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Ryoji Hiraguchi (Meiji University)

Keiichiro Kobayashi (Keio University)

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The Canon Institute for Global Studies

一般財団法人 キヤノングローバル戦略研究所

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Inflation and entry costs in a monetary search model

Ryoji Hiraguchi*[†], Keiichiro Kobayashi [‡§]

August 20, 2024

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*Faculty of Political Science and Economics, Meiji University. 1-1 Kanda-surugadai, Chiyoda-ku, Tokyo, Japan. E-mail: hiraguchi@meiji.ac.jp. Tel: 81-43-290-2399.

†The Canon Institute for Global Studies, Japan

‡Faculty of Economics, Keio University. 2-15-45 Mita, Minato-ku, Tokyo, Japan. Email: kobayasi@econ.keio.ac.jp Tel: 81-3-5418-6703.

§The Canon Institute for Global Studies, Japan

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Keywords: Optimal monetary policy; entry cost; competitive pricing; low inflation.
JEL classification code: E13; E42; E52.

1 Introduction

Free entry is a fundamental property of a market economy, and it seems natural to model any market such that both buyers and sellers make entry decisions. However, as far as we know, there has been no study that incorporates the free entry of both buyers and sellers in the monetary search literature. The difficulty in modeling free entry of two parties is on the fact that one variable (the ratio of sellers' measure to buyers') must satisfy two free entry conditions in the standard search theoretic setting. One purpose of our study here is to overcome this difficulty and construct a model with free entries of all parties. Our innovation is to incorporate a plausible setting of production chains in the otherwise standard money search model. With free entries of both buyers an sellers, we can analyze how changes in their respective entry costs affect the optimal inflation differently.

We construct a monetary model in which both buyers and sellers optimally decide whether to enter the decentralized market by paying fixed entry costs. The model is based on a seminal study of Lagos and Wright (2005). Just like Lagos and Wright (2005), each date is divided into day and night. The day market is decentralized and the night market is centralized. In the decentralized market (DM), the sellers produce the special goods and sell them to the buyers. The buyers then turns the special goods into the intermediate goods, and sell them to the general good firm in the centralized market (CM). The firm uses the intermediate goods and labor as factor inputs and produce the general goods. The general goods are traded in the CM. The probability of entering the DM is determined by the constant returns to scale matching function.

We characterize the stationary equilibrium allocation and show that the Friedman rule of setting nominal interest rate to zero may not be optimal. The optimal inflation rate is derived explicitly for specific functional forms. We show that the optimal inflation rate is lower for lower buyer entry costs, because the lower entry costs generate the congestion of buyers which leads to the lower benefit from holding money, which must be balanced by lower cost of money holdings. It is also shown that the optimal inflation is lower for higher seller entry costs.

To the best of our knowledge, this study is the first attempt to construct a monetary model in which all agents make entry decisions. Constructing such a model is the first contribution of this study. The second contribution of this study is deriving an optimal inflation rate explicitly for specific functional forms.

Related literature: Recent literature on monetary search models studies the entry decisions of the buyers or the sellers, but not both. Rocheteau and Wright (2005) analyze a model in which sellers make entry decisions. Lester, Postlewaite and Wright (2012) construct a model where sellers can expend more cost to acquire the ability to accept more variety of assets as liquidity. In Liu, Wang and Wright (2011), buyers make entry decisions. In Nosal (2008), buyers can choose whether to trade when they are successfully matched with sellers. Amendola, Araujo and Ferraris (2024) assume that buyers can choose search intensity. Our paper studies the monetary model in which both buyers and sellers make entry decisions. We show the suboptimality of the Friedman rule, just as the existing literature. In addition, we explicitly derive the optimal inflation rate and show how it is related to the entry costs.

In the literature of non-monetary search models, there exist the models in which both parties of trades are subject to free-entry conditions, e.g., Wasmer and Weil (2004) for bank loans, Gabrovski and Ortego-Marti (2019, 2022) for housing market, and Pries and Rogerson (2009) for labor market. Gabrovski and Ortego-Marti (2019) construct a search model in which both buyers and sellers optimally enter the housing market and show positive correlations between vacancies and the the number of buyers. Gabrovski and Ortego-Marti (2022) study the efficiency of the equilibrium allocation in that model. The market structure in Wasmer and Weil (2004) is similar to our production chains, and Gabrovski and Ortego-Marti (2019, 2022) do not have production chains, but instead assume that the entry cost depends on the measure of buyers.

The remainder of this paper is organized as follows. The model is described in Section

2. Section 3 presents a welfare analysis in which the optimal monetary policy or optimal inflation rate is explicitly derived. Section 4 presents two extensions of the baseline model, which are the monetary version of the model in Gabrovski and Ortego-Marti (2019) and the Nash bargaining version of our model. Section 5 concludes the paper. The proofs of the propositions and lemmas are provided in Appendix.

2 Model

2.1 Set-up

The set-up is very similar to that of Lagos and Wright (2005). Time is discrete and flows from $t = 0$ to $+\infty$. There is a continuum of the infinitely lived agents with a measure of unity. They consist of the special good buyers and the special good sellers. The measure of the buyers and sellers equal 0.5 respectively. There is also the general good firm who operates competitively. Each date is divided into two subperiods, day and night. In the day subperiod, decentralized markets (DM) open, and in the night subperiod, a centralized market (CM) opens.

The model has three goods: special goods, intermediate goods, and general goods. In the DM, the special good seller produces and sells the special goods to the special good buyer. The special good buyer then transforms the special goods into the intermediate goods. In the CM, the general good firm uses the intermediate goods and labor to produce the general goods. The individuals consume only the general goods. Figure 1 describes the production chain.

Following the monetary search literature, we call the agents who purchase the goods by paying money "buyers" and those who sell the goods by receiving money "sellers." In our model, the sellers produce the special goods, the buyers produce the intermediate goods from the special goods, and the general good firm produces the general goods from the intermediate goods. Thus, focusing on the roles in production, we could call them "producers," "wholesalers," and "retailers," instead of sellers, buyers, and general good firm. Having said that, we still use the names that are commonly used in the monetary

Goods	Inputs	Those who buy	Those who sell
Special good	Effort of the special good seller	Special good buyer	Special good seller
Intermediate good	Special good	General good firm	Special good buyer
General good	Labor and intermediate good	Special good buyer and special good seller	General good firm

1

Figure 1: Production chain

literature for the ease of comparison.

Both buyers and sellers can enter the DM by paying the respective fixed costs at the beginning of the day subperiod. The probabilities of matching for buyers and for sellers are determined endogenously. The buyer and seller trade special goods only if they match successfully. Let e and σ denote the measures of buyers and sellers entering the DM, respectively. These satisfy $0 \leq e \leq \bar{e} \equiv 1/2$ and $0 \leq \sigma \leq \bar{\sigma} \equiv 1/2$. The matching function in the DM is given by $N(e, \sigma)$. The function N has a constant returns to scale and is strictly increasing and strictly concave. In the DM, each buyer can form a successful matching with probability $\pi_b(z) = N(1, z)$, and for each seller, the probability of successful matching is $\pi_s(z) = N(1, z)/z$, where $z = \sigma/e$ denotes the seller-buyer ratio. For simplicity, we use competitive pricing rather than bargaining in terms of pricing in the DM.[1] Let p denote the price of the special good in terms of the general good.

[1] In the literature on the Lagos-Wright framework, competitive pricing is used in Aruoba, Waller and Wright (2011), Berentsen, Camera and Waller (2005, 2007), Lagos and Rocheteau (2005), and Rocheteau and Wright (2005). In these models, the entered agents form matchings and trade the goods at competitive prices, just as in our model. The matching in these and our models should not be interpreted as a bilateral matching between one buyer and one seller. The matching for an agent in these and our models should be interpreted as an entry to the "sub-market," where she can trade competitively with no frictions. That is, an agent who expends the entry cost stochastically enters the competitive sub-market with the "matching" probability.

Individuals are anonymous in the DM, and trade in the DM is mediated by money. Money is intrinsically useless, perfectly divisible, and storable, and provided by the central bank. It controls the money supply M by setting its gross growth rate μ. We assume that in the initial period, the government randomly chooses M_0 units of buyers and gives them one unit of money each, where $M_0 > 0$. As only buyers with money enter the DM, we have $e_0 \leq M_0$.

In the CM, the special good buyers produce the intermediate goods by production technology that transforms q units of the special goods to $g(q)$ units of the intermediate goods. Then, they sell the intermediate goods to the general good firm. The buyers and sellers supply labor, receive wage income, consume general good and adjust money balances. Also, the individuals determine whether to enter the DM in the next period. Each agent obtains utility $U(C)$ from consuming C units of general goods and suffers from the linear disutility H by supplying H units of labor. Figure 2 describes the timing of events in a period.

Let $F(Q, H)$ denote the constant returns to scale production function of the general good firm, where Q denotes the quantity of the intermediate goods and H denotes labor supply. Let w denote the wage rate in terms of general goods, and let R denote the price for the intermediate goods in terms of the general goods. The price mechanisms of the CM are also competitive. The factor prices w and R are determined by

$$w = F_H(Q, H), \tag{1}$$

$$R = F_Q(Q, H). \tag{2}$$

Why do we need production chain? Generally, entering a market is costly for both buyers and sellers. For example, sellers must set up shops by expending fixed and variable costs, and buyers must spend time searching for goods they want to buy. Hence it is natural to model any market such that buyers and sellers make costly entry decisions. However, it is difficult to assume that all agents make entry decisions in the Lagos-Wright model when we use a popular constant return to scale matching function. In that case, the matching probabilities for buyers and sellers are both determined by the seller-buyer

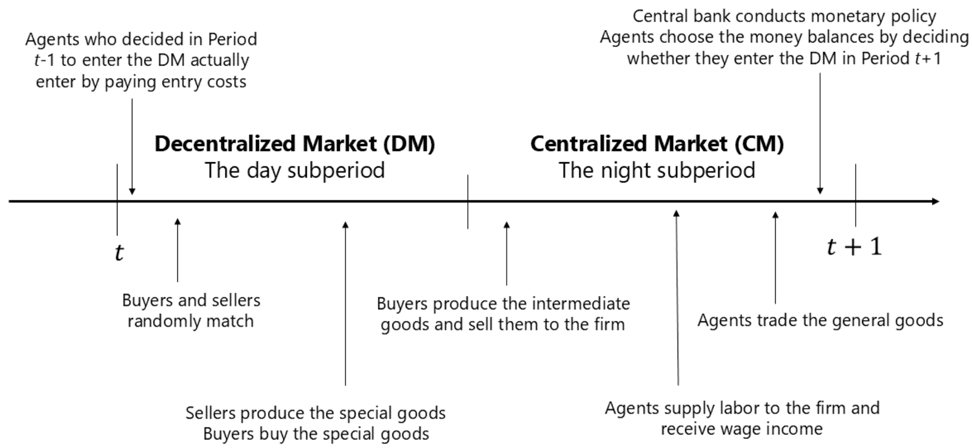

Figure 2: Timing of events

ratio. It is impossible to find the ratio which satisfies both the free-entry condition for buyers and that for sellers. In this study, we overcome this difficulty by assuming that the output in the DM is used as an input in the production of the general goods in the CM. This production chain structure is commonly observed in manufacturing sector such as automobile: Material manufacturers produce inputs for intermediate good firms, that produce parts of final goods for final good firms. The chain structure makes the two matching probabilities depend on two variables (the seller-buyer ratio and the measure of buyers) so that our problem has a solution.

2.2 Problem of the special good buyer

Here we study the problem of the individual who buys the special good in the DM, transforms the good into the intermediate good, and sells the good to the general good firm in the CM. We index the variables for the next period as $+1$.

Problem in the CM We solve the model backward and first investigate the CM. In the CM, the buyer who has q units of the special good sells $g(q)$ units of intermediate goods to the general good firm at the price R. The production function of the intermediate good, g satisfies $g(0) = 0$, $g'(q) > 0$ and $g''(q) < 0$. The buyer also chooses consumption of the general good and the amount of money to be carried to the next period.

Let $W_b^E(m,q)$ denote the value function of the buyer at the beginning of the night subperiod who holds m units of money and q units of special goods, and decides to enter the DM in the next period. Similarly, let $W_b^N(m,q)$ denote the value function of the buyer who decides not to enter the market. Furthermore, let $V_b^E(m)$ denote the value function of the buyer at the beginning of the day subperiod who holds m units of money and decides to enter the DM. Similarly let $V_b^N(m)$ denote the value function of the buyer at the beginning of the day subperiod who holds m units of money and decides *not* to enter the DM. The value functions satisfy

$$\begin{aligned} W_b^j(m,q) &= \max_{C,H,m_{+1}} [U(C) - H + \delta V_b^j(m_{+1})],\ (j = E, N) \\ \text{s.t. } C &= wH + \phi(m - m_{+1} + T) + Rg(q), \end{aligned}$$

where $\delta \in (0,1)$ is the discount factor, C is consumption, H is labor supply, ϕ is the value of money in terms of the general good, and T is the transfer from the government, and m_{+1} is the money balances in the next period.

Since the individual optimally chooses at the beginning of the night subperiod whether to enter the DM in the next period, the value function of the buyer at the end of the day subperiod equals to $W_b(m,q) = \max\{W_b^E(m,q), W_b^N(m,q)\}$. Due to quasi-linearity of the utility function, the function W_b is rewritten as

$$W_b(m,q) = \frac{\phi}{w} m + \frac{R}{w} g(q) + W_b(0,0), \tag{3}$$

where the constant $W_b(0,0)$ is

$$\begin{aligned} W_b(0,0) &= \max_C \left[U(C) - \frac{C}{w} \right] + \frac{\phi}{w} T \\ &\quad + \max \left[\max_{m_{+1}} [-\frac{\phi}{w} m_{+1} + \delta V_b^E(m_{+1})], \max_{m_{+1}^N} [-\frac{\phi}{w} m_{+1}^N + \delta V_b^N(m_{+1}^N)] \right]. \end{aligned}$$

Here m^N is the money balances of the buyer who is not going to enter the DM. Later we show that it is zero. The consumption of the general good is independent of the decision to enter the DM. The first order condition on C is

$$U'(C) = \frac{1}{w}. \tag{4}$$

Later we show that the seller's consumption is the same as that of the buyer.

Problem in the DM The buyer who decides to enter the DM pays the entry cost, and purchases q units of the special good at the price p. To enter the DM, the buyer must pay k_b units of utility cost. Trades in the DM requires money. Therefore the value function of the buyer at the beginning of each period who enters the DM satisfies

$$V_b^E(m) = -k_b + \pi_b \max_{pq \leq \phi m} \{W_b(m - pq, q)\} + (1 - \pi_b)W_b(m, 0),$$

where $\pi_b = \pi_b(z)$. The buyer who skips the DM simply enters the CM in the next subperiod with $q = 0$. Thus his value function $V_b^N(m)$ is equal to $W_b(m, 0)$. Due to the linearity of W_b with respect to m, the value functions V_b^E and V_b^N can be simplified as

$$V_b^E(m) = -k_b + \frac{\pi_b}{w} \max_{pq \leq \phi m} \{-pq + Rg(q)\} + \frac{\phi}{w} m + W_b(0, 0), \tag{5}$$

$$V_b^N(m) = \frac{\phi}{w} m + W_b(0, 0). \tag{6}$$

The surplus from entering the next-period DM is non-negative if

$$\max_{m_{+1}} \left[-\frac{\phi}{w} m_{+1} + \delta V_b^E(m_{+1}) \right] \geq \max_{m_{+1}^N} \left[-\frac{\phi}{w} m_{+1}^N + \delta V_b^N(m_{+1}^N) \right], \tag{7}$$

and $e_{t+1} = 1/2$ if the strict inequality holds. Throughout this study, we focus on equilibria in which nominal interest rate $i = \frac{\phi/w}{\delta \phi_{+1}/w_{+1}} - 1$ is strictly positive. In such an equilibrium, the liquidity constraint binds for the agent who enters the DM:

$$pq = \phi m. \tag{8}$$

The right-hand side of (7) is written as

$$-\frac{\phi}{w} m_{+1}^N + \delta V_b^N(m_{+1}^N) = -\frac{i}{1+i} \frac{\phi}{w} m_{+1}^N + W_b(0, 0).$$

The surplus decreases with m_{+1}^N because $i > 0$.[2] Thus, $m_{+1}^N = 0$.

Since the liquidity constraint binds, (7) is simplified as follows:

$$0 \leq \max_{m_{+1}} \left[-\frac{\phi}{w} m_{+1} + \delta \left\{ -k_b + \pi_{b,+1} \frac{R_{+1}}{w_{+1}} g\left(\frac{\phi_{+1} m_{+1}}{p_{+1}} \right) + (1 - \pi_{b,+1}) \frac{\phi_{+1} m_{+1}}{w_{+1}} \right\} \right]. \tag{9}$$

[2]Throughout this study, we assume that money growth is given, such that this inequality holds. Unless this inequality holds, there exists no equilibrium because, in that case, buyers who do not enter would choose $m_{+1} = +\infty$ to obtain $V_b^N(m_{+1}) = +\infty$.

Using i, the first-order condition on m_{+1} is written as

$$i = \frac{\phi/w}{\delta\phi_{+1}/w_{+1}} - 1 = \pi_{b,+1}\left(\frac{R_{+1}}{p_{+1}}g'\left(\frac{\phi_{+1}m_{+1}}{p_{+1}}\right) - 1\right). \tag{10}$$

Using (10) and (8), we can re-write (9) as

$$k_b \leq \frac{R_{+1}}{w_{+1}}g\left(q_{+1}\right)\pi_{b,+1}\left\{1 - \frac{q_{+1}g'\left(q_{+1}\right)}{g\left(q_{+1}\right)}\right\}, \tag{11}$$

where the right-hand side shows the surplus from the match.

2.3 Problem of the special good seller

We next study the problem of the special good seller who sells the special good to the buyer in the DM. Let $V_s^E(m)$ denote the value function of the seller at the beginning of each period who decides to enter the DM. Similarly, we let $V_s^N(m)$ denote the value function of the seller at the beginning of each period who decides not to enter the DM.

Problem in the CM Let $W_s^E(m)$ denote the value function of the seller at the beginning of the night sub-period who holds m units of money and decides to enter the DM next period. Similarly, let $W_s^N(m)$ denote the value function of the seller at the beginning of the night sub-period who decides not to enter the DM. They are defined as

$$\begin{aligned} W_s^j(m) &= \max_{C,H,m_{+1}}[U(C) - H + \delta V_s^j(m_{+1})],\ (j = E, N) \quad (12)\\ \text{s.t. } C &= wH + \phi(m - m_{+1}). \quad (13) \end{aligned}$$

When the nominal interest rate is strictly positive, the sellers do not carry money to the next period, that is, $m_{+1} = 0$.

Due to the quasi-linearity, the seller's value function at the beginning of second sub-period, $W_s(m) = \max\{W_s^E(m), W_s^N(m)\}$ is expressed as

$$W_s(m) = \frac{\phi}{w}m + \max_C\left[U(C) - \frac{C}{w}\right] + \max\left\{V_s^E(0), V_s^N(0)\right\}.$$

The seller's consumption is also given by (4).

Problem in the DM Subsequently, we investigate the equilibrium condition of the DM. To enter the DM, the seller must pay k_s units of utility cost. The matched seller loses utility $c(q)$ by producing q units of the special good and selling them to the buyer. The function c satisfies $c(0) = 0$, $c' > 0$, and $c'' > 0$. Due to the linearity of W_s, the value functions are expressed as

$$\begin{aligned} V_s^E(m) &= -k_s + \pi_s \max_q \left\{ -c(q) + \frac{p}{w} q \right\} + W_s(m), \\ V_s^N(m) &= W_s(m), \end{aligned}$$

where $\pi_s = \pi_s(z)$. The first order condition on q is

$$wc'(q) = p. \tag{14}$$

The surplus of the sellers from entering the DM is nonnegative if $V_s^E(m) \geq V_s^N(m)$, or equivalently

$$k_s \leq \pi_s \left\{ -c(q) + c'(q)q \right\}. \tag{15}$$

If the strict inequality holds, the all sellers enter the market and then $\sigma_{+1} = \bar{\sigma}(= 1/2)$.

2.4 Feasibility conditions

In the equilibrium, the total amount of intermediate goods is

$$Q = N(e, \sigma)g(q), \tag{16}$$

where $\sigma = ez$. In the CM, all agents choose the same consumption level C defined by (4). The resource constraint is given by

$$F(Q, H) = C, \tag{17}$$

where H is the total labor supply in the CM.

Since number of buyers entering the DM is e, the equilibrium condition on money is

$$M = em, \tag{18}$$

where m is the buyer's nominal balance entering the DM.

2.5 Initial period

In the initial period, buyers enter the DM only if they receive (one unit of) money from the government. In the following, we focus on the equilibria where all the buyers with money enter the DM in the initial period. In such equilibria, we have

$$e_0 = M_0. \tag{19}$$

If the surplus from entering the DM, $V_b^E(1) - V_b^N(1)$, is strictly positive, then all the buyer enters the DM. As is clear from (5) and (6), this occurs if

$$-k_b + \frac{\pi_{b,0}}{w_0}\left(R_0 g\left(\frac{\phi_0}{p_0}\right) - \phi_0\right) > 0. \tag{20}$$

We define the competitive equilibrium as follows:

Definition 1 *The competitive equilibrium is the set of prices $\{\phi_t, w_t, p_t, R_t\}$ and allocation $\{q_t, e_t, z_t, Q_t, H_t, C_t, m_t\}$ that evolves according to (1), (2), (4), (8), (10), (11), (14), (15),(16), (17), and (18), given that the sequence of money supply $\{M_t\}_{t=0}^{\infty}$ is exogenously determined and the variables in the initial period satisfy (19) and (20).*

2.6 Stationary equilibrium

The next proposition characterizes the steady-state allocation and proves existence of a steady state under a specific government policy.

Proposition 1 *Suppose the gross growth rate of the money, μ is greater than δ. The steady state allocation (q, e, z, Q, H, C) is determined by (17) and*

$$\frac{\mu}{\delta} - 1 = \pi_b(z)\left(\frac{g'(q)}{c'(q)}\frac{F_Q(Q,H)}{F_H(Q,H)} - 1\right), \tag{21}$$

$$k_b \leq \pi_b(z) g\left(q\right)\frac{F_Q(Q,H)}{F_H(Q,H)}\left(1 - \frac{qg'(q)}{g(q)}\right), \tag{22}$$

$$k_s \leq \pi_s(z) c(q)\left(\frac{qc'(q)}{c(q)} - 1\right), \tag{23}$$

$$F_H(Q,H)U'(F(Q,H)) = 1, \tag{24}$$

$$Q = eN(1,z)g(q), \tag{25}$$

where $\pi_b(z) = N(1,z)$ and $\pi_s(z) = N(1,z)/z$. Once we determine the steady-state allocation, the stationary price (w, ϕ, p, R) are determined uniquely. In (22), $e = \bar{e}(= 1/2)$ if the strict inequality holds. In (23), $\sigma = \bar{\sigma}(= 1/2)$ if the strict inequality holds. This steady state can be realized from the initial period if the government chooses $M_0 = e$.

Proof. See Appendix. ■

Stability of the equilibrium: The equilibrium values of e and σ, which are determined by the free-entry conditions (11) and (15), are stable for the following reasons. If e is slightly greater than the equilibrium value, then, the buyers' matching probability decreases slightly. Then, given that market prices are invariant, the welfare of the buyer entering the DM, V_b^E, is less than the welfare of the buyer not entering the DM, V_b^N. In this case, the number of entering buyers decreases and the value of e returns to the equilibrium. On the other hand, if e is less than its equilibrium value, $V_b^E > V_b^N$ and more buyers begin to enter the DM. Thus, e increases and returns to the equilibrium. A similar mechanism applies to the sellers. Therefore, the steady-state equilibrium is stable.

Parametric assumptions: To simplify the analysis, we put the following assumptions for functional forms hereafter:

Assumption 1: $F(Q,H) = Q^\alpha H^{1-\alpha}$, $N(e,\sigma) = Ae^\beta \sigma^{1-\beta}$, $c(y) = y^\psi$, $U(c) = \frac{c^{1-\rho}-1}{1-\rho}$ and $g(y) = y^\theta$ where $\psi > 1$, $\alpha \in (0,1)$, $\beta \in (0,1)$, $\theta \in (0,1)$, and $\rho \in (0,1)$.

Under Assumption 1, $\pi_b = N/e = Az^{1-\beta}$ and $\pi_s = N/\sigma = Az^{-\beta}$, where $z = \sigma/e$. One can easily check that (21)-(25) can be re-expressed as follows:

$$\frac{\mu}{\delta} - 1 = Az^{1-\beta}\left(q^{\theta-\psi}\frac{H}{Q}\frac{\alpha}{1-\alpha}\frac{\theta}{\psi} - 1\right), \tag{26}$$

$$k_b \leq \frac{H}{e}\frac{\alpha}{1-\alpha}(1-\theta), \tag{27}$$

$$k_s \leq Az^{-\beta}(\psi - 1)q^\psi, \tag{28}$$

$$Q = Aez^{1-\beta}q^\theta, \tag{29}$$

$$F(Q,H)^{1-\rho} = \frac{H}{1-\alpha}. \tag{30}$$

In the following, we focus on the interior equilibrium in which (27) and (28) hold with equality and the individuals are indifferent between entering the DM and not entering the DM. Later we show that such equilibria exist when the entry costs k_b and k_s are sufficiently high. Let $k = k_b/k_s$ denote the ratio of buyer's entry cost relative to that of the seller. The next proposition characterizes the equilibrium level of seller-buyer ratio z as a function of i and k.

Proposition 2 *Under Assumption 1, the seller-buyer ratio z is determined by*

$$\frac{\mu}{\delta} - 1 = i(z;k), \tag{31}$$

where the function $i(z;k)$ is a decreasing function of z and is defined as

$$i(z;k) = A\left\{z^{-\beta}\frac{\psi-1}{\psi}\frac{\theta}{1-\theta}k - z^{1-\beta}\right\}. \tag{32}$$

We have $q = A^{-1/\psi}(\psi-1)^{-1/\psi}(k_s)^{1/\psi}z^{\beta/\psi}$, $e = \Gamma z^{\epsilon}$, $\sigma = \Gamma z^{\epsilon+1}$ and $H = \frac{1-\alpha}{\alpha(1-\theta)}k_b e$, where $\Gamma = (1-\alpha)^{1/\rho}A^{(1-\theta/\psi)\eta}(\psi-1)^{-\theta\eta/\psi}(\frac{1-\alpha}{\alpha(1-\theta)})^{-\eta-1}(k_b)^{-\eta-1}(k_s)^{\theta\eta/\psi}$, $\eta = \alpha(1/\rho-1) > 0$, and $\epsilon = \eta\left(1-\beta+\beta\frac{\theta}{\psi}\right) > 0$. An increase in the nominal rate i reduces z, q, e, σ, H and Q.

Proof. See Appendix. ■

We note the parameter k as an argument of $i(z;k)$, as we analyze how changes in k affect value of i in what follows. This proposition indicates that a higher nominal interest rate or higher inflation slows down the overall economic activities, including the entry of all agents and amount of production. Thus, entries and outputs are maximized by the Friedman rule, which sets the nominal interest rate at zero. However, in the next section, we show that the Friedman rule does not necessarily maximize social welfare. In some cases, the optimal monetary policy is to set nominal interest rate at a positive value.

We consider the Friedman rule as a limiting case in which i converges to zero from above, i.e., $i \to 0+$. From (31), the level of z under the Friedman rule is proportional to k and is equal to νk where $\nu = \frac{\psi-1}{\psi}\frac{\theta}{1-\theta}$.

Existence of interior equilibrium: The interior equilibria in which (27) and (28) hold with equality and only some of the buyers and sellers enter the DM exist if both

$e = \Gamma z^{\epsilon}$ and $\sigma = \Gamma z^{\epsilon+1}$ are weakly less than $1/2$. As z decreases with the nominal interest rate i, it is sufficient to show that e and σ are less than $1/2$ under the Friedman rule. We let $\bar{k}_b > 0$ and $\bar{k}_s > 0$ be constants that satisfy

$$2\bar{\Gamma}\nu^{\epsilon} = (\bar{k}_s)^{\eta(1-\beta)(1-\theta/\psi)}(\bar{k}_b)^{1+\eta\beta(1-\theta/\psi)}, \tag{33}$$

$$2\bar{\Gamma}\nu^{\epsilon+1} = (\bar{k}_s)^{1+\eta(1-\beta)(1-\frac{\theta}{\psi})}(\bar{k}_b)^{\eta\beta(1-\theta/\psi)}, \tag{34}$$

where $\bar{\Gamma} = (1-\alpha)^{1/\rho}A^{(1-\theta/\psi)\eta}(\psi-1)^{-\theta\eta/\psi}(\frac{1-\alpha}{\alpha(1-\theta)})^{-\eta-1} = \Gamma k_s^{-\theta\eta/\psi}k_b^{\eta+1}$.

We also define constants $\underline{k}_b = A\underline{q}^{\psi}\psi\frac{1-\theta}{\theta}$ and $\underline{k}_s = A\underline{q}^{\psi}\psi\frac{1-\theta}{\theta}$ where $\underline{q}$ is determined by the following equation:

$$\psi^{(1-\rho)(1-\alpha)-1}(1-\alpha)^{(1-\rho)(1-\alpha)} = (A\bar{e}\underline{q}^{\theta})^{\rho}(\underline{q}^{\theta-\psi}\alpha\theta)^{(1-\rho)(1-\alpha)-1}.$$

The next proposition characterize the equilibria with respect to the entry costs k_b and k_s.

Proposition 3 *Under Assumption 1, suppose k_b and k_s satisfy $k_b \geq \bar{k}_b$ and $k_s \geq \bar{k}_s$. Then the interior equilibrium in which the entry conditions (27) and (28) hold with equality exists for all nominal interest rate i. On the other hand, if $k_b < \underline{k}_b$ and $k_s < \underline{k}_s$, then the equilibrium in which all individuals enter the DM exists as long as i is low.*

Proof. See Appendix. ∎

In what follows, we assume the following assumption so that we focus on the interior equilibrium in which the entry conditions (27) and (28) hold with equality.

Assumption 2: The entry costs satisfy $k_b \geq \bar{k}_b$ and $k_s \geq \bar{k}_s$.

3 Welfare

In this section, we characterize the steady state welfare.

3.1 Steady-state welfare

Social welfare in the steady state is expressed as

$$S(e,\sigma,q,H) = -k_s\sigma - k_b e + U(F(Q,H)) - N(e,\sigma)c(q) - H, \tag{35}$$

where $Q = N(e, \sigma)g(q)$. The next proposition characterizes the first best allocation which unconditionally maximizes S with respect to e, σ, q and H.

Proposition 4 *Under Assumptions 1 and 2, the first best allocation is characterized by the following equations.*

$$
\begin{aligned}
k_b &= \beta\pi_b(z)g(q)\frac{F_Q(Q,H)}{F_H(Q,H)}\left\{1-\frac{1}{\psi}\frac{qg'(q)}{g(q)}\right\}, && (36)\\
k_s &= (1-\beta)\pi_s(z)c(q)\left\{\frac{1}{\theta}\frac{qc'(q)}{c(q)}-1\right\}, && (37)\\
1 &= \frac{g'(q)}{c'(q)}\frac{F_Q(Q,H)}{F_H(Q,H)}, && (38)\\
1 &= F_H(Q,H)U'(F(Q,H)). && (39)
\end{aligned}
$$

Proof. See Appendix. ■

The first best conditions (36) - (39) are very much similar to the equilibrium conditions (21) - (24). Especially, the equation on the labor supply (39) is exactly the same as (24). Also, the optimality condition on q, (38) is equivalent to (21) under the Friedman rule. However, the equations on the entry cost, (36) and (37) differ from the corresponding equilibrium conditions (22) and (23) because the matching function $N = e^{\beta}\sigma^{1-\beta}$ and the production function of the intermediate good $g(q) = q^{\theta}$ are strictly concave (i.e., $\beta > 0$ and $\theta < 1$) and the cost function $c(q) = q^{\psi}$ is strictly convex (i.e., $\psi > 1$). Therefore the first best allocation cannot be implemented as an equilibrium allocation even under the Friedman rule. This is mainly because the buyers and sellers take the matching probabilities π_b and π_s as given, while the social planner knows that these depends on the measure of the buyers and the sellers. As we show in the next section, the externalities in the matching make the optimal monetary policy deviate from the Friedman rule.

3.2 Optimal monetary policy

The next proposition characterizes the stationary welfare as a function of z.

Proposition 5 *Under Assumptions 1 and 2, the stationary welfare in the competitive equilibrium is a function of the seller-buyer ratio z:*

$$s(z) = \Gamma k^s \left\{ \frac{1+\eta\theta}{\eta(1-\theta)} k z^{\epsilon} - \frac{\psi}{\psi - 1} z^{\epsilon+1} \right\}. \tag{40}$$

Proof. See Appendix. ■

Function $s(z)$ is maximized when z is equal to

$$z^*(k) = \nu^* k, \tag{41}$$

where $\nu^* = \frac{1+\eta\theta}{\eta(1-\theta)} \frac{\psi-1}{\psi} \frac{\epsilon}{\epsilon+1} > 0$. We define $i^*(k) \equiv i(z^*(k); k)$ as the nominal rate which maximizes the welfare s, although it can be negative at this point. It is written as

$$i^*(k) = A(\nu^*)^{-\beta} \left(\frac{\psi - 1}{\psi} \frac{\theta}{1-\theta} - \nu^* \right) k^{1-\beta}. \tag{42}$$

The following lemma derives a condition to ensure that $i^*(k)$ is positive.

Lemma 1 $i^*(k) > 0$ *if and only if*

$$\theta > \frac{1-\beta}{1-\beta/\psi}. \tag{43}$$

Proof. See Appendix. ■

On the right-hand side of (43), $\frac{1-\beta}{1-\beta/\psi}$ is less than one because $\beta \in (0, 1)$ and $\psi > 1$. Therefore, (43) is satisfied provided that θ is sufficiently close to one. In the following, we assume that (43) holds. We also put the following assumption to ensure that the matching probabilities $\pi_b = Az^{1-\beta}$ and $\pi_s = Az^{-\beta}$ are less than one for all $i \in (0, i^*(k)]$.

Assumption 3: The parameter A satisfies

$$A < \min \left\{ (\nu k)^{\beta-1}, (\nu^* k)^{\beta} \right\}.$$

Note that if $i = 0$, then $z = \nu k$, and if $i = i^*(k)$, then $z = \nu^* k$. Furthermore, $\nu > \nu^*$ because $\partial z / \partial i < 0$ from Proposition 2. We have the following proposition.

Proposition 6 *Suppose Assumptions 1, 2 and 3, and (43) hold. The Friedman rule is suboptimal. The optimal nominal interest rate* $i^*(k) > 0$ *is an increasing function of the relative entry cost of buyer* $k = \frac{k_b}{k_s}$.

Why is the Friedman rule suboptimal? This proposition states that the Friedman rule is suboptimal for a certain range of parameter values. The condition (43) can be satisfied in various cases, such as $\theta < 1$, $\beta < 1$ and $\psi > 1$. One explanation of suboptimality of the rule in an economy with entries is the congestion externality, a buyer's entry decreases the probability of matching for other buyers, and increases that for the sellers. See Rocheteau and Wright (2005), Liu, Wang and Wright (2011), and Berentsen and Waller (2015).[3] This external effect is not internalized in the decision-making of an individual buyer in competitive equilibrium. Thus, a reduction in the number of entering buyers e can improve welfare if the congestion externality is high. As an increase in the nominal rate reduces buyers' entry e by increasing the opportunity cost of holding money, the deviation from the Friedman rule may become optimal policy.

3.3 Response of optimal interest rate to changes in entry costs

Proposition 6 states that the optimal interest rate or the inflation rate is higher in an economy in which the buyers' entry cost is higher or seller's entry costs are lower. In this section, we present a simplified explanation for the response of the optimal interest rate to a change in the buyers' or sellers' entry costs. Considering the optimal response of i, we focus on the changes in e and σ in response to changes in k_b or k_s.

First, suppose that k_b increases but k_s does not. Proposition 6 implies that the optimal interest rate should become higher. This intuition can be written as follows: an increase in the entry cost k_b reduces the entry of buyers and leads to an increase in the matching probability for buyers; an increase in matching probability increases the expected gain of buyer entry, leading to many buyer entries that exacerbate the congestion externality. Thus, the central bank's optimal response is to increase the costs of holding money i to reduce the buyers' entry. This intuition can be explained using the follow-

[3] Some authors distinguish between the effect of one buyer's entry on other buyers, and that on sellers. The former is called the congestion effect and the latter is called the thick-market effect (Rocheteau and Wright 2005; and Shimer and Lones 2001). In this study, we did not distinguish between them and call both congestion externality.

ing equations: (21) indicates that $i = \pi_b\{(R/w)(g'(q)/c'(q)) - 1\}$ and (22) implies that $k_b = \pi_b g(q)(R/w)(1 - qg'(q)/g(q))$. The first equation is the Euler equation for money holdings and the second represents the free-entry condition. An increase in the entry cost k_b decreases the buyers' entry e and increases π_b. For simplicity, we assume that the optimal amounts of q and H do not change. Then, the decrease in e reduces the total number of matches and quantity of the intermediate good Q, leading to an increase in the the relative price, R/w. In other words, the free-entry condition implies that an increase in k_b increases $\pi_b(R/w)$. Now, let us consider the Euler equation above. An increase in $\pi_b(R/w)$ indicates an increase in gains from money holdings, which must be balanced in equilibrium with cost of money holdings i. Therefore, an increase in k_b, assuming that the optimal q and H are invariant, induces an increase in optimal interest rate i.

Second, suppose that k_s increases but k_b does not. Proposition 6 implies that the optimal interest rate should become lower. How can this result be explained? Increase in the entry cost of the sellers, k_s, decreases the entry of sellers and leads to a decrease in the matching probability of the buyers. This decrease reduces the expected gain of buyer entries, leading to too few buyer entries due to congestion externality. Thus, the central bank's optimal response is to decrease the cost of holding money i to increase buyers' entry. This intuition can be explained using the following equations. The sellers' free entry condition (23) can be satisfied only if $\pi_s(z)$ becomes larger in response to increases in k_s. To increase π_s measure of the sellers, σ, should be very small if the measure of the buyers, e, does not change. The central bank can now increase π_s by increasing e through decreasing i. The buyers' entry e increases if the central bank decreases i because i is the opportunity cost of holding money, and a decrease in i indicates an increase in the gain of entry for a buyer. Thus, a decrease in i can make σ not so small and diversify the impact of the increase in k_s to an increase in e. Thus, social welfare should improve when i reduces in response to the higher k_s than when i is invariant.[4]

[4] Under the assumption that q and H are invariant, social welfare $S(e, \sigma, q)$, defined in (35), is a concave function of e and σ. Therefore, the optimal response to a change in constraint k^b or k^s should be a change in both e and σ. A change in either only e or only σ cannot be an optimal response.

In equilibrium, the buyers' entry e may not necessarily increase with k_s, where e is an equilibrium value in the optimal steady state in which the central bank chooses $i = i^*(k)$ because general equilibrium effects exist. The following proposition summarizes the responses of the equilibrium values of variables in the optimal steady state to the changes in k_b and k_s.

Proposition 7 *Suppose that Assumptions 1, 2 and 3 hold. We consider the optimal steady state in which the central bank chooses $i = i^*(k)$. The equilibrium values of e, σ, and π_s decrease, and z, q, and π_b increase in k_b in the optimal steady state. The equilibrium values of e, σ, z and π_b decrease, and q and π_s increase in k_s.*

Proof. See Appendix. ■

4 Extensions

In this section, we study two variants of our model.

4.1 Model with variable buyer entry cost

Our paper is closely related to Gabrovski and Ortego-Marti (2019), who study the search model of the housing market in which both buyers and sellers enter the market. The important assumption in Gabrovski and Ortego-Marti (2019) is that the entry cost of buyer increases with the measure of the buyers. In this section, we incorporate the variable entry cost of the buyer into the canonical model of Lagos and Wright (2005). In the following, we call the model GO-LW model.

Here we abstract from the production chain and simply assume that the special good buyer gets utility from the good just like Lagos and Wright (2005). The variable entry cost in the GO-LW model is somewhat similar to that in Amendola, Araujo and Ferraris (2024). The difference is that the measure of entering buyers is not a choice variable for individual buyers in the GO-LW model, while Amendola, Araujo and Ferraris (2024) assumes that individual buyers choose their search intensity, which is the equivalent to the measure of entering buyers in the GO-LW model. In Amendola, Araujo and Ferraris (2024), the Friedman rule is optimal, while it is not in the GO-LW model as we show in what follows.

We characterize the stationary equilibrium allocation in the GO-LW model, and compare with our basic model. The value functions of the individuals at the beginning of the night sub-period are

$$\begin{aligned}\tilde{W}_n^j(m) &= \max_{C,m_{+1}}[U(C) - H + \delta\tilde{V}_n^j(m_{+1})],\\ \text{s.t.}\quad C &= H + \phi(m - m_{+1} + T),\end{aligned}$$

where the subscript j shows whether the individual enters the DM ($j = E$) or not ($j = N$) in the next period, and the subscript n shows whether the individual is a buyer ($n = b$) or a seller ($n = s$). Due to quasi-linearity, the value functions of the buyers ($n = b$) and sellers ($n = s$) at the end of the day sub-period who optimally choose in the CM whether they enter the DM next period or not, are given by

$$\tilde{W}_n(m) = \max\{\tilde{W}_n^E(m), \tilde{W}_n^N(m)\} = \phi m + \tilde{W}_n(0).\ (n = b, s) \tag{44}$$

The optimal level of the general good consumption C, say C^* is independent of the inflation rate is determined by the first order conditions $U'(C^*) = 1$.

The buyer in the DM purchases q units of the special good and gets utility by $g(q)$. The value function of the buyer who enters the DM is

$$\tilde{V}_b^E(m) = -k_b(e) + \pi_b \max_{pq\leq\phi m}\{-pq + g(q)\} + \phi m + \tilde{W}_b(0),$$

where the entry cost $k_b(e)$ is an increasing function of e. The value function of the buyer who skips the DM is $\tilde{V}_b^N(m) = \phi m + \tilde{W}_b(0)$. The problem of the seller is almost the same as before. The cost function of the seller is the same as before and is given by $c(q)$. The value function of the seller who enters the DM is

$$\tilde{V}_s^E(m) = -k_s + \pi_s \max_q\{pq - c(q)\} + \phi m + \tilde{W}_s(0).$$

We focus on the equilibria in which the nominal interest rate $i = \frac{\phi}{\delta\phi_{+1}} - 1$ is strictly positive and the entry conditions of the buyer and the sellers hold with equality. The next proposition characterizes the steady state equilibrium.

Proposition 8 *The steady state allocation (q, e, z) is determined by*

$$i = \pi_b(z)\left(\frac{g'(q)}{c'(q)} - 1\right), \tag{45}$$

$$k_b(e) = \pi_b(z)g(q)\left(1 - \frac{qg'(q)}{g(q)}\right), \tag{46}$$

$$k_s = \pi_s(z)c(q)\left(\frac{qc'(q)}{c(q)} - 1\right). \tag{47}$$

Proof. See Appendix. ■

Interestingly, the equilibrium conditions of GO-LW model are very similar to the previous ones in our baseline model. If k_b is independent of the measure of the buyer e, then the two entry equations (46) and (47) uniquely determine z and q, and the first order condition on money cannot be satisfied except for a specific value of i. This implies that monetary equilibrium does not exist in general.

However, when k_b depends on e, the monetary equilibrium exists, because for any fixed $i \geq 0$, (45) and (47) uniquely determines (q, z) and for such (q, z), one can find e which satisfies (46). Dependency of k_b on e ensures the equilibrium existence. A similar mechanism works in our baseline model. The monetary equilibrium exists because the buyer's entry condition depends on the aggregate quantity of the intermediate good Q in addition to z and q, and the quantity Q depends on e.

Social welfare in the steady state in the GO-LW model is expressed as

$$\tilde{S} = -k_s\sigma - k_b(e)e + N(e, \sigma)\{g(q) - c(q)\}.$$

Here we ignore the utility in the CM, $U(C^*) - C^*$, which is independent of the policy parameters.

We have the following proposition on the optimal monetary policy.

Proposition 9 *Let $k_b(e) = \bar{k}e^\omega$ where $\bar{k} > 0$ and $\omega > 0$ are constant. Under Assumption 1, the optimal nominal interest rate is always strictly positive and given by*

$$i = (k_s)^{1-1/\beta}A^{1/\beta}(\psi - 1)^{1/\beta-1}\left(\frac{\psi + \tau}{\theta + \tau}\frac{\psi}{\theta}\right)^{\frac{\psi(1/\beta-1)}{\theta-\psi}}\left(\frac{\psi - \theta}{\theta + \tau}\right),$$

where $\tau = \psi(1/\beta - 1)(1/\omega + 1) + \theta/\omega$. The rate is increasing in ω if $\beta > 1/2$. It is decreasing in k_s.

Proof. See Appendix. ■

Here the optimal nominal interest rate is *always* strictly positive. This means that the Friedman rule is never optimal. Furthermore, the optimal rate is decreasing in the seller's entry cost just like our basic model. The optimal rate is increasing in the cost elasticity of buyer's entry (ω) when the buyer's share of matching (β) is large.

4.2 Model with Nash-bargaining

So far we have assumed that trades in the DM is competitive. Here we consider a case where terms of trades in the DM are determined by Nash bargaining. For analytical tractability, here we assume that the measure of sellers is fixed at $\bar{\sigma} = 1/2$ and they do not have a choice of entry. The value functions of the buyer and the seller in the DM, say $\hat{V}_b^E$ and $\hat{V}_s^E$ are written as

$$\begin{aligned}\hat{V}_b^E(m) &= -k_b + \pi_b \left\{ \frac{R}{w} g(q) - \frac{\phi d}{w} \right\} + \frac{\phi m}{w} + K, \\ \hat{V}_s^E(m) &= \pi_s \left\{ \frac{\phi d}{w} - c(q) \right\} + \frac{\phi m}{w} + K.\end{aligned}$$

where d is the money transfer from the buyer to the seller and K is a constant. The terms of trade is determined by the Nash-bargaining. We assume that the bargaining power of the buyer is equal to one, and the buyer makes take-it-or-leave offer to the seller. The problem is written as

$$\max_q \frac{R}{w} g(q) - \frac{\phi d}{w}, \text{ s.t. } c(q) \leq \frac{\phi d}{w}, d \leq m,$$

where the first condition describes the participation constraint of the seller. When the liquidity constraint binds, $c(q) = \frac{\phi m}{w}$. The free entry condition of the buyer is

$$\delta k_b = \max_{m_{+1}} \left[-\frac{\phi}{w} m_{+1} + \delta \pi_{b,+1} \frac{R_{+1}}{w_{+1}} g\left(q_{+1}\right) + \delta(1 - \pi_{b,+1}) \frac{\phi_{+1} m_{+1}}{w_{+1}} \right].$$

The first order conditions on m_{+1} is

$$\frac{1}{\delta} \frac{\phi w_{+1}}{w \phi_{+1}} = \pi_{b,+1} \frac{R_{+1}}{w_{+1}} \frac{g'\left(q_{+1}\right)}{c'(q_{+1})} + 1 - \pi_{b,+1}.$$

The stationary equilibrium allocation (q, Q, H, e) is uniquely determined by

$$\frac{\mu}{\delta} - 1 = \pi_b(\bar{\sigma}/e)\left(\frac{R}{w}\frac{g'(q)}{c'(q)} - 1\right), \tag{48}$$

$$k_b = \frac{H}{e}\frac{\alpha}{1-\alpha}\left(1 - \frac{c(q)}{c'(q)}\frac{g'(q)}{g(q)}\right), \tag{49}$$

$$Q = Ae^{\beta}\bar{\sigma}^{1-\beta}g(q), \tag{50}$$

$$F(Q, H)^{1-\rho} = \frac{H}{1-\alpha}. \tag{51}$$

For simplicity, here we assume that the cost function is linear and given by $c(q) = q$. The stationary welfare is denoted as

$$\hat{S} = -k_b e + \frac{F(Q, H)^{1-\rho}}{1-\rho} - N(e, \bar{\sigma})q - H.$$

The next proposition characterize the optimal monetary policy in the bargaining model.

Proposition 10 *Consider the bargaining model in which the bargaining power of the buyer is equal to one. Suppose Assumption 1 holds. The optimal nominal interest rate is always strictly positive. The rate increases with the buyer's entry cost k_b.*

Proof. See Appendix. ■

The proposition shows that positive relationship between the optimal nominal interest rate and the buyer's entry cost is robust to the trade mechanism in the DM.

5 Conclusion

In this study, we construct a monetary search model in which both buyers and sellers make entry decision. One feature of our model is that goods produced in decentralized markets are used as production inputs in the centralized market. We demonstrate that the Friedman rule is suboptimal and explicitly derive the optimal nominal interest rate or optimal inflation rate for a certain set of functional forms to show that it is lower for a lower cost of buyers' entry or a higher cost of sellers' entry.

There remain many points related to our theory that need to be analyzed further in the future research. It may be necessary to explore the nature of the technological progress in recent decades to determine how these technological changes affect the buyers' and sellers' entry costs, and how they affect the optimal inflation. Another point that has not been fully handled in this paper is the demand externality. The entry of buyers may affect the aggregate demand in a way that an increase in the demand positively affects sellers' production. In that case, the demand externality may induce a positive correlation between the inflation and output. These are examples of the agenda for future research that may deepen our understanding of the relationship between inflation and agents' entry.

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Appendix

In Appendix, we provide proofs of the propositions and lemmas.

A Proofs

A.1 Proof of Proposition 1

We first prove (21)–(25). Because $\mu = \phi/\phi_{+1}$ in the steady state, (10) implies

$$\frac{\mu}{\delta} - 1 = \pi_b \left(g'(q) \frac{R}{p} - 1 \right). \tag{52}$$

From (14) and (1), we have $p = wc'(q) = F_H(Q, H)c'(q)$. Substituting this equality and (2) into (52) yields (21). Next, (11), together with (1) and (2), yields (22). The seller's entry condition (23) is almost the same as (15). Substitution of (1) into (4) yields (24). Finally, since $\sigma = ez$, (16) implies (25).

Next, the prices in the steady state are uniquely determined by $R_t = F_Q(Q, H)$, $w_t = F_H(Q, H)$, $p_t = c'(q)F_H(Q, H)$ and $\phi_t = \frac{eqc'(q)F_H}{M_t}$ where $M_t = e\mu^t$.

We finally show (20). The government chooses e units of buyers and provides each of them one unit of money during the initial period. At time 0, the value function of the buyer with money entering the DM is

$$V_b^E(1) = -k_b + \frac{\pi_b}{w} \max_{p_0 q \le \phi_0} \left(-p_0 q + Rg(q)\right) + \frac{\phi}{w} + W_b(0, 0).$$

Given that the nominal interest rate is positive, we have $p_0 < Rg'(q)$ from (52). Thus, the constraint on money binds at time 0, implying that $q_0 = q$.

On the other hand, value function of the buyer with money who does not enter the DM is $V_b^N(1) = W_b(1, 0)$. The difference between the two value functions is

$$V_b^E(1) - W_b(1, 0) = -k_b + \frac{\pi_b}{w} \left(Rg(q) - p_0 q\right).$$

Equation (9), together with $i = \mu/\delta - 1$, implies that in the steady state

$$k_b \le -\frac{pq}{w} \left(\frac{\mu}{\delta} - 1 \right) + \frac{\pi_b}{w} \left(Rg(q) - p_0 q\right).$$

Hence we have

$$V_b^E(1) - W_b(1,0) \geq \frac{pq}{w}\left(\frac{\mu}{\delta} - 1\right) > 0.$$

Therefore, all buyers with money enter the DM. Thus, $e_0 = e$. Given above price expectations, the economy remains in the steady state where $e_t = e$ for all $t \geq 1$.

A.2 Proof of Proposition 2

We first derive (31) and (32). We have

$$q^{\theta-\psi}\frac{H}{Q} = q^{\theta-\psi}\frac{1}{Az^{1-\beta}q^{\theta}}\frac{H}{e} = \frac{z^{-\beta}(\psi-1)}{z^{1-\beta}k^{s}}\frac{(1-\alpha)k_b}{\alpha\,(1-\theta)} = z^{-1}(\psi-1)\frac{(1-\alpha)k}{\alpha\,(1-\theta)}.$$

Substitution of this equation into (26) yields (31) and (32).

We next derive H, q, e, and σ. From (27), we have $H = \frac{1-\alpha}{\alpha(1-\theta)}k_b e$. From (28), we have $q = A^{-1/\psi}(\psi-1)^{-1/\psi}(k_s)^{1/\psi}z^{\beta/\psi}$. From (29) and (30), we have $\{eF(Q/e, \frac{1-\alpha}{\alpha(1-\theta)}k_b)\}^{1-\rho} = \frac{1}{\alpha(1-\theta)}k_b e$. Thus,

$$e = \left(\frac{1}{\alpha(1-\theta)}k_b\right)^{-1/\rho}\left(\frac{Q}{e}\right)^{\eta}\left(\frac{1-\alpha}{\alpha(1-\theta)}k_b\right)^{1/\rho-1-\eta}.$$

Therefore

$$e = (1-\alpha)^{1/\rho}\left(\frac{Q}{e}\right)^{\eta}\left(\frac{1-\alpha}{\alpha(1-\theta)}k_b\right)^{-1-\eta}.$$

We have

$$Q/e = Az^{1-\beta}q^{\theta} = A^{(1-\theta/\psi)}(\psi-1)^{-\theta/\psi}(k_s)^{\theta/\psi}z^{1-\beta+\theta\beta/\psi}.$$

Hence $e = \Gamma z^{\epsilon}$ and $\sigma = ez = \Gamma z^{\epsilon+1}$ where $\epsilon = \eta(1-\beta+\beta\frac{\theta}{\psi})$ and $\Gamma = (1-\alpha)^{1/\rho}A^{(1-\theta/\psi)\eta}(\psi-1)^{-\theta\eta/\psi}(\frac{1-\alpha}{\alpha(1-\theta)})^{-\eta-1}(k_b)^{-\eta-1}(k_s)^{\theta\eta/\psi}$. Clearly, we have $\partial i(z;k)/\partial z < 0$. It is easy to show that variables $e = \Gamma z^{\epsilon}$, $\sigma = \Gamma z^{\epsilon+1}$, and q are lower for a higher i, given k.

A.3 Proof of Proposition 3

We first show the existence of the interior equilibrium in which only some of the individuals enter the market when $k_b \geq \bar{k}_b$ and $k_s \geq \bar{k}_s$. Since $\theta < 1$ and $\psi > 1$, $1-\theta/\psi > 0$. Therefore if $k_b \geq \bar{k}_b$ and $k_s \geq \bar{k}_s$, we have

$$2\bar{\Gamma}\nu^{\epsilon} \leq (k_s)^{\eta(1-\beta)(1-\theta/\psi)}(k_b)^{1+\eta\beta(1-\theta/\psi)}, \tag{53}$$

$$2\bar{\Gamma}\nu^{\epsilon+1} \leq (k_s)^{1+\eta(1-\beta)(1-\frac{\theta}{\psi})}(k_b)^{\eta\beta(1-\theta/\psi)}. \tag{54}$$

We also have $\eta - \epsilon = \eta\beta\left(1-\theta/\psi\right)$ and $\eta(1-\beta)(1-\theta/\psi) = \epsilon - \theta\eta/\psi$. Thus the inequality (53) is re-written as

$$2\bar{\Gamma}\nu^{\epsilon} \leq (k_s)^{\epsilon-\theta\eta/\psi}(k_b)^{1+\eta-\epsilon}.$$

Under the Friedman rule, $z = \nu k$. Thus we have

$$e = \Gamma z^{\epsilon} = \frac{\bar{\Gamma}\nu^{\epsilon}}{(k_s)^{\epsilon-\theta\eta/\psi}(k_b)^{1+\eta-\epsilon}} \leq \frac{1}{2}.$$

Similarly, the inequality (54) is re-written as

$$2\bar{\Gamma}\nu^{\epsilon+1} \leq (k_s)^{1+\epsilon-\theta\eta/\psi}(k_b)^{\eta-\epsilon}.$$

Under the Friedman rule, $z = \nu k$. Thus we have

$$\sigma = \Gamma(z)^{\epsilon+1} = \frac{\bar{\Gamma}\nu^{\epsilon+1}}{(k_s)^{1+\epsilon-\theta\eta/\psi}(k_b)^{\eta-\epsilon}} \leq \frac{1}{2}.$$

Therefore under the friedman rule, $e \leq 1/2$ and $\sigma \leq 1/2$ and then the interior equilibria exists. As z decreases with i and $\epsilon > 0$, both e and σ decrease with i. Thus the interior equilibria continue to exists for all i.

Next we show that if $k_b < \underline{k}_b$ and $k_s < \underline{k}_s$, the steady state in which all individual enter the DM (i.e., $e = \bar{e} = 1/2$, and $\sigma = \bar{\sigma} = 1/2$) exists if i is small. In this case, $z = 1$. The equilibrium allocation under the Friedman rule, say (q, Q, H), is determined by

$$1 = q^{\theta-\psi}\frac{H}{Q}\frac{\alpha}{1-\alpha}\frac{\theta}{\psi}, \tag{55}$$

$$Q = A\bar{e}q^{\theta}, \tag{56}$$

$$(1-\alpha)F(Q,H)^{1-\rho} = H. \tag{57}$$

It is actually an equilibrium allocation if the entry conditions hold with strict inequalities:

$$k_b < \frac{H}{\bar{e}}\frac{\alpha}{1-\alpha}\left(1-\theta\right), \tag{58}$$

$$k_s < A(\psi-1)q^{\psi}. \tag{59}$$

Equation (57) implies $(1-\alpha)Q^{-\rho}F(1,H/Q)^{1-\rho} = H/Q$. Thus

$$1-\alpha = Q^{\rho}(Q/H)^{(1-\rho)(1-\alpha)-1} = (A\bar{e}q^{\theta})^{\rho}(q^{\theta-\psi}\frac{\alpha}{1-\alpha}\frac{\theta}{\psi})^{(1-\rho)(1-\alpha)-1}.$$

This implies $q = \underline{q}$. By definition, $\underline{k}_s = A(\psi-1)\underline{q}^{\psi}$. Thus the entry conditions on the seller holds if $k_s < \underline{k}_s$. The right-hand side of the buyer's entry condition is written as

$$\frac{H}{\bar{e}}\frac{\alpha}{1-\alpha}(1-\theta) = \frac{H}{Q}\frac{Q}{\bar{e}}\frac{\alpha}{1-\alpha}(1-\theta) = A\underline{q}^{\psi}\psi\frac{1-\theta}{\theta} = \underline{k}_b$$

Thus the buyer's entry condition $k_b < \frac{H}{\bar{e}}\frac{\alpha}{1-\alpha}(1-\theta)$ holds if $k_b < \underline{k}_b$. Therefore the equilibrium in which all individuals enter the DM exists if $k_b < \underline{k}_b$, $k_s < \underline{k}_s$ and i is small.

A.4 Proof of Proposition 4

The first best allocation satisfies

$$\frac{\partial S}{\partial e} = \frac{\partial S}{\partial \sigma} = \frac{\partial S}{\partial q} = \frac{\partial S}{\partial H} = 0.$$

The first order conditions are expressed as

$$k_b = N_e(e,\sigma)\{g(q)F_Q(Q,H)U'(F(Q,H)) - c(q)\}, \tag{60}$$

$$k_s = N_{\sigma}(e,\sigma)g(q)F_Q(Q,H)U'(F(Q,H)) - N_{\sigma}(e,\sigma)c(q), \tag{61}$$

$$0 = g'(q)F_Q(Q,H)U'(F(Q,H)) - c'(q), \tag{62}$$

$$1 = F_H(Q,H)U'(F(Q,H)). \tag{63}$$

Substitution of (62) and (63) into (60) yields

$$k_b = N_e(e,\sigma)g(q)\frac{F_Q(Q,H)}{F_H(Q,H)}\left(1 - \frac{qg'(q)}{g(q)}\frac{c(q)}{qc'(q)}\right)$$

Because $N_e(e,\sigma) = \beta\pi_b(z)$ and $\frac{c(q)}{qc'(q)} = \frac{1}{\psi}$, we get (36). Similarly, substitution of (62) and (63) into (61) yields

$$k_b = N_{\sigma}(e,\sigma)c(q)\left(\frac{g(q)}{qg'(q)}\frac{qc'(q)}{c(q)} - 1\right)$$

Because $N_{\sigma}(e,\sigma) = (1-\beta)\pi_s(z)$ and $\frac{g(q)}{qg'(q)} = \frac{1}{\theta}$, we get (37). Substitution of (62) into (63) yields (38). Finally, (63) is the same as (39).

A.5 Proof of Proposition 5

If (27) holds with equality,

$$k_b = \frac{H}{e}\frac{\alpha}{1-\alpha}(1-\theta).$$

Hence $H = k_b \frac{1-\alpha}{\alpha}\frac{1}{1-\theta}e$. From (30), we can express the third term of S, $U(F(Q,H))$ as

$$U(F(Q,H)) = \frac{H}{(1-\alpha)(1-\rho)} = \frac{1}{\alpha}\frac{1}{1-\rho}\frac{1}{1-\theta}k_b e. \tag{64}$$

If (28) holds with equality, $k_s = Az^{-\beta}(\psi-1)q^{\psi} = \pi_s(\psi-1)q^{\psi}$. Since $N(e,\sigma) = \sigma\pi_s$, the fourth term of S, $Nc(q)$ can be written as

$$Nc(q) = \sigma\pi_s c(q) = \frac{\sigma}{\psi-1}k_s. \tag{65}$$

Thus, $S = -\sigma k_s - ek_b + U(F(Q,H)) - Nc(q) - H$ can be written as

$$\begin{aligned} S &= \frac{1}{1-\theta}\left(\frac{1}{\alpha}\frac{1}{1-\rho} - \frac{1-\alpha}{\alpha} - 1 + \theta\right)k_b e - \left(\frac{1}{\psi-1}+1\right)k_s\sigma \\ &= \frac{1}{1-\theta}\left(\frac{1}{\eta}+\theta\right)k_b e - \frac{\psi}{\psi-1}k_s\sigma. \end{aligned}$$

As $e = \Gamma z^{\epsilon}$ and $\sigma = \Gamma z^{\epsilon+1}$, S is rewritten as $s(z)$ in (40).

A.6 Proof of Lemma 1

$i^*(k) = A(\nu^*)^{-\beta}(\frac{\psi-1}{\psi}\frac{\theta}{1-\theta} - \nu^*)k^{1-\beta} > 0$ if and only if $\frac{\psi-1}{\psi}\frac{\theta}{1-\theta} > \nu^* = \frac{1+\eta\theta}{\eta(1-\theta)}\frac{\psi-1}{\psi}\frac{\epsilon}{\epsilon+1} = (\frac{1}{\eta}+\theta)\frac{1}{1-\theta}\frac{\psi-1}{\psi}\frac{\epsilon}{\epsilon+1}$. The inequality can be simplified as

$$\frac{1}{\eta}+\theta < \theta\frac{\epsilon+1}{\epsilon},$$

which can be further simplified as $\epsilon < \eta\theta$. Because $\epsilon = \eta\left(1-\beta+\beta\frac{\theta}{\psi}\right)$, this inequality is rewritten as (43). As long as the inequality holds, the function $i^*(k)$ is clearly an increasing function of k.

A.7 Proof of Proposition 7

First, we provide the proof of responses to changes in k_b. It is obvious from (41), z increases in k_b as $k = k_b/k_s$. Since π_b increases in z, it increases in k_b in the optimal steady state where $z = z^*(k) \propto k_b$. Similarly, π_s decreases in k_b as it decreases in z. As ψ is independent of k_b, $q = (\psi - 1)^{-1/\psi}(k_s)^{1/\psi} z^{\beta/\psi}$ increases in k_b.

The proof of e decreasing in k_b is as follows. Proposition 2 implies $e = \Gamma z^{\epsilon}$, where $\Gamma = \Gamma_1(k_b)^{-\eta-1}$, where Γ_1 is a constant that is independent of k_b. In the optimal steady state where $z \propto k_b$, e can be rewritten as

$$e = \Gamma_2(k_b)^{-\eta-1+\epsilon},$$

where Γ_2 is a constant that is independent of k_b. Since $\epsilon = \eta(1 - \beta + \beta\theta/\psi)$, we have

$$-\eta - 1 + \epsilon = -1 - \beta\eta(1 - \theta/\psi) < 0.$$

Therefore, e in the optimal steady state decreases in k_b. The seller's entry $\sigma = ez \propto (k_b)^{-\eta+\epsilon}$ also decreases in k_b, because $-\eta + \epsilon = -\beta\eta(1 - \theta/\psi) < 0$.

Second, we provide the proof of responses to changes in k_s. z decreases in k_s. Thus, π_b decreases and π_s increases with increasing k_s. As $(\psi - 1)^{-1/\psi}(k_s)^{1/\psi} \propto (k_s)^{\frac{1}{\psi}}$ and $z = z^*(k) = \nu^* k \propto (k_s)^{-1}$, we have

$$q = (\psi - 1)^{-1/\psi}(k_s)^{1/\psi}(z)^{\frac{\beta}{\psi}} \propto (k_s)^{\frac{1}{\psi}}(k_s)^{-\frac{\beta}{\psi}} = (k_s)^{\frac{1-\beta}{\psi}},$$

which implies that q increases as k_s. Since $\Gamma \propto (k_s)^{\theta\eta/\psi}$, we obtain

$$e = \Gamma z^{\epsilon} \propto (k_s)^{\eta\theta/\psi}(k_s)^{-\epsilon} = (k_s)^{-\eta(1-\theta/\psi)(1-\beta)},$$

which implies that e decreases in k_s. Similarly,

$$\sigma = \Gamma z^{\epsilon+1} \propto (k_s)^{-\eta(1-\theta/\psi)(1-\beta)-1},$$

which implies that σ decreases in k_s.

A.8 Proof of Proposition 8

The entry condition of the buyer is written as follows:

$$0 = \max_{m_{+1}} \left[-\phi m_{+1} - \delta k_b(e) + \delta \pi_{b,+1} g\left(\frac{\phi_{+1} m_{+1}}{p_{+1}}\right) + \delta(1 - \pi_{b,+1})\phi_{+1} m_{+1} \right]. \tag{66}$$

The first-order condition on m_{+1} is written as

$$i = \pi_{b,+1} \left(\frac{1}{p_{+1}} g' \left(\frac{\phi_{+1} m_{+1}}{p_{+1}} \right) - 1 \right), \tag{67}$$

where $i = \frac{\phi}{\delta \phi_{+1}} - 1$ is the nominal interest rate. Substitution of (67) into (66) yields

$$k_b(e) = \pi_{b,+1} g\left(q_{+1}\right) \left\{ 1 - \frac{q_{+1} g'\left(q_{+1}\right)}{g\left(q_{+1}\right)} \right\}. \tag{68}$$

In the steady state, (68) becomes (46). The seller's problem is close to the previous case, and the optimality conditions on q and the entry conditions are respectively written as

$$p = c'(q), \tag{69}$$

$$k_s = \pi_s(z)(-c(q) + pq). \tag{70}$$

Therefore we have (45) and (47).

A.9 Proof of Proposition 9

One can easily check that (45)-(47) can be re-expressed as follows:

$$i = Az^{1-\beta} \left(q^{\theta-\psi} \frac{\theta}{\psi} - 1 \right), \tag{71}$$

$$k_b(e) = \pi_b q^\theta (1 - \theta), \tag{72}$$

$$k_s = \pi_s q^\psi (\psi - 1). \tag{73}$$

Because $N = \sigma \pi_s = e \pi_b$, the welfare $S = -k_s \sigma - k_b e + N\{g(q) - c(q)\}$ is written as

$$S = N \left\{ -\frac{k_s}{\pi_s} - \frac{k_b(e)}{\pi_b} + q^\theta - q^\psi \right\} = \left(\theta q^\theta - \psi q^\psi \right) N.$$

Entry conditions (72) and (73) are expressed as

$$\bar{k} e^\omega = q^\theta A z^{1-\beta} (1 - \theta), \tag{74}$$

$$k_s = A z^{-\beta} (\psi - 1) q^\psi, \tag{75}$$

With some algebra, we get

$$\begin{aligned} N^{\omega} &= A^{\omega}e^{\omega}z^{(1-\beta)\omega} = \bar{k}^{-1}A^{(1+\omega)}z^{(1-\beta)(\omega+1)}q^{\theta}(1-\theta), && (76)\\ z^{\beta} &= (k_s)^{-1}A(\psi-1)q^{\psi}, && (77)\\ z^{(1-\beta)(\omega+1)} &= \left[(k_s)^{-1}A(\psi-1)q^{\psi}\right]^{(1/\beta-1)(1+\omega)}. \end{aligned}$$

Therefore

$$N^{\omega} = \bar{k}^{-1}A^{(1+\omega)}q^{\theta}(1-\theta)[(k_s)^{-1}A(\psi-1)q^{\psi}]^{(1/\beta-1)(1+\omega)}.$$

Hence $N = \xi q^{\tau}$ where $\xi = [A^{(1+\omega)/\beta}(1-\theta)(\psi-1)^{(1/\beta-1)(1+\omega)}\bar{k}^{-1}(k_s)^{-(1/\beta-1)(1+\omega)}]^{1/\omega}$ and $\tau = \psi(1/\beta-1)(1/\omega+1)+\theta/\omega$. The stationary welfare is now a function of q:

$$S = \xi\left(\theta q^{\theta+\tau} - \psi q^{\psi+\tau}\right).$$

This is maximized when $q^{\theta-\psi}\frac{\theta}{\psi} = \frac{\psi+\tau}{\theta+\tau}$. From (77), $z^{\beta} = (k_s)^{-1}A(\psi-1)(\frac{\psi+\tau}{\theta+\tau}\frac{\psi}{\theta})^{\frac{\psi}{\theta-\psi}}$. From (71), the nominal interest rate is written as

$$i = A(z^{\beta})^{(1/\beta-1)}\left(\frac{\psi+\tau}{\theta+\tau}-1\right) = A^{1/\beta}(k_s)^{1-1/\beta}(\psi-1)^{1/\beta-1}\left(\frac{\psi+\tau}{\theta+\tau}\frac{\psi}{\theta}\right)^{\frac{\psi(1/\beta-1)}{\theta-\psi}}\frac{\psi-\theta}{\theta+\tau}.$$

This is strictly positive since $\psi > 1 > \theta$. As τ is decreasing in ω by definition, the optimal interest rate i is increasing in ω iff $\partial i/\partial\tau < 0$. We examine the sign of $\partial\ln i/\partial\tau$ as $\partial i/\partial\tau < 0$ iff $\partial\ln i/\partial\tau < 0$ for any $i > 0$.

$$\begin{aligned} \frac{\partial\ln i}{\partial\tau} &= \frac{\psi(\beta^{-1}-1)}{\theta-\psi}\left[\frac{1}{\psi+\tau}-\frac{1}{\theta+\tau}\right]-\frac{1}{\theta+\tau}\\ &= \frac{\psi\beta^{-1}-2\psi-\tau}{(\theta+\tau)(\psi+\tau)}. \end{aligned}$$

Therefore, for any $\tau > 0$, the sufficient condition for $\partial\ln i/\partial\tau < 0$ is $\beta > 1/2$. This is the sufficient condition for i to be increasing in ω.

A.10 Proof of Proposition 10

The stationary welfare is determined by

$$S = -k_b e + U(F(Q,H)) - N(e,\bar{\sigma})q - H.$$

From (49), we have $H = \frac{k_b(1-\alpha)e}{\alpha(1-\theta)}$. From (51), we can express $U(F(Q,H))$ as

$$U(F(Q,H)) = \frac{H}{(1-\alpha)(1-\rho)} = \frac{1}{\alpha}\frac{1}{1-\rho}\frac{1}{1-\theta}k_b e. \tag{78}$$

Thus, S can be written as

$$S = \left(\frac{1}{\alpha}\frac{1}{1-\rho}\frac{1}{1-\theta} - 1 - \frac{1-\alpha}{\alpha(1-\theta)}\right) k_b e - Nq = \frac{1}{1-\theta}\left(\frac{\rho}{\alpha(1-\rho)} + \theta\right) k_b e - Nq.$$

Since $Q = Ng(q) = (Nq)^\theta N^{1-\theta}$, $(Nq)^{\alpha\theta} = Q^\alpha N^{-\alpha(1-\theta)} = Q^\alpha(Ae^\beta\bar{\sigma}^{1-\beta})^{-\alpha(1-\theta)}$. Moreover, since $F(Q,H)^{1-\rho} = \frac{H}{1-\alpha}$, $Q^\alpha = \frac{H^{1/(1-\rho)-(1-\alpha)}}{(1-\alpha)^{1/(1-\rho)}}$. Therefore

$$(Nq)^{\alpha\theta} = (1-\alpha)^{-1/(1-\rho)}\left[\frac{k_b(1-\alpha)}{\alpha(1-\theta)}\right]^{\rho/(1-\rho)+\alpha}(A\bar{\sigma}^{1-\beta})^{-\alpha(1-\theta)}e^{\rho/(1-\rho)+\alpha(1-\beta)(1-\theta)+\alpha\theta}$$

Hence

$$Nq = \frac{1}{(1-\alpha)^{1/\{(1-\rho)\alpha\theta\}}}\left\{k_b\frac{1-\alpha}{\alpha(1-\theta)}\right\}^{\{\rho/(1-\rho)+\alpha\}/(\alpha\theta)}(A\bar{\sigma}^{1-\beta})^{-(1-\theta)/\theta}e^\lambda,$$

where $\lambda = \frac{1}{\alpha\theta}\frac{\rho}{1-\rho} + \alpha(1-\beta)(1/\theta - 1) + 1$. The optimal level of e is determined by the first order condition

$$\lambda Nq = \frac{1}{1-\theta}\left(\frac{\rho}{\alpha(1-\rho)} + \theta\right) k_b e. \tag{79}$$

The nominal interest rate is

$$i = \pi_b\left(\frac{R}{w}g'(q) - 1\right) = \pi_b\left(\frac{\alpha}{1-\alpha}\frac{H}{Q}g'(q) - 1\right). \tag{80}$$

We have

$$\frac{\alpha}{1-\alpha}\frac{H}{Q}g'(q) = \frac{\alpha}{1-\alpha}\frac{H}{Nq}\frac{qg'(q)}{g(q)} = \frac{\alpha\theta}{1-\alpha}\frac{k_b e}{Nq}\frac{1-\alpha}{\alpha(1-\theta)} = \frac{\alpha\theta\lambda}{\rho/(1-\rho)+\alpha\theta}. \tag{81}$$

Thus $i > 0$ if and only if $\alpha\theta(\lambda - 1) > \rho/(1-\rho)$ which holds if and only if

$$\frac{\rho}{\alpha(1-\rho)} + \alpha(1-\beta)(1-\theta) > \frac{\rho}{\alpha(1-\rho)}.$$

This is always satisfied. Thus the Friedman rule is always suboptimal. From (80) and (81), the optimal nominal interest rate is written as $i = a_1\pi_b(\bar{\sigma}/e)$, where $a_1 > 0$ is independent of k_b. The rate increases with k_b if and only if $\partial e/\partial k_b < 0$. Here (79) is re-written as

$$e^{\lambda-1} = a_2(k_b)^{1-\{\rho/(1-\rho)+\alpha\}/(\alpha\theta)},$$

where $a_2 > 0$ is independent of k_b. Since $\lambda > 1$, $\partial e/\partial k_b < 0$ if and only if $\alpha\theta < \rho/(1-\rho)+\alpha$. This clearly holds since $\theta < 1$.

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