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Growth Promotion Policies When Taxes Cannot Be Raised*

Katsunori Minami† Ryo Horii‡

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Abstract

This paper examines the growth effects of R&D subsidies and public-funded basic research in an R&D-based endogenous growth model under circumstances where the government cannot raise taxes. We show that when individuals have enough life-cycle saving motives and R&D productivity is sufficiently high, $g > r$ holds in equilibrium and the government can finance the required expenses while perpetually rolling over the debt. Whenever possible, debt-financed R&D subsidies always enhance short-run growth. However, long-term growth is promoted only when the initial $g - r$ gap is wide enough. Even when the long-term effect is negative, the economy may benefit from the increased GDP during a long transition to the new BGP. We confirmed that the social return to R&D is always higher than the growth rate even though $g > r$. In an extended model, we examine the effect of enhancing public-funded basic research and find that it is particularly effective for low-growth economies.

Keywords: Endogenous growth, Public debt, Ponzi scheme, R minus G, Research subsidy

JEL Classification Codes: O38, O41

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1 Introduction

Once one starts to think about [growth], it is hard to think about anything else
— Robert Lucas Jr. (1988)

As Lucas (1988) mentioned, economic growth is one of the most important objectives in economics. How can we achieve faster economic growth? Modern theories of endogenous growth provide a surprisingly simple answer. Economic growth, in the long run, is determined by technological change. Technological change is realized by R&D. Therefore, by promoting R&D, e.g., through government subsidies for R&D, economic growth can be accelerated. Nevertheless, many economies are struggling with slower economic growth than they desire. Even when they know that R&D promotion policies enhance growth, there are often insufficient funds with which to implement those policies. Governments typically have a limited ability to raise taxes, which constrains their ability to promote R&D and, therefore, economic growth.

Recent studies on the political economy suggest that it is difficult for the government to raise tax rates for various political reasons. For example, Jiang, Sargent, Wang and Yang (2022) assumed that there is an upper bound for the tax rate on the basis of the political considerations of Keynes (1923).[1] However, existing studies of R&D-based growth typically disregard such constraints by implicitly assuming that the government can levy lump-sum taxes to implement policies. (e.g. Grossman and Helpman, 1991a,b; Jones and Williams, 2000). Some studies include distortions caused by factor-income taxation but still assume that the government can set any tax rate (e.g., Grossmann, Steger and Trimborn, 2013).

In contrast, this paper considers an extreme situation where the government cannot raise (extra) taxes at all. If the government wants to support R&D, it must be financed entirely by public debt. It also does not have the ability to raise taxes in the future to repay the debt. Therefore, the debt must be rolled over infinitely. In other words, we consider the environment in which any growth promotion policies must be financed by a Ponzi scheme. While common sense suggests that such policies would not be sustainable, O'Connell and Zeldes (1988) showed that the government can run a "rational" Ponzi scheme if the rate of economic growth is higher than the interest rate on governmental debt, i.e., $g > r$ for short.

Mehrotra and Sergeyev (2021) reported that in the 1946–2006 period, the median value

[1] In an overlapping generations political economy model, Song, Storesletten and Zilibotti (2012) showed the possibility that intergenerational conflict causes the government to raise no tax if the political power of the old is stronger than that of the young. See also Alesina and Passalacqua (2016) and Yared (2019) for the literature review.

of $g - r$ was 1.0% for the United States and 0.8% for the average of 17 advanced countries. Blanchard (2019) also mentioned that $g > r$ has been more historical than the exception in the U.S. since 1950. Mauro and Zhou (2021) analyzed data on average effective borrowing costs for 55 countries over a span of up to 200 years and reported that $g > r$ prevails for both advanced and emerging economies.

Given these findings, our main model examines whether the growth rate can be enhanced when the government finances R&D subsidies entirely by perpetually rolled-over debt. The result is not obvious because such policies affect both g and r. The direct effect of R&D subsidies is to induce private firms to engage in more R&D, which will speed up technological progress. However, the government's debt may crowd out private R&D investments by increasing the equilibrium interest rate in financial markets. More precisely, a higher interest rate implies that the present value of future profits realized by R&D is lower, thereby discouraging R&D.

We find that the overall effects of R&D subsidies crucially depend on the productivity of R&D, which is defined by how many innovations can be realized per unit of R&D labor. We show that $g > r$ holds in equilibrium if individuals have enough life-cycle saving motives and the productivity of R&D is high enough. However, this does not always mean that the growth rate can be enhanced by debt-financed R&D promotion policies. For these policies to increase the long-term growth rate, the productivity of R&D must be even greater. In other words, the $g - r$ gap needs to be not only positive but also greater than a positive threshold value. If this condition is not met, R&D subsidies enhance the growth rate only temporarily, although the boost to economic growth may continue for several hundred years in the transition.

In an extended model, we introduce public-financed basic research that enhances the productivity of private R&D. The government now chooses the pair of the research subsidy and the addition to basic research while both must be financed by perpetually rolled-over debts. We find that, while R&D subsidy enhances long-term growth only when the initial productivity of private R&D is high enough, putting more resources into basic research is conducive to long-term growth regardless of the initial productivity, unless the effectiveness of basic research on private R&D is very low. This result suggests that enhancing basic research is preferable to R&D subsidies in low-growth economies.

We also find that there are limits to the debt-financed growth promotion policies even when $g > r$. There are threshold levels for research subsidy and basic research above which no balanced growth exists; i.e., the Ponzi scheme becomes unsustainable. We also calculate the maximal debt-to-GDP ratio that can be supported without raising taxes. We find that while debt-financed R&D subsidies may increase the long-run growth rate, they reduce the fiscal space in the sense that the highest level of the debt-to-GDP ratio from which the economy can return to a steady state is now lower.

There is a strand of literature that examines the sustainability of government debt in an environment where $g > r$ holds in equilibrium (e.g. Blanchard, 2019; Reis, 2021; Ball and Mankiw, 2023; Barro, 2023). Similar to some of those studies, we use a continuous-time overlapping generations model to explain why the interest rate can be lower than the growth rate in the long run. However, in most studies, the long-term growth rate is exogenous, and the focus is on the effect of policies on the safe interest rate. Exceptions are Saint-Paul (1992) and King and Ferguson (1993), who developed AK-type endogenous growth models in which $g > r$ holds in equilibrium. These authors showed that the economy is dynamically efficient even when $g > r$ because in endogenous growth models, the social return on investment is higher than the interest rate. We also show that the social return to R&D investment in our model, including the benefits from intertemporal knowledge spillovers, is always higher than the growth rate. The most important difference from those studies is that we consider an R&D-based variety-expansion growth model rather than an AK model, where long-term growth is determined solely by capital accumulation. By explicitly modeling the R&D process, we are able to examine the effect of debt-financed R&D promotion policies on incentives for technological innovations, which unarguably serve as an important source of economic growth. This paper is also related to Angeletos, Lian and Wolf (2024) in that both show that future tax increases are not necessary after debt-financed policies are implemented. While the abovementioned authors consider a short-term stimulus policy in a new Keynesian setting, we consider a long-term growth promotion policy in an R&D-based endogenous growth model.

The rest of this paper is organized as follows. Section 2 presents the model environment. Section 3 explains the equilibrium dynamics of this economy, including the relationship between g and r. In Section 4, we examine the growth effects of R&D subsidies, assuming that they are financed by perpetually rolled-over debts. Both long-term and transitional results are presented. In Section 5, we investigate the social return of R&D and the dynamic efficiency of the economy. Section 6 examines the extended model where public-funded basic research is introduced. Section 7 concludes the paper.

2 Model

2.1 Individuals

We consider a continuous-time overlapping generations model. In the economy, $\overline{N}$ new individuals are born per unit of time, and they face a constant Poisson death rate of $\mu > 0$. This means that the number of individuals who are born at time s and are still alive at time t is

$$N_{s,t} = \overline{N} e^{-\mu(t-s)}. \tag{1}$$

Total population is stationary at $N = \overline{N}/\mu$, which is obtained by integrating $N_{s,t}$ over s from $-\infty$ to t. Each individual supplies inelastically $e^{-\delta(t-s)}$ units of effective labor, where $t-s$ is their age.[2] Effective labor supply decreases with age at the rate of $\delta > 0$ both because of the deterioration of productivity and the declining ability to work longer hours as they age. Because of this aging effect, individuals have an incentive to make lifecycle savings. In this sense, parameter δ represents the strength of individuals' saving motives. Then, the aggregate labor supply is

$$L = \int_{-\infty}^{t} N_{s,t} e^{-\delta(t-s)} ds = \frac{1}{\mu + \delta} \overline{N}. \tag{2}$$

The expected utility of a generation s individual, accounting for their mortality, is given by

$$U_s = \int_{s}^{\infty} (\ln c_{s,t}) e^{-(\rho+\mu)(t-s)} dt, \tag{3}$$

where $c_{s,t}$ is the amount of consumption by a generation-s individual at time t and where $\rho > 0$ is the discount rate. Observe that they further discount the future by their survival probability, $e^{-\mu(t-s)}$. We assume that the discount rate is not too high. In particular, we assume that $\rho < \delta$ so that individuals have enough incentives to save for their later age. (See, Rachel and Summers, 2019).

Let $k_{s,t}$ be the real asset holding by a generation s and r_t be the real interest rate on bonds. Following Blanchard (1985), we assume that there is a perfect market for annuities. Thus, the rate of return from the annuities is $r_t + \mu$ for survivors. Since $r_t + \mu > r_t$, individuals hold all their assets in the form of annuities. We also normalize the price of the final goods to be one. Then, the budget constraint is given by

$$\dot{k}_{s,t} = (r_t + \mu) k_{s,t} + e^{-\delta(t-s)} w_t - c_{s,t}, \tag{4}$$

[2] More precisely, $t-s$ is the period after each individual starts their economic activity. If the starting age is 20, the actual age of an individual is $20 + (t-s)$.

where w_t is the real wage per unit of effective labor. The newborn generation has zero financial assets, which means that $k_{t,t} = 0$. To summarize, each individual maximizes the expedited utility (3) subject to the budget constraint (4), initial condition $k_{t,t} = 0$, and the usual non-Ponzi game condition. In this standard setting, the Euler equation for individuals is $\dot{c}_t = (r_t - \rho)c_t$.

Let us define aggregate consumption and aggregate asset holding by

$$C_t = \int_{-\infty}^{t} c_{s,t} N_{s,t} ds, \quad K_t = \int_{-\infty}^{t} k_{s,t} N_{s,t} ds. \tag{5}$$

In the following, we derive the dynamics for the aggregate consumption, in terms of C_t and K_t. Using the Leibniz integral rule, we can differentiate C_t in (5) as below.

$$\dot{C}_t = \int_{-\infty}^{t} \dot{c}_{s,t} N_{s,t} ds + \int_{-\infty}^{t} c_{s,t} \dot{N}_{s,t} ds + N_{t,t} c_{t,t}. \tag{6}$$

In the right-hand side (RHS), the first term represents the sum of changes in consumption by existing individuals. Using the Euler equation of individuals, $\dot{c}_t = (r_t - \rho)c_t$, this term can be written as $(r_t - \rho)C_t$. In the second term, the change in the cohort size $\dot{N}_{s,t}$ can be written $-\mu N_{s,t}$ from (1). Therefore this term becomes $-\mu C_t$. This value represents the decline in the aggregate consumption due to the dying of portion μ of individuals per unit time. The last term is the sum of the consumption by newly-born individuals. In Appendix A.1, we show that the consumption of a newborn is

$$c_{t,t} = \frac{\mu + \delta}{\mu} \left(\bar{c}_t - (\rho + \mu) \overline{k}_t \right), \tag{7}$$

where $\bar{c}_t = C_t/N$ is average consumption and $k_t = K_t/N$, is average asset holding. Equation (7) can be interpreted as follows. Recall that newborns do not have any financial assets ($k_{t,t} = 0$). Given the log utility, the consumption propensity out of assets is $\rho + \mu$. Therefore, the lack of financial assets induces a newborn to consume $(\rho+\mu)\overline{k}_t$ less than the average consumption, $\bar{c}_t$. However, newborns have a higher ability to work than average individuals. A newborn's working ability (labor supply) is 1, while that of an average individual is $L/N = \mu/(\mu + \delta)$. Therefore, the RHS of (7) is multiplied by $(\mu + \delta)/\mu$. By substituting (7) into (6) and using $N_{t,t} = \overline{N} = \mu N$, we obtain the Euler equation for the aggregate consumption.

$$\dot{C}_t = (r_t - \rho + \delta)\, C_t - (\rho + \mu)(\delta + \mu) K_t. \tag{8}$$

2.2 Supply Side

The supply side of this economy is purposefully close to that of the standard variety-expansion model by Grossman and Helpman (1991a). It consists of three sectors, namely, the final goods sector, the intermediate goods sector, and the R&D sector. In the final goods sector, a representative firm competitively produces final goods X_t from a continuum of varieties of intermediate goods $x_t(i)$, where i is the index of the intermediate goods. The production function is given by

$$X_t = \left[\int_0^{n_t} x_t(i)^{\alpha} di \right]^{1/\alpha}, \tag{9}$$

where n_t is the number of intermediate goods available at time t and where $\alpha \in (0,1)$ is a production parameter. Let $p_t(i)$ be the price of intermediate good i. The representative final goods firm maximizes its profit,

$$X_t - \int_0^{n_t} p_t(i) x_t(i) di. \tag{10}$$

The first-order condition for profit maximization implies that the demand function of the intermediate goods is

$$x_t(i) = p_t(i)^{-\frac{1}{1-\alpha}} X_t. \tag{11}$$

In the intermediate goods sector, there are n_t intermediate goods firms, each of which produces its own variety of goods, $x_t(i)$. The production of one unit of $x_t(i)$ requires one unit of labor; therefore, its profit is given by $\pi_t(i) = (p_t(i) - w_t)x_t(i)$. Given that $x_t(i)$ is determined by the demand function (11), profit-maximizing pricing implies

$$p_t(i) = \frac{w_t}{\alpha}, \quad x_t(i) = \left(\frac{\alpha}{w_t} \right)^{\frac{1}{1-\alpha}} X_t, \tag{12}$$

$$\pi_t(i) = (1-\alpha) \left(\frac{\alpha}{w_t} \right)^{\frac{\alpha}{1-\alpha}} X_t. \tag{13}$$

The above result shows that all intermediate goods firms produce the same amount of output. Therefore, the output of each intermediate goods firm can be written as $x_t(i) = L_t^P / n_t$ for all i, where L_t^P is the total amount of labor employed in this sector. By substituting it into the final goods production function (9), we obtain

$$X_t = n_t^{\frac{1-\alpha}{\alpha}} L_t^P. \tag{14}$$

The R&D sector has a representative R&D firm, which competitively creates new goods according to

$$\dot{n}_t = a n_t L_t^R, \tag{15}$$

where L_t^R is the amount of labor used for R&D and where $a > 0$ is a parameter that specifies the efficiency of R&D. We follow the standard setting in the variety-expansion model and assume that there is an externality from the past R&D to the current R&D. The term n_t in the RHS of (15) reflects this externality.

Equation (15) implies that the creation of a new intermediate good requires $1/an_t$ units of labor. We assume that the government subsidizes a fraction $\theta \in [0, 1)$ of the R&D cost. Then, the private cost of developing a new intermediate good is $(1-\theta)w_t/an_t$. Let v_t be the value of an intermediate goods firm. Then, the free entry condition for R&D is

$$v_t \leq (1-\theta)\frac{w_t}{a n_t} \text{ with equality if } \dot{n}_t > 0. \tag{16}$$

2.3 Government

As explained above, the government subsidizes a fraction θ of the cost of R&D. Since the presubsidy aggregate cost of R&D is $w_t L_t^R$, the amount of government expenditure is $\theta w_t L_t^R$. We assume that the government cannot collect taxes and that all expenditures are financed by government debt. Then, the amount of government debt, B_t, evolves according to

$$\dot{B}_t = r_t B_t + \theta w_t L_t^R. \tag{17}$$

Since the government bond is never repaid (or repaid entirely by issuing new bonds), the government is running a Ponzi scheme.[3] We investigate the possibility that the government can run a rational Ponzi game, similar to the one examined by O'Connell and Zeldes (1988), and use revenue to promote economic growth.

3 Equilibrium

3.1 The $g - r$ Gap

A necessary condition to run a rational Ponzi scheme is that the growth rate is higher than the interest rate. Given the economy described in the previous section, here, we examine

[3] Note that this is in contrast to individuals, who maximize their lifetime utility subject to the no-Ponzi-game condition. The difference in the ability to borrow between the government and individuals reflects reality. Without collateral, individuals usually cannot borrow large amounts of money for various reasons (e.g., the risk of running away). We can rewrite the model with a borrowing constraint for individuals, which yields the same result as the present setting in the steady state.

the gap between the growth rate and the interest rate in equilibrium. First, we derive the real interest rate. The consumers hold all their assets in the form of annuities, and the annuity company invests the assets in government bonds and the shares of intermediate goods firms. Therefore, the equilibrium in the asset market is

$$K_t = B_t + n_t v_t. \tag{18}$$

Since risks are fully diversified, the expected return on holding the shares of an intermediate goods firm should be equal to the interest rate on bonds. This no-arbitrage condition can be written as

$$r_t = \frac{\pi_t + \dot{v}_t}{v_t}, \tag{19}$$

where π_t is given by (13). In the main text, we focus on the case where the amount of R&D is positive, leaving the discussion of the case of $\dot{n}_t = 0$ for Appendix A.3. Then, v_t is given by (16) with equality. Both π_t and v_t depend on the wage level, which we now derive. Since the final goods sector is competitive, the maximized profit of the final goods firm in Equation (10) should be zero. By substituting $x_t(i)$ and $p_t(i)$ into (10), this condition determines the market wage as

$$w_t = \alpha n_t^{\frac{1-\alpha}{\alpha}}. \tag{20}$$

Substituting the values of π_t and v_t into (19) yields the real interest rate in equilibrium.[4]

$$r_t = \frac{a}{\alpha}\left(\frac{1-\alpha}{1-\theta}L_t^P - (2\alpha - 1)L_t^R\right). \tag{21}$$

Next, we turn to the growth rate. The GDP of this economy is defined as the sum of consumption expenditures C_t, private investment expenditures for R&D, $(1-\theta)w_t L_t^R$, and government expenditures, $\theta w_t L_t^R$. Note that the final output is used only for consumption; therefore, the equilibrium in the goods market means $C_t = X_t$. Then, using (14), and (20), the GDP can be written as

$$\text{GDP}_t = n_t^{\frac{1-\alpha}{\alpha}}(L_t^P + \alpha L_t^R). \tag{22}$$

Since $\alpha < 1$, the GDP is greater when more labor is used for production, given the value of n_t. This is because there is a positive markup in the intermediate goods sector, whereas

[4]Using (20), the values of π_t and v_t can be obtained as follows: From (14) and (20), Equation (13) gives $\pi_t = (1-\alpha)n_t^{(1-2\alpha)/\alpha}L_t^P$. From (16) with equality and (20), the value of a firm is $v_t = (1-\theta)w_t/an_t = ((1-\theta)\alpha/a)n_t^{(1-2\alpha)/\alpha}$. Using (15), its derivative is $\dot{v}_t = -((2\alpha-1)/\alpha)aL_t^R v_t$. Substituting these results into (19) yields (21).

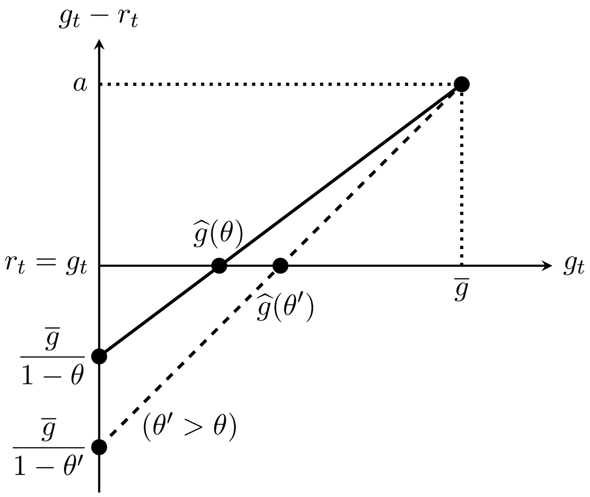


Figure 1: Relationship between the $g_t - r_t$ gap and the growth rate.

there is no markup in the R&D sector. Let us also define the potential GDP, Y_t, as the level of GDP when all labor is used for production, given the number of intermediate goods developed by that time, n_t. With $L_t^P = L$ and $L_t^R = 0$, (22) reduces to

$$Y_t = n_t^{\frac{1-\alpha}{\alpha}} L. \tag{23}$$

This is the upper bound for GDP_t given the state of technology at time t. Y_t can also be viewed as the supply capacity of the economy. From (15), the growth rate of the potential GDP in (23) is given by

$$g_t \equiv \frac{\dot{Y}_t}{Y_t} = \overline{g}\frac{L_t^R}{L}, \text{ where} \tag{24}$$

$$\overline{g} \equiv \frac{1-\alpha}{\alpha} aL \tag{25}$$

is the maximum growth rate of the potential GDP that is realized when all labor is used for R&D. Note that on any balanced growth path (BGP), g_t is constant. Then, (24) means that L_t^R is constant and the same for $L_t^P = L - L_t^R$. In this case, from (22) and (23), the growth rate of GDP_t coincides with $\dot{Y}_t / Y_t$. Therefore, g_t also represents the growth rate of GDP in the long run. Below, we simply call g_t the growth rate unless otherwise noted.

Now, we are ready to derive the gap between the growth rate and the interest rate. The labor market equilibrium condition is

$$L_t^P + L_t^R = L. \tag{26}$$

By eliminating L_t^P and L_t^R from (21) using (24) and (26), we can represent the $g-r$ gap as in terms of g_t:

$$g_t - r_t = s(\theta)(g_t - \widehat{g}(\theta)), \text{ where} \tag{27}$$
$$s(\theta) = \frac{1-\alpha\theta}{(1-\alpha)(1-\theta)} > 1, \quad \widehat{g}(\theta) = \frac{1-\alpha}{1-\alpha\theta}\overline{g} \in (0, \overline{g}].$$

As shown in Figure 1, given parameters, $g_t - r_t$ is positively and linearly related to g_t, with a slope of $s(\theta) > 1$. In particular, g_t is greater than r_t when g_t is greater than the threshold at $\widehat{g}(\theta)$. This means that keeping the growth rate high is crucial for maintaining $g_t > r_t$ and hence running the government's Ponzi scheme.

When the rate of R&D subsidies, θ, is increased, the thick line in Figure 1 rotates counterclockwise, which makes $g_t - r_t$ lower for a given growth rate (as shown by the dashed line). Accordingly, the threshold, $\widehat{g}(\theta)$ in (27), increases with θ. While research subsidies may increase the growth rate, g_t needs to be even higher to maintain $g_t > r_t$. Then, how is g_t determined? In the following subsections, we characterize the equilibrium path of g_t by dynamic equations and phase diagrams, first without the research subsidy and then with it.

3.2 Equilibrium Dynamics

The dynamics of this economy can be examined by focusing on two variables, namely, g_t and $D_t \equiv B_t/Y_t$. Here, D_t is the ratio of government debt to potential GDP.[5] We simply call this the debt-to-GDP ratio. Using (17), (20), (24) and (27), its time derivative is given by

$$\dot{D}_t = -s(\theta)\left(g_t - \widehat{g}(\theta)\right) D_t + \frac{\theta\alpha}{\overline{g}} g_t. \tag{28}$$

The first term of the RHS represents $(r_t - g_t)D_t$. With a balanced budget, the debt-to-GDP ratio would expand or shrink at the rate of $r_t - g_t$. The second term is the ratio of government spending on subsidies to potential GDP.[6] This accelerates the increase in D_t.

Next, we derive the time evolution of g_t. In this model, final goods are used only for consumption. Therefore, $X_t = C_t$ from the equilibrium of the goods market. Using (8),

[5] A benefit of focusing on $D_t \equiv B_t/Y_t$ rather than B_t/GDP_t is that Y_t depends only on the state variable n_t; therefore, D_t is predetermined. This means that we can use D_t as an initial condition for the equilibrium dynamics. In contrast, since GDP_t depends on jump variables L_t^P and L_t^R, it is not possible to use B_t/GDP_t as an initial condition. Note also that g_t is a jump variable because $g_t = \overline{g}L_t^R$ from (24).

[6] The ratio of government spending on subsidies to potential GDP is $\theta w_t L_t^R/Y_t$. Note that $w_t = \alpha Y_t/L$ from (20) and (23), and that $L_t^R = Lg_t/\overline{g}$ from (24). Using these, $\theta w_t L_t^R/Y_t = \theta\alpha g_t/\overline{g}$.

(14), (16), (18), (20), (23), (24) and (27), we obtain[7]

$$\dot{g}_t = (\overline{g} - g_t)\big(s(\theta)\left(g_t - \widehat{g}(\theta)\right) - \delta + \rho\big) + \overline{g}(\rho + \mu)(\delta + \mu)\left(D_t + \frac{\alpha(1-\theta)}{aL}\right). \tag{29}$$

In the first term, $s(\theta)\left(g_t - \widehat{g}(\theta)\right) - \delta + \rho$ represents $g_t - (r_t - \rho + \delta)$. There is a positive effect of g_t on $\dot{g}_t$ because growth in Y_t means that fewer production workers are required to produce a given amount of C_t, and therefore more labor can be used for R&D, increasing $\dot{g}_t$. Additionally, as explained in Section 2.1, the Euler equation for aggregate consumption (Equation 8) implies that an increase in $r_t - \rho + \delta$ raises $\dot{C}_t$, which reduces the labor used for R&D, thereby decreasing $\dot{g}_t$. In the second term, $D_t + \alpha(1-\theta)/aL$ represents the sum of government debt and the value of all firms, divided by the potential GDP (i.e., K_t/Y_t).[8] A larger amount of the aggregate asset negatively affects $\dot{C}_t$ since those who pass away, on average, have assets of $\overline{k}_t = K_t/N$, and they are replaced by newborns who do not have financial assets. Then, $\dot{g}_t$ increases since more labor will be allocated to R&D.

3.3 Dynamics without Research Subsidies

We start the analysis of the phase diagram with the case of no research subsidy ($\theta = 0$) because it clearly gives us the condition under which existing debts can be rolled over perpetually and how much debts can be sustained. We first look at the $\dot{D}_t = 0$ locus. With $\theta = 0$, $\dot{D}_t$ in (28) becomes zero when either $g_t = \widehat{g}(0)$ or $D_t = 0$ holds. On the $g_t = \widehat{g}(0)$ line, $r_t = g_t$ holds, which means that government debt is growing at the same rate as potential GDP; therefore, the debt-to-GDP ratio is stationary. D_t is also stationary on the $D_t = 0$ line because there is no income or expenditure by the government. These lines are drawn in red in Figure 2(a)-(c).

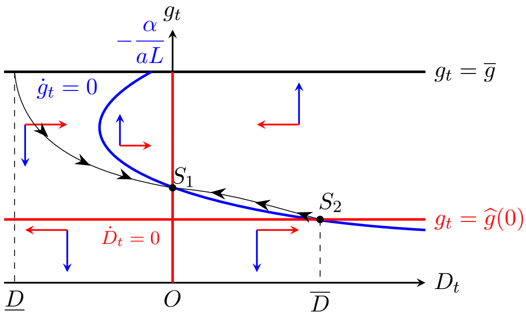
Next, we turn to the $\dot{g}_t = 0$ locus. With $\theta = 0$, $\dot{g}_t$ in (29) becomes zero when

$$D_t = \frac{\overline{g} - g_t}{\overline{g}(\rho + \mu)(\delta + \mu)}\big(\delta - \rho - s(0)\left(g_t - \widehat{g}(0)\right)\big) - \frac{\alpha}{aL}. \tag{30}$$

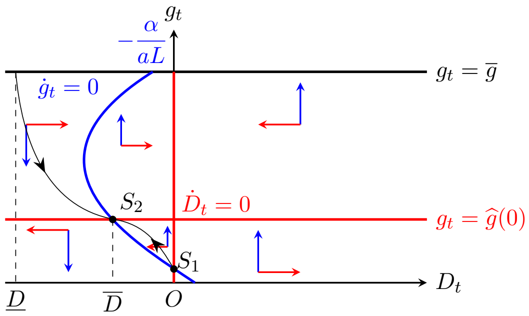
As depicted by the blue curve, the $\dot{g} = 0$ locus is a parabola that opens towards the right. Recall that g_t can only take the values between $[0, \overline{g}]$, where $\overline{g} \equiv (1-\alpha)aL/\alpha$ is the growth rate of potential GDP when all labor is used for R&D. Therefore, we limit the attention

[7] From (24) and (26), $\dot{g}_t = -\overline{g}(\dot{L}_t^P/L)$. Additionally, from (14), (23) and $X_t = C_t$, we have $L_t^P = LX_t/Y_t = LC_t/Y_t$. Since L is constant, the rate of change of this equation is $\dot{L}_t^P/L_t^P = \dot{C}_t/C_t - \dot{Y}_t/Y_t$. Therefore, $\dot{g}_t = -\overline{g}(L_t^P/L)(\dot{L}_t^P/L_t^P) = -\overline{g}(L_t^P/L)(\dot{C}_t/C_t - \dot{Y}_t/Y_t)$. In the RHS, $\dot{C}_t/C_t$ is given by the Euler equation (8), and $\dot{Y}_t/Y_t = g_t$. The Euler equation depends on aggregate assets K_t. From (16), (20), and (23), $n_t v_t = \alpha(1-\theta)Y_t/aL$. Substituting this into (18) yields $K_t/Y_t = D_t + \alpha(1-\theta)/aL$. Finally, we substitute these results and (27) into $\dot{g}_t = -\overline{g}(L_t^P/L)((\dot{C}_t/C_t - g_t)$ and obtain (29).

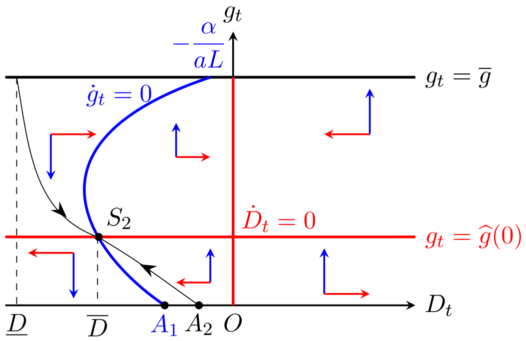
[8] See footnote 7.



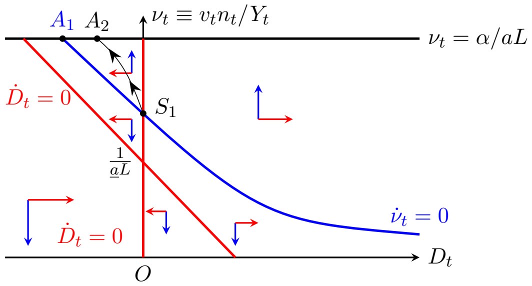
(a) When $a > \overline{a}$, S_1 is a saddle point with $g_1^* > r_1^*$, and S_2 is a source with $g_2^* = r_2^*$.



(b) When $a \in (\widetilde{a}, \overline{a})$, S_2 is a saddle point with $r_2^* = g_2^*$; S_1 is a source with $g_1^* < r_1^*$.



(c) When $a \in (0, \widetilde{a})$, S_2 is a saddle point with $r_2^* = g_2^*$, which is the only steady state in the $g_t > 0$ region.



(d) When $a \in (0, \widetilde{a})$, there is another unstable steady state in the $g_t = 0$ region.

Figure 2: Phase diagram when there is no government expenditure ($\theta = 0$).

to the area of $g_t \in [0, \overline{g}]$.

At the upper end of $g_t = \overline{g}$, the $\dot{g} = 0$ parabola starts from $D_t = -\alpha/aL$, as shown in Figure 2(a)-(c). Since this value is negative, the parabola intersects with the $D_t = 0$ line at most once. When the intersection exists (as in Figure 2(a)-(b)), it is a steady state with $\dot{D}_t = \dot{g}_t = 0$, which we label S_1 and denote its coordinates as $D_1^* = 0$ and $g_1^* \in [0, \overline{g})$. Additionally, the parabola crosses the $g_t = \widehat{g}(0)$ line exactly once before reaching the lower end of 0. We call this crossing point S_2. The coordinates of S_2 are $D_2^* = \overline{D}$ and $g_2^* = \widehat{g}(0)$, where

$$\overline{D} \equiv \frac{\alpha(\delta - \rho)}{(\rho + \mu)(\delta + \mu)} - \frac{\alpha}{aL}. \tag{31}$$

The pattern of the dynamics changes depending on whether $\overline{D}$ is positive or negative and whether S_1 exists. We explain three cases in turn.

Case 1: Saddle-stable steady state with $g_t > r_t$ ($a > \overline{a}$)

Since we assume that $\delta > \rho$, $D_2^* = \overline{D}$ is positive if and only if the research productivity parameter a is higher than

$$\overline{a} \equiv \frac{(\rho + \mu)(\delta + \mu)}{(\delta - \rho)L} > 0. \tag{32}$$

When $a > \overline{a}$, as shown in Panel (a) of Figure 2, the steady state S_1 is above the $g_t = \widehat{g}(0)$ line (i.e., $g_1^* \in (\widehat{g}(0), \overline{g})$). From (27), this implies that the growth rate in the steady state is higher than the interest rate. In Appendix A.2, we show that S_1 is saddle stable, whereas S_2 is totally unstable (a source). Therefore, there is a stable arm that originates from S_2 and converges to S_1. This means that, whenever the initial value of the debt-to-GDP ratio D_0 is less than D_2^*, there is an equilibrium path that converges to S_1, where the debt-to-GDP ratio is zero in the long run.[9] Even when the government has no revenue, its debt-to-GDP ratio can be stabilized as long as the ratio is not too large, given that $a > \overline{a}$.

Case 2: Saddle-stable steady state with $g_t = r_t$ ($a < \overline{a}$)

When $a < \overline{a}$, the steady state S_2 is located in the $D_t < 0$ region.[10] Nevertheless, if the $\dot{g}_t = 0$ parabola crosses the $D_t = 0$ line, another steady state S_1 exists, with $g_1^* \in (0, \widehat{g}(0))$, as depicted in Figure 2(b). This happens when $\widetilde{a} < a < \overline{a}$, where the threshold is given by

$$\widetilde{a} \equiv \frac{-(\delta-\rho) + \sqrt{(\delta-\rho)^2 + 4(1-\alpha)(\rho+\mu)(\delta+\mu)}}{2((1-\alpha)/\alpha)L} > 0. \tag{33}$$

In this case, as formally discussed in Appendix A.2, S_2 is saddle stable, and S_1 is totally unstable (a source). Therefore, the stability property is the opposite of that described in Case 1. There is a stable arm originating from S_1 and converging to S_2. Given that the economy starts from a positive government net asset ($D_t < 0$), there is an equilibrium path that stabilizes the net asset-to-GDP ratio in the long run.[11] In this steady state, the government asset grows at the same rate as the potential GDP ($r_t = g_t$), hence stabilizing the ratio. However, there is no equilibrium path converging to the saddle-stable steady state if the amount of initial net debt is positive. The government will go bankrupt when starting from $D_t > 0$.

When $a < \widetilde{a}$, there is only one steady state in the phase diagram shown in Figure 2(c). Similar to Case 2, S_2 is saddle stable, and a stable arm converges to it. However, the stable arm starts from point A_2, which is located to the left of the origin. This means that the stable arm exists only in the region where D_t is significantly negative, at least in this diagram. However, what happens if the economy starts from an initial debt or

[9] Strictly speaking, the economy can converge to the steady state only when $D_0 \in (\underline{D}, \overline{D})$, where $\underline{D} < 0$ is the point where the downwards-sloping stable arm crosses the $g_t = \overline{g}$ line. Intuitively, if the initial asset/GDP ratio of the government is too large and given that it does not use assets at all, the asset/GDP ratio explodes, and there is no steady state. Numerically, we find that the absolute value of $\underline{D}$ is very large under various parameter values; thus, it is not realistic to consider the case of $D_t < \underline{D}$. Therefore, in the main text, we disregard this possibility.

[10] We ignore the border case of $a = \overline{a}$ because the case has a zero possibility.

[11] Strictly speaking, the economy converges to the stable steady state when $D_0 \in (\underline{D}, 0)$. See the discussion in footnote 9.

asset that is approximately zero? Note that the phase diagram in Figure 2 is drawn under the assumption that the free entry condition (16) holds with equality. In fact, this model economy has another phase diagram in the D_t and $\nu_t = n_t v_t / Y_t$ spaces that applies when the amount of R&D is zero (i.e., $g_t = 0$), where the free entry condition does not need to hold with equality. In Appendix A.3, we show that there is an unstable steady state in this region if $a < \widetilde{a}$. There is a stable arm that originates from this steady state and connects to point A_2, as shown in Figure 2(d). Therefore, if the economy starts from a slightly negative D_t (i.e., a positive asset), it will experience a period of zero growth before arriving at point A_2; then g_t gradually increases until the economy reaches the saddle-stable steady state S_2. Therefore, the economy converges to the stable steady state if $D_0 \in (\underline{D}, 0)$, which is similar to the case of $\widetilde{a} < a < \overline{a}$.

The following proposition summarizes the results.

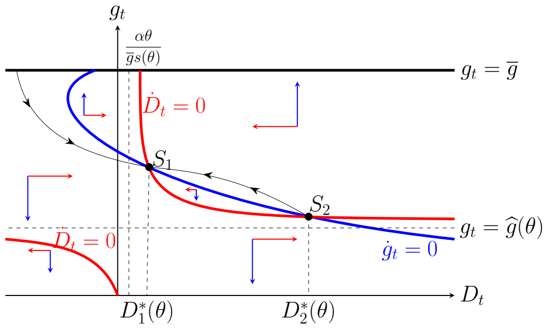
Proposition 1 *Suppose that $\theta = 0$. The growth rate in the saddle-stable steady state is higher than the interest rate if and only if $a > \overline{a}$. In this case, there is an equilibrium path converging to this steady state if the amount of initial debt is less than $\overline{D} > 0$. If $a < \overline{a}$, then the growth rate in the saddle-stable steady state is the same as the interest rate, and there is an equilibrium path converging to this steady state only when the initial debt is less than zero.*

In a simplified setting where there is no revenue or expenditure by the government, the proposition shows that the productivity of R&D, a, is critical for keeping the growth rate higher than the interest rate. If it is below the threshold $\overline{a}$, then the economy can reach a steady state only when the government holds a net positive asset, and the growth rate is equal to the interest rate.

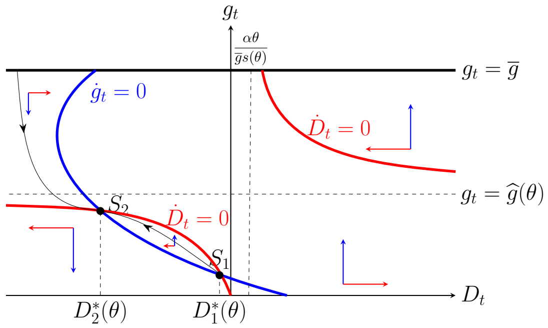
The proposition implies that the government is able to roll over the existing debt infinitely if the productivity of R&D is higher than $\overline{a}$ and the initial debt is less than $\overline{D}$. Given this result, a natural question is whether some government money can be used without collecting taxes. If the answer is yes, revenue could be used to increase growth. The following subsection considers this possibility.

3.4 Dynamics with Research Subsidies

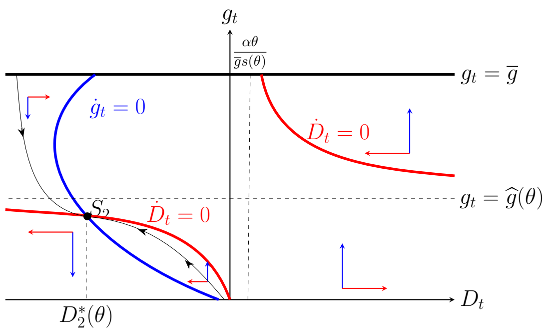
Now, we consider the effect of research subsidies on the dynamics of the economy while keeping the assumption that the government does not have any revenue. With a research



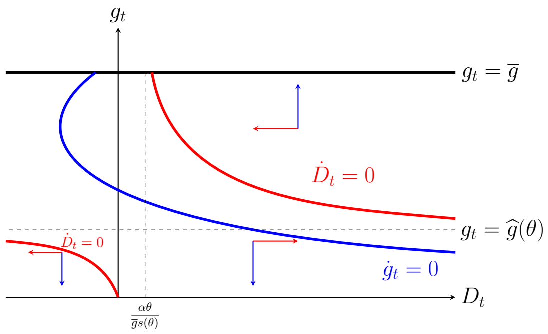
(a) Case 1: When $a > \overline{a}$ and θ is not large, S_1 is a saddle point with $g_1^*(\theta) > r_1^*(\theta)$.



(b) Case 2-1: When $a < \overline{a}$ and θ is not large, S_2 is a saddle point with $g_2^*(\theta) < r_2^*(\theta)$.



(c) Case 2-2: Similar to (b), but there is no S_1 in the $g_t > 0$ region.



(d) Case 3: When θ is too large, there is no steady state.

Figure 3: Phase diagram when the rate of debt-financed research subsidy θ is positive

subsidy rate of $\theta \in [0, 1)$, Equation (29) implies that the $\dot{g} = 0$ locus is as follows:

$$D_t = \frac{\overline{g} - g_t}{\overline{g}(\rho + \mu)(\delta + \mu)} \left(\delta - \rho - s(\theta)\left(g_t - \widehat{g}(\theta)\right)\right) - \frac{\alpha(1-\theta)}{aL}. \tag{34}$$

From (28), the $\dot{D} = 0$ locus is

$$D_t = \frac{\alpha\theta}{\overline{g}s(\theta)} \frac{g_t}{g_t - \widehat{g}(\theta)}. \tag{35}$$

The $\dot{g} = 0$ locus is a parabola, and $\dot{D} = 0$ is a rectangular hyperbola, with asymptotes of $D_t = \alpha\theta/\overline{g}s(\theta)$ and $g_t = \widehat{g}(\theta)$. They may or may not intersect with each other depending on parameter values, particularly θ. Figure 3 shows three possible cases.

Case 1: Saddle-stable steady state with $g_t > r_t$ ($a > \overline{a}$ and θ is not too large)

Recall that if $a > \overline{a}$, the economy has a saddle-stable steady state with $g^* > r^*$ when $\theta = 0$. (See Figure 2(a)).) Because the phase diagram moves continuously with θ, the economy still has a saddle-stable steady state with $g_t > r_t$ if $a > \overline{a}$ and θ is not too large, as shown by S_1 in Figure 3(a). We denote the coordinate of S_1 in the D_t-g_t space by $(D_1^*(\theta), g_1^*(\theta))$ since it changes with θ. As long as the $\dot{g}_t = 0$ locus intersects with the upper right portion of the $\dot{D}_t = 0$ locus, the value of g_t in the steady state is always higher than the horizontal asymptote at $g_t = \widehat{g}(\theta)$. Then, (27) implies that g_t is larger than r_t.

Additionally, note that the long-term level of the debt-to-GDP ratio, $D_1^*(\theta)$, is positive since the stable steady state S_1 is to the right of the asymptote line at $D_t = \alpha\theta/\overline{g}s(\theta) > 0$. Intuitively, the government issues a new bond each year to finance the research subsidy and pays interest on it forever. Nevertheless, the debt-to-GDP ratio converges to a positive constant because the economy is growing faster than the interest rate.

Another steady state S_2 at $(D_2^*(\theta), g_2^*(\theta))$ is a source (totally unstable). There is a saddle path originating from S_2 and converging to S_1. Therefore, the debt-to-GDP ratio can be stabilized in the long run if the initial debt-to-GDP ratio is less than $D_2^*(\theta)$.

Case 2: Saddle-stable steady state with $g_t < r_t$ ($a < \overline{a}$ and θ is not too large)

In contrast to Case 1, if $a < \overline{a}$ and θ is not too large, then the $\dot{g}_t = 0$ locus intersects with the lower left portion of the $\dot{D}_t = 0$ locus, as shown in Figure 2(b)-(c). In this case, S_2 at $(D_2^*(\theta), g_2^*(\theta))$ is a saddle-stable steady state, and there may or may not exist an unstable steady state (S_1) in the $g_t > 0$ region depending on the parameters, as discussed in Case 2 of Section 3.4.

The value of $g_2^*(\theta)$ is always lower than the horizontal asymptote at $\widehat{g}(\theta)$. Therefore, from (27), the interest rate in this steady state, denoted by $r_2^*(\theta)$, is higher than the growth rate $g_2^*(\theta)$. Additionally, $D_2^*(\theta)$ is negative because the hyperbola slopes downwards and moves through the origin. This means that in the steady state S_2, the government holds a positive net asset that grows at the rate of economic growth so that the asset-to-GDP ratio is constant at $|D_2^*(\theta)|$. Given that $r_2^*(\theta) > g_2^*(\theta)$, the government can use the $(r_2^*(\theta) - g_2^*(\theta))\,|D_2^*(\theta)|\,Y_t$ portion of interest revenue while keeping the asset-to-GDP ratio constant. In the steady state, this surplus is just enough to finance the expenditure for the research subsidy.

While the situation might seem desirable, it might be difficult to reach this steady state. Suppose that the unstable steady state S_1 exists in the $g_t > 0$ region, as depicted in Figure 3(b). Then, the stable steady state S_2 can be reached only when the initial value of D_t is less than $D_1^*(\theta)$, with a debt-to-GDP ratio of S_1. Since $D_1^*(\theta) < 0$, the government needs to start with positive net assets, and the asset-to-GDP ratio must be greater than $|D_1^*(\theta)|$. To summarize, when $a < \overline{a}$, it is not possible for the government to subsidize R&D while not collecting taxes unless it has enough initial assets.

Case 3: No steady state (θ is larger than a certain threshold)

If θ is too large, then there is no intersection between the $\dot{g}_t = 0$ locus and the $\dot{D}_t = 0$ locus in the phase diagram, as shown in Figure 3(d). Therefore, regardless of the initial value of D_t, the economy cannot reach a steady state. Starting from any value of g_t, we find that the economy will eventually violate the nonnegativity condition for some variables (e.g., L_t^P or v_t eventually become negative). This means that there is no equilibrium path in this economy. Such a large debt-financed subsidy is not sustainable regardless of the initial debt-to-GDP or asset-to-GDP ratio.

4 The Growth Effects of Debt-Financed Research Subsidies

As discussed in the Introduction, historical data show that the economic growth rate tends to be higher than the interest rate on government bonds in the U.S. and other developed countries (e.g. Blanchard, 2019; Mauro and Zhou, 2021). Given this fact, the analysis outlined in the previous section suggests that the productivity of R&D, a, is greater than $\overline{a}$ in these economies. In this case, the government can provide research subsidies for firms while rolling over their debts, given that the initial debt-to-GDP ratio is less than $D_2^*(\theta)$ (see Case 1 of Section 3.4). Here, we examine whether such a policy can actually enhance economic growth. In the first subsection, we examine the effects on the long-term growth rate by focusing on the steady state. In the second subsection, we investigate the transitional effects.

4.1 Long-Term Effects

A research subsidy has two opposing effects on the growth rate. First, it promotes research activity by reducing the R&D cost incurred by private firms. Second, the government expense for the subsidy will increase the long-term debt-to-GDP ratio $D_1^*(\theta)$, which increases the equilibrium interest rate. A higher interest rate reduces the value of firms and hinders the incentives for R&D. The following proposition shows when the first effect dominates.

Proposition 2 *A marginal increase in θ from $\theta = 0$ increases the growth rate in the saddle-stable steady state if and only if*

$$a > 2\overline{a} + \frac{\delta - \rho}{2L} \equiv \widehat{a}. \tag{36}$$

Proof: In Appendix A.4.

The proposition shows that a high value of a is necessary for the subsidies on R&D to have a positive effect on long-term growth. In Proposition 1, we have shown that $a > \overline{a}$ is the condition for the economy to have a saddle-stable steady state with $g_t > r_t$. Since $\widehat{a} > \overline{a}$, Proposition 2 implies that $g_t > r_t$ is not sufficient for such policies to have positive

Parameter	Description	Value	Source
L	Population	1	Normalization
ρ	Discount rate	0.01	Standard
μ	Mortality rate	1/(80.3 − 20)	OECD (2023b)
δ	Labor deterioration rate	1/(64-20)	OECD (2023a)
α	Inverse of the markup rate	1/1.2	Ball and Mankiw (2023)
a	R&D Productivity	0.546	Calibrated to match $g = 2\%$.
		[0.15]	For comparison ($\overline{a} < a < \widehat{a}$)

Table 1: Parameters for Numerical Simulations

long-run effects. The following corollary shows that research subsidies enhance long-term growth if and only if the $g_t - r_t$ gap is larger than a certain threshold.

Corollary 1 *Suppose that in the absence of research subsidies ($\theta = 0$), the economy has a saddle-stable steady state with $g_1^*(0) > r_1^*(0)$. A marginal increase in θ from $\theta = 0$ increases the growth rate in the saddle-stable steady state if and only if the following holds:*

$$g_1^*(0) - r_1^*(0) > \frac{\delta - \rho}{2}. \tag{37}$$

Proof: In Appendix A.4.

Note that the RHS of (37) is strictly positive because we assume that $\delta > \rho$. In our model environment, $\delta > \rho$ is necessary for the economy to have a saddle-stable steady state with $g_1^*(0) > r_1^*(0)$. When δ is greater, the decline in individual effective labor supply with age is steeper, and the consumer has more incentives to save for their old age. These saving incentives keep the interest rate lower than the growth rate. However, Corollary 1 shows that a larger δ requires a wider gap between g_t and r_t for debt-financed research subsidies to have a positive effect on long-term growth.

In the following, we present the effect of research subsidies by numerical simulations. Table 1 summarizes the parameters used in all simulations in Section 4. We normalize the total labor supply[12] to $L = 1$ and set a standard value for discount rate $\rho = 0.01$. The mortality rate is $\mu = 1/(80.3 - 20)$, where 80.3 is the average life expectancy in the OECD in 2021 (OECD, 2023b), and 20 is the age from which we assume that the agents start economic activities. The deterioration rate of labor productivity δ is set to 1/(64-20), where 64 is the average normal retirement age in the OECD as of 2022 (OECD, 2023a).[13]

[12] As shown in (2), the total labor supply is $L = \overline{N}/(\mu + \delta)$. Therefore, this normalization is the same as assuming $\overline{N} = \mu + \delta$. Note that this choice does not affect the result as long as we calibrate a as explained below. A Doubling of L would result in halving a so that the growth rate is unchanged.

[13] Using similar data from the OECD, Rachel and Summers (2019) developed a probabilistic retirement model, while we consider continuous depreciation of working ability for simplification of the analysis. The evolution of the average productivity is the same in these two specifications.

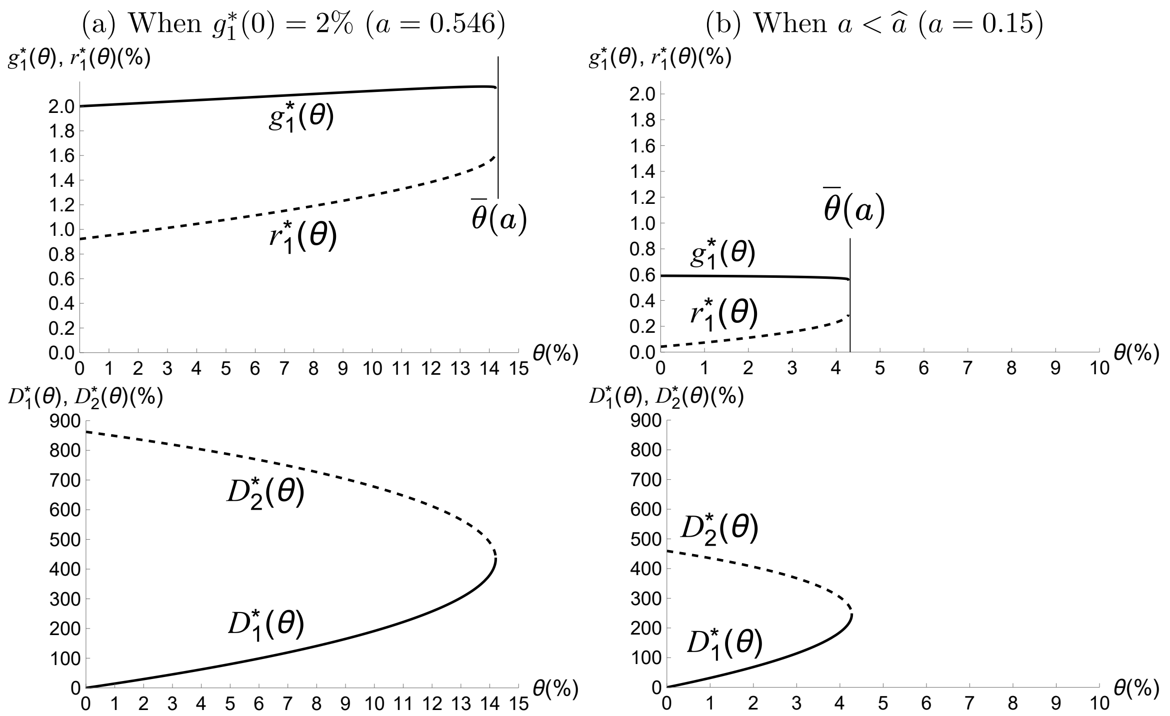


Figure 4: The long-term effect of research subsidy rate θ on the growth rate $D_1^*(\theta)$, the interest rate $r_1^*(\theta)$, the debt-GDP ratio $D_1^*(\theta)$, and the maximum limit of the debt-to-GDP ratio $D_2^*(\theta)$.

We set $\alpha = 1/1.2$ so that the markup rate of intermediate goods is $1/\alpha = 1.2$, following Ball and Mankiw (2023). Given these parameters, we find that the threshold values of a are $\widehat{a} = 0.171$, $\overline{a} = 0.0821$, and $\widetilde{a} = 0.0414$. We calibrate a to $0.546 > \widehat{a}$ so that the growth rate when $\theta = 0$ is 2%, a value close to the historical per capita GDP growth rate in the U.S. For comparison, we also consider the case of $a = 0.15$, which is higher than $\overline{a}$, but lower than $\widehat{a}$. This case corresponds to a low-growth economy where $g^*(0)$ is 0.592%.

Figure 4 numerically plots the long-term growth rate, interest rate and debt-to-GDP ratio against the rate of research subsidies θ. We can confirm the result of Proposition 2 in the upper panels of Figure 4. Column (a) show the baseline case of $a = 0.546 > \widehat{a}$. In this case, the value of $g_1^*(\theta)$ is increasing in θ unless θ is close to the upper limit, where the BGP disappears. We denote this upper limit of θ by $\overline{\theta}(a)$ because it depends on a. When θ exceeds this point, the phase diagram of the dynamics changes from Figure 3(a) to (d), and there is no steady-state level of the debt-to-GDP ratio. This means that it is not possible to increase the rate of research subsidies beyond $\overline{\theta}(a)$ because the government debt subsequently becomes unsustainable. Column (b) shows that $g_1^*(\theta)$ is always decreasing with θ when $a < \widehat{a}$, consistent with Proposition 2. The upper panels also show the response of the interest rate to changes in θ.[14] The interest rate is increasing in θ, but it is always lower than the growth rate as long as the BGP exists. This result confirms analysis in Case 1 of Section 3.4. As shown in Figure 3(a), $g_1^*(\theta)$ is greater than $\widehat{g}(\theta)$; therefore, from (27), $g_t > r_t$ holds. In other words, the government can roll over its debt only when its policy allows the existence of a steady state with $g_t > r_t$.

The lower panels show that the steady-state debt-to-GDP ratio, $D_1^*(\theta)$, also increases with θ especially when θ is near $\overline{\theta}(a)$. Observe that $r_1^*(\theta)$ and $D_1^*(\theta)$ follow the same pattern. This implies that the accumulation of government debt is causing the equilibrium interest rate to rise, which in turn crowds out private R&D. In fact, even when $a > \widehat{a}$ (i.e., in Column (a)), we can observe that $g_1^*(\theta)$ is decreasing in θ when θ is close to the upper limit. This happens because the response of interest rate to θ is sharp when θ is close to $\overline{\theta}(a)$. The lower panels also show $D_2^*(\theta)$, which represents the maximum value of the debt-to-GDP ratio from which the economy can converge to the saddle steady state (see the phase diagram in Figure 3(a)). Observe that $D_2^*(\theta)$ is decreasing in θ, and it eventually connects to $D_1^*(\theta)$ just before the steady state disappears. Therefore, even though a higher θ increases the growth rate given $a > \widehat{a}$, it entails two kinds of costs, namely, it increases the steady-state level of the debt GDP ratio, $D_1^*(\theta)$, and it reduces the highest maintainable debt-to-GDP ratio, $D_2^*(\theta)$.[15] The curves of $D_1^*(\theta)$ and $D_2^*(\theta)$ eventually coincide, which means that the steady state cannot exist beyond this point. This point defines the highest sustainable subsidy rate $\overline{\theta}(a)$.

Figure 5 shows the contour plot of the long-term growth rate, $g_1^*(\theta)$, against a and θ. The growth rate is greater when the color is lighter. The thick black curve shows the relationship between the upper limit of the subsidy rate, $\overline{\theta}(a)$, and the productivity of R&D, a. Since it slopes upwards, a higher rate of research subsidies is sustainable when research productivity is also higher. The area above the $\overline{\theta}(a)$ curve, which is shown in white, indicates that there is no steady state with $g_t > r_t$ for the given combination of a and θ; i.e., such a policy is not sustainable. Recall that according to Proposition 1, the government can roll over the debt only when $a > \overline{a}$. Therefore, the $\overline{\theta}(a)$ curve intersects with the horizontal axis to the right of the origin at $a = \overline{a}$.

[14]From (27), the interest rate in the steady state S_1 is obtained by $r_1(\theta) = s(\theta)\widehat{g}(\theta) - (s(\theta) - 1)g_1^*(\theta)$.

[15]The debt-GDP ratio could be (temporarily) affected by various kinds of shocks, although we do not explicitly consider these. If the current debt-GDP ratio is close to the highest maintainable debt-to-GDP ratio, $D_2^*(\theta)$, these shocks may push up D_t above $D_2^*(\theta)$, which makes the government go bankrupt. Therefore, the decline in $D_2^*(\theta)$ can be viewed as a cost since it negatively affects the sustainability of the government debt.

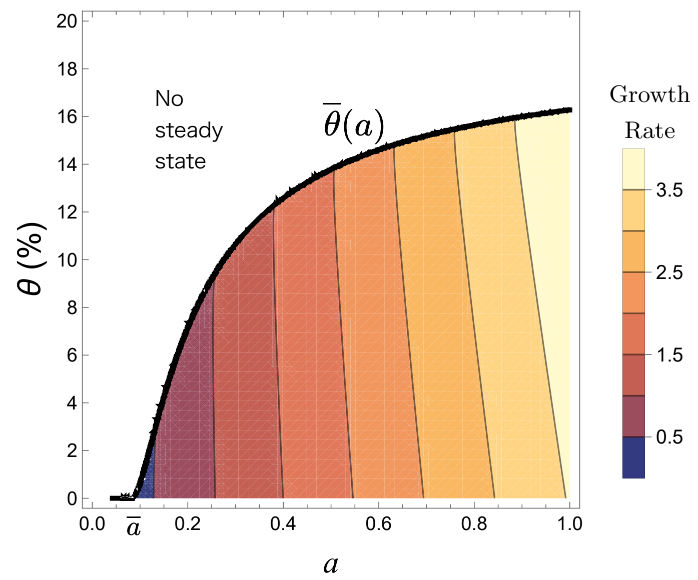


Figure 5: Contour plot of the growth rate in the steady state $g_1^*(\theta)$ against the productivity of R&D, a, and the R&D subsidy rate, θ

4.2 Transitional Effects

So far, we have seen that the debt-financed R&D subsidies enhance long-term growth in economies with high a (which means high $g_1^*(0)$), but they are detrimental to long-term growth when a is low (i.e., $g_1^*(0)$ is low). From this finding, we are tempted to conclude that the policy will be beneficial for economies only when the growth rate is sufficiently high in the first place. However, this conclusion may not hold for two reasons. First, the growth rate may change during the transition from the old steady state to the new steady state. Second, the transition may take a very long time. If the growth rate is higher than the steady-state value for a long time, the eventual difference in the GDP level can be significant.

Figure 6 shows the transitional response of the economy when the rate of subsidy θ is permanently raised from zero.[16] The horizontal axis shows the time after the policy change in years. Column (a) presents the results for the benchmark case of $a = 0.546$, and Column (b) is for a low-growth economy with $a = 0.15 < \widehat{a}$. The economy is assumed to be on the BGP with $\theta = 0$ before time $t = 0$. At time $t = 0$, the subsidy rate is raised to a value close to the maximal sustainable rate, $\overline{\theta}(a)$. In numerical simulations with $a = 0.546$ and $a = 0.15$, we raise θ from 0% to 4% and 14%, respectively.[17] Other parameters are

[16] The transitional paths of g_t and D_t are obtained by numerically solving the differential equations (28) and (29) backwards in time. This method is called the reverse shooting algorithm. We start the calculation from (the neighborhood of) the new steady state with increased θ and terminate the calculation when the state variable D_t reaches the point $D_t = D_1^*(0) = 0$.

[17] These values are slightly below $\overline{\theta}(a)$ (see Figures 4 and 5).

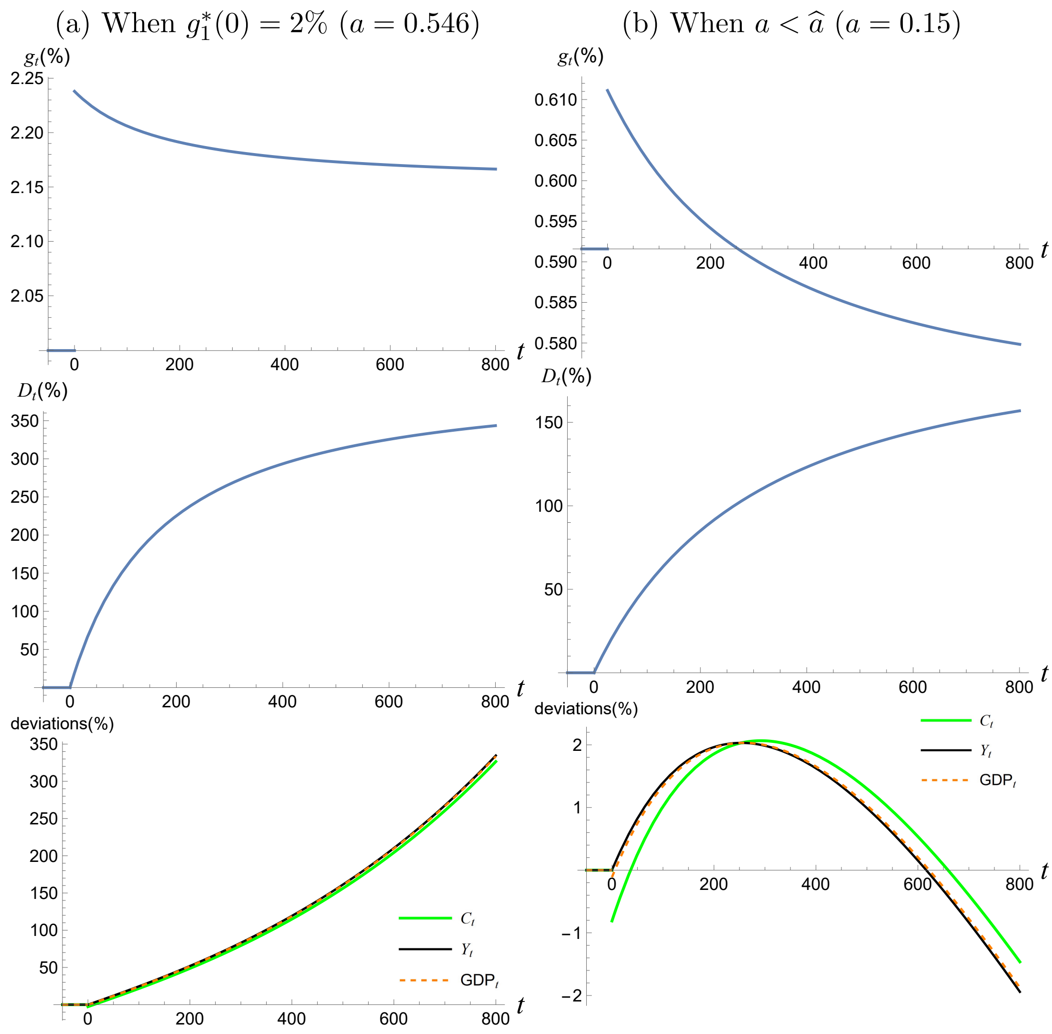


Figure 6: Transitional dynamics after research subsidies are introduced.

the same as in the previous subsection.

The upper panels show the response of g_t, i.e., the growth rate of the potential GDP. It can be seen that g_t jumps up immediately after the R&D subsidy is introduced, even when $a < \widehat{a}$. The growth rate gradually decreases and it takes hundreds of years to converge to the long-term value. The reason for the initial overshoot of g_t and the slow convergence can be seen from the middle panels, which depict the response of D_t, the debt-to-GDP ratio. The panels show that it takes a long time for the debt-GDP ratio to increase, given that $g_t > r_t$. As explained in the previous section, debt-financed R&D subsidies have both positive and negative effects on growth. Among these effects, the positive and direct effects of subsidies on research take effect immediately. In contrast, the negative crowding-out effect comes from the stock of government debt, which only gradually accumulates. Therefore, in the short run, the positive effect always dominates.

The bottom panels show the percentage deviation of paths of potential output Y_t, aggregate consumption C_t, and GDP as defined by (22) from their respective paths without research subsidies.[18] The response of the deviation of Y_t is monotonically increasing when $a > \widehat{a}$ and hump-shaped when $a < \widehat{a}$. Since the growth rate of Y_t is g_t, this pattern directly reflects the results in the upper panels.[19] The amount of aggregate consumption decreases when the research subsidy is introduced.[20] This is because more resources are devoted to R&D than to the production of consumption goods. After approximately 10 to 20 years, the production of consumption goods overtakes the case of no subsidy, owing to increased productivity (i.e., a higher level of potential GDP Y_t). The response of the deviation of the GDP is between those of Y_t and C_t. The GDP is the sum of the values of consumption and R&D, but in our model, consumption goods are produced with a positive markup, where R&D is subject to free entry without markups. Therefore, when more resources are devoted to R&D, the level of GDP temporarily decreases because the total value of markups decreases.

The results in column (b) of Figure 6 suggest that the debt-financed R&D subsidies have sizable positive effects on the levels of GDP, output and consumption for several centuries even though $a < \widehat{a}$. This is because the growth rate is higher in the transition, and the transition takes a very long time. However, the level of GDP will eventually become lower than in the case of no policy change. Since it takes hundreds of years before the negative effect dominates the initial positive response, the desirability of such a policy depends on how we evaluate the utility of current and future generations.

5 Social Rate of Return of R&D and Dynamic Efficiency

We have shown the possibility that R&D subsidies financed by perpetually rolled-over debts may stimulate R&D investments and increase the rate of economic growth, depending on the parameters. Here, we examine whether such increased R&D investments are beneficial to the economy. Since we explicitly consider overlapping generations of consumers, how

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[18] Let $Y_t^{\theta=0}$ be the value of the potential GDP on the BGP without research subsidies. Then, the deviation of Y_t from $Y_t^{\theta=0}$ is defined by $(Y_t - Y_t^{\theta=0})/Y_t^{\theta=0}$. The deviations of C_t and GDP_t are defined similarly.

[19] Note that $Y_t = Y_0 \exp\left(\int_0^t g_\tau d\tau\right)$ and $Y_t^{\theta=0} = Y_0 e^{g_1^*(0)t}$, where $g_1^*(0)$ is the growth rate on the BGP with $\theta = 0$. Based on the above, the deviation of Y_t can be calculated as $(Y_t - Y_t^{\theta=0})/Y_t^{\theta=0} = \exp\left(\int_0^t (g_\tau - g_1^*(0))d\tau\right) - 1$. Therefore, the slope of the deviation of Y_t is positive when $g_t - g_1^*(0) > 0$, and vice versa.

[20] Using Equations (14), (23), (24) and (26), $C_t = X_t = n_t^{(1-\alpha)/\alpha} L_t^P = (L_t^P/L)Y_t = (1 - g_t/\overline{g})Y_t$. From this, the deviation of C_t from the BGP path with $\theta = 0$ is $(C_t - C_t^{\theta=0})/C_t^{\theta=0} = \exp(\log C_t - \log C_t^{\theta=0}) - 1 = \exp\big(\log(\overline{g} - g_t) - \log(\overline{g} - g_1^*(0)) + \int_0^t (g_\tau - g_1^*(0))d\tau\big) - 1$. Similarly, we can write $\mathrm{GDP}_t = n_t^{(1-\alpha)/\alpha}(L_t^P + \alpha L_t^R) = (1 - (1-\alpha)g_t/\overline{g})Y_t$ from (22). Its deviation from the BGP path with $\theta = 0$ is $(\mathrm{GDP}_t - \mathrm{GDP}_t^{\theta=0})/\mathrm{GDP}_t^{\theta=0} = \exp\big(\log(\overline{g} - (1-\alpha)g_t) - \log(\overline{g} - (1-\alpha)g_1^*(0)) + \int_0^t (g_\tau - g_1^*(0))d\tau\big) - 1$.

we should measure the welfare of the whole economy is not obvious. As shown in Column (b) of Figure 6, a policy can be beneficial for the current generation but detrimental to distant future generations. Rather than weighing the utility of different generations, here we consider the most conservative criteria: we examine whether the economy is dynamically efficient or not.

The economy is known to be dynamically inefficient if the rate of return on investment is lower than the rate of economic growth (e.g., Tirole, 1985; Abel, Mankiw, Summers and Zeckhauser, 1989). In this case, the aggregate consumption for all periods can be increased by reducing investment; therefore, stimulating investments is actually welfare-reducing.[21] This possibility is a particular concern in the literature on rolling over debts because most papers focus on the case of $r < g$. Recent studies (e.g., Blanchard, 2019; Ball and Mankiw, 2023; Barro, 2023) have shown that the economy can be dynamically efficient despite $r < g$ when the marginal product of capital is higher than the interest rate for various reasons, including risk premiums and markups on capital production.

In our model, R&D investments have positive externalities on future R&D. In Equation (15), the output of R&D, $\dot{n}_t$, depends positively on today's stock of knowledge capital n_t, i.e., the cumulative number of ideas produced in the past. Similarly, today's R&D creates new knowledge, which will be used in the future. Therefore, the social rate of return on investment r^s, including this externality, is higher than the interest rate r. Thus, we cannot say that $r^s < g$ holds even when the interest rate on the government bond is lower than the growth rate. In fact, in the following, we show that the social rate of return of R&D investments in our model is higher than the rate of economic growth on any BGP. Following Jones and Williams (1998), we define the social rate of return by

Definition 1 *On the BGP, the social rate of return of R&D* (r^s) *is defined as follows:*

$$r^s \equiv \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\frac{dX_{t+\Delta t}}{-dX_t} - 1 \right), \tag{38}$$

where $dX_t < 0$ *and* $dX_{t+\Delta t} > 0$ *are the changes in* X_t *and* $X_{t+\Delta t}$ *from the respective BGP values when the following operations are conducted in the model, where time is discretized*

[21] To understand this point, let us consider a standard neoclassical growth model in discrete time. Suppose that on the BGP, the economy is growing at the rate of g, and the social rate of return on investment is $r^s < g$. Now, reduce the investment by 1% from period t_0 on. Then, the consumption at period t_0 is increased by this amount. In period $t_0 + 1$, the output, including the undepreciated capital, is reduced by $(1 + r^s)$% of the investment of period t. Note that on the BGP before the manipulation, the amount of investment also grows at the rate of g. Therefore, this amount of reduction is $((1 + r^s)/(1 + g))\% < 1\%$ of the investment in period $t_0 + 1$. Recall that in period $t_0 + 1$, the investment is reduced by 1%, which can be used for consumption. In total, the consumption in period $t_0 + 1$ is increased by $((g - r^s)/(1 + g))\% > 0$ of the investment in that period. Similar arguments hold for all $t \geq t_0 + 1$. Therefore, reducing investments is Pareto improving if $r^s < g$.

with an infinitesimally small Δt.

1. *At time t, L_t^R is marginally increased by a small value $dL_t^R > 0$.*

2. *At time $t + \Delta t$, $L_{t+\Delta t}^R$ is decreased by $dL_{t+\Delta t}^R < 0$. The magnitude of $dL_{t+\Delta t}^R$ is determined so that all variables at and after time $t + 2\Delta t$ are on the original BGP.*

Note that in this definition, the final goods production X_t at time t is decreased by $dX_t < 0$ because the economy allocates more resources to R&D. This is counted as an increase in the R&D investment measured in terms of final goods. At time $t+\Delta t$, the return on this R&D investment is harvested by increasing final goods production by $dX_{t+\Delta t} > 0$ so that the reduction in R&D investment at $t + \Delta t$ exactly offsets the effect of the initial increase in R&D. The social rate of return of R&D is measured by how large the magnitude of the harvest ($dX_{t+\Delta t}$) is relative to the initial R&D cost ($-dX_t$). Since this is the social rate of return for the duration of Δt, the rate of return for unit time is given by dividing it by Δt, as in (38). We find the value of r^s as follows.

Proposition 3 *In any BGP of this model, the social rate of return of R&D, defined by Definition 1, is $r^s = \overline{g}$ and, therefore, $r^s \geq g$ holds.*
Proof: in Appendix A.5.

Proposition 3 holds regardless of the value of θ, as long as the BGP exists. Note that $\overline{g}$ is given by (25), which represents the maximum rate of economic growth when all labor is used for R&D. Therefore, the social rate of return $r^s = \overline{g}$ is always higher than the rate of economic growth g. This result implies that the economy satisfies a necessary condition for the dynamic efficiency ($r^s \geq g$) even though $r < g$. The following proposition establishes the essence of dynamic efficiency in a more direct way.

Proposition 4 *For any BGP in this model, it is not possible to increase the aggregate consumption for a short time interval while keeping the aggregate consumption outside this interval unchanged.*
Proof: in Appendix A.6.

Given the state of technology (i.e., n_t), the only way to increase aggregate consumption is by reducing R&D and allocating more resources to production. If this is done, then the proof of proposition shows that it is not possible to maintain the amount of aggregate consumption in the future. In other words, no R&D investments are wasted in any BGP, including the BGP with positive R&D subsidies.

6 Extension: Basic Research

So far, we examined the effect of debt-financed research subsidies on economic growth. The results critically depend on research productivity, a. The R&D subsidies always enhance growth in the short run, but they hinder long-term growth if a is lower than $\widehat{a}$.

Although we treat a as a fixed parameter in our main model, recent studies suggest that the government can influence this parameter by directly funding basic research. For example, Akcigit, Hanley and Serrano-Velarde (2020) considered spillovers across different types of research activities and concluded that the policies geared towards basic public research are welfare-improving. Gersbach, Sorger and Amon (2018b) documented that basic research extends the knowledge base for applied research, and considered a variety expansion model where private applied R&D and publicly funded basic research are complementary in creating new goods.[22] Given these studies, this section considers a simple extension of our model, where the government can directly improve research productivity by putting more resources into public-funded basic research.

Now, we assume that a part of the aggregate labor supply, L, is used for public-funded basic research. We label this amount L^B. They are employed by the government at the market wage w_t. The research productivity of the private R&D depends on the amount of basic research in the following way:[23]

$$a = a_0 \left(\frac{L^B}{L_0^B} \right)^{\phi}, \quad L^B \geq L_0^B, \tag{39}$$

where a_0 is R&D productivity before the policy change, L_0^B is the initial amount of basic research labor, and ϕ is the elasticity of private R&D productivity with respect to the amount of basic research. Public basic research is entirely financed by the government. We assume that there is an existing labor tax that just finances the wages for L_0^B.[24] For political reasons, as we discussed in the introduction, it is not possible to increase the tax rate. Therefore, if the government wants to enhance basic research by increasing L^B above L_0^B, the added cost, $w_t(L^B - L_0^B)$, must be financed by issuing bonds. The other parts of the model are the same as in Section 2. In this setting, we examine how government policies,

[22] In a related study, Gersbach, Schetter and Schneider (2018a) introduced public-funded basic research into a quality-ladder endogenous growth model, where public funding increases the probability of success in quality-upgrading innovation. Leon-Ledesma and Shibayama (2023) extended Romer (1990) by introducing the basic research sector subsidized by the government.

[23] For simplicity, we assume that the government cannot reduce the government-financed basic research labor from L_0^B.

[24] Precisely, we assume that the labor income is taxed at the rate of L_0^B/L. Then, the revenue from this tax is wL_0^B.

Parameter	Description	Value	Explanation
a_0	Ex-ante R&D productivity	0.550	Calibrated to match $g = 2\%$
		[0.15]	For comparison
L_0^B	Ex-ante basic research labor	0.00581	Calibrated to match $\sigma_B = 0.49\%$
		[0.00580]	Same calibration when $a_0 = 0.15$.
ϕ	Elasticity of a to L^B	0.1,0.05,0.03	Free parameter

Table 2: Parameters when the R&D productivity is determined by basic research

characterized by the pair of $\theta \geq 0$ and $L^B \geq L_0^B$, affect long-term economic growth. The details of the analysis of this extended model are presented in Appendix A.7.

Table 2 summarizes the parameters used to specify Equation (39). According to OECD Main Science and Technology Indicators, basic research expenditure as a percentage of GDP in the U.S. is 0.49% in the most recent data (2019).[25] Therefore, as a benchmark, we calibrate $a_0 = 0.550$ and $L_0^B = 0.00581$ to match this value as well as $g_t = 2\%$ on the BGP before the introduction of the policy (i.e., when $\theta = 0$ and $L^B = L_0^B$).[26] We also consider the case of $a_0 = 0.15$, which means that $a = 0.15 \in [\overline{a}, \widehat{a}]$ if $L^B = L_0^B$, corresponding to Column (b) of Figures 4 and 6.[27] In this scenario, we calibrate $L_0^B = 0.00580$ so that the GDP share of the basic research matches the data at 0.49%. We take the elasticity ϕ as a free parameter and consider three modest values: $\phi = 0.1,\ 0.05,\ 0.03$. The remaining parameters are the same as in Section 4 (shown in Table 1).

Figure 7 presents the contour plots of the long-term growth rate against L^B and θ for six combinations of a_0 and θ. The black square in each panel represents the pair of L^B and θ that maximizes the long-term growth rate. The white area means that there is no steady state because the government debt is not sustainable given the pair of L^B and θ. Observe that the border between the colored area and the white area is downward-sloping. This is because the research subsidy (θ) and added basic research ($L^B - L_0^B$) are both financed by perpetually rolled-over government debts. When the government devotes more resources to basic research, the maximum sustainable rate of research subsidy becomes lower.

The three panels in the upper row show the graphs for the baseline case of $a_0 = 0.55$. When ϕ is 0.1 and 0.05, the growth maximizing policy is to combine the research subsidy

[25]In our model, the basic research expenditure as a percentage of GDP is given by $\sigma_B = w_t L^B/\text{GDP}_t$, where GDP_t slightly modified from (22) due to the introduction of L^B. See Appendix A.7.

[26]The value of $a_0 = 0.550$ is slightly higher than $a = 0.546$ in Section 4 for the following reason. In this extended model, total labor $L = 1$ is divided among production, L_t^P, private R&D, L_t^R, and basic research, L^B. While L is unchanged from the main model, the amount of labor that can be allocated for production and R&D is now lower. Therefore, a_0 needs to be slightly higher to maintain $g_t = 2\%$ on the BGP.

[27]Strictly speaking, the threshold values $\overline{a}$ and $\widehat{a}$ slightly change when we explicitly include basic research into the model. We present the equations for the modified threshold values for a_0 in Appendix A.7. With $L_0^B = 0.00580$, we obtain $\overline{a}_0 = 0.0826$ and $\widehat{a}_0 = 0.172$. These values are almost the same as $\overline{a} = 0.0821$ and $\widehat{a} = 0.171$, respectively, in the main model without basic research, given that L_0^B is small relative to L.

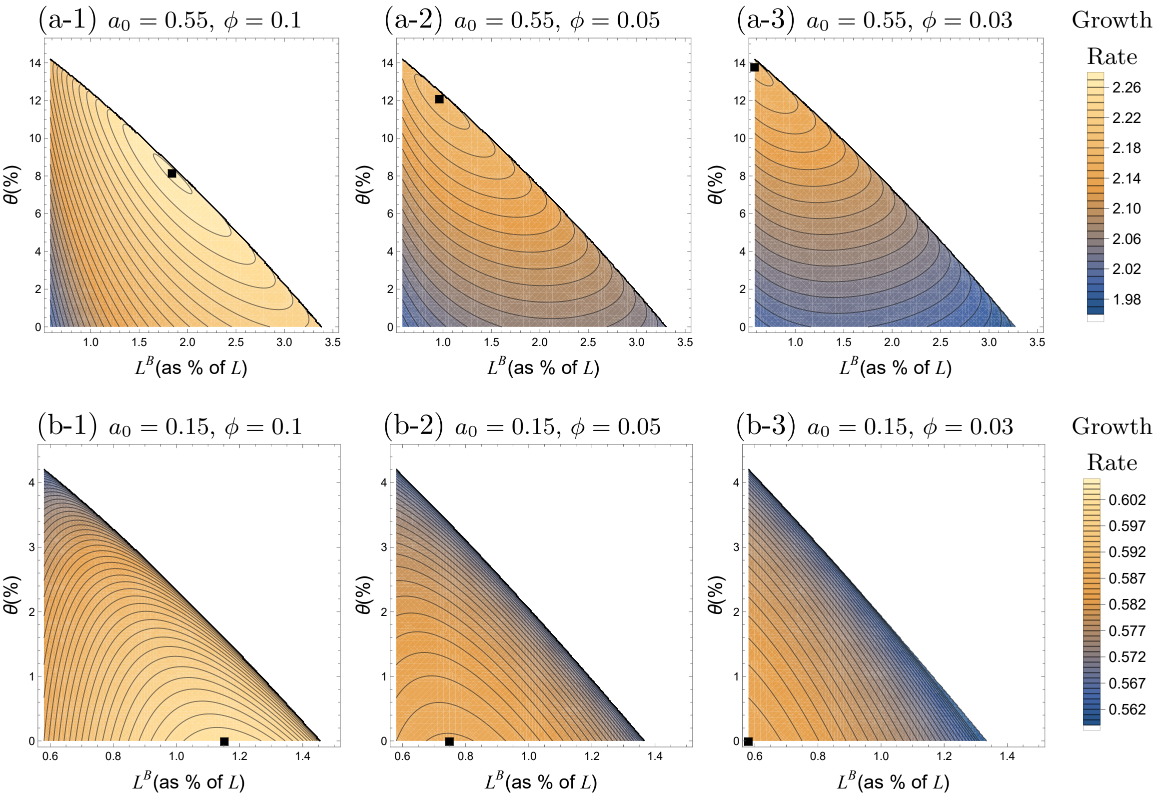


Figure 7: Long-Term Growth Effects of Basic Research, L^B, and Research Subsidy, θ

and the added basic research. It is only when ϕ is 0.03 that the growth maximizing policy only includes the research subsidy. The three panels in the lower row in Figure 7 show the graphs for the low-growth economy with $a_0 = 0.15 < \widehat{a}$. Consistent with Proposition 2, research subsidies does not enhance long-term growth in this case.[28] In other words, research subsidies do not contribute to long-term growth if the growth rate is not sufficiently high in the first place. Still, panels (b-1) and (b-2) show that when $\phi = 0.1$ and 0.05, added basic research financed by perpetually rolled-over debts indeed enhances growth even in these economies. This result highlights the importance of putting more resources into basic research, especially for economies where the initial rate of economic growth is low.

It is also notable that added basic research is beneficial for long-term economic growth even when the elasticity ϕ is at a rather small value of 0.05. Note that $\phi = 0.05$ means that a doubling of basic research labor results only in a 3.5% increase in the private R&D productivity, a.[29] If basic research has at least this modest impact on private R&D productivity, then our numerical simulations show that expanding public-financed basic research

[28]When L^B is increased from L_0^B, a becomes higher than a_0, and may possibly exceed $\widehat{a}$ even when $a_0 < \widehat{a}$. Still, in numerical simulations, we find that research subsidies do not increase growth when $a_0 < \widehat{a}$. We infer that this is because L^B and θ are financed by the same means. When L^B is large, the debt-GDP ratio is already high even when $\theta = 0$, which makes increasing θ less attractive.

[29]This value is obtained by $a = a_0(2L_0^B/L_0^B)^\phi = a_0 2^\phi$. When $\phi = 0.05$, this equation means $a = 1.035a_0$.

should be a part of growth-maximizing policy.

Previous studies tried to estimate this parameter by focusing on the magnitude of local spillovers from basic research activities to private R&D. As mentioned by Bloom, Reenen and Williams (2019), there appears to be a correlation between areas with strong science-based universities and private sector innovation (for example, Silicon Valley in California, Route 128 in Massachusetts, and the Research Triangle in North Carolina). This correlation suggests that there exist significant spillovers from universities to corporate R&D. More specifically, Jaffe (1989) estimated the elasticity of corporate patent numbers with respect to the amount of university research in the U.S. same state, and found that the elasticity ranges between 0.04 to 0.28 across industries. Acs, Audretsch and Feldman (1992) estimated the same elasticity using the innovation index that includes the data on inventions that were not patented but were ultimately introduced into the market. Their result shows that the elasticity is 0.241 when data from all areas are used. These numbers, combined with our simulation results, suggest that a debt-financed increase in basic research is a viable instrument for enhancing long-term growth.

7 Conclusion

This paper has examined the effect of growth promotion policies in an R&D-based endogenous growth model, assuming that the government cannot raise taxes and all expenses must be financed by perpetually rolled-over debts. Such a policy is sustainable only when the interest rate is lower than the growth rate. We find that $g > r$ is realized in the steady state if the productivity of R&D, a, is high enough. In accordance with recent findings in the literature that $g > r$ is more historical than the exception, we have focused on the case where the above-mentioned condition is satisfied.

Given this, we find that subsidies on private R&D always enhance R&D and thus the growth of the potential GDP in the short run. However, they do not necessarily increase the long-term growth rate. When a is high enough to support $g > r$ but is lower than another higher threshold (or, equivalently, the $g - r$ gap is not large enough), then the R&D subsidy initially increases the growth rate but lowers it in the long run because the effect of the increased government debt on the interest rate dominates the positive direct effect of the subsidy. Still, the initial positive response can dominate the negative effect for a long time, and therefore the desirability of such a policy depends on how we evaluate the utility of current and future generations. When a is higher than this threshold (i.e., when the $g - r$ gap is large enough), the policy can indeed increase the long-term growth rate. In both cases, we confirmed that the social return of R&D is higher than the growth rate, and therefore no R&D is wasted.

In an extended model, where the productivity of private R&D depends on the amount of publicly-funded basic research, we find that the growth-maximizing policy includes expansion of basic research unless the dependence of the productivity of private R&D on basic research is very weak. It is notable that expanding basic research enhances long-term growth even in economies where the growth rate is low in the first place and research subsidies are not conducive to growth.

Nevertheless, it is worth remembering that those growth-enhancing policies will reduce the upper bound in the debt-to-GDP ratio (or fiscal space), from which the economy can recover to a steady state. If the government tries to increase the subsidy rate or expense for basic research over certain values, the economy has no equilibrium path. This means that such a policy cannot be implemented in the rational expectation equilibrium. The government will go bankrupt unless the tax rates are raised.

References

Abel, A. B., N. G. Mankiw, L. Summers, and R. Zeckhauser (1989): "Assessing Dynamic Efficiency: Theory and Evidence," *The Review of Economic Studies*, 56, 1–19.

Acs, Z. J., D. B. Audretsch, and M. P. Feldman (1992): "Real Effects of Academic Research: Comment," *The American Economic Review*, 82, 363–367.

Akcigit, U., D. Hanley, and N. Serrano-Velarde (2020): "Back to Basics: Basic Research Spillovers, Innovation Policy, and Growth," *The Review of Economic Studies*, 88, 1–43.

Alesina, A., and A. Passalacqua (2016): "Chapter 33 - The Political Economy of Government Debt," Volume 2 of Handbook of Macroeconomics: Elsevier, 2599–2651.

Angeletos, G.-M., C. Lian, and C. K. Wolf (2024): "Can Deficits Finance Themselves?" *Econometrica*, 92, 1351–1390.

Ball, L., and N. G. Mankiw (2023): "Market Power in Neoclassical Growth Models," *The Review of Economic Studies*, 90, 572–596.

Barro, R. J. (2023): "r Minus g," *Review of Economic Dynamics*, 48, 1–17.

Blanchard, O. (2019): "Public Debt and Low Interest Rates," *American Economic Review*, 109, 1197–1229.

BLOOM, N., J. V. REENEN, AND H. WILLIAMS (2019): "A Toolkit of Policies to Promote Innovation," *The Journal of Economic Perspectives*, 33, 163–184.

GERSBACH, H., U. SCHETTER, AND M. T. SCHNEIDER (2018a): "Taxation, Innovation and Entrepreneurship," *The Economic Journal*, 129, 1731–1781.

GERSBACH, H., G. SORGER, AND C. AMON (2018b): "Hierarchical growth: Basic and applied research," *Journal of Economic Dynamics and Control*, 90, 434–459.

GROSSMAN, G. M., AND E. HELPMAN (1991a): *Innovation and Growth in the Global Economy*, (with Gene M. Grossman) Cambridge, MA: The MIT Press.

——— (1991b): "Quality Ladders in the Theory of Growth," *The Review of Economic Studies*, 58, 43–61.

GROSSMANN, V., T. STEGER, AND T. TRIMBORN (2013): "Dynamically optimal R&D subsidization," *Journal of Economic Dynamics and Control*, 37, 516–534.

JAFFE, A. B. (1989): "Real Effects of Academic Research," *The American Economic Review*, 79, 957–970.

JIANG, W., T. J. SARGENT, N. WANG, AND J. YANG (2022): "A p Theory of Taxes and Debt Management," NBER Working Papers 29931, National Bureau of Economic Research, Inc.

JONES, C. I., AND J. C. WILLIAMS (1998): "Measuring the Social Return to R&D," *The Quarterly Journal of Economics*, 113, 1119–1135.

——— (2000): "Too Much of a Good Thing? The Economics of Investment in R&D," *Journal of Economic Growth*, 5, 65–85.

KEYNES, J. (1923): *A Tract on Monetary Reform*: Macmillan.

KING, I., AND D. FERGUSON (1993): "Dynamic inefficiency, endogenous growth, and Ponzi games," *Journal of Monetary Economics*, 32, 79–104.

LEON-LEDESMA, M., AND K. SHIBAYAMA (2023): "(Endogenous) Growth Slowdowns," Studies in Economics 2303, School of Economics, University of Kent.

LUCAS, R. E. (1988): "On the mechanics of economic development," *Journal of Monetary Economics*, 22, 3–42.

MAURO, P., AND J. ZHOU (2021): "$r - g < 0$: Can We Sleep More Soundly?," *IMF Economic Review*, 69, 197–229.

MEHROTRA, N. R., AND D. SERGEYEV (2021): "Debt sustainability in a low interest rate world," *Journal of Monetary Economics*, 124, 1–18.

O'CONNELL, S. A., AND S. P. ZELDES (1988): "Rational Ponzi Games," *International Economic Review*, 29, 431–450.

OECD (2023a): "Current retirement ages," in *Pensions at a Glance 2023: OECD and G20 Indicators*, Paris: OECD Publishing.

——— (2023b): "Life expectancy at birth," in *Health at a Glance 2023: OECD Indicators*, Paris: OECD Publishing.

RACHEL, C., AND L. H. SUMMERS (2019): "On Secular Stagnation in the Industrialized World," Working Paper 26198, National Bureau of Economic Research.

REIS, R. (2021): "The constraint on public debt when $r < g$," CEPR Discussion Papers 15950, Center for Economic Policy Research.

ROMER, P. M. (1990): "Endogenous Technological Change," *Journal of Political Economy*, 98, S71–S102.

SAINT-PAUL, G. (1992): "Fiscal Policy in an Endogenous Growth Model," *The Quarterly Journal of Economics*, 107, 1243–1259.

SONG, Z., K. STORESLETTEN, AND F. ZILIBOTTI (2012): "Rotten Parents and Disciplined Children: A Politico-Economic Theory of Public Expenditure and Debt," *Econometrica*, 80, 2785–2803.

TIROLE, J. (1985): "Asset Bubbles and Overlapping Generations," *Econometrica*, 53, 1499–1528.

YARED, P. (2019): "Rising Government Debt: Causes and Solutions for a Decades-Old Trend," *Journal of Economic Perspectives*, 33, 115–40.

A Appendix

A.1 Derivation of the Consumption of Newborns

In this section, we derive the consumption of a newly-born individual, (7). As explained in the main text, each individual maximizes the expedited utility (3) subject to the budget constraint (4), initial condition $k_{t,t} = 0$, and the non-Ponzi-game condition:

$$\lim_{T\to\infty} e^{-\int_t^T (r_\tau+\mu)d\tau} k_{s,T} \leq 0. \tag{A.1}$$

From (4), (A.1), and the Euler equation for individuals, $\dot{c}_{s,t} = (r_t - \rho)c_{s,t}$, we obtain

$$c_{s,t} = (\rho + \mu)(k_{s,t} + h_{s,t}), \tag{A.2}$$

where $h_{s,t}$ is the present value of future labor income, or "human wealth," of an individual of generation s evaluated at time t. It is defined by

$$h_{s,t} \equiv \int_t^\infty e^{-\delta(t'-s)} w_{t'} e^{-\int_t^{t'} (r_\tau+\mu)d\tau} dt'. \tag{A.3}$$

By averaging (A.2) for all individuals, we have the average consumption as follows:

$$\overline{c}_t = (\rho + \mu)\left(\overline{k}_t + \overline{h}_t\right), \tag{A.4}$$

where $\overline{h}_t$ is the average human wealth, defined by

$$\overline{h}_t \equiv \frac{1}{N}\int_{-\infty}^t N_{s,t} h_{s,t} ds. \tag{A.5}$$

Note that, from (A.3), (A.5) and (2), $\overline{h}_t$ and the human wealth of a newborn $h_{t,t}$ is related by

$$\begin{aligned} \overline{h}_t &= \frac{1}{N}\int_{-\infty}^t N_{s,t} \int_t^\infty e^{-\delta(t'-s)} w_{t'} e^{-\int_t^{t'} (r_\tau+\mu)d\tau} dt' ds \\ &= \frac{1}{N}\int_{-\infty}^t N_{s,t} e^{-\delta(t-s)} \int_t^\infty e^{-\delta(t'-t)} w_{t'} e^{-\int_t^{t'} (r_\tau+\mu)d\tau} dt' ds \\ &= \frac{1}{N} L h_{t,t} = \frac{\mu}{\mu+\delta} h_{t,t}. \end{aligned} \tag{A.6}$$

This relationship means $h_{t,t} = ((\mu + \delta)/\mu)\overline{h}_t$; i.e., a newborn has $(\mu + \delta)/\mu > 1$ times the human wealth of an average individual. Equation (A.2) with $s = t$ and $k_{t,t} = 0$ implies the consumption of a newborn is

$$c_{t,t} = (\rho + \mu)h_{t,t} = \frac{(\rho + \mu)(\mu + \delta)}{\mu}\overline{h}_t. \tag{A.7}$$

By eliminating $\overline{h}_t$ from (A.4) into (A.7) results in (7).

A.2 Stability of the Steady State When There is no Subsidy

Substituting $\theta = 0$ into (28) and (29) and linearizing the system around a steady state (S_1 or S_2) yields

$$\begin{bmatrix} \dot{D}_t \\ \dot{g}_t \end{bmatrix} = \mathbf{J} \begin{bmatrix} D_t - D^* \\ g_t - g^* \end{bmatrix},$$

where D^* and g^* are the values of D_t and g_t in the steady state, and $\mathbf{J}$ is the Jacobian matrix. $\mathbf{J}$ is given by

$$\mathbf{J} \equiv \begin{bmatrix} -s(0)(g^* - \widehat{g}(0)) & -s(0)D^* \\ \overline{g}(\rho + \mu)(\delta + \mu) & s(0)(\widehat{g}(0) + \overline{g} - 2g^*) + \delta - \rho \end{bmatrix}, \tag{A.8}$$

where $s(0) = 1/(1 - \alpha)$, $\widehat{g}(0) = (1 - \alpha)\overline{g}$, and $\overline{g} = ((1 - \alpha)/\alpha)aL$. Since $D_1^* = 0$ in the steady state S_1, the determinant and the trace of $\mathbf{J}$ in S_1 are

$$\det \mathbf{J}|_{S_1} = -s(0)(g_1^* - \widehat{g}(0)) \{s(0)[(2 - \alpha)\overline{g} - 2g_1^*] + \delta - \rho\}, \tag{A.9}$$

$$\operatorname{tr} \mathbf{J}|_{S_1} = \delta - \rho + s(0)\big(2\widehat{g}(0) + \overline{g} - 3g_1^*\big). \tag{A.10}$$

In the steady state S_2, $D_2^* = \overline{D}$, $g_2^* = \widehat{g}(0)$; therefore,

$$\det \mathbf{J}|_{S_2} = \overline{g}(\rho + \mu)(\delta + \mu)\overline{D}, \tag{A.11}$$

$$\operatorname{tr} \mathbf{J}|_{S_2} = aL + \delta - \rho > 0. \tag{A.12}$$

In the following, we examine the stability of the steady states (S_1 and S_2) in the three cases discussed in the main text.

Case 1: $\boldsymbol{a > \overline{a}}$. First, we show that S_1 is saddle stable. As explained in the main text, $\dot{g}_t = 0$ parabola cuts the $g_t = \overline{g}$ line at $D_t = -\alpha/aL < 0$. This implies that the vertex of the parabola is in the $D_t < 0$ region, and given that S_1 exists, g_1^* is necessarily lower than the g-coordinate of the vertex of the parabola, $g_{\text{vertex}} = [(\delta - \rho)(1 - \alpha) + (2 - \alpha)\overline{g}]/2$. On the RHS of (A.9), the term $s(0)[(2 - \alpha)\overline{g} - 2g_1^*] + \delta - \rho$ is a decreasing function of g_1^* and becomes 0 when $g_1^* = g_{\text{vertex}}$. Since $g_1^* < g_{\text{vertex}}$, as shown above, this term is positive.

Additionally, the term $g_1^* - \widehat{g}(0)$ is positive since $g_1^* > \widehat{g}(0)$ in Case 1. Based on the above, $\det \mathbf{J}|_{S_1}$ in (A.9) is negative. This means that only one eigenvalue is negative; hence, S_1 is saddle stable.

Since $\overline{D} > 0$ in Case 1, (A.11) implies that $\det \mathbf{J}|_{S_2} > 0$. Additionally, since $\delta > \rho$, (A.12) implies that $\operatorname{tr} \mathbf{J}|_{S_2} > 0$. Based on the above, S_2 has two positive eigenvalues and is hence totally unstable.

Case 2-1: $\widetilde{a} < a < \overline{a}$. Similar to Case 1, $s(0)[(2-\alpha)\overline{g} - 2g_1^*] + \delta - \rho > 0$ holds from $g_1^* < g_{\text{vertex}}$. However, since $g_1^* < \widehat{g}(0)$, $\det \mathbf{J}|_{S_1} > 0$ from (A.9). Additionally, substituting $g_1^* < \widehat{g}(0)$ into (A.10), which is a decreasing function of g_1^*, we have $\operatorname{tr} \mathbf{J}|_{S_1} \geq \delta - \rho + \alpha s(0)\overline{g} > 0$. From $\det \mathbf{J}|_{S_1} > 0$ and $\operatorname{tr} \mathbf{J}|_{S_1} > 0$, S_1 has two positive eigenvalues and is therefore totally unstable.

Since $\overline{D} < 0$ in Case 2, (A.11) implies that $\det \mathbf{J}|_{S_2} < 0$ in S_2. Therefore, S_2 has only one negative eigenvalue and is hence saddle stable.

Case 2-2: $a < \widetilde{a}$. Similar to Case 2-1, S_2 is saddle stable. There is no S_1 steady state in the $g_t > 0$ region. However, we show in Appendix A.3 that there is an unstable steady state (S_1) when we consider the dynamics in the $g_t = 0$ region.

A.3 Dynamics in the $g_t = 0$ region

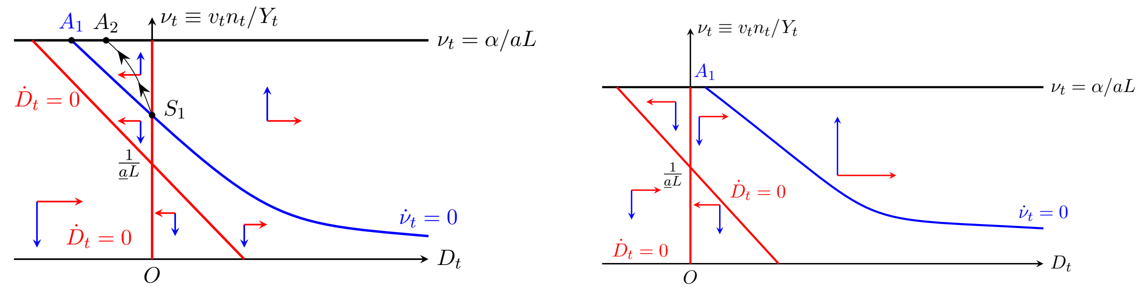
This section explains the dynamics of the economy when g_t becomes 0, which means that $\dot{n}_t = 0$, $L_t^R = 0$ and $L_t^P = L$ from (24) and (26). For simplicity, we focus on the case of $\theta = 0$; however, the analysis can be extended to the case of $\theta > 0$.

When $g_t = 0$, Y_t is constant from (23). Additionally, $L_t^P = L$ means that $X_t = Y_t$ is constant from (14) and (23). Since the final goods are used only for consumption, C_t is also constant. Let $\nu_t \equiv v_t n_t / Y_t$ represent the ratio of the total market value of firms to the potential GDP. When $\dot{n}_t = 0$, the free entry condition (16) holds whenever $\nu_t \leq \alpha / aL$. Substituting $\dot{C}_t = 0$ and $K_t = B_t + v_t n_t$ from the equilibrium of the asset market into the Euler equation (8) determines the interest rate:

$$r_t = (\delta - \rho)\big(\overline{a}L(D_t + \nu_t) - 1\big). \tag{A.13}$$

The dynamics of the economy in the $g_t = 0$ region can be examined by ν_t and D_t.[30] Since the government has no revenue or expense in this region, its debt B_t grows at the

[30] Depending on whether the free entry condition (16) holds with equality or not, either g_t or n_t can move, while the other is fixed.



(a) When $a < \widetilde{a}$, S_1 exists in the $g_t = 0$ region.

(b) When $a > \widetilde{a}$, no steady state exists in the $g_t = 0$ region.

Figure A.1: Phase diagram in the $g_t = 0$ region

rate of r_t. Since Y_t is constant, $\dot{D}_t / D_t = \dot{B}_t / B_t = r_t$. Therefore,

$$\dot{D}_t = (\delta - \rho)\big(\overline{a}L(D_t + \nu_t) - 1\big)D_t. \tag{A.14}$$

Since Y_t and n_t are constant, the definition of $\nu_t \equiv n_t v_t / Y_t$ implies that $\dot{\nu}_t / \nu_t = \dot{v}_t / v_t$. From $L_t^P = L$, $\pi_t = (1-\alpha)Y_t / n_t$.[31] Using these equations, the no-arbitrage condition (19) in the $g_t = 0$ region can be written as

$$\dot{\nu}_t = (\delta - \rho)\big(\overline{a}L(D_t + \nu_t) - 1\big)\nu_t - (1 - \alpha). \tag{A.15}$$

From (A.14) and (A.15), the phase diagram of the economy when $g_t = 0$ can be represented as

$$\text{The } \dot{D}_t = 0 \text{ locus:} \quad D_t + \nu_t = \frac{1}{\overline{a}L}, \text{ and} \tag{A.16}$$

$$D_t = 0.$$

$$\text{The } \dot{\nu}_t = 0 \text{ locus:} \quad D_t + \nu_t = \frac{1}{\overline{a}L} + \frac{1-\alpha}{(\rho + \mu)(\delta + \mu)} \frac{L}{\nu_t}. \tag{A.17}$$

Recall that the free entry condition (16) can be written as $\nu_t \le \alpha / aL$. Therefore, the phase diagram in the D_t-ν_t plane is defined only for $\nu \in [0, \alpha/aL]$. The phase diagram is shown in Figure A.1 for the case when the $\dot{D}_t = 0$ locus intersects with the ν_t axis ($a < \widetilde{a}$) and when it does not ($a > \widetilde{a}$). From (A.17), the D_t coordinate of the $\dot{\nu}_t = 0$ locus at $\nu_t = \alpha / aL$ (denoted as point A_1 in Figure A.1) is

$$\frac{1-\alpha}{(\rho + \mu)(\delta + \mu)} \frac{aL}{\alpha} - \frac{\alpha}{aL} + \frac{1}{\overline{a}L}, \tag{A.18}$$

[31] See footnote 4.

which is positive if and only if $a > \widetilde{a}$. We can confirm that (A.18) coincides with the D_t-intercept of the $\dot{g}_t = 0$ locus in the phase diagram of the $g_t > 0$ region, as shown in Figure 2(b) and 2(c).

We now consider the steady state. According to (A.17), the $\dot{\nu}_t = 0$ locus slopes downwards, and it is always above the sloping part of the $\dot{D}_t = 0$ locus (A.16). Therefore, the $\dot{\nu}_t$ locus intersects with the vertical portion of the $\dot{D}_t = 0$ locus (at $D_t = 0$) given that $a < \widetilde{a}$, i.e., when point A_1 is to the left of the vertical axis, as shown in Figure A.1(a). In this case, we denote the steady state by S_1 and let $\nu^* > 1/\overline{a}L$ represent its ν_t value.[32] If $a > \widetilde{a}$, then there is no steady state, as shown in Figure A.1(b).

Next, we show that S_1 in the $g_t = 0$ region, if it exists, is unstable. Linearizing (A.14) and (A.15) around $(0, \nu^*)$ yields

$$\begin{bmatrix} \dot{D}_t \\ \dot{\nu}_t \end{bmatrix} = \mathbf{J}_0 \begin{bmatrix} D_t \\ \nu_t - \nu^* \end{bmatrix}, \text{ where,}$$

$$\mathbf{J}_0 \equiv \begin{bmatrix} L(\rho+\mu)(\delta+\mu)\nu^* - (\delta-\rho) & 0 \\ L(\rho+\mu)(\delta+\mu)\nu^* & 2L(\rho+\mu)(\delta+\mu)\nu^* - (\delta-\rho) \end{bmatrix}. \tag{A.19}$$

Its determinant is

$$\det \mathbf{J}_0 = 2L^2(\delta-\rho)^2\overline{a}^2 \left(\nu^* - \frac{1}{\overline{a}L}\right)\left(\nu^* - \frac{1}{2\overline{a}L}\right), \tag{A.20}$$

which is positive since $\nu^* > 1/\overline{a}L$. The trace of $\mathbf{J}_0$ is positive:

$$\operatorname{tr} \mathbf{J}_0 = 3L(\rho+\mu)(\delta+\mu)\nu^* - 2(\delta-\rho) > \delta - \rho > 0, \tag{A.21}$$

where the first inequality comes from $\nu^* > 1/\overline{a}L$ and (32). From (A.20) and (A.21), we can conclude that both eigenvalues are positive.

Finally, we explain that there is a saddle path that originates from S_1 and connects to the saddle-stable steady state S_2 in the $g_t \geq 0$ region, as depicted in Figure 2(c). As shown in Figure 2(c), there is a saddle path that originates from a point on the horizontal axis (i.e., $g_t = 0$). This point is located between the origin and the intercept of the $\dot{g}_t = 0$ locus (point A_1); we call it point A_2. Note that point A_2 also belongs to Figure A.1(a) because it satisfies both $g_t = 0$ and $\nu_t = \alpha/aL$.[33] Since the steady state S_1 is completely

[32]We can confirm $\nu^* > 1/\overline{a}L$ by substituting $D_t = 0$ in (A.17).

[33]When $g_t > 0$, the free entry condition for R&D (16) needs to hold with equality. This condition is equivalent to $\nu_t \equiv v_t n_t / Y_t = \alpha/aL$.

unstable, there exists a point in the neighborhood of S_1 from which the path leads to A_2. This is the saddle path of this economy. Once the economy reaches A_2 in Figure A.1(a), it experiences a phase transition to Figure 2(c). If the economy with $a < \widetilde{a}$ starts from a small negative value of D_t, g_t remains at 0 for some time and then starts to increase until it converges to $g_t = \hat{g}(0)$.

The case of $\theta > 0$: Thus far, we have focused on the case of $\theta = 0$. Even when θ is positive, the phase diagram within the $g_t = 0$ region is almost the same as that in the case of $\theta = 0$ because $g_t = 0$ means that no firms do R&D; therefore, the government expenditure is zero. The only difference is the border between the phase diagrams. With $\theta > 0$, the $\nu_t = (1-\theta)\alpha/aL$ line in the diagram of the $g_t = 0$ region will be connected to the $g_t = 0$ line in the phase diagram of the $g_t \geq 0$ region. Intuitively, with $\theta > 0$, firms are more eager to start R&D even when the index of firm values, ν_t, is not as high as in the case of $\theta = 0$. Therefore, the economy transitions from the $g_t = 0$ region to the $g_t \geq 0$ region with a smaller ν_t.

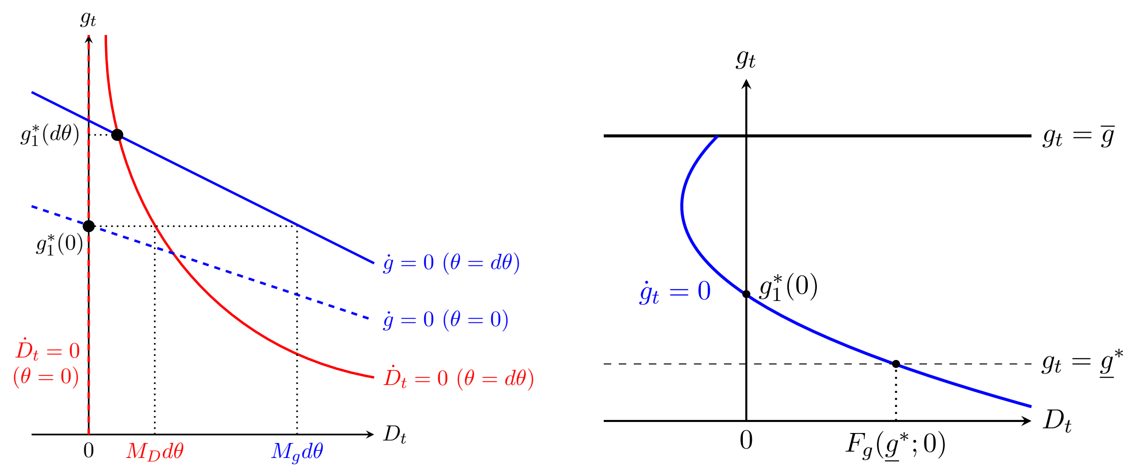
A.4 Proof of Proposition 2 and Corollary 1

A.4.1 Proposition 2

We assume that the economy is in a saddle-stable steady state S_1 before the increase in the subsidy. As illustrated as S_1 in Figure 2(a), this steady state is given by an intersection between the $\dot{g}_t = 0$ locus and the $\dot{D}_t = 0$ locus.

We consider the situation where the rate of the R&D subsidy, θ, is increased marginally by $d\theta$ from 0. Then, both the $\dot{g}_t = 0$ locus and the $\dot{D}_t = 0$ locus shift to the right. The location of the steady state always moves to the right, which means that D_t in the new steady state is higher than before. Whether g_t in the new steady state is higher or lower depends on the relative magnitude of the size of the shifts of the two loci. The slope of the $\dot{D}_0$ locus at S_1 is steeper than the slope of the $\dot{g}_0$ locus. Then, as illustrated in Panel (a) of Figure A.2, g_t in the new steady state should be higher than before if and if the size of the shift of the $\dot{g} = 0$ locus (denoted by $M_g d\theta$), measured at steady state S_1 (located at $D_t = 0,\ g_t = g_1^*(0)$), is larger than the size of the shift of the $\dot{D} = 0$ locus (denoted by $M_D d\theta$).

Let $F_g(g_t; \theta)$ be the RHS of (34), and let $F_D(g_t; \theta)$ be the RHS of (35). $D_t = F_g(g_t; \theta)$ and $D_t = F_D(g_t; \theta)$ represent the $\dot{g} = 0$ locus and the $\dot{D} = 0$ locus, respectively. Then, M_g



(a) The shifts of the $\dot{g}_t = 0$ and $\dot{D}_t = 0$ loci and the movement of the steady state.

(b) Condition $g_1^* > \underline{g}^*$ holds if and only if $F_g(\underline{g}^*; 0) > 0$.

Figure A.2: Proof of Proposition 2

and M_D are calculated as

$$M_g \equiv \frac{\partial F_g(g_1^*(0); 0)}{\partial \theta} = \frac{\alpha}{aL} + \frac{(g_1^*(0) - \overline{g})^2}{\overline{g}(\rho + \mu)(\delta + \mu)}, \tag{A.22}$$

$$M_D \equiv \frac{\partial F_D(g_1^*(0); 0)}{\partial \theta} = \frac{\alpha^2 g_1^*(0)}{aL} \frac{1}{g_1^*(0) - \widehat{g}(0)}. \tag{A.23}$$

(A.22), (A.23) and $\widehat{g}(0) < g_1^* < \overline{g}$ indicate that $M_g - M_D > 0$ holds if and only if

$$(\overline{g} - g_1^*)(g_1^* - \widehat{g}(0)) > (1 - \alpha)^2(\rho + \mu)(\delta + \mu). \tag{A.24}$$

The fact that the steady state before the shift $(D_t = 0,\ g_t = g_1^*(0))$ is on the $\dot{g}_t = 0$ locus means that $F_g(g_1^*(0); 0) = 0$. This equation can be rearranged to

$$(1 - \alpha)^2(\rho + \mu)(\delta + \mu) = (\overline{g} - g_1^*(0))\left((1 - \alpha)(\delta - \rho) + \widehat{g}(0) - g_1^*(0)\right). \tag{A.25}$$

Substituting (A.25) into the RHS of (A.24) and rearranging yields

$$g_1^*(0) > \frac{(1 - \alpha)(\delta - \rho)}{2} + \widehat{g}(0) \equiv \underline{g}^*. \tag{A.26}$$

This inequality shows that when θ is slightly increased from 0, the growth rate in the saddle-stable steady state increases if and only if $g_1^*(0) > \underline{g}^*$.

Recall that for a given value of g_t, $F_g(g_t; 0)$ gives the value of D_t on the $\dot{g}_t = 0$ locus with $\theta = 0$. Therefore, $F_g(\underline{g}^*; 0)$ gives the value of D_t where the $\dot{g}_t = 0$ locus reaches $g_t = \underline{g}^*$, as shown in Figure A.2. Note that the condition $g_1^*(0) > \underline{g}^*$ indicates the intersection of the

$\dot{g}_t = 0$ locus with $\theta = 0$, and the vertical axis ($D_t = 0$) is above the $g_t = \underline{g}^*$ line. Because the $\dot{g}_t = 0$ locus is downwards sloping in the $D_t > 0$ region, this condition holds if and only if $F_g(\underline{g}^*; 0) > 0$ (see Figure A.2). When we solve this condition for a, we can confirm that it is equivalent to (36).

A.4.2 Corollary 1

In this proof, we show that condition (A.26) holds if and only if (37) is satisfied. Note that the relationship between $g_t - r_t$ and g_t in (27) always holds in equilibrium; therefore, it also holds in the saddle-stable steady state with $\theta = 0$:

$$g_1^*(0) - r_1^*(0) = s(0)(g_1^*(0) - \widehat{g}(0)). \tag{A.27}$$

Since $s(0) = 1/(1-\alpha) > 0$,

$$g_1^*(0) > \underline{g}^* \Leftrightarrow s(0)(g_1^*(0) - \widehat{g}(0)) > s(0)(\underline{g}^* - \widehat{g}(0)) = \frac{\delta - \rho}{2}. \tag{A.28}$$

Combining (A.27) and (A.28), we have

$$g_1^*(0) > \underline{g}^* \Leftrightarrow g_1^*(0) - r_1^*(0) > \frac{\delta - \rho}{2}.$$

A.5 Proof of Proposition 3

Here, we derive the social rate of return of R&D. As explained in Definition 1, we discretize time in the model with step size Δt. We later take the limit of $\Delta t \to 0$. We assume that the economy is on the BGP before time t. The state variable n_t at time t is also on the BGP because it is predetermined by time t. Then, we marginally increase L_t^R by $dL_t^R > 0$, which will reduce the production of consumption goods in time t (which is denoted by $dX_t < 0$) but increase the number of goods at $t + \Delta t$ (denoted by $dn_{t+\Delta t} > 0$). Then, at $t + \Delta t$, we marginally decrease $L_{t+\Delta t}^R$ by $dL_{t+\Delta t}^R < 0$ so that $n_{t+2\Delta t}$ returns to the original BGP. The production of consumption goods at $t + \Delta t$ will be higher than that at the BGP. We denote the difference by $dX_{t+\Delta t} > 0$.

In the following, we derive the relationships among $dn_{t+\Delta t}$, dL_t^R, and $dL_{t+\Delta t}^R$. The discrete version of (15) at the time between t and $t + \Delta t$ is

$$n_{t+\Delta t} = n_t + aL_t^R n_t \Delta t. \tag{A.29}$$

We totally differentiate Equation (A.29) while keeping n_t constant, which yields

$$dn_{t+\Delta t} = an_t \Delta t \, dL_t^R. \tag{A.30}$$

Similar to (A.29), the discrete version of (15) at time from $t + \Delta t$ to $t + 2\Delta t$ is

$$n_{t+2\Delta t} = n_{t+\Delta t} + aL_{t+\Delta t}^R n_{t+\Delta t} \Delta t. \tag{A.31}$$

By totally differentiating (A.31), while keeping $n_{t+2\Delta t}$ unchanged from the BGP, we obtain

$$0 = \left(1 + aL_{t+\Delta t}^R \Delta t\right) dn_{t+\Delta t} + an_{t+\Delta t} \Delta t \, dL_{t+\Delta t}^R. \tag{A.32}$$

We substitute $n_{t+\Delta}$ in (A.29) and $dn_{t+\Delta}$ in (A.30) into (A.32), which yields

$$0 = (1 + aL_{t+\Delta t}^R \Delta t) dL_t^R + (1 + aL_t^R \Delta t) dL_{t+\Delta t}^R. \tag{A.33}$$

Recall that the economy is on the BGP before the manipulation of L_t^R, where L_t^R is constant for all t. Let this constant value be L^{R*}. Then, $L_t^R = L^{R*} + dL_t^R$ and $L_{t+\Delta t}^R = L^{R*} + dL_{t+\Delta t}^R$. Substituting these values into (A.33) gives

$$0 = (1 + aL^{R*} \Delta t)(dL_t^R + dL_{t+\Delta t}^R) + 2a\Delta t \, dL_t^R \, dL_{t+\Delta t}^R. \tag{A.34}$$

This equation implies that the ratio of the changes in L_t^R and $L_{t+\Delta t}^R$ is

$$-\frac{dL_{t+\Delta t}^R}{dL_t^R} = 1 + \frac{2a\Delta t \, dL_{t+\Delta t}^R}{1 + aL^{R*} \Delta t}. \tag{A.35}$$

Now, we consider the changes in the final output X_t in periods t and $t + \Delta t$. Note that in period t, n_t is still on the BGP. Therefore, from (14), the changes in X_t and L_t^R are related by

$$dX_t = \frac{\partial X_t}{\partial L_t^R} dL_t^R = n_t^{\frac{1-\alpha}{\alpha}} \left(-dL_t^R\right). \tag{A.36}$$

In period $t + \Delta t$, $n_{t+\Delta t}$ deviates from the BGP by $dn_{t+\Delta t}$. Therefore, the change in $X_{t+\Delta t}$

can be expressed as

$$
\begin{aligned}
dX_{t+\Delta t} &= \frac{\partial X_{t+\Delta t}}{\partial n_{t+\Delta t}} dn_{t+\Delta t} + \frac{\partial X_{t+\Delta t}}{\partial L^R_{t+\Delta t}} dL^R_{t+\Delta t} \\
&= \frac{1-\alpha}{\alpha} n_{t+\Delta t}^{\frac{1-\alpha}{\alpha}-1} (L - L^R_{t+\Delta t}) dn_{t+\Delta t} - n_{t+\Delta t}^{\frac{1-\alpha}{\alpha}} dL^R_{t+\Delta t} \\
&= n_{t+\Delta t}^{\frac{1-\alpha}{\alpha}} \left[\frac{1-\alpha}{\alpha} (L - L^R_{t+\Delta t}) \frac{a\Delta t\, dL^R_t}{1 + aL^R_t \Delta t} - dL^R_{t+\Delta t} \right].
\end{aligned} \tag{A.37}
$$

where we use (A.29) and (A.30) in the third line.

We obtain the social return of R&D investment by substituting (A.36) and (A.37) into Definition 1:[34]

$$
\begin{aligned}
r^s \Delta t &= \frac{dX_{t+\Delta t}}{-dX_t} - 1 \\
&= \left(\frac{n_{t+\Delta t}}{n_t} \right)^{\frac{1-\alpha}{\alpha}} \left(\frac{1-\alpha}{\alpha} (L - L^R_{t+\Delta t}) \frac{a\Delta t}{1 + aL^R_t \Delta t} - \frac{dL^R_{t+\Delta t}}{dL^R_t} \right) - 1.
\end{aligned} \tag{A.38}
$$

In the RHS of the above equation, we can use the Taylor expansion as follows:

$$
\left(\frac{n_{t+\Delta t}}{n_t} \right)^{\frac{1-\alpha}{\alpha}} = (1 + a\, L^R_t \Delta t)^{\frac{1-\alpha}{\alpha}} = 1 + \frac{1-\alpha}{\alpha} aL^R_t \Delta t + o((\Delta t)^2),
$$

where $o((\Delta t)^2)$ is the collection of terms that are of order $(\Delta t)^2$ and higher. We can also eliminate $dL^R_{t+\Delta t}/dL^R_t$ via (A.35). Then, (A.38) becomes

$$
r^s = \frac{1-\alpha}{\alpha} aL^R_t + \frac{\frac{1-\alpha}{\alpha} a(L - L^R_{t+\Delta t}) + 2a\, dL^R_{t+\Delta t}}{1 + aL^R_t \Delta t} + o(\Delta t). \tag{A.39}
$$

Recall that Δt is infinitesimally small. In the continuous-time limit, $\Delta t \to 0$. Additionally, we consider marginal perturbations of dL^R_t and $dL^R_{t+\Delta t}$ from the BGP. Therefore, $dL^R_{t+\Delta t} \to 0$. Applying these to (A.39), we obtain

$$
r^s = \frac{1-\alpha}{\alpha} aL = \overline{g}. \tag{A.40}
$$

Since $g_t \leq \overline{g}$ from (24) and $L^R_t \leq L$, we have $r^s = \overline{g} \geq g_t$.

A.6 Proof of Proposition 4

Let g^* be the growth rate of the aggregate consumption in the original BGP, and let L^{R*} and L^{P*} be the constant amounts of R&D labor and production labor, respectively.

[34] In Definition 1, r^s is defined as the limit of the first line of (A.38) as $\Delta t \to 0$. We take the limit after we rewrite (A.38) as (A.39).

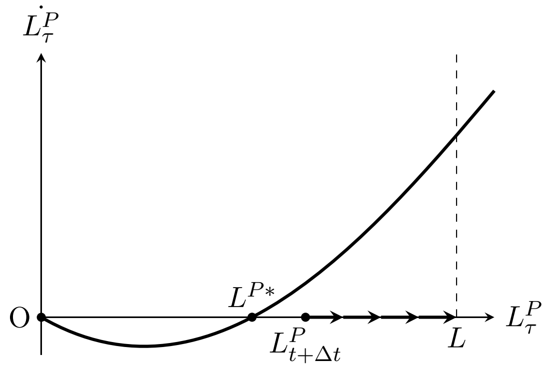


Figure A.3: Movement of L_t^P after $\tau = t + \Delta t$

Suppose that from this BGP, the path of aggregate consumption, $C_t = X_t$, is increased by dX_t from time t to $t + \Delta t$, where Δt is a small time interval. In the following, we show that the aggregate consumption from $t + \Delta t$ cannot be kept at the original BGP path.

As explained in the Proof of Proposition 3 in Appendix A.5, the only way in which to increase aggregate consumption at time t is to reduce R&D labor and increase production labor. Then, at $t + \Delta t$, $n_{t+\Delta t}$ should be lower than the BGP value, which means that the production technology of the final goods is inferior to that of the original BGP case. Then, to keep $X_{t+\Delta t}$ at the original BGP value despite a smaller $n_{t+\Delta t}$, we need to allocate more workers to production, which means that $L^P_{t+\Delta t} > L^{P*}$.

Below, we examine the dynamics of the economy from $t + \Delta t$ on in continuous time. For $\tau \geq t + \Delta t$, the aggregate consumption should be the same as that of the original BGP, which is growing at the rate of g^*. By taking the log of (14) and then differentiating with respect to τ, we obtain

$$\frac{\dot{X}_\tau}{X_\tau} = \frac{1-\alpha}{\alpha}\frac{\dot{n}_\tau}{n_\tau} + \frac{\dot{L}^P_\tau}{L^P_\tau} = g^* \text{ for all } \tau \geq t + \Delta t.$$

Using (15), (24) and (26), the above equation can be rewritten as

$$\begin{aligned}\frac{\dot{L}^P_\tau}{L^P_\tau} &= g^* - \frac{1-\alpha}{\alpha} a L^R_\tau \\ &= \frac{1-\alpha}{\alpha} a (L^{R*} - L^R_\tau) \\ &= \frac{1-\alpha}{\alpha} a (L^P_\tau - L^{P*}).\end{aligned}$$

Therefore, we have the following autonomous differential equation for L^P_τ for $\tau \geq t + \Delta t$:

$$\dot{L}^P_\tau = \frac{1-\alpha}{\alpha} a (L^P_\tau - L^{P*}) L^P_\tau. \tag{A.41}$$

As shown in Figure A.3, the RHS of (A.41) is positive when $L^P_\tau > L^{P*}$. Recall that $L^P_{t+\Delta t} > L^{P*}$. Therefore, $\dot{L}^P_\tau > 0$ for all $\tau \geq t + \Delta t$, and L^P_τ exceeds the total labor supply L in a finite period of time. This means that such a path is not feasible.

A.7 Analysis of the Extended Model

In this section, we analyze the extended model presented in Section 6. Note that the labor supply is fixed at L, and therefore it is not affected by the labor tax. Also, the labor tax does not affect the individual's Euler equation. Therefore, the equation for the dynamics for the aggregate consumption (8) is the same as in the main model. As explained in the main text, the government pays the wages to the basic research workers, the sum of which is $w_t L^B$. Also, there is an existing labor tax at the rate of L^B_0/L, the revenue of which is $w_t L^B_0$. Therefore, the dynamics of the government debt (17) changes to

$$\dot{B}_t = r_t B_t + \theta w_t L^R_t + w_t(L^B - L^B_0). \tag{A.42}$$

The labor demand now consists of that for production, L^P_t, for private R&D, L^R_t, and for basic research, L^B.[35] Therefore, the equilibrium condition of the labor market changes from (26) to

$$L^P_t + L^R_t + L^B = L. \tag{A.43}$$

The GDP of this economy is defined as the sum of consumption expenditures $C_t = X_t$, private investment expenditures for R&D, $(1-\theta)w_t L^R_t$, and government expenditures, $\theta w_t L^R_t + w_t L^B$. Since (14) and (20) are unchanged, the GDP becomes

$$\text{GDP}_t = n_t^{\frac{1-\alpha}{\alpha}} (L^P_t + \alpha L^R_t + \alpha L^B). \tag{A.44}$$

Therefore, the ratio of basic research expenditure to the GDP is given by

$$\frac{w_t L^B}{\text{GDP}_t} = \frac{\alpha L^B}{L^P_t + \alpha L^R_t + \alpha L^B}.$$

In this setting, the $g - r$ gap is written as

$$g_t - r_t = s(\theta)(g_t - \widehat{g}(\theta, L^B)), \text{ where} \tag{A.45}$$
$$s(\theta) = \frac{1-\alpha\theta}{(1-\alpha)(1-\theta)} > 1, \quad \widehat{g}(\theta, L^B) = \frac{1-\alpha}{1-\alpha\theta}\overline{g}\frac{L - L^B}{L} \in (0, \overline{g}].$$

[35]We omit the time subscript on L^B because it is a policy parameter.

Note that the threshold level of growth, $\widehat{g}(\theta, L^B)$, now depends on the pair of policy parameters, θ and L^B. From (A.42) and (A.45), the dynamics for the debt-GDP ratio changes from (28) to

$$\dot{D}_t = -s(\theta)\left(g_t - \widehat{g}(\theta, L^B)\right) D_t + \frac{\theta\alpha}{\overline{g}} g_t + \alpha \frac{L^B - L_0^B}{L}. \tag{A.46}$$

The last term of (A.46) comes from the expenses used for the added basic research. The time evolution of g_t is

$$\dot{g}_t = \left(\overline{g}\frac{L - L^B}{L} - g_t\right)\left(s(\theta)\left(g_t - \widehat{g}(\theta, L^B)\right) - \delta + \rho\right) + \overline{g}(\rho + \mu)(\delta + \mu)\left(D_t + \frac{\alpha(1-\theta)}{aL}\right). \tag{A.47}$$

Note that, from (A.43), the inclusion of basic research means that less labor can be allocated to production and R&D. Therefore, (A.47) differs from (29) in that the first term contains $(L - L_B)/L$ term. We did numerical simulations using (A.46) and (A.47), and the results are presented in the main text.

Given that the equations for the dynamics are changed as above, the threshold values for a in Propositions 1 and 2 are also affected. Since a is given by function (39), the thresholds are now defined in terms of parameter a_0. Specifically, the economy has a stable steady state with $g^* > r^*$ when $\theta = 0$ and $L^B = L_0^B$ if and only if a_0 is larger than

$$\overline{a}_0 = \frac{(\rho + \mu)(\delta + \mu)}{(\delta - \rho)(L - L_0^B)}. \tag{A.48}$$

Also, a marginal increase in θ from 0 increases the long-term growth rate if and only if a_0 is higher than

$$\widehat{a}_0 = 2\overline{a}_0 + \frac{\delta - \rho}{2(L - L_0^B)}. \tag{A.49}$$

Given that L_0^B is small relative to L, as in our calibration presented in Table 2, $\overline{a}_0$ and $\widehat{a}_0$ are close to $\overline{a}$ and $\widehat{a}$ in the main model.