilibria in the Leontief case

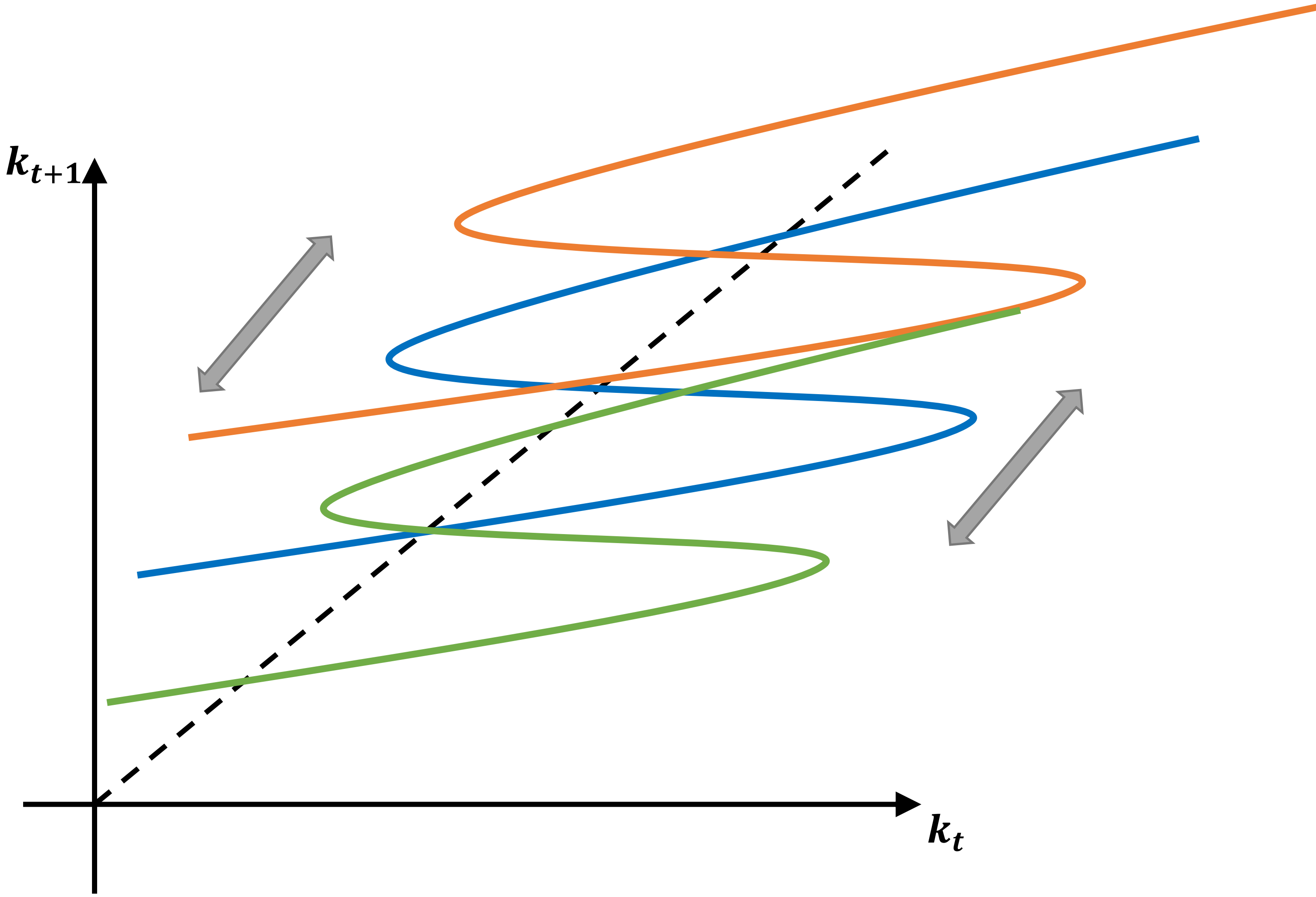

Figure 7: Dynamics of real capital when P_t constantly changes

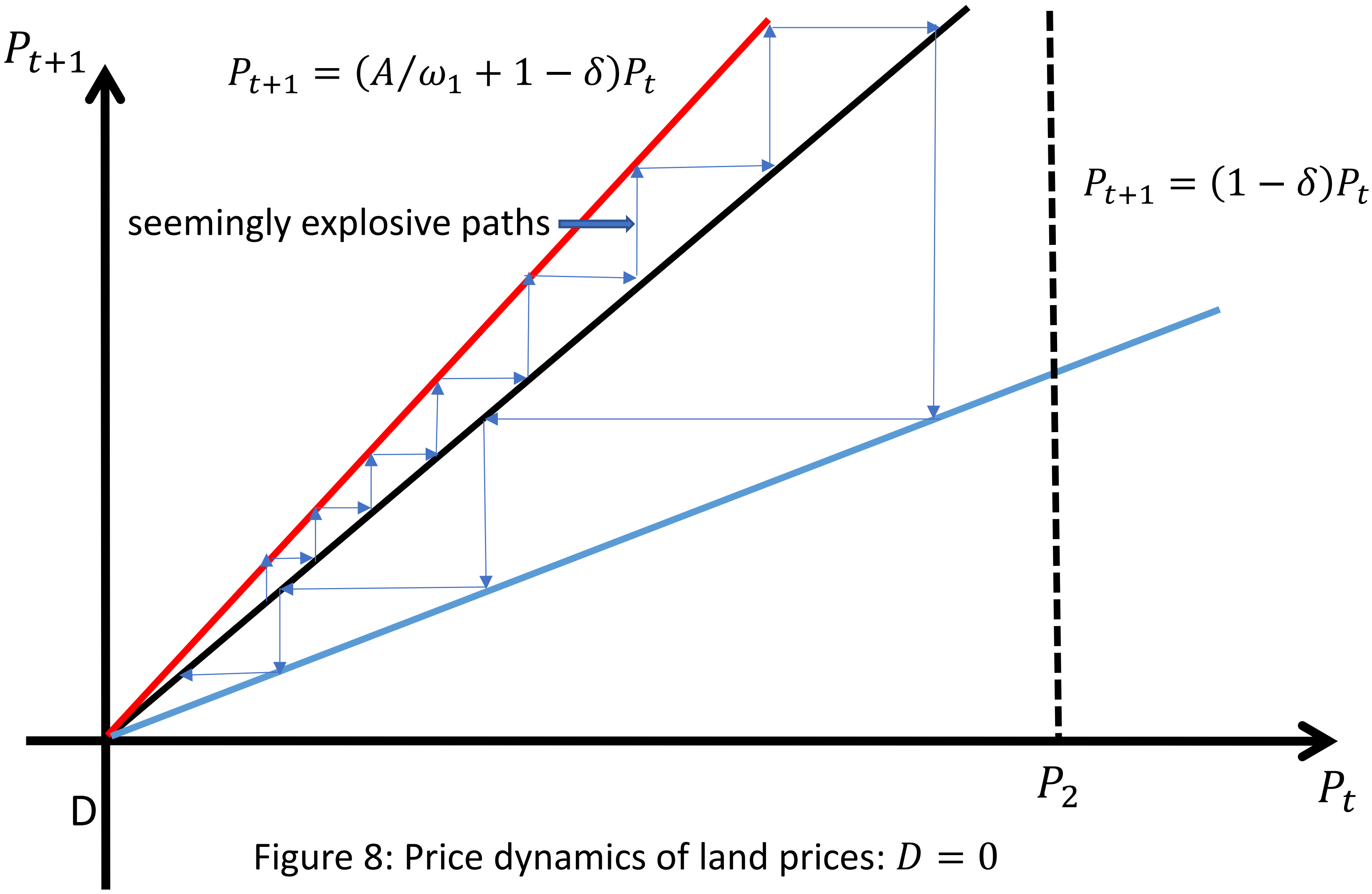

Figure 8: Price dynamics of land prices: $D = 0$

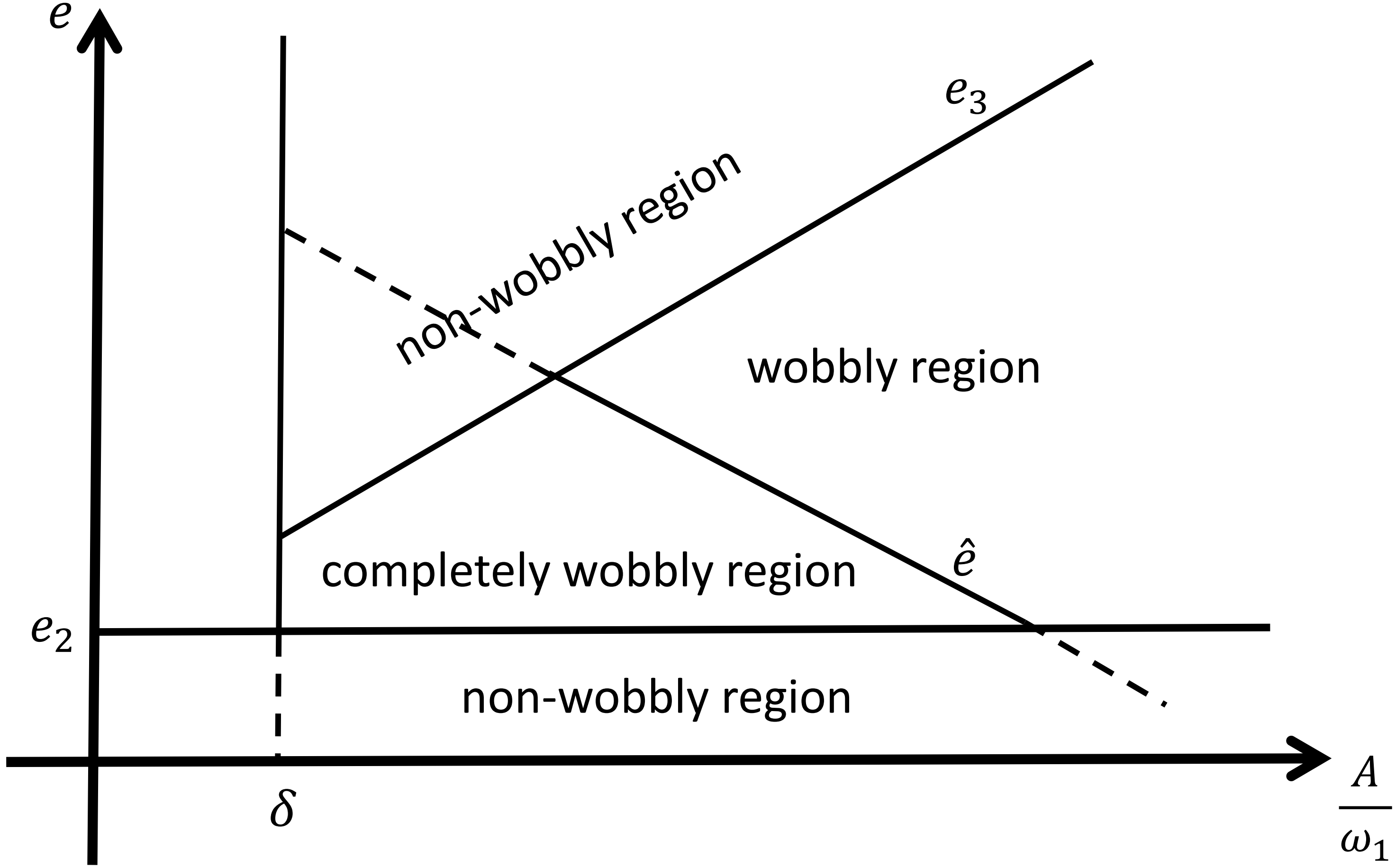

Figure 4: Case (a) $e_2 < e < e_3$ and $\frac{A}{\omega_1} > \delta$

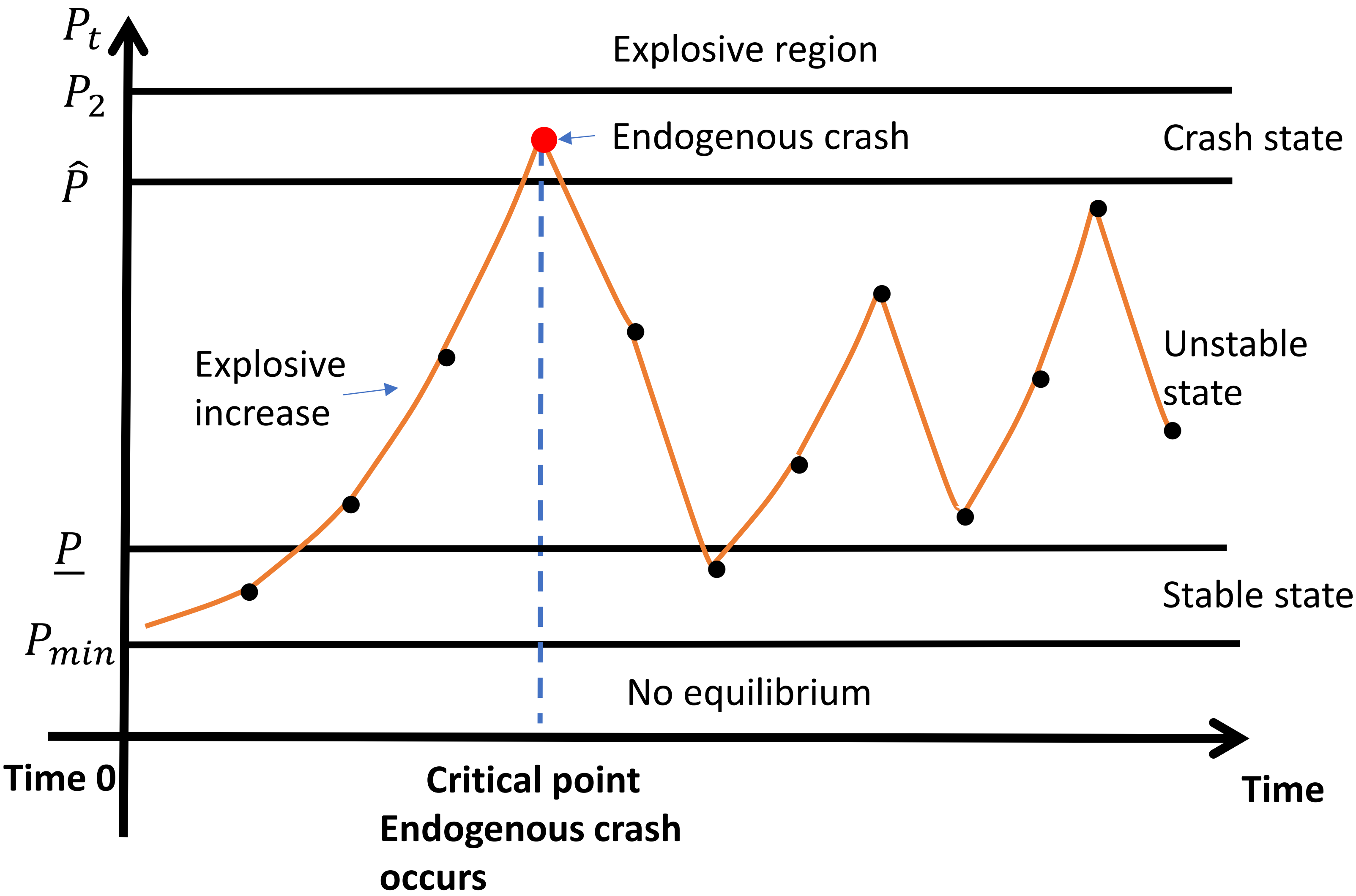

Figure 10: Endogenous Phase Transitions and Endogenous Crash

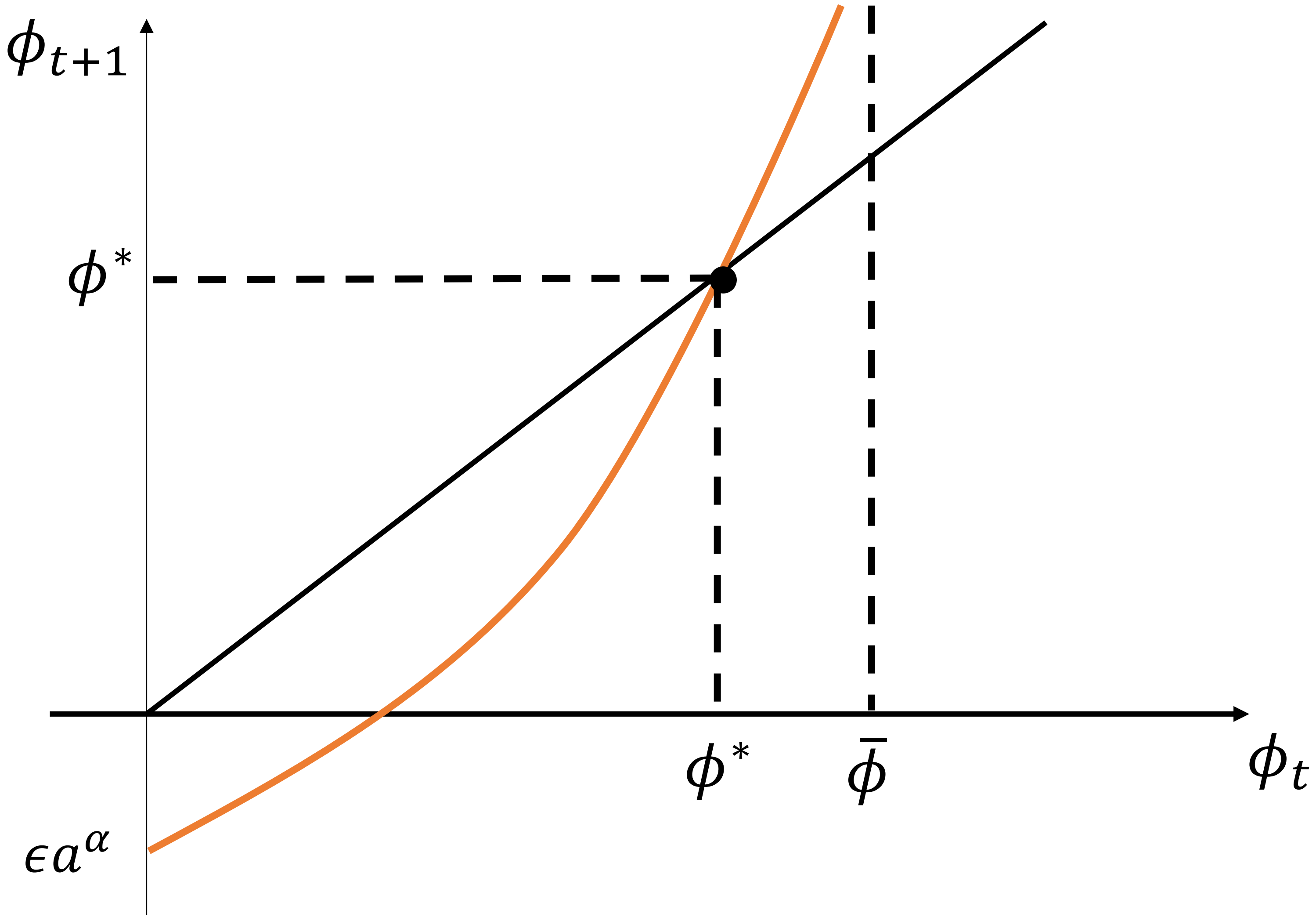

Figure 11:Dynamics of ϕ_t