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Tomohiro Hirano (Royal Holloway, University of London /
Center for Macroeconomics at the London School of Economics /
The Canon Institute for Global Studies (CIGS))
Joseph E. Stiglitz (Columbia University)

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一般財団法人　キヤノングローバル戦略研究所
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The Wobbly Economy; Global Dynamics with Phase Transitions and State Transitions*

Tomohiro Hirano and Joseph E. Stiglitz

Abstract: This paper develops a model providing a markedly different picture of the dynamics of capitalism from the standard model with infinitely lived individuals with rational expectations. Using the standard life-cycle model with production, we show that under not implausible conditions, we show that starting from any initial conditions, there can be a plethora of rational expectations dynamics, including "wobbly macro-dynamics" i.e. the macroeconomy can bounce around infinitely without converging depending on people's beliefs without regular periodicity. As a result, laissez-faire market economies can be plagued by repeated periods of instabilities, inefficiencies, and unemployment.

The characteristics associated with wobbly dynamics is that the state of the economy endogenously changes from a state with a unique momentary equilibrium into a state with multiple momentary equilibria, or vice versa, which we call a phase transition. Depending on how phase transitions occur, various patterns of wobbly dynamics can occur. We identify all possible patterns of dynamics (e.g. unique and multiple, stable and unstable, steady states, with or without wobbly dynamics), providing a complete characterization of the parameter values under which each may occur. Moreover, we provide a complete analytic representation of all the possible state transitions, i.e. how a change in some key parameter changes abruptly the set of feasible global dynamics.

In some cases, if a stable "high output" (an economic boom) benefits from an above trend temporary productivity increase, there is a state transition from a stable regime to an unstable one. The economy enters into a situation where there are multiple equilibria, with the boom now being unstable, leading to the possibility of a large-scale collapse; the economy can enter a stagnation trap characterized by involuntary unemployment. In other cases, an increase in productivity shifts the economy from the economy from the stable boom to a completely wobbly economy in which the economy endogenously fluctuates in both full-employment and involuntary unemployment regions. Thus, the economy can exhibit long run hysteresis effects. There are government interventions which can stabilize the economy and increase societal welfare.

Keywords: Multiplicity of momentary equilibria, Wobbly dynamics, Phase Transitions, State transitions, JEL Classification: C61 (Dynamic Analysis), E32 (Business Fluctuations, Cycles), O11 (Macroeconomic aspects of economic development)

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* First version: February 26, 2017, This version: December 16, 2021 Tomohiro Hirano (Royal Holloway, University of London, and Center for Macroeconomics at the London School of Economics, and Canon Institute for Global Studies in Tokyo), tomohih@gmail.com; Joseph E. Stiglitz (Columbia University), jes322@gsb.columbia.edu. We are grateful to George Akerlof, Behzad Diba, Seppo Honkapohja, Katsuhito Iwai, John Moore, Jose Scheinkman, Pietro Reichlin, Robert Zymek for their discussions. We also thank seminar participants at Columbia U, LUISS, INET Oxford, American U, Georgetown U, U of Edinburgh, Helsinki School of Economics, U of Kent, Aoyama Gakuin U, and Nihon U.

1: Introduction

The 2008 crisis has cast doubt on the relevance of the standard macroeconomic models for explaining economic growth and fluctuations. In the Real Business Cycle (RBC) model (or other representative agent models) or the Solow model, there is a unique momentary equilibrium, a unique steady-state equilibrium, and a unique convergent path to that steady state. These widely used models suggest that a laissez-faire market economy is stable, and, at least in the RBC model, efficient[1], converging smoothly to a well-defined long run equilibrium.

This paper develops a model providing a markedly different picture of the dynamics of capitalism. Using the standard life-cycle model with production (Diamond (1965)), we show that under not implausible conditions, multiplicity of momentary equilibria can easily arise. We explore the implications of multiplicity of momentary equilibrium for global macro-dynamics. Multiplicity of momentary equilibria can generate what we call "wobbly macro-dynamics", i.e. macro-economy can bounce around infinitely without converging depending on people's beliefs. As a result, laissez-faire market economies can be plagued by repeated periods of instabilities, dynamic inefficiencies and unemployment. Moreover, this wobbly macro-dynamics has no regular periodicity.

The key characteristics associated with wobbly dynamics is that the state of the economy endogenously changes from a state with a unique momentary equilibrium into a state with multiple momentary equilibria, or vice versa, which we call a phase transition. Depending on how phase transitions occur, various patterns of wobbly dynamics can occur. We identify all possible patterns of wobbly dynamics, providing a complete characterization of the parameter values under which each may occur.

The intuition behind our analysis is remarkably simple: if individuals' savings decreases as the interest rate increases—they don't have to save as much to smooth consumption, to finance their retirement, or to meet other savings targets, such as buying a home or paying for their children's education—then there can be a low interest equilibrium, where investment is high and individuals save a lot so next period's capital stock is high, and so the interest rate is low; or a high interest equilibrium, where investment is low and individuals don't save much, and so the capital stock next period is low, and the interest rate (as expected) is high.

Moreover, whether there are multiplicity of momentary equilibrium depends on an *endogenous* state variable, the capital stock, which in turn affects how phase transitions occur, leading to several types of wobbly dynamics. In one case, a state with high investment and high capital stock is characterized by multiple equilibria, i.e. an economic boom is fragile and can collapse suddenly, while a state with low investment and low capital stock is characterized by a

[1] Even the Solow model, where the savings rate is arbitrarily specified, is efficient in the Cass-Koopmans sense. Outside the RBC model, with its representative agent, there are typically macroeconomic externalities which imply that the market equilibrium is not in general constrained Pareto efficient. See, e.g. Jeanne and Korinek (2019).

unique equilibrium, i.e. economic stagnation is stable and persistent. In another case, both states are characterized by multiple equilibria, i.e. the economy endogenously fluctuates without converging within a certain area. Or another case is that in which a steady state is locally stable but not globally. If a state of the economy is close to the boundary region with multiple momentary equilibria, with even a small shock to the economy, the economy will fall into the instability region, giving rise to large and persistent business fluctuations.

Further, small changes in the parameters describing the economy can undermine stability: Macroeconomic instabilities suddenly emerge, as some key parameter changes and reaches a critical point. We refer to such critical changes in the patterns of dynamics as "state transitions". We provide a complete analytic representation of all the possible state transitions.

A particularly interesting case that we focus upon is that where a stable "high output" (an economic boom) benefits from an above trend temporary productivity increase, and in which there is as a result a state transition from a stable regime to an unstable one. The economy enters into a situation where there are multiple equilibria, with the boom now being unstable, leading to the possibility of a large-scale collapse; the economy can enter a stagnation trap characterized by involuntary unemployment. As this example illustrates, our model exhibits large hysteresis effects.

In other cases, an increase in productivity leads to a completely wobbly economy in which the economy endogenously fluctuates without converging in either the full-employment and involuntary unemployment regions, even if there is *initially* a unique momentary and a unique high steady-state with full employment.

This paper should largely be viewed as an exercise in pure theory, demonstrating the richness of the macro dynamics that can arise if we move outside the realm of the standard model with an infinitely lived representative agent. Individuals have, of course, finite lives and are heterogeneous. There is ample evidence against the dynastic model, where individuals act *as if* they were infinitely lived. Using the simplest possible model with heterogeneous agents with finite lives, i.e. an overlapping generations model, we show one can generate a rich set of dynamic patterns. The earlier focus (say in Diamond (1965) and Samuelson (1958)) did not expose the full richness of decentralized dynamics.

At the same time, the global dynamics that we identify have *some* properties that are consistent with what has been observed in recent decades—arguably more consistent than that of the standard representative agent model, lending the model a certain degree of plausibility. (though we hasten to add, the major objective of our analysis is *pure* theory, to understand more fully the full range of dynamics that can be exhibited by what has been one of the standard workhorse models in economics for 65 years.) For instance, the macro-instabilities that have been exhibited in the dozens of crises around the world in the last third of a century show that after the collapse of some economic booms, output levels became permanently lower (or at least lower for a very extended period of time) than those on pre-booms and-crises trends (See Ball 2014; Blanchard, Cerutti, and Summers 2014; Cerra and Saxena 2008.). That is, hysteresis occurred.[2]

Moreover, not only do large boom and bust cycles occur frequently, but there are well-established patterns. For instance, economic historians, such as Charles Kindleberger (1978), note that unstable macro-dynamics typically, or at least often, follow technological advances; we provide a model which is at least consistent with that observation.[3] More generally, we characterize under what conditions hysteresis can arise.

One of the purposes of this paper is to provide a basic theoretical framework and a conceptual approach that can be extended into several directions. Our model is not directly intended for being mapped into data for serious quantitative analysis. Hence we abstract from many realistic elements such as credit, money, wage rigidities, or price stickiness etc. In sequels Hirano and Stiglitz (2021b, c) we introduce land or/and credit to capture many of the key aspects of cyclical fluctuations, such as those associated with financial markets and real estate bubbles. We show that these realistic elements can be easily incorporated into the framework we construct here.

To the best of our knowledge, the general question of global dynamics in the presence of multiple momentary equilibrium, and in particular with concepts called “phase transitions” and “state transitions” has been little studied. In this regard, the present paper and our sequel Hirano and Stiglitz (2021b, c) can be thought of as a prototype of how to analyze global dynamics when the existence of multiplicity of momentary equilibrium depends on endogenous state variables.

1.1: Related literature

From a theoretical point of view, our paper is in line with the long literature on nonlinear dynamics showing much richer patterns than exhibited in standard neoclassical models, suggesting that, while such models may be useful in analyzing long run steady states, they have limited insights into shorter run dynamics—even before accounting for short run employment effects. Dynamic complexity has been related to greater heterogeneity in capital goods, distribution,[4] non-separability in utility functions even within infinitely lived representative

[2] There are, of course, other models that have attempted to explain such hysteresis, focusing on capital market imperfections (Greenwald and Stiglitz, 1993, Stiglitz and Greenwald, 2003) or the emergence of asset bubbles and their collapse (Hirano and Yanagawa 2017).

[3] Many of the explanations of this and other unstable aspects of macro dynamics, such as the credit cycle, rest largely on systematic irrationalities in expectations. Our models, by contrast, assume rational expectations. At the same time, the existence of multiple momentary equilibria implies that the assumption of rational expectations may be implausible: there needs to be some coordinating mechanism so that all market participants know the equilibrium which is being selected. Though formally sunspots provide the basis for such coordination, the economic relevance may be questioned. See Guzman and Stiglitz (2021).

[4] See, e.g. Akerlof and Stiglitz (1969).

agent models,[5] endogenous technological change,[6] and credit frictions and distribution.[7] Exploring such complexity is, of course, one of the main objectives of the agent based literature,[8] but that literature, while enriching the model with heterogeneity in many dimensions, drops the assumption of rationality and rational expectations, the focus of our analysis here.

Within the literature on nonlinear dynamics, our paper is in line with the literature exploring macroeconomic implications of multiple equilibria.[9] There has, of course, been a literature in macroeconomic models showing the existence of multiple equilibria in static or two-or-three-period models (see, e.g., Diamond 1982 and Cooper and John 1988 for static models, and see Neary and Stiglitz 1983 and Kiyotaki 1988 and Lamont 1995 for two-period models, and Diamond and Dybvig 1983 for three-period model). These papers, however, did not explore full implications for global macro-dynamics. By contrast, our main focus is to explore the implications of multiplicity of momentary equilibria for global macro-dynamics.

Regarding macro-dynamics with multiple equilibria, in earlier growth literature there was a small literature noting the possibility of multiplicity of momentary equilibria, related to general equilibrium distributional effects (Uzawa 1961, 1963). He did not, however, explore the full implications for global macro-dynamics.[10]

There is also a dynamic literature on the existence of multiple paths converging to the steady-state (see, for instance, Shell et al 1969, Woodford 1986; Reichlin 1986; Benhabib and Farmer 1994; Schmitt-Grohe and Uribe 1997). These papers focused on local analysis around the steady-state. By contrast, while there are multiple dynamic paths *from any initial condition* consistent with rational expectations, they need not converge to *any* steady-state. In particular, our paper focuses on the implications of multiplicity of momentary equilibria for global macro dynamics.[11]

Some recent literature also focuses on a multiplicity of steady states (with the economy converging to one of them) by introducing some frictions, such as search frictions or frictions in nominal wages or prices or a zero lower bound on the interest rates (see Farmer 2020;

[5] See, e.g. Koopmans (1960) and Iwai (1972).
[6] See, e.g. Stiglitz (2014, 2006, 1994) and Acemoglu (2010).
[7] See, e.g. Matsuyama (2013), He and Krishnamurthy (2013), and Brunnermeier and Sannikov (2014).
[8] See, e.g. Farmer and Foley (2009).
[9] There is a vast literature on non-linear dynamics with a unique momentary equilibrium. The literature studies deterministic cycles or chaos in various economic setups by using the bifurcation theory. We do not survey that literature here; our main focus is to explore the implications of multiplicity of momentary equilibria for global macro-dynamics.
[10] Hoff and Stiglitz (2001) a rich set of models in the growth and development literature generating multiple steady states.
[11] In a sequel where we analyze a model with land and capital, unlike the standard model where there is a unique price of land (for any given level of capital stock) that is consistent with rational expectations, there can be a wide range of initial land prices; in this sense, our model is consistent with the earlier results of Shell et al. (1969).

Kocherlakota 2011 and 2020). By contrast, in our paper, in the simplest possible model without such frictions, there still may be a multiplicity of steady states, but the economy may never converge to any of them.

Central to our results are the complex non-linearities that arise in even the simplest overlapping generations models. The possibility of multiplicity of momentary equilibria in the standard life-cycle model developed by Diamond (1965) has been recognized for a long time, but seems little explored. (Stiglitz (1973),[12] Azariadis (1993), De La Croix and Michel (2002), Evans and Honkapohja (2012) and Romer (2019).) These papers, while mentioning the possibility of multiple momentary equilibria, do not provide either necessary or sufficient conditions within a broad class of utility and production functions. Moreover, these papers do not examine implications of multiple momentary equilibria for global macro dynamics, including the possibility explored here of wobbly dynamics.

Most closely related papers are Grandmont (1985), Matsuyama (1991), and Golosov and Menzio (2020). Grandmont's paper, however, shows multiplicity of momentary equilibria in *monetary* economies without capital investment within a two-period overlapping generations framework (see also Azariadis (1981), and Azariadis and Guesnerie (1986)). By contrast, we abstract from fiat money (though, equivalently, in a sequel we introduce land, generating an even richer set of global dynamics than displayed here). Focusing on capital, we show that the existence of multiplicity of momentary equilibria depends on the value of this endogenous state variable. Moreover, the main focus of our paper is wobbly fluctuations, while Grandmont (1985) focuses on deterministic cycles or chaotic dynamics.

Golosov and Menzio (2020) show stochastic fluctuations in unemployment which are driven by the existence of multiplicity of equilibria. However, their models (like that of Grandmont) are based on consumption/endowment economies, with no investment and capital stock.

Matsuyama (1991), while similar to the literature just discussed in not incorporating investment, explores the implications of multiple momentary equilibria for global dynamics in a model with two sectors, agriculture and manufacturing. Without the presence of sufficiently

[12] Stiglitz (1973) investigated dynamic stability within variants of the standard growth models of the time. Stiglitz's paper pointed out that multiple momentary equilibria can arise in the standard life-cycle model when the saving rate is a decreasing function of the interest rates. As we will see, this is a necessary condition. Stiglitz (1973) pointed out that "Whenever momentary equilibrium is not uniquely determined, the economy may wobble". However, Stiglitz did not explore this possibility in greater detail. Azariadis (1993), De La Croix and Michel (2002), Evans and Honkapohja (2012), and Romer (2019) plot one pattern of dynamics, which corresponds to case (a) in the present paper, but their discussions are limited to mentioning the possibility of this type of dynamics. They do not explore the condition under which it might occur or to fully characterizing it, if it did occur. Nor do they explore other patterns of dynamics we uncover, which corresponds to cases (b), (c), and (d) that are crucial in our analyses. These studies indeed impose parameter restrictions so that the momentary equilibrium is unique. (Imposing some restrictions on sets of parameter values that rule out the possibility of multiple equilibria is commonly used across many fields.). Conversely, we put macro dynamics generated by multiple momentary equilibria at the center stage of our analysis.

large increasing returns to scale in manufacturing, multiplicity of momentary equilibria cannot occur in his model. By contrast, as we have noted, our model incorporates all the standard assumptions, showing that even without increasing returns to scale and with no non-convexities, multiplicity of momentary equilibria can occur.

Finally, as He and Krishnamurthy 2013; Matsuyama 2013; and Brunnermeier and Sannikov 2014; and others have noted, a local analysis may not capture the highly nonlinear aspects of crises. The source of the nonlinearity in the global system is, however, crucially different, i.e. in our model large nonlinearities arise from multiplicity of momentary equilibria, while in the other papers just cited, the momentary equilibrium is globally unique but the global dynamic system exhibits nonlinearities due to tighter borrowing constraints or a deterioration in credit allocation.

2. The Basic Model and The Basic Analytical Results

We develop a simple overlapping generations model in which everyone in each generation is identical. In that sense, we are not departing far from the representative agent model. But what is crucial is that at each moment of time, not everyone is identical, i.e. there are heterogeneous agents. We employ a two-period overlapping generations model because it is the simplest model with heterogeneous agents, and heterogeneity is crucial for multiplicity of momentary equilibria to arise.

In each period young agents are born and live for two periods. Each young person is endowed with one unit of labor when young, and supplies it inelastically receiving wage income, w_t. Each young person also has e units of consumption goods as an endowment (e.g., other fixed income, or inheritance from parents), and saves a fraction s_t of the total income ($w_t + e$), generating the first and the second period consumption of

(1) $c_{1t} = (1 - s_t)(w_t + e)$ and $c_{2t} = (1 + r_{t+1})s_t(w_t + e)$,

where $1 + r_{t+1}$ is the gross interest rate between period t and $t + 1$. The holdings of capital by the young at time t becomes the capital stock at $t + 1$. This generates the dynamic equation of aggregate capital stock, i.e.,

(2a) $K_{t+1} = L_t s_t(w_t + e)$,

where L_t is the population of young agents at date t, and it grows at the rate of n, i.e.,

(2b) $L_{t+1} = (1 + n)L_t$

Competitive firms produce output by using capital and labor. Each firm has a standard neoclassical constant return to scale production function. Output per capita, y_t, is a function of capital per capita,

(3) $y_t = f(k_t) = f(K_t/L_t)$,

where K_t and L_t are aggregate capital and labor inputs. We assume a constant fraction rate of depreciation of capital, $\delta \in [0,1]$.

Rental and wage rates, R_t and w_t, satisfy

(4a) $R_t = f'(k_t)$,

(4b) $w_t = f(k_t) - f'(k_t)k_t = w(k_t)$,

with $f'(k_t) < 0$ and $w'(k_t) > 0$. The gross interest rate equals the return to holding capital.

(4c) $1 + r_{t+1} = f'(k_t) + 1 - \delta$

Then the dynamic equation for k_t can be written in per capita terms as

(5) $k_{t+1} = s_t \left(\frac{w(k_t)+e}{1+n}\right)$.

If s were a constant, there is a unique momentary equilibrium, i.e., for any value of k_t, there is a unique value of k_{t+1}, but even then there may not be a unique steady state, i.e. multiple values of k such that $k^* = s\frac{w(k^*)+e}{1+n}$.[13]

This paper focuses on the more interesting case where s is a function of the return on capital, which in turn depends on k_{t+1}. We assume in particular that $s_t = s(r_{t+1}) = s(r(k_{t+1}))$.[14] Define $\Omega(k_{t+1}) \equiv \frac{k_{t+1}}{s_t}$ and $W(k_t) \equiv \frac{w(k_t)+e}{1+n}$. Then the economy's evolution is governed by the equation:

(A) $\Omega(k_{t+1}) = W(k_t)$

Central to this paper is the result that under quite general and plausible conditions, Ω is not monotonic, so there may be, at least for some values of k_t, multiple values of k_{t+1} satisfying (A). We define the correspondence $\Psi(k_t)$ giving the set of k_{t+1} satisfying equation (A). Figure 1-1 illustrates what happens if there are multiple values of k_{t+1} corresponding to any Ω. Given k_t, there is a particular value of $W(k_t)$, but for a wide range of $W(k_t)$ there will be multiple values of k_{t+1}. $\Psi(k_t)$ gives the set of k_{t+1} for any k_t. Most of this paper is an exploration of the various forms Ψ can take and their dynamic implications.

Differentiating $\Omega(k_{t+1})$ with respect to k_{t+1} yields

(6) $\Omega'(k_{t+1}) = \frac{1}{s_t}\left(1 - \frac{dlog(s_t)}{dlog(1+r_{t+1})}\frac{dlog(1+r_{t+1})}{dlog(k_{t+1})}\right)$

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[13] Uniqueness requires $Z(k) \equiv s(w(k) + e)/(1 + n)$ cross the 45 degree line once. Even if s is fixed, $Z' = -skf''/(1 + n) > 0$, and $Z'' = -(s/1 + n)(f'' + kf''')$. Economic theory puts no natural constraints on $f'' + kf'''$.

[14] A still more general savings function would have the savings rate a function of the wage and interest rate. Extending the model to incorporate this is straightforward. What is crucial for our analysis is the dependence of s on k_{t+1} (in our analysis, through the effect on the rationally expected return to capital).

where $\frac{dlog(s_t)}{dlog(1+r_{t+1})}$ is the interest rate elasticity of savings. $\frac{dlog(1+r_{t+1})}{dlog(k_{t+1})} < 0$ is the elasticity of the interest rate with respect to the capital stock. These elasticities depend on the intertemporal elasticity of substitution in consumption (IES) and the elasticity of substitution between capital and labor (ES), respectively. For instance, if individuals have a separable utility function with a constant elasticity of consumption, θ, then

(7) $\frac{dlog(s_t)}{dlog(1+r_{t+1})} = (1 - s_t)(\theta - 1),$

$\frac{dlog(s_t)}{dlog(1+r_{t+1})}$ is negative (positive) if $\theta < 1(\theta > 1)$. The borderline case is the logarithmic utility function ($\theta = 1$), for which $\frac{dlog(s_t)}{dlog(1+r_{t+1})} = 0$.

Similarly,

(8) $\frac{dlog(1+r_{t+1})}{dlog(k_{t+1})} = \frac{k_{t+1}f''(k_{t+1})}{f'(k_{t+1})}\frac{f'(k_{t+1})}{f'(k_{t+1})+1-\delta} = -\frac{h(k_{t+1})s_L(k_{t+1})}{\sigma}$

where σ is the elasticity of substitution, h is the ratio of the rental rate to the return to holding capital, $\frac{f'(k_{t+1})}{f'(k_{t+1})+1-\delta} < 1$, and $s_L(k_{t+1}) < 1$ is the share of labor. Thus, if δ is large, σ is small, and S_L is large, $\frac{dlog(1+r_{t+1})}{dlog(k_{t+1})}$ is more negative.

From (6), a sufficient condition for $\Omega'(k_{t+1}) > 0$ is that $\frac{dlog(s_t)}{dlog(1+r_{t+1})} \geq 0$. That is, if the saving rate is a monotonically increasing function of the interest rate, there is a unique momentary equilibrium.

If, however, for some values of k_{t+1}, $\frac{dlog(s_t)}{dlog(1+r_{t+1})} < 0$, Ω may not be invertible, i.e., for some values of k_t, there may be multiple values of k_{t+1}, all consistent with rational expectations. Figure 1-1 illustrates this. Intuitively, if everyone believes that the interest rate is low (investment is expected to be high), they will save a great deal, and the interest rate will be low (investment will be high).

We have already noted that this is the case for a separable utility function with a constant elasticity of consumption, θ, if $\theta < 1$, i.e. the elasticity of marginal utility decreases strongly with consumption. We now show that if the elasticity of substitution is small enough (sufficiently less than unity) then a multiplicity of momentary equilibria can occur for at least some values of k.

More precisely, if $\sigma < 1$, $\lim_{k_{t+1}\to 0} S_L(k_{t+1}) = 0$. Hence, we have $\lim_{k_{t+1}\to 0} \Omega'(k_{t+1}) > 0$. Also, if $\sigma < 1$, we have $\lim_{k_{t+1}\to\infty} f'(k_{t+1}) = 0$, and thus $\lim_{k_{t+1}\to\infty} h(k_{t+1}) = 0$ for any $\delta \in [0,1)$. Hence, we have $\lim_{k_{t+1}\to\infty} \Omega'(k_{t+1}) > 0$.[15] On the other hand, if σ is small enough, i.e.,

$$\sigma < h(k_{t+1})S_L(k_{t+1})\big(1 - s(k_{t+1})\big)(1-\theta) < 1$$

for some k_{t+1}, $\Omega'(k_{t+1}) < 0$ for some k_{t+1}. With these conditions being satisfied, Ω is not monotonic as illustrated in Figure 1-1.

Define $\underline{\Omega}$ as the minimum value of Ω for which there are multiple values of k solving $\underline{\Omega} = \Omega(k)$; and similarly, $\overline{\Omega}$ as the maximum value of Ω for which there are multiple value of k. Then so long as for some value of k_t,

$$\underline{\Omega} < W(k_t) < \overline{\Omega}\,,$$

there can be indeterminacy in the dynamic trajectory of the economy. Since $W'(k_t) > 0$ under standard assumptions on production functions, and $W(k_t = 0) = \frac{e}{1+n}$, there exists values of $W(k_t)$ for which, for some value of k_t, there exist multiple values of k_{t+1} which satisfies (A).

Steady states

A steady state is defined by $\Omega(k^*) = W(k^*)$. If Ω is monotonic, there is a unique k_{t+1} for any k_t, i.e. a unique momentary equilibrium. Even if $\Omega'(k_{t+1}) > 0$, there may be multiple steady-states, i.e., multiple values of k such that $k^* = \Omega^{-1}(W(k^*))$. Obviously, in the more general case, explored here, there may be multiple steady states. (See Figure 2).

3. Micro foundations for the savings functions and equilibrium aggregate dynamics

We denote the aggregate consumption of young and old at date t as C_{1t} and C_{2t}, respectively, and consumption of each young person by c_{1t} and c_{2t}. The t-th generation chooses c_{1t} and c_{2t} to maximize their utility $u_t = u(c_{1t}, c_{2t})$ subject to their budget constraint.

(9) $c_{1t} + k'_{t+1} = w_t + e$ and $c_{2t} = (1 + r_{t+1})k'_{t+1}$,

where k'_{t+1} is capital investment of each young person. Solving the maximization problem (taking into account the non-negative constraints $c_{1t} \geq 0$, $c_{2t} \geq 0$, and $k'_{t+1} \geq 0$) yields

(10) $k'_{t+1} = s_t(w_t + e)$

(11) $c_{1t} = (1 - s_t)(w_t + e)$ and $c_{2t} = (1 + r_{t+1})k'_{t+1} = (1 + r_{t+1})s_t(w_t + e)$

[15] When $\delta = 1$, if $\theta < 1$, $\lim_{k_{t+1}\to\infty} s(k_{t+1}) = 1$ and hence we have $\lim_{k_{t+1}\to\infty} \Omega'(k_{t+1}) > 0$.

where the savings function obviously depends on the utility function. Maximizing utility subject to the individual's life time budget constraint yields

$$\frac{\partial u(c_{1t},c_{2t})/\partial c_{1t}}{\partial u(c_{1t},c_{2t})/\partial c_{2t}} = 1 + r_{t+1}.$$

Then the market clearing condition for goods is

(12) $$C_{1t} + C_{2t} + K_{t+1} - (1-\delta)K_t = Y_t + L_t e$$

where Y_t is the aggregate output at date t, and $K_t = L_{t-1}k'_t$.

The competitive equilibrium is then defined as a set of prices $\{R_t, w_t\}_{t=0}^{\infty}$ and quantities $\{c_{1t}, c_{2t}, y_t, k'_{t+1}, C_{1t}, C_{2t}, K_{t+1}, Y_{t+1}\}_{t=0}^{\infty}$, given initial K_0 and Y_0, such that (i) each young agent chooses consumption and capital investment to maximize the expected utility under the budget constraints and the non-negative constraints, and (ii) the market clearing condition for goods, capital and labor are all satisfied.

It is, however, much more convenient to represent everything in per capita terms, which we do for the remainder of the paper. In particular, we rewrite (10) as $k_{t+1} = s(k_{t+1})\left(\frac{w(k_t)+e}{1+n}\right)$ which is just (5) above, from which we derived (A) above.

3.1. Dynamic Implications: Phase Transitions

Multiplicity of momentary equilibria translates into an infinity of dynamic paths, all consistent with rational expectations. The economy may wobble, neither settling down to a steady state equilibrium nor diverging in an explosive manner. The right figure in Figure 2 shows an example of a rational expectations trajectory beginning at k_0, oscillating perpetually between some lower and some upper bound (to be described in greater detail below), never converging, never settling into a regular cycle, always moving in a way consistent with rational expectations. In each state where there are multiple momentary equilibria, the outcome depends on beliefs, i.e., bullish (optimistic) or bearish (pessimistic) expectations about investment activity. With bullish expectations, interest rates are low, so savings and investment are high, sustaining that equilibrium; and similarly for bearish expectations. In the figure, we trace out one possible "wobbly" trajectory, where the economy neither converges to a steady state or even a limit cycle. But it should be clear that there are an infinite number of possible rational expectations dynamic trajectories. When there are multiple momentary equilibria, the economy may wobble; the economy can suddenly switch from one momentary equilibrium to another, showing that a laissez-faire market economy can be, in this sense, unstable—though as we shall show, there are bounds within which the economy must oscillate.[16]

[16]A few remarks concerning sunspot probabilities. We could, for instance, assign the following sunspot probability: For any $k_t \in [\underline{k}, \overline{k}]$, there are three values of k_{t+1} consistent with rational expectations. Let the

By contrast, the left figure in Figure 2 represents the typical dynamics of an economy with a unique momentary equilibrium. Given k_t, there is a unique value of k_{t+1}, and that determines, in turn, k_{t+2}, etc. The figure illustrates the standard dynamic process showing convergence to a steady state (for a later purpose, the figure illustrates a situation where there are three steady states, two stable, one unstable).

We focus on the case where the correspondence ψ defined by (A) can take three values of k_{t+1} for a given k_t over an interval $\underline{k} < k < \overline{k}$,[17] where $\underline{k}$ is the solution to $\frac{w(k_t)+e}{1+n} = \underline{\Omega}$ and similarly for $\overline{k}$. This is illustrated in Figure 1-1. There are multiple momentary equilibrium when k is between $\underline{k}$ and $\overline{k}$. There can be three steady states[18], two of which are locally stable in a normal sense, that is, if at those steady states, there are multiple momentary equilibria, and if at the upper steady state, the economy "selects" the upper value of the correspondence, and at the lower one it selects the lower one; then with those selections, the economy converges to the given steady state for a small perturbation from the equilibrium.

Central to the following analysis are four states which are defined by the relationship between the upper steady state, denoted k^H[19] and the upper value of k at which there are multiple momentary equilibria, $\overline{k}$ and the lower steady state, k^L, and the lower value of k at which there are multiple momentary equilibria, $\underline{k}$. (In the parametric model investigated in the next section, these are, in turn, a function of the key parameters in the economy.) The key characteristics associated with wobbly dynamics is that the state of the economy endogenously

distance of each k_{t+1} from the current k_t be $d1$, $d2$, and $d3$, respectively, where $d1 > d2 > d3$. The sunspot probability that each k_{t+1} with distance $d1$, $d2$, and $d3$ is selected is assumed to follow $d3/(d1 + d2 + d3) < d2/(d1 + d2 + d3) < d1/(d1 + d2 + d3)$, respectively. Since $d1$, $d2$, and $d3$ change according to the aggregate state of the economy, the sunspot probability changes over time. Moreover, this inequality means that there is persistence between the current state and the next period state, i.e., the probability that a certain k_{t+1} is selected is higher (lower) if that k_{t+1} is closer (far from) to the current k_t. In other words, even if the current state of the economy experiences economic booms, there is a small probability that the economy suddenly experiences the collapse of the booms and the aggregate economic activities shrink discontinuously. Using this transition probability, we conduct a full welfare analysis and compare welfare under instability with welfare under stability with government policy. See the sequel to this paper Hirano and Stiglitz (2021a) for details.

[17] In the parameterization investigated in the next section, it appears that there are at most three values of k_{t+1} corresponding to any value of k_t, but in the more general case, there can be a larger number of values.

[18] There is also case with a unique stable steady state equilibrium, even though multiple momentary equilibria arise for $\underline{k} < k_t < \overline{k}$. As we have already noted, it is also possible that there can be a unique momentary equilibria (a sufficient condition for which is that $\Omega' > 0$). Even then, it is possible that there are multiple steady states, each of which has its own domain of attraction. In the numerical analyses we have conducted, other dynamic patterns except for these five cases were not found. But clearly, these results are dependent on the particular parameterizations we have employed.

[19] Nothing in our analysis ensures that k^H exceeds the level at which there is overall dynamic inefficiency, in the Cass-Koopmans sense (Cass (1965), Koopmans (1965)). By the same token, nothing in our analysis necessarily implies that the high capital equilibrium, k^H, is better than the low equilibrium k^L, since k^H may be characterized by over-saving. The analysis of this part of the paper is purely descriptive.

changes from a state with a unique momentary equilibrium to a state with multiple momentary equilibria or vice versa, which we call a “phase transition”. Depending on how phase transitions occur, there are four typical patterns of the wobbly dynamics. Figure 3 presents them.

State (a): Three steady states, two stable, unstable wobbly dynamics. This arises when $0 < k^L < \underline{k} < \overline{k} < k^H$. Though there are two steady states—both stable-- rather than converging to either, the economy may fluctuate between $\underline{k}$ and $\overline{k}$. But the wobbly dynamics is not stable; it is possible for the economy to move outside this region; if it moves below $\underline{k}$, it converges to k^L; if it exceeds $\overline{k}$, it converges to k^H. In both trajectories, there is a phase transition from a state with multiple equilibria to a state with a unique equilibrium.

State (b): Three steady states, upper and middle unstable, lower stable, unstable wobbly dynamics. This arises when $0 < k^L < \underline{k} < k^H < \overline{k}$. This case shows that there can be asymmetries in macroeconomic stability between economic booms and stagnation. Again, the economy may fluctuate between $\underline{k}$ and $\overline{k}$, and again there are two steady states, but now the upper equilibrium (which we refer to as the boom) is unstable. Thus, rational expectations economic booms are fragile and can easily collapse by sudden changes in expectations. Moreover, economic booms are not only unstable, but ironically, the utility levels of those experiencing these (rational expectations) booms might have been higher had they not saved so much. By contrast, if k_t ever becomes sufficiently low, there is a phase transition to a state with a unique momentary equilibrium. Once the economy falls into this region (with $k < \underline{k}$) it converges to k^L. It is trapped there. Only a large shock (or a large intervention by the government) can do the trick. That is, economic stagnation is stable and persistent: There is a unique stable steady state k^L. If the economy wobbles, it’s wobbles are bound by $\underline{k}$ and k^H for large t.

State (c): Three steady states, all unstable. Wobbly dynamics stables. This arises when $0 < \underline{k} < k^L < k^H < \overline{k}$. The wobbles are bound by k^L and k^H for large t. Even when the economy is at say k^H, the economy may suddenly jump in a fully rational expectations equilibrium to a smaller value of k. Nothing in the theory ensures that it will remain at k^H. The economy can bounce around infinitely without converging. In this case, a phase transition from a state with a unique momentary equilibrium to a state with multiple momentary equilibria occurs when the economy initially starts from the outer region of $\underline{k}$ or $\overline{k}$.

State (d): Three steady states, higher k stable; other two steady states unstable; unstable wobbly dynamics exists. This case arises when $0 < \underline{k} < k^L < \overline{k} < k^H$. This case shows that animal spirits (entrepreneurial spirits in the private sector) play a key role when economic activity is stagnant (i.e. k_t is low). Even if the economy has been at k^L for an extended period of time, with low investment, wages, and output, if individuals have bullish expectations about investment activity, expecting as a result that the interest rate will fall, they save and invest more, and there will be a phase transition to a state with a unique momentary equilibrium, and as a result the economy can get out of the stagnation.[20] The unique stable steady state is k^H and if the economy wobbles it wobbles between k^L and $\overline{k}$.

For completeness, we note one other case. It is possible as the parameters of the model change that $\overline{\Omega}$ and $\underline{\Omega}$ converge, i.e. that there exists no region of multiplicity of momentary equilibrium, and that k_{t+1} be a monotonically increasing function of k_t. With other parameters, there may be a multiplicity of momentary equilibrium but still a unique steady-state.

Which of the configurations describes the economy depends on the parameters of the production and utility functions as well as the other parameters of the model, n and e, as we will show more clearly in the next section. That means, of course, that changes in those parameters will change the economy's regime.

Of particular interest is state (c), where the only stable dynamics are wobbly dynamics. State (b) is also of interest—fragile booms where, as a result of a change in expectations, the economy enters a period of volatility and eventually settles into a low-level equilibrium trap. We will also discuss the possibility of an economy initially being in state (a), in a stable boom, but a seeming productivity improvement moves it into state (b), so that while the boom is strengthened—so long as it lasts—it becomes fragile, and eventually breaks.

In this wobbly economy, the existence of multiple momentary equilibria depends not only on the parameters describing the economy but also on an endogenous state variable, i.e., the value of k. For instance, in state (a), there is a unique momentary equilibrium around the neighborhood of k^L and k^H, i.e., they are both stable. In state (d) and state (b), k^H and k^L are stable, respectively. In these cases, once the economy settles down into a stable state, it stays there. So long as the size of the exogenous shocks is sufficiently small, macroeconomic fluctuations are small. This can be interpreted as business fluctuations in "normal times". In these circumstances, the presence of multiple momentary equilibria cannot be observed so long as there are only small perturbations. When the size of the shocks is sufficiently large, however, its hidden presence in the global system is suddenly revealed, and exhibited through large and persistent macroeconomic instabilities.[21]

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[20] Note that in this model "bullish" refers to expectations concerning the level of investment, not the returns on those investments. A self-fulfilling expectation of low interest rates drives high savings and investment, in contrast to the usual animal spirits models, where expectations of high returns drives high levels of investment. A natural extension of the rational expectations full equilibrium models explored here entails dropping these two conventional assumptions. Not surprisingly, it is even easier to get multiple equilibria.

[21] One of the criticisms of multiple equilibria is that economic variables are not as volatile as models with multiple equilibria suggest. This criticism may not necessarily apply to our model because the existence of multiple momentary equilibria depends on the endogenous state variable, i.e. capital stock. This means that once the economy settles down into one of the stable steady-states, the macroeconomy exhibits only small changes in economic variables. It is only when the economy is sufficiently away from a stable steady-state that macroeconomic instabilities emerge.

When k^H (or k^L) is near the region with multiple momentary equilibria, with even a small shock to the economy, the economy may fall into the instability region, giving rise to large and persistent business fluctuations.

Moreover, changes in the key parameters of the economy (e.g. technology) not only affect standards of living, wages, output, etc. but they affect the nature of the dynamics. We refer to the movement from one case to another as a *state transition.* As we will see, some changes in technology while increasing incomes *if the economy remains in the boom,* make the boom more fragile.

4. A Parametric Model

4.1 CES utility and production functions

The assumed representative individual's utility function entails constant elasticity of substitution between consumption in the two periods, and is of the form:

(13) $$u_t = \left((a_1)^{\frac{1}{\theta}}(c_{1t})^{\frac{\theta-1}{\theta}} + (a_2)^{\frac{1}{\theta}}(c_{2t})^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}},$$

where θ is intertemporal elasticity of substitution in consumption. c_{1t} and c_{2t} are gross complements (gross substitutes) if $\theta < 1$ ($\theta > 1$). a_1 and a_2 are weights on consumption in working and retirement periods, respectively, and affect the optimal consumption ratio between c_{1t} and c_{2t}.

With this parametric utility function, the saving rate at date t, s_t, is given by

(14) $$s_t = \frac{1}{1+\frac{a_1}{a_2}(1+r_{t+1})^{1-\theta}}.$$

It follows that if the intertemporal elasticity of substitution $\theta < 1$, the income effect dominates the substitution effect, so the saving rate, s_t, decreases as the interest rate increases, while the reverse holds if $\theta > 1$.

We assume a constant elasticity of substitution production function.

(15) $$Y_t = A\left(\alpha(\frac{K_t}{\omega_1})^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(\frac{L_t}{\omega_2})^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}},$$

where σ is the elasticity of substitution (ES). K_t and L_t are gross complements (gross substitutes) if $\sigma < 1$ ($\sigma > 1$). A is a productivity parameter, and $\frac{1}{\omega_1}$ and $\frac{1}{\omega_2}$ are capital productivity and labor productivity, respectively. $\alpha \in (0,1)$ reflects capital intensity in production.

With our CES production function, we can similarly derive factor payments (and hence shares) (equations (4a) and (4b) for this specification):

(16a) $$R_t = A\left(\alpha (k_t)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)\ \left(\frac{\omega_1}{\omega_2}\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} \frac{\alpha}{\omega_1}(k_t)^{\frac{-1}{\sigma}} = R(k_t),$$

(16b) $$w_t = A\left(\frac{\alpha}{\left(\frac{\omega_1}{\omega_2}\right)^{\frac{\sigma-1}{\sigma}}}(k_t)^{\frac{\sigma-1}{\sigma}} + 1-\alpha\right)^{\frac{1}{\sigma-1}} \frac{1-\alpha}{\omega_2} = w(k_t).$$

Our earlier analysis established that $\Omega'(k_{t+1})$ is more likely to be negative for some k_{t+1} if both $\frac{dlog(1+r_{t+1})}{dlog(k_{t+1})} < 0$ and $\frac{dlog(s_t)}{dlog(1+r_{t+1})} < 0$ are more negative. For our CES production and utility functions, we can directly compute $\frac{dlog(s_t)}{dlog(1+r_{t+1})}$ and $\frac{dlog(1+r_{t+1})}{dlog(k_{t+1})}$ and ascertain how various parameters in the production and utility functions affect these terms, obtaining the following Lemma.

Lemma 1: $\frac{\partial^2 log(s_t)}{\partial(a_1/a_2)log(1+r_{t+1})} < 0, \frac{\partial^2 log(s_t)}{\partial(A)log(1+r_{t+1})} < 0, \frac{\partial^2 log(1+r_{t+1})}{\partial(A)log(k_{t+1})} < 0.$

Hence, from Lemma 1, $\Omega'(k_{t+1})$ is more likely to be negative for some k_{t+1} as $\frac{a_1}{a_2}$, or/and A become larger.

By substituting $\frac{dlog(s_t)}{dlog(1+r_{t+1})}$ and $\frac{dlog(1+r_{t+1})}{dlog(k_{t+1})}$ into (6), we have

(17)
$$\Omega'(k_{t+1}) = 1 + \frac{a_1}{a_2}\frac{(1+r_{t+1})^{1-\theta}}{1+r_{t+1}}\left\{A\left(\alpha(k_{t+1})^{\frac{\sigma-1}{\sigma}} + (1-\alpha)\left(\frac{\omega_1}{\omega_2}\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{2-\sigma}{\sigma-1}} \frac{\alpha}{\omega_1}(k_{t+1})^{\frac{-1}{\sigma}}\left[\alpha(k_{t+1})^{\frac{\sigma-1}{\sigma}} + \left(\frac{\omega_1}{\omega_2}\right)^{\frac{\sigma-1}{\sigma}}\left(\frac{1-\alpha}{\sigma}\right)(\sigma-(1-\theta))\right] + 1 - \delta\right\},$$
where $1 + r_{t+1} = R_{t+1} + 1 - \delta$

(17) allows us to establish that a necessary condition for wobbly dynamics is that $\theta < 1-\sigma$. On the other hand, if $\theta \geq 1$ or/and $\sigma \geq 1$, $\Omega'(k_{t+1}) > 0$ for any k_{t+1}.

4.2. Sufficient condition for multiplicity of momentary equilibrium in the CES utility and production functions

It is clear that if $\theta < 1$ and $\sigma < 1$, $\Omega'(k_{t+1})$ may be negative for some k_{t+1}.[22] Indeed, we have the following Lemma 2.

Lemma 2: If $\sigma < 1$, $\lim_{k_{t+1}\to 0} \Omega(k_{t+1}) = 0$, $\lim_{k_{t+1}\to 0} \Omega'(k_{t+1}) = 1 + \frac{a_1}{a_2}\left((\frac{A}{\omega_1}(\alpha)^{\frac{-\sigma}{1-\sigma}} + 1 - \delta) \right)^{1-\theta} > 0$ and $\lim_{k_{t+1}\to \infty} \Omega'(k_{t+1}) = 1 + \frac{a_1}{a_2}((1-\delta))^{1-\theta} > 0$.

By using Lemma 1 and 2, we can prove the following Lemma 3 which provides the sufficient condition for $\Omega'(k_{t+1}) < 0$ for some k_{t+1}.

Lemma 3: If $\sigma < 1 - \theta$ and A is large enough (or given large enough A, if a_1/a_2 is large enough), $\Omega'(k_{t+1}) < 0$ for a given $k_{t+1} \in \left(\frac{\omega_1}{\omega_2}\left[\frac{\alpha}{1-\alpha}\frac{\sigma}{1-\sigma-\theta}\right]^{\frac{\sigma}{1-\sigma}}, \infty \right)$.

A small elasticity of substitution in production means a large (absolute) value for $\frac{dlog(1+r_{t+1})}{dlog(k_{t+1})}$. And a small intertemporal elasticity of substitution in consumption means that $\frac{dlog(s_t)}{dlog(1+r_{t+1})}$ is (more) negative. Also, from Lemma 1, both elasticities are more negative if A is larger.

Lemma 3 provides a sufficient condition that $\Omega'(k_{t+1}) < 0$ for some k_{t+1}: (i) the intertemporal elasticity of substitution in consumption being less than unity; (ii) the elasticity of substitution between capital and labor being less than unity; and (iii) the productivity parameter being sufficiently large. (i) and (ii) mean that c_{1t} and c_{2t}, and k_t and l_t are gross complements.

From Lemma 2 and 3, Ω is, under the stipulated conditions, never a monotonic function of k_{t+1}. A typical shape is illustrated in Figure 1-1. Whether there exists multiple momentary equilibria depends then on the value of W, and the value of W, for any given k_t, depends on e

[22] While this paper should be viewed mainly as a theoretical exercise, we note that the assumed parameters are widely accepted within the literature. In the standard DSGE models, $\theta < 1$ is commonly used. There is more controversy over the value of the elasticity of substitution. For instance, while traditionally, most analyses took $\sigma < 1$, confirmed by more recent studies (Antras (2004), Oberfield and Raval (2014), Chirinko and Mallick (2017)), Piketty and Zucman (2014)'s analysis implies $\sigma > 1$. But Piketty and Zucman's results partially arise out of a confusion between wealth and capital. The difference is the capitalized value of rents, which arguably increased significantly in recent decades, so much so that in some countries arguably the capital output ratio has been declining even as the wealth output ratio has been increasing. See Stiglitz (2015). Recent papers by Best et al. (2019) and by Gechert et al. (2019) show that the average IES is small, around 0.1 and ES between capital and labor is 0.3, respectively.

and n. Hence, for any given k_t, there are multiple values of k_{t+1} for some e and n. We summarize this result in the following Proposition.

Proposition 1: Under sets of parameter values that satisfy Lemma 3, for any given k_t, there are multiple values of k_{t+1} for some e and n.

In our parametric model, it is easy to calculate how changing the parameters of the utility and/or production function or e or n changes the shape of $\Omega(k)$ and the value of $W(k)$. For instance, with greater e, $\underline{k}$ becomes smaller. This is because with greater e, aggregate savings get larger even for small k_t, so that expectations of high investments associated with a low interest rate can be consistent with rational expectations even in the region with small k_t. Likewise, with greater a_1/a_2, $\overline{k}$ becomes larger. This is because each person is more impatient, so that the saving rate gets lower and aggregate savings become small even for large k_t. Expectations of low investments associated with a high interest rate can be self-fulfilling even for large k_t.

4.3. Numerical characterization of the phases in the CES utility and production functions

For the CES utility and production functions, we provide a numerical characterization to determine which of the four cases identified earlier describes the economy, since this case is hard to characterize analytically. Figure 5 focuses on the role of the elasticity of substitution in production (ES) and the intertemporal elasticity of substitution in consumption (IES); it illustrates how different parameter configuration in the (θ, σ)-plane give rise to each of the four cases, given other parameters. As Lemma 2 shows, multiplicity of momentary equilibria can occur in the region below the boundary line of $\sigma = 1 - \theta$, which is a necessary condition. Within this region, in the area relatively nearby the boundary line, state (a) arises. As both θ and σ become smaller, state (b) and state (d) emerge, respectively. If θ and σ become even smaller, then state (c) emerges. This characterization result suggests that if the degree of complementarity between c_{1t} and c_{2t}, and k_t and l_t become stronger, state (b), state (d), and state (c) appear.[23] Figure 4 is constructed for one example of the other parameters. If these other parameters change, the area of each state would change.

[23] In the blue region below the boundary line of $\sigma = 1 - \theta$ in Figure 4, there is a unique stable steady-state and multiplicity of momentary equilibrium may or may not arise in the region of either above or below the 45 degree line. Recall that $\sigma < 1 - \theta$ is a necessary condition. In either case, the economy will converge to the unique steady-state. In this regard, this region is similar to state (e1) or (e2) in the Leontief case, to be described shortly.

5. A complete analytical characterization of a wobbly economy with involuntary unemployment

5.1. The Leontief case

By focusing on a specific case where both utility and production functions are of Leontief forms, we can provide the necessary and sufficient condition for wobbly dynamics, and we can also provide a complete characterization analytically for all possible patterns of wobbly dynamics, state (a)-(d), i.e. under which each may occur. The Leontief case corresponds to the limiting case of $\sigma \to 0$ and $\theta \to 0$. Moreover, unlike the previous analysis showing wobbly dynamics with full employment, in this case, wobbly dynamics with involuntary unemployment can arise if k is small enough. The Leontief case is a very nice tractable case where one can trace out the wobbly paths analytically.

The utility function[24] and the aggregate production function are $u_t = \min\left(\frac{c_{1t}}{a_1}, \frac{c_{2t}}{a_2}\right)$ and $Y_t = A\min(\frac{k_t}{\omega_1}, \frac{L_t}{\omega_2})$. $k_t = \frac{\omega_1}{\omega_2} \equiv k^f$ is per capita capital level required to have full employment. If $k_t < \frac{\omega_1}{\omega_2}$, involuntary unemployment occurs, while if $k_t > \frac{\omega_1}{\omega_2}$, full employment is achieved, but not all capital is utilized. As k becomes lower compared with k^f, there is more involuntary unemployment.

In this specific case, the function $\Omega(k_{t+1})$ and $W(k_t)$ are written as follows:

$$\text{(18a)} \quad \Omega(k_{t+1}) = \left(1 + \frac{a_1}{a_2}(1 + r_{t+1})\right)k_{t+1} = \begin{cases} \left(1 + \frac{a_1}{a_2}\left(\frac{A}{\omega_1} + 1 - \delta\right)\right)k_{t+1} & if\ k_{t+1} < \frac{\omega_1}{\omega_2} \\ \left(1 + \frac{a_1}{a_2}(1 - \delta)\right)k_{t+1} & if\ k_{t+1} > \frac{\omega_1}{\omega_2} \end{cases}$$

and

$$\text{(18b)} \quad W(k_t) = \begin{cases} \frac{e}{1+n} & if\ k_{t+1} < \frac{\omega_1}{\omega_2} \\ \frac{\frac{A}{\omega_2} + e}{1+n} & if\ k_{t+1} > \frac{\omega_1}{\omega_2} \end{cases}$$

For each k_t, there is one or more rational expectations momentary equilibrium. Note that at $k_t = k^f$, k_{t+1} in general depends on r_{t+1} and w_t. We elaborate on this below.

[24] The analysis below makes it clear that what is crucial for analytical tractability is the Leontief production function, which results in the returns to capital either being zero or the full output, and similarly for labor. With fixed returns, savings depend simply on whether k is greater or less than k^f, i.e. Figure 1-2 still describes the economy.

Figure 1-2 illustrates the relationship between $\Omega(k_{t+1})$ and $W(k_t)$. $\Omega(k_{t+1})$ increases linearly with k_{t+1}, with slope $\left(1+\frac{a_1}{a_2}\left(\frac{A}{\omega_1}+1-\delta\right)\right)$ until k^f is reached, then jumps down, and then increases again linearly but now at a lower slope, $1+\frac{a_1}{a_2}(1-\ \delta)$. As we can see, the relationship doesn't change much compared to the general case.

The maximum value of Ω in the capital shortage regime, i.e. $\overline{\Omega}$ is $\left(1+\frac{a_1}{a_2}\left(\frac{A}{\omega_1}+1-\delta\right)\right)\frac{\omega_1}{\omega_2}$ and the minimum value of Ω in the capital surplus regime, i.e. $\underline{\Omega}$ is $\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}$. Note that neither depends on e. On the other hand, W clearly depends on e. There is a critical value of e at which $\underline{\Omega}$ just equals $\frac{e}{1+n}$, i.e. for low k_t there exists a unique momentary equilibrium, and another critical value of e at which $\overline{\Omega}$ just equals $\frac{\frac{A}{\omega_2}+e}{1+n}$, i.e. for high k_t there exists a unique momentary equilibrium.

The wobbly dynamics can be seen by considering what happens if the line $W(k_t <\ k^f)$ lies above $\underline{\Omega}$ or $W(k_t > k^f)$ lies below $\overline{\Omega}$. More precisely, the necessary and sufficient condition for stable wobbly dynamics, i.e. for reverse switching to be possible in both the capital shortage and capital surplus regimes, is that

(19a) $\overline{\Omega}=\left(1+\frac{a_1}{a_2}\left(\frac{A}{\omega_1}+1-\delta\right)\right)\frac{\omega_1}{\omega_2}>\frac{\frac{A}{\omega_2}+e}{1+n}$.

i.e., when there is a capital surplus, the economy can switch to the capital shortage regime; and

(19b) $\underline{\Omega}=\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}<\frac{e}{1+n}$.

i.e., when there is a capital shortage, the economy can switch to the capital surplus regime.

Proposition 2: The necessary and sufficient condition for stable wobbly dynamics in the Leontief case is given by (19a) and (19b).

Given all the other parameters, (19a) and (19b) can be expressed as

(19c) $\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}<\frac{e}{1+n}<\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}+\frac{\frac{A}{\omega_2}\left(\frac{a_1}{a_2}(1+n)-1\right)}{1+n}$

It is easy to see that there exists sets of parameter values for which (19) can be satisfied if $\frac{a_1}{a_2} > \frac{1}{1+n}$. (Later, we will provide still another characterization of the necessary and sufficient conditions for wobbly dynamics.)

Figure 5 depicts the steady states, by plotting k_{t+1} as a function of k_t. A steady state entails $k_{t+1} = k_t$. There are three kinds of steady states: with unemployed labor, unemployed capital, and just full employment of each. And depending on the parameters, there can exist three steady states, one each of the given form, or only one steady state, entailing full employment of only one factor.

When there are wobbly dynamics, there are also three steady states, $k^H = \frac{\frac{\frac{A}{\omega_2}+e}{1+n}}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)}$, $k^L = \frac{\frac{e}{1+n}}{\left(1+\frac{a_1}{a_2}\left(\frac{A}{\omega_1}+1-\delta\right)\right)}$ and k^f, where the latter is supported by a particular distribution of income, i.e. $\{w, r\}$. Each of the steady states is unstable, i.e. the economy can be in k^L, but bullish expectations lead to the belief that there will be high levels of investment and low interest rates, and individuals will save more, supporting those beliefs in a r.e. trajectory. The economy can wobble infinitely between k^L and k^H.

More generally, there are four states (a)-(d), corresponding roughly to the four situations identified in the more general case for wobbly dynamics. We can describe explicitly which arises depending on parameter values. Recall that in the general case, in each of these four states, there were always three steady states, with the middle always unstable. That is true here, and the middle one is given by $k = k^f$, i.e. just full employment. At the particular distribution of income which supports k^f, there are three possible values of k_{t+1}.

If there exists such a steady state, the steady state values of w^* and $1 + r^*$ have to satisfy $k^f = \frac{\omega_1}{\omega_2} = \frac{1}{1+\frac{a_1}{a_2}(1+r^*)} \frac{w^*+e}{1+n}$ with $1 - \delta \leq 1 + r^* \leq \frac{A}{\omega_1} + 1 - \delta$, where

(20a) $\omega_1 R^* + \omega_2 w^* = A$ (product exhaustion equation)[25]

and

(20b) $1 + r^* = R^* + 1 - \delta$.

By rearranging these equations, we have $\Omega(R^*) = \left[1 + \frac{a_1}{a_2}(R^* + 1 - \delta)\right] \frac{\omega_1}{\omega_2} = \frac{\frac{A}{\omega_2} - \frac{\omega_1}{\omega_2} R^* + e}{1+n} = W(R^*)$.

[25] This also defines the factor price frontier.

We can now identify the parameter space where multiplicity of momentary equilibria can occur. We require[26]

$$\Omega(R^* = 0) = \left[1 + \frac{a_1}{a_2}(1-\delta)\right]\frac{\omega_1}{\omega_2} < \frac{\frac{A}{\omega_2}+e}{1+n} = W(R^* = 0)$$

and

$$\Omega\left(R^* = \frac{A}{\omega_1}\right) = \left[1 + \frac{a_1}{a_2}\left(\frac{A}{\omega_1} + 1 - \delta\right)\right]\frac{\omega_1}{\omega_2} > \frac{e}{1+n} = W\left(R^* = \frac{A}{\omega_1}\right).$$

It is clear that the set of conditions under which there exists wobbly dynamics is identical to the set of conditions under which there exists multiple steady states, including the just full employment of both factors steady state.

There are then four states, corresponding to the four states identified earlier in the more general case.

State (a): There are two stable steady state values of k. If workers expect there to be a surplus of capital next period—so the marginal return is zero—they save a lot; and that results in there being a high level of savings. Conversely if they think there will be a surplus of labor. Once the economy moves, however, to either the capital surplus or capital shortage equilibrium at $t+1$, the following period it converges to the low (high) steady state, remaining there forever.[27]

State (b): Unstable (fragile) economic booms with full employment, while economic stagnation associated with involuntary unemployment is stable and persistent

State (c): This is the case where there are multiple values of k_{t+1} for any k_t. There are three steady states, one with capital shortage, one with capital surplus, and k^f. They are all unstable. Hence the economy wobbles without converging between a state with high investments and high output and full employment and a state with low investments and low output and involuntary unemployment.

State (d): Stable booms and fragile recessions.

In addition to these cases where wobbly dynamics arise, there is the situation corresponding to state (e):

[26] This parameter space corresponds to all parameter space generating states (a)-(d) described more fully below.

[27] Even in state (a) in our general case, if the slope of the backward bending curve is almost vertical, even though multiplicity of momentary equilibria arises between $\underline{k}$ and $\overline{k}$, the economy will converge to either k^H or k^L. In the Leontief case, only at $k_t = k^f$, indeterminacy arises but the economy will converge to either k^H or k^L.

State (e): Unique stable steady state, which here, can be divided into two subcases, where that steady state entails surplus labor (e1) or surplus capital (e2). The former steady state is characterized by the stagnation trap with persistent involuntary unemployment.

Figure 5 illustrates all the states. The figure is derived simply from (18) by seeing if the line $W(k_t < k^f)$ lies above $\overline{\Omega}$ or $W(k_t > k^f)$ lies below $\underline{\Omega}$.

The following Proposition summarizes this result.

Proposition 3 (A complete characterization of all possible patterns of wobbly dynamics): The global dynamics are described by one of the five mutually exclusive states (a) to (e). The necessary and sufficient conditions under which each of these arises are provided in Appendix 1.

A closer look at dynamics: wobbles and cycles

After an initial period, k_t where $k_t \neq k^f$ can take on one of seven values: $k^H, k^L, k^f, k^{LH} \equiv s_H \frac{e}{1+n}, k^{HL} \equiv s_L \frac{\frac{A}{\omega_2}+e}{1+n}, k^{MH} \equiv s_H \frac{w^*+e}{1+n}, k^{ML} \equiv s_L \frac{w^*+e}{1+n}$, where $s_H = \frac{1}{1+\frac{a_1}{a_2}(1-\delta)}$ and $s_L = \frac{1}{1+\frac{a_1}{a_2}(\frac{A}{\omega_1}+1-\delta)}$, i.e., the savings rates when net period returns are expected to be low ($k > k^f$) or high ($k < k^f$) respectively.

k^H is the value of k when at t k is high (wages are high) and k is expected (rationally) to be high next period. k^{LH} is the value of k when at t k is low (wages are low) and k is expected to be high next period, so the savings rate and investment will be high. k^{LH} is defined similarly. So too, k^{ML} (k^{MH}) is the value of k when at t $k = k^f$, and at t + 1 k is (rationally) expected to be lower (higher) than k^f. Note that the value of k at any date t+1 depends only on its value at t and (rational) expectations of its value at t+1.

Ignoring for the moment the unstable momentary equilibria associated with k^f, we can see that there is a unique 2-period cycle, with the economy alternating between k^{LH} and k^{HL}; two possible 3-period cycles (k^{LH}, k^H, k^{HL}) and (k^{LH}, k^{HL}, k^L); 4 possible 4-period cycles ((k^{LH}, k^H, k^H, k^{HL}), (k^{LH}, k^H, k^{HL}, k^L), (k^{LH}, k^{HL}, k^L, k^L), ((k^{LH}, k^H, k^{HL}, k^L), etc.

It is easy to see that there can be wobbles of any periodicity—or of no periodicity, e.g. sunspot equilibria where the economy switches regimes with the occurrence of an odd or even number of sunspots.

5.2. Boundary of regions for Parametric Model

In the case of our Leontief model, the value of the parameters in which each of the states described earlier can arise can be derived analytically. We naturally focus on the boundaries of the regions. The key intuition is provided by Figure 1-2: As we lower, for instance, e, $W(k_t)$ is lowered, and when it is lowered enough, it just "touches" the Ω locus, i.e. we shift from three momentary equilibria to one when $W = \underline{\Omega}$, the value of Ω in the capital surplus regime at $k = k^f$.

We can accordingly derive the boundary values for each of the regimes we have identified. Consider, for instance, the most interesting regime, state (c) where there are multiple momentary equilibria. As noted earlier, the necessary and sufficient condition for this state is (19a) and (19b). (19a) and (19b) can be rewritten as providing conditions for $\frac{A}{\omega_2}$:

$$\frac{\frac{e}{1+n}}{\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}-\frac{1}{1+n}\right)} \leq \frac{A}{\omega_2} \leq \frac{\frac{e}{1+n}}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}}.$$

The lower boundary is thus given by

$$\text{(21a)}\quad \frac{\left(\frac{e}{1+n}\right)\frac{A}{\omega_1}}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)+\frac{A}{\omega_1}\left(\frac{a_1}{a_2}-\frac{1}{1+n}\right)} = \frac{A}{\omega_2},$$

and the upper boundary by

$$\text{(21b)}\quad \frac{A}{\omega_2} = \frac{\left(\frac{e}{1+n}\right)\frac{A}{\omega_1}}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)}.$$

(21a) and (21b) can be used to define the limiting values of any of the six parameters in terms of the remaining five parameters.

Figure 6 provides a diagrammatic depiction. For instance, both the upper and lower boundary are depicted as $\frac{1}{\omega_2}$ being a linear function of e, given all the other parameters.

In a similar way we can derive the boundary values for other states. Changes in the parameters, e.g. induced by changes in technology, induce changes in the state of the economy—moving the economy from one phase to another. We now describe these state transitions.

6: State transitions

6.1. Labor Augmenting changes in technology

The changes we are particularly interested in is the movement from a stable unique equilibrium at a high value of k to wobbly dynamics to a stable low k equilibrium. Technological change can take many forms, and as we shall see the different forms have different implications for the nature of the state transitions. We focus our attention on labor augmenting technology progress, where one labor today can do what several workers could do last year. This is the only form of technological change that can give rise (within a broad class of models) to a steady state. We now show how a labor augmenting change in technology could cause a state transition from a stable state to an unstable state. Then we show conditions under which hysteresis can arise after the collapse of economic booms.

Much macroeconomic analysis in recent decades has attempted to interpret economic fluctuations to the impact of shocks, typically i.i.d. and temporary, on aggregate behavior. A single shock, in the standard DSGE model, has an effect on the momentary equilibrium, typically buffered by stabilizing wage and price adjustments and inventory accumulation and decumulation, with the effects diminishing over time, as the economy returns to its (unique) long run equilibrium. On the other hand, a permanent improvement in productivity leads to an increase in per capita income and in the steady state level of capital.

In our wobbly model, there can be markedly different results: (i) Even temporary productivity shocks to the economy, can have permanent effects on investment, output, and wage rate, i.e., there can be hysteresis effects. (ii) Temporary improvements in productivity could lead to a stagnation trap after generating a temporary unstable economic boom. (iii) If the positive productivity shock is permanent, then the stagnation level after the collapse of a fragile economic boom could become more severe, even though the productivity level is increased permanently.

We first provide a general qualitative analysis of how a temporary or permanent shift to $\frac{1}{\omega_2}$ in the CES production function introduced earlier changes the dynamics of the economy (which of the four cases described earlier applies) before providing a complete analytic characterization in the Leontief case.

In both cases, there are two parts to the analysis (a) How changes in $\frac{1}{\omega_2}$ changes Ω and W, and how that in turn changes the other key variables; and (b) how a permanent shift changes the value of the steady state level of k^H or k^L.

An increase in $\frac{1}{\omega_2}$ decreases the saving rate at each value of k_{t+1}, since it increases the return to capital (it increases the effective labor per unit capital). Thus, the Ω curve shifts up. This increases $\overline{\Omega}$ and $\underline{\Omega}$.

At the same time, it may increase or decrease W. Because it decreases (at each k) the *effective* capital labor ratio, the wage per effective labor unit is lower, but each individual "embeds" more labor units. It is easy to show that the wage increases or decreases depends on whether

$1 > or < \frac{1-S_L}{\sigma}$,

where S_L is the share of labor. Thus, with our constant elasticity production function, with an elasticity less than unity, there is a critical k such that above that k the wage increases (below it, it decreases). Depending on the savings elasticity, the value of k, and the elasticity of substitution, $\overline{k}$ and $\underline{k}$ may accordingly either increase of decrease.

At the same time, were wages fixed at any k_t, k_{t+1} would have been smaller. But the increase in $\frac{1}{\omega_2}$ also affects wages, either increasing or decreasing them, depends on k. The net effect is again ambiguous, and depends again on savings elasticities, the elasticity of substitution and the share of labor. What is critical is that k^H and k^L can either increase or decrease.

Moreover, because the strength of each of these effects depends on k_t, if there are multiple steady states, each can be affected differently. That is, $\underline{k}$, $\overline{k}$, k^L, and k^H can each move differently.

The result is that there are a rich set of possible effects of a change in labor productivity, in particular, several different patterns of phase transitions, e.g. from state (a) to (b), or (d) to (c). One that we will be particularly interested in is the following: an increase in labor productivity strengthens the boom, in the sense that k^H increases: an increase in labor productivity leads to higher income per capita, as one might expect for large k. But if $\overline{k}$ increases more than k^H, as well it might, eventually, the upper steady state becomes unstable: there are multiple momentary equilibrium. An example of this case is depicted in Figure 7, i.e. the figure illustrates a state transition from state (a) to (b). At the original steady state, k^H, there is a unique momentary equilibrium. It is stable. But improvements in technology shift k^H up, but shift the upper bound of the value of k for which there are multiple equilibria up more, so that at the new steady state k^{HH} there are multiple momentary equilibrium.

Meanwhile, the increase in productivity could have actually lowered the k^L: the adverse effect of the increased returns so lowers the savings rate that total savings is reduced. But $\underline{k}$ may have increased, or in any case, not been lowered as much *so that the lower equilibrium remains stable—now it is the only stable equilibrium.* Thus, while the economy was initially in a stable “boom,” the improvement in technology, while strengthening the boom, makes it fragile. An expectation of a weaker economy (low k) changes savings behavior in a self-fulfilling way.

There are alternative possible dynamic paths after the collapse of fragile economic booms. If individuals’ expectations change to bullish again in the next period, then there can once again be an economic boom, although these booms are still fragile. The economy may wobble for an extended period of time, before one of the wobbles is sufficiently adverse that the economy gets pulled into the orbit of the low equilibrium trap, where it remains (until some other shock hits the economy. But note that it will take a large shock to move the economy out of the orbit of the stable low equilibrium trap, and even once out, the exit is only temporary—since the k^L steady state is the only stable steady state.

More formally, the changes in k^H, and k^L depend on how the change in $\frac{1}{\omega_2}$ changes wages (described above) and changes savings:

$\frac{dln(s_t)}{dln(x)} = \frac{dln(s_t)}{dln(1+r_{t+1})}\frac{dln(1+r_{t+1})}{dln(x)}$ with, in the case of the constant elasticity preferences,

$\frac{dln(s_t)}{dln(1+r_{t+1})} = -(1-\theta)(1-s(k_{t+1}))$

(See Appendix 3).

Steady state k^H, and k^L increases or decreases if at the previous level, $\frac{dln(s_t)}{dln(x)} + \frac{dln(w_t+e)}{dln(x)} > or < 0$, that is on whether the effect in decreasing the savings *rate* exceeds the effects of the increased wages. Obviously, if wages decrease, then both effects work in the same direction, so that the steady state level of capital is reduced. Thus, if k^L is small enough, the innovation leads not only to a lower wage at any capital labor ratio but also to a smaller steady state capital labor ratio, thus reinforcing the decrease in wages. At the same time, at the upper steady state, wages may have increased—and increased enough that k^H increases.

Similarly, recalling the definition of $\overline{\Omega}$ as the largest value of Ω for which there are multiple momentary equilibria,

$\frac{\partial \ln(\Omega)}{\partial \ln(x)} = -\frac{1}{s}\frac{dln(s_t)}{dln(x)}$.

We have to compare that with $\frac{dln(w_t+e)}{dln(x)}$. If $\frac{\partial \ln(\Omega)}{\partial \ln(x)} > \frac{dln(w_t+e)}{dln(x)}$ then $\overline{k}$ increases, and if $\overline{k}$ increases more than k^H, we move from a regime with a stable upper equilibrium (state (a)) to one where at the upper equilibrium there are multiple momentary equilibrium and the upper steady state is unstable (state (b)). Note too that because $\frac{\partial \ln(\Omega)}{\partial \ln(x)} >> -\frac{dln(s_t)}{dln(x)}$, if k^H increases only a little, i.e. at k^H, $\frac{dln(s_t)}{dln(x)} + \frac{dln(w_t+e)}{dln(x)}$ is small (which could be because the savings elasticity is large or at k^H, $\frac{1-S_L}{\sigma}$ is small), then $\overline{k}$ will increase markedly, so that eventually $\overline{k}$ will exceed k^H. We thus have the unsavory combination described earlier of an unstable economic boom and a stagnation trap. Of course, while the boom lasts, matters are good: because of the increase in productivity, k^H goes up to k^{HH}.

Temporary vs. Permanent Shocks

Even if the productivity level reverts back to the previous level, so the economy goes back to state (a), since k^L is stable, it may remain in the low equilibrium trap: even temporary productivity shocks could have long lasting, and even permanent effects, i.e., the economy can exhibit strong hysteresis.[28]

[28] This result is different from Brunnermeier and Sannikov (2014), and He and Krishnamurthy (2013). In their studies, it is negative shocks that cause the economy to enter into nonlinear regions with a low level of

There are three different scenarios for a temporary technology change. Figure 8 illustrates this.

The first is that the economic boom might persist until the productivity reverts to the previous level. If this is the case, the economy converges back to the original steady-state, k^H, and the economic boom ends with a mild decline (from k^{HH} to k^H). This case can be interpreted as a normal business fluctuation.

The second is that even if the economy experiences the large-scale-collapse, if the decline is not deep enough, the economy can produce self-recovery and eventually converges back to the original steady-state, even if aggregate output is lower than the trend level temporarily. The dotted line in Figure 8 illustrates this scenario.

The third scenario is that if the decline gets sufficiently deep following the large-scale collapse, the economy can no longer generate self-recovery and ends up in the stagnation trap. The temporary boom has resulted in the economy moving from the upper stable equilibrium to the lower stable equilibrium.

6.2. Complete analytic representation of all the possible state transitions: The Leontief Case

By using the analytically tractable Leontief-case, we can provide a complete analytic representation of all the possible state transitions. A state transition is a change induced by an exogenous change in a parameter. Because of the polar assumptions associated with this case, not all the state transitions that could emerge in the general case are possible; nonetheless, this case illustrates some of the many interesting possibilities.

We show that there exists a critical point where an increase in labor productivity changes global dynamics abruptly from a state with a unique equilibrium to a state with multiple momentary equilibria or vice versa. Not surprisingly, there are a number of possible state transitions—as we noted earlier, a change in say labor productivity affects $k^L, k^H, \underline{k}, and\ \overline{k}$ each in a different way, depending on a number of factors some of which we have already identified. If $\overline{k}$ is initially less than k^H, if it increases more than k^H, which may happen under some

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economic activity. Moreover, the economy eventually converges back to the original steady-state. No stagnation trap or no long run hysteresis occurs. By contrast, in our model, it is positive productivity shocks that could lead to the permanent stagnation trap after generating fragile economic booms. The existence of multiple momentary equilibrium plays a critical role in obtaining these results.

In the more standard dynamics where there is a unique momentary equilibrium, as illustrated in the left panel in Figure 2 (which can arise in our model with different parameter values) a positive productivity shock shifts the equilibrium value of k_{t+1} up for each k_t. If the economy is initially at k^H, the economy simply converges to the now higher equilibrium. Similarly, if it is in the low equilibrium, it may remain there, with the equilibrium value of k increased; or the low equilibrium may actually disappear and converges to k^H. In either case and irrespective of the shock is temporary or permanent, a positive productivity shock leads to an increased k, and will not generate a stagnation trap.

parameters, an initially stable steady state may become unstable, and we have a state transition. But under others, this may not happen.

We can characterize all possible patterns of state transitions in the $(\frac{1}{\omega_1}, \frac{1}{\omega_2})$-plane. Recall that A/ω_1 is the output per unit of capital, so that with a fixed A, an increase in $\frac{1}{\omega_1}$ represents an increase in capital productivity. Similarly, an increase in $\frac{1}{\omega_2}$ represents an increase in labor productivity. Given all the other parameters, there are five different patterns (denoted by "Patterns" A-E)

Pattern A: all states other than state (c) and state (e1) arise.

Pattern B: all states other than state (a) and state (e1) occur.

Pattern C: all states other than state (c) arise.

Pattern D: all states other than state (a) arise.

Pattern E: state (a), state (d), and state (e2) arise, while state (b), state (c), and state (e1) cannot arise.

Proposition 4: Given non-negative finite values of $\frac{1}{\omega_1}, \frac{1}{\omega_2}, \delta, A, n, e, a_1, and\ a_2$, there are five patterns of economic configurations, defined by Pattern A-E. Figure 9 illustrates all patterns in the $(\frac{1}{\omega_1}, \frac{1}{\omega_2})$-plane.[29] The condition for each pattern and the boundary values between regions within each pattern are described in Appendix 4.

Figure 9 helps us see clearly what happens as a result of an increase in labor productivity. We illustrate with one of the many possibilities. In Appendix 5, we examine Hicks neutral changes, where $\frac{1}{\omega_2}$ and $\frac{1}{\omega_1}$ change proportionately.

Consider Pattern B. Suppose labor productivity level is initially such that $\frac{e/(1+n)}{\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\frac{a_1}{a_2}} < \frac{A}{\omega_2}$. This means that the initial state of the economy corresponds to state (e2) where there is a unique momentary equilibrium and a unique steady state equilibrium with full employment. Then with an increase in labor productivity, there exist three critical points of labor productivity, that is, $\left(\frac{A}{\omega_2}\right)^{\#} \equiv \frac{e/(1+n)}{\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\frac{a_1}{a_2}}$, $\left(\frac{A}{\omega_2}\right)^{*} = \frac{e/(1+n)}{\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}-\frac{1}{1+n}\right)}$, and

[29] In the $(e, \frac{1}{\omega_2})$-plane, the boundary lines are linearly increasing functions of e.

$\left(\frac{A}{\omega_2}\right)^{\#\#\#} \equiv \frac{e/(1+n)}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}}$, respectively, at which state transitions occur first from state (e2) into state (d) and then from state (d) into state (c) and then from state (c) into state (b). This means that if the economy is initially in the stable high steady-state with full employment (state (e2)), a large enough increase in labor productivity leads to a completely wobbly economy (state (c)) in which the economy endogenously fluctuates between full-employment and involuntary unemployment regions. The steady-state equilibrium in which the economy had been loses its stability abruptly as the result of the existence of the multiple momentary equilibria that then appear. With multiple momentary equilibria, expectations of a low level of investments can be self-fulfilling, and there is the possibility that the boom will collapse. Initially, with labor productivity between the threshold $\left(\frac{A}{\omega_2}\right)^{\#}$ and $\left(\frac{A}{\omega_2}\right)^{*}$ though, there remains the possibility of fluctuations in which the economy recovers. But if labor productivity increases still further, the dynamics change again to state (b): the economy is then characterized by unstable economic booms with full employment and a stagnation trap with involuntary unemployment.

The intuition is the same as before, but the change in $\bar{k}$ takes a particular form—it is just k^f. Thus, an initially stable steady state, where $k^f < k^H$ becomes unstable if the increase in k^f induced by an increase in labor productivity exceeds the increase in k^H. Two conflicting forces are produced by the increase in labor productivity. One effect is that the wage rate increases. And because the return to capital in the capital surplus regime remains unchanged, the savings rate remains unchanged, so k^H increases. k^f increases proportionately with $\frac{A}{\omega_2}$; on the other hand, while w increases proportionately, $w + e$ increases less than proportionately, so that eventually $k^f > k^H$, and the economy moves from a phase with a stable upper steady state to one where at k^H there are multiple momentary equilibria.

This result has one further implication: if the state of the economy is initially close to critical points, even a small increase in labor productivity could produce a large change in global dynamics. .

Proposition 4 shows that an improvement in technology, normally thought to be unambiguously good, may have long run adverse effects: it may end up destabilizing the economy by generating multiplicity of equilibrium or and/or generate dynamic inefficiency. Later, we will identify government policies that can ensure both stability and efficiency.

6.3. Interpretation: Under what conditions can a large-scale collapse and a stagnation trap occur?

Our analysis may provide some insights into the following three related questions. The first concerns the observation of Kindleberger (1978) and Scheinkman (2014) that historically unstable macro dynamics tend to occur when technological innovations arrive. What might be the connections? The second question is that some booms lead to a large-scale collapse or crises; but this is not so for all booms. Under what conditions can economic booms result in a large-scale collapse? The third question relates to hysteresis: after the collapse of some economic booms, output levels and trends became permanently lower than those pre-boom and-crisis. In some cases, a stagnation trap seems to have emerged. Under what conditions can this happen? More generally, when does the economy exhibit strong hysteresis effects?[30]

In our model, small changes in technology can have beneficial effects, as expected, but large changes may change the structural stability properties of the system, especially when substitution effects in consumption are small relative to income effects and there are large distributional effects—so that there exist equilibria where high rates of return induce low levels of savings. The economy enters into a state with multiple equilibria and as a result there is a possibility of the boom being followed by a large-scale collapse and a stagnation trap. There is a discontinuity in the macroeconomic system—the state transition to which we referred in the introduction.

We might be inclined to label a boom followed by stagnation as a bubble: the fundamentals of a strong, sustainable economy were "evidently" missing. But that would be wrong. It was, indeed, possible for the boom to have been sustained—if only the belief that we were in the k^H equilibrium was sustained. The change in technology meant, however, that there were other possible (rational expectations) equilibria, and there was no reason to believe that that equilibrium would be sustained.

Going beyond our model with its strong parameterizations and structures, it is clear that a temporary change in technology can easily create conditions in which (rationally or irrationally) there might be multiple equilibria, and in which the movement to a "new equilibria" has long run effects which persist even after the technology has reverted. Structured finance may have helped create a housing bubble (it was not inevitable that it do so; there were plausibly other equilibria), but the marked changes in wealth distribution that resulted from the breaking of the housing bubble can have long run (indeed, in some models, permanent) effects.

Of course, even if the economy goes through a state transition to a state where it is *possible* that there is a collapse followed by stagnation (what we identified as state (b)), the economic boom does not necessarily result in the large-scale collapse and the stagnation trap, as we have noted.

[30] We emphasize that there are alternative, and in many cases, more persuasive, explanations (Kindleberger, for instance, emphasizes the irrational bubbles often associated with large technology changes (perhaps like the tech bubbles in the late 90's)—markedly different from the rational expectations framework employed in this analysis. Moreover, the critical omission of Keynesian effects means that our model can't provide a full description of the crises that often follow booms. To reiterate the caveat from the introduction: the analysis here is to be seen mostly as an exercise in pure theory. Still, the forces that it identifies will be present in much more realistic and complex models; and the failure of standard macro models to reflect these forces may constitute a serious omission.

7. Increases in e and balanced growth

The previous section considered the effect of a one-time change—either permanent or temporary in labor productivity. But as, say, labor productivity increased, e, each generation's endowment was kept fixed. There is, in this sense, unbalanced growth; the increase in the productivity in one sector, in a sense, induces structural change. The structural change is what gives rise to the instability.

In our analysis, e is introduced exogenously but we could microfound our formulation by considering the following household setting. Each household consists of two members. One supplies one unit of labor to firms which produce output by using capital and labor. The other also supplies one unit of labor totally to a sector where labor is the only input for production, by which e units of consumption goods is produced. Under this setting, total income to a household is $w_t + e$. With this interpretation, the presence of e is equivalent to introducing another sector or another technology, i.e., heterogeneity. Hence Proposition 4 shows possible patterns of state transitions when there is an increase in productivity in one sector, while there is no such a change in another sector, as we shall shortly see.

(There is another interpretation. Assume each household has a parcel of land, which, without labor, yields e. So long as land is not tradeable, our analysis applies. There is no reason that an increase in productivity in manufacturing would be accompanied by a commensurate increase in productivity in agriculture. If, however, land is tradeable, that introduces another asset—and land speculation can both crowd out capital accumulation and lead to new dimensions of instability, and we explore in a sequel to this paper.[31])

Growth

So far, our analysis has focused on the consequence of a one-time change in one of the key parameters (e.g. the level of labor productivity.) But we can also ascertain what happens when the *rate* of (labor augmenting) productivity changes. Interpreting now k as the *effective capital labor ratio* and n as the *sum of the rate of reproduction and the rate of labor augmenting technological progress,* we can easily ascertain the effects of a change in n (or a change in the rate of labor augmenting technological progress). The Ω function does not depend on n, but W does: It follows that an increase in n increases $\overline{k}$ and $\underline{k}$. At the same time, both k^L and k^H are lowered.

[31] Similarly, it might be noted that converting "labor" into an asset, i.e. slavery, can also crowd out productive investment. At the time of the onset of the US Civil War, "slaves" represented a large fraction of the "wealth" in the South.

For our purposes, the most interesting implication is that the economy could initially have been in state (a), with a stable high k equilibrium; an increase in the rate of labor augmenting technological progress, while increasing the pace of increase in standards of living, lowers the steady state value of the effective capital labor ratio; and if the rate of productivity increases enough, the "boom" becomes unstable ($\overline{k}$ exceeds k^H). That is, there is a state transition to state (b). The economy may then go through a period of instability. But even were the economy to revert to the initial level of productivity increase, the economy may nonetheless get stuck in the low equilibrium (k^L)—the economy again exhibit strong hysteresis effects. (see Figure 9 in Appendix for possible patterns of state transitions as n changes.) The result is the possibility that the one time boom in labor productivity, instead of delivering the upward shift in the level of standards of living that might have been expected, does just the opposite.

7: Extensions[32]

7.1: Wobbly dynamics including dynamic efficiency

Earlier studies of life cycle models have emphasized the over-savings problem, i.e., dynamic inefficiency, raised by Diamond (1965). While in the extended example developed in this paper, wobbly dynamics is associated with episodic (but not permanent) dynamic inefficiency, dynamic inefficiency is not a necessary condition for wobbly dynamics to arise.

The easiest way to see this is to assume that capital, rather than depreciating, increases automatically in value, i.e., negative depreciation, $\delta < 0$. The gross interest rates can easily be greater than the economy's growth rate, $1 + n$, so the economy is dynamically efficient. But there is no change mathematically, and so wobbly dynamics can arise.

Proposition 5: Dynamic inefficiency is not a necessary condition for wobbly dynamics to arise. Even if the economy is dynamically efficient, wobbly dynamics can occur.

7.2: Non-homothetic preference

For simplicity, so far we have assumed homothetic preferences. We will now show that it is even easier to get multiplicity of momentary equilibrium when a utility function takes the form of non-homothetic preference. Even with a Cobb-Douglas production function, multiplicity of momentary equilibrium can occur.

Consider the following Stone-Geary preference with $1/\lambda$ being the IES.

[32] In Appendix 6, we also discuss myopic dynamics.

$$u_t = \frac{(c_{1t}-\gamma_1)^{1-\lambda}-1}{1-\lambda} + \beta \frac{(c_{2t}-\gamma_2)^{1-\lambda}-1}{1-\lambda},$$

Solving the maximization problem yields the savings/capital accumulation equation:

$$k_{t+1} = \frac{1}{1+n} \frac{w_t+e-\gamma_1-\left(\beta(1+r_{t+1})\right)^{\frac{-1}{\lambda}}\gamma_2}{1+(\beta)^{\frac{-1}{\lambda}}(1+r_{t+1})^{\frac{\lambda-1}{\lambda}}}.$$

Then the function Ω becomes

$$\Omega(k_{t+1}) \equiv k_{t+1}\left[1+(\beta)^{\frac{-1}{\lambda}}(1+r_{t+1})^{\frac{\lambda-1}{\lambda}}\right] + \gamma_1 - \left(\beta(1+r_{t+1})\right)^{\frac{-1}{\lambda}}\gamma_2$$

while W remains unchanged:

$$W(k_t, e) = \frac{w_t+e}{1+n}$$

Let us consider a Cobb-Douglas production function given by $Y_t = AK_t^{\alpha}L_t^{1-\alpha}$ and for simplicity we assume capital fully depreciates after production ($\delta = 1$). Then the return to holding capital is equal to $A\alpha k_t^{\alpha-1}$, which is equal to the gross interest rate, $1 + r_{t+1} = A\alpha k_t^{\alpha-1}$.

Consider the limiting case $1/\lambda \to 1$; the above utility function becomes

$$u_t = \log(c_{1t} - \gamma_1) + \beta\log(c_{2t} - \gamma_2),$$

where γ_1 and γ_2 can be interpreted as the minimum consumption level when young and old, respectively.

The function Ω can be rewritten as

$$\Omega(k_{t+1}) = \frac{1+\beta}{\beta} k_{t+1} + \gamma_1 - \frac{\gamma_2}{A\alpha\beta} k_{t+1}^{1-\alpha}.$$

It is straightforward to establish that the function $\Omega(k_{t+1})$ has the following property.

$$\Omega(k_{t+1} = 0) = \gamma_1 \quad \text{and} \quad \lim_{k_{t+1}\to 0} \Omega'(k_{t+1}) < 0 \quad \text{and} \quad \Omega''(k_{t+1}) > 0.$$

That is, the function Ω is non-monotonic (a convex function) with respect to k_{t+1}. Hence for some k_t, there are two values of k_{t+1} satisfying $\Omega(k_{t+1}) = W(k_t)$, so that wobbly dynamics can easily arise.

8: Government policy

As we have showed, laissez-faire market economies can be wobbly. Moreover, it can also be inefficient. Here we show how government can increase social welfare.

There are three distinct kinds of welfare losses: The first has been extensively discussed in the life cycle literature--over-saving, to the point where the return to capital is lower than the growth rate of the economy (it will be negative without growth). This leads to *Pareto* inefficiency.

The easiest case to see that that can arise is the Leontief model studied earlier. In that case, if k_t is greater than k^f, there is idle capital and the economy is dynamically inefficient, i.e. $1 - \delta < 1 + n$, which is satisfied if $\delta \geq 0$, since the economy has sacrificed current consumption for low or possibly zero returns.

A government policy to transfer some income from next period's young to the elderly then would induce less savings, making both the t-th and the t+1-th generation better off.

The economic significance of this dynamic inefficiency has been questioned, because in standard models it does not arise when there is an alternative asset—land or money. Consider the case of zero labor growth and technological change. With land bearing positive rents, the value of land becomes infinite as the return to capital goes to zero, and so there cannot be over saving in productive assets. But in a sequel to this paper (Hirano and Stiglitz 2021b), we show that that conclusion is not general: in a slight modification of the model of this paper where we allow land, wobbly dynamics may arise in which, at least for some periods there is dynamic inefficiency.

The second two are associated with a loss of social welfare within a broad class of equalitarian social welfare functions. Volatility in income, and more importantly, consumption gives rise to losses in social welfare that can be of first order importance. And this can be especially so when this is accompanied by volatility in capital accumulation, so that the marginal product of labor is very high in some periods, and low in others. The Leontief model is the extreme case, where the deficiency in savings in some periods is so great that workers suffer from involuntary unemployment.

In that model, we can easily see how interventions might increase social welfare. Consider a "wobble" in which the economy moves from $k > k^f$ (capital surplus) to $k < k^f$ (capital shortage). Consumption in the t-th generation is $s_t\left(e + \frac{A}{\omega_2}\right)$ while consumption in the t+1-th generation is $s_{t+1}e$. Since $s_{t+1} \leq s_t$ and $\frac{A}{\omega_2} > 0$, the utility of the t-th generation is greater than that of the t+1-th generation, and the (social) marginal utility is less: $U_t > U_{t+1}$, $U_t' > U_{t+1}'$. A lump sum tax on the t-th generation with proceeds invested in capital goods and with the proceeds of those investments distributed as lump sum redistributions to the young of the t+1-th generation would accordingly increase $U_t + U_{t+1}$. Thus social welfare is increased if we use a Benthamite social welfare function; and would increase even more if we have a convex equalitarian social welfare function.[33]

[33] Moreover, with appropriate non-linear tax interventions, we can induce the young in the t+1-th generation not to save any more than they would have saved along the original trajectory, so that all other generations consumption is unaffected.

In a sequel (Hirano and Stiglitz 2021a), we provide a full analysis of government monetary and fiscal policies to achieve stability and to maximize social welfare. Here, we want to provide a brief discussion, using the analytically tractable Leontief case, to show how public policy can ensure just the right amount of savings to maintain the economy at k^f. Doing so achieves Phelps' "golden rule," where, for instance, steady state (average) utility (social welfare using a Benthamite social welfare function) is maximized.[34] Earlier, we noted that either k^f was an unstable steady state equilibrium—there were multiple momentary equilibria, in states (a)-(d) or that k^f was not a steady state (states (e1) or (e2), where the only steady states entailed surplus labor or capital).

In states (a)-(d), there existed a momentary equilibrium that just entails full employment but that the full employment momentary equilibrium is not stable. If individuals hold bullish or bearish expectations, the economy will wobble off that steady state. We now show that appropriately chosen government policy can achieve that full employment state as the unique momentary equilibrium and the unique steady state equilibrium.

Assume, in particular, the government announces that with a capital income and a wage tax/subsidy, it fully commits to assuring an after tax return of r^* (and a corresponding wage of w^*), the factor prices described earlier which just sustain the steady state with $k = k^f$ If individuals believe it, then the only possible rational expectations equilibrium is r^* and w^*. That in turn implies that the steady state with full employment k^f can be achieved as the unique r.e. equilibrium.

Next, we consider government policies in those situations where there is a unique steady state and no wobbles. We first examine state (e2) where there is a unique momentary equilibrium and a unique steady state equilibrium with high output and full employment, but the economy is dynamically inefficient as a result of over-saving. With an economy with oversaving (that is, which is dynamically inefficient) every generation can be made better off: if savings/investment is reduced, output in every period can be exactly the same, so that consumption at every date, and utility of every generation, can be increased.

To see the existence of an allocation that generates k^f every period, recall that at k^f, the distribution of income is indeterminate, so that all that is needed is that there exists a solution to s(r) (w(r) + e) = k^f (where w(r) is defined by the product exhaustion equation 20a). With a capital subsidy (which reduces the savings rate) financed by a lump sum tax on workers (the two together ensuring that total savings is reduced), we can achieve just the required savings. Again, we can ensure that this is the unique momentary equilibrium (at k^f) through the imposition of non-linear capital taxes/subsidies described above

Finally, we examine state (e1) where the level of investments and output is low and involuntary unemployment arises. Because the economy is not dynamically inefficient, it is not

[34] Note that if $k^* < k^f$, the return to capital is greater than the rate of growth, and if $k > k^f$, it is less.

possible to make every generation better off. But it is easy to show that social welfare may easily be increased.

Consider the consequences of levying a capital tax τ^c at time t+1, with the proceeds redistributed to young workers. We consider a small perturbation, and assume the government does not intervene at any date other than $\{t, t+1, t+2\}$, so the overall pattern of the trajectory is unchanged.

National income at t is unchanged (it is determined by k_t). But Δk_{t+1} equals the change in savings, which, as a result of the lower after tax return to capital, is $\frac{\frac{a_1}{a_2}\tau^c r W_t L_t}{(1+\frac{a_1}{a_2}(1+r)(1+\frac{a_1}{a_2}(1+(1-\tau^c)r))}$. Moreover, because of the transfers to the young at $t+1$, k_{t+2} increases. Hence the capital tax increases output at time $t+1$ and $t+2$, leaving it, and wages, incomes, and returns to capital unaffected at all other dates. Moreover, since k increases at time $t+1$ and $t+2$, involuntary unemployment decreases. Utility of the t-th generation decreases (its wage income is unchanged, but it faces a lower return on its investments), but the utility of the t+1-th generation increases.

Hence, to ascertain the effect on social welfare, we need only to compare the decrease in consumption in the t-th generation (say when young with the increase in that of the next generation.

$$\Delta C_t = -\frac{\frac{a_1}{a_2}\tau^c \mathrm{r} W_t}{(1+\frac{a_1}{a_2}(1+r)(1+\frac{a_1}{a_2}(1+(1-\tau^c)r))},$$

where ΔC_t is the change in consumption of t-th generation in the first period of his life.

And

$$\Delta C_{t+1} = \frac{\frac{a_1}{a_2}(1+r)}{(1+\frac{a_1}{a_2}(1+r)}\frac{\tau^c r W_t/(1+n)}{(1+\frac{a_1}{a_2}(1+(1-\tau^c)r))}$$

It is thus apparent that $\Delta C_{t+1} > -\Delta C_t$ if (and only if) $1+r > 1+n$, i.e., the economy is dynamically efficient. For a small perturbation around the steady state, with any social welfare function satisfying the condition of non-discrimination (i.e. when $U_t = U_{t+1}$, $SW_t(U_{1\ldots\ldots}U_t, U_{t+1\ldots\ldots}) = SW_{t+1}(U_{1\ldots\ldots}U_t, U_{t+1\ldots\ldots})$) and convexity, it is clear that social welfare is increased.

Again, in this case, we can support the just-full employment equilibrium. We know that $s(r(w = 0))e < k^f$, that $s(r(w))(w + e)$ is increasing in w, and that $s(r(w = \frac{A}{\omega_2})\left(e + \frac{A}{\omega_2}\right) < k^f$. But a tax on the return to capital, with the proceeds distributed to workers, simultaneously lowers s and increases W (workers total income, including the lump sum redistribution). A high enough tax rate will induce a high enough savings rate so that $s(e +$

$\frac{A}{\omega_2} + T) = k^f$, where T is the corresponding lump sum distribution to workers. We summarize in the following Proposition.

Proposition 6: With credible commitment to a system of non-linear capital income taxes/subsidies with corresponding wage subsidies/taxes, the government can achieve just full employment k^f as the unique r.e. equilibrium. In the case where the only steady state entails either unemployment of labor or capital, the government will have to impose in addition in equilibrium either an interest income tax/subsidy with a corresponding wage subsidy/tax. In situations where there is (even temporary) dynamic inefficiency, government interventions can be Pareto improving. In situations where there is unemployment, government interventions can be welfare improving.

9: Concluding remarks

This paper has developed a model providing a markedly different picture of the dynamics of capitalism from that associated with the standard representative agent model. Using the standard life-cycle model, which is the simplest model with heterogeneous agents and finite lives, we have shown that under not implausible conditions, starting from any set of initial conditions k_0 there are in general an infinite number of r.e. trajectories, neither converging nor diverging. In particular, it is possible that all steady states of the economy are unstable; even so, we can define precise bounds within which the economy's wobbles have to occur (along rational expectations trajectories). On such "wobbles" the economy neither converges nor diverges. While there may exist periodic cycles, the wobbles do not necessarily have to have any periodicity. While in the model explored in detail, these trajectories exhibit temporary—but only temporary—dynamic inefficiency, such dynamic inefficiency is not necessary for wobbly dynamics. When there are wobbly dynamics, government intervention can be welfare enhancing.

Underlying the complex dynamics is the multiplicity of momentary equilibria which we have shown can easily arise. We have fully explored the implications of multiplicity of momentary equilibria for global macro-dynamics. The characteristics associated with wobbly dynamics is that the state of the economy endogenously changes from a state with a unique momentary equilibrium into a state with multiple momentary equilibria, or vice versa, which we have called a phase transition. Depending on how phase transitions occur, various patterns of wobbly dynamics can occur. We have identified all possible patterns of wobbly dynamics, providing, in the context of the parametric model we have investigated in detail, a complete characterization of the parameter values under which each may occur.

In particular, we have shown that (a) whether there exists multiple equilibria may depend on the value of k, the key state variable, so that there can exist endogenous transitions from situations where there is a unique equilibrium to one where there are multiple equilibria and vice versa; and (b) the "state of the economy," e.g. whether it is characterized by unique or multiple steady states, whether the steady state is stable or unstable, and whether there are wobbly dynamics, depends critically on key parameters, with the economy going through what we have called state changes as those parameters pass through critical values. (In the special case of Leontief production functions, we have provided a complete analytic representation of all the possible state transitions.)

Innovation and induced instability

We have focused in particular on how changes in technology can change global dynamics abruptly, leading to several types of state transitions, e.g. where the economy switches from having a stable boom to one in which the boom, while stronger, becomes fragile. That is, an economy enters into a region where there are multiple equilibria, leading to the possibility of a large-scale collapse and a stagnation trap characterized by involuntary unemployment. Thus, we have shown, for instance, how a temporary positive productivity shock can have long run adverse effects.

In other cases, an increase in productivity ends up generating a completely wobbly economy, even if there is initially a unique momentary and a unique high steady-state with full employment.

History versus Expectations

Macroeconomics has traditionally emphasized the importance of expectations and history. In our model, both history and expectations play a role not just in fluctuations, but also in growth and development. In our model, with some parameters—when there is a unique momentary equilibrium—only history matters. In such situations, illustrated in the left panel in Figure 2, long run development is determined by history, by initial conditions, whether the initial capital stock is less or greater than k^M. On the other hand, in situations where there is a multiplicity of momentary equilibria, expectations are crucial. In general, both history and expectations matter.

In state (a), for instance, unless the initial capital stock is sufficiently high or sufficiently low, expectations are decisive. In state (d) (state (b)), even if the initial capital stock is small (high), the economy may converge to the high steady state, k^H(the low steady state, k^L). In state (c), economic development can be fragile because it totally depends upon the entrepreneurial spirits.

Moreover, there is an important strand in economics which has emphasized the role of forward looking, rational expectations in stabilizing the economy. In the standard model, rational expectations serve to narrow down the set of possible dynamic paths (indeed, to a single trajectory). While it is clear that there are some contexts in which that is so, our paper has shown that that result is not general: within a complex general equilibrium system, rational expectations can lead to a plethora of equilibria, and dynamics with rational expectations may be more unstable than with, say, myopic expectations.

Finally, we note that in our model *hysteresis abounds.* We have shown, for instance, how as a result of a *temporary* short-lived productivity boom, the economy might move from a seemingly stable "boom" economy with full employment and a high level of k to a stagnation trap.

Generalizations

We have employed the two-period overlapping generations model because it is the simplest model with heterogeneous agents with finite lives. Heterogeneity is crucial for multiplicity of momentary equilibrium to arise. To be sure, a simple model such as that developed here oversimplifies, in particular in assuming that there are only two periods. With a period of half a generation, such theoretical models are not well suited for a quantitative analysis of short-term cyclical fluctuations. Yet, the key intuitions concerning the possibility of multiplicity of momentary equilibria for plausible parameter values of the economy and the dynamic patterns that we have uncovered are robust, as long as the lifetime of individuals is finite. An even richer set of dynamics can emerge from more realistic life-cycle models in which individuals work for N periods, followed by M periods of retirement. The key qualitative dynamic patterns can be demonstrated in an analytically tractable two-periods life cycle model.

It is easy to show that an economy with a mix of individuals—a mass of whom only live a finite life (not matter how long)—exhibits dynamics more akin to that described here than to that where it is assumed that *all* individuals are infinitely lived.[35]

Moreover, as Woodford (1986) showed, the mathematical structure of the overlapping generations model is formally analogous to that of infinitely lived agents models with borrowing constraints in which some agents are liquidity-constrained, while others are not. The behavior of economic agents that expect (never expect) to be financially constrained is much like that of finite (infinite) lived agents as described in the current paper. In this interpretation of our model, the "one period" in the overlapping generations model does not have to be the biological working life span and could be relatively short.[36]

Life cycle models have one important property: the distribution of income matters (here, just between the young and the old). When the distribution of income matters, multiplicity of momentary equilibria can easily arise.[37] The aging process itself gives rise to heterogeneity, and

[35] The intuition is straightforward, and illustrated by the case where the representative infinite lived individuals have logarithmic utility functions.

[36] Woodford (1988) also showed that the mathematical structure for the existence of sunspot equilibria in an overlapping generations monetary exchange economy as shown in Azariadis (1981) is identical to that for the existence of sunspot equilibria in an infinitely lived agents monetary model in which the cash-in-advance constraint is always binding.

[37] As we have already mentioned, this was noted in the early growth literature, by Uzawa (1961,1963), but it

of a kind that can give rise to distributional impacts that are quantitatively significant and which can help support a multiplicity of momentary equilibria. We have established these results in the simplest life cycle model even when individuals within any generation are identical. It is even easier to generate multiplicity of momentary equilibria when there is heterogeneity within an age cohort or when we introduce further complexity such as land and credit.[38] Our sequel Hirano and Stiglitz (2021b) fully explores wobbly dynamics with land speculation in which endogenous phase transitions recurrently occur, and Hirano and Stiglitz (2021c) introduces credit where capital and land are used as collateral and demonstrates how financial deregulations lead to wobbly credit cycles with endogenous phase transitions.

While earlier literature demonstrated the possibility of multiple (sunspot) momentary equilibria in consumption/endowment and monetary economies, we have shown that they arise naturally in a standard production economy with a neoclassical production function and generate complex but still tractable dynamics. At the very least, the analysis provides a warning about taking too seriously the dynamics of economies with infinitely lived individuals generating unique convergent paths to steady state, and in which hysteresis effects are at most short lived. Wide economic fluctuations in economic activity are fully consistent with rational expectations.

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is a theme that has seemingly been lost with the dominant of the representative agent models.

[38] Assume, for instance, that there are two types of workers with different skills. Type A has a skill that is more in demand when investment is high, type B when consumption is high. Then it is easy to show that there may be multiple momentary equilibria even with a fixed savings rate for each type, if type A saves a higher fraction of its income than Type B. There is an equilibrium in which there is a heavy level of investment, a high income for type A workers, and a high average savings rate, because more of the income accrues to type A workers. There is another equilibrium with a low aggregate savings rate and a low investment rate. See Uzawa (1961, 1963), though his analysis relied critically on the effect of the composition of demand on the distribution of income, while our analysis relies critically on the effect of interest rates on the composition of demand.

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Appendix 1: Necessary and sufficient conditions for state (a)-(e), respectively.

■ **State (a):** The necessary and sufficient condition for this case to arise is $\bar{\Omega} > \frac{e}{1+n}$ and $\bar{\Omega} < \frac{\frac{A}{\omega_2}+e}{1+n}$. Solving this condition for $\frac{A}{\omega_2}$ shows that if $\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}-\frac{1}{1+n}\right)>0$, this case arises in $\frac{e/(1+n)}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}} < \frac{A}{\omega_2} < \frac{e/(1+n)}{\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}-\frac{1}{1+n}\right)}$. It is clear that if $\frac{a_1}{a_2} < \frac{1}{1+n}$, there exist parameter values of $\frac{A}{\omega_2}$ for which this can be satisfied. If $\frac{a_1}{a_2} > \frac{1}{1+n}$, this condition cannot be satisfied.

If $\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}-\frac{1}{1+n}\right)<0$, $\bar{\Omega} < \frac{\frac{A}{\omega_2}+e}{1+n}$ is automatically satisfied. Hence solving $\underline{\Omega} > \frac{e}{1+n}$ for $\frac{A}{\omega_2}$ shows that this case arises if $\frac{e/(1+n)}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}} < \frac{A}{\omega_2}$. It is thus clear that if $\frac{a_1}{a_2} \geq \frac{1}{1+n}$, i.e. if there are wobbly dynamics, this case cannot arise.

■ **State (b):** The necessary and sufficient condition for this case to arise is $\underline{\Omega} > \frac{e}{1+n}$ and $\underline{\Omega} < \frac{\frac{A}{\omega_2}+e}{1+n} < \bar{\Omega}$. Solving this condition for $\frac{A}{\omega_2}$ shows that if $\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]-\left(\frac{1}{1+n}\right)>0$, this case arises if

(A.1) $\mathrm{Max}\left\{\frac{e/(1+n)}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}}, \frac{e/(1+n)}{\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}-\frac{1}{1+n}\right)}\right\} < \frac{A}{\omega_2} < \frac{e/(1+n)}{\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]-\left(\frac{1}{1+n}\right)}$.

If $\frac{1}{1+n}-\frac{a_1}{a_2} < \left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1} < \frac{1}{1+n}$, $\underline{\Omega} < \frac{\frac{A}{\omega_2}+e}{1+n}$ is automatically satisfied. Hence, solving $\underline{\Omega} > \frac{e}{1+n}$ and $\frac{\frac{A}{\omega_2}+e}{1+n} < \bar{\Omega}$ for $\frac{A}{\omega_2}$ shows that this case occurs if $\mathrm{Max}\left\{\frac{e/(1+n)}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}}, \frac{e/(1+n)}{\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}-\frac{1}{1+n}\right)}\right\} < \frac{A}{\omega_2}$. It is clear that (i) if $\frac{a_1}{a_2} \geq \frac{1}{1+n}$,so there exists wobbly dynamics, there exists values of parameters satisfying (A.1), since $\frac{e/(1+n)}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}} > \frac{e/(1+n)}{\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}-\frac{1}{1+n}\right)}$. Similarly (ii) if $\frac{a_1}{a_2} < \frac{1}{1+n}$, i.e. there exists no wobbly dynamics, (A.1) is satisfied, since $\frac{e/(1+n)}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}} < \frac{e/(1+n)}{\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}-\frac{1}{1+n}\right)}$On the other hand, if $\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}\right) < \frac{1}{1+n}$, this case cannot occur.

■ **State (c):** The necessary and sufficient condition for this case to arise is $\underline{\Omega} < \frac{e}{1+n}$ and $\frac{\frac{A}{\omega_2}+e}{1+n} < \bar{\Omega}$. Solving this condition for $\frac{A}{\omega_2}$ shows that if $\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}-\frac{1}{1+n}\right) > 0$, this case arises if $\frac{e/(1+n)}{\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}-\frac{1}{1+n}\right)} < \frac{A}{\omega_2} < \frac{e/(1+n)}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}}$. And it is clear that if $\frac{a_1}{a_2} > \frac{1}{1+n}$, there exist parameter values of $\frac{A}{\omega_2}$ for which this can be satisfied. On the other hand, if $\frac{a_1}{a_2} \leq \frac{1}{1+n}$, this case cannot occur.

■ **State (d):** The necessary and sufficient condition for this case to arise is $\underline{\Omega} < \frac{e}{1+n} < \bar{\Omega}$ and $\frac{\frac{A}{\omega_2}+e}{1+n} > \bar{\Omega}$. Solving this condition for $\frac{A}{\omega_2}$ yields the following. If $\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}-\frac{1}{1+n}\right) > 0$, this case occurs if

(A.2) $$\frac{e/(1+n)}{\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}\right)} < \frac{A}{\omega_2} < Min\left\{\frac{e/(1+n)}{\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}-\frac{1}{1+n}\right)}, \frac{e/(1+n)}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}}\right\}.$$

If $\frac{a_1}{a_2} \geq \frac{1}{1+n}$ (there exists wobbly dynamics), it is clear that there exists some values of the parameters for which (A.2) is satisfied, since $\frac{e/(1+n)}{\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}-\frac{1}{1+n}\right)} < \frac{e/(1+n)}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}}$. So too, if $\frac{a_1}{a_2} < \frac{1}{1+n}$, there exists some values of parameters for which (A.2) is satisfied, since $\frac{e/(1+n)}{\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}-\frac{1}{1+n}\right)} > \frac{e/(1+n)}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}}$.

If $\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}-\frac{1}{1+n}\right) < 0$, $\frac{\frac{A}{\omega_2}+e}{1+n} > \bar{\Omega}$ is automatically satisfied. Hence when we solve $\underline{\Omega} < \frac{e}{1+n} < \bar{\Omega}$ for $\frac{A}{\omega_2}$, we learn that this case occurs if $\frac{e/(1+n)}{\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}\right)} < \frac{A}{\omega_2} < \frac{e/(1+n)}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}}$.

■ **State (e1):** The necessary and sufficient condition for this case to arise is $\frac{\frac{A}{\omega_2}+e}{1+n} < \underline{\Omega}$. Solving this condition for $\frac{A}{\omega_2}$ yields the following. If $\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]-\left(\frac{1}{1+n}\right) > 0$, this case occurs in $\frac{A}{\omega_2} > \frac{e/(1+n)}{\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]-\left(\frac{1}{1+n}\right)}$. On the other hand, if $\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]-\left(\frac{1}{1+n}\right) < 0$, $\frac{\frac{A}{\omega_2}+e}{1+n} < \underline{\Omega}$ cannot be satisfied, so this case cannot occur.

■ **State (e2):** The necessary and sufficient condition for this case to arise is $\frac{e}{1+n} > \bar{\Omega}$. Solving this condition for $\frac{A}{\omega_2}$ shows that this case occurs if $\frac{A}{\omega_2} < \frac{e/(1+n)}{\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}\right)}$.

Appendix 2: Patterns of dynamics

The set of possible patterns of dynamics depends on key parameter values. There are just five possible "patterns" of economies. We derive the condition for each type and the boundary values between regions within each type.

Let us define the following.

$$\frac{e/(1+n)}{\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}\right)} \equiv \left(\frac{A}{\omega_2}\right)^{\#}, \frac{e/(1+n)}{\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}-\frac{1}{1+n}\right)} \equiv \left(\frac{A}{\omega_2}\right)^{\#\#}, \frac{e/(1+n)}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}} \equiv \left(\frac{A}{\omega_2}\right)^{\#\#\#},$$

$$\frac{A}{\omega_2} > \frac{e/(1+n)}{\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]-\left(\frac{1}{1+n}\right)} \equiv \left(\frac{A}{\omega_2}\right)^{\#\#\#\#}.$$

The condition for each pattern and the boundary values between regions within each pattern as the value of $\frac{A}{\omega_2}$ increases are given by the following.

■ If $\max\left\{\frac{a_1}{a_2}, \frac{\left(1+\frac{a_1}{a_2}(1-\delta)\right)}{\frac{A}{\omega_1}}\right\} < \frac{1}{1+n} < \frac{\left(1+\frac{a_1}{a_2}(1-\delta)\right)}{\frac{A}{\omega_1}} + \frac{a_1}{a_2}$, we have Pattern A. The boundary values between regions are given by $\left(\frac{A}{\omega_2}\right)^{\#}$, $\left(\frac{A}{\omega_2}\right)^{\#\#\#}$, and $\left(\frac{A}{\omega_2}\right)^{\#\#}$.

■ If $\frac{\left(1+\frac{a_1}{a_2}(1-\delta)\right)}{\frac{A}{\omega_1}} < \frac{1}{1+n} < \frac{a_1}{a_2}$, we have Pattern B. The boundary values between regions given by $\left(\frac{A}{\omega_2}\right)^{\#}$, $\left(\frac{A}{\omega_2}\right)^{\#\#}$, and $\left(\frac{A}{\omega_2}\right)^{\#\#\#}$.

■ If $\frac{a_1}{a_2} < \frac{1}{1+n} < \frac{\left(1+\frac{a_1}{a_2}(1-\delta)\right)}{\frac{A}{\omega_1}}$, we have Pattern C. The boundary values between regions are given by $\left(\frac{A}{\omega_2}\right)^{\#}$, $\left(\frac{A}{\omega_2}\right)^{\#\#\#}$, $\left(\frac{A}{\omega_2}\right)^{\#\#}$, and $\left(\frac{A}{\omega_2}\right)^{\#\#\#\#}$.

■ If $\frac{1}{1+n} < \min\left\{\frac{a_1}{a_2}, \frac{\left(1+\frac{a_1}{a_2}(1-\delta)\right)}{\frac{A}{\omega_1}}\right\}$, we have Pattern D. The boundary values between regions are given by $\left(\frac{A}{\omega_2}\right)^{\#}$, $\left(\frac{A}{\omega_2}\right)^{\#\#}$, $\left(\frac{A}{\omega_2}\right)^{\#\#\#}$, and $\left(\frac{A}{\omega_2}\right)^{\#\#\#\#}$.

■ If $\frac{1}{1+n} > \frac{\left(1+\frac{a_1}{a_2}(1-\delta)\right)}{\frac{A}{\omega_1}} + \frac{a_1}{a_2}$, we have Pattern E. The boundary values between regions are given by $\left(\frac{A}{\omega_2}\right)^{\#}$, and $\left(\frac{A}{\omega_2}\right)^{\#\#\#}$.

Appendix 3: State transitions with labor and capital augmenting technological progress

We first discuss the shift from state (a) to state (b), as described in Figure 7, and then discuss some of the many interesting possibilities on the effect of capital-augmenting technological progress.

■ Consider Pattern A. Suppose the labor productivity level is such that the economy is initially in state (a), i.e., initially $\frac{1}{\omega_2}$ lies in $\frac{e/(1+n)}{\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}} < \frac{A}{\omega_2} < \frac{e/(1+n)}{\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}-\frac{1}{1+n}\right)}$, and the economy initially is in the stable high steady-state k^H and full employment. Then as labor productivity $\frac{1}{\omega_2}$ increases, k^H becomes even higher (since the wage increases and the savings rate remains unchanged). The economic boom is strengthened. If, however, labor productivity increases still further, then there exists a critical value of labor productivity $\frac{A}{\omega_2} = \frac{e/(1+n)}{\left[\left(1+\frac{a_1}{a_2}(1-\delta)\right)/\frac{A}{\omega_1}\right]+\left(\frac{a_1}{a_2}-\frac{1}{1+n}\right)} \equiv \left(\frac{A}{\omega_2}\right)^{\#\#}$ at which there is a state transition from state (a) to state (b). In other words, once the labor productivity reaches a certain threshold, then multiplicity of momentary equilibria emerges in the region of $k_t > k^f$, generating the phase characterized by unstable economic booms with full employment and a stable stagnation trap with involuntary unemployment. The economy remains in state (b) even if labor productivity increases still further.

The intuition is simple: the increase in labor productivity increases k^H. But it also increases $k^f = \frac{\omega_1}{\omega_2}$—more capital is required to fully employ workers. And that increases $\overline{\Omega}$. If $\overline{\Omega}$ increases more than the increase in $W = \frac{e+\frac{A}{\omega_2}}{1+n}$, then the economy moves from a situation with a unique momentary equilibrium to one with multiple equilibria in large k region. More precisely, this can occur if $\left(1+\frac{a_1}{a_2}(1-\delta)\right)\omega_1 + A\frac{a_1}{a_2} > \frac{A}{1+n}$. The left hand side of the inequality captures the increase in $\overline{\Omega}$ and the right hand side represents the increase in W. This condition is satisfied if the economy is of Type A. Note that in the Leontief case, the lower steady-state k^L remains unaffected, since in the lower steady state wages are still just zero and the return to capital is unchanged, so the savings rate is unaltered. If capital productivity increases simultaneously, but that the increase is not large enough so that k^f is increased, then the low steady-state k becomes even lower due to the decrease in the saving rate. This implies that a permanent increase in labor and capital productivity may make the stagnation trap equilibrium k^L even lower, resulting in more involuntary unemployment.

■ Consider Pattern A and suppose the capital productivity level initially lies in $\frac{\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{A}{\omega_2}}{\frac{e}{1+n}+\left(\frac{1}{1+n}-\frac{a_1}{a_2}\right)\frac{A}{\omega_2}} < \frac{A}{\omega_1} < \frac{\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{A}{\omega_2}}{\frac{e}{1+n}}$, in which case the phase diagram corresponds to state (a). Suppose that there is an increase in capital productivity, $\frac{1}{\omega_1}$. Then there exists a critical point of capital productivity $\frac{A}{\omega_1} = \frac{\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{A}{\omega_2}}{\frac{e}{1+n}}$ at which a state transition from state (a) to state (d) occurs. Even if the economy initially stays at the low equilibrium k^L with involuntary unemployment, with bullish expectations the economy can get out of the stagnation trap and can achieve the high equilibrium (the possibly dynamically inefficient equilibrium with full employment arises).

■ Consider Pattern B and suppose the capital productivity level initially lies in $\frac{\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{A}{\omega_2}}{\frac{e}{1+n}+\frac{1}{1+n}\frac{A}{\omega_2}} < \frac{A}{\omega_1} < \frac{\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{A}{\omega_2}}{\frac{e}{1+n}}$, in which case the phase diagram corresponds to state (b). Then with an increase in capital productivity, there exist two critical points of capital productivity, that is, $\frac{A}{\omega_1} = \frac{\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{A}{\omega_2}}{\frac{e}{1+n}}$ and $\frac{A}{\omega_1} = \frac{\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{A}{\omega_2}}{\frac{e}{1+n}+\left(\frac{1}{1+n}-\frac{a_1}{a_2}\right)\frac{A}{\omega_2}}$, respectively, at which state transitions occur first from state (b) into state (c) and then from state (c) into state (d). This means that even if the economy initially stays at the low equilibrium with involuntary unemployment, the increase in capital productivity first makes economy completely wobbly and then changes it to one characterized by unstable stagnation with involuntary unemployment and stable high equilibrium with full employment.

The intuition is that with an increase in capital productivity, the capital required just to have full employment goes down. Expecting this, the saving rate decreases. With the decreased saving rate, expectations of even a low level of investments can more easily be self-fulfilling equilibrium with a low level of k. Hence there is a state transition to state (d) from the initial state (a). Similarly, with an enough of a decrease in the saving rate, expectations of a low level of investments can be self-fulfilling, so a completely wobbly economy will emerge. When there is a further increase in capital productivity, the saving rate decreases even more. As a result, in the region of where total income is high, a high level of investments can be the only rational expectations equilibrium. That is, state (d) will appear.

Appendix 5: State transitions with Hicks neutral changes.

Here we examine how a change in productivity A leads to state transitions.

We first calculate the effects on the existence of multiple equilibria.

We need only calculate

$\frac{dln(s_t)}{dln(A)} + \frac{dln(w_t+e)}{dln(A)}$,

where $\frac{dln(s_t)}{dln(A)} = \frac{dln(s_t)}{dln(1+r_{t+1})} \frac{dln(1+r_{t+1})}{dln(A)}$ with $\frac{dln(s_t)}{dln(1+r_{t+1})} = -(1-\theta)(1-s(k_{t+1}))$ and $\frac{dln(1+r_{t+1})}{dln(A)} = \left(1 - \frac{1-\delta}{1+r(k_{t+1})}\right)$, and $\frac{dln(w_t+e)}{dln(A)} = \frac{w(k_t)}{w(k_t)+e}$.

The interest elasticity of savings becomes more negative as k_{t+1} is lower (the interest rate is higher). Likewise, the elasticity of the interest rate by the change in A becomes higher as k_{t+1} is lower. This means that $\frac{dln(s_t)}{dln(A)}$ becomes more negative as k_{t+1} is lower, meaning that in the low k_{t+1} region, individuals save much less when A increases. By contrast, the elasticity of the wage rate by the change in A becomes higher, i.e., $\frac{dln(w_t+e)}{dln(A)}$ is more positive as k_t is larger.

Hence, for large k_t and large k_{t+1} region, the wage increase effect could dominate, i.e., $\frac{dln(s_t)}{dln(A)} + \frac{dln(w_t+e)}{dln(A)} > 0$, so that k^H could go up. For small k_t and small k_{t+1} region, the saving increase effect could dominate, i.e., $\frac{dln(s_t)}{dln(A)} + \frac{dln(w_t+e)}{dln(A)} < 0$, so that k^L could go down.

The special case of Leontief utility and production functions, explored in greater detail, demonstrates that for large k_t and k_{t+1} region, $\frac{dln(s_t)}{dln(A)} = 0$ and $\frac{dln(w_t+e)}{dln(A)} > 0$. Hence $\frac{dln(s_t)}{dln(A)} + \frac{dln(w_t+e)}{dln(A)} > 0$. For small k_t and k_{t+1} region, $\frac{dln(s_t)}{dln(A)} < 0$ and $\frac{dln(w_t+e)}{dln(A)} = 0$. Hence $\frac{dln(s_t)}{dln(A)} + \frac{dln(w_t+e)}{dln(A)} < 0$.

Analysis on $\overline{\Omega}$ is the same as the one in the main text.

Examining the necessary and sufficient conditions in each state (a)-(e), we note that $\underline{\Omega} = \left(1 + \frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}$ and $W = \frac{e}{1+n}$ will not be affected by a change in A, and only $\overline{\Omega}$ and $W = \frac{\frac{A}{\omega_2}+e}{1+n}$ will be affected. Hence, considering the magnitude of $\underline{\Omega} > or < \frac{e}{1+n}$, we only need investigate if $\overline{\Omega}$ is greater or lower than $\frac{e}{1+n}$ or/and $\frac{\frac{A}{\omega_2}+e}{1+n}$. There are in total just four types (labeled F, G, H, and I.) Figure 11 illustrates how state transitions occur as the technological parameter A changes. In the figure, the further to the right, the larger the value of A becomes. The condition for each type is as follows.

■ If $\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}<\frac{e}{1+n}$ and $\frac{a_1}{a_2}>\frac{1}{1+n}$, we have Pattern F. An increase in A ends up generating a completely wobbly economy, even if there is initially a unique momentary and a unique high steady-state with full employment.

■ If $\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}>\frac{e}{1+n}$ and $\frac{a_1}{a_2}<\frac{1}{1+n}$, we have Pattern G. An increase in A creates asymmetric outcomes in the symmetric environment. That is, even if there is initially a unique momentary and a unique low steady-state, an increase in A ends up generating an asymmetric outcome.

■ If $\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}<\frac{e}{1+n}$ and $\frac{a_1}{a_2}<\frac{1}{1+n}$, we have Pattern H.

■ If $\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}>\frac{e}{1+n}$ and $\frac{a_1}{a_2}>\frac{1}{1+n}$, we have Pattern I. In both patterns H and I, an increase in A eventually lead to unstable global phase, even if the initial phase is stable.

The intuition for why a change in A leads to several types of state transitions is that a productivity increase produces two competing effects. One is that (at any value of k_t) the rise in productivity increases the interest rate (i.e., the return on savings). When the interest rate rises, the saving rate decreases (under the conditions upon which we focus in this paper. This lowers capital stock k_{t+1}, for any value of k_t. The other effect is that wage income increases with the increase in productivity. This leads to increased saving and a larger capital stock k_{t+1} for any k_t.

Moreover, the increase in A can increase $\overline{k}$. This is because the saving rate decrease, and possibly enough that aggregate savings decreases. so that expectations of low investment can be self-fulfilling even for large k_t.

A similar logic helps explain Patterns F and Type G. In Pattern F, there is initially a unique momentary and a unique steady state with full employment. When there is an increase in A, the decrease in the saving rate (from the higher return on capital) dominates the effect of the wage increase if $\frac{a_1}{a_2}\frac{1}{\omega_2}>\frac{1}{1+n}\frac{1}{\omega_2}$ (the left hand side captures the former effect, while the right hand side the latter effect). As a result, there is a critical value of A at which in the region of $k_t<k^f$ in which total income is low, expectations of a low level of investments can be self-fulfilling as well as high and middle levels of investments. That is, there is a state transition from state (e2) to state (d). With a further increase in A, there is another critical value of A at which even in the region of $k_t>k^f$ where total income is high, expectations of the low level of investments can also be self-fulfilling. That is, the economy will enter into a completely wobbly economy region as a result of the technological progress.

On the other hand, consider initially the situation where there is a unique momentary and a unique low steady state equilibrium characterized by involuntary unemployment. With Pattern G, the effect of the wage increase dominates the effect of the decrease in the saving rate. Thus, as A increases, there is a critical value of A at which in the region of $k_t > k^f$, expectations of a high level of investments can be self-fulfilling as well as the middle level of investments. There is a state transition from state (e1) to state (b). With a further increase in A, total income in the region of $k_t > k^f$ becomes so large that the high level of investments can be the only rational expectations equilibrium, while total income in the region of $k_t < k^f$ will not be affected. As a result, there is another state transition from state (b) to state (a). With a sufficiently low level of technology, all economies stay at the low steady-state characterized by involuntary unemployment even as A increases. However, as a result of further technological improvement, an economy may get out of the stagnation trap and may achieve the high steady state with full employment, even with a small shock to the economy. Other economies may continue to stay at the low steady-state and involuntary unemployment persistently occurs.

Appendix 6: Myopic dynamics

Assume that individuals are myopic, and believe that this year's interest rate will be the next year's, i.e., $k_{t+1} = s(k_t)W(k_t)$. For each k_t there is a unique k_{t+1}.

Although myopic expectations ensure a unique momentary equilibrium, there may be more than one (stable) steady state, as illustrated in the left panel in Figure 2. In the figure, for each k_t there is a unique k_{t+1}, with $k < k^M$ being in the domain of attraction of k^L, and $k > k^M$ representing the domain of attraction of k^H. This implies that even though there is local stability, if the economy is perturbed *enough* from say k^H, it may not return to k^H, but converge to k^L. Still, myopic expectations contribute to stability because multiplicity of momentary equilibrium will not arise. By contrast, our wobbly model has instability in two dimensions, i.e., multiple momentary equilibria and multiple steady states.

Of course, myopic expectations imply that consumption-saving decisions are being made on incorrect expectations, and hence the resulting allocations are likely to be inefficient. While a sequel paper investigates the welfare economics of wobbly dynamics, it is perhaps worth noting here that welfare (represented by an equalitarian intergenerational social welfare function) may be higher under myopic dynamics than under at least many of the rational expectations trajectories of wobbly dynamics.[39]

[39] This is a standard result in the theory of the second best; except in this context, there is no market failure other than that inherent in an overlapping generations mode. See the sequel Hirano and Stiglitz (2021b) for a more extensive discussion.

While our wobbly model is markedly different from the standard representative agent models, where the optimum trajectory converges to a unique value of k^*, independent of the initial conditions k_0, that result arises from the strong assumptions made about technology and preferences. For instance, with non-separable Koopmans preferences, initial conditions may matter for long run optimal trajectories, even in a representative agent model. What is distinctive about our analysis is that such multiplicity arises so easily, with relatively standard preferences.

Our wobbly model is also in sharp contrast with Milton Friedman's theory of speculation in which irrational investors destabilize financial markets, while rational investors contribute to stability. The intuition behind Friedman's claim was simple: buying when the price of an asset is ("irrationally") low raises the price then; selling it when it is high lowers the price then; combined, these speculative interventions reduce price volatility. Critically, Friedman's analysis was partial equilibrium. When such interventions occur on a large enough scale, there can be destabilizing feedbacks, where, for instance, the rate of return on capital in the system itself is changed. While the simple general equilibrium model formulated here illustrates, these effects are even more apparent when there are two assets, land and capital. In a sequel, we show how land speculation cannot only lower the average level of incomes, but can also induce greater economic volatility. (Hirano and Stiglitz, 2021b).

Our analysis stands in marked contrast to that of Akerlof and Shiller (2009), who emphasize that human irrationality, i.e., irrational exuberance, is the key source of macroeconomic instability. In our model, an indeterminacy arising in markets with full human rationality is the key source for macroeconomic instability. In this regard, our result is close in spirit to that of Keynes' beauty contest model where what is referred to as higher degree expectations (see e.g., Allen, Morris and Shin 2006) i.e., near full-human rationality, is the main cause for the formation of asset price bubbles in financial markets.[40]

In the discussion above, we simply assumed that all young agents have myopic expectations. Instead, suppose that a fraction π of young agents has rational expectations, and the remaining fraction $1-\pi$ has myopic expectations. The dynamic equation is written as $k_{t+1} = \pi s(k_{t+1})W(k_t) + (1-\pi)s(k_t)W(k_t)$. Then we can show that if π is sufficiently small, but strictly greater than zero, there is a unique momentary equilibrium.

If the population of agents with myopic expectations is sufficiently large, the macroeconomy cannot be "wobbly". To eliminate the high level instabilities discussed in this paper, one only

There is a more extensive literature expanding on the principle that imperfections may be important for the functioning of the stable economy outside the "perfects" market economy—with a full set of markets extending *infinitely* far into the future. Shell and Stiglitz (1967) showed the dynamic instability of the economy with heterogeneous capital goods in the absence of such a full set of futures markets. Beyond that, they showed that the dynamic system was stable under sufficiently inelastic adaptive expectations.

[40] That work in turn is close in spirit to that of Dosi *et al* (2020) who show, in a model with deep uncertainty and a high level of non-linearity, that more sophisticated (seemingly more "rational") expectations are associated with lower levels of economic performance, including greater instability.

needs *enough* individuals with myopic expectations. Conversely, if increasing sophistication starts to pervade society, and an increasing fraction of the population switches to having rational expectations, the economy can switch into a dynamics exhibiting the extremes of instability explored here.[41]

[41] An extension of our model would be to endogenize π by introducing learning dynamics. Even in that case, so long as π is determined at a sufficiently small value, our result would hold, i.e., there would be a unique momentary equilibrium.

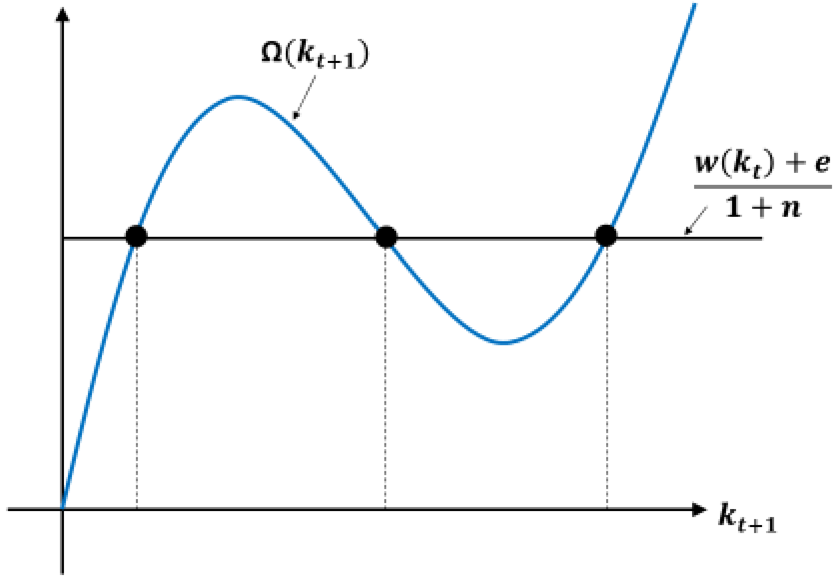


Figure 1-1: Existence of multiplicity of momentary equilibria

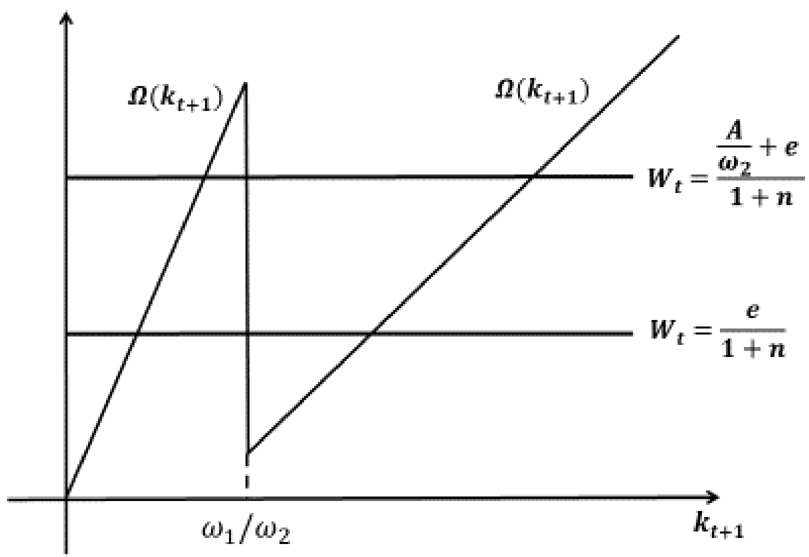


Figure 1-2: Existence of multiplicity of equilibria in the Leontief case

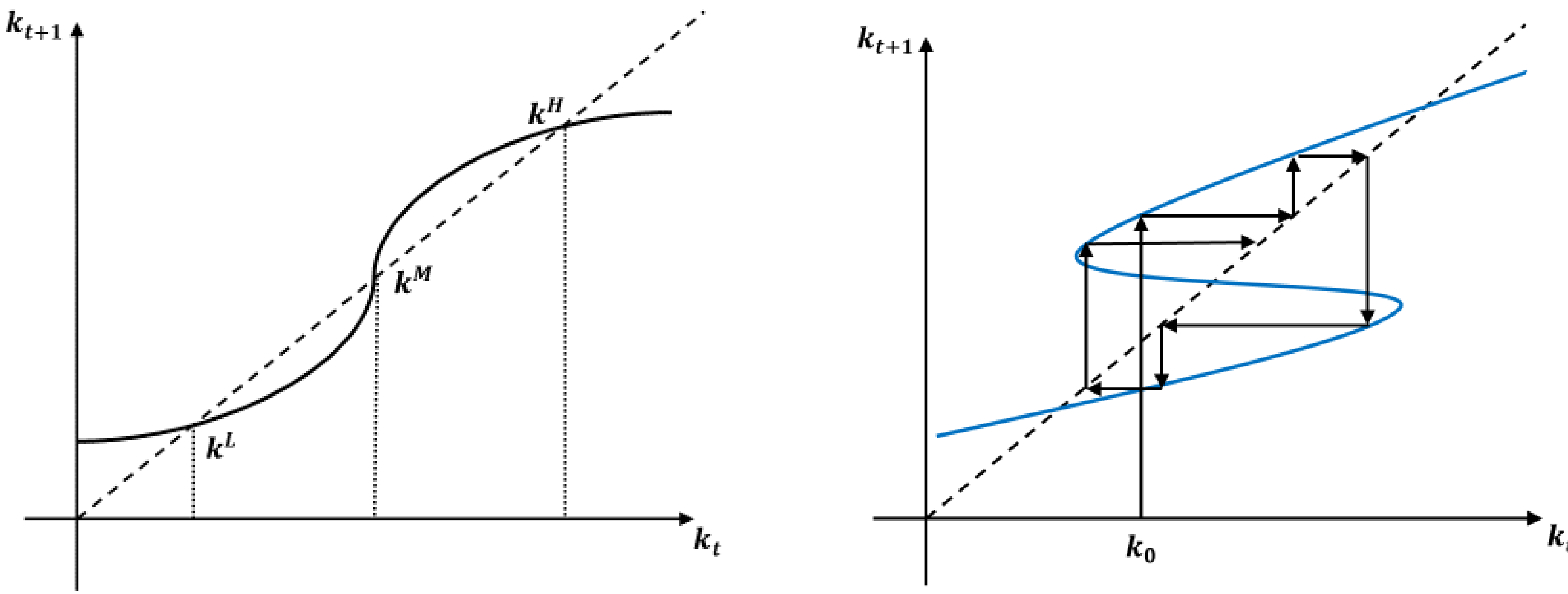


Figure 2: An example of a wobbly trajectory vs the typical dynamics with a unique equilibrium

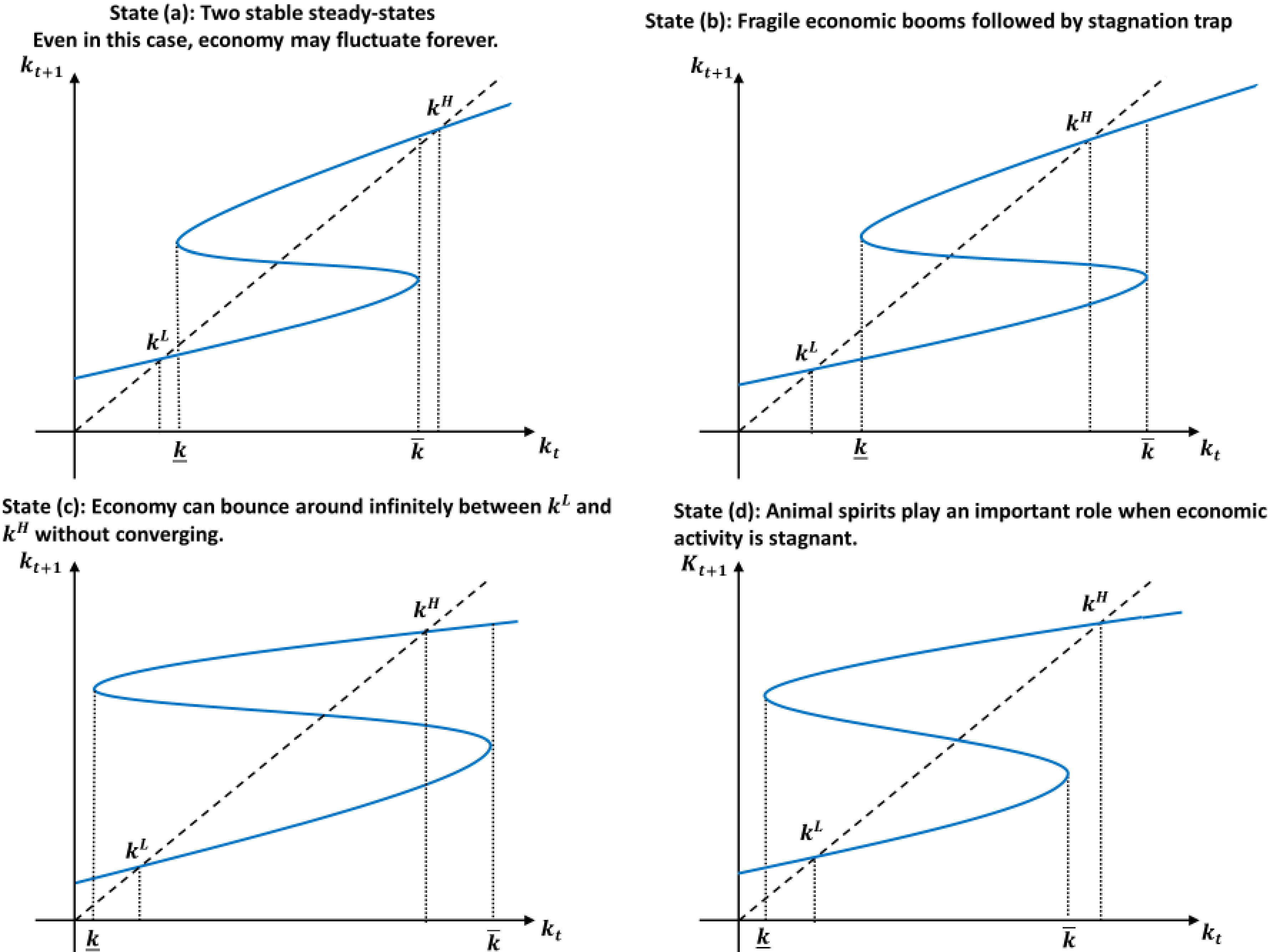


Figure 3: four typical patterns of the wobbly macro-dynamics

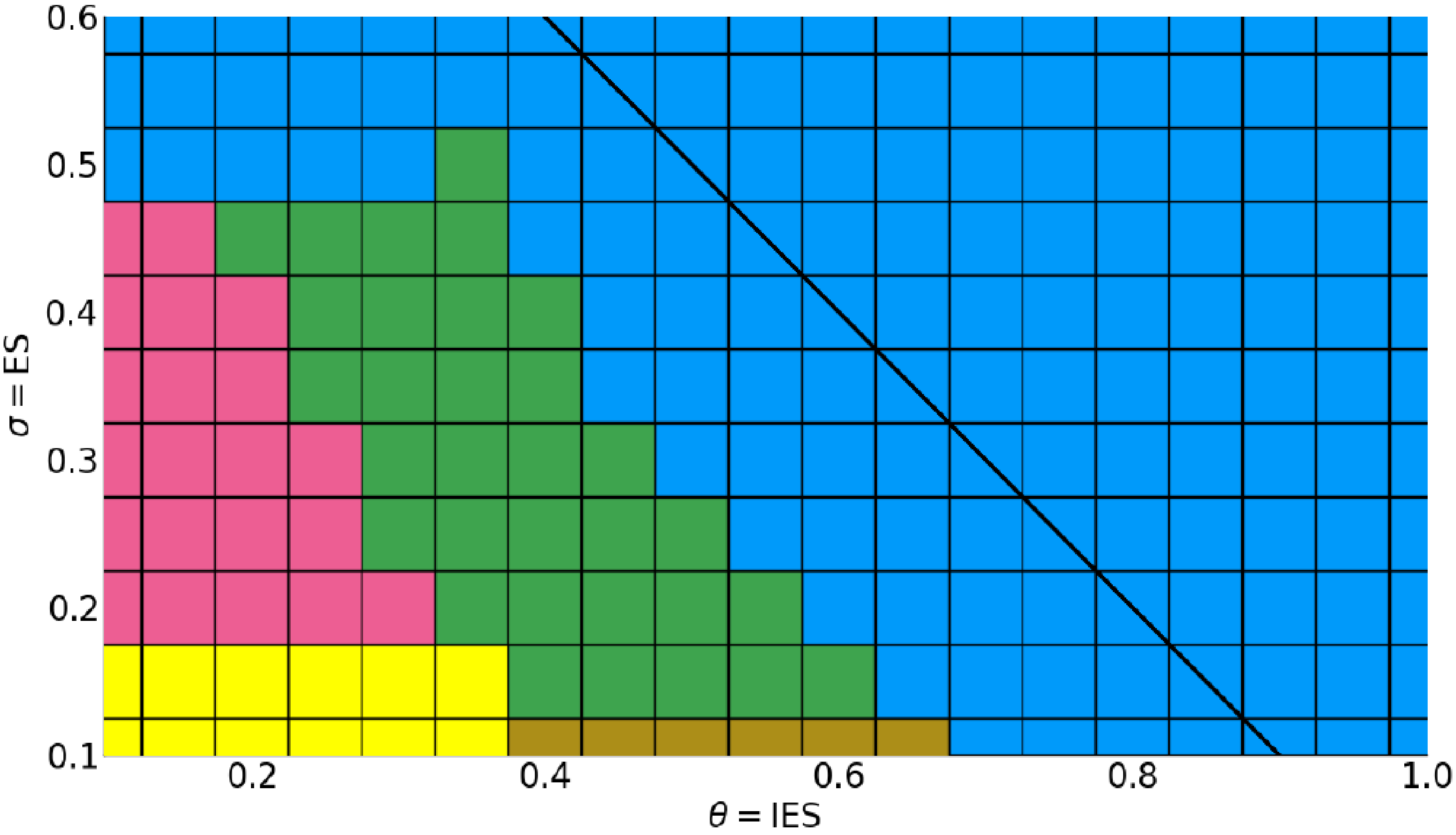


Figure 4: Numerical characterization in the CES utility and production functions
Other parameter values are set as $A = 4.2, \alpha = 0.3, \omega_1 = \omega_2 = 1, a_1 = 8, a_2 = 1, e = 2.633, \delta = 1, n = 0$. Green region=state (a), Pink region=state (b), Brown region=state (d), Yellow region=state (c).

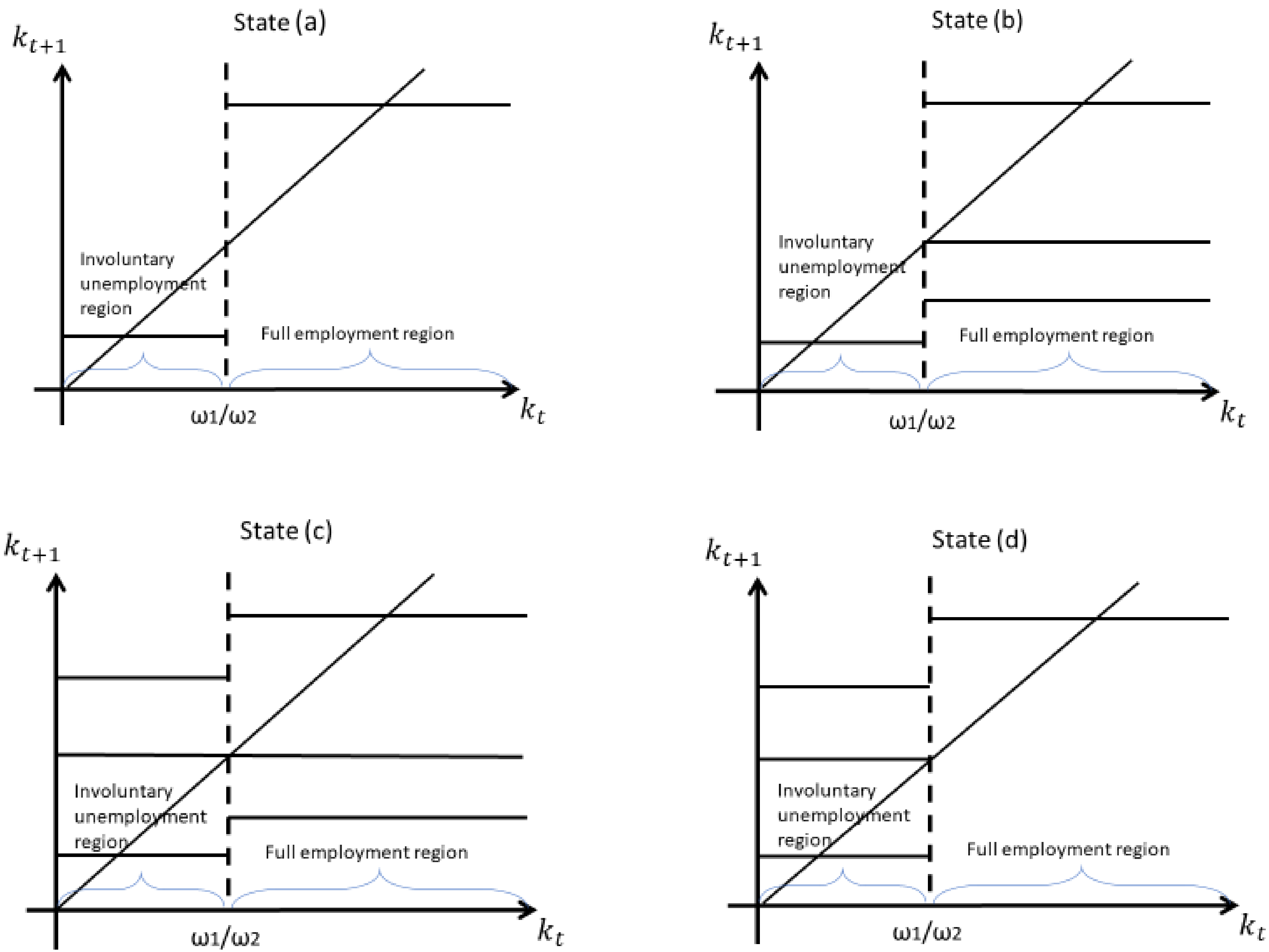


Figure 5-1: Wobbly macro-dynamics in the Leontief case

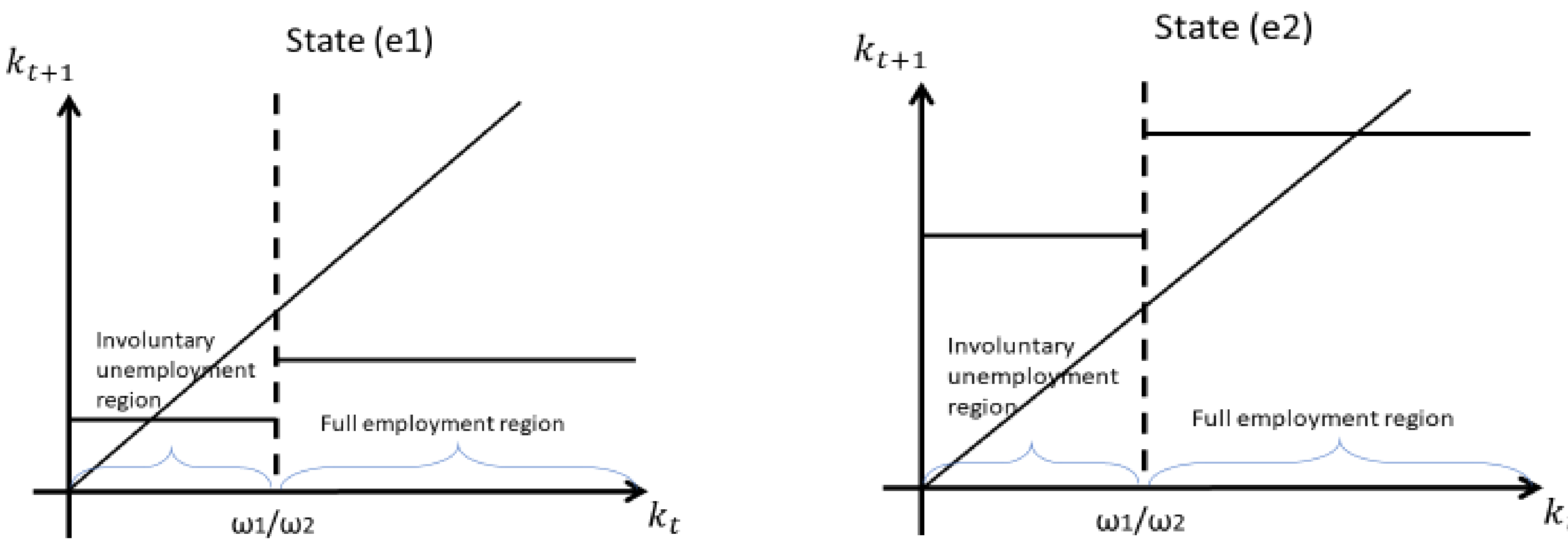


Figure 5-2: wobbly macro-dynamics in the Leontief case

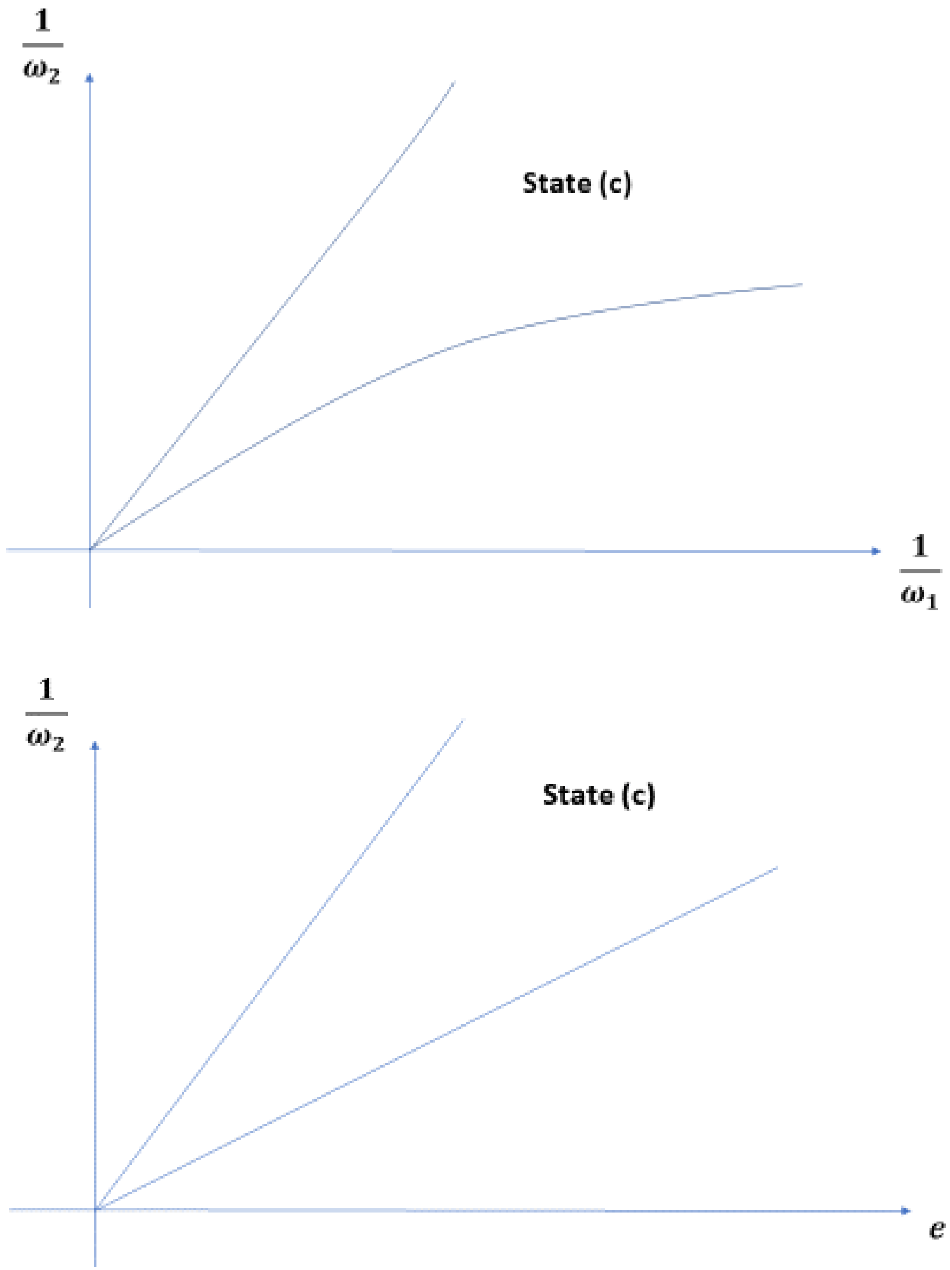


Figure 6: Boundary values in state (c)

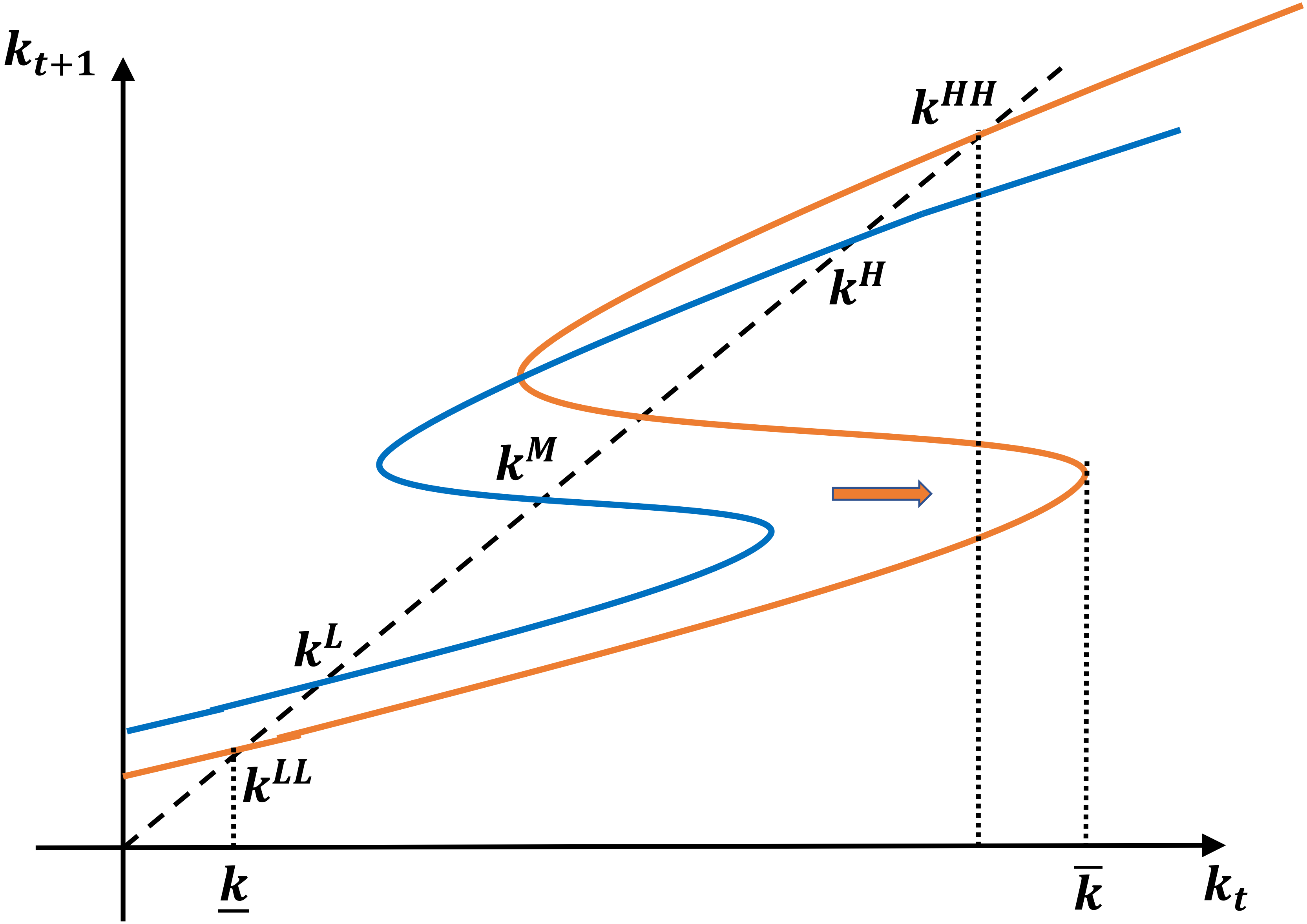


Figure 7: Productivity increase could generate fragile economic booms followed by stagnation trap.

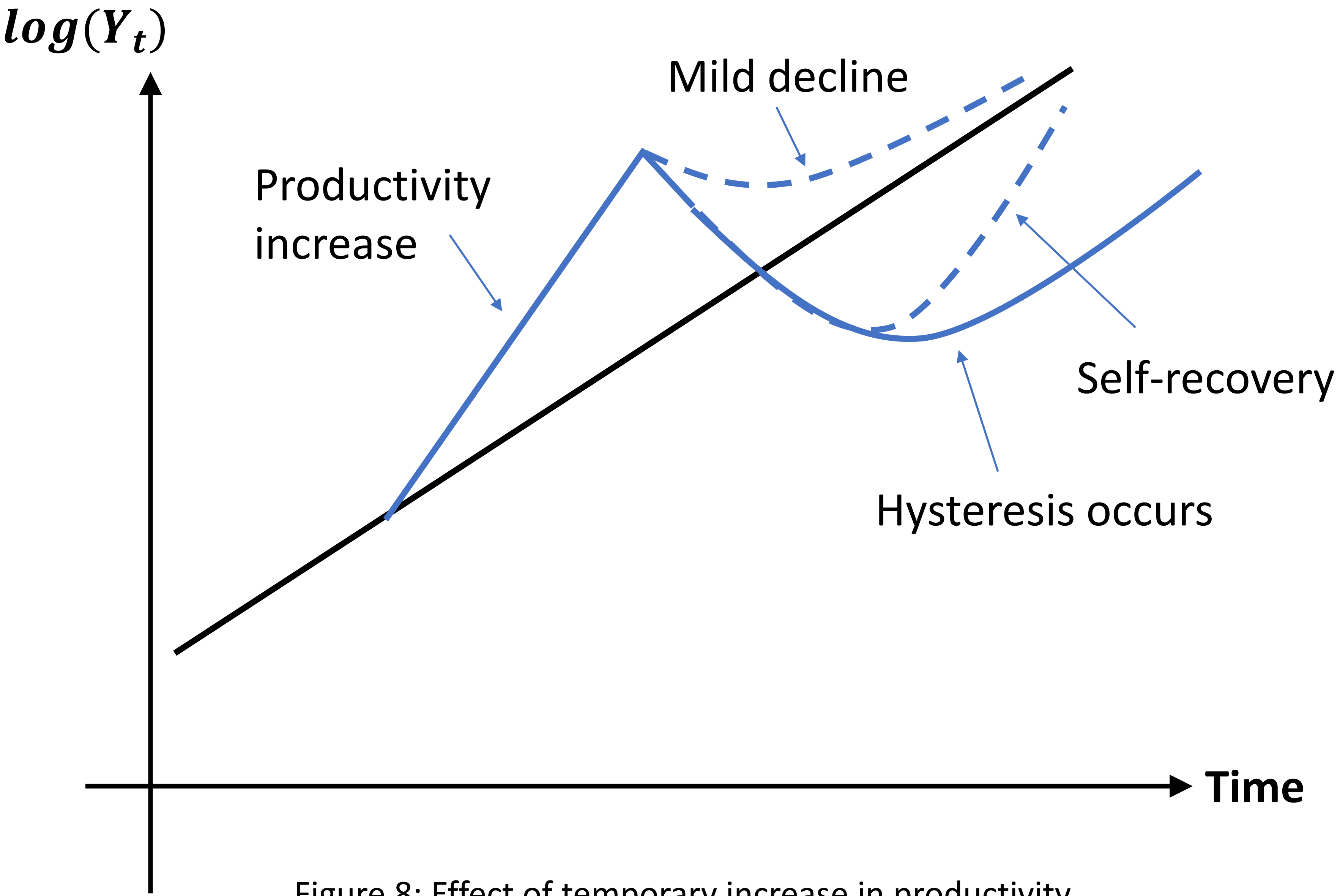


Figure 8: Effect of temporary increase in productivity

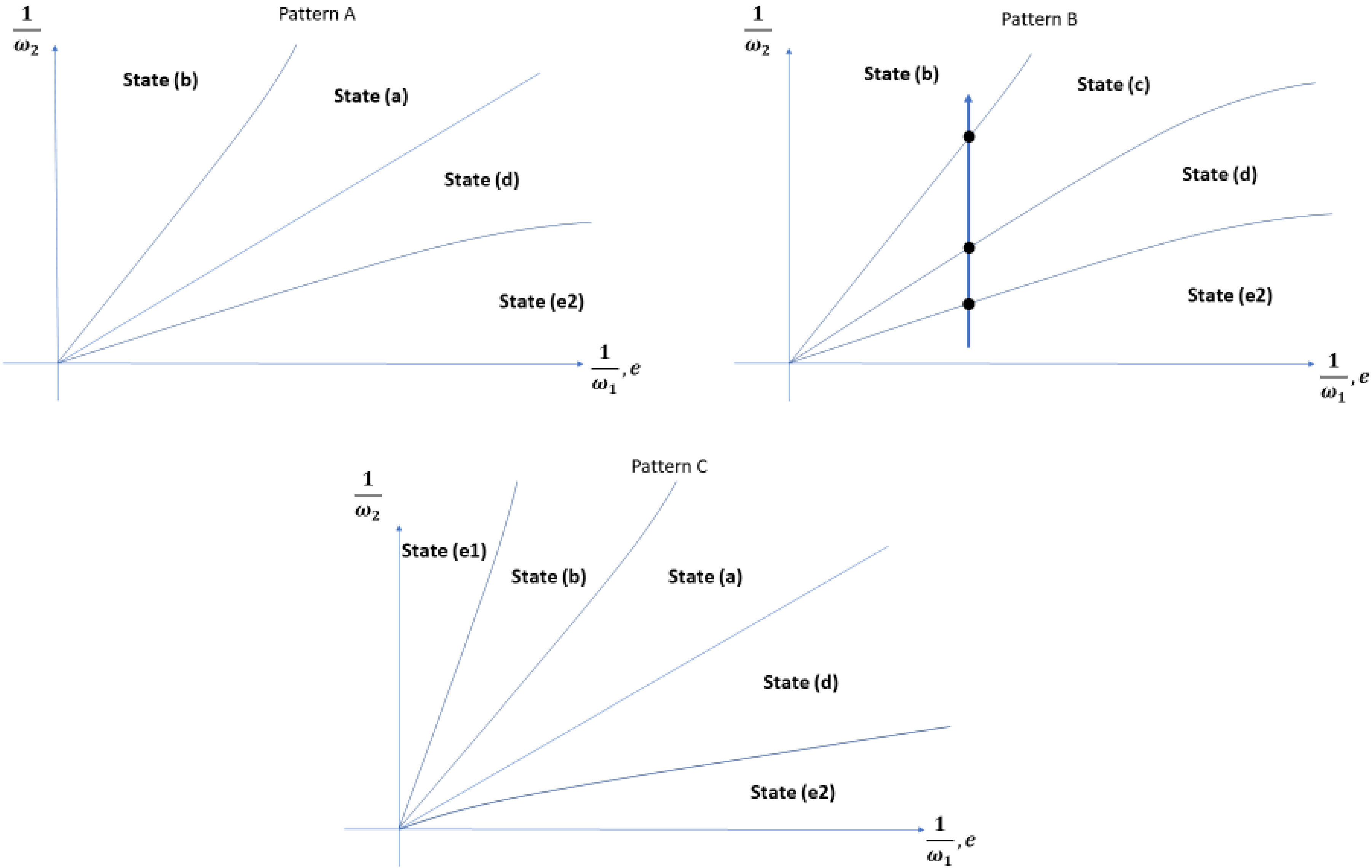


Figure 9-1: Effect of labor-augmenting technological progress on state transitions

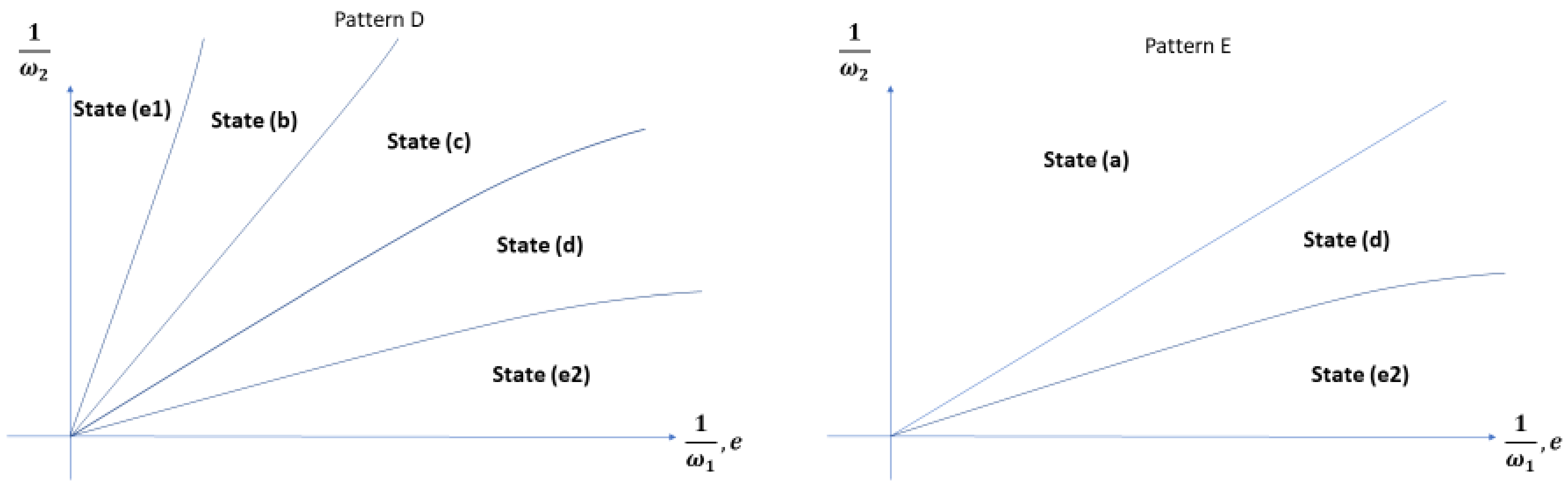
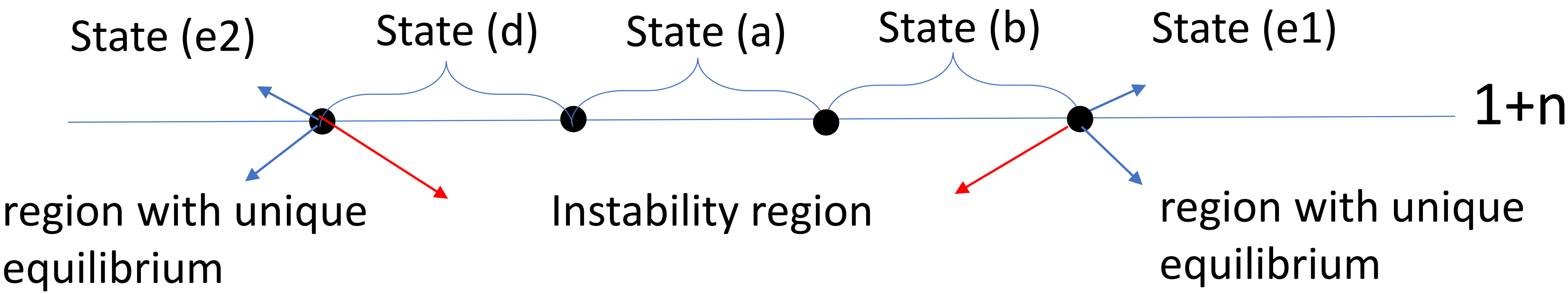


Figure 9-2: Effect of labor-augmenting technological progress on state transitions

Figures for appendix

Pattern O: $\frac{a_1}{a_2}e < 1 + \frac{a_1}{a_2}(1 - \delta)$



Pattern P: $\frac{a_1}{a_2}e > 1 + \frac{a_1}{a_2}(1 - \delta)$

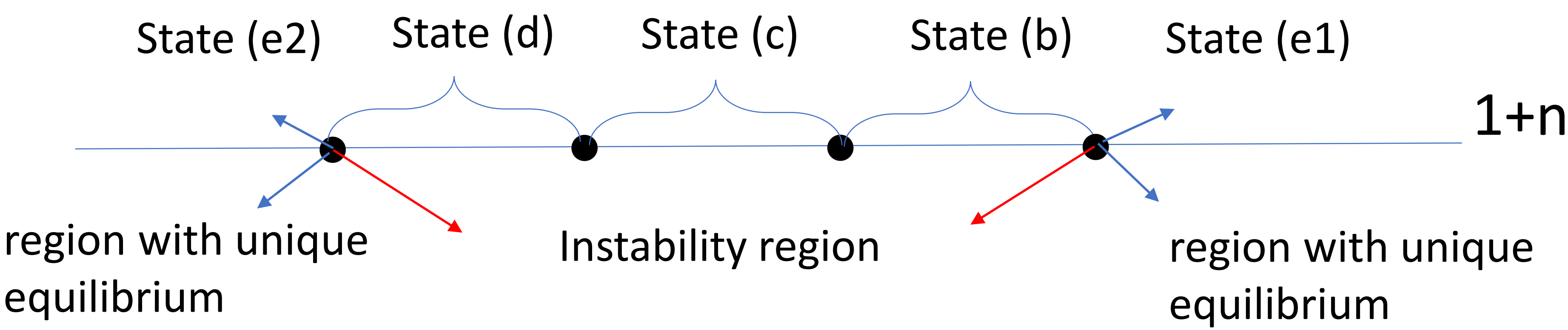


Figure 10: Growth and State Transitions

Pattern F: $\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}<\frac{e}{1+n}$ and $\frac{a_1}{a_2}>\frac{1}{1+n}$

Pattern G: $\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}>\frac{e}{1+n}$ and $\frac{a_1}{a_2}<\frac{1}{1+n}$

Pattern H: $\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}<\frac{e}{1+n}$ and $\frac{a_1}{a_2}<\frac{1}{1+n}$

Pattern I: $\left(1+\frac{a_1}{a_2}(1-\delta)\right)\frac{\omega_1}{\omega_2}>\frac{e}{1+n}$ and $\frac{a_1}{a_2}>\frac{1}{1+n}$

Figure 11: Effect of Hicks neutral change on state transitions