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Price equilibrium with selling constraints[*]

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Abstract

This paper studies how *selling constraints*, which refer to the inability of firms to attend to all the buyers who want to inspect their products, affect the equilibrium price and social welfare. We show that the price that maximizes social welfare is greater than the marginal cost. This is because with selling constraints, a higher price, despite reducing the probability of trade (fewer buyers are willing to pay a higher price) increases the value of trade (only trades generating positive surplus are consummated). We show that the equilibrium price is inefficiently high except in the limit when firms' selling constraints vanish and consumers observe prices before they visit firms. Thus, selling constraints constitute a source of market power.

Keywords: price competition, market power, capacity- and selling-constrained firms

JEL Classification Number: D4, J6, L1, L8, R3

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1 Introduction

Despite the fact that every organization tries to optimize its marketing and sales team,[1] *selling constraints*, which refer to the inability of firms to attend to all interested buyers, are ubiquitous. Waiting at the phone to be attended to, queuing at retail stores, and awaiting landords' replies are common examples of unpleasant situations in everyday shopping that make it difficult for consumers to evaluate alternative options and sometimes even result in that they are unable to buy.[2]

Despite the existence of a relatively large economics literature on competition with capacity-constrained firms (see the early works of Levitan and Shubik, 1972; Kreps and Scheinkman, 1983; and Davidson and Deneckere, 1986), to the best of our knowledge, the literature has not paid any attention to the implication of selling constraints in markets. This omission is very important because these two constraints, capacity and selling, bear quite differently on price formation and efficiency.

This point can easily be illustrated in the case of monopoly. Consider for example a single homeowner who has only one house (and hence is capacity constrained) trying to rent it out to one tenant out of a large set of tenants with random valuations ε drawn from an interval according to some density function f. Suppose the homeowner has an unlimited capacity to let prospective tenants view the house (so he/she does not face any selling constraint). In that case, the homeowner should charge a price equal to the highest tenant valuation and such a high price is efficient because it maximizes the value of trade $E[\varepsilon|\varepsilon > p]$ without compromising the probability of trade (which is equal to 1). By contrast, if the homeowner faces severe selling constraints so that he/she can only attend to one tenant, he/she should charge a price equal to the standard monopoly price (i.e. the inverse of the hazard rate $(1 - F(p))/f(p))$ and such pricing turns out to be inefficient because the socially optimal price equals marginal cost thereby maximizing the probability of trade $1 - F(p)$ and minimizing the value of trade $E[\epsilon|\epsilon > p]$.

In this paper we study how selling constraints affect the functioning of markets with many buyers and sellers. In particular, we ask:

- How does a buyer's buying probability and a firm's selling probability depend on selling constraints?
- How do selling constraints affect the price equilibrium? In other words, do prices increase or

[1] Sales force management, which constitutes a typical course in business programs worldwide (e.g. Chicago Booth, INSEAD and MIS Singapore), has been a research topic in management and marketing for various decades. For a textbook approach to the topic, see for example Johnston and Marshall (2013) and Rich (2017).

[2] The COVID-19 pandemics record-long wait lines to enter stores that recently disrupted retail business have made it to breaking news in many places of the world. A survey conducted by *Qudini* reveals that many upset consumers' complaints are the lack of staff at the store and staff taking too long to attend to customers (see e.g. `https://www.bbc.com/news/business-10866718`). Moreover, in *Airbnb* for example the readiness to reply to tenants' questions and requests is crucial for the ratings landlords and their properties receive.

decrease as selling constraints weaken?

- Are selling constraints a source of marker power? How do selling constraints impact firms' profits and the efficiency of the market?

We address these questions in the simplest possible setting in which the above mentioned trade-off between the probability of trade and the value of trade arises. We consider a market where there is a large number of sellers, each of them selling one unit of a differentiated good, and a large number of buyers, each of them interested in buying one unit of a satisfactory product. An individual buyer picks a seller to visit and, depending on how many other buyers visit the same seller and the ability of the seller to attend to various buyers, she is given the opportunity to inspect the seller's product, in which case she learns her valuation and decides whether to buy it or not.

We start by discussing the price that maximizes social welfare. To do so, we first derive the probability with which a transaction occurs under the natural assumption that buyers' decisions on which seller to visit are independent (i.e. in an uncoordinated fashion). In deriving this probability, one needs to take into account situations in which the number of buyers arriving at a given seller exceeds the number of buyers that the seller can attend to, as well as cases in which this is not the case. This makes this probability an increasing function of the selling capacity of the firms, which implies that the social welfare maximizing price depends on selling constraints.

In standard markets where capacity and selling constraints are absent, marginal cost pricing is a cornerstone of efficiency and thus the shortcut definition of market power is the ability of a firm to sustain prices about marginal cost. The reason for this is that as the price increases away from the marginal cost, low-valuation buyers are excluded from the market without affecting the probability with which high-valuation buyers purchase, which clearly generates a dead-weight loss. Marginal cost pricing is also efficient in our model with capacity-constrained firms provided that firms have maximal selling constraints. However, with weaker selling constraints, the price that maximizes social welfare is greater than the marginal cost. Even though a price higher than the marginal cost excludes low-valuation buyers, the social benefit of it is that high-valuation ones get a higher chance to buy the item, which has a positive effect on surplus. The planner, thus, faces a trade-off. By raising the price, the chance that the product sells goes down but in case of a sale the surplus generated goes up. These two forces operate on social surplus in a way that it first increases in price and then decreases. Welfare maximization consists of balancing these two effects, which drives a wedge between the optimal price and marginal cost. Hence, the shortcut notion of market power as the ability of firms to sustain prices about marginal cost is of no big use here. Instead, market power has to be assessed as the capacity of firms to sustain prices above the efficient level, which differs from the marginal cost.

We show that the above observations on how selling constraints impact the price equilibrium and its efficiency properties in monopoly extend naturally to our search setting if, in the tradition of Diamond (1971), Wolinsky (1986) and Anderson and Renault (1999), deviation prices are only observable upon visiting the firms and hence the price of an individual firm does not have a bearing on the number of buyers who choose to visit that firm.

This motivates us to examine a model of competition in the tradition of Bertrand where price deviations are observed by buyers before they visit firms and hence they face stronger business stealing effects. In such a search setting, we show that a laxer selling constraint operates on the equilibrium price in two ways. On the one hand, because a firm can attend to more buyers, it makes attracting them to its premises more valuable; this gives firms incentives to decrease the equilibrium price. On the other hand, because a firm may offer the opportunity to inspect the product to more buyers, it increases the chance that a higher price is accepted by one of them; this gives firms incentives to increase the equilibrium price. When there are few buyers per seller, the second effect plays a weak role and the equilibrium price decreases as selling constraints become softer. By contrast, when the number of buyers per seller is substantial, the first effect plays little role and the equilibrium price increases as selling constraints weaken.

The efficiency of the pricing equilibrium turns out to hinge upon the firms' ability to let buyers inspect their products. By raising the price, the firm gets fewer visitors and moreover incurs the risk that all the buyers that the firm can attend to choose not to buy the product; however, if one buyer acquires the product, profits go up. We show that when each seller can only attend to a finite number of buyers, equilibrium markups turn out to be inefficiently high. The intuition is that firms have too weak incentives to attract buyers; the fewer the buyers they can attend to, the weaker the incentives to attract them. As the number of buyers an individual firm can attend to goes up, the incentives to attract buyers increase and firms correspondingly lower the price. Only in the limiting case in which all sellers can continue to offer its product to all the buyers that show up at their premises no matter how many, are markups at the efficient level.

We thus conclude that selling constraints constitute a source of market power when it is appropriately measured by the wedge between the equilibrium and the efficient price, rather than marginal cost. An implication of this is that market power need not be associated with higher profits. This has potential implications for the interpretation of all the recent work that measures markups (as differences between prices and marginal costs) and relates them to market inefficiency (see e.g. De Loecker, Eeckhout and Unger (2020)).

To the best of our knowledge, our paper is the first to study the impact of selling constraints on the efficiency of the market equilibrium. In doing so, we model a one-shot search market where products are horizontally differentiated in the tradition of Perloff and Salop (1985) and firms' prices may or may not be observable. Our model is thus connected to two strands of the consumer search literature. The first line of work is the standard literature on consumer search for differentiated products initiated by Wolinsky (1986) and Anderson and Renault (1999). Following the tradition commenced by Diamond (1971), this literature has typically modelled markets in which prices are non-observable before search. More recent contributions include e.g. Armstrong, Vickers and Zhou (2009), Bar-Isaac, Caruana and Cuñat (2012) and Haan and Moraga-González (2011). In our model with capacity-constrained firms, this assumption leads to local monopolies.

The second strand of the consumer search literature to which this paper is related is newer. In recent years, perhaps motivated by the availability of data from the Internet, there has been a surge of interest in the modelling of search markets in which consumers observe prices before search (Armstrong, 2017; Armstrong and Zhou, 2011; Choi, Dai and Kim, 2018; Haan, Moraga-González and Petrikaitė, 2018). This literature has shown that price observability does not combine well with product differentiation to yield a tractable model. The problem is that an equilibrium in pure-strategies does not exist and the mixed-strategy equilibrium is extremely difficult to characterize. To model price-directed search, thus, these authors have modified the standard setting in alternative ways. For example, Choi, Dai and Kim (2018) and Haan, Moraga-González and Petrikaitė. (2018) introduce observable and non-observable product characteristics and show that when the observable product characteristics are sufficiently dispersed an equilibrium in pure strategies exists. Our model in which firms are capacity constrained contributes to this line of work by putting forward yet another way to get a tractable model of price competition in search markets.[3,4]

2 The model

There is a measure B of buyers, and a measure S of sellers. Each buyer has unit demand. Each seller has one unit to sell.[5] Let x be the number of buyers per seller, or number of buyers per firm, i.e. $x \equiv B/S$. The limiting case $x \to 0$ represents a case in which there are infinitely many firms

[3] Other papers that study uncertain product availability in consumer search markets, but without modelling selling constraints, are Janssen and Rasmusen (2002), Lester (2011), Gomis-Porqueras, Julien and Wang (2017), Atabek (2022), and Teh, Wang, and Watanabe (2023).

[4] In the search and matching literature (for a recent survey, see Wright, Kircher, Julien and Guerrieri, 2021) it is standard to model capacity-constrained firms but products (to be sure, jobs in most of the articles) are typically assumed to be homogeneous. When products are homogeneous, however, selling constraints are inconsequential because the first buyer to whom the firm offers the product will buy it. Moreover, because the quality of trade does not matter, the equilibrium price has no bearing on welfare, hence the literature's focus on other aspects of efficiency, in particular the efficiency of entry. An exception is when firms are asymmetric for example because some sellers have more units to sell than others (see Watanabe, 2010, 2018, 2020; Tan, 2012; and Godenhielm and Kultti, 2015).

[5] The insights of our paper carry over to situations where firms have more units to sell but still face selling constraints. For an elaboration of this point, see Section 7.

per buyer so that the market is extremely competitive. The other limiting case $x \to \infty$ refers to a situation in which there are infinitely many buyers per firm so that each firm enjoys a monopoly position. For simplicity, we normalize unit production costs to zero.

Products are horizontally differentiated. We model product differentiation as in the random utility framework of Perloff and Salop (1985).[6] The exact value a buyer ℓ places on the product of a seller i, denoted $\varepsilon_{i\ell}$, depends on how well the product matches the personal tastes of the buyer. Such a match value can only be learnt upon inspection of and interaction with the product. We assume that match values are identically and independently distributed across buyers and sellers. Let F be the distribution of match values, with density f and support $[0, 1]$. From now on, we drop the sub-index of $\varepsilon_{i\ell}$.

With differentiated products, it is important to pay attention to the ability of a seller to attend to its customers and offer its product to a next buyer after an earlier buyer has decided not to buy it because her value falls short of the price. We refer to this (lack of) ability as *selling capacity.* We assume that each seller can attend and (sequentially) offer its product to a total of k buyers, $k = 1, 2, ..., \infty$.[7]

The interaction between buyers and sellers is modeled as a one-shot game.[8] First, each firm chooses its price; then, each buyer picks a seller to visit. Once buyers arrive at the sellers' stores, each seller (with customers) offers its good to a first buyer; after inspecting the good, the buyer decides whether to buy it or not. If the buyer buys the good, she gets utility $\varepsilon_i - p_i$ while the seller gets p_i. If the buyer decides not to buy the good, in which case she gets zero utility, the seller offers the good to the next buyer (if he has more to attend to). The process continues in the same fashion until either the seller does not have more buyers to attend to or the seller can no longer attend to buyers because his selling capacity k is exhausted.

We are interested in the characterization of a symmetric equilibrium price in two distinct informational scenarios. In the first scenario, we assume that firms' prices are not observable by consumers before they pick a seller to visit. This is in the tradition of the random consumer search literature (cf. Wolinksy, 1986; Anderson and Renault, 1999). In the second scenario, we assume that firms'

[6]This assumption is central to our model. Actually, the fact that products are horizontally differentiated makes selling constraints relevant; with homogeneous products, on the contrary, the first buyer to whom a firm offers its product acquires it and selling constraints thus become inconsequential.

[7]The case $k = 1$ represents the extreme case in which each seller can only offer its product to a single buyer. The case $k \to \infty$ represents the also extreme case in which a seller can continue to offer its product to all the buyers who patronize its store, even if infinitely many of them do. In most settings, a small k will reflect better the reality than a large k, for example when firms do not have sufficient salespeople or when products are highly complex so that each consumer takes quite a bit of time to evaluate it (houses, campers, cars, motorbikes, boats, etc.).

[8]The main results of our paper should extend to situations where consumers are allowed to search sequentially till they find a satisfactory match, as it is standard in the consumer search literature. In such a case, the consumers' reservation value plays the role of the equilibrium price. For further details, see Section 7.

prices are observable by consumers before they pick a seller to visit, in which case consumer search is ordered by prices (in a sense to be precised later). Ordered search has recently received much attention (cf. Armstrong, 2017; Armstrong and Zhou, 2011; Ding and Zhang, 2018; Choi, Dai and Kim, 2018; Haan, Moraga-González and Petrikaité, 2018).

3 Buyers' probability of buying

We start the analysis by deriving the probability with which a buyer who visits a seller happens to be attended to, i.e., gets an opportunity to inspect the seller's product and decide whether to buy it or not. Later we provide some useful properties of this probability.

It is well-known (see e.g. Butters, 1977; Peters, 2000) that in large markets in which buyers visit sellers randomly and independently, the number of buyers n that shows up at a given seller follows a Poisson distribution with Poisson parameter equal to the number of buyers per seller x. Thus, the probability that exactly $\ell = 0, 1, 2, ...$ buyers appear at the premises of a given seller is equal to:

$$\Pr(n = \ell) = \frac{x^{\ell} e^{-x}}{\ell!},$$

where the symbol Pr stands for probability.

Let us denote by $\eta(x, p; k)$ the probability with which a given buyer who visits a seller charging a price p gets an opportunity to inspect the product in a market where there are x buyers per seller and each seller can only attend to a maximum of k buyers. This probability will in general depend on the number of buyers per seller x because the more buyers showing up at a seller, the less likely it is that a particular buyer gets an opportunity to inspect the product and interact with it. This likelihood also depends on the price p the firm charges because a higher price makes it more probable that other buyers attended to earlier than the buyer in question choose not to buy the product. Finally, this probability is also affected by the selling capacity k of the seller because the seller may not be able to attend to the given buyer if there are more than k other buyers visiting the firm. Our first contribution is to derive the probability $\eta(x, p; k)$ for an arbitrary selling capacity k:

Proposition 1 *In a market with x buyers per seller, the probability with which a given buyer who visits a firm charging price p gets an opportunity to inspect and buy the product of the firm is equal to:*

$$\eta(x, p; k) = \frac{1}{x(1 - F(p))} m(x, p; k) \tag{1}$$

where

$$m(x,p;k) = 1 - F(p)^k + \left(\frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right) F(p)^k - \frac{\Gamma(k+1,xF(p))}{\Gamma(k+1)} e^{-x(1-F(p))},$$

and $\Gamma(k+1) = k! = \int_0^\infty t^k e^{-t} dt$ *and* $\Gamma(k+1,x) = \int_x^\infty t^k e^{-t} dt.$

Proof. See the Appendix. ■

In deriving this probability, one needs to take into account situations in which the number of buyers arriving at a given seller exceeds the number of buyers that the seller can attend to ($n > k$), as well as cases in which this is not the case ($n \leq k$).

It is now didactic to consider two special cases of interest. The first is the $k = 1$ case, which represents a situation of extreme selling constraints. Setting $k = 1$ in (1) gives the well-known matching probability (see e.g. Butters (1977)):

$$\eta(x,p;1) = \frac{1 - e^{-x}}{x}.$$

Notice that this probability is decreasing in x but independent of the price p. A higher number of buyers per firm x makes it less likely that a particular buyer is offered the product. The price does not matter because if the first buyer to whom the seller offers its product chooses not to buy it, then the seller cannot offer it to anyone else.

The second special case is that in which $k \to \infty$, which represents a situation in which firms do not have selling constraints whatsoever. Taking the limit of the probability in (1) when $k \to \infty$ gives:

$$\eta(x,p;\infty) = \lim_{k\to\infty} \eta(x,p;k) = \frac{1 - e^{-x(1-F(p))}}{x(1-F(p))}$$

This expression is similar in spirit to the one that obtains when $k = 1$ because we can interpret the quantity $x(1-F(p))$ as the "relevant" number of buyers per seller, that is, the number of buyers with a match value above the price per seller. Because F is a distribution function, it is straightforward to verify the intuitive result that $\eta(x,p;\infty) > \eta(x,p;1)$.

In general, it is more realistic to consider environments where selling constraints are neither extreme nor non-existent. Our next result gives some general properties of the probability $\eta(x,p;k)$ for arbitrary k.

Proposition 2 *The probability $\eta(x,p;k)$ (with which a buyer gets an opportunity to inspect and possibly buy the product of the seller he/she visits) is increasing and concave in k, decreasing in x, and increasing in $F(p)$ for $k \geq 2$.*

Proof. See the Appendix. ■

4 Local monopolies

We now move to the characterization of a symmetric equilibrium price when, as in the traditional consumer search literature (cf. Diamond, 1971; Stahl, 1989; Wolinsky, 1986), consumers only discover the actual price a firm charges upon arrival to the firm. This yields a model of local monopolies.

Let p be the symmetric equilibrium price. In order to derive p, consider a deviation by an individual seller i to a price $p_i \neq p$. Because buyers do not observe the actual prices charged by the sellers, price deviations do not affect buyers' visits. Hence, an individual buyer picks a seller at random and the number of buyers an individual seller expects to see at its premises is x.

The deviant's profit function is:

$$\Pi(p_i; p) = p_i \left(\sum_{\ell=1}^{k} \Pr[n_i = \ell](1 - F(p_i)^{\ell}) + \sum_{\ell=k+1}^{\infty} \Pr[n_i = \ell](1 - F(p_i)^{k}) \right).$$

This profit function reflects the fact that the actual number of buyers appearing at the deviant firm, n_i, may be smaller or larger than the maximum number of buyers the firm can attend to, k. In the Appendix we show that this payoff can be written more compactly as:

$$\Pi(p_i; p) = xp_i(1 - F(p_i))\eta(x, p_i; k). \tag{2}$$

The first order condition (FOC) for profits maximization, $\Pi'(p_i; p) = 0$, is:

$$x\left[1 - F(p_i) - p_i f(p_i)\right]\eta_i + p_i x(1 - F(p_i))\frac{\partial \eta_i}{\partial p_i} = 0. \tag{3}$$

After imposing symmetry, i.e. $p_i = p$, which also implies that $\eta_i = \eta$ we can rewrite the FOC (3) as follows:

$$\frac{\partial \eta}{\partial p} p(1 - F(p)) + \eta\left[1 - F(p) - pf(p)\right] = 0. \tag{4}$$

Proposition 3 *Suppose that sellers' prices are not observable before search. Then, for any distribution of match values with increasing density, i.e.* $f' \geq 0$*, there exists a unique symmetric equilibrium price, which is given by the solution to (4) and satisfies:*

$$p = \frac{1 - F(p)^k\left(1 - \frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right) - \frac{\Gamma(k+1,xF(p))}{\Gamma(k+1)}e^{-x(1-F(p))}}{kF(p)^{k-1}f(p)\left(1 - \frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right) + xf(p)\frac{\Gamma(k,xF(p))}{\Gamma(k)}e^{-x(1-F(p))}}. \tag{5}$$

Proof. See the Appendix. ■

4.1 Comparative statics

In this section we examine how the equilibrium price depends on the number of buyers per seller and the number of buyers sellers can attend to. An important fact comes out of the analysis: the monopoly price is sensitive to the ability of the firms to attend to their buyers. A firm that has many buyers interested in its product may not be able to take full advantage of this richness when its ability to attend to buyers is limited. This affects the equilibrium price.

Proposition 4 *The equilibrium price in (5):*

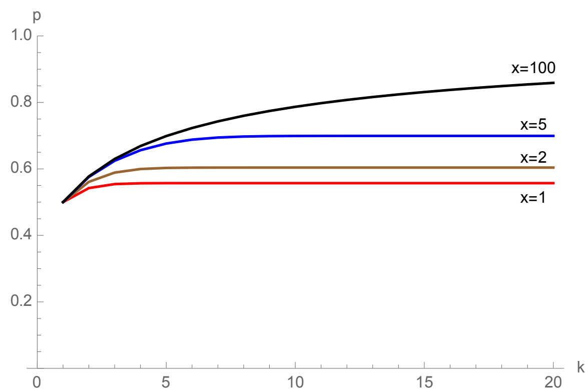
1. *is increasing in the number of buyers per seller x for all $k \geq 1$, satisfying $p \to \frac{1-F(p)}{f(p)}$ as $x \to 0$ and $p \to \frac{1-F^k(p)}{kF^{k-1}(p)f(p)}$ as $x \to \infty$;*
2. *is increasing in the selling capacity k, satisfying $p = \frac{1-F(p)}{f(p)}$ when $k = 1$ and $p \to \frac{1-e^{-x(1-F(p))}}{xe^{-x(1-F(p))}f(p)}$ as $k \to \infty$. The latter limit price approaches the standard monopoly price when $x \to 0$ and the upper bound of the distribution of match values ($= 1$) when $x \to \infty$.*

Proof. See the Appendix. ■

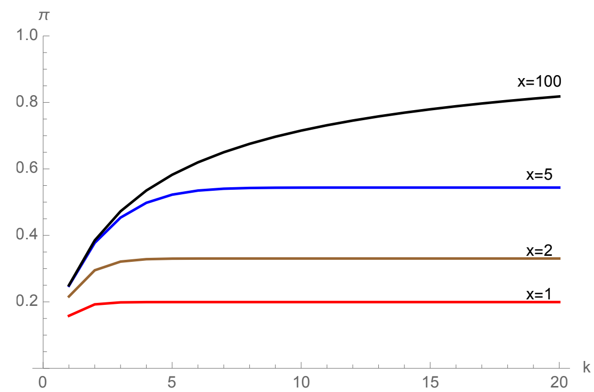
The equilibrium price increases in x and in k. This result is quite intuitive. Recall that the limiting case $x \to 0$ represents a situation in which there are infinitely many firms per buyer. While this tends to create conditions for low prices, firms still hold a significant amount of market power because buyers only visit one seller. Firms, expecting to receive almost no buyers when $x \to 0$ charge the standard monopoly price $p = \frac{1-F(p)}{f(p)}$ no matter k.[9] As x increases, firms expect to get more buyers at their premises. This does not change the equilibrium price if firms' selling constraints are maximal because then expecting to receive more buyers is inconsequential given that firms can only attend to one buyer at most. However, when firms can attend to more buyers, the equilibrium price increases. The reason is that every seller can offer its product to k buyers. When $x \to \infty$ and $k \to \infty$ the equilibrium price converges to the highest consumer valuation. The reason is that an individual firm can afford to continue to increase its price because the expectation is that there is always one more buyer to whom the firm can offer its item.

Figure 1 illustrates the results in Proposition 4. On the LHS we represent the equilibrium price and on the RHS firms' profits as a function of the selling constraint k for various levels of x. Both the price and firms' profits are monotone increasing in k and x.

[9] The "standard" monopoly price is the price that maximizes the payoff $\pi(p) = p(1 - F(p))$.



(a) Price



(b) Profits

Figure 1: Equilibrium price and profits as a function of the selling capacity k.

5 Monopolistic competition

We now move to the characterization of the market equilibrium when, in the tradition of Bertrand competition, consumers observe prices before search. This implies that (deviation) prices have an influence on the number of buyers who visit a given firm, with a low price firm receiving a larger number of visitors (cf. Armstrong, 2017; Armstrong and Zhou, 2011; Ding and Zhang, 2018; Choi, Dai and Kim, 2018; Haan, Moraga-González and Petrikaité, 2018). This yields a model of monopolistic competition.

We proceed in the same way as in Section 4. Let p be the symmetric equilibrium price. In order to derive p, consider a deviation by an individual seller i to a price $p_i \neq p$. The deviant's profit function is the same as that in (2) but after recognizing that the number of buyers visiting firm i is a function of the price charged by firm i and the price charged by the rest of the firms. Let $x(p_i; p)$ denote such function, that is, the (expected) number of buyers showing up at the deviating seller i charging price p_i when the rest of the sellers charge price p. (Sometimes we will use the shorter notation $x_i \equiv x(p_i; p)$ to refer to this expected number of visitors.) Then, the deviant's profit function is:

$$\Pi(p_i; p) = p_i(1 - F(p_i))x(p_i; p)\eta(x_i, p_i; k). \tag{6}$$

To determine the number of buyers $x(p_i; p)$ visiting the deviant firm i we follow Peters (2000) and assume that buyers must be indifferent between the utility they expect to get at the deviant firm and the utility they expect to get at any other seller in the market. The expected utility of a buyer who chooses to visit the deviant seller, denoted by $V(p_i;p)$, is given by:

$$\begin{aligned} V(p_i;p) &= \eta(x_i,p_i;k)[1-F(p_i)][E(\varepsilon \mid \varepsilon \geq p_i) - p_i] \\ &= \eta(x_i,p_i;k)I(p_i), \end{aligned}$$

where

$$I(p_i) \equiv \int_{p_i}^{1} (\varepsilon - p_i) f(\varepsilon) d\varepsilon$$

is the consumer's expected utility conditional on being offered the product of the seller. For later use, note that $\partial I/\partial p_i = -(1-F(p_i))$. Meanwhile the expected utility of a buyer who picks any other seller charging p is:

$$V(p;p) = \eta(x,p;k)[1-F(p)][E(\varepsilon \mid \varepsilon \geq p) - p] = \quad \eta(x,p;k)I(p).$$

Solving the equation $V(p_i;p) = V(p;p)$ for $x(p_i;p)$ gives the expected number of buyers who will visit a deviant seller charging price p_i. Unfortunately, the function $x(p_i;p)$ cannot be obtained in closed form. Nevertheless, in order to study equilibrium pricing we can apply the implicit function theorem to the equation $V(p_i;p) - V(p;p) = 0$ to obtain the sensitiveness of the deviant seller i's number of visitors to its own price:

$$\frac{\partial x(p_i;p)}{\partial p_i} = -\frac{\frac{\partial \eta_i}{\partial p_i} I(p_i) - \eta_i(1-F(p_i))}{\frac{\partial \eta_i}{\partial x_i} I(p_i)}, \tag{7}$$

where, to shorten notation, we have written η_i instead of $\eta(x_i,p_i;k)$.

The first order condition (FOC) for profits maximization, $\Pi'(p_i;p) = 0$, is:

$$[1-F(p_i) - p_i f(p_i)]\, x_i \eta_i + p_i(1-F(p_i)) x_i \frac{\partial \eta_i}{\partial p_i} + p_i(1-F(p_i)) \left(\eta_i + x_i \frac{\partial \eta_i}{\partial x_i}\right) \frac{\partial x_i}{\partial p_i} = 0. \tag{8}$$

After imposing symmetry, i.e. $p_i = p$, which also implies that $x_i = x$ and $\eta_i = \eta$, and using (7), we can rewrite the FOC (8) as follows:

$$
\begin{aligned}
& x\eta\left[1-F(p)-pf(p)\right]+xp(1-F(p))\left.\frac{\partial \eta_i}{\partial p_i}\right|_{p_i=p} \\
& +p(1-F(p))\left(\eta+x\left.\frac{\partial \eta_i}{\partial x_i}\right|_{p_i=p}\right)\frac{\eta(1-F(p))-I(p)\left.\frac{\partial \eta_i}{\partial p_i}\right|_{p_i=p}}{I(p)\left.\frac{\partial \eta_i}{\partial x_i}\right|_{p_i=p}} \\
& =x\eta\left[1-F(p)-pf(p)\right]+\eta p(1-F(p))\frac{\eta(1-F(p))-I(p)\left.\frac{\partial \eta_i}{\partial p_i}\right|_{p_i=p}}{I(p)\left.\frac{\partial \eta_i}{\partial x_i}\right|_{p_i=p}} \\
& +x\eta p\frac{(1-F(p))^2}{I(p)}=0.
\end{aligned}
$$

Rearranging terms, this expression can be rewritten as:

$$
\frac{\partial \eta}{\partial x}x\left\{\left[1-F(p)-pf(p)\right]I(p)+p(1-F(p))^2\right\}+p\eta(1-F(p))^2-\frac{\partial \eta}{\partial p}p(1-F(p))I(p)=0, \tag{9}
$$

where we have replaced $\left.\frac{\partial \eta_i}{\partial p_i}\right|_{p_i=p}$ and $\left.\frac{\partial \eta_i}{\partial x_i}\right|_{p_i=p}$ by $\frac{\partial \eta}{\partial p}$ and $\frac{\partial \eta}{\partial x}$, respectively, for simplicity of notation.

In the Appendix, we establish the following useful relationship between $\frac{\partial \eta}{\partial p}$ and $\frac{\partial \eta}{\partial x}$:

$$
\frac{(1-F(p))}{f(p)}\frac{\partial \eta}{\partial p}=-x\frac{\partial \eta}{\partial x}+d(p), \tag{10}
$$

where

$$
d(p)\equiv-\frac{kF(p)^{k-1}}{x}\left(1-\frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right)\leq 0.
$$

Using relationship (10) we can state that:

Proposition 5 *Suppose that sellers' prices are observable before search. Then, if there exists a symmetric equilibrium price, it is given by the solution to*

$$
\begin{aligned}
& \frac{(1-F(p))^2}{f(p)}\left[\frac{\partial \eta}{\partial p}I(p)+\frac{\partial \eta}{\partial p}p(1-F(p))-p\eta f(p)\right] \\
& \qquad =d(p)\left[\left(1-F(p)-pf(p)\right)I(p)+p(1-F(p))^2\right].
\end{aligned} \tag{11}
$$

When $k=1$ and $f'\geq 0$, and when $k\to\infty$ (and any f), the equilibrium exists and is unique.

Proof. See the Appendix. ■

In the Appendix we show that the FOC (11) has always at least one solution. For such a solution to be a symmetric equilibrium, the payoff function in (6) has to be well-behaved. A sufficient condition is that the function $m(x_i, p_i; k)$ is concave in p_i. Verification of this condition in general is extremely hard because the function $x(p_i; p)$ cannot be computed in closed form. Nevertheless, we can prove the existence and uniqueness of equilibrium for the two most relevant cases. One situation is when $k = 1$, in which case an increasing density of match values suffices. The other situation is when $k \to \infty$, in which case the equilibrium exists and is unique for any arbitrary density of match values. More in general, for fixed primitives x, k and F, it is straightforward to numerically check the concavity of the payoff (see Figures 2 and 3 below).

5.1 Comparative statics

In this section we examine how the equilibrium price depends on the number of buyers per seller and on the number of consumers sellers can attend to when buyers observe deviation prices.

We start by looking at how the equilibrium price depends on the number of buyers per seller. Consider first the case in which the firms' selling constraint is extremely severe so that $k = 1$. Equation (11) simplifies to:

$$p = \frac{1 - F(p)}{f(p) + \frac{(1-F(p))^2}{I(p)} \frac{xe^{-x}}{1-e^{-x}-xe^{-x}}}. \tag{12}$$

Note that the expression $\frac{xe^{-x}}{1-e^{-x}-xe^{-x}}$ is decreasing in x, with

$$\lim_{x \to 0} \frac{xe^{-x}}{1 - e^{-x} - xe^{-x}} = \infty \text{ and } \lim_{x \to \infty} \frac{xe^{-x}}{1 - e^{-x} - xe^{-x}} = 0.$$

This implies that the equilibrium price that solves equation (12) converges to the "standard" monopoly price $p = \frac{1-F(p)}{f(p)}$ as the number of buyers per seller x goes to infinity and approaches marginal cost when the x goes to zero. In the former case, firms do not really compete with one another. In the latter case, firms operate in an extremely competitive environment and we get marginal cost pricing.

For a given number of buyers per seller x and a distribution of match values F, equation (12) can be solved numerically for the equilibrium price. In Figure 2 we plot the payoff of a firm when selling capacity is $k = 1$ assuming match values are uniformly distributed on $[0, 1]$. The red profits function represents a case in which there are very few buyers per seller, concretely $x = 0.5$. The blue profits function represents the case in which there is one buyer per seller, i.e. $x = 1$. Finally, the black profits function represents the case of $x = 5$. The graph also shows the strict concavity of the payoff functions, as per Proposition 5.

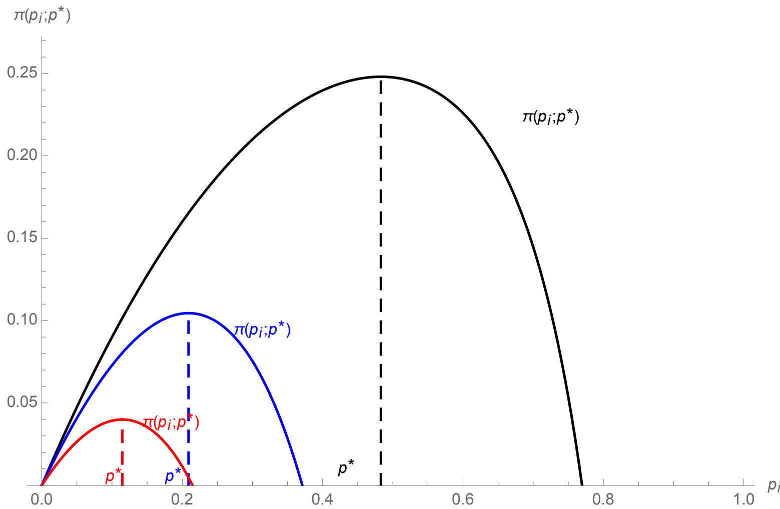


Figure 2: Concavity of the payoff when $k = 1$ and the price equilibrium.

Consider now the case in which firms' selling capacity is not restricted whatsoever. When $k \to \infty$, equation (11) simplifies to:

$$p = \frac{1 - e^{-x(1-F(p))} - x(1 - F(p))e^{-x(1-F(p))}}{1 - e^{-x(1-F(p))}} \frac{\int_p^1 \varepsilon f(\varepsilon) d\varepsilon}{1 - F(p)}.$$

Note that the expression $\frac{1-e^{-z}-ze^{-z}}{1-e^{-z}}$ is increasing in z, with

$$\lim_{z \to 0} \frac{1 - e^{-z} - ze^{-z}}{1 - e^{-z}} = 0 \text{ and } \lim_{z \to \infty} \frac{1 - e^{-z} - ze^{-z}}{1 - e^{-z}} = 1.$$

This implies that, exactly like in the case in which $k = 1$, the price that solves this equation converges to the marginal cost as the number of buyers per firm goes to zero. When the number of buyers per firm goes to infinity, things are quite different though. The equilibrium price converges to a monopoly price that is higher than before, namely, $p = 1$. In fact, using the limit result above, when $x \to \infty$ the equilibrium price solves the equation $p(1 - F(p)) - \int_p^1 \varepsilon f(\varepsilon) d\varepsilon = 0$ which, using the integration by parts formula, can be rewritten as $1 - p - \int_p^1 F(\varepsilon) d\varepsilon = 0$. Because the LHS of this expression is decreasing in p the solution is $p = 1$. In the absence of any selling constraint, an individual firm can charge a price as high as the highest match value; this is because facing a queue of infinitely many buyers and being able to offer its product to as many buyers as it likes, the seller can "pick" a buyer with a match value equal to 1. The notion of market power, here reflected in the monopoly price, is thus clearly linked to the selling capacity of a firm.

In Figure 3 we plot the payoff (2) for the case in which firms do not have selling constraints, again assuming that match values are uniformly distributed on $[0, 1]$. The red profits function represents the case in which there are very few buyers per firm, in particular $x = 0.5$. The equilibrium price is approximately $p = 0.1142$, which is slightly lower than in the case where $k = 1$, but firms obtain higher profits, $\Pi = 0.0408$. The blue profits function represents the case in which $x = 1$. In this case, the price is approximately $p = 0.208$, clearly higher than before. Profits reach $\pi = 0.1138$. Finally, the black profits function represents the case in which $x = 5$. In this case, the price is approximately $p = 0.5645$, clearly higher than before and higher than the "standard" monopoly price of $1/2$. Profits reach $\pi = 0.5005$, significantly higher than when firms do have significant selling constraints.

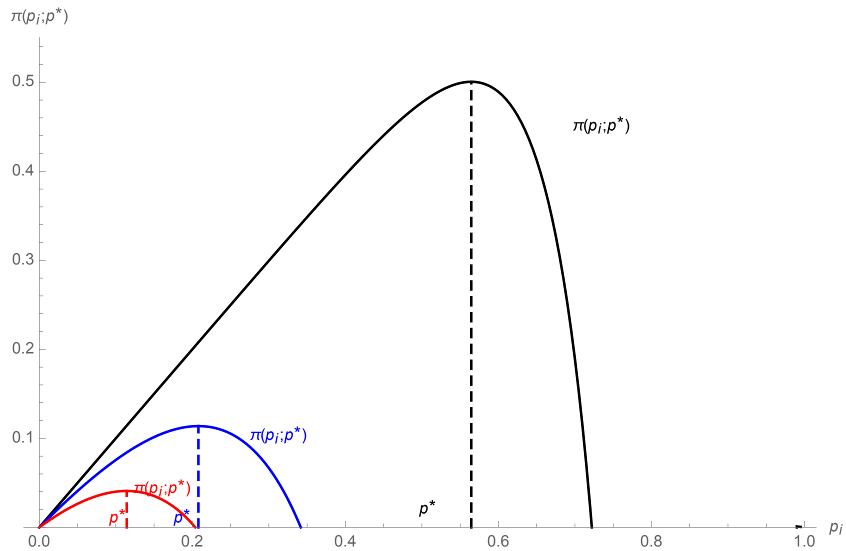
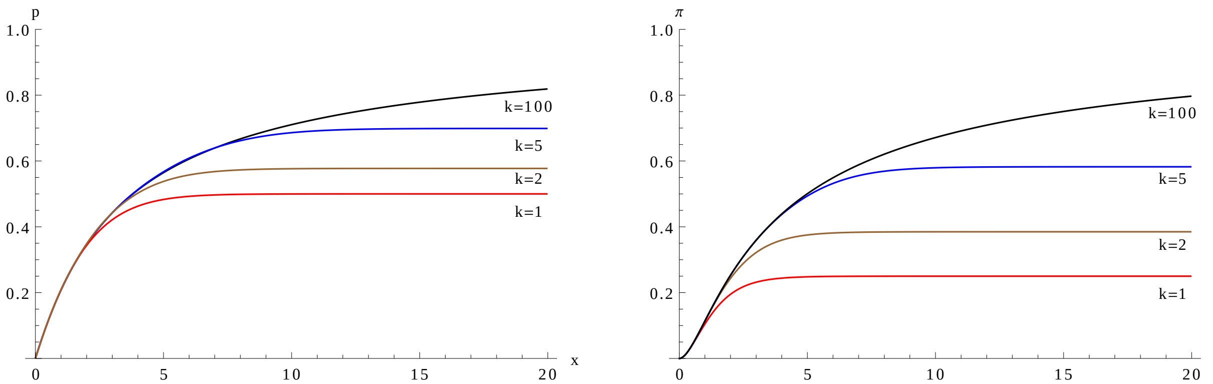


Figure 3: Concavity of the payoff when $k \to \infty$ and the price equilibrium.

The previous results are summarized in the next proposition.

Proposition 6 *(a) Suppose firms' selling constraints are maximal, that is, $k = 1$. Then, the equilibrium price is equal to the marginal cost if $x \to 0$ and approaches the standard monopoly price $p = \frac{1-F(p)}{f(p)}$ if $x \to \infty$.*
(b) Suppose firms do not have selling constraints, that is, $k \to \infty$. Then, the equilibrium price is equal to the marginal cost if $x \to 0$ and approaches the monopoly price $p = 1$ when $x \to \infty$.

For intermediate levels of k, it is quite difficult to derive analytical results on the relationship between the equilibrium price, firms' profits and the number of buyers per firm. We thus proceed to compute the equilibrium numerically. Figure 4 represents the equilibrium price and firms' profits as a function of x for various levels of the selling constraint. We observe that no matter how severe the selling constraint is, both the equilibrium price and the profits of the firms increase in the number of buyers per firm, thus reflecting how sellers take advantage of the buyers as market competitiveness loosens.

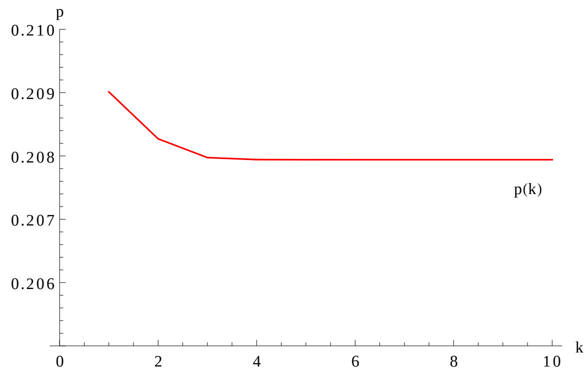


(a) Price (b) Profits

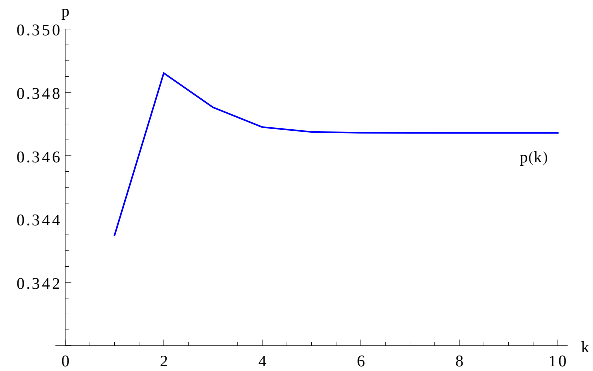
Figure 4: Equilibrium price, profits and the number of buyers per firm.

The graph also shows that the equilibrium price and profits typically increase in k. To be sure, this is clearly visible for large x. When x is small the different curves are too cluttered to be able to distinguish between price levels for different k's. We now analyze this relationship in more detail. In general, an increase in k affects the equilibrium price in two ways that operate in opposite directions. On the one hand, a softer selling constraint makes attracting buyers to its premises more valuable to a firm because it can offer its product to more consumers. By this effect, an increase in k tends to reduce the equilibrium price. On the other hand, a laxer selling constraint increases the maximum of the willingness to pay of the k consumers a firm can offer its product to. By this effect, an increase in k tends to increase the equilibrium price. We have already seen analytically that when $x \to 0$, the equilibrium price is equal to the marginal cost no matter whether $k = 1$ or $k \to \infty$. This is true for any k as a matter of fact. The reason is that when $x \to 0$, the second effect plays no role. When $x \to \infty$, we have the opposite case in which the first effect plays no role. In such a case, the equilibrium price increases in k. In fact, we have already seen that the equilibrium price when $k \to \infty$ is higher than when $k = 1$. When the number of buyers per firm x takes on an intermediate value, the equilibrium price may increase or decrease as the selling constraint becomes laxer.

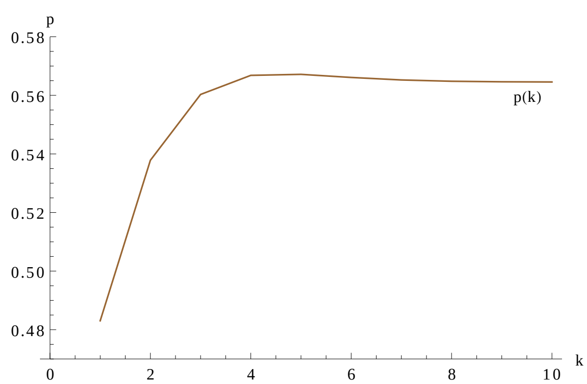
We illustrate these results in Figure 5, where we represent the equilibrium price as a function of the selling constraint for various levels of the number of buyers per firm. The graphs illustrate that the equilibrium price is decreasing in relatively tight markets where there are very few buyers per seller (Figure 5(a)), non-monotonic in markets where the number of buyers per firm is intermediate (Figures 5(a) and 5(b)) and increasing in relatively loose markets where there are many buyers per seller (Figure 5(d)).



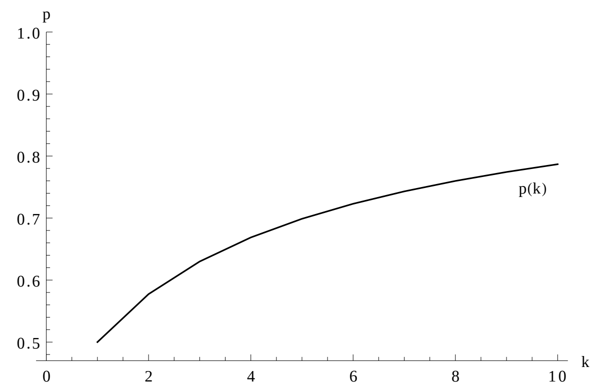
(a) Equal number of buyers and sellers: $x = 1$.



(b) Few buyers per seller: $x = 2$.



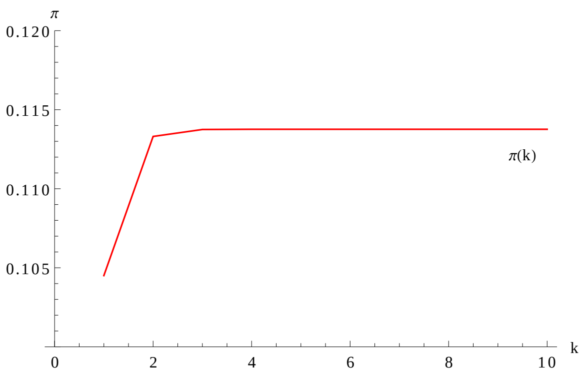
(c) Some buyers per seller: $x = 5$.



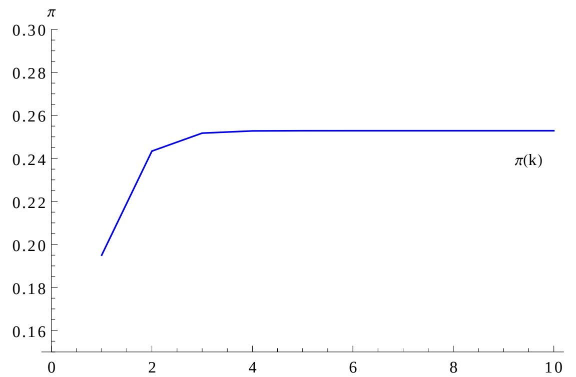
(d) Many buyers per seller: $x = 100$.

Figure 5: Equilibrium price and selling constraints.

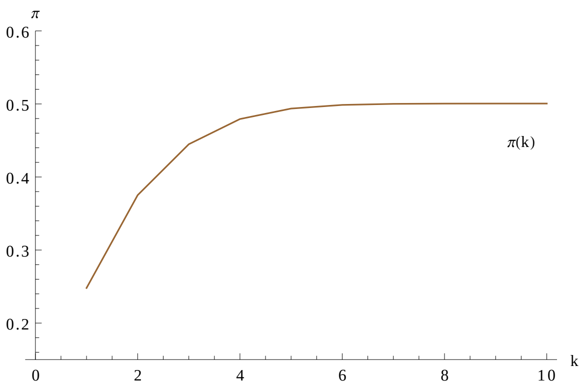
Despite the fact that the equilibrium price may decrease in k when there are few buyers per seller, equilibrium profits are increasing in k. This can be seen in Figure 8 where we have plotted the equilibrium profits corresponding to the equilibrium prices depicted in Figure 5.



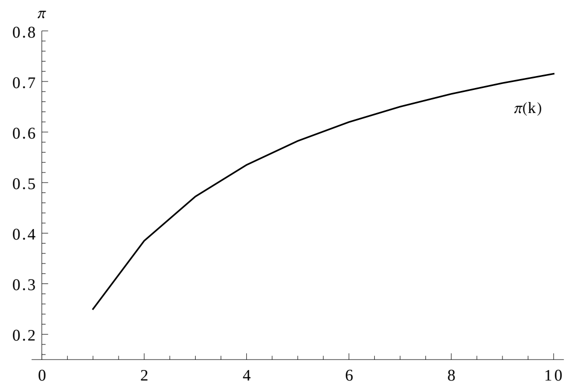
(a) Equal number of buyers and sellers: $x = 1$.



(b) Few buyers per seller: $x = 2$.



(c) Some buyers per seller: $x = 5$.



(d) Many buyers per seller: $x = 100$.

Figure 6: Equilibrium profits and selling constraints.

We conclude this section by summarizing our numerical findings.

Numerical result 1 *(a) The equilibrium price and firms' profits are monotonically increasing in x for any k. (b) For low x, the equilibrium price is decreasing in k; for intermediate x, the equilibrium price is first increasing in k and then decreasing; for large x, the equilibrium price is increasing in k. Firms' profits increase in k.*

6 Welfare

In this section we study the efficiency of the market equilibrium price in the local monopolies and monopolistic competition scenarios of Sections 4 and 5. To do this, we first characterize the efficient price and then we compare it to the market equilibrium.

Social welfare, as usual, equals the sum of buyers' utility and sellers' profits:

$$\begin{aligned} W &= BV + S\Pi \\ &= B\eta(x,p;k)(1-F(p))\left[E(\varepsilon \mid \varepsilon \geq p) - p\right] + Spx\eta(x,p;k)(1-F(p)) \end{aligned}$$

Using the fact that $B = xS$, we can simplify the welfare expression:

$$W = B\underbrace{\eta(x,p;k)(1-F(p))}_{\text{prob. of trade}}\underbrace{E(\varepsilon \mid \varepsilon \geq p)}_{\text{value of trade}}.$$

Inspection of this expression reveals that an increase in the price has both a positive and a negative effect on social welfare. The positive effect is to increase the value of a transaction, that is, the value of $E(\varepsilon \mid \varepsilon \geq p)$. A higher price serves as a selection mechanism: only a consumer with a sufficiently high match value will accept the trade, which generates a higher social surplus. The negative effect is to decrease the quantity of trade, that is, the probability $\eta(x,p;k)(1-F(p))$ with which buyers buy. This is because a higher price makes it less likely that anyone queuing at a seller happens to have a sufficiently high match utility for the product of the seller. Formally, this observation follows from the fact that:

$$\begin{aligned} \frac{x}{f(p)}\frac{\partial \eta(1-F(p))}{\partial p} &= kF(p)^{k-1}\frac{\Gamma(k+1,x)-\Gamma(k+1)}{\Gamma(k+1)} \\ &+ \frac{xe^{-x(1-F(p))}}{\Gamma(k+1)}\left(x^k F(p)^k e^{-xF(p)} - \Gamma(k+1,xF(p))\right) \\ &= kF(p)^{k-1}\frac{\Gamma(k+1,x)-\Gamma(k+1)}{\Gamma(k+1)} - \frac{xe^{-x(1-F(p))}}{\Gamma(k+1)}\int_{xF(p)}^{\infty} kt^{k-1}e^{-t}dt < 0, \end{aligned}$$

where the second equality follows from integrating by parts the Γ function. Hence, in choosing a price the planner faces a trade-off between the quantity and the quality of trade.

Taking the FOC for welfare maximization we can state that:

Proposition 7 *The socially optimal price, denoted p^o, satisfies the FOC:*

$$\frac{\partial \eta}{\partial p}I(p^o) + \frac{\partial \eta}{\partial p}p^o(1-F(p^o)) - p^o\eta f(p^o) = 0. \tag{13}$$

When $k = 1$, η does not depend on price so the socially optimal price is equal to the marginal cost. For $k \geq 2$ the socially optimal price is strictly greater than the marginal cost.

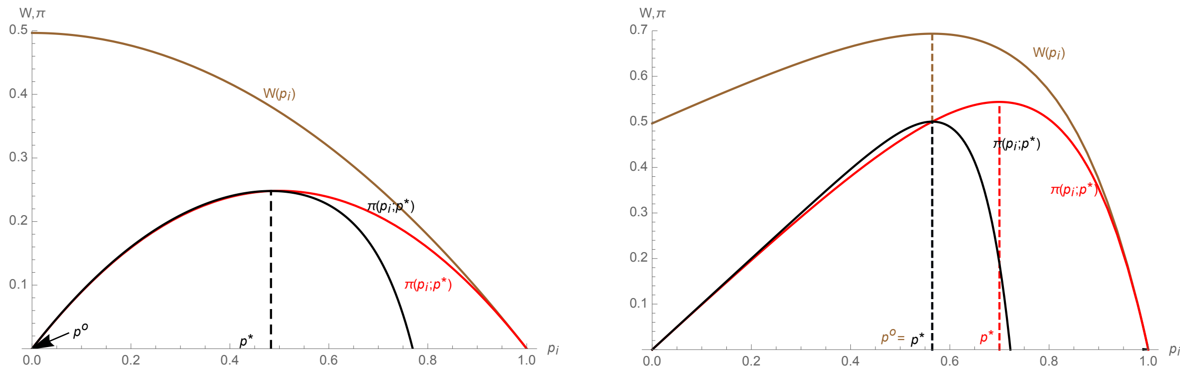
This result is at odds with the standard view in economics that marginal cost pricing is a cornerstone of efficiency. Except in the case in which $k = 1$, in our model the welfare function is typically non-monotonic in price, which implies that *efficient pricing involves positive markups.* What distinguishes our model from the standard model is that firms are capacity constrained and sell differentiated products. Note that it is these two features together that create a trade-off for the planner: a higher price lowers the chance a transaction occurs, but increases its value if it occurs. Welfare maximization consists of balancing these two effects, which drives a wedge between the optimal price and the marginal cost. Hence, the shortcut notion of market power as the ability of firms to sustain prices about marginal cost is not really valid in our setting. Instead, market power has to be assessed as the capacity of firms to sustain prices above the efficient level, which differs from the marginal cost.

Note that equation (13) has surely a solution. This is because equation (13) is exactly the same as the LHS of equation (11) and in the proof of Proposition 5 we show that this expression is strictly positive at $p = 0$ and negative at $p = 1$. Comparing the socially optimal price with the market equilibrium prices of Propositions 3 and 5 leads to the following insight:

Proposition 8

1. *When buyers do not observe prices before visiting sellers, local monopolies result and the equilibrium price is inefficiently high.*
2. *When buyers observe prices before visiting sellers, monopolistic competition results and the equilibrium price is inefficiently high except in the limit when $k \to \infty$ (in which case the market equilibrium price is efficient and greater than the marginal cost).*

Figure 7 illustrates this result by plotting together the payoff functions of the firms and social welfare. The left graph, Figure 7(a), shows the case in which firms face extreme selling constraints so that $k = 1$. The red curve is the profit function when buyers do not observe prices before they visit firms and hence a model of local monopolies result. The black curve is the profit function when they do observe the prices of the firms and so monopolistic competition arises. In this case, the planner just wishes to maximize the probability of trade and sets a price equal to the marginal cost. The equilibrium price, which is the same no matter whether prices are observable before search or not, is clearly excessive. The right graph, Figure 7(b), represents the case in which firms do not face selling constraints whatsoever, $k \to \infty$. Here, again, the red curve is the profit function for local monopolies and the black one is the profit function under monopolistic competition. In this case, welfare is non-monotonic and is maximized at the same price as the equilibrium price under monopolistic competition. In both these graphs, we have set $x = 5$.



(a) Extreme selling constraints, $k = 1$.

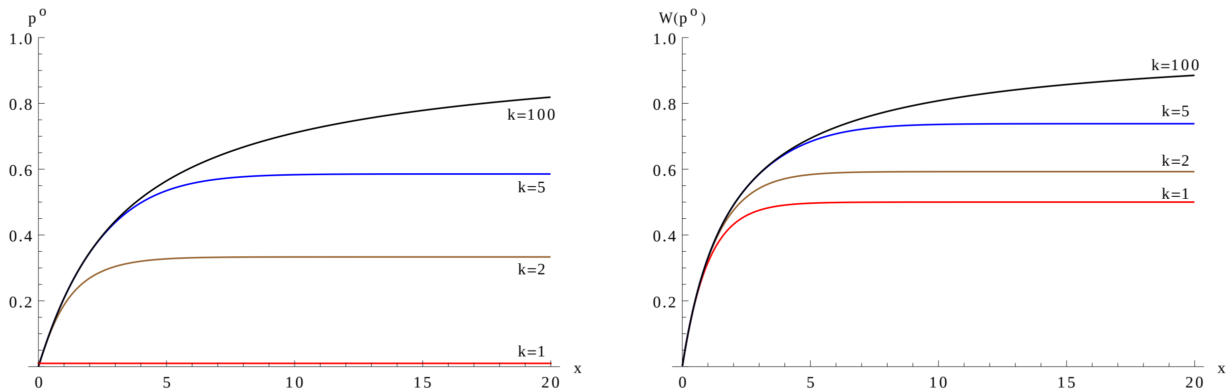
(b) No selling constraints, $k \to \infty$.

Figure 7: Social optimum and equilibrium.

The equilibrium price is generally excessive. It is only efficient in one case, namely when consumers observe prices before they visit sellers and $k \to \infty$ so that selling constraints are completely absent. In such a case, at the efficient price, the social trade-off between the quantity and the quality (or price) of trade is exactly identical to its private counterpart. Comparing the equilibrium condition (11) and the efficiency condition (13), we observe that they become exactly identical if and only if the factor $d(p) = 0$, which implies that firms should be able to continue to show their products to all the buyers who show up at their premises. In fact, only when $k \to \infty$ is the probability of having more buyers than their selling capacity equal to zero ($1 - \frac{\Gamma(k+1,x)}{\Gamma(k+1)} \to 0$ as $k \to \infty$) and therefore the factor $d(p) = 0$.

When firms face non-trivial selling constraints ($k < \infty$), the probability that a firm receives more buyers than it can attend to is strictly positive. That is, $1 - \frac{\Gamma(k+1,x)}{\Gamma(k+1)} > 0$ and thus $d(p) < 0$. This means that an individual firm may find itself in a situation where it fails to sell its product while there still are buyers interested in the product who cannot however be approached. This reduces the firms' incentive to attract buyers by lowering price. Hence, the private benefit of increasing price becomes higher than its social counterpart, leading to an inefficiently high equilibrium price.

The socially optimal price that solves (13), and hence the level of welfare attained in the economy, depends on the number of buyers per firm and the selling constraint. In Figure 8 we plot the efficient price and the corresponding welfare level per seller as a function of the number of buyers per firm for various levels of the selling constraint. It can be seen tat both the efficient price and the maximum welfare level attained are increasing in x and k.



(a) Social welfare maximizing price

(b) Welfare

Figure 8: Social welfare maximizing price and welfare levels.

We conclude this section by summarizing our numerical findings.

Numerical result 2 *The socially optimal price and the level of welfare are monotonically increasing in x and k.*

7 Concluding remarks

Despite the fact that selling constrains are ubiquitous and often firms cannot attend to all the buyers who are interested in inspecting their products, as far as we know, the literature has not paid attention to their impact on the functioning of search markets. This paper has started to close this gap. Our main conclusion has been that, when market power is appropriately measured as the wedge between the market equilibrium price and the price that maximizes social welfare, selling constraints are a source of market power. Moreover, we have seen that the connection between market power and high profits become more loose in the presence of selling constraints. This has potential implications for the interpretation of all the recent work that measures markups (as differences between prices and marginal costs) and relates them to market inefficiency (see e.g. De Loecker, Eeckhout and Unger (2020)).

In reaching this conclusion we have used a model with some specific features. One of the assumptions of the model has been that firms face two types of constraints. First, firms are capacity-constrained and have at their disposal just one unit of a differentiated product. Second, firms face selling constraints and can only attend to a maximum of k buyers. The insights of our paper carry over to situations where firms' capacity constraints are not so stringent. In fact, suppose that sellers have at their disposal ℓ units of the differentiated product but can only attend to k buyers. It is clear that sellers will only be able to sell a maximum number of units lower than or equal to $\min\{\ell, k\}$.

As a result, situations in which $\ell < k$ will be similar to the one studied in this paper because sellers continue to face a trade-off between the quantity of trade and the quality of trade. If $\ell > k$, by contrast, sellers would not face such a trade-off and the equilibrium pricing would be similar to the $k = 1$ case.

Another simplifying assumption of the model has been that sellers and buyers interact for just a single period. This implies that in our search model search costs do not play any role and the only source of search frictions is the potential rationing that buyers may suffer due to the firms' capacity and selling constraints. A more complete depiction of search frictions in markets for differentiated products ought to include both demand- and supply-side frictions. Assuming that in every period the buyers and sellers who transact with one another are replaced by new ones in the economy, it is not very hard to extend our model to allow for consumers' sequential search as it is standard in the consumer search literature. In that case, the search cost becomes the key factor that influences the trade-off between the quantity and the quality of trade. A higher search cost makes consumers less picky, which increases the probability of trade but reduces the value of trade. Welfare is thus non-monotonic in search costs and the social welfare maximizing search cost is typically bounded away from zero. This is akin to our positive efficient markup result.

Finally, we have assumed that firms' selling constraints are exogenous. However, as mentioned in the Introduction, firms not only choose their prices but try to optimize its marketing and sales team to maximize their profits. Though extending our work to allow for the possibility that firms choose k is quite challenging, it would be very interesting to know how firms' marketing and sales teams are influenced by the parameters of the model. We leave the full development of this extension as a topic for further research.

Appendix

Proof of Proposition 1.

Note that the number of buyers visiting a seller n follows a Poisson distribution, Prob.$(n = i) = \frac{x^i e^{-x}}{i!}$. We consider the offer probability to a buyer who visits a seller. Let an index i count the number of the *other* buyers arriving at a seller. Then, we have

$$\begin{aligned}\eta &= \sum_{i=0}^{k-1} \frac{x^i e^{-x}}{i!} \sum_{j=0}^{i} \frac{i!}{j!(i-j)!} F(p)^j (1-F(p))^{i-j} \frac{1}{i+1-j} \\ &\quad + \sum_{i=k}^{\infty} \frac{x^i e^{-x}}{i!} \frac{k}{i+1} \sum_{j=0}^{k-1} \frac{k-1!}{j!(k-1-j)!} F(p)^j (1-F(p))^{k-1-j} \frac{1}{k-j}\end{aligned}$$

The first summation represents cases in which the number of the other buyers is less than the number of buyers that the seller can handle, i.e., $i \leq k-1$. With $j \leq i$ of the other buyers turning out *not* to like the seller's product, which comes in $\frac{i!}{j!(i-j)!}$ ways and occurs with probability $F(p)^j(1-F(p))^{i-j}$, the given buyer will be offered with probability $\frac{1}{i-j+1}$. The second summation represents cases in which $i \geq k$. Note that the seller has to randomly select $k \leq i$ buyers, and the given buyer is selected with probability $\frac{k}{i+1}$. With $j \leq k-1$ of the other *selected* buyers turning out not to like the seller's product, which comes in $\frac{k-1!}{j!(k-1-j)!}$ ways and occurs with probability $F(p)^j(1-F(p))^{k-1-j}$, the given buyer will be offered with probability $\frac{1}{k-j}$.

Note that we can simplify the terms in the first summation,

$$\begin{aligned}&\sum_{j=0}^{i} \frac{i!}{j!(i-j)!} F(p)^j (1-F(p))^{i-j} \frac{1}{i+1-j} \\ =& \frac{1}{(i+1)(1-F(p))} \sum_{j=0}^{i} \frac{(i+1)!}{j!(i+1-j)!} F(p)^j (1-F(p))^{i+1-j} \\ =& \frac{1}{(i+1)(1-F(p))} \left[\sum_{j=0}^{i+1} \frac{(i+1)!}{j!(i+1-j)!} F(p)^j (1-F(p))^{i+1-j} - F(p)^{i+1} \right] \\ =& \frac{1-F(p)^{i+1}}{(i+1)(1-F(p))},\end{aligned}$$

and the terms in the second summation,

$$\begin{aligned}
& \frac{k}{i+1}\sum_{j=0}^{k-1}\frac{k-1!}{j!(k-1-j)!}F(p)^j(1-F(p))^{k-1-j}\frac{1}{k-j} \\
= & \frac{1}{(i+1)(1-F(p))}\sum_{j=0}^{k-1}\frac{k!}{j!(k-j)!}F(p)^j(1-F(p))^{k-j} \\
= & \frac{1}{(i+1)(1-F(p))}\left[\sum_{j=0}^{k}\frac{k!}{j!(k-j)!}F(p)^j(1-F(p))^{k-j}-F(p)^k\right] \\
= & \frac{1-F(p)^k}{(i+1)(1-F(p))}.
\end{aligned}$$

Using these simplifications, we have

$$\begin{aligned}
\eta &= \sum_{i=0}^{k-1}\frac{x^ie^{-x}}{i!}\frac{1-F(p)^{i+1}}{(i+1)(1-F(p))}+\sum_{i=k}^{\infty}\frac{x^ie^{-x}}{i!}\frac{1-F(p)^k}{(i+1)(1-F(p))} \\
&= \frac{1}{x(1-F(p))}\sum_{i=0}^{k-1}\frac{x^{i+1}e^{-x}(1-F(p)^{i+1})}{(i+1)!}+\frac{1-F(p)^k}{x(1-F(p))}\sum_{i=k}^{\infty}\frac{x^{i+1}e^{-x}}{(i+1)!}.
\end{aligned}$$

Setting $h\equiv i+1$, it is further simplified to

$$\begin{aligned}
\eta &= \frac{1}{x(1-F(p))}\sum_{h=0}^{k}\left[\frac{x^he^{-x}}{h!}-\frac{[xF(p)]^he^{-x}}{h!}\right]+\frac{1-F(p)^k}{x(1-F(p))}\sum_{h=k+1}^{\infty}\frac{x^he^{-x}}{h!} \\
&= \frac{1}{x(1-F(p))}\left[\sum_{h=0}^{k}\frac{x^he^{-x}}{h!}-e^{-x(1-F(p))}\sum_{h=0}^{k}\frac{[xF(p)]^he^{-xF(p)}}{h!}\right]+\frac{1-F(p)^k}{x(1-F(p))}\left[1-\sum_{h=0}^{k}\frac{x^he^{-x}}{h!}\right] \\
&= \frac{1}{x(1-F(p))}\left[\frac{\Gamma(k+1,x)}{\Gamma(k+1)}-e^{-x(1-F(p))}\frac{\Gamma(k+1,xF(p))}{\Gamma(k+1)}\right]+\frac{1-F(p)^k}{x(1-F(p))}\left[1-\frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right],
\end{aligned}$$

where we used $\sum_{h=0}^{k}\frac{x^he^{-x}}{h!}=\frac{\Gamma(k+1,x)}{\Gamma(k+1)}$ (i.e., the series definition of the cumulative gamma function), with $\Gamma(k+1)=k!=\int_0^\infty t^ke^{-t}dt$ and $\Gamma(k+1,x)=\int_x^\infty t^ke^{-t}dt$. Rearranging terms, we obtain the expression in (1). ■

Proof of Proposition 2.

(a) In order to show that η is increasing in k, it suffices to show that m is increasing in k. For this, we compute the difference:

$$
\begin{aligned}
m(k+1)-m(k) &= 1-F(p)^{k+1}+\frac{\Gamma(k+2,x)}{\Gamma(k+2)}F(p)^{k+1}-\frac{\Gamma(k+2,xF(p))}{\Gamma(k+2)}e^{-x(1-F(p))} \\
&\quad -\left(1-F(p)^{k}+\frac{\Gamma(k+1,x)}{\Gamma(k+1)}F(p)^{k}-\frac{\Gamma(k+1,xF(p))}{\Gamma(k+1)}e^{-x(1-F(p))}\right) \\
&= F(p)^k-F(p)^{k+1}+\frac{(k+1)\Gamma(k+1,x)+x^{k+1}e^{-x}}{\Gamma(k+2)}F(p)^{k+1} \\
&\quad -\frac{(k+1)\Gamma(k+1,xF(p))+(xF(p))^{k+1}e^{-xF(p)}}{\Gamma(k+2)}e^{-x(1-F(p))}-\frac{\Gamma(k+1,x)}{\Gamma(k+1)}F(p)^k \\
&\quad +\frac{\Gamma(k+1,xF(p))}{\Gamma(k+1)}e^{-x(1-F(p))},
\end{aligned}
$$

where we have used the property of the Gamma function (see Jameson, 2016):

$$
\Gamma(k+2,x)=(k+1)\Gamma(k+1,x)+x^{k+1}e^{-x}. \tag{14}
$$

Because $\Gamma(k+1)=k\Gamma(k)$, we can rewrite the previous expression as follows:

$$
\begin{aligned}
m(k+1)-m(k) &= F(p)^k-F(p)^{k+1}+\frac{\Gamma(k+1,x)}{\Gamma(k+1)}F(p)^{k+1}+\frac{x^{k+1}e^{-x}}{\Gamma(k+2)}F(p)^{k+1}-\frac{\Gamma(k+1,xF(p))}{\Gamma(k+1)}e^{-x(1-F(p))} \\
&\quad -\frac{(xF(p))^{k+1}e^{-xF(p)}}{\Gamma(k+2)}e^{-x(1-F(p))}-\frac{\Gamma(k+1,x)}{\Gamma(k+1)}F(p)^k+\frac{\Gamma(k+1,xF(p))}{\Gamma(k+1)}e^{-x(1-F(p))} \\
&= F(p)^k-F(p)^{k+1}+\frac{\Gamma(k+1,x)}{\Gamma(k+1)}F(p)^{k+1}-\frac{\Gamma(k+1,x)}{\Gamma(k+1)}F(p)^k \\
&= F(p)^k(1-F(p))\left(1-\frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right).
\end{aligned}
$$

The last expression is positive because $\Gamma(k+1,x)=\int_x^\infty t^k e^{-t}dt<\int_0^\infty t^k e^{-t}dt=\Gamma(k+1)$, which completes the proof that m is increasing in k.

To demonstrate that η is concave in k, we compute the difference:

$$
[m(k+2)-m(k+1)]-[m(k+1)-m(k)]=(1-F(p))F(p)^k\left[F(p)\left(1-\frac{\Gamma(k+2,x)}{\Gamma(k+2)}\right)-\left(1-\frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right)\right].
$$

For concavity, this expression must be negative. Becasue $F(p)<1$, it is sufficient that

$$
\frac{\Gamma(k+2,x)}{\Gamma(k+2)}>\frac{\Gamma(k+1,x)}{\Gamma(k+1)},
$$

or that

$$
\frac{\Gamma(k+2,x)}{k+1}>\Gamma(k+1,x),
$$

which is true because of the property (14).

We now use differentiation to show that η is decreasing in x. For this result, notice that it suffices to show that m is decreasing in x. Taking the derivative of m with respect to x, and putting common factors together, gives:

$$\frac{\partial m}{\partial x} = -\frac{e^{-x}(1-F)\left((xF)^k - e^{xF}\Gamma(k+1,xF)\right)}{\Gamma(k+1)}$$

The sign of this expression depends on the sign of $(xF)^k - e^{xF}\Gamma(k+1,xF)$. Using (14), we have:

$$(xF)^k - e^{xF}\Gamma(k+1,xF) = (xF)^k - e^{xF}\left(k\Gamma(k,xF) + (xF)^k e^{-xF}\right) = -e^{xF}k\Gamma(k,xF) < 0.$$

Therefore, m decreases in x and so does η.

We finally show that η is increasing in $F(p)$. For this, we first note that

$$x\frac{\partial \eta}{\partial F(p)} = \frac{\frac{\partial m}{\partial F(p)}(1-F) + m}{(1-F)^2}$$

The sign of $\partial\eta/\partial F(p)$ depends on the sign of the numerator. We note that

$$\frac{\partial m}{\partial F(p)} = \frac{kx\left(-e^{-x(1-F)}\right)\Gamma(k,xF) - kF^{k-1}(\Gamma(k+1) - \Gamma(k+1,x))}{\Gamma(k+1)}.$$

Using this, we calculate:

$$\begin{aligned}\Gamma(k+1)\left(\frac{\partial m}{\partial F(p)}(1-F) + m\right) &= \Gamma(k+1) - (F + k(1-F))F^{k-1}(\Gamma(k+1) - \Gamma(k+1,x)) \\ &\quad - e^{-x(1-F)}((1-F)kx\Gamma(k,xF) + \Gamma(k+1,xF)) \qquad (15)\end{aligned}$$

The RHS of this expression is decreasing in F because its derivative with respect to F can be written as

$$-k(1-F)\left[(k-1)F^{k-2}\left(\Gamma(k+1) - \Gamma(k+1,x)\right) + x^2 e^{-x(1-F)}\left(\Gamma(k,xF) - (xF)^{k-1}e^{-xF}\right)\right],$$

which is negative because the term $\Gamma(k,xF) - (xF)^{k-1}e^{-xF} = (k-1)\Gamma(k-1,xF) > 0$ for $k \geq 2$.

It is straightforward to see that when we set $F = 1$ in the RHS of equation (15) we obtain 0. This means that (15) is positive for all F, which completes the proof that η is increasing in $F(p)$ for $k \geq 2$. ■

Derivation of the profit function in (2)

The expected profit of seller i is given by:

$$\Pi(p_i;p) = p_i \left(\sum_{l=1}^{k} \Pr[n_i = \ell] \left(1 - F(p_i)^\ell\right) + \sum_{\ell=k+1}^{\infty} \Pr[n_i = \ell] \left(1 - F(p_i)^k\right) \right). \tag{16}$$

Because the expected number of buyers visiting a seller n follows a Poisson distribution, $\Pr(n_i = \ell) = \frac{x_i^\ell e^{-x_i}}{\ell!}$.

To obtain the expression in (2), observe that the first term in the bracket of (16) can be simplified as follows:

$$\sum_{\ell=1}^{k} \Pr(n_i = \ell) \left(1 - F(p_i)^\ell\right) = \sum_{\ell=0}^{k} \frac{x_i^\ell e^{-x_i}}{\ell!} (1 - F(p_i)^\ell)$$
$$= \sum_{\ell=0}^{k} \left[\frac{x_i^\ell e^{-x_i}}{\ell!} - \frac{[x_i F(p_i)]^\ell e^{-x_i}}{\ell!} \right] = \sum_{\ell=0}^{k} \frac{x_i^\ell e^{-x_i}}{\ell!} - e^{-x_i(1-F(p_i))} \sum_{\ell=0}^{k} \frac{[x_i F(p_i)]^\ell e^{-x_i F(p_i)}}{\ell!}$$
$$= \frac{\Gamma(k+1, x_i)}{\Gamma(k+1)} - e^{-x_i(1-F(p_i))} \frac{\Gamma(k+1, x_i F(p_i))}{\Gamma(k+1)}$$

where we have used the series definition of the cumulative gamma function: $\sum_{h=0}^{k} \frac{x^h e^{-x}}{h!} = \frac{\Gamma(k+1,x)}{\Gamma(k+1)}$.

Likewise, the second term in the bracket of (16) can be simplified as follows:

$$\sum_{l=k+1}^{\infty} \Pr(n_i = \ell) \left(1 - F(p_i)^k\right) = \left(1 - F(p_i)^k\right) \sum_{\ell=k+1}^{\infty} \frac{x_i^l e^{-x_i}}{\ell!}$$
$$= \left(1 - F(p_i)^k\right) \left[1 - \sum_{\ell=0}^{k} \frac{x_i^\ell e^{-x_i}}{\ell!}\right] = \left(1 - F(p_i)^k\right) \left[1 - \frac{\Gamma(k+1, x_i)}{\Gamma(k+1)}\right].$$

Hence, the payoff expression in (16) can be written as

$$\Pi(p_i;p) = p_i \left(\frac{\Gamma(k+1, x_i)}{\Gamma(k+1)} - e^{-x_i(1-F(p_i))} \frac{\Gamma(k+1, x_i F(p_i))}{\Gamma(k+1)} + \left(1 - F(p_i)^k\right) \left[1 - \frac{\Gamma(k+1, x_i)}{\Gamma(k+1)}\right] \right)$$

Using the expression for η in equation (1), it is now straightforward to obtain the payoff in (2).

Proof of Proposition 3.

To prove this result, we first observe that the LHS of (11) evaluated at $p = 0$ is equal to η, which is strictly positive; moreover, the LHS of (11) evaluated at $p = 1$ is equal to $-\eta f(1)$, which is strictly negative. Because the LHS of (11) is a continuous function of p, equation (11) has a solution in p.

Such a solution is a symmetric Nash equilibrium is the payoff of a firm i is concave in p_i. For this it is sufficient that the function $m(x, p_i; k)$ is concave in p_i. Notice that $m(x, p_i; k)$ is a sum of functions and the sum of concave functions is concave.

Further notice that the first three summands of the function $m(x, p_i; k)$ are:

$$1 + \left(\frac{\Gamma(k+1,x)}{\Gamma(k+1)} - 1 \right) F(p)^k$$

These three summands are concave in p_i because $\frac{\Gamma(k+1,x)}{\Gamma(k+1)} - 1 < 0$.

Consider now the last summand:

$$-\frac{\Gamma(k+1, xF(p))}{\Gamma(k+1)} e^{-x(1-F(p))}$$

For this summand to be concave it is sufficient that $\Gamma(k+1, xF(p))e^{-x(1-F(p))} = \left(\int_{xF(p)}^{\infty} t^k e^{-t} dt \right) e^{-x(1-F(p))}$ is convex.

Taking the second derivative of $\Gamma(k+1, xF(p))e^{-x(1-F(p))}$ gives:

$$\frac{e^{-x} \left(xF(p)e^{xF(p)} \Gamma(k+1, xF(p)) \left(xf(p)^2 + f'(p) \right) - x(xF(p))^k \left(f(p)^2 (xF(p) + k) + F(p)f'(p) \right) \right)}{F(p)}$$

This expression is positive when

$$xF(p)e^{xF(p)} \Gamma(k+1, xF(p)) \left(xf(p)^2 + f'(p) \right) - x(xF(p))^k \left(f(p)^2 (xF(p) + k) + F(p)f'(p) \right) > 0 \quad (17)$$

To show that this inequality holds, we use the property (see Jameson, 2016):

$$\Gamma(k+1, x) = k\Gamma(k, x) + x^k e^{-x} = k \left((k-1)\Gamma((k-1), x) + x^{k-1} e^{-x} \right) + x^k e^{-x}$$

Therefore:

$$\Gamma(k+1, xF(p)) > k(xF(p))^{k-1} e^{-xF(p)} + (xF(p))^k e^{-xF(p)} = e^{-xF(p)} \left(k(xF(p))^{k-1} + (xF(p))^k \right),$$

which can be used in (17) to get

$$xF(p)e^{xF(p)} \underbrace{\Gamma(k+1, xF(p))}_{> e^{-xF(p)} \left(k(xF(p))^{k-1} + (xF(p))^k \right)} \left(xf(p)^2 + f'(p) \right) - x(xF(p))^k \left(f(p)^2 (xF(p) + k) + F(p)f'(p) \right) >$$

$$xF(p) \left(k(xF(p))^{k-1} + (xF(p))^k \right) \left(xf(p)^2 + f'(p) \right) - x(xF(p))^k \left(f(p)^2 (xF(p) + k) + F(p)f'(p) \right) =$$

$$xF(p)(xF(p))^{k-1} \left[(k + xF(p)) \left(xf(p)^2 + f'(p) \right) - x \left(f(p)^2 (xF(p) + k) + F(p)f'(p) \right) \right] =$$

$$k(xF(p))^k \left[f'(p) \right],$$

which is positive for all density functions with $f'(p) > 0$.

Finally, we show how to get the expression in (5). For this, note that the equilibrium price satisfies $p = -\frac{m(p)}{m'(p)}$ where

$$m(p) = 1 - F(p)^k \left(1 - \frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right) - \frac{\Gamma(k+1,xF(p))}{\Gamma(k+1)} e^{-x(1-F(p))}. \tag{18}$$

Taking the derivative of $m(p)$ with respect to p gives:

$$m'(p) = -kF(p)^{k-1} f(p) \left(1 - \frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right) - xf(p)\frac{\Gamma(k,xF(p))}{\Gamma(k)} e^{-x(1-F(p))}. \tag{19}$$

Plugging the expressions for $m(p)$ and $m'(p)$ in $p = -\frac{m(p)}{m'(p)}$ we obtain (5). ■

Proof of Proposition 4.

⊙ Claim 1. Differentiation yields

$$\frac{\partial p}{\partial x} = \frac{-m'(p)\frac{\partial m(p)}{\partial x} + m(p)\frac{\partial m'(p)}{\partial x}}{(m'(p))^2}.$$

Observe that

$$\begin{aligned}
\frac{\partial m(p)}{\partial x} &= -F(p)^k \frac{x^k e^{-x}}{\Gamma(k+1)} + \frac{(xF(p))^k e^{-xF(p)}}{\Gamma(k+1)} F(p) e^{-x(1-F(p))} + (1-F(p))\frac{\Gamma(k+1,xF(p))}{\Gamma(k+1)} e^{-x(1-F(p))} \\
&= (1-F(p)) \left[\frac{\Gamma(k+1,xF(p))}{\Gamma(k+1)} e^{-x(1-F(p))} - \frac{(xF(p))^k e^{-x}}{\Gamma(k+1)}\right] \\
&= (1-F(p)) \frac{\Gamma(k,xF(p))}{\Gamma(k)} e^{-x(1-F(p))} > 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial m'(p)}{\partial x} &= -kF(p)^{k-1} f(p) \frac{x^k e^{-x}}{\Gamma(k+1)} + xf(p) e^{-x(1-F(p))} \frac{(xF(p))^{k-1} e^{-xF(p)}}{\Gamma(k)} \\
&\quad - (1 - x(1-F(p)))\, e^{-x(1-F(p))} f(p) \frac{\Gamma(k,xF(p))}{\Gamma(k)} \\
&= -(1 - x(1-F(p)))\, e^{-x(1-F(p))} f(p) \frac{\Gamma(k,xF(p))}{\Gamma(k)}.
\end{aligned}$$

To derive these expressions, we use $\frac{\partial}{\partial x}\frac{\Gamma(k+1,x)}{\Gamma(k+1)} = \frac{\partial}{\partial x}\frac{\int_x^\infty t^k e^{-x} dt}{\Gamma(k+1)} = -\frac{x^k e^{-x}}{\Gamma(k+1)}$ and $\frac{\Gamma(k+1,x)}{\Gamma(k+1)} = \frac{\Gamma(k,x)}{\Gamma(k)} + \frac{x^k e^{-x}}{\Gamma(k+1)}$.

If $1 < x(1-F(p))$ then $\frac{\partial m'(p)}{\partial x} > 0$ and so $\frac{\partial p}{\partial x} > 0$ (because $m'(p) < 0$ and $m(p) > 0$). Suppose

$1 \geq x(1-F(p))$ and define

$$\Omega(x) \equiv \left[kF(p)^{k-1}\left(1-\frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right)+x\frac{\Gamma(k,xF(p))}{\Gamma(k)}e^{-x(1-F(p))}\right](1-F(p)) - \left[1-F(p)^k\left(1-\frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right)-\frac{\Gamma(k+1,xF(p))}{\Gamma(k+1)}e^{-x(1-F(p))}\right](x(1-F(p))-1).$$

Note that $\Omega(x) = \frac{\left[-m'(p)\frac{\partial m(p)}{\partial x}+m(p)\frac{\partial m'(p)}{\partial x}\right]}{f(p)e^{-x(1-F(p))}\frac{\Gamma(k,xF(p))}{\Gamma(k)}}$ and so $\frac{\partial p}{\partial x} > 0$ if and only if $\Omega(x) > 0$. Observe that: $\Omega(x) \to 0$ as $x \to 0$;

$$\frac{\partial \Omega(x)}{\partial x} = \left[1-F(p)^k\left(1-\frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right)-\frac{\Gamma(k+1,xF(p))}{\Gamma(k+1)}e^{-x(1-F(p))}\right](1-F(p)) > 0.$$

Hence, $\Omega(x) > 0$ and so $\frac{\partial p}{\partial x} > 0$ for all $1 \geq x(1-F(p))$.

Finally, the limits are immediate.

⊙ Claim 2. To show the comparative statics with respect to k, observe in (5) that since $m(p;k)$ is increasing in k, it suffices to show that $m'(p;k)$ (< 0) is increasing in k.

$$m'(p;k+1)-m'(p;k) = kF^{k-1}(p)f(p)\left[\left(1-\frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right)-\frac{k+1}{k}F(p)\left(1-\frac{\Gamma(k+2,x)}{\Gamma(k+2)}\right)\right] + xe^{-x(1-F(p))}f(p)\left[\frac{\Gamma(k,xF(p))}{\Gamma(k)}-\frac{\Gamma(k+1,xF(p))}{\Gamma(k+1)}\right].$$

Suppose that $F(p) < \frac{k}{k+1}$, which we will verify below. Then,

$$\begin{aligned} m'(p;k+1)-m'(p;k) &> kF^{k-1}(p)f(p)\left[\frac{\Gamma(k+2,x)}{\Gamma(k+2)}-\frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right] \\ &\quad + xe^{-x(1-F(p))}f(p)\left[\frac{\Gamma(k,xF(p))}{\Gamma(k)}-\frac{\Gamma(k+1,xF(p))}{\Gamma(k+1)}\right] \\ &= \frac{x^{k+1}F^{k-1}e^{-x}}{\Gamma(k+1)}f(p)\left(\frac{k}{k+1}-F(p)\right) > 0, \end{aligned}$$

where the first and the last inequalities hold because $F(p) < \frac{k}{k+1}$, and the second equality is obtained by using $\frac{\Gamma(k+2,x)}{\Gamma(k+2)} - \frac{\Gamma(k+1,x)}{\Gamma(k+1)} = \frac{x^{k+1}e^{-x}}{\Gamma(k+2)}$ and $\frac{\Gamma(k,xF(p))}{\Gamma(k)} - \frac{\Gamma(k+1,xF(p))}{\Gamma(k+1)} = -\frac{(xF(p))^k e^{-xF(p)}}{\Gamma(k+1)}$. Hence, if the inequality $F(p) < \frac{k}{k+1}$ holds true, then $m'(p;k)$ is increasing in k.

We now prove $F(p) < \frac{k}{k+1}$. Note that it suffices to prove $p < \frac{k}{k+1}$ because under our assumption of the monotone increasing density $f'(p) \geq 0$, we must have $F(p) < p$ for any $p \in (0,1)$. We therefore prove

$$p = \frac{1-p^k\left(1-\frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right)-\frac{\Gamma(k+1,xp)}{\Gamma(k+1)}e^{-x(1-p)}}{kp^{k-1}\left(1-\frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right)+x\frac{\Gamma(k,xp)}{\Gamma(k)}e^{-x(1-p)}} < \frac{k}{k+1}.$$

Define

$$\Phi(p) \equiv -(k+1)\left[1 - p^k\left(1 - \frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right) - \frac{\Gamma(k+1,xp)}{\Gamma(k+1)}e^{-x(1-p)}\right]$$
$$+k\left[kp^{k-1}\left(1 - \frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right) + x\frac{\Gamma(k,xp)}{\Gamma(k)}e^{-x(1-p)}\right].$$

Note that $p < \frac{k}{k+1} \Longleftrightarrow \Phi(p) > 0$. Observe that: $\Phi(0) = -(k+1)(1-e^{-x}) + kxe^{-x} < -k(1-e^{-x} - xe^{-x}) < 0$; $\Phi(1) = k\left[k\left(1 - \frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right) + x\frac{\Gamma(k,x)}{\Gamma(k)}\right] > 0$;

$$\Phi'(p) = (k+1)kp^{k-1}\left(1 - \frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right) + (k+1)xe^{-x(1-p)}\frac{\Gamma(k,xp)}{\Gamma(k)}$$
$$+k^2(k-1)p^{k-2}\left(1 - \frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right) + kx^2\frac{\Gamma(k-1,xp)}{\Gamma(k-1)}e^{-x(1-p)} > 0.$$

To identify the sign of $\Phi\left(\frac{k}{k+1}\right)$, we define

$$\phi(x) \equiv \Phi\left(\frac{k}{k+1}\right) = -(k+1)\left[1 - \left(\frac{k}{k+1}\right)^k\left(1 - \frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right) - \frac{\Gamma(k+1,x\frac{k}{k+1})}{\Gamma(k+1)}e^{-x(1-\frac{k}{k+1})}\right]$$
$$+k\left[k\left(\frac{k}{k+1}\right)^{k-1}\left(1 - \frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right) + x\frac{\Gamma(k,x\frac{k}{k+1})}{\Gamma(k)}e^{-x(1-\frac{k}{k+1})}\right].$$

It satisfies: $\phi(0) = 0$; $\phi(\infty) = -(k+1)\left[1 - \left(\frac{k}{k+1}\right)^k\right] + k^2\left(\frac{k}{k+1}\right)^{k-1} = (k+1)\left[(k+1)\left(\frac{k}{k+1}\right)^k - 1\right] > 0$; $\phi'(x) = \frac{\Gamma(k,x\frac{k}{k+1})}{\Gamma(k)}e^{-x(1-\frac{k}{k+1})}\left[k\left(1 - \frac{x}{k+1}\right) - 1\right]$, which implies $\phi(x)$ is increasing in $x < k^2 - 1$ and decreasing in $x > k^2 - 1$, and so $\phi(x) > 0$ for all $x \in (0, \infty)$. Therefore, $\Phi\left(\frac{k}{k+1}\right) > 0$.

We are now ready to prove $p < \frac{k}{k+1}$. The above analysis shows $\Phi(p) < 0$ when p is low and $\Phi(p) > 0$ when p is high. Suppose $p \geq \frac{k}{k+1}$. Then, by definition, we must have $\Phi(p) \leq 0$ for all $p \geq \frac{k}{k+1}$. However, this contradicts to $\Phi\left(\frac{k}{k+1}\right) > 0$ and $\Phi'(p) > 0$, since they imply that $\Phi(p) > 0$ for all $p \geq \frac{k}{k+1}$. Hence, we must have $p < \frac{k}{k+1}$.

To obtain the case $k = 1$, noting $\frac{\Gamma(k+1,x)}{\Gamma(k+1)} = \sum_{h=0}^{k}\frac{x^h e^{-x}}{h!} = (1+x)e^{-x}$ when $k = 1$, we apply $m(p) = (1 - F(p))(1 - e^{-x})$ when $k = 1$ and $m'(p) = -f(p)(1 - e^{-x})$ when $k = 1$ to p in (5).

To obtain the limit as $k \to \infty$, noting $\frac{\Gamma(k+1,x)}{\Gamma(k+1)} \to 1$ as $k \to \infty$, we apply $m(p) \to 1 - e^{-x(1-F(p))}$ as $k \to \infty$ and $m'(p) \to -xf(p)e^{-x(1-F(p))}$ as $k \to \infty$ to p in (5). The limit as $x \to 0$ is obtained by applying l'Hopital's rule once. The limit $x \to \infty$ is immediate. ■

Proof of relationship in equation (10).

Computing the derivatives involved in this relationship gives:

$$\frac{\partial \eta}{\partial x} = -\frac{m(x,p;k)}{x^2(1-F(p))} \\ + \frac{1}{x(1-F(p))}\left[\frac{F(p)^k}{\Gamma(k+1)}\frac{\partial \Gamma(k+1,x)}{\partial x} - \frac{e^{-x(1-F(p))}}{\Gamma(k+1)}\left(\frac{\partial \Gamma(k+1,xF(p)}{\partial x} - (1-F(p))\Gamma(k+1,xF(p)\right)\right],$$

and

$$\frac{\partial \eta}{\partial p} = \frac{f(p)m(x,p;k)}{x(1-F(p))^2} + \frac{1}{x(1-F(p))}\left[\frac{\Gamma(k+1,x)-\Gamma(k+1)}{\Gamma(k+1)}kF(p)^{k-1}f(p)\right] \\ + \frac{1}{x(1-F(p))}\left[-\frac{e^{-x(1-F(p))}}{\Gamma(k+1)}\left(\frac{\partial \Gamma(k+1,xF(p)}{\partial p} + xf(p)\Gamma(k+1,xF(p)\right)\right].$$

Notice that:

$$\frac{\partial \Gamma(k+1,x)}{\partial x} = -x^k e^{-x}$$
$$\frac{\partial \Gamma(k+1,xF(p)}{\partial x} = -x^k F(p)^{k+1} e^{-xF(p)}$$
$$\frac{\partial \Gamma(k+1,xF(p)}{\partial p} = -x^{k+1} F(p)^k f(p) e^{-xF(p)}.$$

Using these, we can rewrite $\frac{\partial \eta}{\partial x}$ and $\frac{\partial \eta}{\partial p}$ as follows:

$$\frac{\partial \eta}{\partial x} = -\frac{m(x,p;k)}{x^2(1-F(p))} \\ - \frac{1}{x(1-F(p))}\left[\frac{F(p)^k}{\Gamma(k+1)}x^k e^{-x} - \frac{e^{-x(1-F(p))}}{\Gamma(k+1)}\left(x^k F(p)^{k+1} e^{-xF(p)} + (1-F(p))\Gamma(k+1,xF(p)\right)\right],$$

and

$$\frac{\partial \eta}{\partial p} = \frac{f(p)m(x,p;k)}{x(1-F(p))^2} + \frac{f(p)}{x(1-F(p))}\left[\frac{\Gamma(k+1,x)-\Gamma(k+1)}{\Gamma(k+1)}kF(p)^{k-1}\right] \\ + \frac{f(p)}{x(1-F(p))}\left[\frac{e^{-x(1-F(p))}}{\Gamma(k+1)}\left(x^{k+1} F(p)^k e^{-xF(p)} - x\Gamma(k+1,xF(p)\right)\right].$$

It is convenient to multiply and divide the squared bracket of $\frac{\partial \eta}{\partial x}$ by x, and that of $\frac{\partial \eta}{\partial p}$ by $1-F(p)$.

This gives:

$$
\begin{aligned}
\frac{\partial \eta}{\partial x} &= -\frac{m(x,p;k)}{x^2(1-F(p))} \\
&\quad - \frac{1}{x^2(1-F(p))}\left[\frac{F(p)^k x^{k+1} e^{-x}}{\Gamma(k+1)} - \frac{e^{-x(1-F(p))}}{\Gamma(k+1)}\left(x^{k+1}F(p)^{k+1}e^{-xF(p)} + x(1-F(p))\Gamma(k+1,xF(p)\right)\right],
\end{aligned}
\tag{20}
$$

and

$$
\begin{aligned}
\frac{\partial \eta}{\partial p} &= \frac{f(p)m(x,p;k)}{x(1-F(p))^2} + \frac{f(p)}{x(1-F(p))^2}\left[\frac{\Gamma(k+1,x)-\Gamma(k+1)}{\Gamma(k+1)}kF(p)^{k-1}(1-F(p)\right] \\
&\quad + \frac{f(p)}{x(1-F(p))^2}\frac{e^{-x(1-F(p))}}{\Gamma(k+1)}\left[x^{k+1}F(p)^k e^{-xF(p)}(1-F(p) - x(1-F(p)\Gamma(k+1,xF(p)\right] \\
&= \frac{f(p)m(x,p;k)}{x(1-F(p))^2} + \frac{f(p)}{x(1-F(p))^2}\left[\frac{\Gamma(k+1,x)-\Gamma(k+1)}{\Gamma(k+1)}kF(p)^{k-1}(1-F(p)\right] \\
&\quad + \frac{f(p)}{x(1-F(p))^2}\left[\frac{F(p)^k x^{k+1} e^{-x}}{\Gamma(k+1)} - \frac{e^{-x(1-F(p))}}{\Gamma(k+1)}\left(x^{k+1}F(p)^{k+1}e^{-xF(p)} + x(1-F(p)\Gamma(k+1,xF(p)\right)\right]
\end{aligned}
\tag{21}
$$

where, to establish the second equality, we have rewritten the term $x^{k+1}F(p)^k e^{-xF(p)}(1-F(p)$ as a sum.

Finally, close inspection of (20) and (21) reveals that:

$$
\frac{1-F(p)}{f(p)}\frac{\partial \eta}{\partial p} = -x\frac{\partial \eta}{\partial x} + \frac{1}{x(1-F(p))}\left[\frac{\Gamma(k+1,x)-\Gamma(k+1)}{\Gamma(k+1)}kF(p)^{k-1}(1-F(p)\right],
$$

which is relationship (10). ■

Proof of Proposition 5.

Using (10), the FOC for profits maximization in expression (9) can be rewritten as follows

$$
\begin{aligned}
&\left(-\frac{(1-F(p))}{f(p)}\frac{\partial \eta}{\partial p} + d(p)\right)\left[(1-F(p)-pf(p))\,I(p) + p(1-F(p))^2\right] \\
&\qquad\qquad + p\eta(1-F(p))^2 - \frac{\partial \eta}{\partial p}p(1-F(p))I(p) = 0.
\end{aligned}
$$

Rearranging terms gives:

$$\left(\frac{(1-F(p))}{f(p)}\frac{\partial\eta}{\partial p}\right)\left[\left(1-F(p)-pf(p)\right)I(p)+p(1-F(p))^2\right]-p\eta(1-F(p))^2+\frac{\partial\eta}{\partial p}p(1-F(p))I(p)$$
$$=d(p)\left[\left(1-F(p)-pf(p)\right)I(p)+p(1-F(p))^2\right],$$

which can be simplified to:

$$\left(\frac{\partial\eta}{\partial p}I(p)\right)\frac{(1-F(p))^2}{f(p)}+\left(\frac{\partial\eta}{\partial p}\right)\frac{(1-F(p))}{f(p)}p(1-F(p))^2-p\eta(1-F(p))^2$$
$$=d(p)\left[\left(1-F(p)-pf(p)\right)I(p)+p(1-F(p))^2\right].$$

This can be rewritten as in the Proposition:

$$\frac{(1-F(p))^2}{f(p)}\left[\frac{\partial\eta}{\partial p}I(p)+\frac{\partial\eta}{\partial p}p(1-F(p))-p\eta f(p)\right]$$
$$=d(p)\left[\left(1-F(p)-pf(p)\right)I(p)+p(1-F(p))^2\right],$$

which completes the proof of the first statement of the proposition.

We now prove the second statement of the proposition. We first show that (11) has at least one solution. For this, it is convenient to rewrite (11) as follows:

$$\frac{\partial\eta}{\partial p}I(p)+\frac{\partial\eta}{\partial p}p(1-F(p))-p\eta f(p)$$
$$=d(p)\frac{f(p)}{(1-F(p))^2}\left[\left(1-F(p)-pf(p)\right)I(p)+p(1-F(p))^2\right]. \tag{22}$$

Consider the LHS of (22). Observe that it is strictly positive at $p=0$. This follows easily from using (10) and noting that $d(0)=0$ and

$$\frac{\partial\eta}{\partial x}=\frac{1}{x(1-F(p))}\left(\frac{\partial m}{\partial x}-\frac{m}{x}\right),$$

where

$$\frac{\partial m}{\partial x}=\frac{e^{-x}(1-F(p))e^{xF(p)}\Gamma(k,xF(p))}{\Gamma(k)}.$$

To evaluate the LHS of (22) at $p=1$, note that $F(1)=1$, $I(1)=0$ and $\eta(1)=1$. Observe also that, again using (10) and the fact that, by the L'Hopital rule,

$$\lim_{p\to 1}\frac{I(p)}{1-F(p)}=\lim_{p\to 1}\frac{-(1-F(p))}{-f(x)}=0,$$

we have:

$$\lim_{p\to 1}\frac{\partial\eta}{\partial p}I(p)=\lim_{p\to 1}\frac{f(p)I(p)}{1-F(p)}\left(\frac{-\frac{\partial m}{\partial x}}{1-F(p)}+\eta+d(p)\right)=0$$

$$\lim_{p\to 1}\frac{\partial\eta}{\partial p}p(1-F(p))=f(1)\left(1-\frac{\Gamma(k,x)}{\Gamma(k)}-d(1)\right)$$

Hence, altogether the LHS of (22) takes on value $-f(1)\left(\frac{\Gamma(k,x)}{\Gamma(k)}-d(1)\right)<0$ at $p=1$.

Consider now the RHS of (22). Note that it is equal to 0 at $p=0$. Therefore, for the existence of a candidate equilibrium it suffices to show that at $p=1$ the LHS of (22) is lower than the RHS of (22). Taking the limit of the RHS of (22) when $p\to 1$ gives

$$d(1)f(1)\left(-f(1)\lim_{p\to 1}\frac{I(p)}{(1-F(p))^2}+1\right)=\frac{1}{2}d(1)f(1) \tag{23}$$

because by the L'Hopital rule,

$$\lim_{p\to 1}\frac{I(p)}{(1-F(p))^2}=\frac{1}{2f(1)}.$$

Therefore, the existence of a candidate equilibrium is guaranteed if

$$-f(1)\left(\frac{\Gamma(k,x)}{\Gamma(k)}-d(1)\right)<\frac{1}{2}d(1)f(1),$$

or

$$-f(1)\left(\frac{\Gamma(k,x)}{\Gamma(k)}-\frac{d(1)}{2}\right)<0,$$

which is always true because $d(1)<0$.

We now show that the equilibrium exists and is unique when $k=1$ and $f'>0$. Recall that when $k=1$ we have a demand function $m(x(p),p)=(1-F(p))\left(1-e^{-x(p)}\right)$. Differentiation with respect to p yields

$$\frac{dm(x(p),p)}{dp} = -f(p)\left(1-e^{-x(p)}\right)+(1-F(p))e^{-x(p)}\frac{dx(p)}{dp}.$$

Taking the second derivative with respect p gives:

$$\frac{d^2m}{dp^2}=-f'(1-e^{-x})+\frac{d^2x}{dp^2}(1-F)e^{-x}-2\frac{dx}{dp}fe^{-x}-\left(\frac{dx}{dp}\right)^2(1-F)e^{-x}, \tag{24}$$

where we have omitted the argument "(p)" to shorten the expression. In this expression we need to plug in dx^2/dp^2. Differentiating (7) yields

$$\frac{d^2x}{dp^2} = -\frac{2\frac{dx}{dp}\left(\frac{\partial^2\eta}{\partial x\partial p}I - \frac{\partial\eta}{\partial x}(1-F)\right) + \left(\frac{dx}{dp}\right)^2 \frac{\partial^2\eta}{\partial x^2}I + \frac{\partial^2\eta}{\partial p^2}I - 2\frac{\partial\eta}{\partial p}(1-F) + \eta f}{\frac{\partial\eta}{\partial x}I}. \tag{25}$$

Noting $\frac{\partial\eta}{\partial p} = 0$ when $k = 1$ and plugging this derivative into (27), we get

$$\frac{d^2m}{dp^2} = -f'(1-e^{-x}) + \frac{dx}{dp}e^{-x}\left[-3f + \frac{(1-F)^2}{I}\left(2 - \frac{\eta}{\frac{\partial\eta}{\partial x}}\right)\right] + 2\left(\frac{dx}{dp}\right)^2(1-F)e^{-x}\frac{\frac{\partial^2\eta}{\partial x^2}}{\frac{\partial\eta}{\partial x}}. \tag{26}$$

In expression (28), the first term is negative by the assumption $f' > 0$, and the third term is negative since $\frac{\partial\eta}{\partial x} < 0$ and $\frac{\partial^2\eta}{\partial x^2} > 0$. To determine the sign of the second term, observe that

$$-3f + \frac{(1-F)^2}{I}\left(2 - \frac{\eta}{\frac{\partial\eta}{\partial x}}\right) > 3\left(-f + \frac{(1-F)^2}{I}\right)$$

because

$$-\frac{\eta}{\frac{\partial\eta}{\partial x}} = \frac{\frac{1-e^{-x}}{x}}{\frac{1-e^{-x}-xe^{-x}}{x^2}} > 1.$$

We now identify the sign of the term

$$Z(p) \equiv -f + \frac{(1-F)^2}{I},$$

where $I = \int_p^1(\varepsilon - p)f(\varepsilon)d\varepsilon$. Observe that: $Z(1) = -f(1) + \left.\frac{(1-F(p))^2}{I(p)}\right|_{p=1} = -f(1) + \frac{-2(1-F(p))f(p)|_{p=1}}{-(1-F(p))|_{p=1}} = f(1) > 0$. Moreover,

$$Z'(p) = -f' - \frac{1-F}{I}\left[f - Z(p)\right],$$

which implies that if $Z(p) \leq 0$ then $Z'(p) < 0$.

Suppose now that there exists a $p \in [0,1)$ such that $Z(p) < 0$. Then, since $Z(1) > 0$, there must be a non-empty interval of $p \in (0,1)$ such that $Z(p) < 0$ and $Z'(p) > 0$. However, this contradicts the above statement that $Z'(p) < 0$ if $Z(p) \leq 0$. Therefore, we must have $Z(p) \geq 0$ for all $p \in [0,1]$. Hence, the second term in (28) is negative (since $\frac{dx}{dp} < 0$).

Altogether, we conclude that when $k = 1$ and $f' > 0$, $\frac{d^2m(x(p),p)}{dp^2} < 0$ for all $p \in [0,1]$.

Finally, we show that the equilibrium exists and is unique when $k \to \infty$ for any density of match values. First, recall that in the limit as $k \to \infty$, we have a demand function

$$m(x(p), p) \to 1 - e^{-x(p)(1-F(p))}.$$

Differentiation with respect to p yields

$$\frac{dm(x(p),p)}{dp} = -x(p)f(p)e^{-x(p)(1-F(p))} + (1-F(p))e^{-x(p)(1-F(p))}\frac{dx(p)}{dp}$$

and

$$\frac{d^2m}{dp^2} = e^{-x(1-F)}\left[-xf' - (xf)^2 + \frac{d^2x}{dp^2}(1-F(p)) + \frac{dx}{dp}2f\left[-1+x(1-F)\right] - \left(\frac{dx}{dp}\right)^2(1-F)^2\right]. \quad (27)$$

Plugging (25) into the previous expression gives:

$$\begin{aligned}\frac{d^2m}{dp^2}e^{x(1-F)} &= -x(f'+xf^2) - (1-F(p))\frac{\frac{\partial^2\eta}{\partial p^2}}{\frac{\partial\eta}{\partial x}} + \frac{dx}{dp}2\left[f\left(-1+x(1-F)\right) - \frac{\frac{\partial^2\eta}{\partial p\partial x}}{\frac{\partial\eta}{\partial x}}(1-F) + \frac{(1-F)^2}{I}\right] \\ &\quad - \left(\frac{dx}{dp}\right)^2(1-F)\left[1-F+\frac{\frac{\partial^2\eta}{\partial x^2}}{\frac{\partial\eta}{\partial x}}\right] - \frac{1-F}{\frac{\partial\eta}{\partial x}I}\left[-2\frac{\partial\eta}{\partial p}(1-F)+\eta f\right]. \quad (28)\end{aligned}$$

We now show that the sign of (28) is negative.

Denote $\Lambda \equiv 1 - e^{-\tilde{x}} - \tilde{x}e^{-\tilde{x}}$ and $\tilde{x} \equiv x(1-F)$. In the limit as $k \to \infty$, we can compute the terms as follows: $\eta \to \frac{1-e^{-\tilde{x}}}{\tilde{x}}$; $\frac{\partial\eta}{\partial p} \to \frac{f\Lambda}{\tilde{x}(1-F)}$; $\frac{\partial\eta}{\partial x} \to -\frac{\Lambda}{\tilde{x}x}$; $\frac{dx}{dp} \to -\frac{1-e^{-\tilde{x}}}{\Lambda I}\tilde{x} + \frac{x^2 f}{\tilde{x}}$; $\frac{\partial^2\eta}{\partial x\partial p} \to f\left(-\frac{\Lambda}{\tilde{x}^2}+e^{-\tilde{x}}\right)$; $\frac{\partial^2\eta}{\partial p^2} \to \frac{\Lambda}{\tilde{x}(1-F)^2}(2f^2+(1-F)f') - \frac{e^{-\tilde{x}}}{\tilde{x}}(xf)^2$; $\frac{\partial^2\eta}{\partial x^2} \to \frac{2\Lambda}{x^2\tilde{x}} - \frac{\tilde{x}e^{-\tilde{x}}}{x^2}$. Using these expressions and collecting terms in (28), we get in the limit as $k \to \infty$, $\frac{d^2m}{dp^2}e^{\tilde{x}} \to$

$$\begin{aligned}&-x(f'+xf^2) + \frac{\tilde{x}^2}{\Lambda}\left[\frac{\Lambda}{\tilde{x}(1-F)^2}(2f^2+(1-F)f') - \frac{e^{-\tilde{x}}}{\tilde{x}}(xf)^2\right] + \frac{dx}{dp}2f\left(\tilde{x}+\frac{\tilde{x}^2e^{-\tilde{x}}}{\Lambda}\right) \\ &-\left(\frac{dx}{dp}\right)^2(1-F)\left[1-F-\frac{2}{x}+\frac{\tilde{x}^2e^{-\tilde{x}}}{x\Lambda}\right] + \frac{1-F}{I}\left[2(1-F)\frac{dx}{dp}+\frac{x}{\Lambda}f\left(-2\Lambda+1-e^{-\tilde{x}}\right)\right]. (29)\end{aligned}$$

In what follows, we identify the sign of these terms step by step.

First, the sum of the first two terms of (29) can be simplified to

$$-\frac{(xf)^2}{\tilde{x}\Lambda}\left[\tilde{x}(1+e^{-\tilde{x}}) - 2(1-e^{-\tilde{x}})\right] < 0.$$

Second, we show $\frac{dx}{dp} < 0$ as $k \to \infty$. Since $\frac{dx}{dp} \to -\frac{1-e^{-\tilde{x}}}{\Lambda I}\tilde{x} + \frac{x^2f}{\tilde{x}} < 0 \iff \frac{1-e^{-\tilde{x}}}{\Lambda}\frac{\tilde{x}^2}{I} > x^2 f$, it is sufficient to show recall the observation above that $Z(p) \equiv \frac{(1-F)^2}{I} - f \geq 0$ (because $\frac{1-e^{-\tilde{x}}}{\Lambda} > 1$).

Third, since $\frac{dx}{dp} \to -\frac{1-e^{-\tilde{x}}}{\Lambda I}\tilde{x} + \frac{x^2 f}{\tilde{x}} < 0$, the sum of the third term and the forth term of (29) can be computed as

$$\frac{dx}{dp}\left[\tilde{x}f + 2f + \frac{\tilde{x}^2 e^{-\tilde{x}}}{\Lambda}f + \frac{1-e^{-\tilde{x}}}{\Lambda^2 I}(1-F)^2\left[\tilde{x}(1+e^{-\tilde{x}}) - 2(1-e^{-\tilde{x}})\right]\right] < 0.$$

Finally, the last term of (29) can be simplified to

$$\frac{1-F}{I}\frac{1-e^{-\tilde{x}}}{\Lambda}x\left[-2\frac{(1-F)^2}{I} + f\right] < 0,$$

since $-2\frac{(1-F)^2}{I} + f = -\frac{(1-F)^2}{I} - Z(p) < 0$.

Altogether, we conclude that $\frac{d^2 m(x(p),p)}{dp^2} < 0$ as $k \to \infty$ for all $p \in [0,1]$. The proof is now complete. ■

Proof of Proposition 7.

Welfare is given by

$$W = B\eta(x,p;k)\int_p^1 \varepsilon f(\varepsilon)d\varepsilon.$$

Taking the FOC gives

$$\frac{1}{B}\frac{\partial W}{\partial p} = \frac{\partial \eta}{\partial p}\int_p^1 \varepsilon f(\varepsilon)f\varepsilon - \eta p f(p) = 0.$$

Using the expression for $I(p) = \int_p^1 (\varepsilon - p)f(\varepsilon)f\varepsilon$, this can be rewritten as

$$\frac{1}{B}\frac{\partial W}{\partial p} = \frac{\partial \eta}{\partial p}I(p) + \frac{\partial \eta}{\partial p}p(1-F(p)) - \eta p f(p) = 0,$$

which is the expression given in the proposition.

When $p = 0$, this expression is positive. When $p = 1$, it is negative. This ensures that p^o exists. ■

Proof of Proposition 8.

We first prove the second claim. Recall that the SNE price p is given by the solution to:

$$\frac{(1-F(p))^2}{f(p)}\left[\frac{\partial \eta}{\partial p}I(p) + \frac{\partial \eta}{\partial p}p(1-F(p)) - p\eta f(p)\right] - d(p)\left[(1-F(p)-pf(p))\,I(p) + p(1-F(p))^2\right] = 0. \quad (30)$$

while the socially optimal price p^o satisfies the FOC:

$$\frac{\partial \eta}{\partial p}I(p^o) + \frac{\partial \eta}{\partial p}p^o(1-F(p^o)) - p^o\eta f(p^o) = 0. \quad (31)$$

Comparing (30) and (31), we immediately see that the equilibrium and the optimum coincide when $d(p) = 0$. This occurs in the limit when $k \to \infty$.

To prove the claim for finite k, we show that

$$\left.\frac{\partial \Pi_i}{\partial p_i}\right|_{p_i = p^o} > 0,$$

which implies that the payoff of a firm increases at $p_i = p^o$ so that $p > p^o$. For this, define

$$\Psi(p) \equiv (1 - F(p) - pf(p))\, I(p) + p(1 - F(p))^2$$

for $p \in [0, 1]$. In what follows, we show that $\Psi(p) > 0$ for all $p \in (0, 1)$, which implies that, since $d(p) < 0$ for finite k, the RHS of (11) is negative.

Observe that: $\Psi(0) = I(0) > 0$; $\Psi(1) = 0$;

$$\Psi'(p) = -\left(2f(p) + pf'(p)\right) I(p) - p(1 - F(p))f(p). \tag{32}$$

and notice that

$$\lim_{p \to 1} \frac{\Psi'(p)}{1 - F(p)} = -f(p) < 0.$$

To establish a contradiction, suppose there exists a region of prices $p \in (0, 1)$ for which $\Psi(p) \leq 0$. Then, because $\Psi(p)$ is decreasing at $p = 1$, there must exist some $\tilde{p} \in (0, 1)$ such that $\Psi'(\tilde{p}) = 0$ and $\Psi(\tilde{p}) > 0$. Using (32), the condition $\Psi'(\tilde{p}) = 0$ gives:

$$\tilde{p}(1 - F(\tilde{p}) = -\frac{(2f(\tilde{p}) + \tilde{p}f'(\tilde{p}))\, I(\tilde{p})}{f(\tilde{p})}$$

Using this relation in the condition $\Psi(\tilde{p}) > 0$ gives

$$\begin{aligned} \Psi(\tilde{p}) &= (1 - F(\tilde{p}) - \tilde{p}f(\tilde{p}))\, I(\tilde{p}) + \tilde{p}(1 - F(\tilde{p}))^2 \\ &= -\left(1 - F(\tilde{p}) + \tilde{p}\frac{f(\tilde{p})^2 + f'(\tilde{p})(1 - F(\tilde{p}))}{f(\tilde{p})}\right) I(\tilde{p}) > 0. \end{aligned}$$

But this is impossible because, by the log-concavity of $f(p)$, the hazard rate $\frac{f(p)}{1-F(p)}$ is increasing in p and the expression in brackets is positive. We therefore reach the desired contradiction and so we must have $\Psi(p) > 0$ for all $p \in (0, 1)$.

We now prove the first claim. For this is enough to compare the FOCs (4):

$$[1 - F(p) - pf(p)]\, x\eta + p(1 - F(p))x\frac{\partial \eta}{\partial p} = 0$$

and (11):

$$[1 - F(p) - pf(p)]\, x\eta + p(1 - F(p))x\frac{\partial \eta}{\partial p} + p(1 - F(p))\left(\eta + x\frac{\partial \eta}{\partial x}\right)\frac{\partial x}{\partial p} = 0.$$

Notice that the difference between these two FOCs is the term

$$p(1 - F(p))\left(\eta + x\frac{\partial \eta}{\partial x}\right)\frac{\partial x}{\partial p}. \tag{33}$$

The sign of $\partial x/\partial p$ is negative. Moreover,

$$\eta + x\frac{\partial \eta}{\partial x} = \frac{m}{x(1 - F(p)} + x\frac{1}{1 - F(p)}\frac{\frac{\partial m}{\partial x}x - m}{x^2} = \frac{1}{1 - F(p)}\frac{\partial m}{\partial x} < 0,$$

where the sign follows from the proof of Proposition 2. Therefore, the term (33) is positive. ■

References

[1] Anderson, S.P. and Renault, R. (1999). Pricing, product diversity, and search costs: a Bertrand-Chamberlin-Diamond model. *RAND Journal of Economics*, 30, 719–735.

[2] Armstrong, M. (2017). Ordered consumer search. *Journal of European Economic Association*, 15, 989–1024.

[3] Armstrong, M., Vickers, J. and Zhou, J., 2009. Prominence and consumer search. *RAND Journal of Economics*, 40 (2), 209–233.

[4] Armstrong, M. and Zhou, J. (2011). Paying for prominence. *Economic Journal*, 121 (556), F368–F395.

[5] Atayev, A. (2022). Uncertain product availability in search markets. *Journal of Economic Theory*, 204, 105524.

[6] Bar-Isaac, H., Caruana, G. and Cuñat, V. (2012). Search, design and market structure. *American Economic Review*, 102(2), 1140–1160.

[7] Butters, G.R. (1977). Equilibrium distributions of sales and advertising prices. *Review of Economic Studies*, 44 (3), 465–491.

[8] Choi, M., Dai, A.Y., and Kim, K. (2018). Consumer search and price competition. *Econometrica*, 86 (4), 1257–1281.

[9] Davidson, C. and Deneckere, R. (1986). Long-run competition in capacity, short-run competition in price, and the Cournot model. *The RAND Journal of Economics*, 17(3), 404–415.

[10] De Loecker, J., Eeckhout, J. and Unger, G. (2020). The rise of market power and the macroeconomic implications. *The Quarterly Journal of Economics*, 135(2), 561-644.

[11] Diamond, P.A. (1971). A model of price adjustment. *Journal of Economic Theory*, 3, 156–168.

[12] Ding, Y. and Zhang, T. (2018), Price-directed consumer search, *International Journal of Industrial Organization*, 58, 106–135.

[13] Godenhielm, M. and Kultti, K. (2015), Directed search with endogenous capacity, *B.E. Journal of Theoretical Economics*, 15, 211–249.

[14] Gomis-Porqueras, P., Julien, B., and Wang, C. (2017). Strategic advertising and directed search. *International Economic Review*, 58(3), 783–805.

[15] Haan, M.A., Moraga-González, J.L. and Petrikaitė, V. (2018), A model of directed consumer search, *International Journal of Industrial Organization*, 61, 223–255.

[16] Haan, M. A. and Moraga-González, J.L. (2011). Advertising for attention in a consumer search model. *Economic Journal*, 121 (552), 552–579.

[17] Haan, M. A., Moraga-González, J.L. and Petrikaitė, V. (2018). A model of directed consumer search. *International Journal of Industrial Organization*, 61, 223–255.

[18] Jameson, G.J.O. (2016), The incomplete gamma function. *Math. Gazette*, 100 (548), 298–306.

[19] Janssen, M. and Rasmusen, E. (2002). Bertrand competition under uncertainty. *Journal of Industrial Economics*, 50 (1), 11-21.

[20] Johnston, M.W. and Marshall, G.W. (2013), *Sales Force Management: Leadership, Innovation, Technology*, Routledge, New York.

[21] Kreps, D.M. and Scheinkman, J.A. (1983). Quantity precommitment and Bertrand competition yield Cournot outcomes. *The Bell Journal of Economics*, 14 (2), 326–337.

[22] Lester, B. (2011). Information and prices with capacity constraints. *American Economic Review* 101(4), 1591–1600.

[23] Levitan, R. and Shubik, M. (1972). Price duopoly and capacity constraints. *International Economic Review*, 13(1), 111–122.

[24] Perloff, J.M. and Salop, S.C. (1985). Equilibrium with product differentiation. *Review of Economic Studies*, 52, 107–120.

[25] Peters, M. (2000). Limits of exact equilibria for capacity-constrained sellers with costly search. *Journal of Economic Theory* 95, 139–168.

[26] Rich, G.A. (2017). *Sales Force Management*. Chicago Business Press, Chicago.

[27] Tan, S. (2012). Directed search and firm size. *International Economic Review*, 53(1), 95–113.

[28] Teh, C., Wang, C. and Watanabe, M. (2023), Strategic limitation of market accessibility: search platform design and welfare, *Journal of Economic Theory*, forthcoming.

[29] Watanabe, M. (2010). A Model of merchants. *Journal of Economic Theory*, 145, 1865–1889.

[30] ——— (2018). Middlemen: the visible market makers, *Japanese Economic Review*, 69 (2), 156–170.

[31] ——— (2020). Middlemen: a directed search equilibrium approach. *BE Journal of Macroeconomics (Advances)*, 20-2, 1–37

[32] Wright, R., Kircher, P., Julien, B. and Guerrieri, V. (2021). Directed search: a guided tour. *Journal of Economic Literature*, 59(1), 90–148.