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Bubble Necessity Theorem*

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Abstract

Asset price bubbles are situations where asset prices exceed the fundamental values defined by the present value of dividends. This paper presents a conceptually new perspective: the *necessity* of asset price bubbles. We establish the Bubble Necessity Theorem in a plausible general class of economic models: with faster long run economic growth (G) than dividend growth (G_d) and counterfactual long run autarky interest rate (R) below dividend growth, *all* equilibria are bubbly with non-negligible bubble sizes relative to the economy. The bubble necessity condition $R < G_d < G$ naturally arises in multi-sector economies with uneven productivity growth with sufficiently high savings motive.

Keywords: bubble, fundamental value, possibility versus necessity.

JEL codes: D53, G12.

1 Introduction

A rational asset price bubble is a situation in which the asset price (P) exceeds its fundamental value (V) defined by the present value of dividends (D). This paper asks whether asset price bubbles *must* arise, that is, the necessity of bubbles. This question is of fundamental importance. Economists have long held the view that bubbles are either not possible in rational equilibrium models or even if they are, a situation in which asset price bubbles occur is a special circumstance and hence

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fragile.[1] If bubbles are inevitable, it would challenge the conventional wisdom and economic modeling. In this paper, we establish a theorem showing that there is a general plausible class of economic models in which asset price bubbles arise in *all* equilibria, implying the necessity of bubbles.

Our question of whether asset price bubbles *must* arise (the necessity of bubbles) is conceptually different from whether asset price bubbles *can* arise (the possibility of bubbles). As is well known from the seminal work of Samuelson (1958) and Tirole (1985), the answer to the latter question is affirmative. These fundamental theoretical contributions have generated a large literature on bubbles and financial conditions as well as policy and quantitative analyses.[2] To date, this literature has almost exclusively focused on pure bubbles, namely assets that pay no dividends ($D = 0$) and hence are intrinsically worthless ($V = 0$). In these models, there always exists an equilibrium in which the asset price equals its fundamental value (*fundamental equilibrium*), which is zero, as well as a bubbly steady state. In addition, there also exist a continuum of bubbly equilibria converging to the fundamental steady state (asymptotically bubbleless equilibria). In the existing rational bubble models, the equilibrium is indeterminate and there is no need for bubbles to arise. This literature is limited to showing the *possibility* of bubbles.

We present a conceptually new perspective on thinking about asset price bubbles: their *necessity*. The main message of our paper can be summarized as follows. Let G be the long run endowment growth rate, G_d the long run dividend growth rate, and R the counterfactual long run autarky interest rate. When the bubble necessity condition

$$R < G_d < G \tag{1.1}$$

holds, we prove that all equilibria feature asset price bubbles with non-negligible

[1]This view is summarized well by the abstract of Santos and Woodford (1997): "Our main results are concerned with nonexistence of asset pricing bubbles in those economies. These results imply that the conditions under which bubbles are possible—including some well-known examples of monetary equilibria—are relatively fragile."

[2]This more applied literature is too large to review comprehensively. Since the 2008 financial crisis, the literature has intensively focused on the relationship between asset bubbles and financial conditions. Examples include Farhi and Tirole (2012), Aoki, Nakajima, and Nikolov (2014), Hirano and Yanagawa (2017), Bengui and Phan (2018), and Miao and Wang (2018). For monetary and fiscal policy analysis in rational bubble models, see Kocherlakota (2009), Galí (2014, 2021), Aoki and Nikolov (2015), Hirano, Inaba, and Yanagawa (2015), Kocherlakota (2023), Plantin (2021), and Amol and Luttmer (2022). Allen, Barlevy, and Gale (2022) examine monetary policy and asset price bubbles using a risk-shifting approach. Barlevy (2018) provides a comprehensive survey on economic theory of bubbles and policy perspectives. For quantitative analysis in rational bubble models, see Miao, Wang, and Xu (2015), Domeij and Ellingsen (2018), and Guerron-Quintana, Hirano, and Jinnai (2023).

bubble sizes relative to the economy (*asymptotically bubbly equilibria*). The intuition for this result is straightforward. If a fundamental equilibrium exists, in the long run the asset price (the present value of dividends) must grow at the same rate of G_d. Then the asset price becomes negligible relative to endowments because $G_d < G$ and the equilibrium consumption allocation approaches autarky. With an autarky interest rate $R < G_d$, the present value of dividends (the fundamental value of the asset) becomes infinite, which is of course impossible in equilibrium. Therefore there exist no fundamental equilibria, and all equilibria must be bubbly.

We emphasize that the bubble necessity condition (1.1) naturally arises in plausible economic models. Regarding the inequality $R < G_d$, note that R is the *counterfactual autarky* interest rate, not the *actual equilibrium* interest rate. In models with sufficiently high savings motives, it is not difficult to create a low interest rate environment in the absence of trade. To clearly and convincingly show that the other inequality $G_d < G$ is also natural, in Section 3 we present an example economy with a unique equilibrium in closed-form. The model is a two-sector production economy with uneven productivity growth. In one sector, labor is the primary input for production such as the labor-intensive service sector. In the other, labor and land are inputs such as the land-intensive agricultural sector. We show that when the productivity growth rate is lower in the land-intensive sector, then the bubble necessity condition (1.1) is satisfied and the only possible equilibrium is one that features a land price bubble. Note that in an economy with multiple sectors, unlike a situation with $G_d = G$ that all sectors have the same growth rate, it is natural to suppose that different sectors have different productivity growth rates, which implies $G_d < G$ for some sectors.

Given the results in Section 3, we establish the Bubble Necessity Theorem using workhorse models in macroeconomics. In Section 4, we provide it in a classical two-period overlapping generations (OLG) model under minimal assumptions on preferences, endowments, and dividends. In Section 5, we establish the Theorem in an infinitely-lived heterogeneous-agent model in the tradition of Bewley (1977) and Scheinkman and Weiss (1986). We show that under some conditions on technological innovations that enhance the overall productivity, the conditions $R < G_d$ and $G_d < G$ are simultaneously satisfied, necessarily generating asset price bubbles. We emphasize that while establishing the Theorem and providing its proof is highly nontrivial, our bubble necessity result is not merely a theoretical curiosity but economically relevant and can naturally arise in modern macroeconomic models.

1.1 Related literature

The theoretical possibility of asset price bubbles in overlapping generations (OLG) models is well known since Samuelson (1958) and Tirole (1985). In a model with infinitely-lived agents, Bewley (1980) and Scheinkman and Weiss (1986) show that an intrinsically worthless asset like fiat money can have a positive value if agents are subject to financial constraints. Following these seminal work, the theoretical literature has studied both necessary and sufficient conditions for the existence of asset price bubbles. Regarding necessary conditions, Kocherlakota (1992) considers a deterministic economy with infinitely-lived agents subject to shortsales constraints and proves in Proposition 4 that in any bubbly equilibrium, the present value of the aggregate endowment must be infinite. Santos and Woodford (1997, Theorem 3.3) significantly extend this result to an abstract general equilibrium model with uncertainty. Regarding sufficient conditions, in an OLG model with a stochastic saving technology, Aiyagari and Peled (1991) show that whenever the equilibrium without fiat money is Pareto inefficient, there exists an equilibrium with valued fiat money. See Barbie and Hillebrand (2018) and Bloise and Citanna (2019) for recent extensions.

A crucial difference of our paper from this literature on the possibility of bubbles is that we prove the necessity of bubbles. In this respect, our paper is related to Wilson (1981) and Tirole (1985). Wilson (1981) studies an abstract general equilibrium model with infinitely many agents and commodities. Under standard assumptions, he proves in Theorem 1 that an equilibrium with transfer payments exists and that the transfers can be set to zero (so budgets balance exactly) for agents endowed with only finitely many commodities. However, he shows in Section 7 through an example that an equilibrium without transfer payments may not exist and notes that an equilibrium with positive transfer could be interpreted as an equilibrium with money as in Samuelson (1958). (He never uses the word "bubble".) To our knowledge, this is the first (and only) example of the nonexistence of fundamental equilibria. Relative to Wilson (1981), our contribution is that we provide a general nonexistence theorem in workhorse models, and the nonexistence applies not only to fundamental equilibria but also to bubbly equilibria that are asymptotically bubbleless.

Proposition 1(c) of Tirole (1985) recognizes the possibility that bubbles are necessary for equilibrium existence if the interest rate without bubbles is negative in an OLG model with positive population growth and constant rents, which corresponds to $R < G_d = 1 < G$. Although he gives some explanations on p. 1506 in the sentence starting with "The intuition behind this fact roughly runs as follows," he did not necessarily provide a formal proof.[3] To the best of our understanding, his Proposition 1(c) is limited to providing an important conjecture. Indeed, proving the nonexistence of fundamental or asymptotically bubbleless equilibria in a robust way is not straightforward and is highly nontrivial. From a purely theoretical point of view, without a rigorous mathematical proof, the possibility of the nonexistence of fundamental equilibria remains a conjecture and an open question. We resolve this long-standing unsolved conjecture affirmatively and establish the Bubble Necessity Theorem.[4]

2 Definition and characterization of bubbles

This section defines asset price bubbles and provides an exact characterization.

We consider an infinite-horizon, deterministic economy with a homogeneous good and time indexed by $t = 0, 1, \dots$. Consider an asset with infinite maturity that pays dividend $D_t \geq 0$ and trades at ex-dividend price P_t, both in units of the time-t good. In the background, we assume the presence of rational, perfectly competitive investors. Free disposal of the asset implies $P_t \geq 0$.[5] Let $q_t > 0$ be the Arrow-Debreu price, i.e., the date-0 price of the consumption good delivered at time t, with the normalization $q_0 = 1$. The absence of arbitrage implies

$$q_t P_t = q_{t+1}(P_{t+1} + D_{t+1}). \tag{2.1}$$

[3] In Tirole (1985), the proof of the nonexistence of fundamental equilibria appears at the bottom of p. 1522 and the top of p. 1523. The proof uses a convergence result discussed in Lemma 2. However, this convergence heavily relies on the monotonicity condition on the function ψ defined in Equation (7) on p. 1502. This monotonicity/stability condition, which Tirole adopts from Diamond (1965), is a high-level assumption that need not be satisfied in a general setting. In fact, Tirole does not provide any example for which this assumption is satisfied, and in OLG models it is well known that there are robust examples of equilibrium with cycles (Geanakoplos and Polemarchakis, 1991, §5), which necessarily violates this assumption.

[4] In mathematics, a conjecture and a theorem are obviously completely different. Providing an important conjecture is highly valuable. Proving the conjecture rigorously and promoting it to a theorem is equally valuable. A famous example is Fermat's Last Theorem. Andrew Wiles proved it for the first time by proving the Taniyama-Shimura conjecture. The Taniyama-Shimura conjecture had been known for many decades as a conjecture until Wiles proved it.

[5] If $P_t < 0$, by purchasing one additional share of the asset at time t and immediately disposing it, an investor can increase consumption at time t by $-P_t > 0$ with no cost, which violates individual optimality.

Iterating the no-arbitrage condition (2.1) forward and using $q_0 = 1$, we obtain

$$P_0 = \sum_{t=1}^{T} q_t D_t + q_T P_T. \tag{2.2}$$

Noting that $P_t \geq 0$, $D_t \geq 0$, and $q_t > 0$, the infinite sum of the present value of dividends

$$V_0 \coloneqq \sum_{t=1}^{\infty} q_t D_t \tag{2.3}$$

exists, which is called the *fundamental value* of the asset. Letting $T \to \infty$ in (2.2), we obtain

$$P_0 = \sum_{t=1}^{\infty} q_t D_t + \lim_{T\to\infty} q_T P_T = V_0 + \lim_{T\to\infty} q_T P_T. \tag{2.4}$$

We say that the *transversality condition* for asset pricing holds if

$$\lim_{T\to\infty} q_T P_T = 0. \tag{2.5}$$

When the transversality condition (2.5) holds, the identity (2.4) implies that $P_0 = V_0$ and the asset price equals its fundamental value. If $\lim_{T\to\infty} q_T P_T > 0$, then $P_0 > V_0$, and we say that the asset contains a *bubble*.

Note that in deterministic economies, for all t we have

$$P_t = \underbrace{\frac{1}{q_t} \sum_{s=1}^{\infty} q_{t+s} D_{t+s}}_{=:V_t} + \frac{1}{q_t} \lim_{T\to\infty} q_T P_T. \tag{2.6}$$

Therefore either $P_t = V_t$ for all t or $P_t > V_t$ for all t, so the economy is permanently in either the bubbly or the fundamental regime. Thus, a bubble is a permanent overvaluation of an asset, which is a feature of rational expectations.

In general, checking the transversality condition (2.5) directly could be difficult because it involves q_T. The following lemma provides an equivalent characterization. This lemma, although quite simple, may be of independent interest because it significantly facilitates checking the presence or absence of bubbles.

Lemma 2.1 (Bubble Characterization)**.** *If $P_t > 0$ for all t, the asset price exhibits a bubble if and only if $\sum_{t=1}^{\infty} D_t/P_t < \infty$.*

We prove Lemma 2.1 in the main text as it is short and has many applications.

Proof. Because the economy is deterministic, the interest rate is defined by the asset return, so $R_t = (P_{t+1} + D_{t+1})/P_t$. Using the definition of q_t, we obtain

$$\begin{aligned} q_T P_T &= P_T \prod_{t=0}^{T-1} \frac{1}{R_t} = P_T \prod_{t=0}^{T-1} \frac{P_t}{P_{t+1} + D_{t+1}} \\ &= P_0 \prod_{t=1}^{T} \frac{P_t}{P_t + D_t} = P_0 \left(\prod_{t=1}^{T} \left(1 + \frac{D_t}{P_t} \right) \right)^{-1}. \end{aligned}$$

Expanding terms and using $1 + x \le \mathrm{e}^x$, we obtain

$$P_0 \exp\left(-\sum_{t=1}^{T} \frac{D_t}{P_t} \right) \le q_T P_T \le P_0 \left(1 + \sum_{t=1}^{T} \frac{D_t}{P_t} \right)^{-1}.$$

Letting $T \to \infty$, we have $\lim_{T\to\infty} q_T P_T > 0$ if and only if $\sum_{t=1}^{\infty} D_t/P_t < \infty$. $\square$

Lemma 2.1 implies that there is an asset price bubble if and only if the infinite sum of dividend yields D_t/P_t is finite. Because $\sum_{t=1}^{\infty} 1/t = \infty$ but $\sum_{t=1}^{\infty} 1/t^\alpha < \infty$ for any $\alpha > 1$, roughly speaking, there is an asset price bubble if the price-dividend ratio P_t/D_t grows faster than linearly.

3 Example with only bubbly equilibria

In this section, to convince the reader of the necessity of asset price bubbles in plausible economic models, we present an example economy with a unique equilibrium in which asset price bubbles necessarily arise.

Consider a two-period overlapping generations model with production. Each agent has the Cobb-Douglas utility function $U(y, z) = (1-\beta)\log y + \beta \log z$, where $\beta \in (0, 1)$ and y, z denote the consumption when young and old. The initial old is endowed with a unit supply of land, which is durable and non-reproducible. Each period, the young are endowed with one unit of labor and the old none. There are two production technologies with time t production functions given by

$$F_{1t}(H, L) = G_1^t H, \tag{3.1a}$$

$$F_{2t}(H, L) = (G_2^t H)^\alpha L^{1-\alpha}, \tag{3.1b}$$

where H, L denote the inputs of labor and land and $\alpha \in (0, 1)$. Sector 1 uses labor as the primary input and can be interpreted as the labor-intensive service industry with labor-augmenting productivity growth rate G_1. Sector 2 uses both labor and land as inputs and can be interpreted as the land-intensive agriculture with a Cobb-Douglas production function and labor-augmenting productivity growth rate G_2.[6]

A competitive equilibrium with sequential trading is defined by a sequence

$$\{(P_t, r_t, w_t, x_t, y_t, z_t, H_{1t}, H_{2t})\}_{t=0}^{\infty}$$

of land price P_t, land rent r_t, wage w_t, land holdings of young x_t, consumption of young and old (y_t, z_t), and labor allocation (H_{1t}, H_{2t}) such that, (i) the young maximize utility subject to the budget constraints $y_t + P_t x_t = w_t$ and $z_{t+1} = (P_{t+1} + r_{t+1})x_t$, (ii) firms maximize profits, (iii) commodity market clears: $y_t + z_t = G_1^t H_{1t} + (G_2^t H_{2t})^\alpha$, (iv) labor market clears: $H_{1t} + H_{2t} = 1$, and (v) land market clears: $x_t = 1$. In this model, we can prove the necessity of land bubbles under some conditions on productivity growth rates in different sectors.

Proposition 3.1. *If $G_2^\alpha < G_1$, then the unique equilibrium land price is $P_t = \beta G_1^t$, and there is a bubble.*

Proof. In equilibrium, it must be $H_{2t} > 0$ by the Inada condition. Therefore profit maximization of Sector 2 implies

$$w_t = \alpha (G_2^t H_{2t})^{\alpha-1} G_2^t = \alpha G_2^{\alpha t} H_{2t}^{\alpha-1}.$$

If $H_{2t} = 1$, then $w_t = \alpha G_2^{\alpha t} < G_1^t$ because $\alpha < 1$ and $G_2^\alpha < G_1$. Then firms in Sector 1 make infinite profits, which is a contradiction. Therefore in equilibrium it must be $H_{2t} \in (0, 1)$, and profit maximization of Sector 1 implies $w_t = G_1^t$. Therefore labor in Sector 2 satisfies

$$\alpha G_2^{\alpha t} H_{2t}^{\alpha-1} = w_t = G_1^t \iff H_{2t} = \alpha^{\frac{1}{1-\alpha}} (G_2^\alpha / G_1)^{\frac{t}{1-\alpha}}.$$

By profit maximization in Sector 2 and $L = 1$, land rent satisfies

$$r_t = (1-\alpha)(G_2^t H_{2t})^\alpha = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} (G_2/G_1)^{\frac{\alpha t}{1-\alpha}}.$$

Due to log utility, the optimal consumption of the young is $y_t = (1-\beta) w_t$. Because land is the only store of value, the budget constraint and $x_t = 1$ imply

$$P_t = P_t x_t = w_t - y_t = \beta w_t = \beta G_1^t.$$

[6] In Sector 2, in addition to a labor-augmenting technological progress, we can also include a land-augmenting technological progress. Since the production function is Cobb-Douglas, we obtain the same result by redefining G_2^t.

Clearly, the equilibrium is unique. The dividend yield on land is then

$$\frac{r_t}{P_t} = \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}(G_2/G_1)^{\frac{\alpha t}{1-\alpha}}}{\beta G_1^t} = \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{\beta}(G_2^\alpha/G_1)^{\frac{t}{1-\alpha}},$$

which decays geometrically and is summable because $G_2^\alpha < G_1$ by assumption. Therefore by Lemma 2.1, there is a land bubble. □

The essential point of this example can be understood as follows. Let G be the asymptotic growth rate of the economy, which equals G_1 in this example. Let G_d be the asymptotic dividend growth rate, which equals $(G_2/G_1)^{\frac{\alpha}{1-\alpha}}$ in this example. The condition $G_2^\alpha < G_1$ in Proposition 3.1 is then equivalent to the bubble necessity condition $G_d < G$ discussed in the introduction. Whenever (1.1) holds, we claim that bubbles must arise in equilibrium. Although the proof depends on the context (in Proposition 3.1, we proved it directly), the intuition is straightforward and is the same as the one we stated in the introduction.

Interestingly, along the equilibrium path, the wage (income of the young) grows at the same rate as the productivity growth of Sector 1. At the same time, since Sector 1 has higher productivity growth, labor moves into Sector 1. During this transitional dynamics, to attract labor in Sector 2 under rising wages, land rents will be suppressed. Then the growth rate of land rents (dividends) will become lower than the growth rate of incomes. On the other hand, since the land price rises together with the income of the young, it will eventually exceed its fundamental value, necessarily generating a bubble. From a backward induction argument, the price of land contains a bubble at all dates.

As this example shows, bubbles naturally arise when there are multiple sectors with different productivity growth rates, which implies $G_d < G$ for some sectors. For instance, the productivity growth rate of production sectors (say manufacturing, information/technology, or service) could be higher in the long run than that of land sectors (say agriculture, construction, or real estate industries), which is exactly the case in the model of Proposition 3.1.[7]

4 Necessity of asset price bubbles

In Section 3, we showed through an example that asset price bubbles must arise in all equilibria under the bubble necessity condition $R < G_d < G$ (1.1) in plausible

[7]In fact, according to Acemoglu (2009, p. 698, Figure 20.1), the employment share of agriculture in U.S. has declined from about 80% to below 5% over the past two centuries, so the technological growth rate in the "land" sector has been lower than the whole economy.

economic models. In this section, we establish the Bubble Necessity Theorem in an abstract two-period OLG model.

4.1 Model

We consider an OLG endowment economy with a long-lived asset.

Agents At each date $t = 0, 1, \dots$, a unit mass of a new generation of agents are born, who live for two dates. An agent born at time t has utility function $U_t(y_t, z_{t+1})$, where y_t, z_{t+1} are the consumption when young and old. At $t = 0$, there is a unit mass of old agents, who care only about their consumption z_0.

Commodities and asset There is a single perishable good. The endowments of the young and old at time t are denoted by $a_t, b_t > 0$. There is a unit supply of a dividend-paying asset with infinite maturity. Let $D_t \geq 0$ be the dividend of the asset at time t.

Budget constraints Letting $P_t \geq 0$ be the asset price and x_t the number of asset shares demanded by the young, the budget constraints are

$$\text{Young:} \qquad y_t + P_t x_t = a_t, \tag{4.1a}$$

$$\text{Old:} \qquad z_{t+1} = b_{t+1} + (P_{t+1} + D_{t+1})x_t. \tag{4.1b}$$

That is, at time t the young decide to spend the income a_t on consumption y_t and asset purchase $P_t x_t$, and at time $t + 1$ the old liquidate all wealth to consume. The asset demand x_t is arbitrary (positive or negative) as long as consumption is nonnegative.

Equilibrium Our equilibrium notion is the competitive equilibrium with sequential trading.

Definition 1. A competitive equilibrium consists of a sequence of prices $\{P_t\}_{t=0}^{\infty}$ and allocations $\{(x_t, y_t, z_t)\}_{t=0}^{\infty}$ satisfying the following conditions.

(i) (Individual optimization) The initial old consume $z_0 = b_0 + P_0 + D_0$; for all t, the young maximize utility $U_t(y_t, z_{t+1})$ subject to the budget constraints (4.1).

(ii) (Commodity market clearing) $y_t + z_t = a_t + b_t + D_t$ for all t.

(iii) (Asset market clearing) $x_t = 1$ for all t.

Note that, because at each date there are only two types of agents (the young and old) and the old exit the economy, market clearing forces that the young buy the entire shares of the asset, which explains $x_t = 1$.

We define fundamental and bubbly equilibria in the obvious way.

Definition 2. An equilibrium is *fundamental* (*bubbly*) if $P_0 = V_0$ ($P_0 > V_0$).

By definition, a bubbly equilibrium is an equilibrium in which the asset price exceeds its fundamental value. However, as discussed in the introduction, pure bubble models with $D_t = 0$ often admit a fundamental equilibrium and a bubbly steady state as well as a continuum of bubbly equilibria converging to the fundamental steady state.[8] Therefore, proving the nonexistence of fundamental equilibria alone may not be convincing because it does not rule out bubbly equilibria converging to the fundamental steady state.

We thus define a more demanding equilibrium concept, *asymptotically bubbly equilibria*, which are bubbly equilibria with non-negligible bubble sizes relative to the economy.

Definition 3 (Asymptotically bubbly equilibria)**.** Let $\{P_t\}_{t=0}^{\infty}$ be equilibrium asset prices. The asset is *asymptotically relevant (irrelevant)* if

$$\liminf_{t\to\infty} \frac{P_t}{a_t} > 0 \quad (= 0). \tag{4.2}$$

A bubbly equilibrium is *asymptotically bubbly (bubbleless)* if the asset is asymptotically relevant (irrelevant).

By Definition 3, the set of asymptotically bubbly equilibria is a subset of bubbly equilibria. The relevance condition (4.2) implies that there exists $p > 0$ such that $P_t/a_t \geq p$ for all large enough t, so the asset price is always a non-negligible fraction of the endowment of the young. Because the young are the natural buyer of the asset, this condition implies that there will be trade in the asset in the long run, which motivates the term "relevance".

4.2 Existence and characterization of equilibrium

We introduce the following assumptions.

[8] See Appendix B for a detailed analysis of a specific example using a pure bubble model.

Assumption 1. *For all t, the utility function $U_t : \mathbb{R}^2_+ \to [-\infty, \infty)$ is continuous, quasi-concave, and continuously differentiable on $\mathbb{R}^2_{++}$ with positive partial derivatives.*

Assumption 1 is standard. We define the marginal rate of substitution

$$M_t(y, z) := \frac{(U_t)_z(y, z)}{(U_t)_y(y, z)} > 0, \tag{4.3}$$

where we denote partial derivatives with subscripts, e.g., $F_y = \partial F/\partial y$.

Assumption 2. *The dividends satisfy $D_t > 0$ infinitely often.*

The following theorem shows that an equilibrium always exists under the maintained assumptions. Furthermore, it characterizes the equilibrium as a solution to a nonlinear difference equation.

Theorem 1 (Existence of equilibrium)**.** *If Assumption 1 holds, an equilibrium exists. The asset prices satisfy $0 \leq P_t \leq a_t$ and*

$$P_t = \min \{M_t(y_t, z_{t+1})(P_{t+1} + D_{t+1}), a_t\}, \tag{4.4}$$

where $(y_t, z_{t+1}) = (a_t - P_t, b_{t+1} + P_{t+1} + D_{t+1})$. If in addition Assumption 2 holds, then $P_t > 0$.

Note that if $P_t < a_t$, (4.4) reduces to the familiar asset pricing equation

$$P_t = M_t(y_t, z_{t+1})(P_{t+1} + D_{t+1}).$$

In general, the min operator appears in (4.4) due to the nonnegativity constraint on consumption: $y_t = a_t - P_t x_t \geq 0$. If we assume the Inada condition $U_y(0, z) = \infty$, this constraint never binds. If $0 < P_t < a_t$, because the economy is deterministic, we can define the interest rate by

$$R_t := \frac{P_{t+1} + D_{t+1}}{P_t} = \frac{1}{M(y_t, z_{t+1})}. \tag{4.5}$$

4.3 Bubble Necessity Theorem

The question of fundamental importance is whether bubbles must arise in equilibrium, that is, whether it is possible that fundamental equilibria or asymptotically bubbleless equilibria may fail to exist. We now state and prove the Bubble Necessity Theorem under additional assumptions.

Assumption 3. *The endowments $\{(a_t, b_t)\}_{t=0}^{\infty}$ satisfy*

$$\lim_{t\to\infty} \frac{a_{t+1}}{a_t} =: G \in (0, \infty), \tag{4.6a}$$

$$\lim_{t\to\infty} \frac{b_t}{a_t} =: w \in (0, \infty). \tag{4.6b}$$

Condition (4.6a) implies that the endowments of the young grow at rate $G > 0$ in the long run. Although we used the term "growth", it could be $G \leq 1$, so stationary or shrinking economies are also allowed. Furthermore, (4.6a) is an assumption only at infinity, so the endowments are arbitrary for arbitrarily long finite periods. Condition (4.6b) implies that the old-to-young endowment ratio approaches w in the long run. Again, the income ratio is arbitrary for arbitrarily long finite periods.

Using the marginal rate of substitution (MRS) (4.3), we define the scaled MRS $m_t : \mathbb{R}_{++}^2 \to \mathbb{R}_{++}$ by

$$m_t(y, z) := M_t(a_t y, a_t z). \tag{4.7}$$

We impose the following uniform convergence condition on m_t.

Assumption 4. *There exists a continuous function $m : \mathbb{R}_{++}^2 \to \mathbb{R}_{++}$ such that $m_t \to m$ uniformly on compact sets, that is, for any nonempty compact set $K \subset \mathbb{R}_{++}^2$, we have*

$$\lim_{t\to\infty} \sup_{(y,z)\in K} |m_t(y, z) - m(y, z)| = 0. \tag{4.8}$$

There are two interpretations of Assumption 4. If the utility function U_t is homothetic, then the marginal rate of substitution (4.3) is homogeneous of degree 0, so $m_t = M_t$ by (4.7). Alternatively, if there is population growth as in Tirole (1985), a_t denotes the population size, and $u_t(y, z) := U_t(a_t y, a_t z)$ denotes the utility of an agent consuming (y, z), then m_t in (4.7) is exactly the MRS of u_t. In either case, condition (4.8) reduces to the uniform convergence of MRS instead of the scaled MRS.

The following theorem is the main result of this paper. It shows that, under some assumptions on the model parameters, all equilibria are asymptotically bubbly. To state the result, we define the asymptotic dividend growth rate by

$$G_d := \limsup_{t\to\infty} D_t^{1/t}. \tag{4.9}$$

Theorem 2 (Bubble Necessity Theorem)**.** *Suppose Assumptions 1–4 hold and let*

G_d be as in (4.9). If

$$\frac{1}{m(1, Gw)} < G_d < G, \tag{4.10}$$

then all equilibria are asymptotically bubbly.

Note that the statement of Theorem 2 is not vacuous because an equilibrium always exists by Theorem 1. Although the proof of Theorem 2 is highly nontrivial (we shall describe the steps momentarily), its intuition is straightforward. For simplicity, suppose that the endowments grow at a constant rate G, so $a_t = a_0 G^t$. Suppose also that the dividends grow at a constant rate $G_d < G$, so $D_t = D_0 G_d^t$. If a fundamental equilibrium exists, the price P_t must asymptotically grow at rate G_d because (by the definition of a fundamental equilibrium) it must equal the present value of dividends, which grow at rate G_d. Because $G_d < G$, the price-income ratio P_t/a_t grows at rate $G_d/G < 1$ and hence converges to zero, so the asset becomes asymptotically irrelevant. Hence the budget constraint (4.1) together with the asset market clearing condition $x_t = 1$ implies that the consumption allocation (y_t, z_{t+1}) approaches autarky in the long run, and the equilibrium interest rate (4.5) converges to

$$R_t = \frac{1}{M_t(y_t, z_{t+1})} \to \frac{1}{m(1, Gw)} < G_d \tag{4.11}$$

by Assumption 4 and condition (4.10). However, (4.11) implies that the interest rate is asymptotically lower than the dividend growth rate, so the fundamental value of the asset is infinite, which is of course impossible in equilibrium. Thus a fundamental equilibrium cannot exist.

Of course this argument is heuristic, so here we explain the steps of the proof more clearly. Let $p_t = P_t/a_t$ be the price-income ratio and $d_t = D_t/a_t$ be the dividend-income ratio. In Lemma A.3, we show that d_t is summable and hence $d_t \to 0$, which uses $G_d < G$. In Lemma A.4, we show that in any equilibrium, the growth rate of the price-income ratio p_{t+1}/p_t is bounded above, which uses the asset pricing equation (4.4). In Lemma A.5, we show that this upper bound can be chosen to be less than 1 (so $p_{t+1}/p_t < 1$) whenever p_t is small enough, which relies on the condition (4.10). In Lemma A.6, we show that in any equilibrium in which the asset is asymptotically irrelevant, $\{p_t\}$ always converges to 0. This step is immediate, because if the asset is asymptotically irrelevant, p_t can get arbitrarily close to 0 by Definition 3, and once p_t gets close enough to 0, it decays geometrically by Lemma A.5. In Lemma A.7, we show that in any equilibrium, the asset must be asymptotically relevant. This claim holds, because if the asset is asymptotically irrelevant, then $p_t \to 0$ by Lemma A.6, the interest rate R_t converges to the autarky value (4.11), and the argument in the preceding paragraph leads to a contradiction. In the last step, we show that all equilibria are asymptotically bubbly, which is where we use the Bubble Characterization Lemma 2.1 together with Lemma A.3.

4.4 Examples

We present three examples to illustrate Theorem 2.

Example 1 (Wilson, 1981)**.** Let the utility function be $U(y,z) = y + \beta z$, endowments $(a_t, b_t) = (aG^t, bG^t)$, and dividends $D_t = DG_d^t$. Since the marginal rate of substitution $M(y,z) = \beta$ is constant, Assumptions 1–4 clearly hold. The bubble necessity condition (4.10) holds if $1/\beta < G_d < G$. The example in Wilson (1981, §7) is a special case with $\beta = 3$, $G_d = 1/2$, and $G = 1$.

In fact, for Example 1 we can prove the uniqueness of equilibrium.

Proposition 4.1. *If $1/\beta < G_d < G$, then the unique equilibrium asset price is $P_t = aG^t$, and there is a bubble.*

Example 2 (Two-sector production economy)**.** Recall the OLG two-sector production economy in Section 3. As discussed after Proposition 3.1, under the condition $G_2^\alpha < G_1$, we have

$$0 = R < G_d = (G_2/G_1)^{\frac{\alpha}{1-\alpha}} < G_1 = G,$$

which reduces to (4.10).

The next example is perhaps the most canonical.

Example 3. Suppose agents have the constant relative risk aversion (CRRA) utility function

$$U(y,z) = \frac{y^{1-\gamma}}{1-\gamma} + \beta \frac{z^{1-\gamma}}{1-\gamma},$$

where $\beta > 0$ and $\gamma > 0$ (the case $\gamma = 1$ corresponds to log utility). Suppose the endowments grow at a constant rate $G > 0$, so $(a_t, b_t) = (aG^t, bG^t)$, and the dividends are positive but constant: $D_t = D > 0$. Then $G_d = 1$ using (4.9).

Assumptions 1–3 are clearly satisfied. Regarding Assumption 4, using (4.7), we obtain

$$m_t(y,z) = M(a_t y, a_t z) = \beta \left(\frac{a_t z}{a_t y} \right)^{-\gamma} = \beta (z/y)^{-\gamma} =: m(y,z),$$

so (4.8) clearly holds. Since $G_d = 1$, (4.10) reduces to

$$\frac{1}{\beta(bG/a)^{-\gamma}} < 1 < G \iff a > \beta^{-1/\gamma} G b. \tag{4.12}$$

Thus if (4.12) holds, all assumptions of Theorem 2 are satisfied and all equilibria are asymptotically bubbly.

The bubble necessity condition (4.12) implies that, if the young are sufficiently rich, all equilibria are asymptotically bubbly. The intuition is that, when the young are sufficiently rich, they have a high demand for savings, which pushes down the interest rate, but the interest rate cannot fall below the dividend growth rate, for otherwise the asset price would be infinite. This implies that when the economy falls into a region characterized by sufficiently low interest rate, the only possible outcome is an asset price bubble.

Although an application of Theorem 2 to Example 3 implies that all equilibria are asymptotically bubbly, Theorem 2 does not say anything about how to analyze them. For this purpose, we may apply the local stable manifold theorem (essentially linearization around the steady state); see Appendix C for details.

5 Necessity of bubbles in Bewley-type economy

Section 4 provides sufficient conditions for the necessity of asset price bubbles in overlapping generations endowment economies with exogenous dividends. Although we chose this model for simplicity and clarity, there is nothing special about this apparently restrictive setting. In fact, the model in Section 3 is an OLG two-sector production economy with endogenous dividends. In this section, to further show the robustness of the necessity of bubbles, we present a Bewley-type endogenous growth model with infinitely-lived heterogeneous agents.

Agents There are countably infinitely many agents indexed by $i = 1, 2, \dots$.[9] The agents are infinitely-lived and have utility function

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \log c_t, \tag{5.1}$$

[9]Although the countability assumption may appear nonstandard, it is convenient to avoid the measurability issue associated with a continuum of agents, while making the mathematical argument simple yet rigorous. See Sun and Zhang (2009) for an approach with a continuum of agents.

where $\beta \in (0,1)$ is the discount factor and $c_t \geq 0$ is consumption. Every period, each agent could be one of $N+1$ productivity types indexed by $n \in \{0,1,\dots,N\}$. The type of agent i at time t is denoted by n_{it}, which evolves over time according to a Markov chain with transition probability matrix $\Pi = (\pi_{nn'})$, where $\pi_{nn'} = \Pr(n_{i,t+1} = n' \mid n_{it} = n)$. Productivity is independent across agents, so there is no aggregate uncertainty. In what follows, to simplify the notation, we suppress the i subscript whenever no confusion arises.

Investment Agents have two ways to transfer wealth across time. One is through the purchase of a dividend-paying asset (land) yielding dividend $D_t \geq 0$ at time t, whose aggregate supply is normalized to 1. The other is through investment in a production technology. The gross return on production for a type n agent is $z_n \geq 0$. That is, if an agent invests capital $k_t \geq 0$ at time t, it yields an output of $y_{t+1} = z_{n_t} k_t$ at time $t+1$. We assume $0 = z_0 < z_1 < \cdots < z_N$, so type 0 can be interpreted as savers (who have no entrepreneurial skill) and type $n \geq 1$ can be interpreted as investors, with higher types being more productive.

Budget constraint The time $t \geq 1$ budget constraint of a typical agent is

$$c_t + k_t + P_t x_t = w_t =: z_{n_{t-1}} k_{t-1} + (P_t + D_t) x_{t-1}, \tag{5.2}$$

where $c_t \geq 0$ is consumption, $k_t \geq 0$ is capital investment, P_t is the land price, and x_t is land holdings. That is, the wealth of an agent at time t consists of production $z_{n_{t-1}} k_{t-1}$ and the market value of land, which the agent spends on consumption, investment, and purchase of land. At $t = 0$, the budget constraint (5.2) holds by setting $z_{n_{t-1}} k_{t-1} = y_0$, where $y_0 > 0$ is the initial endowment of the good. We suppose that land cannot be shorted, so $x_t \geq 0$.[10]

Equilibrium The economy starts at $t = 0$ with an initial specification of types and endowments of the good and land $\{(n_{i0}, y_{i0}, x_{i,-1})\}_{i \in \mathbb{N}}$, which is an IID draw from some distribution. Applying the strong law of large numbers, we define aggregate quantities by the cross-sectional average. For instance, the aggregate wealth at time t is denoted by $W_t := \lim_{I \to \infty} \frac{1}{I} \sum_{i=1}^{I} w_{it}$.

[10] The Santos and Woodford (1997) Bubble Impossibility Theorem implies that in order for bubbles to arise, the present value of the aggregate endowment must be infinite. Unlike in OLG models where agents have finite lives, in models with infinitely-lived agents, to prevent agents from capitalizing infinite wealth, some financial constraints such as shortsales or borrowing constraints are necessary. See also the discussion in Kocherlakota (1992).

The equilibrium concept is the rational expectations equilibrium, defined as follows.

Definition 4. A *rational expectations equilibrium* consists of sequences of land prices $\{P_t\}_{t=0}^{\infty}$ and allocations $\{(c_{it}, k_{it}, x_{it})_{i \in \mathbb{N}}\}_{t=0}^{\infty}$ such that

(i) each agent maximizes utility (5.1) subject to the budget constraint (5.2) and the nonnegativity constraints $c_{it}, k_{it}, x_{it} \geq 0$, and

(ii) the land market clears, so $\lim_{I \to \infty} \frac{1}{I} \sum_{i=1}^{I} x_{it} = 1$ almost surely.

We define a bubbly equilibrium as in Definition 2. We introduce the following assumption.

Assumption 5. *The transition probability matrix* $\Pi = (\pi_{nn'})$ *as well as its* $N \times N$ *submatrix* $\Pi_1 := (\pi_{nn'})_{n,n'=1}^{N}$ *are irreducible.*

The irreducibility of Π implies that the agent types are not permanent. The irreducibility of Π_1 implies that once agents become investors, with positive probability, they could visit all productivity states before returning to savers.

The following theorem shows the necessity of asset price bubbles under the condition $R < G_d < G$.

Theorem 3 (Necessity of bubbles in Bewley-type model)**.** *Let* $A := (\beta z_n \pi_{nn'})$ *with* $G := \rho(A)$ *its spectral radius and* $G_d := \limsup_{t \to \infty} D_t^{1/t}$*. If Assumption 5 holds and* $0 < G_d < G$*, then all equilibria are bubbly.*

Some remarks are in order. Since by assumption $z_0 = 0$, type 0 agents have no entrepreneurial skill and can save only through the purchase of land. Under this condition, the counterfactual autarky interest rate is $R = 0$. The numbers $G = \rho(A)$ and G_d can be interpreted as (a lower bound of) the long run economic growth rate and the long run dividend growth rate, respectively. (The actual growth rate of the economy is of course endogenously determined because the risk-free rate is endogenous.) Thus the assumption of Theorem 3 exactly parallels that of Theorem 2.

Using $G = \rho(A)$, we can derive some comparative statics. Note that by Theorem 8.1.18 of Horn and Johnson (2013), $G = \rho(A)$ is increasing in each z_n. Therefore if the productivity of capital investment gets sufficiently high, the condition $G > G_d$ will be satisfied. This implies that technological innovations that enhance the overall productivity will inevitably generate asset price bubbles, which is consistent with the view in Scheinkman (2014, p. 22) that highlights the importance of the relationship between technological progress and asset price bubbles. Furthermore, applying an intermediate step of the proof of Proposition 5(iv) in Beare and Toda (2022), $G = \rho(A)$ is also increasing in the persistence of the Markov chain,[11] so if the persistence of productivity gets sufficiently high, bubbles inevitably arise.

As is obvious from the proof of Theorem 3, the assumption of log utility (5.1) is unimportant; we maintained this assumption only to obtain a closed-form solution to the individual optimization problem. In general, all we need is that the asymptotic propensity to save is bounded below by a positive constant, which holds in a general setting (Ma and Toda, 2021, 2022).

6 Concluding remarks

In this paper we presented a conceptually new perspective on thinking about asset price bubbles: their *necessity*. We showed a plausible general class of economic models in which the emergence of bubbles is a necessity by proving that all equilibria are asymptotically bubbly. This surprising insight of the necessity of bubbles is fundamentally different from economists' long-held view that bubbles are either not possible in rational equilibrium models or even if they are, a situation in which bubbles occur is a special circumstance and hence fragile. This is also conceptually different from the rational bubble literature that shows the possibility that a bubble may occur. We emphasize that the necessity of bubbles naturally arises in workhorse models in macroeconomics. Hence, our Bubble Necessity Theorem challenges the conventional wisdom on bubbles and may open up a new direction for research.

As we showed in the model of Section 5, there is nothing special about the apparently restrictive environment of overlapping generations endowment economies. In a series of working papers, we and our collaborators apply the idea of the necessity of asset price bubbles to macro-finance (Hirano, Jinnai, and Toda, 2022) and to housing (Hirano and Toda, 2023). Hirano, Jinnai, and Toda (2022) study an endogenous growth model with two sectors in which in the production sector, heterogeneous entrepreneurs invest capital using leverage and the land sector generates positive dividends. They find that when leverage gets sufficiently high, the production sector grows faster than the land sector, which inevitably makes land

[11] By "increasing persistence", we mean that we parameterize the transition probability matrix as $\tau I + (1-\tau)\Pi$ for $\tau \in [0, 1)$ and increase τ.

prices higher and disconnected from fundamentals, generating a land price bubble. Hirano and Toda (2023) construct a model with housing, in which rents—the dividends of housing—are endogenously determined by the demand and supply of housing service. They prove that as the incomes of home buyers rise together with economic development, a housing bubble inevitably occurs through a two-stage phase transition. We also have ongoing work on asset price bubbles with aggregate risk and production. Interestingly and importantly, in all of these models, bubbles naturally and necessarily arise.

A Proofs

A.1 Proof of Theorem 1

We need several lemmas to prove Theorem 1.

Lemma A.1. *In equilibrium, the asset pricing equation (4.4) holds.*

Proof. Take any equilibrium. To simplify notation, let $U_t = U$, $P_t = P$, $P_{t+1} = P'$, and $D_{t+1} = D'$, etc. Using the budget constraint (4.1) to eliminate y, z', the young seek to solve

$$\begin{aligned} &\text{maximize} && U(a - Px, b' + (P' + D')x) && \text{(A.1a)}\\ &\text{subject to} && a - Px \geq 0, && \text{(A.1b)}\\ &&& b' + (P' + D')x \geq 0. && \text{(A.1c)} \end{aligned}$$

In equilibrium, market clearing forces $x = 1$. Since $b' > 0$ and $P' + D' \geq 0$, the nonnegativity constraint (A.1c) never binds at $x = 1$. Let $\lambda \geq 0$ be the Lagrange multiplier associated with the nonnegativity constraint (A.1b) and let

$$L(x, \lambda) = U(a - Px, b' + (P' + D')x) + \lambda(a - Px)$$

be the Lagrangian. The first-order condition implies

$$-PU_y + (P' + D')U_z - \lambda P = 0 \iff P = M(P' + D') - \frac{\lambda}{U_y} P, \tag{A.2}$$

where $M = U_z/U_y$ and U_y, U_z are evaluated at $(a - P, b' + P' + D')$. If $P < a$, then the nonnegativity constraint (A.1b) does not bind, $\lambda = 0$, and (A.2) reduces to $P = M(P' + D')$ and (4.4) holds. If $P = a$, then the nonnegativity constraint

(A.1b) binds, $\lambda \geq 0$, and (A.2) implies $M(P' + D') = a + (\lambda/U_y)a \geq a$, so (4.4) holds. $\square$

Lemma A.2. *For all $P_{t+1} \geq 0$, there exists $P_t \in [0, a_t]$ that satisfies (4.4).*

Define the function $f : [0, a] \to \mathbb{R}$ by

$$f(P) = (P' + D')M(a - P, b' + P' + D') - P.$$

By Assumption 1, f is continuous. Since U is quasi-concave, the marginal rate of substitution M in (4.3) is increasing in y. Therefore $M(a - P, b' + P' + D')$ is decreasing in P, so f is strictly decreasing. Clearly

$$f(0) = (P' + D')M(a, b' + P' + D') \geq 0.$$

If $f(a) > 0$, then the definition of f implies that

$$a < (P' + D')M(0, b' + P' + D'),$$

so (4.4) holds with $P = a$. If $f(a) \leq 0$, by the intermediate value theorem, there exists $P \in [0, a]$ such that $f(P) = 0$, which clearly satisfies (4.4).

Proof of Theorem 1. Although the existence of equilibrium follows from Wilson (1981, Theorem 1), to make the paper self-contained, we present a standard truncation argument as in Balasko and Shell (1980). Define the set $\mathcal{A} \coloneqq \prod_{t=0}^{\infty}[0, a_t]$ endowed with the product topology induced by the Euclidean topology on $[0, a_t] \subset \mathbb{R}$ for all t. By Tychonoff's theorem, $\mathcal{A}$ is nonempty and compact.

Define a *T-equilibrium* by a sequence $\{P_t\}_{t=0}^{\infty}$ such that $P_t \in [0, a_t]$ for all t and the asset pricing equation (4.4) holds for $t = 0, \dots, T - 1$. Let $\mathcal{P}_T \subset \mathcal{A}$ be the set of all T-equilibria. For any sequence $\{P_t\}_{t=T}^{\infty}$ such that $P_t \in [0, a_t]$ for all $t \geq T$, repeatedly applying Lemma A.2, by backward induction we can construct a T-equilibrium $\{P_t\}_{t=0}^{\infty}$. Therefore $\mathcal{P}_T \neq \emptyset$.

Since U_t is continuously differentiable, the marginal rate of substitution M_t is continuous, and hence $\mathcal{P}_T$ is closed. Furthermore, by the definition of the T-equilibrium, we have $P_T \supset P_{T+1}$ for all T. Since $\mathcal{P}_T \subset \mathcal{A}$ and $\mathcal{A}$ is compact, we have $\mathcal{P} \coloneqq \bigcap_{t=0}^{\infty} \mathcal{P}_t \neq \emptyset$. If we take any $\{P_t\}_{t=0}^{\infty} \in \mathcal{P}$, by definition (4.4) holds for all t. The quasi-concavity of U_t implies that we have an equilibrium.

Finally, let us show $P_t > 0$ if Assumption 2 holds. By Assumption 2, we can take an arbitrarily large T such that $D_T > 0$. Since $P_T \geq 0$, we have $P_T + D_T > 0$. Then (4.4) and $M_{T-1} > 0$ imply $P_{T-1} > 0$, so $P_{T-1} + D_{T-1} > 0$. Continuing this argument, we have $P_t > 0$ for all $t = 0, \dots, T-1$. Since $T > 0$ is arbitrary, we have $P_t > 0$ for all t. $\square$

A.2 Proof of Theorem 2

Take any equilibrium $\{P_t\}_{t=0}^{\infty}$. Let $p_t = P_t/a_t$ and $d_t = D_t/a_t$ be the asset price and dividend detrended by the endowment of the young. Let $G_t = a_{t+1}/a_t$ be the endowment growth rate and $w_t = b_t/a_t$ be the endowment ratio. Theorem 1 implies $0 < P_t \le a_t$. Dividing both sides by $a_t > 0$, we obtain $0 < p_t \le 1$. Since $D_t \ge 0$, we have $d_t = D_t/a_t \ge 0$.

We need several lemmas to prove Theorem 2.

Lemma A.3. *We have $\sum_{t=1}^{\infty} D_t/a_t < \infty$. In particular, $\lim_{t\to\infty} d_t = 0$.*

Proof. Since $G_d < G$ by (4.10), we can take $\epsilon > 0$ such that $G_d + \epsilon < G - \epsilon$. Using (4.6a) and (4.9), we can take $T > 0$ such that $D_t^{1/t} < G_d + \epsilon$ and $a_{t+1}/a_t > G - \epsilon$ for $t \ge T$. Therefore

$$\frac{D_t}{a_t} = \frac{D_t}{a_T}\left(\prod_{s=T}^{t-1}\frac{a_{s+1}}{a_s}\right)^{-1} < \frac{(G_d+\epsilon)^t}{a_T}(G-\epsilon)^{T-t} = \frac{(G-\epsilon)^T}{a_T}\left(\frac{G_d+\epsilon}{G-\epsilon}\right)^t,$$

which is summable because $G_d + \epsilon < G - \epsilon$. $\square$

Lemma A.4. *The following statement is true:*

$$(\exists r > 0)(\exists T > 0)(\forall t \ge T) \quad p_t \in (0, 1/2) \implies \frac{p_{t+1}}{p_t} \le r. \tag{A.3}$$

In other words, there exists a universal constant $r > 0$ such that $p_{t+1}/p_t \le r$ for all large enough t whenever $p_t < 1/2$.

Proof. Suppose $p_t < 1/2$. Then in particular $p_t < 1$ and $P_t < a_t$, so (4.4) holds without the min operator with a_t. Dividing both sides by $a_t > 0$ and using the definition of the scaled MRS in (4.7), we obtain

$$p_t = G_{t+1}(p_{t+1} + d_{t+1})m_t(1 - p_t, G_{t+1}(w_{t+1} + p_{t+1} + d_{t+1})). \tag{A.4}$$

Using $d_t \ge 0$ and (A.4), we obtain

$$\begin{aligned} 0 < \frac{p_{t+1}}{p_t} &\le \frac{p_{t+1} + d_{t+1}}{p_t} \\ &= \frac{1}{G_{t+1}m_t(1 - p_t, G_{t+1}(w_{t+1} + p_{t+1} + d_{t+1}))}. \end{aligned} \tag{A.5}$$

With a slight abuse of notation, let

$$(y_t, z_t) := (1 - p_t, G_{t+1}(w_{t+1} + p_{t+1} + d_{t+1})). \tag{A.6}$$

By Assumption 3, we can take $0 < \underline{G} < G < \bar{G}$ and $0 < \underline{w} < w < \bar{w}$ such that $G_t \in (\underline{G}, \bar{G})$ and $w_t \in (\underline{w}, \bar{w})$ for large enough t. Define the compact set

$$K := [1/2, 1] \times [\underline{G}\underline{w}, \bar{G}(\bar{w} + 1)] \subset \mathbb{R}^2_{++}. \tag{A.7}$$

Since $0 < p_{t+1} \le 1$ and $d_{t+1} \to 0$ by Lemma A.3, it follows from the definitions of (y_t, z_t) in (A.6) and K in (A.7) that

$$(\exists T_1)(\forall t \ge T_1) \quad p_t < 1/2 \implies (y_t, z_t) \in K, \tag{A.8}$$

that is, $(y_t, z_t) \in K$ for all large enough t whenever $p_t < 1/2$. In general, for any nonempty compact set $K \subset \mathbb{R}^2_{++}$, define

$$0 < \underline{m}(K) := \min_{(y,z)\in K} m(y, z), \tag{A.9}$$

which is well defined because m is continuous and positive by Assumption 4. Let $\delta := \underline{m}(K)/2 > 0$ for K in (A.7). By Assumption 4, we have

$$(\exists T_2 > 0)(\forall t \ge T_2) \quad (y, z) \in K \implies |m_t(y, z) - m(y, z)| < \delta.$$

In particular, if $t \ge T_2$, by the definition of $\underline{m}$ is (A.9) and δ, we have

$$(\forall t \ge T_2) \quad (y_t, z_t) \in K \implies m_t(y_t, z_t) \ge \delta. \tag{A.10}$$

Define $T = \max\{T_1, T_2\}$. If $t \ge T$ and $p_t < 1/2$, then (A.8) implies $(y_t, z_t) \in K$. Therefore putting all the pieces together, we obtain

$$\begin{aligned} \frac{p_{t+1}}{p_t} &\le \frac{1}{G_{t+1} m_t(y_t, z_t)} && (\because \text{(A.5), (A.6)}) \\ &\le 1/(\delta \underline{G}), && (\because G_{t+1} \ge \underline{G}, \text{(A.10)}) \end{aligned}$$

so we may choose the universal constant $r := 1/(\delta \underline{G})$. □

Lemma A.5. *The following statement is true:*

$$(\exists \epsilon > 0)(\exists T > 0)(\forall t \ge T) \quad p_t < \epsilon \implies \frac{p_{t+1}}{p_t} \le 1. \tag{A.11}$$

Proof. Take $r > 0$ and $T > 0$ as in Lemma A.4. For $\epsilon \in (0, 1/2)$, define the compact set

$$K(\epsilon) := [1-\epsilon, 1] \times [\underline{G}\underline{w}, \bar{G}(\bar{w} + r\epsilon)]. \tag{A.12}$$

If $t \geq T$ and $p_t < \epsilon$, then in particular $p_t < 1/2$. Therefore by Lemma A.4, we have

$$\frac{p_{t+1}}{p_t} \leq r \implies p_{t+1} \leq rp_t \leq r\epsilon.$$

Therefore by the definition of $K(\epsilon)$ in (A.12), we have $(y_t, z_t) \in K(\epsilon)$. Take any $\delta < \underline{m}(K(\epsilon))$, where $\underline{m}$ is defined in (A.9). By Assumption 4, we have

$$\sup_{(y,z)\in K(\epsilon)} |m_t(y, z) - m(y, z)| \leq \delta \tag{A.13}$$

for $t \geq T$ (by choosing a larger T if necessary). Since $(y_t, z_t) \in K(\epsilon)$, it follows from (A.5) that

$$\begin{aligned}
\frac{p_{t+1}}{p_t} &\leq \frac{1}{G_{t+1}m_t(y_t, z_t)} && (\because \text{(A.5), (A.6)}) \\
&\leq \frac{1}{\underline{G}} \frac{1}{\underline{m}(y_t, z_t) - \delta} && (\because \text{(A.13)}) \\
&\leq \frac{1}{\underline{G}} \frac{1}{\underline{m}(1-\epsilon, \bar{G}(\bar{w} + r\epsilon)) - \delta},
\end{aligned} \tag{A.14}$$

where the last line follows from the quasi-concavity of U_t (hence $m(y, z)$ is increasing in y and decreasing in z) and the fact that $(y_t, z_t) \in K(\epsilon)$ with $K(\epsilon)$ defined as in (A.12).

By Assumption 3, we may take $\underline{G}, \bar{G}$ arbitrarily close to G and $\bar{w}$ arbitrarily close to w. Clearly, we can take $\epsilon, \delta > 0$ arbitrarily close to zero. Therefore the right-hand side of (A.14) can be made arbitrarily close to

$$\frac{1}{Gm(1, Gw)} < 1$$

by condition (4.10). Therefore (A.11) holds. □

Lemma A.6. *In any equilibrium in which the asset is asymptotically irrelevant, we have* $\lim_{t\to\infty} p_t = 0$.

Proof. Let $\bar{\epsilon}, \bar{T} > 0$ be the ϵ, T in Lemma A.5. Take any $\epsilon \in (0, \bar{\epsilon})$. Then by Definition 3, there exists $T > \bar{T}$ such that $p_T < \epsilon$. Let us show by induction that $p_t < \epsilon$ for all $t \geq T$. The claim is trivial when $t = T$. Suppose $p_t < \epsilon$ for some $t \geq T > \bar{T}$. Then by Lemma A.5, we have $p_{t+1}/p_t \leq 1$, so $p_{t+1} \leq p_t < \epsilon$. By induction, we have $p_t < \epsilon$ for all $t \geq T$. By the definition of convergence, we have $\lim_{t\to\infty} p_t = 0$. $\square$

Lemma A.7. *In all equilibria, the asset is asymptotically relevant.*

Proof. By way of contradiction, suppose there exists an equilibrium in which the asset is asymptotically irrelevant. By Lemma A.6, we have $p_t \to 0$. Multiplying both sides of (A.5) by G_{t+1} and letting $t \to \infty$, by Assumptions 3 and 4, we obtain

$$\begin{aligned} R_t &= G_{t+1}\frac{p_{t+1} + d_{t+1}}{p_t} \\ &= \frac{1}{m_t(1 - p_t, G_{t+1}(w_{t+1} + p_{t+1} + d_{t+1}))} \to \frac{1}{m(1, Gw)}. \end{aligned} \tag{A.15}$$

By (4.10) and (A.15), we can take $\epsilon > 0$ and $T > 0$ such that

$$R_t < \frac{1}{m(1, Gw)} + \epsilon < G_d - \epsilon$$

for $t \geq T$. Furthermore, by (4.9), we have $D_t^{1/t} > G_d - \epsilon$ infinitely often. Therefore for such t, the present value of D_t can be bounded from below as

$$q_t D_t = q_T (q_t/q_T) D_t \geq q_T (G_d - \epsilon)^{T-t} (G_d - \epsilon)^t = q_T (G_d - \epsilon)^T.$$

Since the lower bound is positive and does not depend on t, and there are infinitely many such t, we obtain $P_0 \geq V_0 = \sum_{t=1}^{\infty} q_t D_t = \infty$, which is a contradiction. $\square$

Proof of Theorem 2. Take any equilibrium. By Lemma A.7, the asset is asymptotically relevant. Since $P_t > 0$, by Definition 3, we can take $p > 0$ such that $P_t/a_t \geq p$ for all t. Then

$$\sum_{t=1}^{\infty} \frac{D_t}{P_t} \leq \sum_{t=1}^{\infty} \frac{D_t}{p a_t} < \infty$$

by Lemma A.3. Lemma 2.1 implies that the equilibrium is bubbly, so by Definition 3 the equilibrium is asymptotically bubbly. $\square$

A.3 Proof of Proposition 4.1

Let $P_t > 0$ be any equilibrium asset price. (P_t must be nonnegative by Footnote 5, and it must be positive because the asset pays positive dividends.) Because the old exit the economy, market clearing forces the young to buy the entire asset in equilibrium. Therefore the equilibrium consumption allocation is

$$(y_t, z_t) = (aG^t - P_t, bG^t + P_t + D_t).$$

Nonnegativity of consumption implies $P_t \le aG^t$. Define the gross interest rate between time t and $t+1$ by the asset return

$$R_t := \frac{P_{t+1} + D_{t+1}}{P_t}.$$

The first-order condition for optimality together with the nonnegativity of consumption implies that $R_t \ge 1/\beta$, with equality if $P_t < aG^t$. Suppose $R_t > 1/\beta$. Then $P_t = aG^t$, so

$$\begin{aligned}
R_{t-1} := \frac{P_t + D_t}{P_{t-1}} = \frac{aG^t + DG_d^t}{P_{t-1}} &\ge \frac{aG^t + DG_d^t}{aG^{t-1}} && (\because P_{t-1} \le aG^{t-1}) \\
= \frac{aG^{t+1} + DGG_d^t}{aG^t} &> \frac{aG^{t+1} + DG_d^{t+1}}{P_t} && (\because G > G_d, P_t = aG^t) \\
\ge \frac{P_{t+1} + D_{t+1}}{P_t} &= R_t > \frac{1}{\beta}. && (\because P_{t+1} \le aG^{t+1})
\end{aligned}$$

Therefore by induction, if $R_t > 1/\beta$, then $R_s > 1/\beta$ for all $s \le t$. This argument shows that, in equilibrium, either (i) there exists $T > 0$ such that $R_t = 1/\beta$ for all $t \ge T$, or (ii) $R_t > 1/\beta$ for all t. In Case (i), using (2.6), $1/R_t = \beta$ for $t \ge T$, and $1/\beta < G_d$, the asset price at time $t \ge T$ can be bounded from below as

$$P_t \ge V_t = \sum_{s=1}^{\infty} \beta^s DG_d^{t+s} = \sum_{s=1}^{\infty} DG_d^t (\beta G_d)^s = \infty,$$

which is impossible in equilibrium. Therefore it must be Case (ii) and hence $P_t = aG^t$ and $y_t = 0$ for all t. In this case, we have

$$R_t = \frac{aG^{t+1} + DG_d^{t+1}}{aG^t} \ge G > \frac{1}{\beta},$$

so the first-order condition holds and we have an equilibrium, which is unique. Using $P_t = aG^t$, $D_t = DG_d^t$, and applying Lemma 2.1, we immediately see that there is a bubble. □

A.4 Proof of Theorem 3

We need several lemmas to prove Theorem 3.

Lemma A.8. *Let W_{nt} be the aggregate wealth held by type n agents at time t and $v_t' = (W_{0t}, \dots, W_{Nt})$ be the row vector of aggregate wealth. Then $v_t' \ge v_0' A^t$.*

Proof. Let $R_t = (P_{t+1} + D_{t+1})/P_t$ be the gross risk-free rate between time t and $t+1$. Due to log utility, the optimal consumption rule is $c_t = (1-\beta)w_t$. Savings is thus βw_t. Because the productivity is predetermined, it is optimal for an agent to invest entirely in the technology (land) if $z_{n_t} > R_t$ ($z_{n_t} < R_t$). If $z_{n_t} = R_t$, the agent is indifferent between the technology and land. Therefore using the budget constraint (5.2), individual wealth evolves according to

$$w_{t+1} = \beta \max \{z_{n_t}, R_t\} w_t. \tag{A.16}$$

Let W_{nt} be the aggregate wealth held by type n agents. Then aggregating (A.16) across agents and applying the strong law of large numbers, we obtain

$$W_{n',t+1} = \sum_{n=0}^{N} \pi_{nn'} \beta \max \{z_n, R_t\} W_{nt} \ge \sum_{n=0}^{N} \pi_{nn'} \beta z_n W_{nt}. \tag{A.17}$$

Collecting the terms in (A.17) into a row vector and using the definitions of v_t' and A, we obtain $v_{t+1}' \ge v_t' A$. Iterating this inequality, we obtain $v_t' \ge v_0' A^t$. $\square$

Lemma A.9. *There exists a constant $w_0 > 0$ such that $W_{0t} \ge w_0 G^t$ for all t.*

Proof. Noting that $z_0 = 0$, partition the matrix $A = (\beta z_n \pi_{nn'})$ as

$$A = \begin{bmatrix} 0 & 0 \\ b_1 & A_1 \end{bmatrix}, \tag{A.18}$$

where $A_1 = (\beta z_n \pi_{nn'})_{n,n'=1}^N$ is the $N \times N$ submatrix. Since Π is irreducible and $z_n > 0$ for all $n \ge 1$, we have $b_1 > 0$. Similarly, since $\Pi_1 = (\pi_{nn'})_{n,n'=1}^N$ is irreducible by Assumption 5 and $z_n > 0$ for all $n \ge 1$, A_1 is irreducible.

Since A is nonnegative, by Theorem 8.3.1 of Horn and Johnson (2013), the spectral radius $\rho(A)$ is an eigenvalue with a corresponding nonnegative left eigenvector u'. Partition u' as $u' = (u_0, u_1')$. Multiplying u' to A in (A.18) from the left and comparing entries, we obtain

$$\rho(A) u_0 = u_1' b_1, \tag{A.19a}$$

$$\rho(A) u_1' = u_1' A_1. \tag{A.19b}$$

If $u_1 = 0$, then (A.19a) and $\rho(A) = G > 0$ implies $u_0 = 0$. Then $u = 0$, which contradicts the fact that u is an eigenvector of A. Therefore $u_1 \ne 0$, and

(A.19b) implies that u'_1 is a nonnegative left eigenvector of A_1. Since A is block lower triangular, we have $0 < \rho(A) = \rho(A_1)$. Since A_1 is irreducible, by the Perron-Frobenius theorem (Horn and Johnson, 2013, Theorem 8.4.4), u'_1 must be the left Perron vector of A_1 and hence $u_1 \gg 0$. Therefore (A.19a) implies $u_0 = u'_1 b_1/\rho(A) > 0$, so $u' = (u_0, u'_1) \gg 0$.

Since agents are endowed with positive endowments, the vector of initial aggregate wealth v'_0 is positive. Therefore we can take $\epsilon > 0$ such that $v'_0 \geq \epsilon u'$. By Lemma A.8, we obtain

$$v'_t \geq v'_0 A^t \geq \epsilon u' A^t = \epsilon \rho(A)^t u' = \epsilon G^t u'.$$

Comparing the 0-th entry, we obtain $W_{0t} \geq \epsilon u_0 G^t$, so we can take $w_0 := \epsilon u_0$. □

Proof of Theorem 3. Using the definition of G_d and the assumption $G_d < G$, it follows from Lemma A.9 that we can take $\epsilon > 0$ such that $G > G_d + \epsilon$ and

$$W_{0t} \geq w_0 G^t > (G_d + \epsilon)^t \geq D_t \tag{A.20}$$

for large enough t. Since $z_0 = 0$, type 0 agents invest all wealth in land, so the market capitalization of land (which equals the land price because land is in unit supply) must exceed the aggregate savings of type 0: $P_t \geq \beta W_{0t}$. Therefore using (A.20), for large enough t we can bound the dividend yield from above as

$$\frac{D_t}{P_t} \leq \frac{D_t}{\beta W_{0t}} \leq \frac{1}{\beta w_0} \left(\frac{G_d + \epsilon}{G} \right)^t.$$

Since $G_d + \epsilon < G$, we have $\sum_{t=1}^{\infty} D_t/P_t < \infty$, so by Lemma 2.1 the equilibrium is bubbly. □

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Online Appendix

B Asymptotically bubbleless equilibria

This appendix presents a canonical example of bubbly equilibria that are asymptotically bubbleless (Definition 3).

Suppose agents have the Cobb-Douglas utility function

$$U(y,z)=\log y+\beta\log z,$$

where $\beta>0$. Suppose the endowments grow at a constant rate $G>0$, so $(a_t,b_t)=(aG^t,bG^t)$, and the dividends are zero: $D_t=0$ (pure bubble).

Clearly the fundamental value of the asset is $V_t=0$, and $P_t=0$ (autarky) is a fundamental equilibrium. Suppose there exists an equilibrium with $P_t>0$ for all t, which is bubbly. The asset pricing equation (4.4) holds, so

$$P_t=\beta\frac{aG^t-P_t}{bG^{t+1}+P_{t+1}}P_{t+1}. \tag{B.1}$$

Letting $p_t:=P_t/a_t=P_t/(aG^t)$ be the price-income ratio, we can rewrite (B.1) as

$$p_t=\beta\frac{1-p_t}{b/a+p_{t+1}}p_{t+1}\iff\frac{1}{p_{t+1}}=\frac{\beta a}{b}\frac{1}{p_t}-\frac{(1+\beta)a}{b}. \tag{B.2}$$

Because (B.2) is a linear difference equation in $1/p_t$, it is straightforward to solve. The general solution is

$$\frac{1}{p_t}=\begin{cases}\frac{(1+\beta)a}{\beta a-b}+\left(\frac{\beta a}{b}\right)^t\left(\frac{1}{p_0}-\frac{(1+\beta)a}{\beta a-b}\right) & \text{if } \beta a\neq b,\\ \frac{1}{p_0}-\frac{(1+\beta)a}{b}t & \text{if } \beta a=b.\end{cases} \tag{B.3}$$

If $\beta a<b$, (B.3) converges to $\frac{(1+\beta)a}{\beta a-b}<0$, which contradicts $p_t>0$. If $\beta a=b$, (B.3) diverges to $-\infty$, which contradicts $p_t>0$. Therefore a bubbly equilibrium can exist only if $\beta a>b$. Under this condition, because the term $(\beta a/b)^t$ diverges to ∞, in order for $p_t>0$ for all t, it is necessary and sufficient that

$$\frac{1}{p_0}-\frac{(1+\beta)a}{\beta a-b}\geq 0\iff 0<p_0\leq\frac{\beta a-b}{(1+\beta)a}.$$

Therefore if $\beta a > b$, the equilibrium price-income ratio can be solved as

$$\frac{P_t}{a_t} = p_t = \begin{cases} \left(\frac{(1+\beta)a}{\beta a-b} + \left(\frac{\beta a}{b}\right)^t \left(\frac{1}{p_0} - \frac{(1+\beta)a}{\beta a-b}\right)\right)^{-1} & \text{if } 0 < p_0 < \frac{\beta a-b}{(1+\beta)a}, \\ \frac{\beta a-b}{(1+\beta)a} & \text{if } p_0 = \frac{\beta a-b}{(1+\beta)a}. \end{cases} \tag{B.4}$$

Note that when $p_0 < \frac{\beta a-b}{(1+\beta)a}$, because $(\beta a/b)^t \to \infty$, we have $p_t \to 0$. We can thus summarize all possible equilibria as follows.

(i) The fundamental equilibrium $p_t = 0$ always exists. If $\beta a/b \leq 1$, this is the only equilibrium.

(ii) If $\beta a/b > 1$, there exists a unique asymptotically bubbly equilibrium, which is given by $p_t = \frac{\beta a-b}{(1+\beta)a}$.

(iii) If $\beta a/b > 1$, there exist a continuum of bubbly but asymptotically fundamental equilibria parameterized by $0 < p_0 < \frac{\beta a-b}{(1+\beta)a}$, which are given by (B.4).

C Global dynamics of Example 3

This appendix provides a step-by-step analysis of the asymptotically bubbly equilibrium in Example 3.

Start with the asset pricing equation (4.4), which is

$$P_t = \beta \left(\frac{bG^{t+1} + P_{t+1} + D}{aG^t - P_t}\right)^{-\gamma} (P_{t+1} + D). \tag{C.1}$$

Define the detrended variable $\xi = (\xi_{1t}, \xi_{2t})$ by $\xi_{1t} := P_t/(aG^t)$ and $\xi_{2t} := D/(aG^t)$. Then (C.1) can be rewritten as the system of autonomous nonlinear implicit difference equations

$$H(\xi_t, \xi_{t+1}) = 0, \tag{C.2}$$

where $H : \mathbb{R}^4 \to \mathbb{R}^2$ is defined by

$$H_1(\xi, \eta) = \beta G^{1-\gamma} \left(\frac{w + \eta_1 + G\xi_2}{1 - \xi_1}\right)^{-\gamma} (\eta_1 + G\xi_2) - \xi_1, \tag{C.3a}$$

$$H_2(\xi, \eta) = \eta_2 - \frac{1}{G}\xi_2 \tag{C.3b}$$

with $w := b/a$. Let $\xi^* = (\xi_1^*, \xi_2^*)$ be a steady state of the difference equation (C.2), so $H(\xi^*, \xi^*) = 0$. Since $G > 1$ and hence $1/G \in (0, 1)$, (C.3b) implies that $\xi_2^* = 0$.

Using (C.3a), we can solve for ξ_1^* as

$$\beta G^{1-\gamma}\left(\frac{w+\xi_1^*}{1-\xi_1^*}\right)^{-\gamma}\xi_1^*-\xi_1^*=0 \iff \xi_1^*=\frac{(\beta G^{1-\gamma})^{1/\gamma}-w}{1+(\beta G^{1-\gamma})^{1/\gamma}}>0 \tag{C.4}$$

because (4.12) holds. (Note that the other (fundamental) steady state $\xi_1^*=0$ is ruled out by Theorem 2.)

We apply the implicit function theorem at $(\xi,\eta)=(\xi^*,\xi^*)$ to express (C.2) as $\xi_{t+1}=h(\xi_t)$ for ξ_t close to ξ^*. Noting that $\xi_2^*=0$, a straightforward calculation using (C.3) and (C.4) implies that

$$D_\xi H(\xi^*,\xi^*)=\begin{bmatrix}H_{1,\xi_1} & H_{1,\xi_2}\\ 0 & -1/G\end{bmatrix} \quad \text{and} \quad D_\eta H(\xi^*,\xi^*)=\begin{bmatrix}H_{1,\eta_1} & 0\\ 0 & 1\end{bmatrix},$$

where

$$\begin{aligned}
H_{1,\xi_1}&=-\gamma\beta G^{1-\gamma}(w+\xi_1^*)^{-\gamma}(1-\xi_1^*)^{\gamma-1}\xi_1^*-1=-1-\gamma\frac{\xi_1^*}{1-\xi_1^*},\\
H_{1,\eta_1}&=\beta G^{1-\gamma}\left(\frac{w+\xi_1^*}{1-\xi_1^*}\right)^{-\gamma}\left(1-\gamma\frac{\xi_1^*}{w+\xi_1^*}\right)=1-\gamma\frac{\xi_1^*}{w+\xi_1^*},
\end{aligned}$$

and H_{1,ξ_2} is unimportant. Therefore except the special case with $\gamma=1+w/\xi_1^*$, we may apply the implicit function theorem, and for (ξ,η) sufficiently close to (ξ^*,ξ^*), we have $H(\xi,\eta) \iff \eta=h(\xi)$ for some C^1 function h with

$$Dh(\xi^*)=\begin{bmatrix}\lambda_1 & *\\ 0 & \lambda_2\end{bmatrix}, \quad \text{where} \quad (\lambda_1,\lambda_2)=\left(\frac{1+\gamma\frac{\xi_1^*}{1-\xi_1^*}}{1-\gamma\frac{\xi_1^*}{w+\xi_1^*}},\frac{1}{G}\right). \tag{C.5}$$

We thus obtain the following proposition.

Proposition C.1. *Let everything be as in Example 3 and define* $\kappa:=(\beta G^{1-\gamma})^{1/\gamma}$. *If* $\frac{1}{\gamma}\neq\frac{\kappa-w}{\kappa(1+w)}$, *then there exists an asymptotically bubbly equilibrium such that* $P_t/(aG^t)$ *converges to* ξ_1^* *in (C.4). If in addition*

$$\frac{1}{\gamma}>\frac{1}{2}\frac{\kappa-w}{\kappa}\frac{1-\kappa}{1+w}, \tag{C.6}$$

then such an equilibrium is unique.

Proof. By the implicit function theorem, the equilibrium dynamics can be expressed as $\xi_{t+1}=h(\xi_t)$ if ξ_t is sufficiently close to ξ_1^*. To study the local stability, we apply the local stable manifold theorem. Since $G>1$, one eigenvalue of $Dh(\xi^*)$

is $\lambda_2 = 1/G \in (0,1)$. If $1-\gamma\frac{\xi_1^*}{w+\xi_1^*} > 0$, then clearly $\lambda_1 > 1$. If $1-\gamma\frac{\xi_1^*}{w+\xi_1^*} < 0$, then

$$\lambda_1 = \frac{1+\gamma\frac{\xi_1^*}{1-\xi_1^*}}{1-\gamma\frac{\xi_1^*}{w+\xi_1^*}} < -1 \iff 1+\gamma\frac{\xi_1^*}{1-\xi_1^*} > -1+\gamma\frac{\xi_1^*}{w+\xi_1^*}$$
$$\iff \frac{1}{\gamma} > \frac{(\kappa-w)(1-\kappa)}{2\kappa(1+w)}$$

using the definition of κ and ξ_1^*. Furthermore, we have

$$1-\gamma\frac{\xi_1^*}{w+\xi_1^*} = 0 \iff \frac{1}{\gamma} = \frac{\kappa-w}{\kappa(1+w)}.$$

Therefore if $\frac{1}{\gamma} \neq \frac{\kappa-w}{\kappa(1+w)}$, the eigenvalues of $Dh(\xi^*)$ are not on the unit circle, so the local stable manifold theorem (see Irwin (1980, Theorems 6.5 and 6.9) and Guckenheimer and Holmes (1983, Theorem 1.4.2)) implies that for sufficiently large T (so that $\xi_{2T} = D/(aG^T)$ is sufficiently close to the steady state value 0), there exists an equilibrium path $\{\xi_t\}_{t=T}^{\infty}$ starting at $t = T$ converging to ξ^*. A backward induction argument similar to Lemma A.2 implies the existence of an equilibrium path $\{\xi_t\}_{t=0}^{\infty}$ starting at $t = 0$. Finally, if (C.6) holds, then $|\lambda_1| > 1 > \lambda_2 > 0$, so the number of free initial conditions (1, because P_0 is endogenous) agrees with the number of unstable eigenvalues (1, because $|\lambda_1| > 1$) and the equilibrium path is unique. □