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Debt-Ridden Borrowers and Persistent Stagnation

Keiichiro Kobayashi and Daichi Shirai *

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Abstract

Persistent stagnation often follows a financial crisis. We construct a model in which a debt buildup in the corporate sector can persistently depress the economy, even when there are no changes in structural parameters. We consider endogenous borrowing constraints on short-term (intra-period) and long-term (inter-period) debt. A firm is referred to as *debt-ridden* when its long-term debt is so large that it can never decrease, even if the firm pays all income in each period to the lender. A debt-ridden firm continues inefficient production permanently, and the emergence of a substantial number of debt-ridden firms causes a persistent recession. Further, if the initial debt exceeds a certain threshold, the firm may opt to increase borrowing intentionally and, thus, may become debt-ridden. We numerically show that successive productivity shocks or a large wealth shock can generate debt-ridden firms. Relieving debt-ridden borrowers from excessive debt may be effective for economic recovery.

JEL Classification Numbers: E30, G01, G30

Keywords: borrowing constraint, debt overhang, secular stagnation, nonlinear solution methods.

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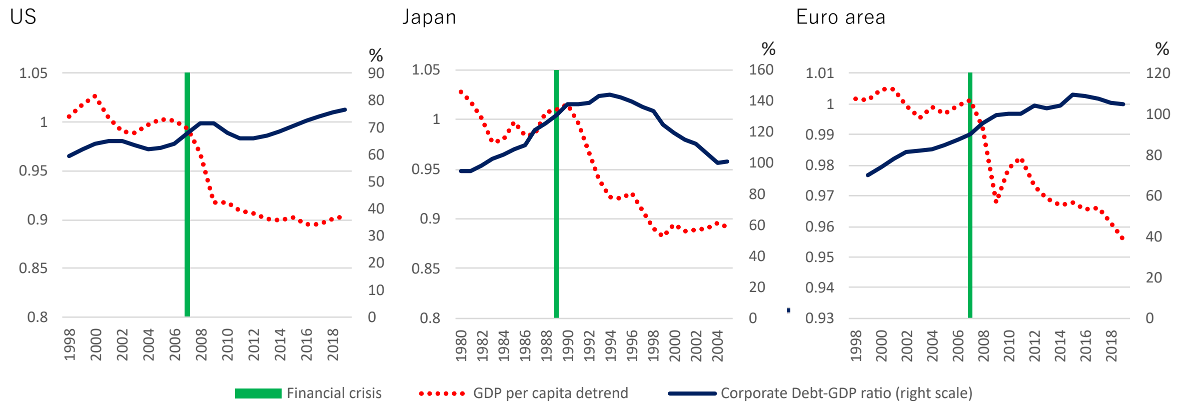
1 Introduction



Notes: The data frequency is annual. The series of GDP per capita is detrended by a linear time trend. We measured the trend from 1990 to 2007 for the US, from 1982 to 1990 for Japan, and from 1995 to 2007 for the Euro area.
Sources: BIS total credit statistics; US: OECD, *Quarterly National Accounts*; Japan: Cabinet Office, Government of Japan, *Annual Report on National Accounts*; Euro area: World bank, *World Development Indicators*.

Figure 1: Non-financial corporate debt ratio and GDP per capita

The decade after a financial crisis tends to be associated with low economic growth (Cerra and Saxena, 2008; Reinhart and Rogoff, 2009; Reinhart and Reinhart, 2010). Cerra and Saxena (2008) show that economies tend to slow economic growth for an extended period after banking and/or currency crises. Financial constraints were tightened during and after the Global Financial Crisis (GFC). See, for example, Altavilla, Darracq Paries, and Nicoletti (2015). However, the factors contributing to the tightening of these financial constraints and whether this tightening can cause a persistent slowdown in economic growth remain unclear.

The data from developed economies indicates a positive correlation between debt accumulation and the occurrence of financial crises. Figure 1 shows the buildup of corporate debt and per-capita GDP in the US, Japan, and the Euro area. Per-capita GDP is the level detrended from the pre-crisis trend. In these three developed economies, the increase in the debt ratio is followed by the financial crisis, and the buildup of debt and the lower level of GDP continues persistently in the subsequent periods.

Recent empirical studies also show that sizeable corporate debt negatively affects GDP growth (e.g., Cecchetti, Mohanty, and Zampolli, 2011; Mian, Sufi, and Verner, 2017). Giroud and Mueller (2017) find that more highly leveraged firms experienced more significant employment losses during and after the GFC in the US. Duval, Hong, and Timmer (2017) show that highly leveraged firms experienced significant and persistent drops in total factor productivity (TFP) growth in the aftermath of the GFC.

This study proposes a theoretical model in which the buildup of debt induces an endogenous tightening of the borrowing constraints and prolongs stagnation persistently in a stochastic economy where the productivity shocks or wealth shocks hit firms. Our theory demonstrates that inefficiency due to debt buildup can continue persistently, which is consistent with the debt super-cycle hypothesis (Rogoff, 2016; Lo and Rogoff, 2015). Our model also shows a theoretical possibility that the borrowers

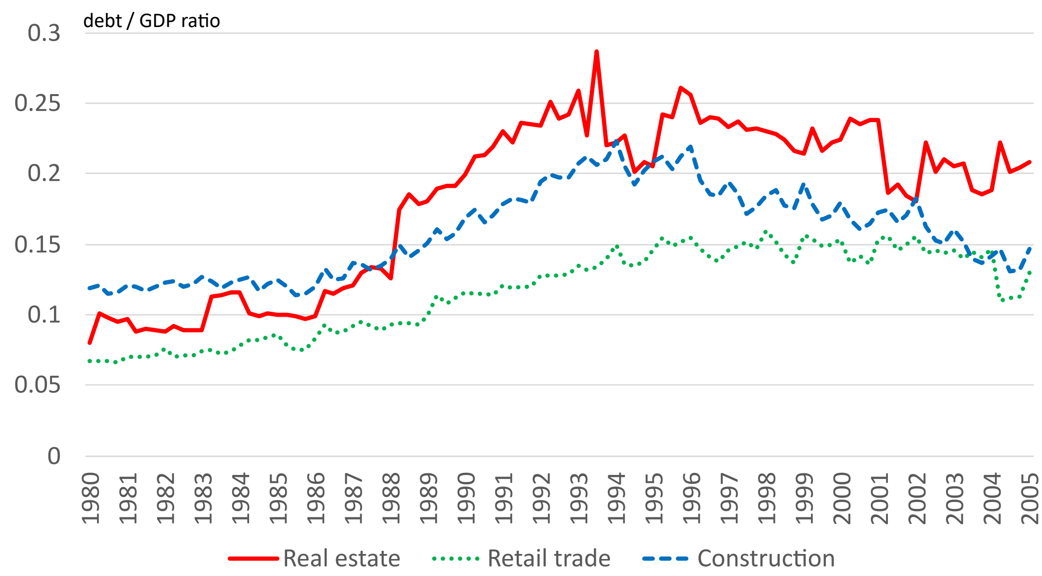


Figure 2: Debt by industry in Japan

Note: The data are seasonally adjusted by X12-ARIMA.

Sources: Ministry of Finance, Policy Research Institute, *Financial Statements Statistics of Corporations by Industry*; Cabinet Office, Government of Japan, *Annual Report on National Accounts*;

may intentionally choose to increase debt and stay debt-ridden for an extended period when the initial debt exceeds a certain threshold.

Our theory explains the behavior of Japanese corporate borrowers in the 1990s who accrued unrepayable debt and stayed debt-ridden for a decade. The cases of Sogo and Daiei, major general merchandise stores, exemplify this trend. Despite the collapse of the real estate bubble in the early 1990s, both companies significantly increased their borrowing and land purchases in the mid-1990s. Sogo eventually went bankrupt in 2000 with JPY 1,870 billion in debt, and Daiei was effectively nationalized in 2004 with JPY 1,630 billion in debt. Our theory offers insights into the debt accumulation by Japanese industries after the economic downturn in 1991. As shown in Figure 2, the real estate, retail trade, and construction sectors saw substantial debt increases in 1992-93, sustained for nearly a decade. These cases deviate from existing models, such as Khan, Senga, and Thomas (2017), which assume firms accumulate debt anticipating future good shocks. In contrast, Japanese firms knowingly increased unrepayable debt, a puzzle for the existing models.

Our study contributes to the theoretical literature by demonstrating that a buildup of debt can persistently tighten borrowing constraints and cause aggregate inefficiency that can continue indefinitely. Thus, our theory provides a rationale for government interventions to facilitate partial debt forgiveness in the private sector, which aligns with heterodox policy recommendations (see Geanakoplos, 2014). In standard models of financial friction, such as those by Carlstrom and Fuerst (1997) and Bernanke, Gertler, and Gilchrist (1999), the debt buildup generates inefficiency only for a few periods. Jermann and Quadrini (2012) and Albuquerque and Hopenhayn (2004) show in their models of long-term debt that inefficiency due to debt buildup can continue for finite periods. Our result that inefficiency can continue indefinitely thus contrasts sharply with these prior findings and suggests new causality from a financial crisis to persistent stagnation in the post-crisis period.

Our model of financial contracts has endogenous borrowing constraints that arise because of borrowers' lack of commitment. Lenders can choose whether to liquidate defaulting firms or forgive them. The market is incomplete, and debt is the only available financial instrument to raise external funds. A firm may pay dividends to the firm owner, while it cannot issue equity to external investors because of market frictions that prevent them from quickly issuing new equity. The model distinguishes between inter- and intra-period loans in this economy. If much debt is carried over from previous periods, the borrowing constraint will be tighter, making it harder to borrow more in the current period. As the borrowing constraint tightens, firms cannot raise sufficient intra-period debt for working capital, which leads to inefficient production. When inter-period debt exceeds the maximum repayable amount, firms fall into a *debt-ridden* state in which they can repay no more than the interest payments, even if they pay all of their income in each period. Consequently, the amount of debt does not decrease. Therefore, debt-ridden firms continue inefficient production permanently. Moreover, when the debt exceeds a certain threshold, a firm may choose to increase borrowing and intentionally become debt-ridden because the gain from additional borrowing can exceed the inefficiency of further tightening the borrowing constraint. This result implies that an overly indebted firm may rationally choose to become and then stay debt-ridden.

Although our model is a close variant of that of Jermann and Quadrini (2012), there is a significant difference in that the debt-ridden state arises naturally in our model. By contrast, it does not exist in Jermann and Quadrini (2012). This distinction arises because the models have different settings: In our model, a portion of output can serve as the collateral for borrowing, whereas in Jermann and Quadrini's model, it cannot.

In our model, stagnation can be permanent, even without permanent changes in structural parameters, whereas in existing models, permanent changes in parameters usually cause persistent recessions. See, for example, Christiano, Eichenbaum, and Trabandt (2015) and Bianchi, Kung, and Morales (2019) for the GFC, Cole and Ohanian (2004) for the Great Depression, and Kaihatsu and Kurozumi (2014b) for Japan's lost decade. Several authors argue that permanent shocks that cause persistent recessions are exogenous changes in the structural parameters, such as the risk shock in Christiano, Motto, and Rostagno (2014) and the financial shock in Jermann and Quadrini (2012). In this study, we consider temporary shocks to TFP, whereas there is no change in the parameters: TFP evolves with the Markov process, and the debt builds up if the low productivity continues for an extended period.

Our model proposes a unique policy recommendation that differs from most existing models, attributing persistent recessions to exogenous shocks on the structural parameters. Policymakers can only mitigate these shocks by adopting accommodative monetary and fiscal policies or designing ex-ante financial regulations. In our model, debt restructuring or forgiveness for heavily indebted borrowers improves aggregate efficiency permanently. In the existing models, such as Bernanke et al. (1999), debt forgiveness may improve the borrower's efficiency but only temporarily. In our model, restoring economic efficiency does not necessitate the physical liquidation of debt-ridden firms but rather their relief from excessive debt. This argument aligns with the policy recommendations of partial debt forgiveness by Geanakoplos (2014).

1.1 Related literature

Our theory is related to the literature on debt overhang, such as Myers (1977), Krugman (1988), and Lamont (1995). While debt overhang typically arises from a coordination failure between incumbent and new lenders, our model demonstrates that inefficiency can still occur even when incumbent lenders provide new money. Additionally, debt overhang typically causes inefficiency in the short run, whereas inefficiency can continue permanently in our study. Jungherr and Schott (2022) analyze the persistent inefficiency due to debt overhang in the economy where long-term debt exists. In their framework, the borrowing firm's default decision becomes inefficient because it considers only the cost for the buyers of newly issued bonds and neglects the cost for the existing bondholders. This externality causes high debt levels to reduce only gradually during recessions. The externality is due to the multiplicity of lenders in Jungherr and Schott (2022), while our model generates persistent stagnation despite no coordination failure among lenders. Our model is also closely related to that of Kobayashi, Nakajima, and Takahashi (2023), who analyze a version of the debt overhang effect, in which a substantial debt makes the lender lose the commitment, which in turn discourages the borrower from investing.

Our study is also closely related to the work of Caballero, Hoshi, and Kashyap (2008). They define "zombie lending" is the provision of a de facto subsidy from banks to unproductive firms, and it is argued that congestion by zombie firms hinders the entry of more productive firms and lowers aggregate productivity. In this study, we complement their argument by demonstrating that even an intrinsically productive firm can become inefficient when it is debt-ridden. Our theory offers a notably different policy implication. Caballero et al. (2008) imply that the physical liquidation of zombie firms is desirable, whereas our theory implies that zombie firms can restore high productivity if they are relieved of their excessive debt. Fukuda and Nakamura (2011) report that the majority of firms identified as zombies by Caballero et al. (2008) recovered substantially in the first half of the 2000s. This observation seems consistent with our model.

In the macroeconomic literature, endogenous borrowing constraints have been introduced by the seminal works of Kiyotaki and Moore (1997), Carlstrom and Fuerst (1997), and Bernanke et al. (1999). These works spawned a large body of literature on dynamic stochastic general equilibrium (DSGE) models with financial frictions. The borrowing constraints in an economy in which intra-period and inter-period loans exist are analyzed by Albuquerque and Hopenhayn (2004), Cooley, Marimon, and Quadrini (2004), and Jermann and Quadrini (2006, 2007, 2012). The modeling method in this study is closest to that of Jermann and Quadrini (2012). The endogenous borrowing constraint in our model can be classified as a hybrid of the asset-based borrowing constraint and the earnings-based borrowing constraint. Drechsel (2023) introduces the concept of the earnings-based borrowing constraint, which limits borrowing as a multiple of the borrower's current earnings. This serves as a proxy of the borrower's future earnings and can be seized by the lender in the future. Conversely, Drechsel (2023) argues that there is another type of borrowing constraint, the collateral constraint, which limits debt as a portion of the collateral asset, such as land. Lian and Ma (2021) show that both asset- and earnings-based borrowing are widely used in developed economies. They report that for large nonfinancial firms in the US, the share of asset-based lending is 11%, while that of earnings-based lending is 85% in the US.

Furthermore, our model is similar to that of Guerrón-Quintana and Jinnai (2014) in that a temporary shock affects economic performance persistently, although there is a significant difference in the policy implications. In our model, the emergence of debt-ridden borrowers due to negative productivity shocks causes a persistent recession. Thus, debt restructuring (i.e., wealth redistribution from lenders to borrowers) restores aggregate efficiency. By contrast, debt restructuring has no effect on the model of Guerron-Quintana and Jinnai because, in their model, the financial crisis is caused by a shock to the parameters of financial technology.

Khan et al. (2017) also quantitatively analyzes a persistent recession in the aftermath of a financial crisis. Their heterogeneous firm model conveys that a credit shock leads to persistent stagnation. This happens because the shock tightens the borrowing constraints that give rise to the persistence of low productivity due to less entry of new firms and more exits of incumbents. Ottonello and Winberry (2020) build a heterogeneous agent New Keynesian model that extends the Khan et al. (2017)'s framework. They show that monetary policy's aggregate effect depends on the default risk distribution.

Another study closely related to ours is Ikeda and Kurozumi (2014). They build a medium-scale DSGE model with financial friction à la Jermann and Quadrini (2012) and endogenous productivity growth à la Comin and Gertler (2006). Their study differs from ours in that Ikeda and Kurozumi (2014) also posit that a financial crisis is an exogenous shock on financial parameters.

These studies argue that endogenous productivity slowdowns in the endogenous growth models generate persistent recessions after financial crises. Other similar studies, such as Ates and Saffie (2021), argue that sudden stops reduce firm entries but raise the average productivity of entrants. Further, Queraltó (2020) argues that financial frictions reduce firm entries and innovations in an expanding variety model. Our model differs from these endogenous growth models in that stagnation can be persistent in the aftermath of a financial crisis, even in a model without endogenous productivity growth.

Our study is also closely related to the literature on the secular stagnation hypothesis. This hypothesis shows that a persistent stagnation can be caused by, for example, a sunspot shock under nominal rigidities (Benigno and Fornaro, 2017), a change in debt limit (Eggertsson, Mehrotra, and Robbins, 2019), and wealth inequality, wherein rich people having lower propensities to consume (Mian, Straub, and Sufi, 2021). By contrast, our hypothesis in this paper is that debt buildup due to a financial crisis can cause persistent inefficiency and lower output for years.

We solve our model using a fully nonlinear method. As our model has a kink at the switch of the borrowing constraint, we cannot solve it via standard linearization. Following Hirose and Sunakawa (2019), we adopt a Smolyak algorithm developed by Judd, Maliar, Maliar, and Valero (2014) and an index function approach to deal with occasionally binding constraints. Prior studies apply this approach mainly in analyses of the effective lower bound (ELB) of monetary policy (e.g., Hirose and Sunakawa, 2023). This study is one of the few works to use it to analyze a theme besides the ELB.

The remainder of this paper is organized as follows. In the next section, we present the model. In Section 3, we analyze the debt dynamics. In Section 4, we adjust the model for quantitative simulation. Simulation results are provided in Section 5. Section 6 presents our concluding remarks.

2 The model

Time is discrete and continues from zero to infinity: $t = 0, 1, 2, \cdots, \infty$. We consider a closed economy in which the final good is produced competitively from a variety of intermediate goods. The intermediate goods firms are monopolistic competitors, producing their varieties of intermediate goods from capital and labor inputs. The main players are intermediate goods firms, and they face a borrowing constraint that limits financing intra-period borrowing for working capital and inter-period debt. Increased debt tightens the borrowing constraint for working capital. The household is a lender and supplies labor, capital, intra-period loans for working capital, inter-period loans at market prices, and buys consumer goods from the firm. Real prices $\{w_t, r_t^K, r_t, m_t\}$ are taken as given, where w_t is the wage rate, r_t^K is the rental rate of capital, r_t is the inter-period interest rate for safe assets, and m_t is the stochastic discount factor. These prices are later determined in the general equilibrium.

2.1 Final goods firm

The final good is produced competitively from intermediate goods $y_{i,t}$, where $i \in [0,\ 1]$, by the following production function:

$$Y_t = \left(\int_0^1 y_{i,t}^{\eta} di \right)^{\frac{1}{\eta}},$$

where $0 < \eta < 1$. Since the final good producer maximizes $Y_t - \int_0^1 p_{i,t} y_{i,t} di$, where $p_{i,t}$ is the real price of intermediate good i, perfect competition in the final goods market implies that

$$p_{i,t} = p(y_{i,t}) = Y_t^{1-\eta} y_{i,t}^{\eta-1},$$

In the numerical simulations in Section 4, we divide intermediate goods firms into risky and safe firms. We consider an experiment in which risky firms are subject to a common productivity shock, while the productivity level of safe firms is time-invariant.

2.2 Intermediate goods firm

A representative household owns a mass of intermediate goods firms. Firm i produces a variety i monopolistically and can borrow funds from the household. Given the productivity $a_{i,t}$, firm i produces intermediate good i from capital $k_{i,t}$ and labor $l_{i,t}$ by the following production function:

$$y_{i,t} = a_{i,t} k_{i,t}^{\alpha} l_{i,t}^{1-\alpha}.$$

Each firm i employs labor $l_{i,t}$ and capital $k_{i,t}$, and produces intermediate goods $y_{i,t}$. The productivity shock a_t evolves stochastically. The transition of productivity is a Markov process, which is determined exogenously and is taken as given by the household and firms. Hereafter, we omit the subscript i for simplicity. The firm's gross revenue in period t is given by

$$F(A_t, k_t, l_t) = p_t y_t = A_t k_t^{\alpha\eta} l_t^{(1-\alpha)\eta},$$

where $A_t \equiv a_t^{\eta} Y_t^{1-\eta}$. Firms use debt to raise external funds, where debt is not state-contingent. They use debt to finance inputs and dividends to the firm's owners. The firm i, owned by household i, cannot borrow debt from the household i but from other households. Note that all of firm i's net worth is paid out as a dividend to the owner-household i every period, and nothing but debt is left for firm i at the end of every period. We focus on the case with initial debt stock b_{-1} at $t = 0$, where $\frac{b_{-1}}{R_{-1}}$ is the amount borrowed in the previous period, and R_t is the gross rate of corporate loans. This study assumes that firms hold inter-period debt because it offers tax advantages.[1] The firms borrow inter-period loans to pay dividends to their owners and earn tax advantage. Thus, R_t is determined by

$$R_t = 1 + (1 - \tau) r_t,$$

where τ represents the tax benefit. The inter-period debt $\frac{b_{t-1}}{R_{t-1}}$ at the end of period $t - 1$ grows at the gross rate R_{t-1} to become b_{t-1} at the beginning of period t. Specifically, the firm owes $(1 + r_{t-1})\frac{b_{t-1}}{R_{t-1}}$ to the lender. Hence, the firm has to pay this amount to the lender, whereas it obtains a transfer from the government as a tax advantage, amounting to $\tau r_{t-1}\frac{b_{t-1}}{R_{t-1}}$. Thus, the net payment by the firm is $(1 + r_{t-1})\frac{b_{t-1}}{R_{t-1}} - \tau r_{t-1}\frac{b_{t-1}}{R_{t-1}} = b_{t-1}$.

The cost of the capital and labor inputs for the firm is given by q_t:

$$q_t \geq r_t^K k_t + w_t l_t.$$

The firm needs to borrow working capital, q_t, from the bank as an intra-period loan and pay the household in advance of production as in Albuquerque and Hopenhayn (2004), Cooley, Marimon, and Quadrini (2004), and Jermann and Quadrini (2006, 2007, 2012). We define $f_t(q_t)$ by

$$f_t(q_t) = \max_{k_t,\ l_t} F(A_t, k_t, l_t),$$
$$\text{subject to} \quad r_t^K k_t + w_t l_t \leq q_t,$$

where the subscript t of $f_t(\cdot)$ represents the dependence of $f_t(\cdot)$ on A_t, r_t^K, and w_t. The solution to the above problem implies

$$f_t(q_t) = A_t \left(\frac{\alpha}{r_t^K}\right)^{\alpha\eta} \left(\frac{1-\alpha}{w_t}\right)^{(1-\alpha)\eta} q_t^{\eta}.$$

The budget constraint for the firm is given by

$$\pi_t \leq f_t(q_t) - q_t - b_{t-1} + \frac{b_t}{R_t},$$

[1] This assumption is a shortcut to formulating the motivation for holding debt.

where π_t is the dividend payment to the firm's owner. The firm's owner has no liquid assets and cannot pay any positive amount to the firm, as in Albuquerque and Hopenhayn (2004). Therefore, the dividend must be non-negative:

$$\pi_t \geq 0. \tag{1}$$

The firm cannot avoid the non-negativity constraint (1) by soliciting equity investment from outside investors because of market frictions such as a lack of commitment and asymmetric information. We do not specify the details of the market frictions in this analysis; for simplicity, we assume that the firm cannot issue new equity, even if the new money can generate a positive surplus by relaxing the borrowing constraint and even if outside investors are willing to buy new equity. Relaxing the constraint (1) to $\pi_t \geq c$, where c is a negative number, reduces the persistence of the inefficiencies in our results but does not change the results qualitatively.

The intra-period loan $q_t = w_t l_t + r_t^K k_t$ is subject to the following borrowing constraint (derived in Appendix A):

$$q_t \leq \phi f_t(q_t) + \max\left\{\xi S_t - \frac{b_t}{R_t},\ 0\right\}, \tag{2}$$

where $0 \leq \phi < 1$, $0 \leq \xi \leq 1$, and S_t is the present discounted value of future earnings of the firm, which is the value that the lender can obtain by taking control of the firm. S_t is taken as given by the lender and borrower. The equilibrium condition that S_t must satisfy is later given by (12), and ξ is the probability that the lender can successfully take control of the firm when it defaults on the debt.

The intuition of the borrowing constraint: Theoretically, we derive the borrowing constraint above from a no-default condition in Appendix A. The intuition or outline of the Appendix is given as follows. Suppose counterfactually that the firm defaults on the intra-period debt q_t. The lender immediately seizes a part of the output $\phi f_t(q_t)$. Then, the lender and borrower initiate renegotiation to reduce repayment and continue operation. If the lender takes over the firm, she will obtain the firm's value S_t with probability ξ, while she loses the claim of $\frac{b_t}{R_t}$. If the lender allows the borrower to continue operation, she can obtain the present value of $\frac{b_t}{R_t}$ next period. Thus if $\xi S_t \geq \frac{b_t}{R_t}$, the lender takes over the firm, and if $\xi S_t < \frac{b_t}{R_t}$, the lender allows the firm to continue operation. In sum, when the firm defaults on q_t, the lender obtains $\phi f_t(q_t) + \xi S_t - \frac{b_t}{R_t}$ if $\xi S_t \geq \frac{b_t}{R_t}$, and she obtains $\phi f_t(q_t)$ holding the claim of $\frac{b_t}{R_t}$ if $\xi S_t < \frac{b_t}{R_t}$. Therefore, the no default condition for q_t gives (2). See Appendix A for more details.

The nature of the borrowing constraint: Next, we argue the relevance of constraint (2). First of all, we argue that S_t is the "asset" value. Although the firm does not possess capital or labor, it employs them every period and owns the production technology. We can regard the firm as an asset that generates future earnings, and the value of this asset is S_t, as given by (12) later on. Given that S_t is the asset value, the constraint (2) can be seen as a hybrid of the asset-based borrowing constraint and the earnings-based borrowing constraint, which are widely observed in the US and other developed economies (see Drechsel, 2023; Lian and Ma, 2021). Lian and Ma (2021) report that nonfinancial firms usually use both asset-based borrowings, such as bank loans, and earnings-based borrowings, such as corporate bonds. They report that the median share of asset-based lending is 11% for large nonfinancial firms in the US, while the share of earnings-based lending is 85%. Thus, we can say that the external funds available for a firm are constrained by the sum of the collateralizable asset and the earnings. Therefore, the hybrid of asset- and earnings-based constraints is more relevant than either asset- or earnings-based borrowing constraints. In our model, the long-term debt b_t is covered by a part of the future earnings ξS_t, while the intra-period debt q_t is covered by the current period's output $\phi f_t(q_t)$ and the future periods' net earnings $\max\left\{\xi S_t - \frac{b_t}{R_t},\ 0\right\}$. We can interpret the output $f_t(q_t)$ here as receivables, constituting one type of collateralizable asset used in asset-based lending (Lian and Ma, 2021). Thus, this constraint is a hybrid of the asset- and earnings-based borrowing constraints.

Throughout this analysis, we assume that

$$\phi < \eta,$$

which means that production becomes inefficient when the borrowing constraint is

$$q_t \leq \phi f_t(q_t). \tag{3}$$

This constraint corresponds to the case where $\frac{b_t}{R_t} > \xi S_t$. If $\frac{b_t}{R_t}$ is smaller than ξS_t, then the borrowing constraint becomes

$$q_t \leq \phi f_t(q_t) + \xi S_t - \frac{b_t}{R_t}. \tag{4}$$

We define $q_{z,t}$ as the solution to $q_{z,t} = \phi f_t(q_{z,t})$.[2] Given the assumption that $\phi < \eta$, we show that $q_{z,t} < q^*$, where q^* is the first-best value that solves $f'(q) = 1$, and production is inefficient.

2.3 Optimizations by the firms (and their lenders)

The lenders in this model play a passive but crucial role. Usually, a borrowing firm can choose input and output to maximize its profit if it can repay the debt. The lender is indifferent to the borrower's action as long as the debt is fully repaid. This is what happens in normal states of the economy. On the other hand, if the debt is too large to be repaid, the lender overrides the firm in choosing the input and output to maximize the recovery of her claim. This is what happens when the borrower is debt-ridden.[3] Here, we formulate the optimization problems under a unified view that

- the lender can choose the input and output to maximize her recovery of debt but refrains from intervening when full repayment of debt is expected and
- the borrower can choose the input and output freely only when the lender refrains from intervening.

[2]Note that $q_{z,t}$ depends on t because $f_t(\cdot)$ depends on A_t, r_t^K, and w_t, as the definition of $f_t(\cdot)$ indicates.

[3]In this model, the lender does not liquidate the borrower firm nor forgive a part of the debt because she is worse off by doing so.

Thus, we view that the lender can intervene anytime but refrain from doing so only when the borrower repays the debt entirely. Under this view, we formulate the firm's problems as sub-problems included in the lender's optimization, as follows. Formal descriptions are given in Appendix B.

Lender's problem: We briefly explain the outline of the lender's optimization problem here and relegate the details to Appendix B. We denote the repayment of debt that the lender receives by ρ_t^L and the present value of the lender's payoff by $M_t(b_{t-1})$. Later, we use ρ_t as the repayment that the borrower pays net of tax advantage: $\rho_t = \rho_t^L - \tau \min\{r_{t-1}\frac{b_{t-1}}{R_{t-1}},\ f_t(q_t) - q_t\}$. The second term of the right-hand side is tax advantage from the government. Note that the maximum amount of tax benefit is $\tau \times \{f_t(q_t) - q_t\}$ when all the earnings $f_t(q_t) - q_t$ are used as an interest payment of the debt. Note that the contractual amount of the inter-period debt, b_{t-1}, is a state variable, which evolves by

$$\frac{b_t}{R_t} = (1 + r_{t-1})\frac{b_{t-1}}{R_{t-1}} - \rho_t^L.$$

where the outstanding debt at the end of period $t-1$ is $\frac{b_{t-1}}{R_{t-1}}$, which grows to $(1 + r_{t-1})\frac{b_{t-1}}{R_{t-1}}$ at the beginning of period t, which is covered by the repayment ρ_t^L and refinancing $\frac{b_t}{R_t}$.[4] Subscript t for $M_t(\cdot)$ represents the dependence on $a_{i,t}$, r_{t-1}, and other prices $\{r_t, r_t^K, w_t, m_t\}$. Since in equilibrium the prices are given as functions of the aggregate amount of capital stock K_{t-1}, the distributions of productivity $\{a_{j,t}\}_{j\in[0,1],j\neq i}$ and debt $\{b_{j,t-1}\}_{j\in[0,1],j\neq i}$, we can regard that the subscript t of $M_t(\cdot)$ indicates the dependence on $\{a_{i,t}, s_{i,t}\}$, where

$$s_{i,t} \equiv (K_{t-1}, r_{t-1}, \{a_{j,t}, b_{j,t-1}\}_{j\in[0,1],\ j\neq i}).$$

$M_t(b_{t-1})$ is given by

$$M_t(b_{t-1}) = \rho_t^L + \mathbb{E}_t[m_{t+1}M_{t+1}(b_t)],$$

where ρ_t^L can be negative, $\mathbb{E}_t$ is the expectation operator as of period t, and m_{t+1} is the stochastic discount factor given as an outcome of the household's problem and is defined later by (13). The lender maximizes $M_t(b_{t-1})$ by choosing $\{\rho_{t+j}^L\}_{j=0}^{\infty}$ under the resource constraint and other constraints, among which there is the constraint that the lender has no right to obtain repayments that exceeds the amount of the current debt. This constraint is written as follows, noting that b_{t-1} is not the amount the lender is eligible to receive ($(1 + r_{t-1})b_{t-1}/R_{t-1}$), but it is the amount the borrower pays net of tax advantage ($b_{t-1} = (1 + r_{t-1})b_{t-1}/R_{t-1} - \tau r_{t-1}b_{t-1}/R_{t-1}$):

$$M_t(b_{t-1}) \leq (1 + r_{t-1})\frac{b_{t-1}}{R_{t-1}}, \tag{5}$$

which we call the *No-Overpayment Constraint* (NOC), meaning that the lender has no right to take from the borrower more than the contractual amount $(1 + r_{t-1})b_{t-1}/R_{t-1}$. The debt, b_{t-1}, is a state variable that can grow indefinitely, while M_t is finite because the repayment ρ_t is finite due to the

[4] This constraint can be written as follows when $r_{t-1}b_{t-1}/R_{t-1} < f_t(q_t) - q_t$:

$$\frac{b_t}{R_t} = b_{t-1} - \rho_t.$$

resource constraint. For small b_{t-1}, the constraint (5) is binding, and b_{t-1} becomes a payoff-relevant state variable. On the other hand, when b_{t-1} is large enough, (5) is not binding, and b_{t-1} becomes a payoff-irrelevant state variable. We can easily show that when (5) is binding, the lender allows the borrowing firm to take action to maximize its own value as long as it can repay b_{t-1}. In this case, the firm is in the normal state. When (5) is nonbinding, then b_{t-1} is no longer payoff-relevant, and neither the lender nor borrower can make their actions contingent on the variable b_{t-1}. Under this condition, the lender takes action to maximize her payoff or makes the borrower take action to maximize the repayment to the lender. The inability to make actions state-contingent generates huge inefficiency. In this case, the borrowing firm is in the debt-ridden state. Below, we explain what happens in debt-ridden state and normal state.

Debt-ridden firm: As we indicate above, there are two states for the firm: the *normal state* and the *debt-ridden state*. In the normal state, the firm continues to repay debt and obtains positive payoffs with positive probabilities. The firm in the normal state has the option to borrow an additional amount in the current period to enter the debt-ridden state. In the debt-ridden state, the firm's debt is so large that the firm can only partially repay. Instead of fully repaying the inter-period debt, b_t, the firm pays all earnings to the lender every period. Thus, we can define the debt-ridden state as follows: *The debt-ridden state is the state where the firm continues to pay all earnings to the bank every period indefinitely, and the payoff to the firm stays at zero forever.* We define a debt-ridden firm as a firm in the debt-ridden state. A debt-ridden firm is not taken over by the lender and is allowed to continue operation independently, while all earnings are taken by the lender as debt repayment forever. Thus, the value for the firm-owner of the debt-ridden firm is zero, permanently. At the beginning of period t, the value of the debt-ridden firm that all accrues to the lender, Z_t, is given by the following.

$$Z_t = (1+\tau)(1-\phi)f_t(q_{z,t}) + E_{Z,t}, \tag{6}$$

$$\text{where} \quad E_{Z,t} \equiv \mathbb{E}_t[m_{t+1}Z_{t+1}],$$

$$q_{z,t} = \phi f_t(q_{z,t}), \tag{7}$$

where $E_{Z,t}$ is the present value of the expected earnings that the lender obtains in the future periods. $E_{Z,t}$ gives the upper limit of the inter-period debt for the normal firms $\frac{b_t}{R_t}$, and we focus in this paper on the parameter region where

$$E_{Z,t} > \xi S_t.$$

As explained in Appendix A, the borrowing constraint is (7) permanently. The intuition is the following: The renegotiation after the counterfactual default on the working capital $q_{z,t}$ results in the lender obtains $\phi f_t(q_{z,t})$ and allows the firm to continue operation because the continuation value for the lender is $E_{Z,t}$, while the liquidation (takeover) value is ξS_t, which is lower than $E_{Z,t}$. The earnings of the debt-ridden firm are given by $(1+\tau)(1-\phi)f_t(q_{z,t})$ as it includes the tax advantage.[5] Given

[5]Note that the maximum amount of tax benefit is $\tau(1-\phi)f_t(q_{z,t})$ when all the earnings $(1-\phi)f_t(q_{z,t})$ are used as an interest payment of the debt.

these results, the condition for a firm with b_{t-1} being categorized as a debt-ridden firm is

$$(1+r_{t-1})\frac{b_{t-1}}{R_{t-1}} > Z_t, \tag{8}$$

which implies that the NOC, (5), is nonbinding.

Normal firm: The firm in the normal state has a binary choice of whether to stay in the normal state or enter the debt-ridden state, whereas the firm in the debt-ridden state has no choice but to stay debt-ridden.[6] The firm may choose to become debt-ridden because there are cases where the gain from tax advantage of having a large debt is strictly larger than the marginal cost of tightening the borrowing constraint. We denote the repayment net of tax advantage by ρ_t, that is,

$$\rho_t = \rho_t^L - \tau \min\left\{ r_{t-1}\frac{b_{t-1}}{R_{t-1}},\ f_t(q_t) - q_t \right\}.$$

The net amount that the borrower pays as debt repayment is ρ_t in period t. Given that the firm was in the normal state in period $t-1$, the value of the firm $V_t(b_{t-1})$ is given by the following dynamic programming equation. See also Appendix B for the derivation. Note also that the subscript t of $V_t(\cdot)$ indicates the dependence of $V_t(\cdot)$ on $a_{i,t}$ and $s_{i,t}$, just the same as the case of $M_t(\cdot)$.

$$V_t(b_{t-1}) = \max_{\rho_t,\ q_t} \pi_t + \mathbb{E}_t\left[m_{t+1}V_{t+1}(b_t)\right], \tag{9}$$

$$\begin{aligned}
\text{subject to}\quad & \pi_t + \rho_t = f_t(q_t) - q_t, \\
& \frac{b_t}{R_t} = \tilde{b}_{t-1} - \rho_t, \\
& q_t \le \phi f_t(q_t) + \max\left\{\xi S_t - \frac{b_t}{R_t},\ 0\right\}, \\
& \pi_t \ge 0, \\
& \rho_t = \tilde{b}_{t-1} - \frac{b_t}{R_t} \ge \min\{\tilde{b}_{t-1} - E_{Z,t},\ 0\},
\end{aligned} \tag{10}$$

where

$$\tilde{b}_{t-1} \equiv \max\left\{ b_{t-1},\ (1+r_{t-1})\frac{b_{t-1}}{R_{t-1}} - \tau(f_t(q_t) - q_t) \right\}, \tag{11}$$

and the last condition (10) is the participation constraint for the lender, which says that the lender is willing to lend additional funds up to $\max\{E_{Z,t} - \tilde{b}_{t-1},\ 0\}$. The lender is willing to lend this amount because the maximum value that the lender can recover from period $t+1$ on is $E_{Z,t}$ in terms of the present value as of period t.[7] In other words, this condition gives the natural debt limit of

[6] Specifically, the debt-ridden firm has the option to exit. The value of exiting is zero, while the value of staying debt-ridden is also zero because all earnings are taken by the bank forever. We assume that the firm owner obtains nonpecuniary utility from continuing the operation so that the firm chooses to stay debt-ridden rather than to exit, even though both options give zero as the pecuniary payoffs.

[7] To be more specific, there may be some parameter values under which $E_{Z,t}$ is not the maximum value that the lender can get. But $E_{Z,t}$ is likely the maximum, and we focus on the parameter values that make $E_{Z,t}$ the maximum throughout this paper.

inter-temporal debt: $\frac{b_t}{R_t} \leq \max\{E_{Z,t},\ \tilde{b}_{t-1}\}$. If $\frac{b_t}{R_t} \geq E_{Z,t}$, condition (8) implies that the firm will be in the debt-ridden state in period $t+1$. It is numerically shown in Section 4 that there exists a threshold $B_{Z,t}$, where $B_{Z,t} < E_{Z,t}$, that satisfies the following two conditions: First, $\tilde{b}_{t-1} = b_{t-1}$ for $b_{t-1} \leq B_{Z,t}$; Second, if $b_{t-1} \in [B_{Z,t}, E_{Z,t})$, then the firm chooses to borrow $E_{Z,t} - \tilde{b}_{t-1}$ additionally to make $\frac{b_t}{R_t} = E_{Z,t}$ to go to the debt-ridden state, whereas if $b_{t-1} \in [0, B_{Z,t})$, then the firm chooses not to go to the debt-ridden state. If a normal firm intentionally falls into the debt-ridden state, it can obtain a cash flow of $(1-\phi)f_t(q_{z,t}) + E_{Z,t} - b_{t-1}$ to consume as the dividend in the current period, but will receive zero cash flows in all future periods.[8]

We denote by V_t^Z the value of a firm intentionally falling into the debt ridden state, that is, $V_t(b_{t-1})$ for $b_{t-1} \in [B_{Z,t}, E_{Z,t})$. We also denote by V_t^N the value of a firm continuing repayment, that is, $V_t(b_{t-1})$ for $b_{t-1} \in [0, B_{Z,t})$.

Seizure value of the firm: The seizure value S_t is the firm's value when the bank seizes it. Since S_t is the value of a brand-new firm without existing debt, the condition that determines S_t is

$$S_t = \max_{b_t} \mathbb{E}_t \left[m_{t+1} V_{t+1}(b_t)\right] + \frac{b_t}{R_t}. \tag{12}$$

Difference between inter- and intra-period debt: Regarding the difference between inter-period debt b_{t-1} and intra-period debt q_t, the firm can default on inter-period debt at the beginning of period t. If the firm defaults on b_{t-1} at the beginning of period t, the default value for the firm is zero because the firm possesses nothing at the beginning of a period. Therefore, the firm will default only if the continuation value of the firm is negative, $V_t < 0$. However, this outcome never occurs because the firm's dividend is non-negative ($\pi_t \geq 0$), as is the continuation value ($V_t \geq 0$). Thus, the firm never defaults on its inter-period debt b_{t-1}. The firm has the chance to default on intra-period debt q_t at the end of period t, which we analyze in the Appendix A, in which the borrowing constraint (2) for q_t is given as the no-default condition. Thus, the firm does not default on q_t in equilibrium.

Timing of events: The events in a given period t occur in the following way. The firm and bank enter period t with outstanding debt of b_{t-1}. At the beginning of the period, the firm has the chance to default on b_{t-1}, and it will do so if the continuation value is negative (which never happens). Subsequently, the firm borrows intra-period debt q_t, employs labor and capital by paying q_t, and produces output $f_t(q_t)$. The firm repays b_{t-1} and borrows new inter-period debt $\frac{b_t}{R_t}$ by paying $b_{t-1} - \frac{b_t}{R_t}$. Finally, it repays intra-period debt q_t to the bank. At this point, the firm has the chance to default on q_t. After repaying q_t, the firm pays out the remaining amount, $\pi_t = f_t(q_t) - q_t - b_{t-1} + \frac{b_t}{R_t}$, to the firm owner as a dividend.

[8] The amount of the working capital in the period when the normal firm falls into debt-ridden is $q_{z,t}$ because the borrowing constraint becomes (7) as it chooses $\frac{b_t}{R_t} = E_{Z,t} > \xi S_t$.

2.4 Household

A representative household solves the following problem: Given the state $\{K_{t-1}, D_{t-1}\}$, the transfers $\{T, \{\pi_{i,t}\}_{i\in[0,1]}\}$ and prices $\{r_t, r_t^K, w_t\}$,

$$\max_{C_t, L_t, D_t, K_t} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t U(C_t, L_t) \right],$$

subject to the budget constraint

$$C_t + K_t + \frac{D_t}{1+r_t} + T_t \leq w_t L_t + (r_t^K + 1 - \delta) K_{t-1} + D_{t-1} + \int_0^1 \pi_{i,t} di,$$

where β is the subjective discount factor, C_t is consumption, L_t is total labor supply, K_t is capital stock, δ is the depreciation rate of capital, D_t is inter-period lending to the firms, and T_t is a lump-sum tax. The period utility is

$$U(C_t, L_t) = \left[\ln C_t - \gamma_L \frac{L_t^{1+\nu}}{1+\nu} \right],$$

where $\nu > 0$ is the elasticity of labor supply, and γ_L is the coefficient of labor disutility relative to contemporaneous consumption utility.

Let m_t be $\frac{\lambda_{t+1}}{\lambda_t}$, where λ_t is the Lagrange multiplier associated with the budget constraint for the representative household. The FOC with respect to C_t implies

$$m_{t+1} = \frac{\beta^{t+1} \partial U(C_{t+1}, L_{t+1}) / \partial C_{t+1}}{\beta^t \partial U(C_t, L_t) / \partial C_t}. \tag{13}$$

The FOC with respect to K_t and D_t implies

$$\frac{1}{1+r_t} = \mathbb{E}_t \left[m_{t+1} \right],$$
$$\mathbb{E}_t \left[m_{t+1} (r_{t+1}^K + 1 - \delta) \right] = 1.$$

Thus, m_t is the stochastic discount factor.

2.5 Competitive equilibrium

The market-clearing conditions are

$$C_t + K_t - (1-\delta) K_{t-1} = Y_t,$$
$$\int_0^1 l_{i,t} di = L_t,$$
$$\int_0^1 k_{i,t} di = K_{t-1},$$
$$\int_0^1 \frac{b_{i,t}}{R_t} di = \frac{D_t}{1+r_t}.$$

The laws of motion for state variables, $b_{i,t}$ and a_{it}, are

$$\frac{b_{i,t}}{R_t} = \tilde{b}_{i,t-1} - \rho_{i,t}, \quad \text{for } i \in [0,1],$$

$$\{a_{i,t}\}_{i\in[0,1]} = \Psi(\mathbf{a}_{t-1}),$$

where $\mathbf{a}_{t-1}$ is the set of $a_{j,s}$ for all $j \in [0,1]$ and all $s \leq t-1$, and $\Psi(\cdot)$ is an exogenous and stochastic law of motion for productivity. A competitive equilibrium consists of sequences of prices $\{r_t, r_t^K, w_t, m_t\}$, a household's decisions $\{C_t, L_t, K_t, D_t\}$, firms' decisions $\{\pi_t, l_t, k_t, b_t\}$, such that *(i)* the representative household and firms solve their respective optimization problems, taking prices as given, and *(ii)* the market-clearing conditions and the laws of motion for state variables are all satisfied.

Kobayashi and Shirai (2022) proves the existence of the competitive equilibrium with certain restrictions on parameters for a partial equilibrium version of our model, in which the prices are given exogenously.

The observed TFP is defined as follows:

$$TFP_t = \frac{Y_t}{K_t^{\alpha} L_t^{1-\alpha}}.$$

In Section 5, we will see that the observed TFP declines when negative productivity shocks or a wealth shock hits the economy. In the simulation of Section 5, the observed TFP remains low even after the negative productivity shock is gone, as long as the borrowing constraint is tightened. This is because the shocks hit only a subgroup of intermediate firms and cause the misallocation of labor and capital between the firms hit by the shocks and the other firms. As it is well known in the literature (e.g., Chari, Kehoe, and McGrattan, 2007), the misallocation of inputs lowers the observed TFP.

3 Debt dynamics

In this section, we characterize the debt dynamics of our model. In our model, state variables for firm $i \in [0,1]$ are its own debt ($b_{i,t-1}$) and productivity ($a_{i,t}$), the other firms' debt ($b_{j,t-1}$ for $j \neq i$) and productivity ($a_{j,t}$), the aggregate amount of capital stock(K_{t-1}), and the interest rate r_{t-1}.[9] These state variables govern the policy functions. This section presents the policy functions and value functions as functions of debt, $b_{i,t-1}$, given that other state variables are constant. This is equivalent to assuming prices, $\{r_t, r_t^K, w_t, m_t\}$, are constant when we derive the policy functions and value functions. The severity of the borrowing constraint varies with the size of the debt, which affects the shape of policy functions. Kobayashi and Shirai (2022) provides the detailed analytical results.

Figure 3 shows the policy functions $b_t = b(b_{t-1})$ and $q_t = q(b_{t-1})$ and the value function $V_t = V(b_{t-1})$ when $a_t = 1$, whereas Figure 4 illustrates the policy function $b_t = b(b_{t-1})$ for both $a_t = 0$ and $a_t = 1$. Although the model in the previous section allows an idiosyncratic productivity shock $a_{i,t}$ with a general probability distribution, in this section, we focus on the case where productivity

[9] Although the aggregate capital K_{t-1} does not appear in the firm's optimization problem, it affects the values of the variables r_t, r_t^K, R_t, and w_t, which appear in the firm's problem. Therefore, we include K_{t-1} in the state variables of the firm's problem.

shock follows the binary distribution where a_t takes on either 0 or 1. In the simulation of Section 4, we divide the firms into two groups, i.e., safe firms and risky firms, and assume that the productivity of safe firms is always 1, i.e., $a_{n,t} = 1$. In Section 4, we will also assume that the productivity shock hits the risky firms. It is the aggregate shock: The productivity is always the same for all risky firms, and it takes on either $a_{d,t} = 0$ or $a_{d,t} = 1$, following a binary Markov process.

To characterize the debt dynamics, we divide the debt level into four levels: debt-ridden, large debt, medium-sized debt, and small debt. The horizontal axis in Figure 3 represents these levels as regions.

Debt-ridden where $\xi S_t < \frac{b_t}{R_t}$, $\pi_t = 0$, and $V_t = 0$: When the debt b_{t-1} satisfies $b_{t-1} \in [B_Z, +\infty)$, the firm intentionally borrows additional money and falls into the debt-ridden state. B_Z is determined as the minimum value of b_{t-1} that makes $b_t = E_{Z,t}$ the solution to (9). As a result of the new borrowing, the firm obtains a positive payoff when it moves to the debt-ridden state, where the payoff of becoming debt-ridden is $V_t^Z > 0$. The firm's payoff is zero forever once it falls into the debt-ridden state, that is, $V_{t+j} = 0$ for $j \geq 1$, where the firm falls into debt-ridden in period t. In Section 5.2, we will numerically confirm that firms intentionally move to the debt-ridden state when negative productivity shocks hit the economy over a long period. The inefficiency due to the tighter borrowing constraint continues indefinitely once the firm enters the debt-ridden state. The intuition of this result is the following: By increasing the amount of debt by borrowing additionally, the firm can enjoy additional tax advantage immediately, whereas reducing the debt by repaying more will improve efficiency by relaxing the borrowing constraint only in far future because the current amount of debt is too large; the firm chooses to go debt-ridden because the immediate tax advantage of additional borrowing is bigger than the future gains from debt repayment.

Large debt where $\xi S_t < \frac{b_t}{R_t}$, $\pi_t = 0$, and $V_t > 0$: There exists B_L such that for $b_{t-1} \in [B_L, B_Z]$ it is the case that $\xi S_t < \frac{b_t}{R_t}$ and $\pi_t = 0$ in equilibrium. In this region, we call b_{t-1} a large debt, and the firm is severely inefficient because the borrowing constraint is tight ($q_t \leq \phi f_t(q_t)$) and the nonnegativity constraint (1) is binding. In this region, the borrowing firm pays all earnings to the bank to reduce its debt, but the decrease is slow because the output $f_t(q_{z,t})$ and the earnings $(1-\phi)f_t(q_{z,t})$ are both small. The intuition is similar to that of Albuquerque and Hopenhayn (2004): When the debt is large, but not too large, and the borrowing constraint is tight, the efficiency gains for the firm of relaxing the borrowing constraint by repaying debt as much as possible is more significant than the gains from consuming the dividend in the current period; therefore, the optimal choice for the firm when the debt is large is to repay the debt as much as possible.[10]

Lemma 1. *When b_{t-1} is large, that is, $b_{t-1} \in [B_L,\ B_Z]$, the policy function for debt is $b_t = R_t[b_{t-1} - (1-\phi)f_t(q_{z,t})]$.*

Proof. As $\pi_t = 0$ and $q_t \leq \phi f_t(q_t)$, the budget constraint can be written as $b_t = R_t[b_{t-1} - (1-\phi)f_t(q_{z,t})]$. □

[10] In the model of Albuquerque and Hopenhayn (2004), the firms with too large debt have no option to go to debt-ridden state, but they will be just liquidated.

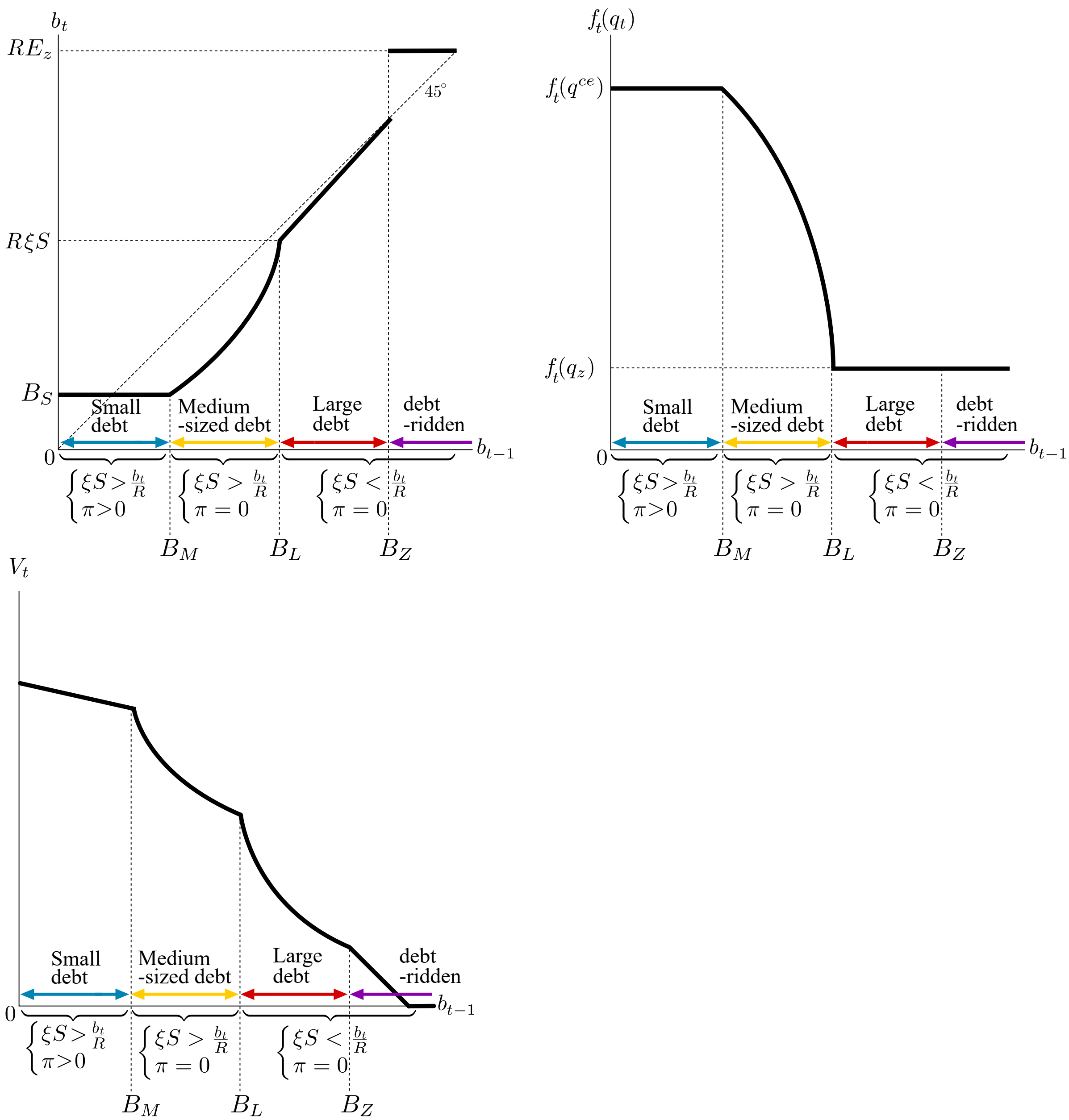


Figure 3: Policy functions and value function

This lemma implies that the speed of the decrease in debt is slow when debt is large. Because $(1-\phi)f(q_z)$ is considerably small, and R_t is larger than one, the policy function of b_t in Figure 3 is close to the 45-degree line for $b_{t-1} \in [B_L, B_Z]$. The figure indicates that the economy can suffer from extreme persistence of inefficiency if b_{t-1} falls into the region where $B_L < b_{t-1} \leq B_Z$. This mechanism is a key ingredient for persistent stagnation that appears when the economy is deeply indebted in a time of financial crisis.

Medium-sized debt where $\xi S_t > \frac{b_t}{R_t}$, $\pi_t = 0$, and $V_t > 0$: There exists B_M such that for $b_{t-1} \in [B_M, B_L]$ it is the case that $\xi S_t \geq \frac{b_t}{R_t}$ and $\pi_t = 0$ in equilibrium. Although the nonnegativity constraint (1) is binding, the borrowing constraint is not so tightly binding as for the large debt, and the production inefficiency is not as severe as for the large debt. Thus, the policy function $b_t = b(b_{t-1})$ in Figure 3 shows that debt decreases rapidly in the region where $b_{t-1} \in [B_M, B_L]$. The reason why the firm pays no dividend and reduces the debt by repaying as much amount as possible is the same as for the large debt; the immediate gains from paying out dividends are smaller than the gains from relaxing the borrowing constraint in the (near) future.

Small debt where $\xi S_t > \frac{b_t}{R_t}$, $\pi_t > 0$, and $V_t > 0$: For $b_{t-1} \in [0,\ B_M]$, it is the case that $\xi S_t \geq \frac{b_t}{R_t}$ and $\pi_t > 0$ in equilibrium. First, we define the constrained-efficient equilibrium. The constrained-efficient equilibrium is where the marginal benefit of tax advantage from an additional dividend payout equals the marginal cost of tightening the borrowing constraint. We denote the constrained-efficient production and debt by q^{ce} and B_S, respectively. For $b_{t-1} \in [0, B_M]$, the nonnegativity constraint (1) is nonbinding, and the firm attains the constrained-efficient production q^{ce} and debt B_S. The firm optimally chooses $b_t = B_S$ for all $b_{t-1} \in [0,\ B_M]$. The following conditions must be satisfied on the equilibrium:

$$\xi S_t > B_S, \tag{14}$$

$$B_S > 0. \tag{15}$$

The first condition requires that the borrowing constraint (2) must be (4) rather than (3) in the constrained-efficient equilibrium. The second condition requires that firms are not net lenders to other firms or households in the constrained-efficient equilibrium. The firm has no incentive to deviate from the constrained-efficient equilibrium because the marginal gains from consuming additional dividends in the current period are equal to the marginal losses in efficiency by tightening the borrowing constraint by increasing the debt. Unless bad shocks hit the firm and make it unable to stay, it will never deviate from the constrained-efficient equilibrium.

Can debt increase to debt-ridden level? To see that the debt increases when productivity is low, we consider the case in which productivity is zero in the low-productivity state. In this case, $q_t = f_t(q_t) = 0$. When $\xi S_t < \frac{b_t}{R_t}$, $\pi_t = 0$ and $b_t = R_t b_{t-1}$. Figure 4 compares the low and high-productivity states for the policy function of debt and shows that the debt increases in the low-productivity state (the blue dotted line). This figure shows that the debt increases gradually and

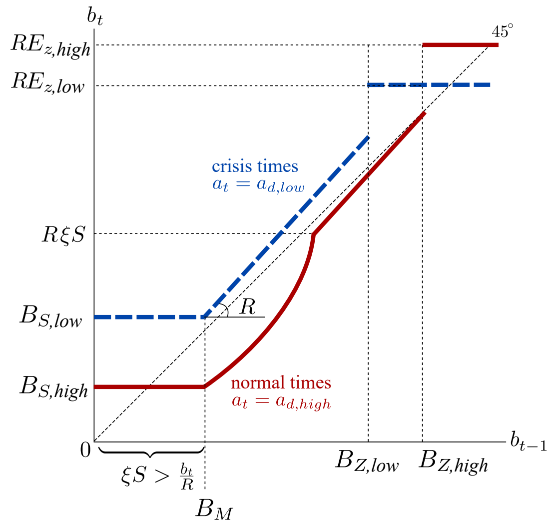


Figure 4: Policy function for debt: low and high-productivity-state

eventually reaches the debt-ridden level if the low productivity continues. As shown in the Online Appendix F, this result holds even when productivity is not zero but below a certain level in the low-productivity state.

4 Settings for numerical simulation

Following Khan and Thomas (2013) we divide intermediate goods firms into two groups: *risky firms* with measure ζ and *safe firms* with measure $1-\zeta$. The productivity level of safe firms is time-invariant at all times, $a_{n,t} = 1$, where the subscript n denotes variables associated with safe firms. By contrast, the productivity level of risky firms, $a_{d,t}$, is time-variant and follows a two-state Markov chain process, where we put the subscript d for risky firms. The market-clearing conditions are modified as follows:

$$\zeta l_{d,t} + (1-\zeta) l_{n,t} = L_t,$$
$$\zeta k_{d,t} + (1-\zeta) k_{n,t} = K_{t-1},$$
$$\zeta \frac{b_{d,t}}{R_t} + (1-\zeta)\frac{b_{n,t}}{R_t} = \frac{D_t}{1+r_t}.$$

When risky firms' productivity is low, their debt will increase as the amount repaid is smaller than the interest. When risky firms' debt $b_{d,t}$ increases to $\xi S_t < b_{d,t}/R_t$, the borrowing constraint becomes (3) and working capital financing becomes severely constrained. If the level of debt is $b_{d,t} < B_{Z,t}$ and the low state does not continue, then the debt will decrease to the optimal level over time, and the borrowing constraint will return to (4). If the low state is prolonged, debt may exceed the threshold, that is, $b_{d,t} \geq B_{Z,t}$. In this case, risky firms borrow new money $E_{Z,t}$ to increase the debt to gain the tax advantage, and risky firms become debt-ridden firms. The level of debt is too large, and paying all earnings to the bank cannot cover the interest payment. As a result, the borrowing constraint remains (3), and risky firms will remain permanently inefficient due to the severe constraint. ζ is the percentage of firms that can be trapped in the debt-ridden state.

By contrast, the debt level of safe firms is always small and never exceeds the threshold ($B_{L,t}$). Hence, the borrowing constraint for safe firms is always (4).

The detail of calibration strategy is given in Online Appendix C.

5 Simulation

The full set of equilibrium conditions is available in Online Appendix D, and the method details are described in Online Appendix E.

5.1 Temporary shocks can induce a persistent recession

In this subsection, we compare our model to a frictionless real business cycle (RBC) model to demonstrate that the financial friction in our model leads to a more persistent inefficiency following a negative productivity shock. Firstly, we construct a version of the standard RBC model with no financial frictions. The final goods firm and representative household settings are the same as in Section 2.1 and 2.4. The difference between the main model and the RBC model is that the intermediate goods firm does not face the borrowing constraints or non-negative constraint for the dividend in the RBC model. We assume that interest payments do not confer any tax advantages. Thus, the RBC model has no distinction between debt and equity. Hence, we omit debt in the RBC model. The following dynamic programming equation determines the value of the intermediate goods firm:

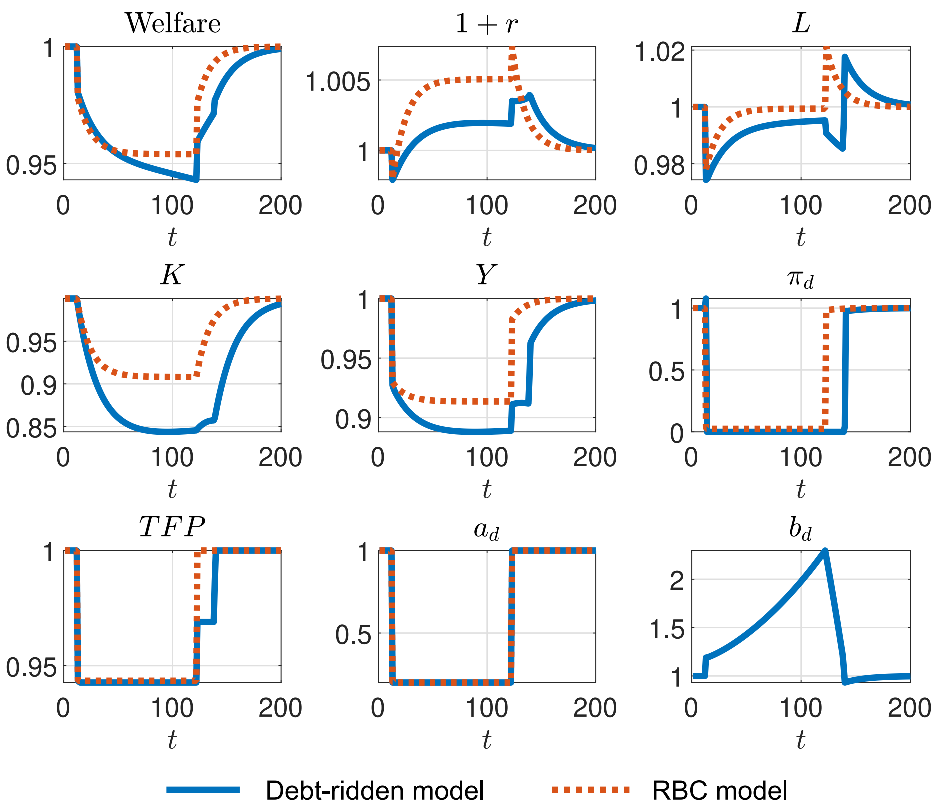
$$\begin{aligned} V_t^F &= \max \pi_t^F + \mathbb{E}_t \left[m_{t+1}^F V_{t+1}^F \right], \\ \text{subject to} \quad \pi_t^F &= f_t(q_t^F) - q_t^F, \end{aligned}$$

where superscript F denotes variables associated with the RBC model.

Figure 5 shows the responses to a negative productivity shock in the main model and the RBC model. The vertical axis is an index of deviation from the steady-state equilibrium. It is normalized to 1 when the value of the corresponding variable is equal to that in the steady-state equilibrium.

Initially, the economy was in the steady-state equilibrium with $a_{d,t} = 1$. In period 10, the low state is realized, and the productivity level of firms of the ratio ζ (i.e., risky firms) falls to zero. Since the firms' productivity falls to zero, they cannot repay the debt, and their debt increases each period in the main model.[11] In period 110, the high state is realized, and the productivity level recovers to $a_{d,t} = a_{d,high} = 1$. However, in the main model, the production of risky firms remains inefficient for some periods because the borrowing constraint is (4) due to a large increase in debt during the low state. In addition, they repay as much debt as possible by setting the dividend to zero. The decreases in the total output, the observed TFP, and the social welfare are more persistent, reflecting that inefficiencies are more prolonged than in the RBC model due to the large debt and the tighter

[11] Online Appendix F shows that the debt accumulation also occurs even if $a_{d,low} > 0$.



Notes: Notes: The vertical axis represents X_t/X as the deviation from the steady-state equilibrium, where X is the steady-state equilibrium value of a corresponding variable.
π_d: Risky firm's dividend, f_d: Risky firm's revenue (production), a_d: Risky firm's productivity, b_d: Risky firm's debt

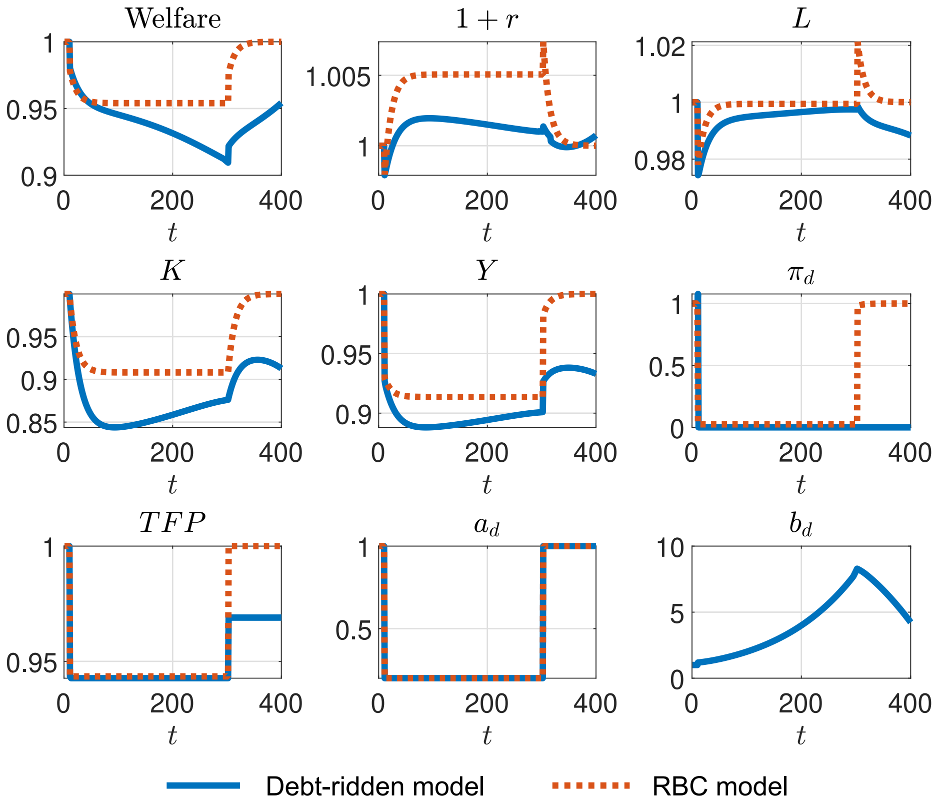
Figure 5: RBC model versus the debt-ridden model

borrowing constraint. Misallocations of labor and capital between the risky and safe firms cause a decrease in the observed TFP.

Thus, the debt accumulation during the low state results in the tighter borrowing constraint (3) and a long-lasting inefficient production. The greater the amount of debt accumulated, the more inefficient production continues.

5.2 Prolonged shocks lead to the debt-ridden state

In this subsection, we show that firms become debt-ridden endogenously when the low state persists for a considerably long period. In our model, the low productivity state is modeled as the financial crisis, and the transition probabilities are calibrated to the duration of the financial crisis. Figure 6 shows the simulation results when the low state is prolonged. From the 10th period to the 259th period, the productivity a_d is low, and the firm's output remains zero. The debt continues to increase at the rate of interest because the debt cannot be repaid at all. As the level of debt increases, the value of V^Z comes closer to the value of V^N. Eventually, V^Z exceeds V^N in the 226th period, and the risky



Notes: The vertical axis represents X_t/X as the deviation from the steady-state equilibrium, where X is the steady-state equilibrium value of a corresponding variable.
π_d: Risky firm's dividend, f_d: Risky firm's revenue (production), a_d: Risky firm's productivity, b_d: Risky firm's debt

Figure 6: Long-term depression and debt-ridden firms

firms increase their borrowing in the 226th period and become debt-ridden firms from then on. Even if productivity returns to the high state in the 260th period, macro variables do not return to their pre-crisis levels because the production of debt-ridden firms remains inefficient permanently. Thus, the total output, the observed TFP, and the social welfare all stay lower than the pre-crisis levels. Note that in Figure 6, we show Z_t defined in (6) as the debt (b_d) after the firm becomes debt-ridden.[12]

The figure suggests that this situation is one explanation for the failure of GDP to return to its pre-crisis trend in many countries since the GFC. Recent empirical studies have shown evidence of a downturn due to corporate debt (e.g., Cecchetti, Mohanty, and Zampolli, 2011; Mian, Sufi, and Verner, 2017), and our model provides one mechanism that explains the recession due to the buildup of corporate debt.

The figure shows that firms become debt-ridden only after they experience more than 200 periods (50 years) of low productivity. This requirement may seem unrealistic because the required period of low productivity is too long. The requirement of an unrealistically long duration of low productivity

[12]The value of $b_{z,t}$ is not the face value of debt but the present discounted value of the payoff for the lender in the debt-ridden state. See the arguments about b_z in Section 2.2

is an artifact due to our simplifying assumption that the productivity shock is the only exogenous shock to the economy. As in many DSGE models, introducing various shocks may shorten the period required to make the risky firms debt-ridden. For example, a buildup of debt due to a collapse of an asset-price bubble would induce $V^N < V^Z$ and make the firms debt-ridden immediately, and a significantly large decrease in capital stock due to a capital quality shock would also have the same result.

In addition to these results, we conduct an analysis of impulse response to the TFP shock in Online Appendix G, and that of persistent response to the wealth shock in Online Appendix H.

6 Conclusion

Persistent stagnation in the aftermath of financial crises is often observed and requires a convincing theoretical explanation and sensible policy recommendations. This study examines debt dynamics under borrowing constraints, distinguishing between short-term (intra-period) and long-term (inter-period) borrowing. This distinction reveals how increasing long-term debt tightens the borrowing constraint for short-term debt and perpetuates inefficiency. Our model shows that when long-term debt exceeds a certain threshold, borrowers intentionally accumulate new debt, leading to a debt-ridden state where inefficiency persists indefinitely.

Our numerical simulations indicate that a succession of many negative productivity shocks can drive borrowers to become debt-ridden, resulting in prolonged economic stagnation. This suggests that temporary shocks can lead to intentional debt accumulation, causing persistent stagnation often seen after financial crises.

The policy implications of our theory are straightforward: debt restructuring or forgiveness for heavily indebted borrowers after a financial crisis may help escape persistent stagnation. This is crucial since no technological or structural changes drive such stagnation; debt accumulation alone can cause prolonged recessions.

Our theory emphasizes corporate debt buildup and restructuring, potentially increasing government debt through intervention. Future research should integrate private and public debt analysis, highlighting the importance of debt restructuring as a recovery policy in the aftermath of financial crises.

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For Online Publication: Appendices

A Derivation of the borrowing constraint

Here, we describe the events that follow a counterfactual default on q_t and derive the borrowing constraint (2) as the no-default condition. Our argument is similar to that of Jermann and Quadrini (2012).

As described in the text, the firm owes an inter-period debt $\frac{b_t}{R_t}$ and intra-period debt q_t at the end of period t, where b_t is to be repaid in period $t+1$ and q_t is to be repaid in period t. At the end of period t, the firm has the chance to default on q_t.

Now, we consider what would happen if the firm defaults on q_t. Once the firm defaults, the bank unilaterally seizes a part of the firm's revenue, $\phi f_t(q_t)$, where $0 \leq \phi < 1$.[13] The amount of seizures, $\phi f_t(q_t)$, may be interpreted ex-ante as collateral that the bank can legitimately seize when the firm defaults. After the seizure, the firm and bank renegotiate the conditions for the firm to continue to operate. Following Jermann and Quadrini (2012), we assume the firm has all the bargaining power in the renegotiation. The bank has acquired the right to seize the firm at this stage. Here, the seizure of the firm means that the bank takes control of the firm. Recall that S_t defined by (12) is the present discounted value of a firm's future earnings. When the bank chooses seizure, it successfully operates the firm by itself and recovers value S_t with probability ξ, whereas the firm is destroyed with probability $1-\xi$. When the firm is destroyed, the bank obtains nothing. Thus, the expected value that the bank can obtain by liquidation is ξS_t. By contrast, if the bank decides to allow the firm to continue to operate, it can recover its inter-period debt in the next period, the present value of which is $\frac{b_t}{R_t}$. The renegotiation agreement depends on whether ξS_t is larger or smaller than $\frac{b_t}{R_t}$.

A.1 Normal state

First, we consider the case where the firm is in a normal state; that is, the firm is not debt-ridden. In this case, $\frac{b_t}{R_t}$ is smaller than $E_{Z,t}$, defined by (6), and thus it is feasible to repay $\frac{b_t}{R_t}$ fully.

- **Case where $\xi S_t > \frac{b_t}{R_t}$:** The firm has to make a payment that leaves the bank indifferent between liquidation and allowing the firm to continue to operate. Thus, the firm has to make payment $\xi S_t - \frac{b_t}{R_t}$ and promise to pay $(1+r_t)\frac{b_t}{R_t}$ at the beginning of the next period. Therefore, the ex-post default value of the firm is

$$(1-\phi)f_t(q_t) - b_{t-1} + \frac{b_t}{R_t} - \left\{\xi S_t - \frac{b_t}{R_t}\right\} + \mathbb{E}_t\left[m_{t+1}V_{t+1}^N\right].$$

[13]Because the firm has paid $b_{t-1} - \frac{b_t}{R_t}$, the remaining value of the resources it possesses is $f_t(q_t) - b_{t-1} + \frac{b_t}{R_t}$ after defaulting on q_t. Thus, if the bank were to seize $\phi f(q)$ from the remaining output only, then the seizure should have been feasible only if

$$\phi f_t(q_t) \leq f_t(q_t) - b_{t-1} + \frac{b_t}{R_t}. \tag{16}$$

However, we assume for simplicity of the analysis that the bank can take $\phi f_t(q_t)$ from the firm owner's pocket and not just from the remaining output of the firm.

Thus, here, we assume that the bank seizure is not constrained by (16).

- **Case where $\xi S_t \le \frac{b_t}{R_t}$:** In this case, the optimal choice for the bank is to wait until the next period when $(1+r_t)\frac{b_t}{R_t}$ is due. In period t, the bank receives no further payments. Thus, the ex-post default value of the firm is

$$(1-\phi)f(q_t) - b_{t-1} + \frac{b_t}{R_t} + \mathbb{E}_t\left[m_{t+1}V_{t+1}^N\right].$$

Therefore, the default value is

$$(1-\phi)f_t(q_t) - b_{t-1} + \frac{b_t}{R_t} - \max\left\{\xi S_t - \frac{b_t}{R_t},\ 0\right\} + \mathbb{E}_t\left[m_{t+1}V_{t+1}^N\right].$$

Enforcement requires that the value of not defaulting is no smaller than the value of default; that is,

$$f_t(q_t) - b_{t-1} + \frac{b_t}{R_t} - q_t \ge (1-\phi)f_t(q_t) - b_{t-1} + \frac{b_t}{R_t} - \max\left\{\xi S_t - \frac{b_t}{R_t},\ 0\right\},$$

which can be rearranged as (2).

A.2 Debt-ridden state

In the debt-ridden state, the value the bank can expect to obtain by waiting until the next period is $E_{Z,t}$, larger than ξS_t. Therefore, the optimal choice for the bank in response to the firm's default on q_t is to allow the firm to continue and wait until the next period. Noting that the value for the firm in the debt-ridden state is zero, the ex-post default value for the firm is

$$(1-\phi)f_t(q_t) - b_{t-1} + \frac{b_t}{R_t} + \mathbb{E}_t[m_{t+1}\times 0].$$

Enforcement implies that

$$f_t(q_t) - b_{t-1} + \frac{b_t}{R_t} - q_t + \mathbb{E}_t[m_{t+1}\times 0] \ge (1-\phi)f_t(q_t) - b_{t-1} + \frac{b_t}{R_t} + \mathbb{E}_t[m_{t+1}\times 0],$$

which can be rearranged to $q_t \le \phi f_t(q_t)$.

B Lender's Optimization Problem

In this Appendix, we state the unified optimization problem for the lender and borrower, which reduces to the optimization problem for the normal firm or the debt-ridden firm, depending on whether the *no-overpayment constraint* (NOC), which is defined shortly, is binding or not.

First, we summarize the evolution of debt. The outstanding amount of debt at the end of period $t-1$ is $\frac{b_{t-1}}{R_{t-1}}$, which grows to $(1+r_{t-1})\frac{b_{t-1}}{R_{t-1}}$ at the beginning of period t, which is covered by the repayment and refinancing ($\frac{b_t}{R_t}$). As the borrower obtains tax benefit $\tau \times \min\{r_{t-1}\frac{b_{t-1}}{R_{t-1}}, f_t(q_t)-q_t\}$, the net liability for the borrower at the beginning of period t is

$$(1+r_{t-1})\frac{b_{t-1}}{R_{t-1}} - \tau\min\left\{r_{t-1}\frac{b_{t-1}}{R_{t-1}}, f_t(q_t)-q_t\right\} = \tilde{b}_{t-1},$$

where $\tilde{b}_{t-1}$ is defined by (11). The lender maximizes her payoff $M_t(b_{t-1})$ by solving the following dynamic problem:

$$
\begin{aligned}
M_t(b_{t-1}) = &\max_{\rho_t^L, q_t} \rho_t^L + \mathbb{E}_t[m_{t+1}M_{t+1}(b_t)], && (17)\\
\text{s.t. } & M_t(b_{t-1}) \leq (1+r_{t-1})\frac{b_{t-1}}{R_{t-1}}, && (18)\\
& \pi_t + \rho_t^L \leq f_t(q_t) - q_t + \tau \min\left\{ f_t(q_t) - q_t,\ \frac{r_{t-1}b_{t-1}}{R_{t-1}} \right\}, && (19)\\
& q_t \leq \phi f_t(q_t) + \max\left\{\xi S_t - \mathbb{E}_t[m_{t+1}M_{t+1}(b_t)],\ 0\right\}, && (20)\\
& \pi_t \geq 0,\\
& \frac{b_t}{R_t} = (1+r_{t-1})\frac{b_{t-1}}{R_{t-1}} - \rho_t^L,
\end{aligned}
$$

where ρ_t^L is the debt repayment the lender receives in period t. The first constraint (18) is the NOC which says that the present value of the repayments $\{\rho_{t+j}^L\}_{j=0}^{\infty}$ must be no greater than the contractual value of debt, $(1+r_{t-1})b_{t-1}/R_{t-1}$. In other words, this constraint says that the lender has no right to take more than the pre-determined amount of debt $(1+r_{t-1})b_{t-1}/R_{t-1}$ from the borrower. The second constraint (19) is the budget constraint that says the dividend to the firm-owner (π_t) and the repayment to the lender (ρ_t^L) must be paid out from the output ($f_t(q_t) - q_t$) plus the tax benefit from the government, which cannot exceed $\tau(f_t(q_t) - q_t)$. The third constraint (20) is the borrowing constraint that we derived in Appendix A, while we replace $\frac{b_t}{R_t}$ in (2) to $\mathbb{E}_t[m_{t+1}M_{t+1}(b_t)]$. This constraint (20) is derived by a similar argument as Appendix A.

We denote the set of solutions (ρ_t^L, q_t) to the above problem by $\Omega_t(b_{t-1})$. Thus, all $(\rho_t^L, q_t) \in \Omega_t(b_{t-1})$ are the solutions to (17). Given $\Omega_t(b_{t-1})$, the borrowing firm's problem is

$$
\begin{aligned}
V_t(b_{t-1}) = &\max_{(\rho_t^L, q_t)\in\Omega_t(b_{t-1})} \pi_t + \mathbb{E}_t[m_{t+1}V_{t+1}(b_t)], && (21)\\
\text{s.t. } & \frac{b_t}{R_t} = (1+r_{t-1})\frac{b_{t-1}}{R_{t-1}} - \rho_t^L.
\end{aligned}
$$

We can divide this problem into two cases where the NOC, $M_t(b_{t-1}) \leq (1+r_{t-1})b_{t-1}/R_{t-1}$, is binding, and where it is nonbinding.

Case where NOC is nonbinding: We consider the case where b_{t-1} is too large so that the NOC is nonbinding, i.e., $M_t(b_{t-1}) < (1+r_{t-1})b_{t-1}/R_{t-1}$. In this case, we guess and verify later that

$$
\begin{aligned}
&\min\left\{ f_t(q_t) - q_t,\ \frac{r_{t-1}b_{t-1}}{R_{t-1}} \right\} = f_t(q_t) - q_t,\\
&\max\left\{\xi S_t - \mathbb{E}_t[m_{t+1}M_{t+1}(b_t)],\ 0\right\} = 0.
\end{aligned}
$$

Given this guess, it is obvious that the state variable b_{t-1} becomes payoff-irrelevant and that the lender's problem (17) is reduced to (6), and $M_t(b_{t-1}) = Z_t$ and $(\rho_t^L, q_t) = ((1+\tau)(1-\phi)f_t(q_{z,t}), q_{z,t})$, where Z_t does not depend on b_{t-1}. Since b_{t+j} is increasing in j where $j = 0, 1, 2, \cdots$, and $\xi S_t < E_{Z,t}$, the above guess can be verified, once b_{t-1} is bigger than $\frac{R_{t-1}}{r_{t-1}}(1-\phi)f_t(q_{z,t})$.

Parameters	Values	Description	Source or Target
α	0.3	Cobb–Douglas production function	
β	0.99	Subjective discount factor	
δ	0.025	Depreciation rate	
η	0.7	Intermediate goods elasticity, $1/(1-\eta)$	
ν	1	Labor supply elasticity	Ikeda and Kurozumi (2019)
γ_L	5.039	Labor disutility weight	Steady state labor supply $\overline{L} = 1/3$
ϕ	0.3577	Collateral ratio of revenue	average debt/GDP ratio, NIPA
ξ	0.065	Collateral ratio of foreclosure value	average debt/GDP ratio, NIPA
ζ	0.13	Debt-ridden firms ratio	Banerjee and Hofmann (2018)
τ	0.35	Corporate tax rate	Jermann and Quadrini (2012)

Table 1: Calibrated parameters

Case where NOC is binding: Next, we consider the case where b_{t-1} is not too large so that the NOC is binding, i.e., $M_t(b_{t-1}) = (1 + r_{t-1})\frac{b_{t-1}}{R_{t-1}}$. In this case, the lender is indifferent to the choice of (ρ_t^L, q_t) as long as the contractual value of debt $(1 + r_{t-1})\frac{b_{t-1}}{R_{t-1}}$ is repaid eventually. Thus, it is evident that the firm's problem (21) is reduced to the normal firm's problem (9).

C Calibration

The parameter values are set as shown in Table 1. We set the capital share in the Cobb–Douglas production function at $\alpha = 0.3$, the subjective discount factor at $\beta = 0.99$, and the depreciation rate at $\delta = 0.025$, as the economy is modeled at quarterly frequencies. The parameter for the elasticity of substitution is set at $\eta = 0.7$, which is a standard value as most studies set at the value between [0.6 0.9]. The elasticity of the labor supply is set at $\nu = 1$, following the literature. The coefficient of labor disutility relative to contemporaneous consumption utility $\gamma_L = 5.039$ is chosen to make a steady-state labor supply 1/3. These are the standard settings used in prior studies. The collateral ratio of foreclosure value ξ and of revenue ϕ are calibrated for the US economy. We set $\xi = 0.065$ as the working capital borrowing is usually nearly 7% of the corporate value in reality (see, e.g., Galindo, 2021).[14] Subsequently, ϕ is chosen to have a steady-state ratio of total debt over value-added equal to 1.648. This value is the average ratio over the period 1984:I-2017:IV for liability of the non-financial corporate business from the Board of Governors of the Federal Reserve System, *Financial Accounts of the United States* and the Bureau of Economic Analysis, *NIPA Tables*. The required value is $\phi = 0.3577$.

Following Jermann and Quadrini (2012), the mean tax rate is set to $\tau = 0.35$. The borrowing constraint is always binding because the firm borrows inter-period debt to exploit the tax advantage.

The ratio of risky firms is set to $\zeta = 0.13$, estimated as the average zombie firm ratio of 14 advanced

[14] In addition, ϕ and ξ are chosen to satisfy the equilibrium conditions, that is, (14) and (15), and fit the data. The value ξ that can satisfy these conditions is limited to a narrow range of [0.0540, 0.0763] and is set to 0.065.

countries in 2016 by Banerjee and Hofmann (2018). We use the zombie firms as a proxy of the debt-ridden firms in our model, and as we see later in the simulation, risky firms become debt-ridden firms when low productivity shocks hit them. Thus, we can approximate the ratio of risky firms to that of zombie firms.[15]

The economy evolves through changes in the productivity of risky firms. The productivity shocks, $a_{d,t}$, follow the two-state Markov chain, with realizations $\{a_{d,high}, a_{d,low}\}$ and transition matrix:

$$\begin{bmatrix} p_{high} & 1-p_{high} \\ 1-p_{low} & p_{low} \end{bmatrix}.$$

The realization $a_{d,high}$ corresponds to the productivity level during normal times in business cycles, and $a_{d,low}$ corresponds to low productivity level (financial crises). p_{high} is the probability of continuing normal time, $Pr[a_{t+1} = a_{d,high}|a_t = a_{d,high}]$, while $1 - p_{low}$ is the probability of escape from crisis conditions, $Pr[a_{t+1} = a_{d,high}|a_t = a_{d,low}]$. Previous empirical studies have shown that sizeable negative productivity shocks are one of the leading causes of financial crises. Accordingly, we set $p_{high} = 0.9941$ and $p_{low} = 0.9219$ so that the average duration of the financial crisis in our quarterly model is 12.8 quarters, and the economy spends 7 percent of the time in the crisis state. These numbers are the facts about financial crises summarized by Reinhart and Rogoff (2009) as well as Khan and Thomas (2013). Note that unconditional probability distributions evolve according to

$$[Pr(a_{t+1} = a_{d,high}), \quad Pr(a_{t+1} = a_{d,low})] = [Pr(a_t = a_{d,high}), \quad Pr(a_t = a_{d,low})] \begin{bmatrix} p_{high} & 1-p_{high} \\ 1-p_{low} & p_{low} \end{bmatrix}.$$

Concerning the realizations, $a_{d,high}$ and $a_{d,low}$, we calibrate our model to capture the aggregate inefficiency of the financial crises in the US economy. We set $a_{d,high} = 1$ and $a_{d,low} = 0$ using evidence on banking crises from Reinhart and Rogoff (2014). They show that the average peak-to-trough decline for the US real per capita GDP across nine major financial crises is about 9 percent, which is replicated when $a_{d,t}$ changes from $a_{d,high} = 1$ to $a_{d,low} = 0$. In Online Appendix F, we perform the simulation using an alternative setting for $a_{d,low}$ greater than zero and show that the main results are robust.

D Equilibrium Conditions

This Appendix lists the complete set of equilibrium conditions for the model.

[15]There may be a slight difference between zombie firms and debt-ridden firms. The zombie firms include the firms that are intrinsically unproductive but kept afloat by banks, whereas the debt-ridden firms are intrinsically productive, but the debt burden makes them inefficient.

However, we could argue that most zombie firms are debt-ridden because, in Japan, a substantial proportion of the firms classified as "zombie firms" in the 1990s recovered to become non-zombie firms in 2000 (see Fukuda and Nakamura, 2011). This observation implies that most zombie firms are intrinsically productive and debt-ridden.

D.1 Household optimality conditions

The optimality conditions for the household problem described in subsection 2.4 are

$$w_t = \gamma_L C_t L_t^{\nu},$$

$$\frac{1}{C_t} = \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} (1 - \delta + r_{t+1}^K) \right], \tag{22}$$

$$\frac{1}{(1 + r_t)} = \mathbb{E}_t [m_{t+1}], \tag{23}$$

where $m_{t+1} \equiv \beta \frac{C_t}{C_{t+1}}$.

D.2 Intermediate goods firms optimality conditions

The conditions for the intermediate goods firm problem described in subsection 2.2 are

$$V_{i,t} = \max \left\{ V_{i,t}^N, V_{i,t}^Z \right\},$$

$$V_{i,t}^N = \max \ \pi_{i,t} + \mathbb{E}_t \left[m_{t+1} V_{i,t+1} \right], \tag{24}$$

$$\pi_{i,t} = f_t(q_{i,t}) - q_{i,t} - b_{i,t-1} + \frac{b_{i,t}}{R_t},$$

$$q_{i,t} \leq \phi f_t(q_{i,t}) + \max \left\{ \xi S_t - \frac{b_{i,t}}{R_t},\ 0 \right\},$$

$$\pi_{i,t} \geq 0,$$

$$\begin{cases} \text{if } \xi S_t - \dfrac{b_{i,t}}{R_t} > 0, & \dfrac{1}{R_t} \left(1 - \dfrac{\mu_{i,t}}{\lambda_{i,t}} \right) = \mathbb{E}_t \left[m_{t+1} \dfrac{\lambda_{i,t+1}}{\lambda_{i,t}} \right], \\ \text{if } \xi S_t - \dfrac{b_{i,t}}{R_t} \leq 0, & \dfrac{1}{R_t} = \mathbb{E}_t \left[m_{t+1} \dfrac{\lambda_{i,t+1}}{\lambda_{i,t}} \right], \end{cases} \tag{25}$$

$$1 + \lambda_{\pi_{i,t}} - \lambda_{i,t} = 0,$$

$$f_t(q_{i,t}) = A_{i,t} \left(\frac{\alpha}{r_t^K} \right)^{\alpha \eta} \left(\frac{1 - \alpha}{w_t} \right)^{(1-\alpha)\eta} q_{i,t}^{\eta}.$$

$$r_t^K = \alpha \eta \frac{1 + \phi \frac{\mu_{i,t}}{\lambda_{i,t}}}{1 + \frac{\mu_{i,t}}{\lambda_{i,t}}} \frac{f_t(q_{i,t})}{k_{i,t}},$$

$$w_t = (1 - \alpha) \eta \frac{1 + \phi \frac{\mu_{i,t}}{\lambda_{i,t}}}{1 + \frac{\mu_{i,t}}{\lambda_{i,t}}} \frac{f_t(q_{i,t})}{l_{i,t}},$$

$$\begin{cases} \pi_{i,t} > 0, & \lambda_{\pi,t} = 0, \\ \pi_{i,t} = 0, & \lambda_{\pi,t} > 0, \end{cases}$$

$$y_{i,t} = tfp_t \cdot a_{i,t} k_{i,t}^{\alpha} l_{i,t}^{1-\alpha},$$

$$R_t = 1 + (1 - \tau) r_t,$$

$$S_t = \mathbb{E}_t \left[m_{t+1} V_{n,t+1} \right] + \frac{b_{n,t}}{R_t},$$

where $i \in \{n, d\}$.

$$
\begin{aligned}
&V_t^Z = \max f_t(q_{z,t}) - q_{z,t} - b_{t-1} + E_{Z,t}, \\
&Z_t = (1+\tau)(1-\phi) f_t(q_{z,t}) + E_{Z,t}, \\
&q_{z,t} = \phi f_t(q_{z,t}), \\
&k_{z,t} = \left[\phi A_{d,t} \left(\frac{r_t^K}{\alpha} \right)^{(1-\alpha)\eta - 1} \left(\frac{1-\alpha}{w_t} \right)^{(1-\alpha)\eta} \right]^{\frac{1}{1-\eta}}, \\
&l_{z,t} = \frac{(1-\alpha) r_t^K k_{z,t}}{\alpha w_t}.
\end{aligned} \tag{26}
$$

D.3 Final goods firm optimality conditions

The conditions for the final goods firm problem are described in subsection 2.1 are

$$
\begin{aligned}
Y_t &= \left[\zeta y_{d,t}^{\eta} + (1-\zeta) y_{n,t}^{\eta} \right]^{\frac{1}{\eta}}, \\
A_{i,t} &= (tfp_t \cdot a_{i,t})^{\eta} Y_t^{1-\eta}.
\end{aligned}
$$

D.4 Exogenous Processes

The productivity shock for risky firms $a_{d,t}$ follows the two-state Markov process. The productivity level of safe firms is time-invariant at all times, $a_{n,t} = 1$. The TFP shock follows AR(1) process:

$$
\ln tfp_{t+1} = \omega \ln tfp_t + (1-\omega) \ln \overline{tfp} + \epsilon_{t+1}.
$$

D.5 Market clearing conditions

$$
\begin{aligned}
C_t + K_t - (1-\delta) K_{t-1} &= Y_t, \\
\zeta l_{d,t} + (1-\zeta) l_{n,t} &= L_t, \\
\zeta k_{d,t} + (1-\zeta) k_{n,t} &= K_{t-1}, \\
\zeta \frac{b_{d,t}}{R_t} + (1-\zeta) \frac{b_{n,t}}{R_t} &= \frac{D_t}{1+r_t}.
\end{aligned}
$$

E Solving the model

To simulate the model numerically, we need to solve the model using a non-linear global solution method to handle four occasionally binding constraints: the borrowing constraint (2), the non-negative constraint for dividends (1), the upper limit of tax advantage $\min\{r_{t-1}\frac{b_{t-1}}{R_{t-1}},\ f_t(q_t) - q_t\}$, and the debt-ridden condition $b_t/R_t \lesseqgtr E_{Z,t}$.[16] These occasionally binding constraints generate policy functions with kinks and non-linearity. Since a standard numerical approximation method cannot solve the non-linear

[16]More precisely, the borrowing constraint is always binding in the sense that it holds with equality. We abuse the usage of the term "binding" in a way that the borrowing constraint is called binding if it is (3) with equality, and non-binding if it is (4) with equality.

policy function, we apply a Smolyak-based projection method proposed by Judd et al. (2014) and the index function approach to account for non-linearity in policy functions. We modified the simulation code developed by Shirai (2021), who extended the method of Hirose and Sunakawa (2019) to account for two occasionally binding constraints.

To solve our DSGE model, we closely follow Shirai (2021). This paper analyzes the effectiveness of fiscal policy under high debt levels and extends the algorithm provided by Hirose and Sunakawa (2019) to handle two occasionally binding constraints (OBC). The algorithm combines various methods developed in recent years to solve the model nonlinearly, considering the zero lower bound of the nominal interest rate. For more detail on the solution method, see Appendix C on Shirai (2021), and Hirose and Sunakawa (2019).

In our model, there are four OBCs: the borrowing constraint (2), the non-negativity constraint on dividend (1), the upper limit of the tax advantage $\min\left\{\frac{r_{t-1}}{R_{t-1}}b_{t-1}, f_t(q_t)-q_t\right\}$, and the debt-ridden condition $b_t/R_t \lesseqgtr E_{Z,t}$. Addressing the scenarios where these four constraints bind or do not bind entails considering sixteen distinct cases. However, due to the definitions of B_M, B_L, and B_Z, we need to consider only five cases as follows.

	borrowing constraint	non-negativity constraint	$\min\left\{r_{t-1}\frac{b_{t-1}}{R_{t-1}},\ f_t(q_t)-q_t\right\}$	debt-ridden condition
nnn	$q_t = \phi f_t(q_t) + \xi S_t - \frac{b_t}{R_t}$	Not bind	$r_{t-1}\frac{b_{t-1}}{R_{t-1}}$	$\frac{b_t}{R_t} < E_{Z,t}$
nbn	$q_t = \phi f_t(q_t) + \xi S_t - \frac{b_t}{R_t}$	bind	$r_{t-1}\frac{b_{t-1}}{R_{t-1}}$	$\frac{b_t}{R_t} < E_{Z,t}$
bbn	$q_t = \phi f_t(q_t)$	bind	$r_{t-1}\frac{b_{t-1}}{R_{t-1}}$	$\frac{b_t}{R_t} < E_{Z,t}$
bbb	$q_t = \phi f_t(q_t)$	bind	$f_t(q_t)-q_t$	$\frac{b_t}{R_t} < E_{Z,t}$
dr	$q_t = \phi f_t(q_t)$	bind		$\frac{b_t}{R_t} \geq E_{Z,t}$

The labels *nnn*, *nbn*, *bbn*, *bbb*, and *dr* correspond to the state of each constraint. Note that the min operator is employed in computing π_t in the *dr* state. This is because changing relative magnitudes of the two arguments within the min operator, $\min\left\{r_{t-1}\frac{b_{t-1}}{R_{t-1}},\ f_t(q_t)-q_t\right\}$, does not lead to the occurrence of a kink.

It is well known that the OBC generates policy functions with kinks. The standard approximation method, such as the Chebyshev polynomials function, has difficulty dealing with kinks. This study adapts an index function approach to deal with kinks. The index function combines the five policy functions, $\psi_{x,nnn}$, $\psi_{x,nbn}$, $\psi_{x,bbn}$, $\psi_{x,bbb}$ and $\psi_{x,dr}$, to generate one new policy function ψ_x for each endogenous variable,

$$\psi_x = \mathbb{1}_{nnn}\psi_{x,nnn} + \mathbb{1}_{nbn}\psi_{x,nbn} + \mathbb{1}_{bbn}\psi_{x,bbn} + \mathbb{1}_{bbb}\psi_{x,bbb} + \left(1 - \mathbb{1}_{nnn} - \mathbb{1}_{nbn} - \mathbb{1}_{bbn} - \mathbb{1}_{bbb}\right)\psi_{x,dr},$$

where x represents each endogenous variable, and $\mathbb{1}$ is an index function and defined by:

$$\begin{aligned}
\mathbb{1}_{nnn} &= 1 \quad \text{if } \xi S_t - b_t > 0,\ \ \pi_t > 0,\ \ r_{t-1}\frac{b_{t-1}}{R_{t-1}} < f_t(q_t) - q_t,\ \text{and } \frac{b_t}{R_t} < E_{Z,t} \\
&= 0 \quad \text{otherswise}, \\
\mathbb{1}_{nbn} &= 1 \quad \text{if } \xi S_t - b_t > 0,\ \ \pi_t \leq 0,\ \ r_{t-1}\frac{b_{t-1}}{R_{t-1}} < f_t(q_t) - q_t,\ \text{and } \frac{b_t}{R_t} < E_{Z,t}, \\
&= 0 \quad \text{otherswise},
\end{aligned}$$

$$
\begin{aligned}
\mathbb{1}_{bbn} &= 1 \quad \text{if } \xi S_t - b_t \le 0, \quad \pi_t \le 0, \quad r_{t-1}\frac{b_{t-1}}{R_{t-1}} < f_t(q_t) - q_t, \text{ and } \frac{b_t}{R_t} < E_{Z,t}, \\
&= 0 \quad \text{otherswise}, \\
\mathbb{1}_{bbb} &= 1 \quad \text{if } \xi S_t - b_t \le 0, \quad \pi_t \le 0, \quad r_{t-1}\frac{b_{t-1}}{R_{t-1}} \ge f_t(q_t) - q_t, \text{ and } \frac{b_t}{R_t} < E_{Z,t}, \\
&= 0 \quad \text{otherswise}, \\
1 - \mathbb{1}_{nnn} - \mathbb{1}_{nbn} - \mathbb{1}_{bbn} - \mathbb{1}_{bbb} &= 1 \quad \text{if } \xi S_t - b_t \le 0, \quad \pi_t \le 0, \quad \text{and } \frac{b_t}{R_t} \ge E_{Z,t}, \\
&= 0 \quad \text{otherswise}.
\end{aligned}
$$

The five policy functions, $\psi_{x,nnn}$, $\psi_{x,nbn}$, $\psi_{x,bbn}$, $\psi_{x,bbb}$ and $\psi_{x,dr}$ are smooth functions without kinks, assuming that the OBC does not switch even if the state variables change. For example, $\psi_{x,nnn}$ is assumed that the borrowing constraint is always (4) and the non-negative constraint for π_t never binds even if the debt level is so high and $\xi S_t - \frac{b_t}{R_t} < 0$. In addition, even when b_i is so large, we suppose that $\min\left\{\frac{r_{t-1}}{R_{t-1}}b_{t-1}, f_t(q_t) - q_t\right\} = \frac{r_{t-1}}{R_{t-1}}b_{t-1}$. Subsequently, $\psi_{x,nnn}$ is a smooth function and has no kinks. Even though each policy function is smooth, kinks can emerge at the switchings of the policy function.

E.1 The total value of the firm

The fixed point iteration is a method of finding a solution by computing iteration until the fixed point is found. This method is similar to the value function iteration. As explained above, to deal with occasionally binding constraints, we solve five models assuming that the status of each constraint (binding or non-binding) is invariant. One of these five is a model in which the non-negativity constraint for the dividend is always binding. In this model, the dividend is always zero, and the firm value V_t converges to zero even though it should be positive.

In our theory, as long as the level of debt does not exceed B_Z, the firm's equity value is strictly greater than zero ($V_t > 0$) because the debt level eventually returns to its optimal level. However, the numerical result that $V_t = 0$ is inconsistent with the theory. To avoid this inconsistency, we define the total value of the firm W_t as the sum of the values of the firm owner and the lender:

$$W_t = \frac{(1 + r_{-1})}{R_{-1}} b_{-1} + V_t.$$

The firm's total value is the sum of debt and the present discounted values of dividends. Note that the bank receives $\frac{1+r_{-1}}{R_{-1}}b_{-1}$ from the firm, whereas the net payment for the firm is b_{-1} because the government provides it with a tax advantage $\frac{\tau r_{-1}}{R_{-1}}b_{-1}$.

As with V_t, the firm must choose between staying in the normal state or falling into the debt-ridden state. Once in the debt-ridden state, the firm has no choice but to remain in that state. Thus, given that the firm was in the normal state in period $t-1$, the total value of the firm W_t is given by:

$$W_t = \max\left\{W_t^N,\ W_t^Z\right\},$$

where W_t^N is the total value of the firm in the normal state, and W_t^Z is the total value of the firm that is initially in the normal state to move into the debt-ridden state by borrowing additional funds in period t. W_t^N evolves by the following dynamic programming equation:

$$W_t^N = \max_{q_t, b_t} f_t(q_t) - q_t + \tau \min\left\{\frac{r_{t-1}}{R_{t-1}} b_{t-1}, f_t(q_t) - q_t\right\} + \mathbb{E}_t\left[m_{+1} W_{t+1}\right], \tag{27}$$

$$\text{subject to} \quad f_t(q_t) - q_t - b_{t-1} + \frac{b_t}{R_t} \geq 0,$$

$$q_t \leq \phi f_t(q_t) + \max\left\{\xi S_t - \frac{b_t}{R_t},\ 0\right\}.$$

W_t^Z, the total value of the firm that borrows additionally to become debt-ridden, is determined by the following dynamic programming equation:

$$W_t^Z = f_t(q_{z,t}) - q_{z,t} + \tau \min\left\{\frac{r_{t-1}}{R_{-1}} b_{-1}, f_t(q_{z,t}) - q_{z,t}\right\} + \mathbb{E}_t\left[m_{t+1} Z_{t+1}\right],$$

$$\text{subject to} \quad Z_t = (1+\tau)(1-\phi) f_t(q_{z,t}) + \mathbb{E}_t\left[m_{t+1} Z_{t+1}\right],$$

$$q_{z,t} \leq \phi f_t(q_{z,t}).$$

Thus, these dynamic programming problems are equivalent to (6) and (9). We solve the numerical optimization problem using W_t instead of V_t. Hence, the endogenous variables in the system are determined by the state variables $\{K_{t-1}, b_{n,t-1}, b_{d,t-1}, r_{t-1}, tfp_t\}$. The social surplus (27) is determined by the lagged interest rate, and r_{t-1} works as an additional state variable to solve the dynamics in this numerical simulation.

E.2 Fixed-point iteration

In this subsection, we solve our DSGE model using the fixed-point iteration with the index function approach. We approximate expectation terms in Euler equations and the value functions and solve them by the fixed-point iteration. To approximate functions, we use a Smolyak-based projection method proposed by Judd et al. (2014) and construct the Smolyak polynomials using extrema of second-order Chebyshev polynomials and unidimensional second-order Chebyshev polynomials.

Following Gust, Herbst, López-Salido, and Smith (2017) and Hirose and Sunakawa (2019), we define the expectation functions for expectation terms of the right-hand side in Euler equations (Equation (22), (23), and (25)) and the value function (Equation (24), (26), and (27)) as follows:

$$e_{C,jj}(h) \equiv \frac{1}{\beta}\mathbb{E}\left[\frac{C'}{1-\delta+r^{K\prime}}\right], \qquad jj = nnn, nbn, bbn, bbb, dr,$$

$$e_{r,jj}(h) \equiv \frac{\mathbb{E}\left[C'\right]}{\beta C} - 1, \qquad jj = nnn, nbn, bbn, bbb, dr,$$

$$e_{\mu_n,jj}(h) \equiv \mathbb{E}\left[m' \frac{\lambda'}{\lambda}\right] R, \qquad jj = nnn, nbn, bbn, bbb, dr,$$

$$e_{V,jj}(h) \equiv \mathbb{E}\left[m' V'\right], \qquad jj = nnn, nbn, bbn, bbb, dr,$$

$$e_{W,jj}(h) \equiv \mathbb{E}\left[m' W'\right], \qquad jj = nnn, nbn, bbn, bbb, dr,$$

$$e_{Z,jj}(h) \equiv \mathbb{E}\left[m' Z'\right], \qquad jj = nnn, nbn, bbn, bbb, dr,$$

where h denotes the vector of state variables such that $h = [K, b_n, b_d, r_{-1}, tfp]$ and jj is an index for regimes. In this Appendix, hereafter, to clarify notation, we represent current period values using letters without time subscripts, denote the values of the next period with a prime ($'$), and denote the values of the previous period with a -1 subscript.

Step 1 Set an upper bound and a lower bound for each state variable.

Step 2 Set realizations $\{a_{d,high}, a_{d,low}\}$ and transition matrix:

$$\begin{bmatrix} p_{high} & 1-p_{high} \\ 1-p_{low} & p_{low} \end{bmatrix}.$$

of the technology shocks, a_d, which follow a 2-state Markov chain.

Step 3 Make an initial guess for expectation functions and value functions:

$$\begin{aligned}
e^{(0)}_{C,j,jj,g} &= \frac{\overline{C}}{\beta(1-\delta+\overline{r})}, && \text{for } j=1,2,\cdots,J, \quad jj = nnn, nbn, bbn, bbb, dr, \quad g = high, low,\\
e^{(0)}_{r,j,jj,g} &= \overline{r}, && \text{for } j=1,2,\cdots,J, \quad jj = nnn, nbn, bbn, bbb, dr, \quad g = high, low,\\
e^{(0)}_{\mu_n,j,jj,g} &= \beta\overline{R} && \text{for } j=1,2,\cdots,J, \quad jj = nnn, nbn, bbn, bbb, dr, \quad g = high, low,\\
e^{(0)}_{V,j,jj,g} &= \beta\overline{V}, && \text{for } j=1,2,\cdots,J, \quad jj = nnn, nbn, bbn, bbb, dr, \quad g = high, low,\\
e^{(0)}_{W,j,jj,g} &= \beta\overline{W}, && \text{for } j=1,2,\cdots,J, \quad jj = nnn, nbn, bbn, bbb, dr, \quad g = high, low,\\
e^{(0)}_{Z,j,jj,g} &= \beta\overline{Z}, && \text{for } j=1,2,\cdots,J, \quad jj = nnn, nbn, bbn, bbb, dr, \quad g = high, low,
\end{aligned}$$

where j is an index for state variables, J is the total number of grid points and equal to 9 in our setting, and overbars indicate the steady-state value of the corresponding variable.

Step 4 Compute the coefficients for Smolyak polynomials $\boldsymbol{\theta}$:

$$\begin{aligned}
\boldsymbol{\theta}_{C,jj,g} &= \mathfrak{T}(\mathcal{H})^{-1} e^{(i-1)}_{C,jj,g},\\
\boldsymbol{\theta}_{r,jj,g} &= \mathfrak{T}(\mathcal{H})^{-1} e^{(i-1)}_{r,jj,g},\\
\boldsymbol{\theta}_{\mu_n,jj,g} &= \mathfrak{T}(\mathcal{H})^{-1} e^{(i-1)}_{\mu_n,jj,g},\\
\boldsymbol{\theta}_{V,jj,g} &= \mathfrak{T}(\mathcal{H})^{-1} e^{(i-1)}_{V,jj,g},\\
\boldsymbol{\theta}_{W,jj,g} &= \mathfrak{T}(\mathcal{H})^{-1} e^{(i-1)}_{W,jj,g},\\
\boldsymbol{\theta}_{Z,jj,g} &= \mathfrak{T}(\mathcal{H})^{-1} e^{(i-1)}_{Z,jj,g},
\end{aligned}$$

where $\boldsymbol{\theta}_{*,jj,g} = [\theta_{*,jj,g,0},\ \theta_{*,jj,g,1},\ \cdots\ ,\theta_{*,jj,g,J}]'$, $e^{(i-1)}_{*,jj,g} = [e^{(i-1)}_{*,1,jj,g},\ \cdots\ e^{(i-1)}_{*,J,jj,g}]'$, $* = \{C,\ r,\ \mu_n,\ V,\ W,\ Z\}$, $\mathcal{H}$ is a Smolyak grid point, and $\mathfrak{T}(\mathcal{H})$ is a Smolyak basis function. For more detail on the Smolyak grid points and the Smolyak basis function, see Judd et al. (2014) and Appendix C.1 of Shirai (2021).

Step 5 Choose a grid: $h_j = [K_j, b_{d,j}, b_{n,j}, r_j, tfp_j]$. Exogenous variables are set using the grid: $K_{-1} = K_j,\ b_{d,-1} = b_{d,j},\ b_{n,-1} = b_{n,j},\ r_{-1} = r_j,\ tfp = tfp_j$.

Step 6 Taking as given the productivity for risky firms ($a_{d,g}$) and the expectation functions previously obtained,

$$
\begin{aligned}
&C_{j,jj,g} = e^{(i-1)}_{C,j,jj,g}(h_j),\\
&r_{j,jj,g} = e^{(i-1)}_{r,j,jj,g}(h_j),\\
&\frac{\mu_{n,j,jj,g}}{\lambda_{n,j,jj,g}} = 1 - e^{(i-1)}_{\mu_n,j,jj,g}(h_j),\\
&\mathbb{E}[m'_{j,jj,g}V'_{n,j,jj,g}] = e^{(i-1)}_{V,jj,g}(h_j),\\
&\mathbb{E}[m'_{j,jj,g}W^{N'}_{j,jj,g}] = e^{(i-1)}_{W,jj,g}(h_j),\\
&\mathbb{E}[m'_{j,jj,g}Z'_{j,jj,g}] = e^{(i-1)}_{Z,j,jj,g}(h_j),
\end{aligned}
$$

Step 7 Solve the dynamics equations for each regime $jj = nnn,\ nbn,\ bbn,\ bbb,\ dr, \quad g = high, low$:

$$
\begin{cases}
\text{If regime is in } nnn, & \dfrac{\mu_{d,j,jj,g}}{\lambda_{d,j,jj,g}} = \dfrac{\mu_{n,j,jj,g}}{\lambda_{n,j,jj,g}}\\
\text{If regime is in } nbn, & \dfrac{\mu_{d,j,jj,g}}{\lambda_{d,j,jj,g}} \text{ is given by solving the nonlinear equation (28).}\\
\text{If regime is in } bbn, bbb \text{ or } dr, & \begin{cases} \dfrac{1+\phi\frac{\mu_{d,j,jj,g}}{\lambda_{d,j,jj,g}}}{1+\frac{\mu_{d,j,jj,g}}{\lambda_{d,j,jj,g}}} = \dfrac{\phi}{\eta},\\ \dfrac{\mu_{d,jj,g}}{\lambda_{d,jj,g}} = \dfrac{(\eta-\phi)}{\phi(1-\eta)} \end{cases}
\end{cases}
$$

Calculate each equation sequentially at time t.

$$
\begin{aligned}
&\Omega_{j,jj,g} \equiv \frac{f(q_{d,j,jj,g})}{f(q_{n,j,jj,g})} = \left(\frac{\frac{1+\phi\frac{\mu_{d,j,jj,g}}{\lambda_{d,j,jj,g}}}{1+\frac{\mu_{d,j,jj,g}}{\lambda_{d,j,jj,g}}}}{\frac{1+\phi\frac{\mu_{n,j,jj,g}}{\lambda_{n,j,jj,g}}}{1+\frac{\mu_{n,j,jj,g}}{\lambda_{n,j,jj,g}}}} a^{\eta}_{d,g} \right)^{\frac{1}{1-\eta}}\\
&\Psi_{y,j,jj,g} \equiv \left[\zeta\Omega^{\eta}_{j,jj,g}a^{\eta}_{d,g} + 1 - \zeta \right]^{\frac{1}{\eta}},\\
&\Psi_{k,j,jj,g} \equiv \zeta\Omega_{j,jj,g} + 1 - \zeta,\\
&k_{n,j,jj,g} = \frac{K_j}{\Psi_{k,j,j}},\\
&l_{n,j,jj,g} = \left[\frac{(1-\alpha)\eta\frac{1+\phi\frac{\mu_{n,j,jj,g}}{\lambda_{n,j,jj,g}}}{1+\frac{\mu_{n,j,jj,g}}{\lambda_{n,j,jj,g}}}\Psi^{1-\eta}_{y,j,jj,g}a_j k^{\alpha}_{n,j,jj,g}}{\Psi^{\nu}_{k,j,jj,g}\gamma_L C_{j,jj,g}} \right]^{\frac{1}{\alpha+\nu}},\\
&L_{j,jj,g} = \Psi_{k,j,jj,g} l_{n,j,jj,g},\\
&w_{j,jj,g} = \gamma_L C_{j,jj,g} L^{\nu}_{j,jj,g},\\
&y_{n,j,jj,g} = tfp_j \cdot k^{\alpha}_{n,j,jj,g} l^{1-\alpha}_{n,j,jj,g},\\
&Y_{j,jj,g} = \Psi_{y,j,jj,g} y_{n,j,jj,g},\\
&A_{j,jj,g} = tfp^{\eta}_j \cdot Y^{1-\eta}_{j,jj,g},\\
&A_{d,j,jj,g} = (tfp_j \cdot a_{d,g})^{\eta} Y^{1-\eta}_{j,jj,g},
\end{aligned}
$$

$$r^K_{j,jj,g} = \frac{\alpha\eta\left(1+\phi\frac{\mu_{n,j,jj,g}}{\lambda_{n,j,jj,g}}\right)}{1+\frac{\mu_{n,j,jj,g}}{\lambda_{n,j,jj,g}}}\frac{A_{j,jj,g}k^{\alpha\eta}_{n,j,jj,g}l^{(1-\alpha)\eta}_{n,j,jj,g}}{k_{n,j,jj,g}},$$

$$K'_{j,jj,g} = Y_{j,jj,g} - C_{j,jj,g} + (1-\delta)K_j,$$

$$q_{n,j,jj,g} = w_{j,jj,g}l_{n,j,jj,g} + r^K_{j,jj,g}k_{n,j,jj,g},$$

$$f(q_{n,j,jj,g}) = A_{j,jj,g}k^{\alpha\eta}_{n,j,jj,g}l^{(1-\alpha)\eta}_{n,j,jj,g},$$

$$R_{j,jj,g} = 1 + r_{j,jj,g}(1-\tau),$$

$$k_{d,j,jj,g} = \left[\frac{1+\phi\frac{\mu_{d,j,jj,g}}{\lambda_{d,j,jj,g}}}{1+\frac{\mu_{d,j,jj,g}}{\lambda_{d,j,jj,g}}}\eta A_{d,j,jj,g}\left(\frac{r^K_{j,jj,g}}{\alpha}\right)^{(1-\alpha)\eta-1}\left(\frac{1-\alpha}{w_{j,jj,g}}\right)^{(1-\alpha)\eta}\right]^{\frac{1}{1-\eta}},$$

$$l_{d,j,jj,g} = \frac{(1-\alpha)r^K_{j,jj,g}k_{d,j,jj,g}}{\alpha w_{j,jj,g}},$$

$$k_{z,j,jj,g} = \left[\phi A_{d,j,jj,g}\left(\frac{r^K_{j,jj,g}}{\alpha}\right)^{(1-\alpha)\eta-1}\left(\frac{1-\alpha}{w_{j,jj,g}}\right)^{(1-\alpha)\eta}\right]^{\frac{1}{1-\eta}},$$

$$l_{z,j,jj,g} = \frac{(1-\alpha)r^K_{j,jj,g}k_{z,j,jj,g}}{\alpha w_{j,jj,g}},$$

$$q_{d,j,jj,g} = w_{j,jj,g}l_{d,j,jj,g} + r^K_{j,jj,g}k_{d,j,jj,g},$$

$$f(q_{d,j,jj,g}) = A_{d,j,jj,g}k^{\alpha\eta}_{d,j,jj,g}l^{(1-\alpha)\eta}_{d,j,jj,g},$$

$$f(q_{z,j,jj,g}) = A_{d,j,jj,g}k^{\alpha\eta}_{z,j,jj,g}l^{(1-\alpha)\eta}_{z,j,jj,g},$$

$$b'_{n,j,jj,g} = \frac{R_{j,jj,g}}{1-\xi}\left[\phi f(q_{n,j,jj,g}) - q_{n,j,jj,g} + \xi e^{(i-1)}_{V,jj,g}(h_j)\right],$$

$$S_{j,jj,g} = e^{(i-1)}_{V,jj,g}(h_j) + \frac{b'_{n,j,jj,g}}{R_{j,jj,g}},$$

$$\pi_{n,j,jj,g} = f(q_{n,j,jj,g}) - q_{n,j,jj,g} - b_{n,j} + \frac{b'_{n,jj,g}}{R_{j,jj,g}},$$

$$V_{n,j,jj,g} = \pi_{n,j,jj,g} + \mathbb{E}[m'_{j,jj,g}V'_{n,j,jj,g}],$$

$$R_j = 1 + r_j(1-\tau), \qquad \iff R_{t-1} = 1 + r_{t-1}(1-\tau),$$

$$\begin{cases} \text{If } jj = nnn, & b'_{d,j,jj,g} = R_{j,jj,g}\left[\phi f(q_{d,j,jj,g}) - q_{d,j,jj,g} + \xi S_{j,jj,g}\right], \\ & \pi_{d,j,jj,g} = f(q_{d,j,jj,g}) - q_{d,j,jj,g} - b_{d,j} + \frac{b'_{d,j,jj,g}}{R_{j,jj,g}}, \\ \text{If } jj = nbn, & b'_{d,j,jj,g} = R_{j,jj,g}\left[\phi f(q_{d,j,jj,g}) - q_{d,j,jj,g} + \xi S_{j,jj,g}\right], \\ & \pi_{d,j,jj,g} = 0, \\ \text{If } jj = bbn, & b'_{d,j,jj,g} = R_{j,jj,g}\left[b_{d,j} - f(q_{d,j,jj,g}) + q_{d,j,jj,g}\right], \\ & \pi_{d,j,jj,g} = 0, \\ \text{If } jj = bbb, & b'_{d,j,jj,g} = R_{j,jj,g}\left[\frac{1+r_{j,jj,g}}{R_{j,jj,g}}b_{d,j} - (1+\tau)(1-\phi)f(q_{d,j,jj,g})\right], \\ & \pi_{d,j,jj,g} = 0, \\ \text{If } jj = dr, & b'_{d,j,jj,g} = R_{j,jj,g}\mathbb{E}[m'_{j,jj,g}Z'_{j,jj,g}], \\ & \pi_{d,j,jj,g} = 0, \end{cases}$$

$$Z_{j,jj,g} = (1+\tau)(1-\phi)f(q_{z,j,jj,g}) + \mathbb{E}[m'_{j,jj,g}Z'_{j,jj,g}],$$

$$V^Z_{j,jj,g} = (1-\phi)f(q_{z,j,jj,g}) + \mathbb{E}[m'_{j,jj,g} Z'_{j,jj,g}] + \tau \min\left\{ \frac{r_j \tau}{R_j} b_{d,j}, (1-\phi)f(q_{z,j,jj,g}) \right\},$$

$$\begin{cases} \text{If } jj = nnn,\ nbn, \text{ or } bbn & W^N_{j,jj,g} = \frac{r_j\tau}{R_j} b_{d,j} + f(q_{d,j,jj,g}) - q_{d,j,jj,g} + \mathbb{E}[m'_{j,jj,g} W^{N'}_{j,jj,g}], \\ & W^Z_{j,jj,g} = \frac{r_j\tau}{R_j} b_{d,j} + (1-\phi)f(q_{z,j,jj,g}) + \mathbb{E}[m'_{j,jj,g} Z'_{j,jj,g}], \\ \text{If } jj = bbb, & W^N_{j,jj,g} = (1+\tau)(1-\phi)f(q_{d,j,jj,g}) + \mathbb{E}[m'_{j,jj,g} W^{N'}_{j,jj,g}], \\ & W^Z_{j,jj,g} = (1+\tau)(1-\phi)f(q_{z,j,jj,g}) + \mathbb{E}[m'_{j,jj,g} Z'_{j,jj,g}], \\ \text{If } jj = dr, & V_{j,jj,g} = V^Z_{j,jj,g}, \\ & W^N_{j,jj,g} = V_{j,jj,g} + \frac{1+r_j}{R_j} b_{d,j}, \\ & W^Z_{j,jj,g} = (1-\phi)f(q_{z,j,jj,g}) + \tau \min\left\{ \frac{r_j\tau}{R_j} b_{d,j}, (1-\phi)f(q_{z,j,jj,g}) \right\} \\ & \qquad + \mathbb{E}[m'_{j,jj,g} Z'_{j,jj,g}], \end{cases}$$

where V^Z is the value of the firm that borrows additionally to become debt-ridden.

If regime is in nbn, solve for $\mu_{d,j,jj,g}/\lambda_{d,j,jj,g}$ with the equation below numerically: [17]

$$0 = f(q_{d,j,jj,g}) - q_{d,j,jj,g} - b_{d,j} + \frac{b'_{d,j,jj,g}}{R_{j,jj,g}}. \tag{28}$$

Compute the next period productivity, government expenditures, and the corporate tax rate for $m = 1, 2, \cdots, M$:

$$\ln tfp'_{j,m} = \omega \ln a_j + (1-\omega) \ln \overline{tfp} + \epsilon'_m,$$

where ϵ'_m is a structural TFP shock and approximated by the Gauss–Hermite quadrature, m is an index for grid points of the shock, and M is the total number of grid points for the shock.

Interpolate between grids using Smolyak polynomials:

$$\begin{aligned} \hat{e}_C(h'_{j,jj,g}; \boldsymbol{\theta}_{C,jj,g}) &= \mathfrak{T}(\varphi(h'_{j,jj,g}))\boldsymbol{\theta}_{C,jj,g}, \\ \hat{e}_r(h'_{j,jj,g}; \boldsymbol{\theta}_{r,jj,g}) &= \mathfrak{T}(\varphi(h'_{j,jj,g}))\boldsymbol{\theta}_{r,jj,g}, \\ \hat{e}_{\mu_n}(h'_{j,jj,g}; \boldsymbol{\theta}_{\mu_n,jj,g}) &= \mathfrak{T}(\varphi(h'_{j,jj,g}))\boldsymbol{\theta}_{\mu,jj,g}, \\ \hat{e}_V(h'_{j,jj,g}; \boldsymbol{\theta}_{V,jj,g}) &= \mathfrak{T}(\varphi(h'_{j,jj,g}))\boldsymbol{\theta}_{V,jj,g}, \\ \hat{e}_W(h'_{j,jj,g}; \boldsymbol{\theta}_{W,jj,g}) &= \mathfrak{T}(\varphi(h'_{j,jj,g}))\boldsymbol{\theta}_{W,jj,g}, \\ \hat{e}_Z(h'_{j,jj,g}; \boldsymbol{\theta}_{Z,jj,g}) &= \mathfrak{T}(\varphi(h'_{j,jj,g}))\boldsymbol{\theta}_{Z,j,jj,g}, \end{aligned}$$

where $h'_{j,jj,g} = [K'_{j,jj,g},\ b'_{n,j,jj,g},\ b'_{d,j,jj,g},\ r_{j,jj,g},\ tfp'_{j,m}]$. The domain of Chebyshev polynomials is the interval $[-1, 1]$, and in order to approximate a function by the Chebyshev polynomials, it is necessary to transform the interval $h_j \in [h^{\min},\ h^{\max}]$ into the interval of $x_j \in [-1, 1]$, $h^{\min}$ and $h^{\max}$ are each state variable's maximum and minimum values chosen to encompass a wide interval. For each state variable in h, we use $\varphi : [h^{\min},\ h^{\max}] \to [-1,\ 1]$ for $\{K, b_n, b_d, r, tfp\}$,

$$x_j = \varphi(h_j) = \frac{2(h_j - h^{\min}) - (h^{\max} - h^{\min})}{h^{\max} - h^{\min}}.$$

[17] For example, fsolve is a numerical solver in Matlab.

Step 8 In calculating for period $t+1$, $a_{d,g'}$ is given. If $\frac{b'_{d,j,jj,g}}{R_{j,jj,g}} > \mathbb{E}[m'_{j,jj,g} Z'_{j,jj,g}]$ and $jj \neq dr$, skip this step and go to Step 9 . If $jj = dr$ or $\frac{b'_{d,j,jj,g}}{R_{j,jj,g}} \geq \mathbb{E}[m'_{j,jj,g} Z'_{j,jj,g}]$, the state in period $t+1$ is always in the debt-ridden state (i.e., $jj = dr$). This state represents one of the steady-state equilibria within the model, and once reached, it persists indefinitely. Hence, if $jj = dr$ in period t, it will persist as $jj = dr$ in period $t+1$, rendering consideration of alternative states unnecessary for expected value computation. Therefore, the expected value is computed utilizing the policy function with $jj = dr$ in the $t+1$ period as well:

$$
\begin{aligned}
C'_{j,jj,g'} &= \hat{e}_C(h'_{j,jj,g'}; \boldsymbol{\theta}_{C,dr,g'}),\\
r'_{j,jj,g'} &= \hat{e}_r(h'_{j,jj,g'}; \boldsymbol{\theta}_{r,dr,g'}),\\
\frac{\mu'_{n,j,jj,g'}}{\lambda'_{n,j,jj,g'}} &= 1 - \hat{e}_{\mu_n}(h'_{j,jj,g'}; \boldsymbol{\theta}_{\mu,dr,g'}),\\
\mathbb{E}[m''_{j,jj,g'} V''_{n,j,jj,g'}] &= \hat{e}_V(h'_{j,jj,g'}; \boldsymbol{\theta}_{V,dr,g'}),\\
\mathbb{E}[m''_{j,jj,g'} W^{N''}_{j,jj,g'}] &= \hat{e}_W(h'_{j,jj,g'}; \boldsymbol{\theta}_{W,dr,g'}),\\
\mathbb{E}[m''_{j,jj,g'} Z''_{j,jj,g'}] &= \hat{e}_Z(h'_{j,jj,g'}; \boldsymbol{\theta}_{Z,dr,g'}).
\end{aligned}
$$

Calculate Step 7 for time $t+1$, and go to Step 10.

Step 9 For $jj \neq dr$ and $\frac{b'_{d,j,jj,g}}{R_{j,jj,g}} < \mathbb{E}[m'_{j,jj,g} Z'_{j,jj,g}]$, firstly, we assume that $\xi S'_{j,jj,g'} - \frac{b''_{d,j,jj,g'}}{R'_{j,jj,g'}} \leq 0$, $\pi'_{d,j,jj,g'} = 0$, $\frac{r_{j,jj,g}\tau}{R_{j,jj,g}} b'_{d,j,jj,g} \geq (1-\phi) f'(q'_{z,j,jj,g})$, $\frac{b'_{d,j,jj,g}}{R_{j,jj,g}} < \mathbb{E}[m'_{j,jj,g} Z'_{j,jj,g}]$, and the regime is bbb, and the expectation term is given by the followings:

$$
\begin{aligned}
C'_{j,jj,g'} &= \hat{e}_C(h'_{j,jj,g'}; \boldsymbol{\theta}_{C,bbb,g'}),\\
r'_{j,jj,g'} &= \hat{e}_r(h'_{j,jj,g'}; \boldsymbol{\theta}_{r,bbb,g'}),\\
\frac{\mu'_{n,j,jj,g'}}{\lambda'_{n,j,jj,g'}} &= 1 - \hat{e}_{\mu_n}(h'_{j,jj,g'}; \boldsymbol{\theta}_{\mu,bbb,g'}),\\
\mathbb{E}[m''_{j,jj,g'} V''_{n,j,jj,g'}] &= \hat{e}_V(h'_{j,jj,g'}; \boldsymbol{\theta}_{V,bbb,g'}),\\
\mathbb{E}[m''_{j,jj,g'} W^{N''}_{j,jj,g'}] &= \hat{e}_W(h'_{j,jj,g'}; \boldsymbol{\theta}_{W,bbb,g'}),\\
\mathbb{E}[m''_{j,jj,g'} Z''_{j,jj,g'}] &= \hat{e}_Z(h'_{j,jj,g'}; \boldsymbol{\theta}_{Z,bbb,g'}).
\end{aligned}
$$

Calculate Step 7 for time $t+1$. If the condtion $\frac{r_{j,jj,g}\tau}{R_{j,jj,g}} b'_{d,j,jj,g} \geq (1-\phi) f'(q'_{z,j,jj,g})$ is satisfied, go to Step 10. Else, if the condition $\frac{b''_{d,j,jj,g'}}{R'_{j,jj,g'}} > \mathbb{E}[m''_{j,jj,g'} Z''_{j,jj,g'}]$ is satisfied, go back to Step 8. If these conditions are not satisfied, next we assume that $\frac{r_j\tau}{R_j} b_{d,j} < (1-\phi) f'(q'_{z,j,jj,g})$, $\xi S'_{j,jj,g'} - \frac{b''_{d,j,jj,g'}}{R'_{j,jj,g'}} \leq 0$, $\pi'_{d,j,jj,g'} = 0$, the regime is in bbn, and calculate as the following:

$$
\begin{aligned}
C'_{j,jj,g'} &= \hat{e}_C(h'_{j,jj,g'}; \boldsymbol{\theta}_{C,bbn,g'}),\\
r'_{j,jj,g'} &= \hat{e}_r(h'_{j,jj,g'}; \boldsymbol{\theta}_{r,bbn,g'}),\\
\frac{\mu'_{n,j,jj,g'}}{\lambda'_{n,j,jj,g'}} &= 1 - \hat{e}_{\mu_n}(h'_{j,jj,g'}; \boldsymbol{\theta}_{\mu,bbn,g'}),\\
\mathbb{E}[m''_{j,jj,g'} V''_{n,j,jj,g'}] &= \hat{e}_V(h'_{j,jj,g'}; \boldsymbol{\theta}_{V,bbn,g'}),
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}[m''_{j,jj,g'} W^{N''}_{j,jj,g'}] = \hat{e}_W(h'_{j,jj,g'}; \boldsymbol{\theta}_{W,bbn,g'}),\\
&\mathbb{E}[m''_{j,jj,g'} Z''_{j,jj,g'}] = \hat{e}_Z(h'_{j,jj,g'}; \boldsymbol{\theta}_{Z,bbn,g'}).
\end{aligned}
$$

Calculate Step 7 for time $t+1$. If two condtions $\xi S'_{j,jj,g'} - \frac{b''_{d,j,jj,g'}}{R'_{j,jj,g'}} < 0$ and $\pi'_{d,j,jj,g'} = 0$ are satisfied, go to Step 10. Else, if the condition $\frac{b''_{d,j,jj,g'}}{R'_{j,jj,g'}} > \mathbb{E}[m''_{j,jj,g'} Z''_{j,jj,g'}]$ is satisfied, go back to Step 8. If these conditions are not satisfied, next, we assume that $\xi S'_{j,jj,g'} - \frac{b''_{d,j,jj,g'}}{R'_{j,jj,g'}} > 0$, $\pi'_{d,j,jj,g'} > 0$, $\frac{r_{j,jj,g}\tau}{R_{j,jj,g}} b'_{d,j,jj,g} \geq (1-\phi) f'(q'_{z,j,jj,g})$, the regime is in nnn, and the followings give the expectation term:

$$
\begin{aligned}
&C'_{j,jj,g'} = \hat{e}_C(h'_{j,jj,g'}; \boldsymbol{\theta}_{C,nnn,g'}),\\
&r'_{j,jj,g'} = \hat{e}_r(h'_{j,jj,g'}; \boldsymbol{\theta}_{r,nnn,g'}),\\
&\frac{\mu'_{n,j,jj,g'}}{\lambda'_{n,j,jj,g'}} = 1 - \hat{e}_{\mu_n}(h'_{j,jj,g'}; \boldsymbol{\theta}_{\mu,nnn,g'}),\\
&\mathbb{E}[m''_{j,jj,g'} V''_{n,j,jj,g'}] = \hat{e}_V(h'_{j,jj,g'}; \boldsymbol{\theta}_{V,nnn,g'}),\\
&\mathbb{E}[m''_{j,jj,g'} W^{N''}_{j,jj,g'}] = \hat{e}_W(h'_{j,jj,g'}; \boldsymbol{\theta}_{W,nnn,g'}),\\
&\mathbb{E}[m''_{j,jj,g'} Z''_{j,jj,g'}] = \hat{e}_Z(h'_{j,jj,g'}; \boldsymbol{\theta}_{Z,nnn,g'}).
\end{aligned}
$$

Calculate Step 7 for time $t+1$. If the condition $\pi'_{d,j,jj,g'} > 0$ is satisfied, go to Step 10. Else, if the condition $\frac{b''_{d,j,jj,g'}}{R'_{j,jj,g'}} > \mathbb{E}[m''_{j,jj,g'} Z''_{j,jj,g'}]$ is satisfied, go back to Step 8.

Next, if these conditions are not satisfied, we assume that $\xi S'_{j,jj,g'} - \frac{b''_{d,j,jj,g'}}{R'_{j,jj,g'}} > 0$, $\pi'_{d,j,jj,g'} < 0$, the regime is in nbn, and calculate as the following:

$$
\begin{aligned}
&C'_{j,jj,g'} = \hat{e}_C(h'_{j,jj,g'}; \boldsymbol{\theta}_{C,nbn,g'}),\\
&r'_{j,jj,g'} = \hat{e}_r(h'_{j,jj,g'}; \boldsymbol{\theta}_{r,nbn,g'}),\\
&\frac{\mu'_{n,j,jj,g'}}{\lambda'_{n,j,jj,g'}} = 1 - \hat{e}_{\mu_n}(h'_{j,jj,g'}; \boldsymbol{\theta}_{\mu,nbn,g'}),\\
&\mathbb{E}[m''_{j,jj,g'} V''_{n,j,jj,g'}] = \hat{e}_V(h'_{j,jj,g'}; \boldsymbol{\theta}_{V,nbn,g'}),\\
&\mathbb{E}[m''_{j,jj,g'} W^{N''}_{j,jj,g'}] = \hat{e}_W(h'_{j,jj,g'}; \boldsymbol{\theta}_{W,nbn,g'}),\\
&\mathbb{E}[m''_{j,jj,g'} Z''_{j,jj,g'}] = \hat{e}_Z(h'_{j,jj,g'}; \boldsymbol{\theta}_{Z,nbn,g'}).\\
&\frac{\mu'_{d,j,jj,g'}}{\lambda'_{d,j,jj,g'}} \text{ is given by solving the nonlinear equation (28).}
\end{aligned}
$$

Calculate Step 7 for time $t+1$. If the condition $\frac{b''_{d,j,jj,g'}}{R'_{j,jj,g'}} > \mathbb{E}[m''_{j,jj,g'} Z''_{j,jj,g'}]$ is satisfied, go back to Step 8, and this condition is not satisfed, go to Step 10.

Step 10 Calculate expectation terms:

$$
\mathbb{E}\left[\frac{C'_{j,jj,g}}{\beta(1-\delta+r^{K'}_{j,jj,g})}\right] = \sum_{g'=\{high,\ low\}} prob(a_{d,g'}|a_{d,g}) \frac{C'_{j,jj,g'}}{\beta(1-\delta+r'_{j,jj,g'})},
$$

$$
\begin{aligned}
\mathbb{E}\left[\frac{C'_{j,jj,g}}{\beta C_{j,jj,g}}\right]-1 &= \sum_{g'=\{high,\ low\}} prob(a_{d,g'}|a_{d,g})\left(\frac{C'_{j,jj,g'}}{\beta C_{j,jj}}-1\right), \\
\mathbb{E}\left[\frac{1}{C'_{j,jj,g}}\right]\beta C_{j,jj,g}R_{j,jj,g} &= \sum_{g'=\{high,\ low\}} prob(a_{d,g'}|a_{d,g})\frac{1}{C'_{j,jj,g'}}\beta C_{j,jj,g}R_{j,jj,g}, \\
\mathbb{E}\left[m'_{j,jj,g}V'_{n,j,jj,g}\right] &= \sum_{g'=\{high,\ low\}} prob(a_{d,g'}|a_{d,g})\frac{\beta C_{j,jj,g}}{C'_{j,jj,g'}}V'_{j,jj,g'}, \\
\mathbb{E}\left[m'_{j,jj,g}W'_{j,jj,g}\right] &= \sum_{g'=\{high,\ low\}} prob(a_{d,g'}|a_{d,g})\frac{\beta C_{j,jj,g}}{C'_{j,jj,g'}}W'_{j,jj,g'}, \\
\mathbb{E}\left[m'_{j,jj,g}Z'_{j,jj,g}\right] &= \sum_{g'=\{high,\ low\}} prob(a_{d,g'}|a_{d,g})\frac{\beta C_{j,jj,g}}{C'_{j,jj,g'}}Z'_{j,jj},
\end{aligned}
$$

Step 11 Next, substitute in the policy functions:

$$
\begin{aligned}
\psi^{(i)}_{*,j,jj,g} &= *_{j,jj,g}, \\
e^{(i)}_{C,j,jj,g} &= \mathbb{E}\left[\frac{C'_{j,jj,g}}{\beta(1-\delta+r^{K'}_{j,jj,g})}\right], \\
e^{(i)}_{r,j,jj,g} &= \mathbb{E}\left[\frac{C'_{j,jj,g}}{\beta C_{j,jj,g}}\right]-1 \\
e^{(i)}_{\mu_n,j,jj,g} &= \mathbb{E}\left[\frac{1}{C'_{j,jj,g}}\right]\beta C_{j,jj,g}R_{j,jj,g} \\
e^{(i)}_{V,j,jj,g} &= \mathbb{E}\left[m'_{j,jj,g}V'_{n,j,jj,g}\right], \\
e^{(i)}_{W,j,jj,g} &= \mathbb{E}\left[m'_{j,jj,g}W'_{d,j,jj,g}\right], \\
e^{(i)}_{Z,j,jj,g} &= \mathbb{E}\left[m'_{j,jj,g}Z'_{j,jj,g}\right],
\end{aligned}
$$

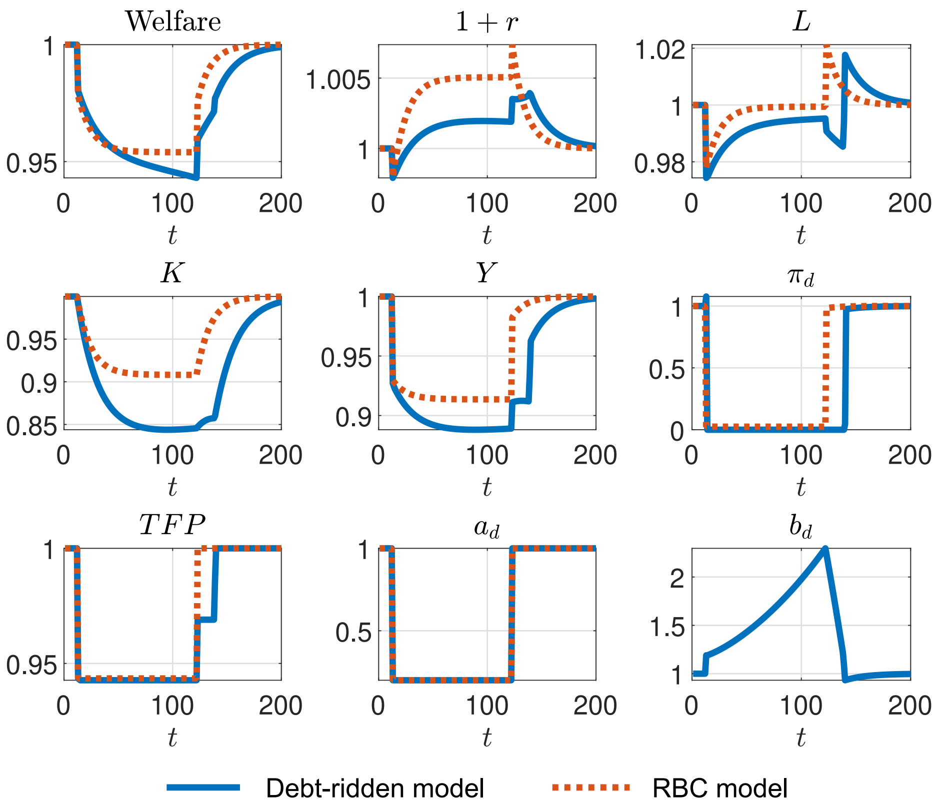
where $\psi^{(i)}_{*,jj}(h;\boldsymbol{\theta})$ is a policy function, $*_{j,jj,g} = \Big\{C_{j,jj,g},\ V_{n,j,jj,g},\ K'_{j,jj,g},\ r_{j,jj,g},\ w_{j,jj,g},\ \pi_{n,j,jj,g},\ \pi_{d,j,jj,g},\ b'_{n,j,jj,g},\ b'_{d,j,jj,g},\ y_{n,j,jj,g},\ y_{d,j,jj,g},\ \mu_{n,j,jj,g},\ \mu_{d,j,jj,g}\Big\}$.

Step 12 If $||\psi^{(i)}-\psi^{(i-1)}|| > 10^{-6}$, update the policy functions and expectation functions by $\psi(i) = \delta_\psi\psi^{(i-1)}+(1-\delta_\psi)\psi^{(i)}$ and $e^{(i)} = \delta_\psi e^{(i-1)}+(1-\delta_\psi)e^{(i)}$, respectively, where δ_ψ is set to 0.8, and go back to Step 5. If $||\psi^{(i)}-\psi^{(i-1)}|| \leq 10^{-6}$, end.

F Robustness analysis

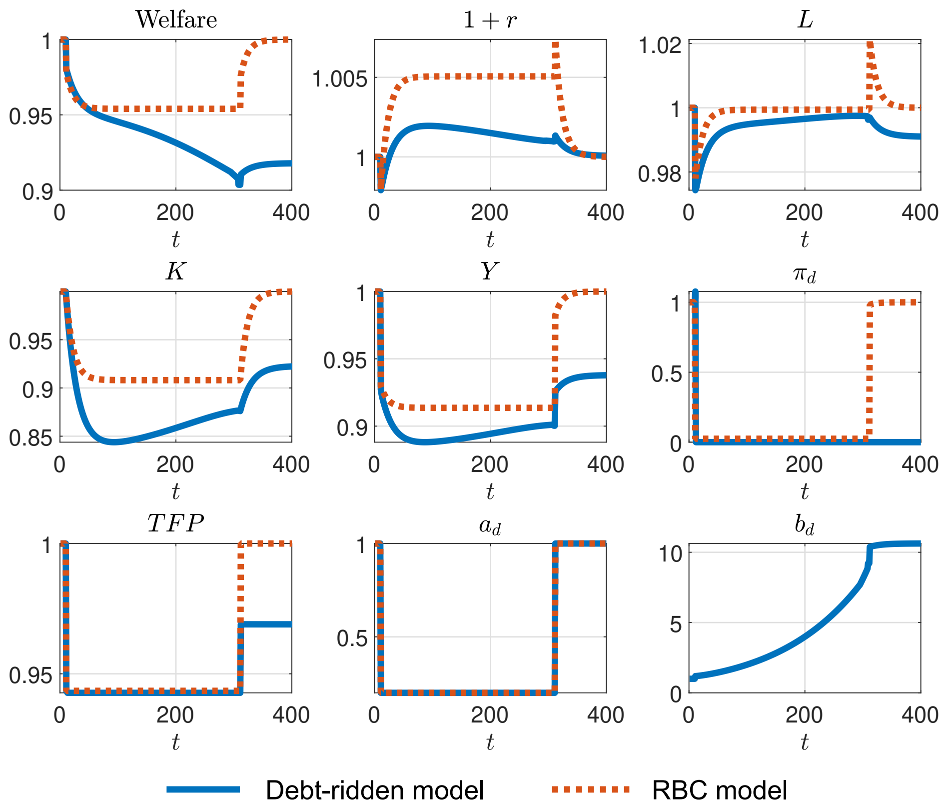
In the main text of this paper, we calibrate $a_{d,low} = 0$ to fit the average peak-to-trough decline for the US real per capita GDP across nine major financial crises. However, this setting might seem extreme. This Appendix shows that the main results are robust even if we set that $a_{d,low} = 0.2$.

Figure 7 and Figure 8 look almost identical to Figure 5 and Figure 6, respectively. Figure 7 also shows persistence due to debt accumulation. Figure 8 also shows that risky firms fall into the debt-ridden state. However, if $a_{d,low} > 0.2903$, risky firms do not fall into the debt-ridden state because $W^N_t \leq W^Z_t$ do not happen.



Notes: The vertical axis represents X_t/X as the deviation from the steady-state equilibrium, where X is the steady-state equilibrium value of a corresponding variable.
π_d: Risky firm's dividend, f_d: Risky firm's revenue (production), a_d: Risky firm's productivity, b_d: Risky firm's debt

Figure 7: RBC model versus the debt-ridden model: $a_{d,low} = 0.2$



Notes: The vertical axis represents X_t/X as the deviation from the steady-state equilibrium, where X is the steady-state equilibrium value of a corresponding variable.
π_d: Risky firm's dividend, f_d: Risky firm's revenue (production), a_d: Risky firm's productivity, b_d: Risky firm's debt

Figure 8: Long-term depression and debt-ridden firms: $a_{d,low} = 0.2$

G Analysis of impulse response

To evaluate our model quantitatively, we introduce an aggregate total factor productivity (TFP) shock, which is a common shock that hits both safe firms and risky firms. In this subsection, to introduce the TFP shock, we modify the production function as follows:

$$y_{i,t} = tfp_t \cdot a_{i,t} k_{i,t}^{\alpha} l_{i,t}^{1-\alpha},$$

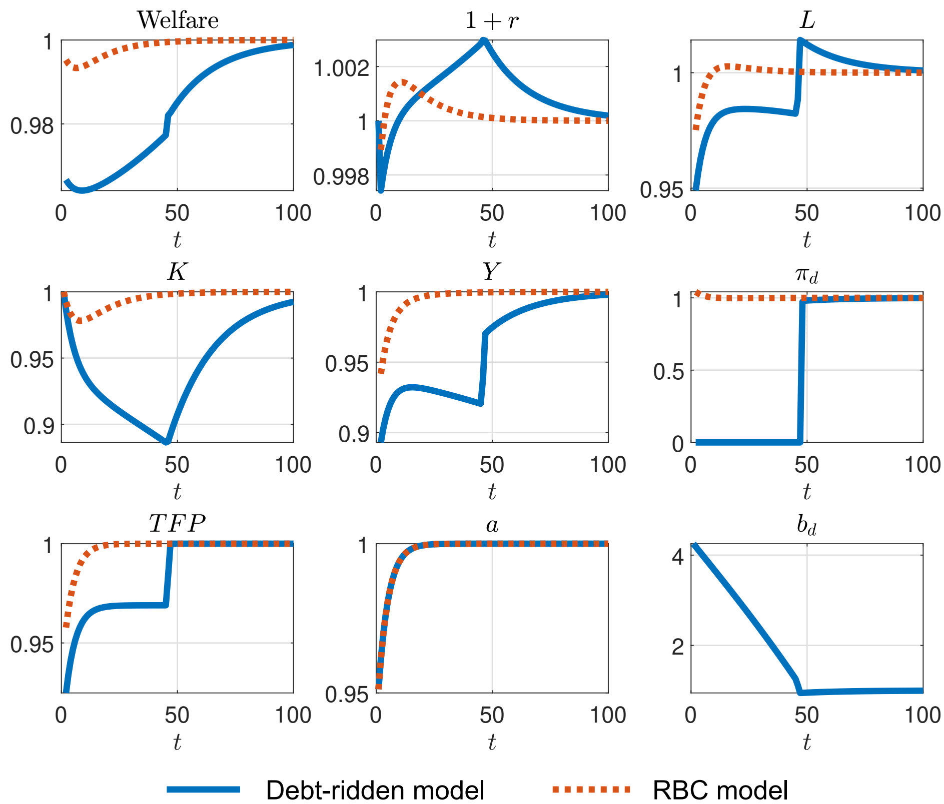
where tfp_t is the TFP shock. The TFP shock follows independent AR(1) process:

$$\ln tfp_{t+1} = \omega \ln tfp_t + (1-\omega) \ln \overline{tfp} + \epsilon_{t+1},$$

where ω is the parameter for persistence, and $\overline{tfp}$ is the steady-state value of TFP. We normalize this level to be 1. The disturbance term ϵ_t is normally distributed with zero mean and variances σ^2. The parameters for TFP shock are chosen from Bi, Shen, and Yang (2016) (i.e., $\omega = 0.79$, $\sigma = 0.007$).

Figure 9 shows impulse responses to a negative TFP shock when the risky firms (measure z firms) owe a large debt at $t = 0$. The economy was in the steady state with all firms having the constrained-efficient amount of debt in period -1, and suddenly, the debts of risky firms jump up to a large debt in period 0. The figure compares results between our model (debt-ridden model) and the standard RBC model, as in the previous subsection. The large debt owed by risky firms negatively affects the macro variables. When a negative TFP shock occurs in this situation, the debt increases further, further depressing the macro variables.

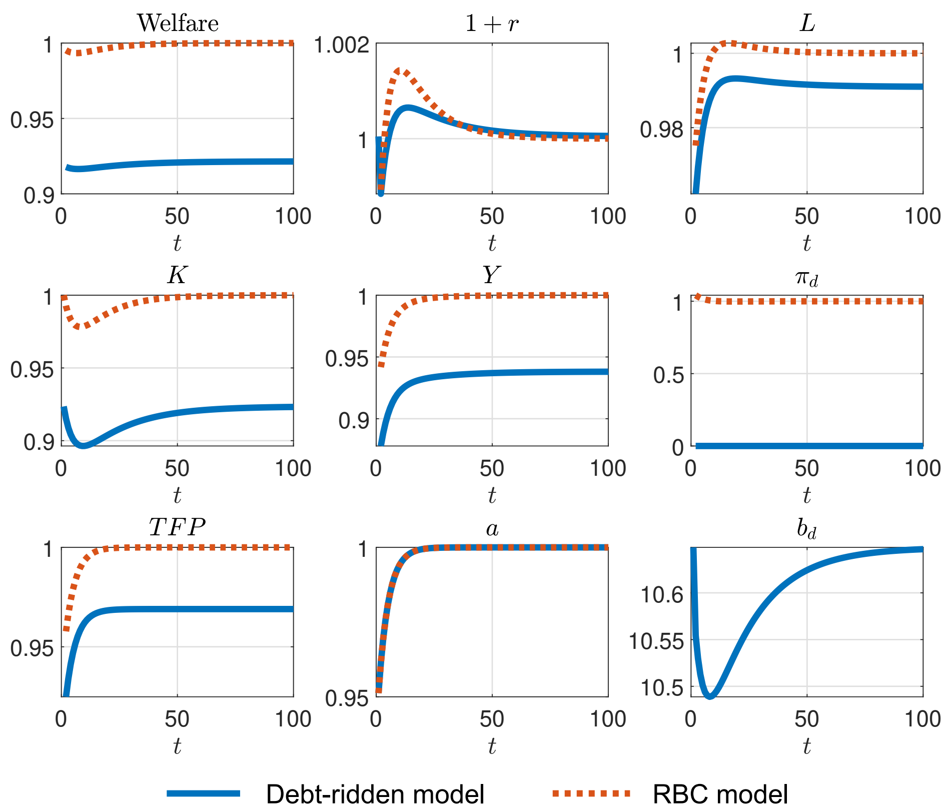
Next, we consider a situation where risky firms are debt-ridden. The production of debt-ridden firms is permanently inefficient because of huge unrepayable debts. The response to negative TFP shocks under such circumstances is illustrated in Figure 10. The vertical axis of the figure indicates the extent of deviation from the steady state where all firms hold optimal debt levels. There is little difference between the two models in the extent to which the macro variables are depressed by the TFP shock. In the debt-ridden model, macro variables converge to an inefficient steady-state equilibrium.



Notes: The vertical axis represents X_t/X as the deviation from the steady-state equilibrium, where X is the value of a corresponding variable in the steady-state equilibrium where all firms owe the constrained-efficient amount of debt. In the debt-ridden model, the ratio of z firms owe a large debt in period 0.

π_d: Risky firm's dividend, f_d: Risky firm's revenue (production), a_d: Risky firm's productivity, b_d: Risky firm's debt

Figure 9: Impulse Responses of TFP shocks to firms with large debt



Notes: The vertical axis represents X_t/X as the deviation from the steady-state equilibrium, where X is the value of a corresponding variable in the steady-state equilibrium where all firms owe the constrained-efficient amount of debt. In the debt-ridden model, the risky firms with measure z are always debt-ridden firms.

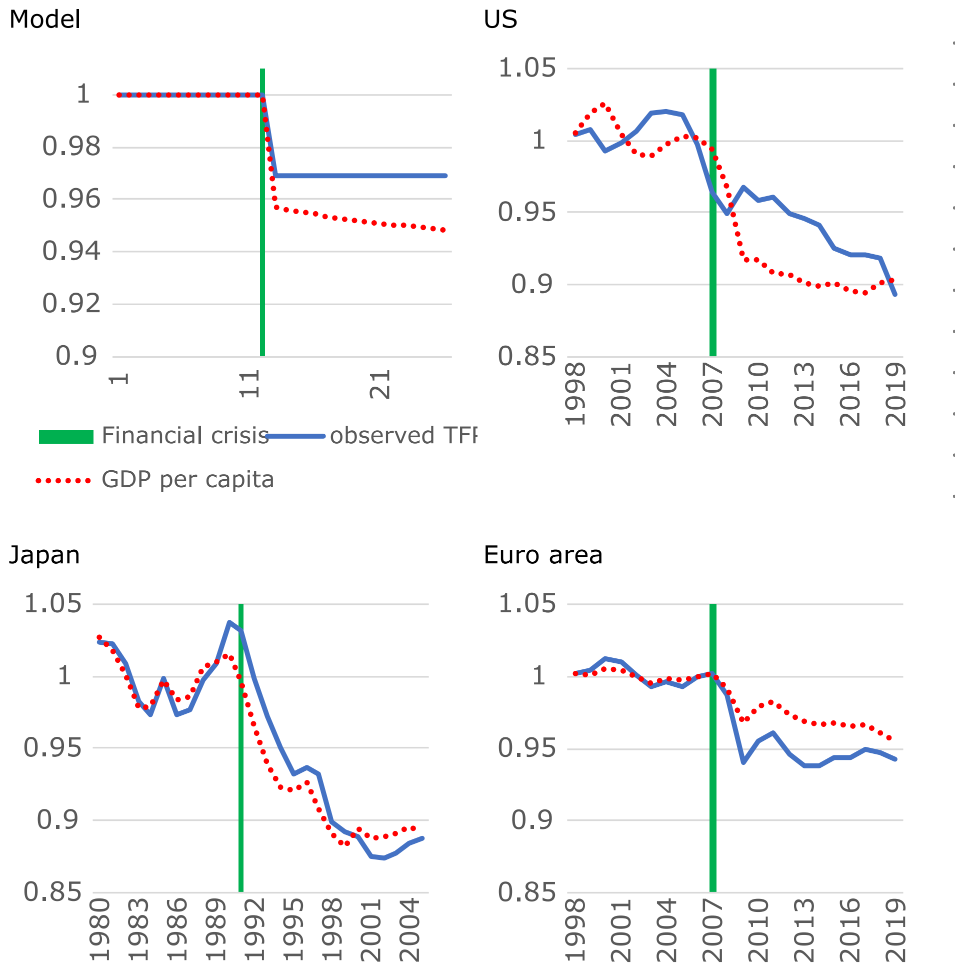
π_d: Risky firm's dividend, f_d: Risky firm's revenue (production), a_d: Risky firm's productivity, b_d: Risky firm's debt

Figure 10: Impulse Responses of TFP shocks to debt-ridden firms

H Persistent effect of a wealth shock

In Section 5.2, we have shown that a negative productivity shock can cause intermediate goods firms to accumulate debt, and the accumulation of debt has a persistent negative impact. We also showed that prolonged low productivity could lead firms to a debt-ridden state. In this section, we show that if a wealth shock in the financial crisis causes many firms to become debt-ridden, our model can explain that the economy can suddenly fall into a persistent stagnation after the crisis, as we observe in the aftermath of the GFC.

Existing literature has examined the exogenous redistribution of assets and liabilities due to wealth shocks as a driving force of the financial crisis (Carlstrom and Fuerst, 1997; Bernanke et al., 1999; Christiano, Motto, and Rostagno, 2010; Christiano, Trabandt, and Walentin, 2011; Kaihatsu and Kurozumi, 2014a). A typical example of a wealth shock is a collapse of the asset-price bubble, which reduces the borrowers' net worth. Following this literature, we examine the response of the economy to a wealth shock that redistributes wealth from intermediate goods firms to households (lenders). Risky firms become excessively indebted, surpassing a threshold of B_Z. In this exercise, we introduce wealth shocks instead of productivity shocks and assume that the level of productivity is constant over time. It is common knowledge that wealth shocks follow the two-state Markov chain. In the low state, the lender owes an additional large debt. The transition probabilities are assumed to be the same as in section C. The wealth shock causes risky firms to become overly indebted. Figure 11 shows the results. Additionally, the figure illustrates the actual detrended GDP and the observed TFP for the US, Japan, and the Euro area. The negative shocks in these series correspond to the bursting of the land price in 1991 in Japan and the housing price in 2007 in the US and Euro area. The simulation result in Figure 11 shows that GDP and the observed TFP are permanently stagnant when risky firms become debt-ridden due to a negative wealth shock. Our model shows that the financial crisis, modeled as a negative wealth shock, can cause prolonged stagnation in the aftermath of the crisis.



Notes: The actual data frequency is annual. These series are detrended by a linear time detrend. We measured the trend from 1990 to 2007 for the US, from 1982 to 1990 for Japan, and from 1995 to 2007 for the Euro area. In Japan, TFP is classified as the "market economy" sector, which excludes education, medical services, government activities, and imputed housing rent.

Sources: US: Fernald (2012); OECD, *Quarterly National Accounts*. Japan: Cabinet Office, Government of Japan, *Annual Report on National Accounts*; The Research Institute of Economy, Trade and Industry, *JIP 2014 database*. Euro area: World Bank, *World Development Indicators*; European Commission, *AMECO database*.

Figure 11: Wealth shock