Debt overhang and lack of lender's commitment [*]

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Abstract

The debt overhang of sovereigns or firms is modeled in the recent literature as a constrained efficient outcome of dynamic debt contracts under the lack of the borrower's commitment, where debt relief is not Pareto improving. The early literature observes another type of debt overhang where the borrower is discouraged from expending effort, anticipating that the lender will take all output ex post. We show that this inefficiency is due to the lack of the lender's commitment and that debt relief is Pareto improving. Nevertheless, debt overhang may persist, as frictional bargaining over debt relief can take a long time.

Key words: Backloading, debt Laffer curve, two-sided lack of commitment.

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1 Introduction

In this study, we analyze the relationship between the amount of debt and the borrower's economic activity in a model for a long-term debt contract between a sovereign or private borrower and lenders. We assume that debt is not state contingent and evolves at a fixed interest rate; thus, debt can grow too large. Debt can accumulate beyond the repayable amount as a result of repeated and/or large negative shocks. However, debt relief, though possible, takes time, as it involves time-consuming bargaining, such as a war of attrition among lenders. It is demonstrated that large debt can hinder the borrower's economic activity in two respects: One is the standard debt overhang, which we call the *first type* of debt overhang, and the other is the *second type* of debt overhang that we highlight in this study.

The first type of debt overhang is caused by a lack of commitment on the borrower side, which is well known in the literature on optimal debt contracts. The borrower can default on the debt at any time; in the constrained optimal debt contract, the borrower's economic activity measured by output is smaller when debt is larger. This is because a larger output tempts the borrower to default when the output is the borrower's payoff of defaulting (Albquerque and Hopenhayn 2004; Kovrijnykh and Szentes 2007; Aguiar, Amador, and Gopinath 2009).

We show that, as debt increases and exceeds a certain threshold, the economy that was originally in the first type of debt overhang enters the second type. It is the debt overhang that Sachs (1988) observes: The borrower hesitates to invest in a new project because the debt is so large that the borrower expects that all return on the investment will be taken by the lenders ex post (see pp. 29–31 of Sachs [1988]). This line of thinking emerges because lenders cannot credibly commit to making the repayment smaller than the face value of debt. This inability of lenders to commit causes inefficiency, which we call the second type of debt overhang. The first type of debt overhang in, for example, Albuquerque and Hopenhayn (2004), is a situation wherein the lender cannot lend the first-best amount because the borrower would abscond with the borrowed money. In the second type of debt overhang, the borrower cannot expend the first-best amount of effort because the lender would take all output, as the existing debt is too large. Suppose that lenders keep the contractual amount of debt unchanged, and, nevertheless, promise that they will give a sufficient amount to the borrower. The lenders' promise is not trustworthy when the debt is larger than the borrower's output, because lenders have the legitimate right to take all as repayment. The contractual amount of debt works as a commitment device when it is small, as it gives the upper bound for the amount that lenders can demand. The upper bound no longer works as a commitment device for the lenders when it is so large that it exceeds the maximum repayable amount. Note that the lack of commitment on the

borrower side always exists regardless of how large the debt is, whereas the lender loses credibility when debt becomes large. Therefore, debt relief can be effective in restoring the commitment of lenders and reducing the inefficiency caused by the lack of lenders' commitment.

Let us explain how the lack of lenders' commitment emerges in the following intuitive example. Suppose, for example, that borrowers owe lenders D dollars, and the borrowers can earn USD 1,000 if they work hard, but earn nothing if they do not work hard. They will work hard if their payoff is no less than USD 200, but will not work if their payoff is less than USD 200. Obviously, the maximum repayable amount is USD 800 (=1000-200). Now, suppose that D is smaller than USD 800, say, $D = 500$. In this case, the lenders' promise to the borrowers, that is, "You pay D dollars and you get the remaining," is trustworthy because the lenders have no legitimate right to take more than D dollars. Therefore, the lenders' commitment to the repayment plan is credible, and the contractual amount of debt, D, is a payoff-relevant state variable on which the repayment plan depends. Given this promise, as the borrowers can take $1000 - D$ (> 200) if they work hard, they actually choose to work hard and repay D.

Next, suppose that D is larger than USD 800, say, $D = 10,000$. Here, we assume that the debt relief that reduces D from $10,000$ to 800 is impossible to implement, because bargaining frictions make it prohibitively costly to reach an agreement on debt relief.[1] In this case, the lenders will offer the following plan if they can: "You pay USD 800 and you take the remaining USD 200." However, this repayment plan is not credible, because D is larger than USD 800, and lenders have the legitimate right to demand that the borrowers repay D. Suppose that the lenders make the above promise: "You pay USD 800, and you get the remaining USD 200," and the borrowers trust this promise and work hard. Subsequently, they earn USD 1,000 and, ex post, the lenders will demand the borrowers to pay the full USD 1,000, as they have the legitimate right to demand full repayment. Thus, the ex-ante promise that lenders will not demand more than USD 800 is not credible. Therefore, if $D = 10,000$, any offer made by the lenders is not trustworthy, the borrowers will not work hard, and they earn nothing. Ultimately, both lenders and borrowers obtain zero dollars. Note also that, when D is larger than the maximum repayable amount, it is no longer a payoff-relevant state variable.

In short, when the contractual amount of debt is very large, the lenders' offer of the repayment plan is not credible ex ante, because they can and will demand the borrowers to pay more ex post. This lack of a lender's commitment makes the borrower reluctant to work hard and reduces output. This is the second type of debt overhang, which Sachs (1988) observes.

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[1]An episode relating to and the literature on frictional bargaining on debt restructuring are discussed later.

Theoretically, we can explain that the inefficiency of the second type of debt overhang emerges owing to the fact that the contractual amount of debt, D, is no longer a payoff-relevant state variable, if it exceeds a certain threshold. As we see in the simple example in section 2 or in the model of section 3, when D is small, the lenders and borrower agree on a constrained optimal repayment plan contingent on D that backloads the borrower's payoff. That is, they agree that the borrower pays as much as possible until the debt is fully repaid; only after the full repayment of the debt, the borrower will be given the dividend. Ray (2002) demonstrates that the repayment plan that backloads the borrower's payoff is constrained optimal under the borrower's lack of commitment. When D is too large and no longer a payoff-relevant state variable, the backloading of the borrower's payoff is infeasible. The feasible repayment plan should be static or independent of D when D is too large. As long as the dynamic plan that backloads the borrower's payoff is constrained optimal, the repayment plan that does not depend on D is Pareto inferior.

The policy implication of debt overhang in our study is in stark contrast to those of the recent literature on optimal contracts, that is, our model implies that debt relief increases the payoffs for both the lender and the borrower in the second type of debt overhang, while it is not Pareto improving in the first type of debt overhang, which has been the focus of recent literature. In contrast to the recent literature that emphasizes that debt relief is not necessarily a desirable policy response to debt overhang, our study implies that it could be desirable for both borrowers and lenders, as Sachs (1988) argues.

In other words, the second type of debt overhang provides a formal explanation of the debt Laffer curve, that is, the lenders' payoff may decrease with the contractual amount of debt. In the existing models of long-term debt, such as Albuquerque and Hopenhayn (2004) and Aguiar, Amador, and Gopinath (2009), the lenders' value is a weakly increasing function of the contractual amount of debt. In our model, however, the lenders' value has an inverted U-shaped relationship with debt because the lenders' payoff may strictly decrease when the contractual amount of debt exceeds a certain threshold. The inverted U-shape is interpreted as a debt Laffer curve. It illustrates that the payoff for lenders can be increased by debt restructuring when the debt is too large. There are several episodes wherein sovereign debt restructuring increases the payoff for lenders. Asonuma, Niepelt, and Ranciere (2021a, b) show that the exchange recovery rate (i.e., the ratio of the market value of the new debt to that of the old debt) for the 2003 external debt restructuring of Uruguay exceeds 100% (Figure D1 in Asonuma, Niepelt, and Ranciere [2021 b]).[2] Cruces and Trebesch (2013) also show that there existed several debt restructuring episodes in the 1980s wherein the market values of debt increased when the contractual values were reduced. Our model is consistent with these episodes and provides a theoretical explana-

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[2]In a private communication, Asonuma notes that there are several other debt restructuring cases wherein the exchange recovery rate exceeds 100%, such as Dominican Republic in 2004–2005.

tion.

Can debt relief delay even if it is Pareto improving? Our answer is yes: Bargaining frictions can cause a significant delay in negotiations over debt restructuring. Delays in debt relief are widely observed in episodes of sovereign debt restructuring, such as in the case of the Argentine bond in the 2000s. Argentina defaulted on sovereign bonds in 2001, and it took 16 years to complete negotiations over debt restructuring. There are several explanations for the delay in debt restructuring. A major theory is that it takes a considerable amount of time to reach an agreement on debt relief, as the negotiations among stakeholders in many cases are frictional bargaining (e.g., Benjamin and Wright 2009; Pitchford and Wright 2012). In the above Argentine case, the *pari passu* litigation raised by one of the bondholders (Argentine v. NML Capital) overturned an otherwise-agreed settlement and caused a year-long delay. In the simple model in section 2, we assume that the borrower owes two banks equally and that the negotiation on debt relief is a war of attrition game between the two banks. In this case, it can take a very long time on average to reach the agreement of debt relief, and thus the second type of debt overhang can continue for a considerable length of time.

Literature: Our model builds on that of Albuquerque and Hopenhayn (2004)(hereafter, AH), who study the constrained efficient debt contract between a borrower and lender in a model where the borrower is unable to commit to repaying the debt. We show that lenders lose the ability to commit to a variant of AH when the amount of debt becomes too large and debt relief cannot be implemented immediately (owing to bargaining frictions).[3] In AH, when the debt is large, the output produced by the borrower is smaller than the first-best level, although the equilibrium outcome is constrained efficient. The constrained efficient allocation is implemented by backloading payoffs to the borrower: The borrower's earnings in each period are all paid to the lenders and the borrower receives no dividends until the first-best allocation is attained.[4] The constrained efficient outcome in the recent literature on sovereign debt (e.g., Kovrijnykh and Szentes 2007; Aguiar, Amador, and Gopinath 2009) can be interpreted as debt overhang owing to the

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[3]Similar to AH, we use recursive contracts to formulate the equilibrium. Golosov, Tsyvinski, and Werquin (2016) survey the literature on recursive contracts. Although we employ dynamic programming to solve our equilibrium, the Lagrange multiplier method may also be used, as described by Marcet and Marimon (2019). Commonly used frameworks for borrowing constraints in macroeconomics are provided by Kiyotaki and Moore (1997); Carlstrom and Fuerst (1997); and Bernanke, Gertler, and Gilchrist (1999). The financial contracts in these studies are essentially static, whereas in our model, they are dynamic.

[4]In a more general model of long-term relationship between two parties with one-sided lack of commitment, Ray (2002) demonstrates that the optimal contract involves backloading the payoff of the party who lacks the ability to commit. Furthermore, as demonstrated by Clementi and Hopenhayn (2006), backloading payoffs to the borrower also plays a crucial role in the dynamic optimal contract problem with asymmetric information.

lack of the borrower's commitment, which we call the first type of debt overhang. Since the first type of debt overhang is constrained efficient, the recent literature stresses that debt relief is not Pareto improving, implying that debt relief is not a desirable policy response to overly accumulated debt. In contrast, the early literature on debt overhang emphasized that debt relief is a desirable policy (Krugman, 1988; Sachs, 1988).[5] Our contribution is to formulate a formal model to justify the intuition in the early literature; in other words, to formulate a formal model of a debt Laffer curve (Krugman, 1988) for the first time in the optimal contract literature. The inefficiency in the second type of debt overhang will disappear if debt relief is possible. Then why does debt relief take time? In the simple model in section 2, the negotiation over debt relief takes time because it is a war of attrition among lenders. The intention of this setting is to capture bargaining frictions in a reduced form. An example of a structural model that gives inefficient delays in bargaining is provided by Abreu and Gull (2000), who demonstrate that the belief that the opponent of negotiation may be irrational causes a delay in the settlement, even among rational players. Another example is Fuchs and Skrzypacz (2010), who demonstrate that asymmetric information with a stochastic arrival of new players creates a delay in bargaining. In the context of sovereign debt restructuring, Benjamin and Wright (2009) demonstrate that the option value of waiting creates a delay in debt restructuring. Pitchford and Wright (2012) examine a model wherein sovereign debt restructuring is delayed by creditors' holdouts. We often observe persistent recessions for years in the aftermath of financial crises (e.g., Reinhart and Rogoff 2008). In particular, the global financial crisis of 2008–2009 and the subsequent recession raised growing concerns about *secular stagnation* (Summers 2013, Eggertsson and Mehrotra 2014). Because our theory predicts persistent inefficiency caused by the accumulation of debt, we expect it to be helpful in understanding the persistence of a recession following a financial crisis. In particular, it suggests that policies that facilitate debt restructuring can be effective for recovery from a persistent recession with debt overhang. Geanakoplos (2014) provides similar policy implications.

The remainder of this paper is organized as follows. In section 2, the example discussed above is elaborated further to demonstrate how overaccumulation of debt impairs the credibility of lenders and leads to an inefficient outcome. It is also demonstrated that a war of attrition between lenders delays debt relief. In section 3, we describe the baseline model wherein debt relief never occurs. The results of the numerical simulations are presented in this section. In section 4, the model is extended in such a way that debt restructuring can occur stochastically with a constant probability. This is a reduced-form

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[5]For early and other models on debt overhang, see also Myers (1977) in corporate finance, and Lamont (1995) and Philippon (2009) in macroeconomics.

model for frictional bargaining. Section 5 concludes.

2 Simple Model

Why do lenders become unable to offer a credible repayment plan when the debt is too large? Why does this distrust of lenders lead to inefficient outcomes? The simple model in this section provides an intuitive account for these questions, which may be helpful for proceeding to the full model in section 3.

2.1 Setting

Suppose that there exists a borrower and two banks, bank 1 and bank 2, in an economy where time goes $t = 0, 1, 2, \cdots$. The borrower respectively owes each bank $\frac{1}{2}D_0$ in $t = 0$, where the total amount of debt owed is D_0. In each period t, the borrower earns y_t by expending effort e_t, where

$$y_t = \begin{cases} y_H & \text{if } e_t = e, \\ y_L & \text{if } e_t = 0, \end{cases}$$

where e_t is non-pecuniary disutility, and $0 < e < y_L < y_H - e$. The contractual amount of debt evolves by

$$D_{t+1} = \beta^{-1}(D_t - b_t),$$

where β is the subjective time discount factor with $0 < \beta < 1$, and b_t is the total amount repaid in period t, where each bank obtains $\frac{1}{2}b_t$.

In each period t, the borrower chooses e_t, and then the banks (collectively) choose b_t, after which e_t is chosen. We focus on the Markov equilibrium with a state variable D_t, that is, both the borrower and banks can make their actions contingent only on D_t, but not on their actions in the previous periods. In other words, we assume that neither the borrower nor the banks have the ability to implement credible threats (Cole and Kehoe 2000). Thus, it is not possible for the borrower and lenders to credibly agree that one agent will be penalized forever by the other agents if that agent deviates from the agreement.

The borrower chooses e_t, given the expectations for b_t and $D_{t+1} = \beta^{-1}(D_t - b_t)$ to solve the following problem:

$$V(D_t) = \max_{e_t} y_t - e_t - b_t + \beta V(D_{t+1}). \tag{1}$$

where $V(D_t)$ is the borrower's value. The banks (collectively) solve the following problem, given that e_t is already chosen:

$$d(D_t) = \max_{b_t} b_t + \beta d(D_{t+1}), \tag{2}$$

$$\text{s.t. } \begin{cases} b_t \leq y_t, \\ b_t \leq D_t, \end{cases}$$

where $d(D_t)$ is the total payoff for the two banks.

Before proceeding to the analysis of the equilibrium in the next subsection, we briefly describe the intuition. As banks choose the amount of repayment (b_t) after the borrower produces the output (y_t), the banks cannot credibly commit to making $b_t < y_t$, unless $D_t < y_t$. Only if $D_t < y_t$ can the banks credibly commit to making b_t strictly less than y_t because they are subject to the constraint $b_t \leq D_t$. Thus, D_t works as a commitment device for lenders if D_t is sufficiently small. In the next subsection, we show that, when the initial debt D_0 is no greater than a certain threshold, $D_{\max}$, the lenders can credibly commit to the repayment plan that gives a positive value to the borrower. This is because both the borrower and lenders know that the constraint $b_t \leq D_t$ will eventually bind, and the repayment will be terminated within a finite period. In this case, the lenders' commitment can induce the first-best outcome $(e_t, y_t) = (e, y_H)$ for all t. It is also shown that, when D_0 exceeds $D_{\max}$, the lenders are unable to commit to make the borrower's value positive, because the constraint $b_t \leq D_t$ never binds and nothing can prevent the banks from taking all output y_t as a repayment b_t for all t. In this case, the outcome becomes inefficient, $(e_t, y_t) = (0, y_L)$.

2.2 Equilibrium

The initial value of debt D_0 changes the equilibrium outcome drastically if it exceeds a threshold value, $D_{\max}$, which is defined as the maximum amount of debt that makes the borrower's value nonnegative, given that the borrower fully repays $D_{\max}$.[6] Thus,

$$D_{\max} = \frac{y_H - e}{1 - \beta}.$$

We show by the guess-and-verify method that

$$V(D) = \begin{cases} D_{\max} - D & \text{for } D \leq D_{\max}, \\ 0 & \text{for } D > D_{\max}, \end{cases} \tag{3}$$

and

$$d(D) = \begin{cases} D & \text{for } D \leq D_{\max}, \\ d_L & \text{for } D > D_{\max}, \end{cases} \tag{4}$$

where $d_L \equiv \frac{1}{1-\beta} y_L$. In the following, we show that, given the assumption that $V(D_{t+1})$ and $d(D_{t+1})$ satisfy (3) and (4), respectively, the solutions to (1) and (2), that is, $V(D_t)$ and $d(D_t)$, also satisfy (3) and (4), respectively.

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[6]Suppose the borrower repays $D_{\max}$ by paying y_H from period 0 to period $\tau - 1$. Next, $D_{\max} = \frac{1-\beta^\tau}{1-\beta} y_H$. Because V_0 should be zero if the borrower repays $D_{\max}$, the parameter τ must satisfy $V_0 = \beta^\tau \frac{y_H - e}{1-\beta} - \frac{1-\beta^\tau}{1-\beta} e = 0$. Thus, τ satisfies $\beta^\tau = \frac{e}{y_H}$, implying that $D_{\max} = \frac{y_H - e}{1-\beta}$.

Let us begin with the banks' problem (2). Given the expectation that $d(D_{t+1})$ satisfies (4), the weakly dominant strategy for banks is $b_t = \min\{y_t, D_t\}$, where it is strictly dominant if $D_{t+1} > D_{\max}$.

Next, we consider the borrower's problem (1). Given the expectation that $V(D_{t+1}) = \max\{D_{\max} - D_{t+1}, 0\}$ and $b_t = \min\{y_t, D_t\}$, (1) can be written as

$$V(D_t) = \begin{cases} \max_{e_t} \ y_t - e_t + \beta D_{\max} - D_t, & \text{if } D_{t+1} \leq D_{\max}, \\ \max_{e_t} \ y_t - e_t - b_t, & \text{if } D_{t+1} > D_{\max}, \end{cases}$$

the solution to which is $e_t = e$ if $D_t \leq D_{\max}$ and $e_t = 0$ if $D_t > D_{\max}$.[7] The resulting value function $V(D_t)$ satisfies (3). Note that for $D_t > D_{\max}$, the borrower maximizes $y_t - e_t - b_t$, given the expectation that $b_t = y_t$.

Strategy $b_t = \min\{y_t, D_t\}$, together with the borrower's action $(e_t, y_t) = (e, y_H)$ for $D_t \leq D_{\max}$ and $(e_t, y_t) = (0, y_L)$ for $D_t > D_{\max}$, gives rise to the fact that $d(D_t)$ also satisfies (4), given that $d(D_{t+1})$ satisfies (4).

In the case where $D_t > D_{\max}$, if the banks could commit to $b_t = y_t - e_t$, they could have induced the first-best outcome, that is, $e_t = e$ and $y_t = y_H$, because the borrower chooses $(e_t, y_t) = (e, y_H)$ as long as the borrower's instantaneous payoff $y_t - e_t - b_t$ is expected to be nonnegative. However, the lenders cannot commit to $b_t = y_t - e_t$, because they choose $b_t = \min\{y_t, D_t\}$ by solving (2) optimally.

The borrower cannot make banks accept b_t strictly less than $\min\{y_t, D_t\}$. This is because the borrower cannot punish the lenders contingent on their past actions; for example, the borrower cannot make a credible threat that the borrower will choose $e_{t+j} = 0$ for all $j \geq 1$ if the banks promise to make $b_t \leq y_t - e_t$, and then violate the promise after the borrower chooses e_t. The reason is that this threat is not consistent with the optimization, (1), that the borrower solves from $t + 1$ on.

In summary, given the initial value of debt D_0, the equilibrium outcome is characterized as follows: For simplicity, we assume that the initial value of debt D_0 satisfies $D_0 = \frac{1-\beta^j}{1-\beta} y_H$, where j is an integer.

- For $D_0 \leq D_{\max}$, the solution is $(e_t, y_t, b_t) = (e, y_H, y_H)$ as long as $D_t > 0$, and $(e_t, y_t, b_t) = (e, y_H, 0)$ when $D_t = 0$. The present value of the total amount that the banks obtain is D_0, that is, the initial debt D_0 will be fully repaid, as long as $D_0 \leq D_{\max}$. The value for the borrower is $V(D_0) = D_{\max} - D_0$. The borrower obtains the dividends only after all debt is repaid, meaning that the borrower's payoff is backloaded.

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[7]For $D_t \leq D_{\max}$, choosing $e_t = e$ maximizes $y_t - e_t + \beta D_{\max} - D_t = D_{\max} - D_t \geq 0$. For $D_t > D_{\max}$, choosing $e_t = e$ makes $y_t - e_t + \beta D_{\max} - D_t < 0$, whereas choosing $e_t = 0$ leads to $y_t - e_t - b_t = 0$ as $b_t = y_t$, and the choice of $e_t = 0$ also results in $D_{t+1} = \beta^{-1}(D_t - y_L) > D_{\max}$. Therefore, it is optimal to choose $e_t = 0$ for $D_t > D_{\max}$ because choosing $e_t = e$ makes the borrower's value strictly negative, while choosing $e_t = 0$ makes it zero.

- If $D_0 > D_{\max}$, the solution is $(e_t, y_t, b_t) = (0, y_L, y_L)$ for all t. The present value of the total repayments that banks obtain is $\frac{1}{1-\beta} y_L \equiv d_L$. Clearly, d_L is strictly smaller than $D_{\max}$. The value for the borrower is $V(D_0) = 0$. The contractual amount of debt, D_t, grows monotonically according to the law of motion: $D_{t+1} = \beta^{-1}(D_t - y_L)$.

2.3 Why is debt relief delayed?

The above results show that the lenders' payoff is $d(D_t) = D_t$ for $D_t \leq D_{\max}$, whereas $d(D_t) = d_L$ for $D_t > D_{\max}$, with $d_L < D_{\max}$. The borrower's payoff is $V(D_t) = \max\{D_{\max} - D_t, \, 0\}$. Thus, social welfare, defined as $d(D) + V(D)$, is $D_{\max}$ for $D_t \leq D_{\max}$ and d_L for $D_t > D_{\max}$. Therefore, when D_t exceeds $D_{\max}$, the debt relief that reduces the debt from D_t to $D_{\max}$ is obviously Pareto improving. A simple question is, why then they do not reduce the debt immediately, even though debt relief makes both the borrower and banks better off? One reason is that negotiation over debt relief often takes a considerable or infinite amount of time if it is frictional bargaining. For simplicity, we assume the following war of attrition game between the two banks.

The borrower has debt D_t ($> D_{\max}$) in total, where the borrower owes $\frac{1}{2} D_t$ each to the two banks, bank 1 and bank 2. Banks negotiate over shares of the debt relief payoff. The two banks choose either strategy C (concession) or N (no concession) simultaneously at the beginning of period t. If one of the banks or both of them choose C, they can reduce D_t to $D_{\max}$. If bank 1 (bank 2) chooses C and bank 2 (bank 1) chooses N, bank 1 (bank 2) obtains $\alpha D_{\max}$ and bank 2 (bank 1) obtains $(1 - \alpha) D_{\max}$, where $0 < \alpha < \frac{1}{2}$. If both of them choose C, they reduce their debt to $D_{\max}$ and split it equally. If both of them choose N, the debt remains at D_t and they split the repayment b_t equally and the debt evolves by $D_{t+1} = \beta^{-1}(D_t - b_t)$.

This war of attrition game has a unique symmetric mixed-strategy Nash equilibrium wherein each bank chooses C with the same probability p. The value of p is characterized by the condition that the bank's payoff of choosing C and that of choosing N are equal, given that the opponent chooses C with probability p. We can show the following proposition.

Proposition 1. *For $\alpha \in (\frac{y_L}{2(y_H - e)}, \frac{1}{2})$, there exists a unique mixed-strategy Nash equilibrium wherein banks choose C with a positive probability. For $\alpha \in [0, \frac{y_L}{2(y_H - e)}]$, there exists a unique Nash equilibrium in which both banks always choose N.*

Proof. The sum of the payoffs for the two banks is $D_{\max}$ when at least one bank chooses

C, and $d(p)$ when both banks choose N, where

$$d(p) = b_t + \beta[(1-p)^2 d(p) + \{1 - (1-p)^2\} D_{\max}]$$

$$= y_L + \beta(1-p)^2 d(p) + \beta\{1 - (1-p)^2\}\frac{y_H - e}{1 - \beta}$$

$$= \frac{y_L}{1 - \beta(1-p)^2} + \left(\frac{\beta - \beta(1-p)^2}{1 - \beta(1-p)^2}\right)\frac{y_H - e}{1 - \beta},$$

because $(e_t, y_t, b_t) = (0, y_L, y_L)$ when $D_t > D_{\max}$, under the condition that debt relief will be implemented with a positive probability in the future.[8] Given the opponent chooses C with probability p, the payoff of choosing N is $p(1-\alpha)D_{\max} + (1-p)\frac{1}{2}d(p)$ and that of choosing C is $\frac{p}{2}D_{\max} + (1-p)\alpha D_{\max}$. The equilibrium condition that determines p is that the two payoffs are equal: $g(p) = 0$, where

$$g(p) = p(1-\alpha)D_{\max} + (1-p)\frac{1}{2}d(p) - \frac{p}{2}D_{\max} - (1-p)\alpha D_{\max}.$$

For $\alpha \in (\frac{y_L}{2(y_H - e)}, \frac{1}{2})$, we have

$$g(0) = \frac{1}{2}\frac{y_L}{1 - \beta} - \alpha D_{\max} < 0,$$

$$g(1) = \left(1 - \alpha - \frac{1}{2}\right)D_{\max} > 0.$$

Differentiating $g(p)$, it is easily confirmed that

$$g'(p) = \frac{y_H - e - y_L}{2[1 - \beta(1-p)^2]} + \frac{\beta(1-p)^2(y_H - e - y_L)}{[1 - \beta(1-p)^2]^2} > 0.$$

Therefore, there exists a unique $p^* \in (0, 1)$ such that $g(p^*) = 0$. The equilibrium wherein each bank chooses C with probability p^* is the unique mixed-strategy Nash equilibrium of the war of attrition game between the two banks.

For $\alpha \in [0, \frac{y_L}{2(y_H - e)}]$, the payoff of choosing N is larger than that of choosing C, that is, $g(p) \geq 0$, given that the opponent chooses C with probability p for any value of $p \in [0, 1]$. In this case, there exists a unique Nash equilibrium in which each bank always chooses N. □

The simple example in this section is a game wherein the borrower chooses the output y_t, and then the lenders (collectively) choose the repayment b_t in every period t. We chose this setting to demonstrate the lack of lenders' commitment intuitively, that is, the lender

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[8]It is shown as follows that the equilibrium with a positive probability of debt relief is $(e_t, y_t, b_t) = (0, y_L, y_L)$, when $D_t > D_{\max}$. In the equilibrium $(e_t, y_t, b_t) = (0, y_L, y_L)$ for all t, the borrower's payoff is $V(D_t) = 0$. In addition, we know that $V(D_{\max}) = 0$ when debt relief is implemented. As the borrower expects that the bank chooses $b_t = y_t$ when $D_t > D_{\max}$, the borrower's deviation $(e_t = e)$ makes $V_t = y_H - b_t - e + \beta V_{t+1} = -e < 0$ because $V_{t+1} = 0$ in any case. Thus, the borrower never deviates from $e_t = 0$.

cannot commit to make b_t strictly less than y_t, unless $y_t > D_t$. In the full model in the next section, the lender chooses both the output y_t and repayment b_t, and the loss of the lender's commitment is reflected only in the future values of the borrower, $V(D_{t+j})$, that is, the lender can commit to make the future values $V(D_{t+j})$ contingent on D_{t+j} for $j = 1, 2, \cdots$, if D_{t+j} is small, whereas it cannot commit to make the future value $V(D_{t+j})$ contingent on D_{t+j} if D_{t+j} is larger than the threshold.

3   Model

We modify the AH model of long-term debt contracts by restricting the possibility of debt restructuring. Here, we focus on debt relief or debt forgiveness as a means of debt restructuring. In the baseline model, we consider a case wherein debt forgiveness is not feasible. This is extended in section 4 to allow for stochastic debt restructuring. As in AH, the borrowing constraint arises because the borrower may default at any time. The amount of debt can accumulate over time if negative productivity shocks hit the borrower repeatedly. If the debt exceeds a threshold value, it is no longer repayable. Then, as we discuss, the lender loses credibility regarding the future repayment plans it offers, which leads to an equilibrium outcome that is constrained inefficient. The loss of the lender's credibility is permanent in the baseline model, whereas the credibility can be restored stochastically with a constant probability in the extended model in section 4.

3.1   Setup

We consider an economy wherein time is discrete and goes from zero to infinity, that is, $t = 0, 1, 2, \cdots, \infty$. There is a lender (bank) and a borrower (firm or sovereign) with a common discount factor β, where $0 < \beta < 1$. At the beginning of period 0, the borrower owes D_0 to the lender as the initial debt. The interest rate for debt D_0 is fixed at r in the debt contract. We assume that the value of r is given exogenously, satisfies $\beta \geq \frac{1}{1+r}$ as there exists a default risk, and is constant over time. Our assumption that the interest rate r is a given constant can be interpreted as that in a small open economy model, where the borrower is a sovereign and the lender is a group of international banks, and r is the world interest rate. Alternatively, we can interpret that the borrower is a firm and the lender is a domestic bank in the partial equilibrium model of corporate debt, where r is the market rate. In the general equilibrium model of corporate debt, the value of r is determined by the bank's zero-profit condition, as described in footnote 10.

The debt at the beginning of period t, D_t, evolves as

$$D_{t+1} = (1+r)(D_t - b_t), \quad \text{for } t \geq 0, \tag{5}$$

where b_t is the repayment in period t. In each period t, the borrower needs to borrow

capital service (working capital), k_t, to generate revenue, $F(s_t, k_t)$, where $s_t \in \mathbb{R}_+$ is the borrower's productivity in period t.

The revenue function $F(s, k)$ is a continuously differentiable function that satisfies $F(s, 0) = 0$, and $F_k(s, k) > 0$, $F_s(s, k) > 0$, $F_{sk}(s, k) > 0$, and $F_{kk}(s, k) < 0$ for $k > 0$, where $F_k \equiv \frac{\partial F}{\partial k}$, $F_s \equiv \frac{\partial F}{\partial s}$, $F_{sk} \equiv \frac{\partial^2 F}{\partial k \partial s}$, and $F_{kk} \equiv \frac{\partial^2 F}{\partial k^2}$. Productivity s_t is either s_H or s_L, where $0 \leq s_L < s_H$ and changes over time following a stationary Markov process with $\Pr(s_{t+1} = s_j | s_t = s_i) = \pi_{ij}$, where $\pi_{ij} > 0$ for $i, j \in \{L, H\}$. The borrower finances the input k_t by borrowing the amount Rk_t of an intra-period loan from the bank.[9] The borrower borrows Rk_t at the beginning of period t and repays Rk_t at the end of the same period t, where the price of capital input R is constant.

The borrower: The dividend to the borrower is $F(s_t, k_t) - Rk_t - b_t$. The borrower in our economy is protected by limited liability, so that the dividend is nonnegative:

$$F(s_t, k_t) - Rk_t - b_t \geq 0, \quad \forall t \geq 0. \tag{6}$$

Let V_t denote the expected value of the PDV of dividends:

$$V_t \equiv \mathbb{E}_t \left\{ \sum_{j=0}^{\infty} \beta^j [F(s_{t+j}, k_{t+j}) - Rk_{t+j} - b_{t+j}] \right\} = F(s_t, k_t) - Rk_t - b_t + \beta \mathbb{E}_t V_{t+1}, \tag{7}$$

where $\mathbb{E}_t$ is the expectation operator as of time t.

In any period t, the borrower can choose to default after receiving working capital k_t. If the borrower defaults on debt $D_t + Rk_t$, it has the outside opportunity to use k_t and earn $G(s_t, k_t)$. The value of the outside opportunity $G(s, k)$ is a continuously differentiable function that satisfies $G(s, 0) = 0$, and $G_k(s, k) > 0$, $G_s(s, k) \geq 0$, and $G_{kk}(s, k) \leq 0$ for $k > 0$. Furthermore, it is assumed that $F(s, k)$ and $G(s, k)$ satisfy

$$F_{kk}(s, k) - G_{kk}(s, k) < 0, \quad \text{and} \quad F_{ks}(s, k) - G_{ks}(s, k) > 0,$$

for all s and k (> 0). To prevent the borrower from defaulting, the value of the borrower, V_t, must satisfy

$$V_t \geq G(s_t, k_t), \qquad \forall t \geq 0, \tag{8}$$

which yields the borrowing limit on k_t.

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[9]In this study, we assume for simplicity that the borrower borrows the intra-period loan, Rk_t, from the same bank from which it borrowed the initial loan, D_0. It can be easily confirmed that our result does not change even if the borrower borrows Rk_t from other banks.

The bank: Given the market rates of interest for the inter-period debt, r, and the intra-period loan, R, the bank chooses an offer $\{b_{t+j}, k_{t+j}\}_{j=0}^{\infty}$ in period t to maximize the expected value of the present discouted value (PDV) of repayments, d_t, which is defined as

$$d_t = \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j b_{t+j}. \tag{9}$$

The offer is made to the borrower in a take-it-or-leave-it manner. If the borrower declines the offer, it will be liquidated. For simplicity, we assume that the liquidation value of the borrower is zero. In equilibrium, the bank never chooses an offer that will be rejected by the borrower.

As explained above, the borrower has an option to default after it accepts an offer and receives working capital k_t. We assume that the bank obtains nothing when the borrower chooses to default. In this case, $G(s_t, k_t)$ can be interpreted as the liquidation value of the borrower, which is obtained by the borrower at the time of default.

As the contractual value of the debt, D_t, is verifiable, the bank has no legal right to require a repayment that exceeds the outstanding debt. Thus, the following constraint must be satisfied:

$$b_t \leq D_t, \tag{10}$$

for all $t \geq 0$.

Debt restructuring: In this paper, by the term *debt restructuring* we mean debt forgiveness or debt relief, that is, a reduction in the contractual amount of debt, D_t. In the baseline model, we consider a case wherein debt restructuring is not possible. Thus, D_t cannot deviate from the law of motion (5), and, thus, it is generally different from the PDV of repayments, d_t. This is the most crucial difference between our model and AH model. In the AH model, there is no distinction between D_t and d_t, which reflects the assumption that debt restructuring occurs immediately at every instant of time.

3.2 The bank's problem and the equilibrium

Throughout this paper, we focus on the Markov perfect equilibrium, in which all agents' actions and the value functions in each period t are functions of the state variables $(s_t, D_t) \in \{s_L, s_H\} \times \mathbb{R}_+$. This is formulated in a recursive manner. In doing so, we omit the time subscript and use the subscript $+1$ for the variables in the next period, and the subscript, -1, for the variables in the previous period.

The bank's problem: Given a belief in the borrower's value, $V^e(s, D)$, the bank solves the following problem,

$$d(s, D) = \max_{b,k} \quad b + \beta \mathbb{E} d(s_{+1}, D_{+1}) \tag{11}$$

$$\text{s.t.} \quad \begin{cases} F(s, k) - Rk - b + \beta \mathbb{E} V^e(s_{+1}, D_{+1}) \geq G(s, k), \\ F(s, k) - Rk - b \geq 0, \\ D_{+1} = (1 + r)(D - b), \\ b \leq D, \end{cases}$$

where the first constraint is the borrowing constraint (8), the second constraint is the limited liability constraint (6), and the third is the law of motion for debt (5). The solution to this problem is written as

$$b = b(s, D),$$

$$k = k(s, D).$$

We define

$$D_{+1}(s, D) \equiv (1 + r)(D - b(s, D)),$$

$$V(s, D) \equiv F(s, k(s, D)) - Rk(s, D) - b(s, D) + \beta \mathbb{E} V^e(s_{+1}, D_{+1}(s, D)).$$

Assuming rational expectations, the following condition must be satisfied in equilibrium:

$$V(s, D) = V^e(s, D), \tag{12}$$

and

$$V^e(s, D) \leq \frac{1}{1 - \beta} \{ F(s_H, k^*(s_H)) - Rk^*(s_H) \},$$

where $k^*(s)$ is the first-best level of working capital at $s \in \{s_H, s_L\}$.

$$k^*(s) \equiv \arg \max_k F(s, k) - Rk. \tag{13}$$

Assumption 1. If there exist multiple solutions to the maximization problem in (11) for some (s, D), the bank selects the solution that maximizes $k(s, D)$. Thus, if both (b_1, k_1) and (b_2, k_2) solve the problem and $k_2 < k_1$, then $k(s, D) = k_1$ and $b(s, D) = b_1$.

Definition 1. Rational expectations equilibrium is a solution to (11), $\{k(s, D), b(s, D), d(s, D), V(s, D), D_{+1}(s, D)\}$, which satisfies (12).

Note that our model is a partial equilibrium model wherein r is given exogenously.[10]

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[10] In the general equilibrium model, the value of r is determined by the zero-profit condition for the bank, given the initial amount of lending:

$$D_0 = d(s_0, D_0),$$

where D_0 is an exogenous parameter that represents the initial amount of bank lending, and $d(s_0, D_0)$ is the bank's payoff, defined as the solution to (11). The interest rate r is determined such that the above equation holds.

As discussed in section 3.4 below, there may be a fundamental difficulty in proving the existence of an equilibrium and characterizing it for general values of D. In the Online Appendix, we discretize the model to overcome this difficulty and prove the existence of the equilibrium. Here, we provide a brief description of the equilibrium, which is a summary of the online appendix. There exists a threshold $D_{\max}(> 0)$ for the contractual amount of debt, and the equilibrium path is dynamic and constrained efficient, similar to the AH model, for $D_t \leq D_{\max}$, whereas it is static and inefficient for $D_t > D_{\max}$. We describe this in more detail.

There are threshold values D^* and $D_{\max}$, where $0 < D^* < D_{\max}$ and the production attains the first-best value for $D \leq D^*$, that is, $k(s, D) = k^*(s)$. When $D \in [0, D^*]$, $d(s, D) = D$, and $V(s, D) = V^*(s) - D$, where $V^*(s) = F(s, k^*(s)) - Rk^*(s) + \beta \mathbb{E}[V^*(s_{+1})|s]$. The constraints $V^*(s) \geq G(s, k^*(s))$ and $F(s, k^*(s)) - Rk^*(s) - b \geq 0$ are nonbinding.

In the case where $D_t \in (D^*, D_{\max}]$, the economy that consists of the borrower and lender falls into the first type of debt overhang: $k(s, D)$ and $V(s, D)$ decrease with D, and $k(s, D) < k^*(s)$ and $V(s, D) < V^*(s) - D$. The borrower's payoff is backloaded, so that the dividend to the borrower is zero, and $b(s, D) \leq F(s, k(s, D)) - Rk(s, D)$ is binding in this region. The borrowing constraint $V(s, D) \geq G(s, k)$ is also binding. This borrowing constraint is imposed because the borrower lacks the ability to commit to repaying the debt. Furthermore, the bank can commit to take no more than D, and borrower can believe that the payoff will be nonnegative, as the borrower can take the remaining value after repaying D. The bank's ability to commit makes the dynamic provision of incentives feasible so that the constrained efficient outcome, $k(s, D)$, is achieved.

In the case where $D_t > D_{\max}$, the economy falls into the second type of debt overhang and the equilibrium path becomes static: $\{k(s, D), b(s, D), d(s, D), V(s, D)\} = \{k^{npl}(s), b^{npl}(s), d^{npl}(s), G^{npl}(s)\}$, where the superscript "npl" stands for "nonperforming loans." These are the values of the variables chosen under the two-sided lack of commitment wherein neither the borrower nor the lender can commit to any dynamic repayment plan. Because D is larger than the feasible amount of repayment, D_t monotonically increases over time, and the bank's commitment to take no more than D is meaningless as a constraint to the bank's action. For example, banks cannot commit to terminating repayments within a finite period. As we prove in the next subsection, the repayment path becomes static because the lender becomes unable to commit to any dynamic repayment plan when D_t exceeds a certain threshold.

3.3 Second Type of Debt Overhang

Now, we restrict attention to the behavior of the equilibrium for large values of D. Define $\bar{D}$ by

$$\bar{D} \equiv \frac{1+r}{r}\{F(s_H, k^*(s_H)) - Rk^*(s_H)\},$$

where $k^*(s_H)$ is the first-best level of working capital at s_H, as defined in (13). Clearly, there is no way for the firm to repay more than $\bar{D}$. In this subsection, we show that the equilibrium path does not depend on D when

$$D > \bar{D}. \tag{14}$$

We focus on this case, although the threshold $D_{\max}$, which is characterized in the Online Appendix, may be less than $\bar{D}$. We define $k^{npl}(s)$, and $G^{npl}(s) \equiv G(s, k^{npl}(s))$ as follows. First, define $\tilde{k}^{npl}(s)$ as

$$\tilde{k}^{npl}(s) \equiv \arg\max_k F(s, k) - Rk - G(s, k).$$

Then, if the following inequality holds for each $s \in \{s_L, s_H\}$

$$G(s, \tilde{k}^{npl}(s)) \geq \beta\mathbb{E}[G(s_{+1}, \tilde{k}^{npl}(s_{+1}))|s], \tag{15}$$

then set $k^{npl}(s) = \tilde{k}^{npl}(s)$. Note that (15) is necessarily satisfied for $s = s_H$ under our assumption, as we see below. This is because $F_{kk} - G_{kk} < 0$ and $F_{ks} - G_{ks} > 0$, $\tilde{k}^{npl}(s_H) > \tilde{k}^{npl}(s_L)$.[11] Then, because $G(s, k)$ is increasing in both s and k, $G(s_H, \tilde{k}^{npl}(s_H)) > G(s_L, \tilde{k}^{npl}(s_L))$. It follows that (15) is satisfied for $s = s_H$.

If (15) is not satisfied for $s = s_L$, then we redefine $k^{npl}(s)$ by

$$k^{npl}(s_H) = \tilde{k}^{npl}(s_H), \tag{16}$$

$$G(s_L, k^{npl}(s_L)) = \beta\mathbb{E}[G(s_{+1}, k^{npl}(s_{+1}))|s_L]. \tag{17}$$

Given $k^{npl}(s_H) = \tilde{k}^{npl}(s_H)$, there exists a unique solution $k^{npl}(s_L)$ that solves equation (17). The next lemma demonstrates that k is no less than k^{npl} in equilibrium.

Lemma 2. *In equilibrium, $k(s, D) \geq k^{npl}(s)$ for all $s \in \{s_L, s_H\}$ and $D \in \mathbb{R}_+$.*

Proof. Suppose that $k(s, D) < k^{npl}(s)$ for some $(s, D) \in \{s_L, s_H\} \times \mathbb{R}_+$. Subsequently, the bank can increase both k and b without violating any constraints. This contradicts Assumption 1. Thus, $k(s, D) \geq k^{npl}(s)$. □

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[11] By differentiating the first-order condition for the definition of $\tilde{k}^{npl}(s)$, we obtain

$$\frac{d\tilde{k}^{npl}}{ds} = -\frac{F_{ks} - G_{ks}}{F_{kk} - G_{kk}} > 0.$$

The next proposition is one of the main results of this study. This implies that, if the contractual amount of debt D exceeds a threshold value $\bar{D}$, then (i) the equilibrium values $\{k(s,D), b(s,D), d(s,D), V(s,D)\}$ do not depend on D; (ii) their values correspond to $k^{npl}(s)$ defined above; and (iii) the contractual amount of debt D will never decrease. A similar but stronger result is obtained for the discrete version of the model in Proposition 8 in the online appendix.

Proposition 3. *For $D > \bar{D}$, the equilibrium values of the variables do not depend on D, and satisfy $\{k(s,D), b(s,D), d(s,D), V(s,D)\} = \{k^{npl}(s), b^{npl}(s), d^{npl}(s), G^{npl}(s)\}$, where*

$$b^{npl}(s) \equiv F(s, k^{npl}(s)) - Rk^{npl}(s) - G^{npl}(s) + \beta\mathbb{E}G^{npl}(s_{+1}),$$
$$d^{npl}(s) \equiv b^{npl}(s) + \beta\mathbb{E}d^{npl}(s_{+1}).$$

The proof is provided in Appendix A. As Lemma 2 shows, k^{npl} is the lowest level of working capital provision that can occur in equilibrium. Thus, once D becomes greater than $\bar{D}$, the equilibrium level of production falls to the lowest level *permanently*. This creates a sharp contrast with the property of the constrained efficient equilibrium analyzed by AH, where the first-best provision of working capital is attained in a finite period of time with probability one in the absence of liquidation.

Intuitively, the proposition follows from the fact that the contractual amount of debt D is no longer payoff-relevant if it becomes so large that there is no way for the borrower to pay it back in full. Thus, the offer $\{k(s,D), b(s,D), d(s,D), V(s,D)\}$ made by the bank cannot depend on D in the region where $D > \bar{D}$. In other words, the bank loses its ability to commit to future repayment plans when the debt becomes "too large." More specifically, when D is too large, the bank cannot commit to any dynamic repayment plan that offers a feasible amount of the PDV of the repayments, which is strictly smaller than the contractual amount of debt, D. The loss of the bank's credibility forces its offer to be "static," depending solely on the current exogenous state s. As discussed by AH, constrained efficiency requires the offer to be dynamic. In particular, the payoff to the borrower must be backloaded until the amount of debt becomes sufficiently small. In the absence of debt restructuring, too much debt makes the dynamic provision of incentives infeasible, leading to an inefficiently low level of production.

3.4 Note on equilibrium for a small D

We have established that the economy falls into the second type of debt overhang for sufficiently large D. However, for a smaller value of D, it is difficult to provide a rigorous characterization of the equilibrium. There are two reasons for this, which are discussed in this subsection.

Discontinuity of the value functions: One of the difficulties is that a solution to (11) involves many (possibly an infinite number of) jumps in $\{b(s, D), k(s, D), V(s, D), d(s, D)\}$. First, there is a discontinuous jump in the variables when D equals the threshold value for the second type of debt overhang. Second, the above threshold in the initial round defines another, smaller threshold for D, at which the above variables make a discontinuous jump because the next period's debt is just equal to the threshold in the initial round. Third, the above threshold in the second round defines another, smaller threshold at which the variables jump discontinuously because the next period's debt is just equal to the threshold in the second round. Fourth, the above threshold in the third round defines another, smaller threshold at which the variables jump because the next period's debt is just equal to the threshold in the third round, and so on. This repetition continues indefinitely. Such discontinuities make the application of the standard results for dynamic programming difficult. To overcome this difficulty, we consider a discrete version of the model in the online appendix.

Competing forces of back loading and front loading: Another difficulty arises if we assume that $\beta > \frac{1}{1+r}$. This is a natural assumption because debt may not be repaid in full. However, it provides an incentive for the bank to front load the payment to the firm—at least when the debt is sufficiently small. To see this, suppose that D is sufficiently small in period 0, so that the firm can repay D at once. If D is repaid in period 0, then the value of the bank is D. On the other hand, if the firm repays nothing in period 0 and $(1 + r)D$ in period 1, then the present value for the bank in period 0 is $\beta(1 + r)D$, which is larger than D, because $\beta(1+r) > 1$. Thus, for a small value of D, the bank may choose repayment b such that D_{+1} is greater than D. Therefore, if $\beta > \frac{1}{1+r}$, there are competing forces that induce backloading and frontloading the borrower's payoff. This complicates the dynamics. Because of this difficulty, we assume that $\beta = \frac{1}{1+r}$ to obtain the analytical results in the online appendix. The case of $\beta > \frac{1}{1+r}$ is demonstrated numerically in section 3.5.

3.5 Numerical experiment

In this subsection, we report numerical solutions to our model. We obtain the equilibrium $(d(s, D), V(s, D), D_{\max}(s))$ as a fixed point of operator T, which is defined in the online appendix, by iterating $(d^{(n+1)}(s, D), V^{(n+1)}(s, D), \bar{D}^{(n+1)}(s)) = T(d^{(n)}(s, D), V^{(n)}(s, D), \bar{D}^{(n)}(s))$. The numerical examples illustrate the properties of our model discussed in the previous subsections. Furthermore, they demonstrate that our model generates a *debt Laffer curve*, that is, the bank's value $d(s, D)$ has an inverted U-shaped relationship with the contractual amount of debt D.

We assume the following functional forms: $F(s, k) = sAk^{\alpha}$ and $G(s, k) = Bk$. The

Table 1: Parameter values

	Description	Value
A	Normalization	0.1
B	Outside option	0.1
α	Production function	0.8
R	Rental rate of capital	0.1
β	Discount factor	0.96
s_H, s_L	Productivity	1.15, 0.85
π_{HH}, π_{LL}	Transition probability	0.9, 0.9

parameter values are set as shown in Table 1. Our purpose here is to confirm the properties of the model, and thus the parameter values are set somewhat arbitrarily, without much empirical grounding.[12]

3.5.1 Baseline case with $1 + r = \beta^{-1}$

Figure 1 plots the bank's value function, $d(s, D)$, the borrower's value function, $V(s, D)$, the schedule for the working capital provision, $k(s, D)$, and the repayment schedule, $b(s, D)$. The reader may be puzzled about the discrete jumps in these functions. These jumps at smaller values of D are due to discretization—for example, $b = (m - \beta n)\delta$—where $D = m\delta$ and $D_{+1} = n\delta$, with $n, m \in \mathbb{Z}$.[13] In addition, some jumps at larger values of D are caused by the discontinuity at the boundary of the second type of debt overhang, as discussed in section 3.4.

The bank's value function $d(s, D)$ in Figure 1 displays a debt Laffer curve for each s. For a sufficiently small value of D, $d(s, D) = D$, that is, D is repaid in full with probability

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[12]In particular, the value of α may appear too high. However, note that α does not correspond to the capital share. Specifically, suppose that k finances the capital input K and the production function exhibits a decreasing return to scale, that is, $Y = K^\beta L^\gamma$, where $\beta + \gamma < 1$ and L is the labor input. The borrower's revenue is then given by:

$$F(K) = \max_L K^\beta L^\gamma - wL = CK^{\frac{\beta}{1-\gamma}},$$

where $C > 0$ is a constant and $\frac{\beta}{1-\gamma} < 1$. Then, α in our model is given by $\alpha \equiv \frac{\beta}{1-\gamma}$, which is greater than the capital share.

[13] To illustrate, suppose that b is selected as a function of $D = m\delta$ to solve $b = \max_n (m - \beta n)\delta$ subject to $b < C$. Suppose that $\delta = 1$, $\beta = 0.9$, and C is an integer. Then, $D = m$, $D_{+1} = n$, and the solution to the above problem becomes

$$b(D) = C - 0.9 + 0.1x,$$

where $x = D - C \mod 9$; that is, x is an integer with $0 \le x \le 8$, and there exists an integer i such that $D = C + 9i + x$. Thus, $b(D)$ has discrete jumps at $D = C + 9i$, where $i = 1, 2, 3, \cdots$.
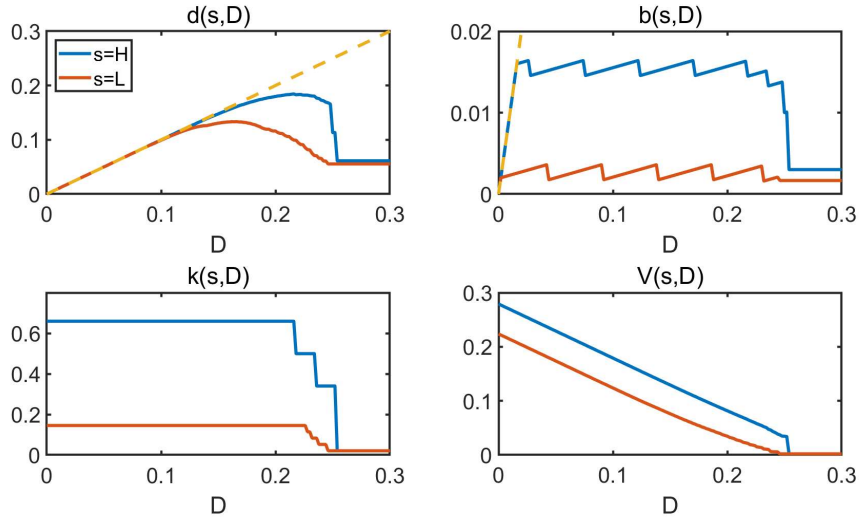
Figure 1: The case with $1 + r = \beta^{-1}$.

one, and the economy never falls into the second type of debt overhang. When D is in this region, working capital is provided at the first-best level, $k(s, D) = k^*(s)$, and the firm repays as much as possible to the bank, $b(s, D) = \min(D, F(s, k^*(s)) - Rk^*(s))$. As D increases, $k(s, D)$ and $b(s, D)$ start to decrease with D, and $d(s, D)$ exhibits an inverted-U shape in D. When D exceeds the threshold, the economy falls into the second type of debt overhang. In this example, $d(s_H, D) = d^{npl}(s_H)$ for $D > 0.252$ and $d(s_L, D) = d^{npl}(s_L)$ for $D > 0.244$. Note that in this example, the difference between $k^{npl}(s)$ and $k^*(s)$ is very large. This may be too large to be justified by the evidence. One reason for this is that the second type of debt overhang continues permanently under the assumption that debt restructuring never occurs. In section 4, we see that stochastic debt restructuring makes the difference between $k^*(s)$ and $k^{npl}(s)$ for large D much more modest.

3.5.2 Case with $1 + r > \beta^{-1}$

Thus far, we have assumed that $1 + r = \beta^{-1}$ for ease of theoretical analysis. Here, we numerically examine the case wherein $1 + r > \beta^{-1}$. We set $r = 0.05$, and $\beta = 0.96$. All other parameters were given the same values as before. The bank's value function, $d(s, D)$, the borrower's value function $V(s, D)$, the schedule for working capital provision, $k(s, D)$, and the repayment schedule, $b(s, D)$, in this case are plotted in Figure 2.

Figure 2 also demonstrates the debt Laffer curve relationship between the bank's value and the contractual amount of debt. Major differences from Figure 1 are that $b(s, D) = 0$ and $d(s, D) > D$ for small values of D. Setting $b(s, D) = 0$ is optimal because with
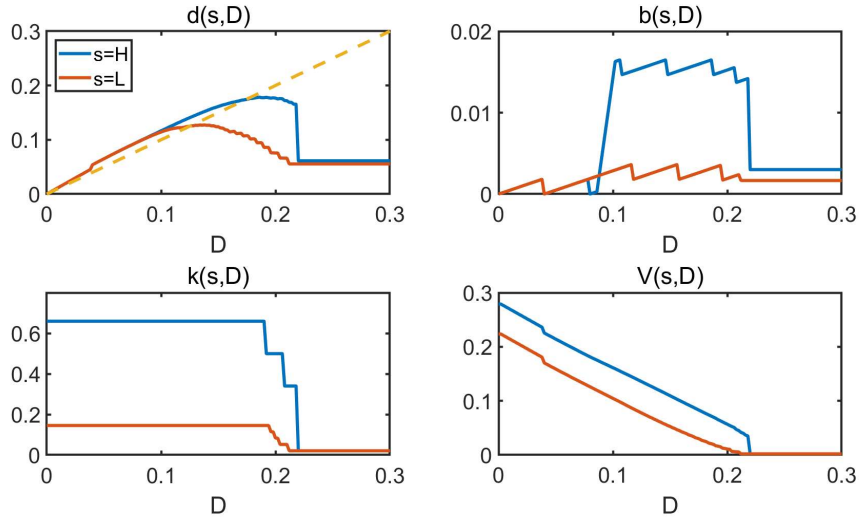
21

Figure 2: The case with $1 + r > \beta^{-1}$.

$1 + r > \beta^{-1}$, the bank can increase $d(s, D)$ by delaying the repayment when D is small, as discussed in section 3.4. As a result, $d(s, D) > D$ for small values of D. Except for these two differences, the qualitative features of the model with $1 + r > \beta^{-1}$ are the same as the model with $1 + r = \beta^{-1}$. Here, the second type of debt overhang occurs when $D > 0.218$ for $s = s_H$, and when $D > 0.210$ for $s = s_L$.

4 Extension with Stochastic Debt Restructuring

In the baseline model, debt restructuring is prohibited. We modify the model in this section such that debt restructuring is feasible with some friction. For simplicity, we assume that the bank has a chance of debt restructuring in each period t with probability $p \in (0, 1)$. If this chance arrives, the bank can reduce D_t to any value $D \in [0, D_t]$. The probability p is a fixed parameter and represents the bargaining frictions in debt restructuring. This setting can be understood as a reduced form of the war of attrition in the simple model in section 2.[14]

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[14]When D_t is so large that the economy is in the second type of debt overhang, the war of attrition (between the banks) results in a constant probability of concession, p, as the values of the borrower and lenders in the second type of debt overhang do not depend on D_t. Furthermore, when D_t is small, the probability of concession resulting from the war of attrition is generally dependent on D_t. Therefore, the setting wherein the probability of debt restructuring is constant for any D_t is not rigorously derived from the war of attrition. However, this simplification is almost innocuous because, for a small D_t, the bank chooses not to reduce D_t when it has a chance of debt restructuring.

When a bank with a contractual amount of debt D_t restructures debt, it reduces D_t to $\hat{D}(s, D_t)$ defined by

$$\hat{D}(s, D_t) = \arg \max_{0 \leq D \leq D_t} d(s, D).$$

Here, $d(s, D)$ is the PDV of repayments, given as the solution to (20) below. Clearly, $\hat{D}(s, D) = D$ for a small value of D because the bank has no incentive to reduce the debt if it is sufficiently small.

Definitions: Given the possibility of debt restructuring, we modify the formulation of the baseline model because the second type of debt overhang, $\{k^{npl}(s),\ b^{npl}(s),\ d^{npl}(s),\ G^{npl}(s)\}$ now depends on when and by how much debt is reduced.

In what follows, we formulate the bank's problem, given the beliefs $\{V^e(s, D), \hat{D}^e(s, D)\}$, where $V^e(s, D)$ describes the expected value of the borrower, and $\hat{D}^e(s, D)$ is the expected amount of debt remaining after debt relief. We used the same parameter values as those in the baseline model. For a probability p of a certain size, the candidate for $k^{npl}(s)$ makes the enforcement constraint nonbinding, that is, $\tilde{k}^{npl}(s) \equiv \arg \max_k F(s, k) - Rk - G(s, k)$ does not satisfy

$$G(s, k) > \beta \mathbb{E}[(1 - p)V^{npl}(s_{+1}) + pV^e(s_{+1}, \hat{D}^e_{+1})|s], \tag{18}$$

where we define $V^{npl}(s_{+1})$ by

$$V^{npl}(s) = F(s, k^{npl}(s)) - Rk^{npl}(s) - b^{npl}(s) + \beta \mathbb{E}[(1 - p)V^{npl}(s_{+1}) + pV^e(s_{+1}, \hat{D}^e_{+1})|s],$$

and $\hat{D}^e_{+1} = \hat{D}^e(s_{+1}, D_{+1})$.[15] Therefore, we define $k^{npl}(s) \equiv \tilde{k}^{npl}(s)$ if it satisfies (18), and define $k^{npl}(s)$ for the case wherein $\tilde{k}^{npl}(s)$ does not satisfy (18) as the solution to

$$G(s, k^{npl}(s)) = \beta \mathbb{E}[(1 - p)V^{npl}(s_{+1}) + pV^e(s_{+1}, \hat{D}^e_{+1})|s]. \tag{19}$$

Note that $k^{npl}(s)$ depends on the given beliefs $\{V^e(s, D), \hat{D}^e(s, D)\}$. We define $b^{npl}(s)$ by

$$b^{npl}(s) = F(s, k^{npl}(s)) - Rk^{npl}(s) + \beta \mathbb{E}[(1 - p)V^{npl}(s_{+1}) + pV^e(s_{+1}, \hat{D}^e_{+1})|s] - G(s, k^{npl}(s)).$$

As stated above, the values of the variables in the second type of debt overhang, $\{k^{npl}(s),\ b^{npl}(s),\ d^{npl}(s),\ V^{npl}(s)\}$, are defined, given the beliefs $\{V^e(s, D),\ \hat{D}^e(s, D)\}$.

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[15]Note that in the second type of debt overhang, where $D > D_{\max}(s)$, $\hat{D}(s, D)$ is independent of D, that is, $\hat{D}(s, D) = \hat{D}(s)$, which is defined by $\hat{D}(s) \equiv \arg \max_D d(s, D)$. Thus, for $D > D_{\max}(s)$, $\hat{D}^e(s, D)$ should also be independent of D.

The bank's problem: Given beliefs $\{V^e(s, D), \hat{D}^e(s, D)\}$, the bank solves

$$d(s, D) = \max_{b,k} \quad b + \beta\mathbb{E}[(1-p)d(s_{+1}, D_{+1}) + pd(s_{+1}, \hat{D}^e_{+1})], \tag{20}$$

$$\begin{cases} D_{+1} = (1+r)(D-b), \\ F(s, k) - Rk - b + \beta\mathbb{E}[(1-p)V^e(s_{+1}, D_{+1}) + pV^e(s_{+1}, \hat{D}^e_{+1})] \geq G(s, k), \\ F(s, k) - Rk - b \geq 0, \\ b \leq D. \end{cases} \tag{21}$$

The bank decides on (b, k), and thus $V(s, D)$:

$$V(s, D) = F(s, k) + \beta\mathbb{E}[(1-p)V^e(s_{+1}, D_{+1}) + pV^e(s_{+1}, \hat{D}^e_{+1})]. \tag{22}$$

$\hat{D}(s, D)$ is determined by

$$\hat{D}(s, D) = \arg\max_{D' \leq D} d(s, D'),$$

and $d^{npl}(s)$ is

$$d^{npl}(s) = b^{npl}(s) + \beta\mathbb{E}[(1-p)d^{npl}(s_{+1}) + pd(s_{+1}, \hat{D}^e_{+1})].$$

For consistency, we require that

$$V(s, D) = V^e(s, D), \quad \text{and} \quad \hat{D}(s, D) = \hat{D}^e(s, D). \tag{23}$$

4.1 Numerical experiment

Here, we report the results of numerical experiments for the extended model. Except for the probability of debt restructuring, p, all parameter values and functional forms are set in the same way as in the baseline model with $1 + r > \beta^{-1}$.

Figure 3 plots the main equilibrium functions for the case with $p = 0.2$. In this case, $k^{npl}(s)$ is defined by (19) for both s_H and s_L. The bank's value function $d(s, D)$ increases with D when D is small and remains constant when D is large. Thus, the debt-Laffer curve is not inverted U-shaped but inverted L-shaped. The economy enters the second type of debt overhang when $D > 0.234$ for $s = s_H$ and when $D > 0.226$ for $s = s_L$. Thus, the thresholds become larger than in the baseline model, where the second type of debt overhang arises when $D > 0.218$ for $s = s_H$ and $D > 0.210$ for $s = s_L$. In addition, $d(s, D)$ increases for each s and D compared with the baseline model. These results are consistent with our intuition because the possibility of debt restructuring increases the firm's value, relaxes the borrowing constraint, and thus raises the amount of debt that the firm can repay. In addition, the difference between $k^*(s)$ and $k^{npl}(s)$ is more modest than that in the baseline model: $k^{npl}(s_H)/k^*(s_H) = 0.313$ and $k^{npl}(s_L)/k^*(s_L) = 1$.
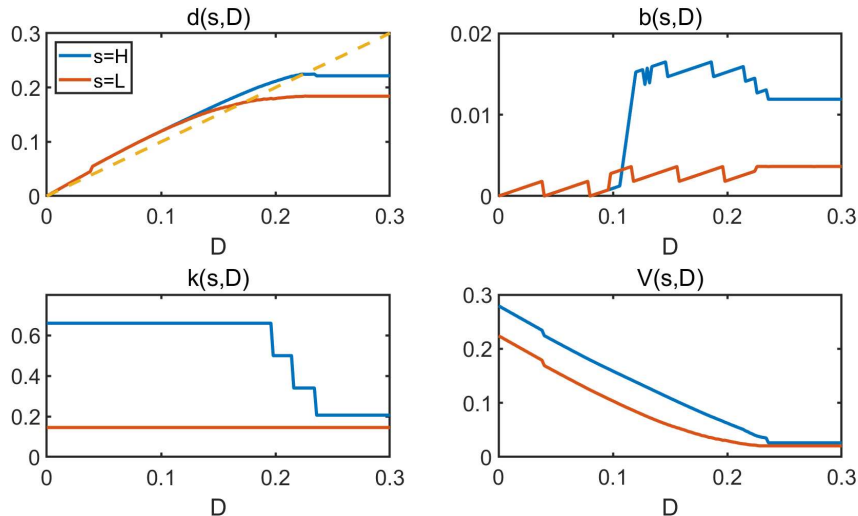
Figure 3: The case with frictional debt restructuring ($p = 0.2$).

Figure 4 shows how the equilibrium is affected by the possibility of debt restructuring, where we compare three values of p: 0, 0.002, 0.2. The left panels show the equilibrium functions corresponding to state s_H, and the right panels show those corresponding to state s_L. Although $k^{npl}(s)$ is defined by (19) for both s_H and s_L in the case where $p = 0.002$, the variables in this case are almost the same as those in the baseline case with $p = 0$. They show that an increase in the possibility of debt restructuring leads to upward shifts in the bank's value function, working capital provision, and the firm's value function.

5 Concluding remarks

In this study, a model of long-term debt contracts has been analyzed, and it has been demonstrated that the accumulation of debt can cause persistent inefficiency by impairing the lender's commitment to dynamic repayment plans. To the extent that debt restructuring is delayed due to political and/or bargaining frictions, a borrower's debt may grow to an unrepayable level. Without reducing the debt to some repayable level, the lender loses its credibility with respect to any future repayment plans it may offer to the borrower. This is because the borrower anticipates that the lender will not keep the promise that the lender will not take all output, as the amount of debt is so large that the lender has a legitimate right to demand all as the repayment. This impairment of the lender's credibility discourages the borrower from expending effort, leading to an inefficiently low level of activity and persistent inefficiency. Although the optimal contract features a backloaded payoff to the borrower until the amount of debt becomes sufficiently small, it is no longer possible to provide incentives dynamically when the debt becomes "too large," because
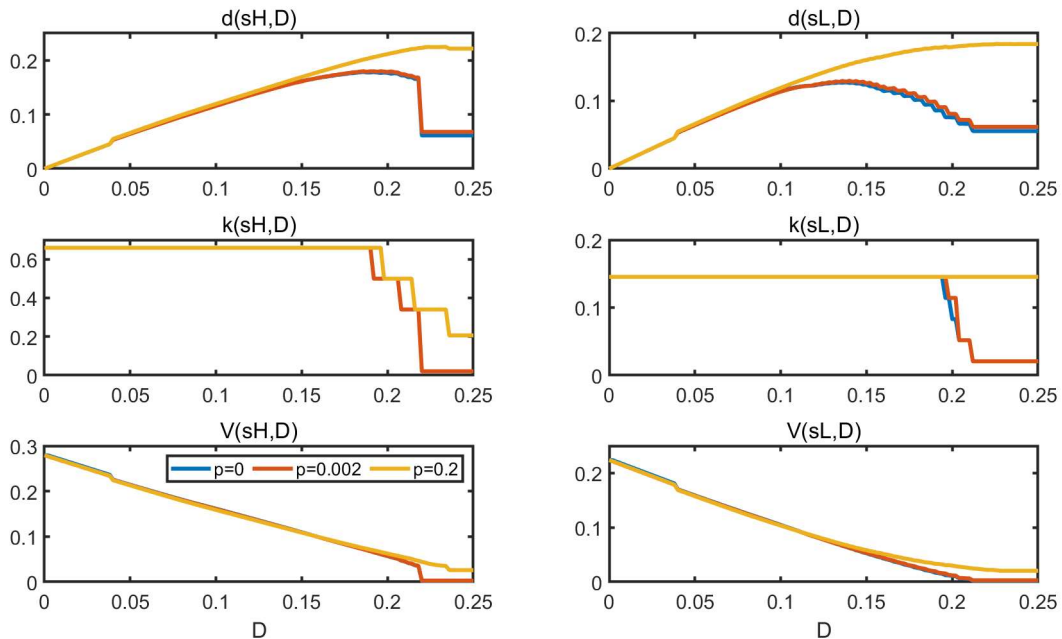
Figure 4: Frictional debt restructuring ($p = 0.0,\ 0.002,\ 0.2$).

of this lack of lender's credibility. Our model generates a debt Laffer curve, that is, the payoff to the lender, which is the present discounted value of repayments, can decrease with the contractual amount of debt if it exceeds a certain threshold value. The efficiency of equilibrium can be improved by debt relief, implying that policy measures that reduce bargaining frictions and facilitate debt restructuring are Pareto improving.

This study has several limitations. As the focus of our analysis is solely theoretical, the model used is simplistic and stylized. Thus, further elaboration is needed to be applicable to real episodes of financial crises and business fluctuations—for instance, the possibility of secular stagnation in the aftermath of the global financial crisis. The bargaining frictions of debt restructuring could be modeled more explicitly, as opposed to the reduced-form approach adopted in this study. All these extensions should be explored in future research.

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A Proof of Proposition 3

In this appendix, we characterize that the values of the variables in the second type of debt overhang are not dependent on the contractual amount of debt, D. First, we prove the following lemma.

Lemma 4. *If $k(s, D) < k^*(s) \equiv \arg\max_k F(s, k) - Rk$, then*

$$\beta\mathbb{E}G(s_{+1}, k(s_{+1}, D_{+1})) \leq G(s, k(s, D)),$$
$$V(s, D) = G(s, k(s, D)).$$

Proof. For any s, any b that is feasible for $k < k^{npl}(s)$ is also feasible for $k = k^{npl}(s)$. Thus, $k(s, D) \geq k^{npl}(s)$. Suppose that $\beta\mathbb{E}G(s_{+1}, k(s_{+1}, D_{+1})) > G(s, k(s, D))$. Then, there exists $\varepsilon > 0$, such that $k = k(s, D) + \varepsilon$ ($\leq k^*(s)$) is feasible under the borrowing constraint ($F(s, k) - Rk - b + \beta\mathbb{E}V^e(s_{+1}, D_{+1}) \geq G(s, k)$) because $V^e(s, D) = V(s, D) \geq G(s, D)$ in equilibrium. Then, Assumption 1 implies that the equilibrium value of k should be $k(s, D) + \varepsilon$, not $k(s, D)$. This is a contradiction. Therefore, $\beta\mathbb{E}G(s_{+1}, k(s_{+1}, D_{+1})) \leq G(s, k(s, D))$. Suppose that $V(s, D) > G(s, k(s, D))$. Then, there exists $\varepsilon > 0$ such that $k = k(s, D) + \varepsilon$ ($\leq k^*(s)$) is feasible. Assumption 1 implies that $k = k(s, D) + \varepsilon$ should be the equilibrium value. This is a contradiction. Thus, $V(s, D) = G(s, k(s, D))$ in equilibrium. $\square$

Because $D > \bar{D}$ implies that $D_{+1} = (1 + r)(D - b) > D$ for any feasible b, the lender's commitment constraint ($b \leq D$) never binds to D. Thus, the bank's problem can be rewritten as

$$d(s, D) = \max_{b,k} \ b + \beta\mathbb{E}d(s_{+1}, D_{+1}) \tag{24}$$
$$\text{s.t.} \quad V = F(s, k) - Rk - b + \beta\mathbb{E}V^e(s_{+1}, D_{+1}),$$
$$V \geq G(s, k),$$
$$F(s, k) - Rk - b \geq 0,$$

with the equilibrium conditions $V(s, D) = V^e(s, D)$ and $V^e(s, D) \leq V_{\max}$, where

$$V_{\max} \equiv \frac{1}{1 - \beta}\{F(s_H, k^*(s_H)) - Rk^*(s_H)\}.$$

Lemma 5. *Consider the case where $D > \bar{D}$. Suppose that $F(s, k(s, D)) - Rk(s, D) - b(s, D) > 0$ for some (s, D). Then, $k(s, D) = k^{npl}(s)$ for the same (s, D).*

Proof. The proof of this claim is by contradiction. Suppose that $k(s, D) \neq k^{npl}(s)$ for a particular value of (s, D), for which $F(s, k(s, D)) - Rk(s, D) - b(s, D) > 0$. Then, Lemma 2 implies $k(s, D) > k^{npl}(s)$. Define $\varepsilon(s, D) \equiv F(s, k(s, D)) - Rk(s, D) - b(s, D)$. We define $k^\varepsilon(s, D) = \max\{k(s, D; \varepsilon), k^{npl}(s)\}$, where $k(s, D; \varepsilon)$ is the solution to $F(s, k) - $

$Rk - b(s, D) = \frac{1}{2}\varepsilon(s, D)$. Obviously, $k^\varepsilon(s, D) < k(s, D)$. Now, we define $b^\varepsilon(s, D) = \min\{b_1(s, D), b_2(s, D)\}$, where

$$b_1(s, D) = F(s, k^\varepsilon(s, D)) - Rk^\varepsilon(s, D),$$

and

$$b_2(s, D) = \max\{b \mid F(s, k^\varepsilon(s, D)) - Rk^\varepsilon(s, D) - b + \beta\mathbb{E}V^e(s_{+1}, (1+r)(D-b)) \geq G(s, k^\varepsilon(s, D))\}.$$

Note that $b_2(s, D) = +\infty$ may be possible for some (s, D). Obviously, $b_1(s, D) > b(s, D)$, because $b_1(s, D) = b(s, D) + \frac{1}{2}\varepsilon$ when $k^\varepsilon(s, D) = k(s, D; \varepsilon)$, and $b_1(s, D) > b(s, D) + \frac{1}{2}\varepsilon$ when $k^\varepsilon(s, D) = k^{npl}(s) > k(s, D; \varepsilon)$. Furthermore, it is easily confirmed that $b_2(s, D) > b(s, D)$, because $b_2(s, D)$ is the maximum value of b that satisfies

$$b \leq F(s, k^\varepsilon(s, D)) - Rk^\varepsilon(s, D) - G(s, k^\varepsilon(s, D)) + \beta\mathbb{E}V^e(s_{+1}, (1+r)(D-b)),$$

and $b(s, D)$ is the maximum value of b that satisfies

$$b \leq F(s, k(s, D)) - Rk(s, D) - G(s, k(s, D)) + \beta\mathbb{E}V^e(s_{+1}, (1+r)(D-b)),$$

where $F(s, k) - Rk - G(s, k)$ is decreasing in k for $k > k^{npl}(s)$. Here, we used $k(s, D) > k^{npl}(s)$ to show that $b_2(s, D) > b(s, D)$. Because $b_1(s, D) > b(s, D)$ and $b_2(s, D) > b(s, D)$, it is obvious that $b^\varepsilon(s, D) > b(s, D)$ for a particular (s, D). Because $\{b^\varepsilon(s, D), k^\varepsilon(s, D)\}$ satisfies all the constraints of (24), it is feasible. As formally stated in Claim 1, $\{b^\varepsilon(s, D), k^\varepsilon(s, D)\}$ should be the solution to (24) instead of $\{b(s, D), k(s, D)\}$, which contradicts the fact that $\{b(s, D), k(s, D)\}$ is the solution to (24). Therefore, $k(s, D) = k^{npl}(s)$ should hold if $F(s, k(s, D)) - Rk(s, D) - b(s, D) > 0$.

The reason why $\{b^\varepsilon(s, D), k^\varepsilon(s, D)\}$ should be the solution to (24) is formally described in the following Claim 1. First, we define the sequential problem corresponding to the recursive problem (24) as follows: For $(s_0, D_0) = (s, D)$, the bank's value, $d^*(s, D)$, is defined as the solution to the sequential problem

$$d^*(s, D) \equiv \max_{\{b_t, k_t\}_{t=0}^\infty} \mathbb{E}_0\left[\sum_{t=0}^\infty \beta^t b_t\right], \tag{25}$$
$$\text{s.t.} \quad F(s_t, k_t) - Rk_t - b_t + \beta\mathbb{E}_t V^e(s_{t+1}, D_{t+1}) \geq G(s_t, k_t),$$
$$F(s_t, k_t) - Rk_t - b_t \geq 0,$$
$$D_{t+1} = (1+r)(D_t - b_t).$$

We know that the solution to (24) is $\{b(s, D), k(s, D)\}$, and $d(s, D)$ is written as

$$d(s, D) = \mathbb{E}_0\left[\sum_{t=0}^\infty \beta^t b(s_t, D(s^t))\right], \tag{26}$$

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where $s^t = \{s_0, s_1, s_2, \cdots, s_t\}$, $s_0 = s$, $s^0 = \{s\}$, $D(s^0) = D$, and

$$D(s^t) = (1 + r)\{D(s^{t-1}) - b(s_{t-1}, D(s^{t-1}))\}$$

for $t \geq 1$. We define $d^\varepsilon(s, D)$ by

$$d^\varepsilon(s, D) = \mathbb{E}_0\left[\sum_{t=0}^\infty \beta^t b^\varepsilon(s_t, D^\varepsilon(s^t))\right], \tag{27}$$

where $D^\varepsilon(s^0) = D$, $D^\varepsilon(s^t) = (1 + r)\{D^\varepsilon(s^{t-1}) - b^\varepsilon(s_{t-1}, D^\varepsilon(s^{t-1}))\}$, for $t \geq 1$, and for $t = 0$, $b^\varepsilon(s_0, D(s^0)) = b^\varepsilon(s, D) > b(s, D)$ for a particular (s, D) and

$$b^\varepsilon(s_t, D_t^\varepsilon(s^t)) = b(s_t, D(s^t)), \tag{28}$$

for $t \geq 1$. Note that the right-hand side of (28) is $b(s_t, D(s^t))$, and not $b(s_t, D^\varepsilon(s^t))$. The claim is as follows.

Claim 1. For the particular (s, D) where $F(s, k(s, D)) - Rk(s, D) - b(s, D) > 0$, it must be the case that $d^*(s, D) \geq d^\varepsilon(s, D) > d(s, D)$.

(Proof of Claim 1) We have shown that $b^\varepsilon(s_0, D(s^0))$ is feasible for the particular $(s_0, D(s^0)) = (s, D)$. For $t \geq 1$, it is obvious that $D^\varepsilon(s^t) < D(s^t)$ because $b^\varepsilon(s_0, D_0) > b(s_0, D_0)$ for $t = 0$. As we assumed that $V^e(s_t, D_t)$ is a (weakly) decreasing function of D_t, it is the case that

$$V^e(s_t, D^\varepsilon(s^t)) \geq V^e(s_t, D(s^t)).$$

Then, in the state $(s_t, D^\varepsilon(s^t))$, the pair $\{b(s_t, D(s^t)), k(s_t, D(s^t))\}$ is feasible because it satisfies both constraints (Note that this argument holds for all t because the constraint $b_t \leq D_t$ never binds for $D_t > \bar{D}$). Therefore, $d^\varepsilon(s, D)$ is feasible and, by definition of $d^*(s, D)$, (25), it must be the case that $d^*(s, D) \geq d^\varepsilon(s, D)$. Clearly, $d^\varepsilon(s, D) > d(s, D)$, from (26) and (27) by the definition of $b^\varepsilon(s_t, D^\varepsilon(s^t))$ and $b^\varepsilon(s, D) > b(s, D)$ for the particular (s, D). *(End of the proof of Claim 1)*

This claim contradicts the theorem of dynamic programming that the solutions to the recursive problem (24) and the sequential problem (25) are identical, that is, $d(s, D) = d^*(s, D)$. Thus, $k(s, D) = k^{npl}(s)$ should hold if $F(s, k(s, D)) - Rk(s, D) - b(s, D) > 0$.

$\square$

For any s and $D > \bar{D}$, we consider a stochastic sequence $\{s_t, k_t, b_t, D_t\}$, where $k_t = k(s_t, D_t)$, $b_t = b(s_t, D_t)$, $D_t = (1 + r)(D_{t-1} - b_{t-1})$, $s_0 = s$, and $D_0 = D$, given that s_t is an exogenous stochastic variable. We will prove $k(s_0, D_0) = k^{npl}(s_0)$ in what follows.

For $s_t = s_H$, we have the following lemma.

Lemma 6. *Consider the case where $D > \bar{D}$. For all $t \geq 0$, if $s_t = s_H$, then*

$$G(s_H, k(s_H, D_t)) > \beta\mathbb{E}_t[G(s_{t+1}, k(s_{t+1}, D_{t+1})].$$

This inequality implies from Lemma 5 that $k(s_H, D_t) = k^{npl}(s_H)$ for all t.

Proof. The latter half of the lemma is proven as follows. Lemma 4 states that $V(s, D) = G(s, k)$, which implies that $F(s, k) - Rk - b \geq G(s, k) - \beta\mathbb{E}G(s_{+1}, k_{+1})$. The inequality $G(s, k) - \beta\mathbb{E}G(s_{+1}, k_{+1}) > 0$ implies that $F(s, k) - Rk - b > 0$, which implies that $k = k^{npl}(s)$ by Lemma 5.

The proof of the inequality $G(s, k) - \beta\mathbb{E}G(s_{+1}, k_{+1}) > 0$ is by contradiction. Suppose that this inequality does not hold. Then, for $s_0 = s_H$, Lemma 4 implies

$$G(s_H, k(s_H, D_0)) = \beta\mathbb{E}_0[G(s_1, k(s_1, D_1))],$$

where $D_1 \geq D_0$ as $D_0 > \bar{D}$. Then, in the case where $s_1 = s_H$,

$$G(s_H, k(s_H, D_0)) = \beta[\pi_{HH}G(s_H, k(s_H, D_1)) + (1 - \pi_{HH})G(s_L, k(s_L, D_1))] < \beta G(s_H, k(s_H, D_1)),$$

because $k(s_H, D) > k(s_L, D)$ for any D, $G(s, k) > G(s, k')$ for $k > k'$, and $G(s_H, k) \geq G(s_L, k)$. As $G(s_H, k(s_H, D_0)) \geq G^{npl}(s_H)$, it is the case that $G(s_H, k(s_H, D_1)) > G^{npl}(s_H)$. This inequality implies that $k(s_H, D_1) > k^{npl}(s_H)$. Then,

$$G(s_H, k(s_H, D_1)) = \beta\mathbb{E}_1[G(s_2, k(s_2, D_2))],$$

because it should be the case that $k(s_H, D_1) = k^{npl}(s_H)$ and $G(s_H, k(s_H, D_1)) = G^{npl}(s_H)$ if $G(s_H, k(s_H, D_1)) > \beta\mathbb{E}_1[G(s_2, k(s_2, D_2))]$. Iterating this argument, it is easily shown that for any integer t,

$$G(s_H, k(s_H, D_0)) < \beta^t G(s_H, k(s_H, D_t)).$$

This inequality holds for an arbitrarily large t, as the above iteration can continue indefinitely because $D_t \geq D_{t-1} \geq \bar{D}$ for any t, and the constraint, $b_t \leq D_t$, never binds. Then, for a sufficiently large t,

$$G(s_H, k(s_H, D_t)) > \beta^{-t}G(s_H, k(s_H, D_0)) > V_{\max},$$

which contradicts that $k(s_H, D_t)$ is the equilibrium value. Therefore, this lemma must hold. $\square$

For $s_t = s_L$, we have the following lemma.

Lemma 7. *Consider the case where $D > \bar{D}$. For all $t \geq 0$, if $s_t = s_L$, then $k(s_L, D_t) = k^{npl}(s_L)$.*

Proof. If $G(s_L, k(s_L, D_t)) > \beta\mathbb{E}_t[G(s_{t+1}, k(s_{t+1}, D_{t+1}))]$, the limited liability constraint is nonbinding, and $k(s_L, D_t)$ must be $k^{npl}(s_L)$ from Lemma 5.

Suppose that $G(s_L, k(s_L, D_t)) > \beta\mathbb{E}_t[G(s_{t+1}, k(s_{t+1}, D_{t+1}))]$ is not satisfied. Then, Lemma 4 implies that $G(s_L, k(s_L, D_t)) = \beta\mathbb{E}_t[G(s_{t+1}, k(s_{t+1}, D_{t+1}))]$. In this case, because

Lemma 6 implies that $k(s_H, D_t) = k^{npl}(s_H)$ for $t \geq 0$, the following equation must hold for all $t \geq 0$:

$$G(s_L, k_t^L) = \beta[\pi_{LL}G(s_L, k_{t+1}^L) + \pi_{LH}G^{npl}(s_H)],$$

where $\pi_{LL} = \Pr(s_{t+1} = s_L | s_t = s_L)$, $\pi_{LH} = 1 - \pi_{LL}$, and $k_t^L = k(s_L, D_t)$. Here, we define $\bar{k}_L$ as the fixed point of this equation, that is, $G(s_L, \bar{k}_L) = \beta[\pi_{LL}G(s_L, \bar{k}_L) + \pi_{LH}G^{npl}(s_H)]$. Let us consider what would happen if $k_0^L \neq \bar{k}_L$. If $k_0^L < \bar{k}_L$, then the sequence $\{k_t^L\}_{t=0}^{\infty}$ that satisfies the above equation and $k_t^L \geq 0$ for all t cannot exist, because k_t^L becomes a negative number for a finite t. If $k_0^L > \bar{k}_L$, then $\lim_{t\to\infty} k_t^L = +\infty$, which cannot satisfy the condition for an equilibrium, $G(s_L, k_t^L) \leq V_{\max}$ for all t. Therefore, it must be that $k_t^L = \bar{k}_L$ for all t. We assume that (15) is satisfied by $k^{npl}(s_L)$, that is, $G(s_L, k^{npl}(s_L)) \geq \beta\mathbb{E}[G(s_{+1}, k^{npl}(s_{+1})) | s_L]$, it is the case that $k^{npl}(s_L) \geq \bar{k}_L$. If $k_t^L < k^{npl}(s_L)$ for any t, Lemma 2 is violated, which is a contradiction. Therefore, it must be the case that either $G(s_L, k(s_L, D_t)) > \beta\mathbb{E}_t[G(s_{t+1}, k(s_{t+1}, D_{t+1}))]$ or $\bar{k}_L = k^{npl}(s_L)$. The former case implies that $k(s_L, D_t) = k^{npl}(s_L)$ from Lemma 5. Thus, $k_t^L = k^{npl}(s_L)$ for any $D_0 > \bar{D}$. $\qquad \square$

Lemmas 6 and 7 imply that $k(s, D) = k^{npl}(s)$, for any s and D that satisfy $D > \bar{D}$. Since the no-default constraint is binding, $V(s, D) = G(s, k^{npl}(s)) = G^{npl}(s)$ for all s. and $D > \bar{D}$. The equilibrium condition implies that $V^e(s, D) = G^{npl}(s)$. Thus, $b(s, D) = b^{npl}(s)$ and $d(s, D) = d^{npl}(s)$.