Online Appendix to "Debt overhang and lack of lender's commitment"

Keiichiro Kobayashi[*], Tomoyuki Nakajima[†], Shuhei Takahashi[‡]

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This appendix proves the existence of the equilibrium in the discrete version of the model in Kobayashi, Nakajima, and Takahashi (2021).

1 Discretization of the model

Discretization: Denote the set of integers by $\mathbb{Z}$, and define

$$\Delta = \{0, \delta, 2\delta, \cdots, N_{\max}\delta\},$$
$$\Delta_{+1} = \{0, \delta, 2\delta, \cdots, n_\delta[(1+r)N_{\max}\delta]\}.$$

Here, δ is the minimum unit of debt, $N_{\max} \in \mathbb{Z}$ is a sufficiently large integer, and $n_\delta(x) = n\delta$ for $x > 0$, where n is the integer satisfying $(n-1)\delta < x \leq n\delta$. We assume that the amount of debt, D, must be an element of Δ:

$$D \in \Delta.$$

For each $s \in \{s_L, s_H\}$, the set of possible values of k, $\Delta_k(s)$, is defined as

$$\Delta_k(s) = \left\{ k \;\middle|\; \exists n \in \mathbb{Z}, \text{ s.t. } F(s,k) - Rk - G(s,k) = n \times \frac{\delta}{1+r} \right\}.$$

Then, $k^*(s)$ and $k^{npl}(s)$ are defined as

$$k^*(s) = \arg\max_{k \in \Delta_k(s)} F(s,k) - Rk,$$
$$k^{npl}(s_H) = \arg\max_{k \in \Delta_k(s_H)} F(s_H,k) - Rk - G(s_H,k),$$
$$k^{npl}(s_L) = \arg\max_{k \in \Delta_k(s_L)} F(s_L,k) - Rk - G(s_L,k),$$

Here, we are assuming that the parameter values are selected such that

$$G^{npl}(s_H) > \beta[\pi_{HH}G^{npl}(s_H) + \pi_{HL}G^{npl}(s_L)], \tag{1}$$
$$G^{npl}(s_L) > \beta[\pi_{LL}G^{npl}(s_L) + \pi_{LH}G^{npl}(s_H)], \tag{2}$$

[*]Keio University, CIGS, TKFD

[†]University of Tokyo, CIGS

[‡]Kyoto University, CIGS

where $\pi_{HH} = \Pr(s_{t+1} = s_H | s_t = s_H)$, $\pi_{HL} = 1 - \pi_{HH}$, $\pi_{LL} = \Pr(s_{t+1} = s_L | s_t = s_L)$, and $\pi_{LH} = 1 - \pi_{LL}$. We also let $G^{npl}(s) \equiv G(s, k^{npl}(s))$.

Our arguments in this paper can be easily modified for the case where the inequalities (1) and/or (2) do not hold.[1] For each $s \in \{s_L, s_H\}$, the repayment in the NPL equilibrium, $b^{npl}(s)$, is defined by

$$b^{npl}(s) = F(s, k^{npl}(s)) - Rk^{npl}(s) - G^{npl}(s) + \beta \mathbb{E}[G^{npl}(s_{+1})|s].$$

The set of possible values of repayments, $\Delta_b(s, D)$, depends on D:

$$\Delta_b(s, D) = \left\{ b \in \mathbb{R} \;\middle|\; \exists \tilde{D}_{+1} \in \Delta_{+1} \;\text{ s.t. }\; b = D - \frac{1}{1+r}\tilde{D}_{+1}, \;\text{ and } b \geq 0 \right\} \cup \{b^{npl}(s)\}.$$

At each state (s, D), b and k must satisfy

$$b \in \Delta_b(s, D), \quad \text{and} \quad k \in \Delta_k(s).$$

Bank's problem: Let $V^e(s, D)$ denote the bank's expectation regarding the value of the firm as a function of the current state (s, D). Then, the bank's profit maximization is formulated as the Bellman equation:

$$d(s, D) = \max_{b \in \Gamma(s,D)} \quad b + \beta \mathbb{E} d(s_{+1}, D_{+1}), \tag{3}$$

where

$$\begin{aligned}\Gamma(s, D) = \{ & b \in \Delta_b(s, D) \mid \exists k \in \Delta_k(s) \text{ s.t.} \\ & D_{+1} = \min\{N_{\max}\delta, \; n_\delta[(1+r)(D-b)]\}, \\ & F(s,k) - Rk - b + \beta \mathbb{E} V^e(s_{+1}, D_{+1}) \geq G(s,k), \\ & F(s,k) - Rk - b \geq 0\}.\end{aligned}$$

Here, $n_\delta[(1+r)(D-b)] = n \times \delta$, where n is the integer that satisfies $(n-1)\delta < (1+r)(D-b) \leq n\delta$.

Let $\Sigma(s, D)$ denote the set of (b, D_{+1}) that solves the maximization problem in (3). The bank then decides k and $V(s, D)$ by solving the following problem:

$$V(s, D) = \max_{k \in \Delta_k(s), \; (b, D_{+1}) \in \Sigma(s,D)} F(s,k) - Rk - b + \beta \mathbb{E} V^e(s_{+1}, D_{+1}), \tag{4}$$

subject to

$$\begin{aligned} & F(s,k) - Rk - b + \beta \mathbb{E} V^e(s_{+1}, D_{+1}) \geq G(s,k), \\ & F(s,k) - Rk - b \geq 0. \end{aligned}$$

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[1] For this purpose, it suffices to redefine
$k^{npl}(s_H) = \max\{k \in \Delta_k(s_H) | \; G(s, k_H) \leq \beta[\pi_{HH} G(s_H, k) + \pi_{HL} G(s_H, k^{npl}(s_L))]\}$,
and/or
$k^{npl}(s_L) = \max\{k \in \Delta_k(s_L) | \; G(s, k_L) \leq \beta[\pi_{LL} G(s_L, k) + \pi_{LH} G(s_H, k^{npl}(s_H))]\}$. In the case where $k^{npl}(s)$ is redefined, $b^{npl}(s)$ is also redefined as $b^{npl}(s) = F(s, k^{npl}(s)) - Rk^{npl}(s)$.

Let $\Lambda(s, D)$ denote the set of (k, b, D_{+1}) that solves the maximization problem in (4).

Given $\Lambda(s, D)$, the equilibrium values of (k, b, D_{+1}) at (s, D) are selected as follows. First, $b(s, D)$ and $D_{+1}(s, D)$ are decided as

$$b(s, D) = \max_{(k,b,D_{+1})\in\Lambda(s,D)} b, \tag{5}$$

$$D_{+1}(s, D) = \min\{N_{\max}\delta,\ n_\delta[(1 + r)\{D - b(s, D)\}]\}. \tag{6}$$

Then, $k(s, D)$ is determined by

$$k(s, D) = \max_{(k,b(s,D),D_{+1}(s,D))\in\Lambda(s,D)} k.$$

Then, the value of the firm must satisfy

$$V(s, D) = F(s, k(s, D)) - Rk(s, D) - b(s, D) + \beta\mathbb{E}V^e(s_{+1}, D_{+1}(s, D)). \tag{7}$$

Assuming rational expectations, the bank's belief $V^e(s, D)$ should be consistent with $V(s, D)$ given in (7):

$$V(s, D) = V^e(s, D). \tag{8}$$

Definition of the threshold, $D_{\max}(s)$: Given the existence of an equilibrium, we define $D_{\max}(s)$ as follows:

$$D_{\max}(s_H) \equiv \max\{D \in \Delta \mid D_{+1}(s_H, D) < D\}, \tag{9}$$

$$D_{\max}(s_L) \equiv \max\{D \in \Delta \mid D_{+1}(s_L, D) < D_{\max}(s_H)\}. \tag{10}$$

Thus, if D exceeds $D_{\max}(s_H)$ at s_H, the amount of debt in the next period is greater than or equal to D. Similarly, if D exceeds $D_{\max}(s_L)$ at state s_L, the next period's debt is greater than or equal to $D_{\max}(s_H)$. The following lemma demonstrates that if $D > D_{\max}(s_L)$, then $D_{+1}(s_L, D) \geq D$. As a result, once D exceeds $D_{\max}(s)$ at each s, D will never decrease.

Lemma 1. *If $D > D_{\max}(s_L)$, then $D_{+1}(s_L, D) \geq D$.*

Proof. Let $D > D_{\max}(s_L)$, and suppose, for the sake of contradiction, that $D_{+1}(s_L, D) < D$. Then,

$$D_{+1}(s_L, D_{+1}(s_L, D)) < D_{+1}(s_L, D).$$

However, since $D > D_{\max}(s_L)$, $D_{+1}(s_L, D) \geq D_{\max}(s_H)$. By the definition of $D_{\max}(s_H)$, we have

$$D_{+1}(s_H, D_{+1}(s_L, D)) \geq D_{+1}(s_L, D).$$

We also have

$$D_{+1}(s_H, D_{+1}(s_L, D)) \leq D_{+1}(s_L, D_{+1}(s_L, D)).$$

Combining these inequalities, we obtain

$$D_{+1}(s_L, D) \leq D_{+1}(s_H, D_{+1}(s_L, D)) \leq D_{+1}(s_L, D_{+1}(s_L, D)) < D_{+1}(s_L, D),$$

which is a contradiction. □

We can confirm that $D_{\max}(s) < \infty$ as follows. For $D > \bar{D}$, it is obvious that, for any $b \leq \max_k\{F(s,k) - Rk\}$, the debt never decreases over time, i.e., $D_{+1} = (1+r)(D-b) > D$. Thus, there exists $D_{\max}(s_H)$ such that $D_{\max}(s_H) \leq \bar{D} < \infty$. As $D_{\max}(s_H) < \infty$, it follows from (9)-(10) that $D_{\max}(s_L) \leq D_{\max}(s_H)$.

2 Equilibrium of the discrete model

In this section, we assume that the interest rate in the debt contract is equal to the market rate for the risk-free bond:

$$\beta = \frac{1}{1+r}. \tag{11}$$

As discussed in Section **??**, it simplifies the analysis on the equilibrium dynamics in our model. Note, however, that even under assumption (11), the bank can still make the expected payoff nonnegative, by adjusting the initial amount of the principal of the loan.[2] In Sections 2.1, 2.2, and 2.3, we characterize the equilibrium, taking the existence of an equilibrium as given. In Section 2.4, we prove the existence. In Section **??** we show numerical results. There, we also consider the case where $\beta > \frac{1}{1+r}$ and confirm the robustness of the results.

2.1 The repayment in the case of small D

Two working assumptions: In Sections 2.1 and 2.2, we proceed by making the following two assumptions. They are verified later in Lemma 12 in Section 2.4. All proofs are provided in the Appendix.

[2] The initial principal of the debt D_0 may not be fully repaid in equilibrium, so that the expected PDV of repayments, $d(s_0, D_0) \equiv \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t b_t$, may be smaller than D_0. Let I_0 denote the initial amount of lending. The zero profit condition for the bank is satisfied if the contractual amount of initial debt, D_0, is set as

$$I_0 = d(s_0, D_0).$$

Assumption 1. For $D < D_{\max}(s)$, $V^e(s, D + \delta) \le V^e(s, D) - \delta$.

Assumption 2. For all s and $D \ge \delta$, $b(s, D)$ satisfies

$$b(s, D) \ge \delta. \tag{12}$$

We first characterize the equilibrium repayment function $b(s, D)$ for $D \le D_{\max}(s)$.

Lemma 2. *For all $D \ge 0$, $d(s, D + \delta) \le d(s, D) + \delta$.*

Lemma 3. *For $D \le D_{\max}(s)$, $b(s, D) = \bar{b}(s, D)$, where $\bar{b}(s, D)$ is the maximum feasible value, i.e., $\bar{b}(s, D) = \max\{b \mid b \in \Gamma(s, D)\}$. It also holds that $k(s, D) > k^{npl}(s)$ for $D \le D_{\max}(s)$.*

Lemma 3 directly implies the following corollary.

Corollary 4. *If (s, D) is a state such that $k(s, D) = k^*(s)$, then*

$$b(s, D) = \min\ \{D,\ b^*(s, D)\},$$

where

$$\begin{aligned} b^*(s, D) = &\max_{n \in \mathbb{Z}} D - \beta n \delta, \\ &s.t. \quad D - \beta n \delta \le F(s, k^*(s)) - Rk^*(s). \end{aligned}$$

Now, we define

$$\begin{aligned} &f(s, k) \equiv F(s, k) - Rk - G(s, k), \\ &\delta_f \equiv \max_{k \in \Delta_k(s),\ k^{npl}(s) \le k \le k^*(s)} F'(s, k) - R, \\ &\delta_k \equiv \max\{k' - k \mid k \in \Delta_k(s),\ k' \in \Delta_k(s), \\ &\qquad\qquad k^{npl}(s) \le k < k' < k^*(s),\ |f(s, k) - f(s, k')| = \beta\delta\}, \\ &\delta_g \equiv \max\{G(s, k') - G(s, k) \mid k \in \Delta_k(s),\ k' \in \Delta_k(s), \\ &\qquad\qquad k^{npl}(s) \le k < k' < k^*(s),\ |f(s, k) - f(s, k')| = \beta\delta\}. \end{aligned}$$

Note that $\delta_f = O(1)$, $\delta_k = O(\delta)$, and $\delta_g = O(\delta)$. Then, the following lemma holds.

Lemma 5. *For (s, D) such that $k^{npl}(s) < k(s, D) < k^*(s)$, it holds that $0 \le F(s, k(s, D)) - Rk(s, D) - b(s, D) < \xi + \beta\delta$, where $\xi = \delta_f \delta_k$.*

As $\xi = O(\delta)$, Corollary 4 and Lemma 5 implies that $b(s, D) \approx \min\{D,\ F(s, k(s, D)) - Rk(s, D)\}$ for small δ. This means that the optimal contract involves backloaded payment to the firm; that is, the firm repays debt as fast as possible by setting its dividend at almost zero, i.e., $b \approx \min\{D,\ F(s, k) - Rk\}$, when D is smaller than or equal to $D_{\max}(s)$.

2.2 Equilibrium at large D

Here, we demonstrate that when D is large so that $D > D_{\max}(s)$, the equilibrium exhibits the feature that we call the NPL equilibrium. For that, the minimum unit δ is sufficiently small such that the following assumption is satisfied.

Assumption 3. The value of δ and the function $G(s,k)$ satisfy

$$\min_{s} \ G^{npl}(s) > \frac{\xi + \beta(\delta + \delta_g)}{1-\beta},$$

where $\xi = \delta_f \delta_k$.

Lemma 6. *For $k(s,D) < k^*(s)$, the binding no-default constraint implies that*

$$V(s,D) - \delta_g < G(s,k(s,D)) \leq V(s,D).$$

Proof. The first inequality holds because otherwise the bank can obtain a positive gain by changing $k(s,D)$ to k', where $k' > k(s,D)$ and $|f(k(s,D)) - f(k')| = \beta\delta$. □

Lemma 7. *For all $D > D_{\max}(s)$, it holds that $k(s,D) = k^{npl}(s)$.*

Proposition 8. *For all (s,D) with $D > D_{\max}(s)$, $d(s,D) = d^{npl}(s)$, $k(s,D) = k^{npl}(s)$, $b(s,D) = b^{npl}(s)$, and $V(s,D) = G^{npl}(s)$.*

This proposition[3] is similar to Proposition **??** in Section **??**, but stronger because $D_{\max}(s) \leq \bar{D}$. Once D exceeds $D_{\max}(s)$ at any s, the contractual amount of debt will keep on growing and the constraint $b \leq D$ will never bind. Thus, D becomes irrelevant for the choice of k and b, and the equilibrium variables depend solely on the exogenous state s, given as the NPL equilibrium. The intuition is that when D is larger than $D_{\max}(s)$, it becomes impossible to pay back D in full, and thus the contractual amount of debt becomes payoff irrelevant. It follows that the lender can no longer commit to any future repayment plans. The loss of the bank's credibility leads to an inefficient outcome referred to as the NPL equilibrium.

[3] In Proposition 8, we have assumed that the parameter values are restricted such that $k^{npl}(s)$ is defined by $k^{npl}(s) \equiv \arg\max_{k \in \Delta_k(s)} F(s,k) - Rk - G(s,k)$. It is generalized as follows, in the case where $k^{npl}(s_L)$ is defined by $k^{npl}(s_L) = \max\{k \in \Delta_k(s_L)| \ G(s,k_L) \leq \beta[\pi_{LL} G(s_L,k) + \pi_{LH} G(s_H, k^{npl}(s_H))]\}$: We define $V^{npl}(s)$ by

$$V^{npl}(s_H) = G^{npl}(s_H),$$
$$V^{npl}(s_L) = \beta \mathbb{E}[V^{npl}(s_{+1}) | s = s_L].$$

Then, we redefine $b^{npl}(s)$ by $b^{npl}(s) = F(s,k^{npl}(s)) - Rk^{npl}(s) - G^{npl}(s) + \beta\mathbb{E}[V^{npl}(s_{+1})|s]$. Then, the modified version of Proposition 8 states: For all (s,D) with $D > D_{\max}(s)$, $d(s,D) = d^{npl}(s)$, $k(s,D) = k^{npl}(s)$, $b(s,D) = b^{npl}(s)$, and $V(s,D) = V^{npl}(s)$. The proof of the modified version is similar to that of Proposition 8.

2.3 Characterization of the equilibrium

Here, we summarize the analytical results obtained for the discrete model with $1+r = \beta^{-1}$. First, there exist endogenously determined thresholds, $D_{\max}(s)$, which are defined by (9) and (10).

Define $D_{\min}(s_L)$ by

$$D_{\min}(s_L) = \max \left\{ D \in \Delta \,|\, \forall D' \leq D, D_{+1}(s_L, D') < D' \right\}.$$

Since $D_{+1}(s_H, D) \leq D_{+1}(s_L, D)$ for all D, once D becomes sufficiently small that $D \leq D_{\min}(s_L)$, D declines over time thereafter, regardless of the realization of the exogenous state s.

Thus, if the initial debt D_0 satisfies $D_0 \leq D_{\min}(s_L)$, there is no chance that the economy will fall into the NPL equilibrium. In this case, the equilibrium dynamics are qualitatively the same as those of the AH model. The borrower repays as much debt as possible in every period by setting dividend (almost) zero, i.e., $F(s,k) - Rk - b \approx 0$ (Lemma 5), where the qualification "almost" is required because of the discretization. Functions $k(s, D)$ and $V(s, D)$ are both non-increasing in D.[4] As the current debt D satisfies $D \leq D_{\min}(s_L)$, the next period debt D_{+1} is smaller than D. Thus, along the equilibrium path, $D_{t+1} = \beta^{-1}[D_t - b(s_t, D_t)]$ converges to 0 within finite periods. When $D = 0$, the bank takes 0 because $b \leq D$ binds at $D = 0$, and the problem (for the bank) is to maximize the firm's profits by selecting $k = k^*(s) = \arg\max_k F(s,k) - Rk$. Thus, the economy converges to a first-best allocation, $\{D, k\} = \{0,\, k^*(s)\}$, within finite periods. In this case, the state variable, D, remains payoff-relevant along the whole equilibrium path.

If the initial debt satisfies $D_0 \geq D_{\max}(s_H)$, debt D_t always increases regardless of the exogenous state s, i.e., $D_{t+1} \geq D_t$ with probability one for all t. Then, D_t is no longer a payoff-relevant state variable, and the bank is unable to make a commitment to future repayment plans. As a result, the economy falls into the NPL equilibrium: $\{k(s, D), b(s, D),\, d(s, D),\, V(s, D)\} = \{k^{npl}(s),\, b^{npl}(s),\, d^{npl}(s),\, V^{npl}(s)\}$. In the NPL equilibrium, the firm's output is "minimized" in the sense that $k^{npl}(s) = \min_{D \in \Delta} k(s, D)$.

For initial debt D_0 in the intermediate region, $D_{\min}(s_L) < D_0 \leq D_{\max}(s_H)$, the economy may end up with either the first best or NPL equilibrium. Both can occur with a positive probability. While D is in this region, the dividend to the firm is $F(s,k) - Rk - b \approx 0$ (Lemma 5). D remains to be payoff-relevant.

2.4 Existence of equilibrium

In this subsection, we demonstrate the existence of an equilibrium, which is characterized as a fixed point of an operator, T, on the functions of (s, D). As the space for (s, D) is

[4] First, Lemma 12 in Section 2.4 implies that $V(s, D)$ is non-increasing in D. Second, $k(s, D)$ is non-increasing in D, because $k(s, D) = \max\{k \in \Delta_k(s) \,|\, V(s, D) \geq G(s, k)\}$ and $V(s, D)$ is non-increasing.

discrete and finite, the existence of an equilibrium is proved by finding a fixed point of the operator T in a finite-dimensional vector space.

Define the operator T by

$$(d^{(n+1)}(s,D), V^{(n+1)}(s,D), \bar{D}^{(n+1)}(s)) = T(d^{(n)}(s,D), V^{(n)}(s,D), \bar{D}^{(n)}(s)),$$

where $(d^{(n+1)}(s,D), V^{(n+1)}(s,D), \bar{D}^{(n+1)}(s))$ is generated from $(d^{(n)}(s,D), V^{(n)}(s,D), \bar{D}^{(n)}(s))$, as follows. Define $\Gamma^{(n+1)}(s,D)$ by

$$\begin{aligned}\Gamma^{(n+1)}(s,D) \equiv \{b \in \Delta_b(s,D) \mid \exists k \in \Delta_k(s) \ \text{ s.t.}\\ & D_{+1} = \min\{N_{\max}\delta,\ n_\delta[(1+r)(D-b)]\},\\ & F(s,k) - Rk - b + \beta \mathbb{E} V^{(n)}(s_{+1}, D_{+1}) \geq G(s,k),\\ & F(s,k) - Rk - b \geq 0\}.\end{aligned}$$

Given state (s,D) and expectations $(V^{(n)}(s,D),\ d^{(n)}(s,D))$, the bank solves

$$d^{(n+1)}(s,D) = \max_{b \in \Gamma^{(n+1)}(s,D)} \quad b + \beta \mathbb{E} d^{(n)}(s_{+1}, D_{+1}). \tag{13}$$

Denote by $\Sigma^{(n+1)}(s,D)$ the set of (b, D_{+1}) that solves the maximization in (13). The bank decides k and $V^{(n+1)}(s,D)$ by solving the following problem.

$$V^{(n+1)}(s,D) = \max_{k \in \Delta_k(s),\ (b,D_{+1}) \in \Sigma^{(n+1)}(s,D)} F(s,k) - Rk - b + \beta \mathbb{E} V^{(n)}(s_{+1}, D_{+1}), \tag{14}$$

subject to

$$\begin{aligned} & F(s,k) - Rk - b + \beta \mathbb{E} V^{(n)}(s_{+1}, D_{+1}) \geq G(s,k),\\ & F(s,k) - Rk - b \geq 0.\end{aligned}$$

Let $\Lambda^{(n+1)}(s,D)$ denote the set of (k, b, D_{+1}) that solves the maximization in (14).

The equilibrium values of (k, b, D_{+1}) are selected as follows. First, $b^{(n+1)}(s,D)$ and $D_{+1}^{(n+1)}(s,D)$ are determined as

$$b^{(n+1)}(s,D) = \max_{(k,b,D_{+1}) \in \Lambda^{(n+1)}(s,D)} b, \tag{15}$$

$$D_{+1}^{(n+1)}(s,D) = \min\{N_{\max}\delta,\ n_\delta((1+r)[D - b^{(n+1)}(s,D)])\}. \tag{16}$$

Then, $k^{(n+1)}(s,D)$ is decided as

$$k^{(n+1)}(s,D) = \max_{(k,\ b^{(n+1)}(s,D),\ D_{+1}^{(n+1)}(s,D)) \in \Lambda^{(n+1)}(s,D)} k,$$

and $\bar{D}^{(n+1)}(s)$ is provided by

$$\begin{aligned}\bar{D}^{(n+1)}(s_H) &= \max\left\{D \in \Delta \mid D_{+1}^{(n+1)}(s_H, D) < \bar{D}^{(n)}(s_H)\right\},\\ \bar{D}^{(n+1)}(s_L) &= \max\left\{D \in \Delta \mid D_{+1}^{(n+1)}(s_L, D) < \bar{D}^{(n)}(s_H)\right\}.\end{aligned}$$

Define $V_H^* \equiv \frac{1}{1-\beta}[F(s_H, k^*(s_H)) - Rk^*(s_H)]$.

We set the initial values $(\bar{D}^{(0)}(s), d^{(0)}(s,D), V^{(0)}(s,D))$ as follows.

$$\bar{D}^{(0)}(s) = \bar{D}^{(0)} \equiv V_H^* - G^{npl}(s_H),$$
$$d^{(0)}(s,D) = \begin{cases} D & \text{for } D \le \bar{D}^{(0)}, \\ d^{npl}(s) & \text{for } D > \bar{D}^{(0)}, \end{cases}$$
$$V^{(0)}(s,D) = \begin{cases} V_H^* - D & \text{for } D \le \bar{D}^{(0)}, \\ G^{npl}(s) & \text{for } D > \bar{D}^{(0)}. \end{cases}$$

Now, the existence of a fixed point of operator T is established by demonstrating the convergence of the sequence $\{d^{(n)}, V^{(n)}, \bar{D}^{(n)}\}_{n=0}^{\infty}$.

Theorem 9. *There exists a fixed point $(d(s,D), V(s,D), D_{\max}(s))$ of the operator T, that is, $(d, V, D_{\max}) = T(d, V, D_{\max})$.*

This fixed point is an equilibrium of the economy. The proof of this theorem is as follows. The following lemmas demonstrate that $(d^{(n)}(s,D),\ V^{(n)}(s,D),\ \bar{D}^{(n)}(s))$ satisfies

$$\begin{aligned}(d^{npl}(s), G^{npl}(s), d^{npl}(s)) &\le (d^{(n+1)}(s,D), V^{(n+1)}(s,D), \bar{D}^{(n+1)}(s)) \\ &\le (d^{(n)}(s,D), V^{(n)}(s,D), \bar{D}^{(n)}(s))\end{aligned}$$

for $D > d^{npl}(s)$, and that

$$\begin{aligned}(0,\ G^{npl}(s), d^{npl}(s)) &\le (d^{(n+1)}(s,D), V^{(n+1)}(s,D), \bar{D}^{(n+1)}(s)) \\ &\le (d^{(n)}(s,D), V^{(n)}(s,D), \bar{D}^{(n)}(s))\end{aligned}$$

for $D \le d^{npl}(s)$. Thus, the sequence $\left\{d^{(n)}(s,D), V^{(n)}(s,D), \bar{D}^{(n)}(s)\right\}_{n=0}^{\infty}$ at any fixed (s,D) converges pointwise, because it is a weakly decreasing sequence of real numbers, which is bounded from below: $\exists (d(s,D), V(s,D), D_{\max}(s))$ such that

$$(d^{(n)}(s,D), V^{(n)}(s,D), \bar{D}^{(n)}(s)) \to (d(s,D), V(s,D), D_{\max}(s))$$

as $n \to \infty$. This $(d(s,D), V(s,D), D_{\max}(s))$ is a fixed point of the operator T by construction.

The proof is by induction. The first step of the induction is provided by the following lemma.

Lemma 10. *Denote $(d^{(1)}(s,D), V^{(1)}(s,D), \bar{D}^{(1)}(s)) = T(d^{(0)}(s,D), V^{(0)}(s,D), \bar{D}^{(0)}(s))$. Let $(b^{(1)}(s,D), k^{(1)}(s,D))$ be the value of (b,k) that solves (13) and (14) with $n = 0$. Then, $(d^{(1)}(s,D),\ V^{(1)}(s,D),\ \bar{D}^{(1)}(s),\ b^{(1)}(s,D),\ k^{(1)}(s,D))$ satisfies*

(i) $d^{(1)}(s, D+\delta) \le d^{(1)}(s,D) + \delta$,

(ii) $d^{npl}(s) \leq d^{(1)}(s,D) \leq d^{(0)}(s,D)$ *for* $D > d^{npl}(s)$*, and* $0 \leq d^{(1)}(s,D) \leq d^{(0)}(s,D)$ *for* $D \leq d^{npl}(s)$*,*

(iii) $\forall D > \bar{D}^{(1)}(s)$*,* $d^{(1)}(s,D) = d^{npl}(s)$*,* $V^{(1)}(s,D) = V^{npl}(s)$*,* $b^{(1)}(s,D) = b^{npl}(s)$*,* $k^{(1)}(s,D) = k^{npl}(s)$*,*

(iv) $V^{(1)}(s,D+\delta) \leq -\delta + V^{(1)}(s,D)$ *for* $D < \bar{D}^{(1)}(s)$*,*

(v) $\forall (s,D), \quad G^{npl}(s) \leq V^{(1)}(s,D) \leq V^{(0)}(s,D)$*,*

(vi) $d^{npl}(s) < \bar{D}^{(1)}(s) < \bar{D}^{(0)}$*.*

The second step of the induction is provided by the following lemma.

Lemma 11. *Denote* $(d^{(n+1)}(s,D), V^{(n+1)}(s,D), \bar{D}^{(n+1)}(s)) = T(d^{(n)}(s,D), V^{(n)}(s,D), \bar{D}^{(n)}(s))$*. Let* $(b^{(n+1)}(s,D),\ k^{(n+1)}(s,D))$ *be the value of* (b,k) *that solves* (13) *and* (14)*. Suppose that* $(d^{(n)}(s,D),\ V^{(n)}(s,D),\ \bar{D}^{(n)}(s),\ b^{(n)}(s,D),\ k^{(n)}(s,D))$ *satisfies*

(i') $d^{(n)}(s,D+\delta) \leq d^{(n)}(s,D) + \delta$*,*

(ii') $d^{npl}(s) \leq d^{(n)}(s,D) \leq d^{(n-1)}(s,D)$ *for* $D > d^{npl}(s)$*, and* $0 \leq d^{(n)}(s,D) \leq d^{(n-1)}(s,D)$ *for* $D \leq d^{npl}(s)$

(iii') $\forall D > \bar{D}^{(n)}(s)$*,* $d^{(n)}(s,D) = d^{npl}(s)$ *and* $V^{(n)}(s,D) = V^{npl}(s)$*,*

(iv') $V^{(n)}(s,D+\delta) \leq -\delta + V^{(n)}(s,D)$ *for* $D < \bar{D}^{(n)}(s)$*,*

(v') $\forall (s,D), \quad G^{npl}(s) \leq V^{(n)}(s,D) \leq V^{(n-1)}(s,D)$*,*

(vi') $0 < \bar{D}^{(n)}(s) \leq \bar{D}^{(n-1)}(s)$*.*

Then, $(d^{(n+1)}(s,D),\ V^{(n+1)}(s,D),\ \bar{D}^{(n+1)}(s),\ b^{(n+1)}(s,D),\ k^{(n+1)}(s,D))$ *satisfies*

(i) $d^{(n+1)}(s,D+\delta) \leq d^{(n+1)}(s,D) + \delta$*,*

(ii) $d^{npl}(s) \leq d^{(n+1)}(s,D) \leq d^{(n)}(s,D)$ *for* $D > d^{npl}(s)$*, and* $0 \leq d^{(n+1)}(s,D) \leq d^{(n)}(s,D)$ *for* $D \leq d^{npl}(s)$*,*

(iii) $\forall D > \bar{D}^{(n+1)}(s)$*,* $d^{(n+1)}(s,D) = d^{npl}(s)$ *and* $V^{(n+1)}(s,D) = V^{npl}(s)$*,*

(iv) $V^{(n+1)}(s,D+\delta) \leq -\delta + V^{(n+1)}(s,D)$ *for* $D < \bar{D}^{(n+1)}(s)$*,*

(v) $\forall (s,D), \quad G^{npl}(s) \leq V^{(n+1)}(s,D) \leq V^{(n)}(s,D)$*,*

(vi) $0 < \bar{D}^{(n+1)}(s) \leq \bar{D}^{(n)}(s)$*.*

In Sections 2.1 and 2.2, we have assumed Assumptions 1 and 2 to establish some equilibrium properties. The next lemma demonstrates that those assumptions are indeed satisfied by the equilibrium constructed as the fixed point of T.

Lemma 12. *For $D \leq D_{\max}(s)$, $V(s, D+\delta) \leq V(s,D) - \delta$. For all $D \geq \delta$, $b(s,D)$ satisfies $b(s,D) \geq \delta$.*

3 Discrete model with stochastic debt restructuring

In the baseline model, debt restructuring is prohibited. We modify the model in this section such that debt restructuring is feasible with some friction. For simplicity, we adopt a reduced-form approach: In each period t, the bank may be able to reduce the contractual amount of debt D_t. However, this option of debt restructuring arrives with an exogenously given probability $p \in (0,1)$ in each period. With this option in hand, the bank can reduce D_t to any value $D \in [0, D_t]$. The probability p is a fixed parameter and represents the friction in debt restructuring.

When the bank with contractual amount of debt D_t restructures debt, it reduces D_t to $\hat{D}(s, D_t)$ defined by

$$\hat{D}(s, D_t) = \arg \max_{0 \leq D \leq D_t} d(s, D).$$

Here, $d(s,D)$ is the PDV of repayments, given as the solution to (20) below. Clearly, $\hat{D}(s,D) = D$ for a small value of D, because the bank has no incentive to reduce the debt if it is sufficiently small.

Definitions: Given the possibility of debt restructuring, we modify the formulation of the discrete model, because the NPL equilibrium, $\{k^{npl}(s),\ b^{npl}(s),\ d^{npl}(s),\ G^{npl}(s)\}$ now depends on when and by how much debt is reduced. The grid points for D, D_{+1}, and k are the same as in the previous sections, but we modify the grid points for b, $\Delta_b(s,D)$.

Take as given the beliefs $\{V^e(s,D), k^e_{npl}(s), \hat{D}^e(s,D)\}$, where $V^e(s,D)$ describes the expected value of the firm, $k^e_{npl}(s)$ the expected value of working capital in the NPL equilibrium, and $\hat{D}^e(s,D)$ the expected amount of debt after debt restructuring. We use the same parameter values as in the baseline model. For the probability p of a certain size, the candidate for $k^{npl}(s)$ makes the enforcement constraint nonbinding, that is, $\tilde{k}^{npl}(s) \equiv \arg\max_{k \in \Delta_k(s)} F(s,k) - Rk - G(s,k)$ does not satisfy

$$G(s,k) > \beta \mathbb{E}[(1-p)V^{npl}(s_{+1}) + pV^e(s_{+1}, \hat{D}^e_{+1})|s], \tag{17}$$

where we define $V^{npl}(s_{+1})$ by

$$V^{npl}(s) = F(s, k^{npl}(s)) - Rk^{npl}(s) - b^{npl}(s) + \beta \mathbb{E}[(1-p)V^{npl}(s_{+1}) + pV^e(s_{+1}, \hat{D}^e_{+1})|s],$$

and $D^e_{+1} = \hat{D}^e(s_{+1}, D_{+1})$.[5] Therefore, not as in the baseline case, we define $k^{npl}(s)$ for

[5] Note that in the NPL equilibrium where $D > D_{\max}(s)$, $\hat{D}(s,D)$ is independent of D, i.e., $\hat{D}(s,D) = \hat{D}(s)$, which is defined by $\hat{D}(s) \equiv \arg\max_{D \in \Delta} d(s,D)$. Thus, for $D > D_{\max}(s)$, $\hat{D}^e(s,D)$ should also be independent of D.

the case where $\tilde{k}^{npl}(s)$ does not satisfy (17) as

$$k^{npl}(s) = \max\{k \in \Delta_k(s) | \; G(s,k) \leq \beta \mathbb{E}[(1-p)V^{npl}(s_{+1}) + pV^e(s_{+1}, \hat{D}^e_{+1})|s]. \tag{18}$$

Note that $k^{npl}(s)$ depends on the given beliefs $\{V^e(s,D), k^e_{npl}(s), \hat{D}^e(s,D)\}$. Of course, $k^{npl}(s) = k^e_{npl}(s)$ must hold in equilibrium. We define $b^{npl}(s)$ by

$$b^{npl}(s) = F(s, k^{npl}(s)) - Rk^{npl}(s) + \beta \mathbb{E}[(1-p)V^{npl}(s_{+1}) + pV^e(s_{+1}, \hat{D}^e_{+1})|s] - G^{npl}(s),$$

in the case where $k^{npl}(s) = \tilde{k}^{npl}(s)$, and by

$$b^{npl}(s) = F(s, k^{npl}(s)) - Rk^{npl}(s), \tag{19}$$

in the case where $k^{npl}(s)$ is defined by (18).

Now, we define the grid points for b as

$$\Delta_b(s,D) = \left\{ b \in \mathbb{R} \mid \exists D_{+1} \in \Delta_{+1} \;\text{ s.t. }\; b = D - \frac{1}{1+r} D_{+1}, \;\; \text{and } b \geq 0 \right\} \cup \{b^{npl}(s)\}.$$

As stated above, the NPL equilibrium, $\{k^{npl}(s),\, b^{npl}(s),\, d^{npl}(s),\, V^{npl}(s)\}$, is defined given the beliefs $\{V^e(s,D),\, k^e_{npl}(s),\, \hat{D}^e(s,D)\}$.

The bank's problem: Given beliefs $\{V^e(s,D),\, k^e_{npl}(s),\, \hat{D}^e(s,D)\}$, the bank solves

$$d(s,D) = \max_{b \in \Gamma(s,D)} \;\; b + \beta \mathbb{E}[(1-p)d(s_{+1}, D_{+1}) + pd(s_{+1}, \hat{D}^e_{+1})], \tag{20}$$

where

$$\begin{aligned}\Gamma(s,D) = \{ & b \in \Delta_b(s,D) \mid \exists k \in \Delta_k(s) \;\text{ s.t.} \\ & D_{+1} = \min\{N_{\max}\delta,\; (1+r)(D-b)\}, \\ & F(s,k) - Rk - b + \beta \mathbb{E}[(1-p)V^e(s_{+1}, D_{+1}) + pV^e(s_{+1}, \hat{D}^e_{+1})] \geq G(s,k), \\ & F(s,k) - Rk - b \geq 0\}.\end{aligned}$$

Let $\Sigma(s,D)$ denote the set of (b, D_{+1}) that solves the maximization problem in (20). The bank decides on k and $V(s,D)$ by solving the following problem:

$$\begin{aligned}V(s,D) = \max_{k \in \Delta_k(s),\; (b,D_{+1}) \in \Sigma(s,D)} & F(s,k) - Rk - b \\ & + \beta \mathbb{E}[(1-p)V^e(s_{+1}, D_{+1}) + pV^e(s_{+1}, \hat{D}^e_{+1})],\end{aligned} \tag{21}$$

subject to

$$\begin{aligned}& F(s,k) - Rk - b + \beta \mathbb{E}[(1-p)V^e(s_{+1}, D_{+1}) + pV^e(s_{+1}, \hat{D}^e_{+1})] \geq G(s,k), \\ & F(s,k) - Rk - b \geq 0.\end{aligned}$$

Let $\Lambda(s,D)$ denote the set of (k, b, D_{+1}) that solves the maximization problem in (21).

The equilibrium values of (k, b, D_{+1}) are determined as follows. First, $b(s, D)$ and $D_{+1}(s, D)$ are given by

$$b(s, D) = \max_{(k,b,D_{+1}) \in \Lambda(s,D)} b, \tag{22}$$

$$D_{+1}(s, D) = \min\{N_{\max}\delta,\ (1 + r)\{D - b(s, D)\}\}. \tag{23}$$

Then, $k(s, D)$ is determined by

$$k(s, D) = \max_{(k,b(s,D),D_{+1}(s,D)) \in \Lambda(s,D)} k,$$

$\hat{D}(s, D)$ by

$$\hat{D}(s, D) = \arg\max_{D' \leq D} d(s, D'),$$

and $d^{npl}(s)$ is

$$d^{npl}(s) = b^{npl}(s) + \beta\mathbb{E}[(1 - p)d^{npl}(s_{+1}) + pd(s_{+1}, \hat{D}^e_{+1})].$$

For consistency, we require that

$$V(s, D) = V^e(s, D), \quad k^{npl}(s) = k^e_{npl}(s), \quad \text{and} \quad \hat{D}(s, D) = \hat{D}^e(s, D). \tag{24}$$

A Proof of Lemma 2

There exists $D_{+1} \in \Delta$ such that

$$d(s, D+\delta) = b' + \beta \mathbb{E} d(s_{+1}, D_{+1}),$$
$$b' = D + \delta - \beta D_{+1}.$$

Note that Assumption 2 implies that $b' \geq \delta$. Consider $b = D - \beta D_{+1}$. Then, $b \geq 0$, and therefore, $b \in \Delta_b(s, D)$, while b may not be an element of $\Delta_b(s, D+\delta)$. It is easily confirmed that $b \in \Gamma(s, D)$. Thus,

$$\begin{aligned} d(s, D+\delta) &= b + \delta + \beta \mathbb{E} d(s_{+1}, D_{+1}) \\ &= \delta + [b + \beta \mathbb{E} d(s_{+1}, D_{+1})] \\ &\leq \delta + \max_{\tilde{b} \in \Gamma(s,D)} [\tilde{b} + \beta \mathbb{E} d(s_{+1}, \beta^{-1}(D - \tilde{b})] \\ &= \delta + d(s, D). \end{aligned}$$

B Proof of Lemma 3

Suppose that $b(s, D)$ is not the maximum feasible value. Then, $b(s, D) + \beta\delta \in \Gamma(s, D)$. We compare $d(s, D)$ and $X(b(s, D) + \beta\delta, s, D)$, where $X(b, s, D) \equiv b + \beta \mathbb{E} d(s_{+1}, \beta^{-1}[D - b])$. Lemma 2 implies that

$$\begin{aligned} X(b(s, D) + \beta\delta, s, D) &= b(s, D) + \beta\delta + \beta \mathbb{E} d(s_{+1}, \beta^{-1}(D - b(s, D)) - \delta) \\ &= b(s, D) + \beta \mathbb{E}\{\delta + d(s_{+1}, \beta^{-1}(D - b(s, D)) - \delta)\} \\ &\geq b(s, D) + \beta \mathbb{E} d(s_{+1}, \beta^{-1}(D - b(s, D))) \\ &= d(s, D) = \max_b X(b, s, D). \end{aligned}$$

If $X(b(s, D) + \beta\delta, s, D) > d(s, D)$, it contradicts (3), which defines $b(s, D)$. If $X(b(s, D) + \beta\delta, s, D) = d(s, D)$, Assumption 1 implies that $F(s, k(s, D)) - Rk(s, D) - b(s, D) - \beta\delta + \beta \mathbb{E} V^e(s_{+1}, D_{+1}(s, D) - \delta) \geq F(s, k(s, D)) - Rk(s, D) - b(s, D) + \beta \mathbb{E} V^e(s_{+1}, D_{+1}(s, D)) = V(s, D)$. Then, $b(s, D) + \beta\delta$ should be the equilibrium value of b. This is a contradiction. Therefore, $b(s, D)$ is the maximum feasible value in $\Gamma(s, D)$, i.e., $b(s, D) = \bar{b}(s, D)$.

Next, we prove $k(s, D) > k^{npl}(s)$ for $D \leq D_{\max}(s)$. For $D \leq D_{\max}(s)$, we have $V(s, D) \geq G^{npl}(s) + \delta$, as $V(s, D) \geq V(s, D+\delta) + \delta$ from Assumption 1 and $V(s, D+\delta) \geq G^{npl}(s)$ due to Lemma ?? in Appendix ??. Now, we prove $k(s, D) > k^{npl}(s)$ by contradiction. Suppose that $k(s, D) = k^{npl}(s)$. Then, since $(b(s, D),\ k(s, D))$ satisfy the above inequality and the limited liability constraint, we have

$$V(s, D) = F(s, k^{npl}(s)) - Rk^{npl}(s) - b(s, D) + \beta \mathbb{E} V(s_{+1}, D_{+1}(s, D)) \geq G^{npl}(s) + \delta,$$
$$F(s, k^{npl}(s)) - Rk^{npl}(s) - b(s, D) \geq 0.$$

Pick $k^{npl+}(s)$ $(> k^{npl}(s))$, which is defined by $f(s,k^{npl}(s)) - f(s,k^{npl+}(s)) = \beta\delta$, where $f(s,k) \equiv F(s,k) - Rk - G(s,k)$. Then, $k^{npl+}(s)$ satisfies

$$F(s,k^{npl+}(s)) - Rk^{npl+}(s) - b(s,D) + \beta\mathbb{E}V(s_{+1},D_{+1}(s,D)) \geq G(s,k^{npl+}(s)) + (1-\beta)\delta,$$
$$F(s,k^{npl+}(s)) - Rk^{npl+}(s) - b(s,D) \geq 0.$$

Therefore, $k(s,D)$ should be $k^{npl+}(s)$, not $k^{npl}(s)$, because $k^{npl+}(s)$ is feasible without changing $b(s,D)$ and $D_{+1}(s,D)$. This is a contradiction. Thus, we have demonstrated that for $D \leq D_{\max}(s)$, $k(s,D) > k^{npl}(s)$.

C Proof of Lemma 5

Suppose that $F(s,k(s,D)) - Rk(s,D) - b(s,D) \geq \xi + \beta\delta$ for $k(s,D) \in (k^{npl}(s),\ k^*(s))$. In this case, the bank can choose $\hat{k} < k(s,D)$, where $\hat{k} \in \Delta_k(s)$, so that $F(s,\hat{k}) - R\hat{k} - b(s,D) \geq \beta\delta$. We know that $F(s,k(s,D)) - Rk(s,D) - G(s,k(s,D)) - b(s,D) + \beta\mathbb{E}V^e(s,D_{+1}(s,D)) \geq 0$, where $D_{+1}(s,D) = \beta^{-1}[D - b(s,D)]$. As $F(s,k) - Rk - G(s,k)$ is strictly decreasing in k for $k > k^{npl}(s)$, it must be the case that

$$F(s,\hat{k}) - R\hat{k} - G(s,\hat{k}) \geq F(s,k(s,D)) - Rk(s,D) - G(s,k(s,D)) + \beta\delta.$$

Thus, $\hat{b} = b(s,D) + \beta\delta$ satisfies

$$F(s,\hat{k}) - R\hat{k} - \hat{b} \geq 0,$$
$$F(s,\hat{k}) - R\hat{k} - \hat{b} - G(s,\hat{k}) + \beta\mathbb{E}V^e(s_{+1},\beta^{-1}(D-\hat{b})) \geq 0.$$

Then, $\hat{b} = b(s,D) + \beta\delta$ is feasible and Lemma 3 implies that $\hat{b}$ should be the solution to (3). This is a contradiction.

D Proof of Lemma 7

For any s and $D > D_{\max}(s)$, we consider a stochastic sequence $\{s_t, k_t, b_t, D_t\}$, where $k_t = k(s_t,D_t)$, $b_t = b(s_t,D_t)$, $D_t = n_\delta[(1+r)(D_{t-1} - b_{t-1})]$, $s_0 = s$, and $D_0 = D$, given that s_t is an exogenous stochastic variable.

First, we consider the case where $s = s_H$. Suppose there exists D, which satisfies $D > D_{\max}$, such that $k(s,D) \neq k^{npl}(s)$. Then, Lemma **??** implies $k(s,D) > k^{npl}(s)$. Then, Lemma 5 implies that $0 \leq F(s,k) - Rk - b < \xi + \beta\delta$, which implies, together with $V \geq G(s,k)$, that

$$G(s,k(s,D)) \leq V(s,D) \leq \xi + \beta\delta + \beta\mathbb{E}V(s_{+1},D_{+1})$$

As it is obvious that $V(s_L,D) \leq V(s_H,D)$, it must be the case that $\mathbb{E}V(s_{+1},D_{+1}) \leq V(s_H,D_{+1})$. Then,

$$G(s,k(s,D)) \leq V(s,D) \leq \xi + \beta\delta + \beta V(s_H,D_{+1}), \tag{25}$$

where $D_{+1} > D$ as $D > D_{\max}(s)$. Lemma 6 implies that $V(s_H, D_{+1}) < \delta_g + G(s_H, k(s_H, D_{+1}))$. Thus,

$$G(s_H, k(s_H, D)) < \xi + \beta(\delta + \delta_g) + \beta G(s_H, k(s_H, D_{+1})). \tag{26}$$

Assumption 3 and the inequality (26) imply that $G(s_H, k(s_H, D)) < (1-\beta)G^{npl}(s) + \beta G(s_H, k(s_H, D_{+1})) \leq G(s_H, k(s_H, D_{+1}))$, because $G^{npl}(s) \leq G(s_H, k(s_H, D_{+1}))$. Thus, $k(s_H, D) < k(s_H, D_{+1})$. Let us set $(s_0, D_0) = (s, D)$ and consider the sequence $\{s_t, D_t, k(s_t, D_t)\}$. Given (26), we can prove the following inequality:

$$k^{npl}(s_H) < k(s_H, D_t) < k(s_H, D_{t+1}), \tag{27}$$

$$G(s_H, k(s_H, D_0)) < \frac{\{\xi + \beta(\delta + \delta_g)\}(1-\beta^t)}{1-\beta} + \beta^t G(s_H, k(s_H, D_t)) \tag{28}$$

The proof is by induction. The above argument has proven (27) and (28) for $t = 0$. Suppose that (27) holds for $t-1$. (26) applies for D_t and implies that

$$G(s_H, k(s_H, D_t)) < \xi + \beta(\delta + \delta_g) + \beta G(s_H, k(s_H, D_{t+1})), \tag{29}$$

which, together with Assumption 3, implies that $G(s_H, k(s_H, D_{t+1})) > G(s_H, k(s_H, D_t))$, or $k(s_H, D_{t+1}) > k(s_H, D_t)$. Thus, (27) has been proven for t. Suppose that (28) holds for t. This inequality, together with (29), implies that

$$\begin{aligned}
G(s_H, k(s_H, D_0)) &< \frac{\{\xi + \beta(\delta + \delta_g)\}(1-\beta^t)}{1-\beta} + \beta^t G(s_H, k(s_H, D_t)) \\
&< \frac{\{\xi + \beta(\delta + \delta_g)\}[1-\beta^t + \beta^t(1-\beta)]}{1-\beta} + \beta^{t+1} G(s_H, k(s_H, D_{t+1})) \\
&= \frac{\{\xi + \beta(\delta + \delta_g)\}(1-\beta^{t+1})}{1-\beta} + \beta^{t+1} G(s_H, k(s_H, D_{t+1})).
\end{aligned}$$

Thus, (28) has been proven for $t+1$. We have demonstrated that (27) and (28) hold for all t.

Assumption 3 and (28) imply that, in the limit of $t \to \infty$, we have $V(s_t, D_t) \to \infty$. This is a contradiction because $V(s, D)$ is bounded from above: $V(s, D) < V_{\max}$. Thus, it cannot be the case that $k(s_H, D) \neq k^{npl}(s_H)$.

Next, we consider the case where $s = s_L$. Suppose that $k(s_L, D) \neq k^{npl}(s_L)$. Then, Lemma **??** implies that $k(s_L, D) > k^{npl}(s_L)$. In this case, Lemmas 5 and 6 imply that for $D_0 = D$ and the sequence $\{s_t, D_t, k(s_t, D_t)\}$,

$$\begin{aligned}
G(s_L, k(s_L, D_t)) &< \xi + \beta(\delta + \delta_g) + \beta \mathbb{E}_t G(s_{t+1}, k(s_{t+1}, D_{t+1})) \\
&= \xi + \beta(\delta + \delta_g) + \beta[p_L G(s_L, k(s_L, D_{t+1})) + (1-p_L)G^{npl}(s_H)],
\end{aligned}$$

where $p_L = \Pr(s_{t+1} = s_L | s_t = s_L)$ and $G(s_H, k(s_H, D_{t+1})) = G^{npl}(s_H)$ for $D_{t+1} > D_{\max}$, as shown above. Let $k(s_L, D) = k_0$ and define $\{k_t\}_{t=0}^{\infty}$ by the following law of motion,

$$G(s_L, k_t) = \xi + \beta(\delta + \delta_g) + \beta[p_L G(s_L, k_{t+1}) + (1-p_L)G^{npl}(s_H)].$$

Lemma **??** implies that $k(s_L, D_t) \geq k^{npl}(s_L)$ for all $t \geq 1$. In the case where $k(s_L, D) = k_0 > k^{npl}(s_L)$, the sequence $\{k_t\}_{t=0}^{\infty}$ is such that $\lim_{t\to\infty} k_t = \infty$. Thus, $V(s_L, D_t) > G(s_L, k(s_L, D_t)) - \delta_g$ goes to infinity, and eventually violates the condition $V(s_L, D_t) < V_{\max}$. This is a contradiction. Thus, $k(s_L, D)$ must be $k^{npl}(s_L)$.

Therefore, if $D > D_{\max}$, then $k(s, D) = k^{npl}(s)$ for all $s \in \{s_L, s_H\}$.

E Proof of Proposition 8

The proof consists of two parts. First, we prove the existence of one equilibrium, in which $V^e(s, D) = G(s, k^{npl}(s)) \equiv G^{npl}(s)$. Second, we demonstrate that this equilibrium is the unique equilibrium that maximizes $d(s, D)$ subject to the no-default condition.

Existence: we guess and later verify that $V^e(s, D) = G^{npl}(s)$. Given this expectation, the bank solves

$$d(s, D) = \max_{b\in\Delta_b(s,D), k\in\Delta_k(s)} b + \beta\mathbb{E}d(s_{+1}, D_{+1}),$$
$$\text{s, t.} \begin{cases} F(s, k) - Rk - b + \beta\mathbb{E}G(s_{+1}, k^{npl}(s_{+1})) \geq G(s, k), \\ F(s, k) - Rk - b \geq 0. \end{cases}$$

Given that $V^e(s, D) = G^{npl}(s)$, it is easily shown that $\Gamma(s, D) = \{b \mid b \in \Delta_b(s, D),\ 0 \leq b \leq b^{npl}(s)\}$.

Claim: The solution to the bank's problem is $b(s, D) = b^{npl}(s)$ and $k(s, D) = k^{npl}(s)$.
(Proof of Claim)
Because $b(s, D) \leq b^{npl}(s)$, there exists a nonnegative integer m and a nonnegative real number ε, where $0 \leq \varepsilon < \beta\delta$, such that $b(s, D) = b^{npl}(s) - \varepsilon - m\beta\delta$. Then, $D_{+1}(s, D) = \min\{N_{\max}\delta, \beta^{-1}[D - b(s, D)]\} = D_{+1}^{npl} + m'\delta$, where $0 \leq m' \leq m$ and we define $D_{+1}^{npl} = \min\{N_{\max}\delta, n_\delta(\beta^{-1}[D - b^{npl}(s)])\}$. Thus,

$$\begin{aligned} d(s, D) &= b(s, D) + \beta\mathbb{E}d(s_{+1}, D_{+1}^{npl} + m'\delta) \\ &= b^{npl}(s) - \varepsilon - m\beta\delta + \beta\mathbb{E}d(s_{+1}, D_{+1}^{npl} + m'\delta) \\ &= b^{npl}(s) - \varepsilon - (m - m')\beta\delta + \beta\mathbb{E}[-m'\delta + d(s_{+1}, D_{+1}^{npl} + m'\delta)] \\ &\leq b^{npl}(s) - \varepsilon - (m - m')\beta\delta + \beta\mathbb{E}d(s_{+1}, D_{+1}^{npl}) \\ &\leq b^{npl}(s) + \beta\mathbb{E}d(s_{+1}, D_{+1}^{npl}). \end{aligned}$$

The first inequality is from Lemma 2. Therefore, $b(s, D) = b^{npl}(s)$ and $k(s, D) = k^{npl}(s)$.
(End of Proof of Claim)
Thus, the solution to the bank's problem is $k = k^{npl}(s)$ and $b = b^{npl}(s)$. It is also easily confirmed that $V(s, D) = F(s, k^{npl}(s)) - Rk^{npl}(s) - b^{npl}(s) + \beta\mathbb{E}G(s_{+1}, k^{npl}(s_{+1})) = G(s, k^{npl}(s))$, which verifies the expectation.

Uniqueness: In what follows, we demonstrate that $d^{npl}(s)$ is the maximum amount of the present discounted value (PDV) of repayments that satisfies the enforcement constraint, and the above equilibrium is the unique equilibrium that attains $d^{npl}(s)$. We consider the following planner's problem, assuming that $k(s,D) = k^{npl}(s)$. We set this assumption because Lemma 7 shows that $k(s,D) = k^{npl}(s)$ for $D > D_{\max}(s_H)$ in any equilibrium that exists. Given $k(s,D) = k^{npl}(s)$, the planner's problem is

$$
\begin{aligned}
d(s,D) = \max_{b,V(s,D)} \; & b + \beta\mathbb{E}d(s_{+1}, \beta^{-1}(D-b)), \\
\text{s. t.} \quad & V(s,D) = F(s,k^{npl}(s)) - Rk^{npl}(s) - b + \beta\mathbb{E}V(s_{+1}, \beta^{-1}(D-b)) \geq G^{npl}(s), \\
& F(s,k^{npl}(s)) - Rk^{npl}(s) - b \geq 0.
\end{aligned}
$$

Define $W^{npl}(s) = F(s,k^{npl}(s)) - Rk^{npl}(s) + \beta\mathbb{E}W^{npl}(s_{+1})$. Then, $d(s,D) = W^{npl}(s) - V(s,D)$. Thus, the planner's problem can be rewritten as

$$
\begin{aligned}
\max_{b,V(s,D)} \; & d(s,D) = W^{npl}(s) - V(s,D), \\
\text{s. t.} \quad & d(s,D) \leq W^{npl}(s) - G^{npl}(s), \\
& F(s,k^{npl}(s)) - Rk^{npl}(s) - b \geq 0.
\end{aligned}
$$

We temporarily omit the limited liability constraint, $F(s,k^{npl}(s)) - Rk^{npl}(s) - b \geq 0$, and later justify that it is satisfied. Without this constraint, it is obvious that the maximum PDV of repayments is $W^{npl}(s) - G^{npl}(s) = d^{npl}(s)$, and it is attained by setting $b = d(s,D) - \beta\mathbb{E}d(s_{+1}, D_{+1}) = W^{npl}(s) - G^{npl}(s) - \beta\mathbb{E}[W^{npl}(s_{+1}) - G^{npl}(s_{+1})] = F^{npl}(s) - Rk^{npl}(s) - G^{npl}(s) + \beta\mathbb{E}G^{npl}(s_{+1}) = b^{npl}(s)$. Therefore, the value of the firm becomes $V(s,D) = G^{npl}(s)$. By definition of $k^{npl}(s)$, it is obvious that the limited liability constraint is satisfied in this equilibrium. Thus, the unique equilibrium that maximizes the PDV of repayments is the NPL equilibrium.

F On the proof of Theorem 9

F.1 Proof of Lemma 10

We prove Lemma 10 by explicitly deriving $\{d^{(1)}(s,D), V^{(1)}(s,D), b^{(1)}(s,D), k^{(1)}(s,D)\}$. For $D < D^{**}(s) \equiv F(s,k^*(s)) - Rk^*(s)$,

$$
\begin{aligned}
& d^{(1)}(s,D) = D, \\
& V^{(1)}(s,D) = F(s,k) - Rk + \beta V_H^* - D,
\end{aligned}
$$

as $d^{(1)}(s,D) = \max_b b + \beta[\beta^{-1}(D-b)]$ and $b = D$ is feasible because $F(s,k) - Rk + \beta V_H^* - D \geq G(s,k)$ is satisfied at $k = k^*(s)$. Thus, for $0 \leq D \leq D^{**}(s)$, $(d^{(1)}(s,D), V^{(1)}(s,D))$ are given as above, with $k = k^*(s)$ and $b = D$.

For $D \in (D^{**}(s),\ D^{*}(s)]$, where $D^{*}(s)$ is the solution to $D^{**}(s)+\beta[\beta^{-1}(D-D^{**}(s))] = D = F(s,k^{*}(s)) - Rk^{*}(s) + \beta V_H^{*} - G(s,k^{*}(s))$,

$$
\begin{aligned}
&d^{(1)}(s,D) = D,\\
&V^{(1)}(s,D) = F(s,k) - Rk + \beta V_H^{*} - D,
\end{aligned}
$$

where $k = k^{*}(s)$ and $b = D^{**}(s)$.

For $D \in (D^{*}(s),\ \hat{D}^{(1)}(s)]$, where $\hat{D}^{(1)}(s) = F(s,k^{npl}(s)) - Rk^{npl}(s) - G(s,k^{npl}(s)) + \beta V_H^{*}$, the solution $(d^{(1)}(s,D), V^{(1)}(s,D))$ is given as follows.

$$
\begin{aligned}
&d^{(1)}(s,D) = D,\\
&V^{(1)}(s,D) = F(s,k(s,D)) - Rk(s,D) + \beta V_H^{*} - D,
\end{aligned}
$$

where

$$
\begin{aligned}
k(s,D) = \arg\max_{k\in\Delta_k(s)} &\ F(s,k) - Rk - D + \beta V_H^{*},\\
\text{s.t.}\ &\ F(s,k) - Rk - D + \beta V_H^{*} \geq G(s,k).
\end{aligned} \tag{30}
$$

Then, it is obvious that $k(s,D)$ is decreasing in D. $D_{+1}(s,D)$ is given by

$$
\begin{aligned}
D_{+1}(s,D) = \min_{D_{+1}\in\Delta} &\ D_{+1},\\
\text{s. t.}\ &\ D - \beta D_{+1} \leq F(s,k(s,D)) - Rk(s,D).
\end{aligned}
$$

Note that if $D = \hat{D}^{(1)}(s)$, then $D_{+1} = V_H^{*} - \beta^{-1}G^{npl}(s) < \bar{D}^{(0)}$. Note that if $D > \hat{D}^{(1)}(s)$, the enforcement constraint (30) is never satisfied for any value of k, if $V^{(1)}(s,D) = F(s,k(s,D)) - Rk(s,D) + \beta V_H^{*} - D$.

For $D > \hat{D}^{(1)}(s)$, it must be the case that $D_{+1} \geq \bar{D}^{(0)}$, since otherwise $V^{(1)}(s,D)$ becomes $F(s,k(s,D)) - Rk(s,D) + \beta V_H^{*} - D$ and the enforcement constraint (30) is never satisfied because $\hat{D}^{(1)}(s)$ is the maximum value that is feasible under (30). $D_{+1} \geq \bar{D}^{(0)}$ is feasible for D ($> \hat{D}^{(1)}(s)$), because $\beta^{-1}\hat{D}^{(1)}(s) > \bar{D}^{(0)}$ is easily shown. Given that $D_{+1} > \bar{D}^{(0)}$, we have $d^{(0)}(s,D_{+1}) = d^{npl}(s)$ and $V^{(0)}(s,D_{+1}) = G^{npl}(s)$. Thus, the values of $(d^{(1)}(s,D),\ V^{(1)}(s,D),\ b(s,D), k(s,D))$ are given as the solution to the following problem.

$$
\begin{aligned}
d^{(1)}(s,D) = &\max_{b\in\Delta_b(s,D),\ k\in\Delta_k(s)} b + \beta\mathbb{E}d^{npl}(s),\\
&\text{s.t.} \begin{cases} F(s,k) - Rk - b + \beta\mathbb{E}G^{npl}(s) \geq G(s,k),\\ F(s,k) - Rk \geq b. \end{cases}
\end{aligned}
$$

Then,

$$
V^{(1)}(s,D) = F(s,k(s,D)) - Rk(s,D) - b(s,D) + \beta\mathbb{E}G^{npl}(s).
$$

The solution is

$$b(s,D) = b^{npl}(s), \quad k(s,D) = k^{npl}(s), \quad d^{(1)}(s,D) = d^{npl}(s), \quad V^{(1)}(s,D) = G^{npl}(s),$$

for $D > \hat{D}^{(1)}(s)$. It is also easily confirmed that

$$\hat{D}^{(1)}(s) = \bar{D}^{(1)}(s),$$

where $\bar{D}^{(1)}(s)$ is defined by

$$\begin{aligned}\bar{D}^{(1)}(s_H) &= \max D, \\ &\quad \text{s.t. } D_{+1}(s_H, D) < \bar{D}^{(0)}, \\ \bar{D}^{(1)}(s_L) &= \max D, \\ &\quad \text{s.t. } D_{+1}(s_L, D) < \bar{D}^{(0)}.\end{aligned}$$

Now, we can show the following claim.

Claim 1. $\bar{D}^{(1)}(s_L) \le \bar{D}^{(1)}(s_H) < \bar{D}^{(0)}$.

(Proof of Claim 1)
We have $\bar{D}^{(1)}(s_L) \le \bar{D}^{(1)}(s_H)$, and

$$\begin{aligned}\bar{D}^{(1)}(s_H) &= F(s_H, k^{npl}(s_H)) - Rk^{npl}(s_H) - G(s_H, k^{npl}(s_H)) + \beta V_H^* \\ &< F(s_H, k^*(s_H)) - Rk^*(s_H) + \beta V_H^* - G(s_H, k^{npl}(s_H)) \\ &= V_H^* - G(s_H, k^{npl}(s_H)) = \bar{D}^{(0)}.\end{aligned}$$

(End of proof of Claim 1)

Note that $d^{npl}(s) < \bar{D}^{(1)}(s)$ because $V_H^* > G^{npl}(s_H) + d^{npl}(s_H)$ implies that $d^{npl}(s) = b^{npl}(s) + \beta \mathbb{E} d^{npl}(s_{+1}) = F(s, k^{npl}(s)) - Rk^{npl}(s) - G^{npl}(s) + \beta \mathbb{E}[G^{npl}(s_{+1}) + d^{npl}(s_{+1})] < F(s, k^{npl}(s)) - Rk^{npl}(s) - G^{npl}(s) + \beta V_H^* = \bar{D}^{(1)}(s)$.

These explicit solutions directly imply (i)–(vi) of Lemma 10.

F.2 Proof of Lemma 11

Proof of (ii). The assumption (ii') implies that $\mathbb{E} d^{(n)}(s_{+1}, D_{+1}) \le \mathbb{E} d^{(n-1)}(s_{+1}, D_{+1})$, and the assumption (v') implies that $\Gamma^{(n+1)}(s,D) \subset \Gamma^{(n)}(s,D)$. These facts imply that

$$d^{(n+1)}(s,D) = \max_{b \in \Gamma^{(n+1)}(s,D)} b + \beta \mathbb{E} d^{(n)}(s_{+1}, D_{+1}) \le \max_{b \in \Gamma^{(n)}(s,D)} b + \beta \mathbb{E} d^{(n-1)}(s_{+1}, D_{+1}) = d^{(n)}(s,D).$$

Since $b^{npl}(s) \in \Gamma^{(n+1)}(s,D)$ and $d^{(n)}(s,D) \ge d^{npl}(s)$ for $D > d^{npl}(s)$,

$$d^{(n+1)}(s,D) = \max_{b \in \Gamma^{(n+1)}(s,D)} b + \beta \mathbb{E} d^{(n)}(s_{+1}, D_{+1}) \ge b^{npl}(s) + \beta \mathbb{E} d^{npl}(s_{+1}) = d^{npl}(s),$$

for $D > d^{npl}(s)$. It is obvious that $d^{(n+1)}(s, D) \geq 0$ for $D \leq d^{npl}(s)$.

Proof of (iii). Assumption (iii') implies that for $D \geq \bar{D}^{(n+1)}(s)$, the values of $(d^{(n+1)}(s, D)$, $V^{(n+1)}(s, D)$, $b^{(n+1)}(s, D)$, $k^{(n+1)}(s, D))$ are given as the solution to the following problem.

$$d^{(n+1)}(s, D) = \max_{b \in \Delta_b(s,D),\ k \in \Delta_k(s)} b + \beta \mathbb{E} d^{npl}(s),$$
$$\text{s.t.} \begin{cases} F(s, k) - Rk - b + \beta \mathbb{E} G^{npl}(s) \geq G(s, k), \\ F(s, k) - Rk \geq b. \end{cases}$$

Then,

$$V^{(n+1)}(s, D) = F(s, k(s, D)) - Rk(s, D) - b(s, D) + \beta \mathbb{E} G^{npl}(s).$$

It is easily shown that the solution is given by

$$b(s, D) = b^{npl}(s), \quad k(s, D) = k^{npl}(s), \quad d^{(n+1)}(s, D) = d^{npl}(s), \quad V^{(n+1)}(s, D) = G^{npl}(s).$$

Proof of (i). For $D \geq \bar{D}^{(n+1)}(s)$, it is the case that $d^{(n+1)}(s, D + \delta) = d^{npl}(s) \leq d^{(n+1)}(s, D) + \delta$ by the part (iii) above. Next, we consider the case where $D < \bar{D}^{(n+1)}(s)$. We can prove the following claim.

Claim 2. For $D < \bar{D}^{(n+1)}(s)$, $b^{(n+1)}(s, D + \delta)$ is the maximum feasible value, i.e.,

$$b^{(n+1)}(s, D + \delta) = \max_{b \in \Gamma^{(n+1)}(s, D+\delta)} b.$$

(Proof of Claim 2). Suppose that $b^{(n+1)}(s, D + \delta)$ is not the maximum feasible value. Then, $b^{(n+1)}(s, D+\delta) + \beta\delta \in \Gamma^{(n+1)}(s, D+\delta)$. We compare $d^{(n+1)}(s, D+\delta)$ and $X^{(n+1)}(b^{(n+1)}(s, D+\delta) + \beta\delta, s, D + \delta)$, where $X^{(n+1)}(b, s, D) \equiv b + \beta \mathbb{E} d^{(n)}(s_{+1}, \beta^{-1}(D - b))$. Assumption (i') implies that

$$\begin{aligned}
&X^{(n+1)}(b^{(n+1)}(s, D + \delta) + \beta\delta, s, D + \delta) \\
&= b^{(n+1)}(s, D + \delta) + \beta\delta + \beta \mathbb{E} d^{(n)}(s_{+1}, \beta^{-1}(D + \delta - b^{(n+1)}(s, D + \delta)) - \delta) \\
&= b^{(n+1)}(s, D + \delta) + \beta \mathbb{E}\{\delta + d^{(n)}(s_{+1}, \beta^{-1}(D + \delta - b^{(n+1)}(s, D + \delta)) - \delta)\} \\
&\geq b^{(n+1)}(s, D + \delta) + \beta \mathbb{E} d^{(n)}(s_{+1}, \beta^{-1}(D + \delta - b^{(n+1)}(s, D + \delta))) \\
&= d^{(n+1)}(s, D + \delta) = \max_b X^{(n+1)}(b, s, D + \delta).
\end{aligned}$$

Assumption (iv') implies that

$V^{(n+1)}(s, D + \delta) =$
$F(s, k(s, D + \delta)) - Rk(s, D + \delta) - b(s, D + \delta) + \beta \mathbb{E} V^{(n)}(s_{+1}, D_{+1}^{(n+1)}(s, D + \delta)) \leq$
$F(s, k(s, D + \delta)) - Rk(s, D + \delta) - b(s, D + \delta) + \beta \mathbb{E}(-\delta + V^{(n)}(s_{+1}, D_{+1}^{(n+1)}(s, D + \delta) - \delta)) =$
$F(s, k(s, D + \delta)) - Rk(s, D + \delta) - b(s, D + \delta) - \beta\delta + \beta \mathbb{E} V^{(n)}(s_{+1}, D_{+1}^{(n+1)}(s, D + \delta) - \delta).$

Assumption (iv') applies here as $D+\delta \leq \bar{D}^{(n+1)}(s)$, which implies $D_{+1}^{(n+1)}(s, D+\delta) \leq \bar{D}^{(n)}(s)$. These two inequalities imply that the equilibrium value of b should be $b(s, D+\delta)+\beta\delta$. This contradicts the definition of $b^{(n+1)}(s, D+\delta)$. Therefore, $b^{(n+1)}(s, D+\delta)$ is the maximum feasible value. *(End of proof of Claim 2)*

This claim implies that it suffices to consider the region $b \geq \delta$, when we evaluate $d^{(n+1)}(s, D+\delta)$. If $b+\delta \in \Gamma^{(n+1)}(s, D+\delta)$ then $b \in \Gamma^{(n+1)}(s, D)$ for $D > F(s, k^*(s)) - Rk^*(s)$.[6] Defining $\hat{b}$ by $\hat{b} = b(s, D+\delta) - \delta$, it is easily demonstrated that $\hat{b} \in \Gamma^{(n+1)}(s, D)$. Thus,

$$\begin{aligned} d^{(n+1)}(s, D+\delta) &= b(s, D+\delta) + \beta \mathbb{E} d^{(n)}(s_{+1}, \beta^{-1}(D+\delta - b(s, D+\delta))) \\ &= \delta + \hat{b} + \beta \mathbb{E} d^{(n)}(s_{+1}, \beta^{-1}(D-\hat{b})), \\ &\leq \delta + \max_{b \in \Gamma^{(n+1)}(s,D)} b + \beta \mathbb{E} d^{(n)}(s_{+1}, \beta^{-1}(D-b)) = \delta + d^{(n+1)}(s, D). \end{aligned}$$

Proof of (iv). We consider the case where $D+\delta \leq \bar{D}^{(n+1)}(s)$. Define $\tilde{\Delta}_b(s, D) = \{b \in \mathbb{R} | b = D - \beta D_{+1},\ \text{where } D_{+1} \in \Delta_{+1},\ \text{ and } b \geq 0\} \cup \{b^{npl}(s) - \delta\}$. Define $\tilde{\Gamma}^{(n+1)}(s, D) = \{b \in \tilde{\Delta}_b(s, D) \mid \exists k \in \Delta_k(s),\ \text{s.t. } F(s,k) - Rk - b - \delta + \beta \mathbb{E} V^{(n)}(s_{+1}, \beta^{-1}(D-b)) \geq G(s,k),\ \text{ and } F(s,k) - Rk - b - \delta \geq 0\}$. Let $\tilde{b}(s, D)$ be the maximum value of $\tilde{\Gamma}^{(n+1)}(s, D)$. It is obvious that $\tilde{b}(s, D) \leq b(s, D)$, as $b(s, D)$ is the maximum value of $\Gamma^{(n+1)}(s, D)$. $V^{(n+1)}(s, D+\delta)$ can be written as

$$V^{(n+1)}(s, D+\delta) = -\delta + \tilde{V}^{(n+1)}(s, D), \tag{31}$$

where

$$\begin{aligned} \tilde{V}^{(n+1)}(s, D) = \max_{k \in \Delta_k(s)} &F(s,k) - Rk - \tilde{b}(s, D) + \beta \mathbb{E} V^{(n)}(s_{+1}, \beta^{-1}(D - \tilde{b}(s, D))), \quad (32) \\ \text{s.t. } &F(s,k) - Rk - \tilde{b}(s, D) - \delta + \beta \mathbb{E} V^{(n)}(s_{+1}, \beta^{-1}(D - \tilde{b}(s, D))) \geq G(s,k), \\ &F(s,k) - Rk - \tilde{b}(s, D) - \delta \geq 0. \end{aligned}$$

Let $\tilde{k}(s, D)$ be the solution to (32). The following claim holds:

Claim 3. $\tilde{b}(s, D)$ and $\tilde{k}(s, D)$ satisfy $\tilde{b}(s, D) \leq b(s, D)$ and $\tilde{k}(s, D) \leq k(s, D)$.

(Proof of Claim 3). We know $\tilde{b}(s, D) \leq b(s, D)$ from the above argument. Now, $k(s, D)$ is the maximum k that satisfies

$$\begin{aligned} &F(s,k) - Rk - b(s, D) + \beta \mathbb{E} V^{(n)}(s_{+1}, \beta^{-1}(D - b(s, D))) \geq G(s,k), \\ &F(s,k) - Rk - b(s, D) \geq 0, \end{aligned}$$

[6] For $D \leq F(s, k^*(s)) - Rk^*(s)$, $(b, D_{+1}) = (D, 0)$ is feasible. Let $d^{(n+1)}(s, D) = b + \beta \mathbb{E} d^{(n)}(s_{+1}, \beta^{-1}(D-b))$. Assumption (i') implies that, for any $b \geq 0$, $\beta \mathbb{E} d^{(n)}(s_{+1}, \beta^{-1}(D-b)) \leq \beta[\beta^{-1}(D-b)] + \beta \mathbb{E} d^{(n)}(s_{+1}, 0)$. Thus, it must be the case that $d^{(n+1)}(s, D) = D + \beta \mathbb{E} d^{(n)}(s_{+1}, 0)$. Therefore, $d^{(n+1)}(s, D+\delta) = \delta + d^{(n+1)}(s, D)$, for $D \leq F(s, k^*(s)) - Rk^*(s)$.

while $\tilde{k}(s,D)$ is the maximum k that satisfies

$$F(s,k) - Rk - \tilde{b}(s,D) - \delta + \beta \mathbb{E} V^{(n)}(s_{+1}, \beta^{-1}(D - \tilde{b}(s,D))) \geq G(s,k),$$
$$F(s,k) - Rk - \tilde{b}(s,D) - \delta \geq 0.$$

We will demonstrate that $\tilde{k}(s,D) \leq k(s,D)$ by contradiction. Suppose that $\tilde{k}(s,D) > k(s,D)$. Then, $F(s,\tilde{k}(s,D)) - R\tilde{k}(s,D) - b(s,D) \geq 0$ is satisfied. The condition for $\tilde{b}(s,D)$ implies

$$F(s,\tilde{k}(s,D)) - R\tilde{k}(s,D) - \tilde{b}(s,D) - \delta + \beta \mathbb{E} V^{(n)}(s_{+1}, \beta^{-1}(D - \tilde{b}(s,D))) \geq G(s,\tilde{k}(s,D)). \tag{33}$$

By definition of $\tilde{\Gamma}^{(n+1)}(s,D)$, the fact that $\tilde{b}(s,D) \leq b(s,D)$ implies that there exists an integer $m\,(\geq 0)$ such that $\tilde{b}(s,D) + m\beta\delta = b(s,D)$. Then,

$$\begin{aligned} &- \tilde{b}(s,D) + \beta \mathbb{E} V^{(n)}(s_{+1}, \beta^{-1}(D - \tilde{b}(s,D))) = -b(s,D) + m\beta\delta + \beta \mathbb{E} V^{(n)}(s_{+1}, \beta^{-1}(D - b(s,D) + m\beta\delta)) \\ &\leq -b(s,D) + \beta \mathbb{E} V^{(n)}(s_{+1}, \beta^{-1}(D - b(s,D))), \end{aligned}$$

where the inequality is due to assumption (iv'). This inequality together with (33) implies that

$$F(s,\tilde{k}(s,D)) - R\tilde{k}(s,D) - b(s,D) - \delta + \beta \mathbb{E} V^{(n)}(s_{+1}, \beta^{-1}(D - b(s,D))) \geq G(s,\tilde{k}(s,D)).$$

This condition and the nonnegativity condition $(F(s,\tilde{k}(s,D)) - R\tilde{k}(s,D) - b(s,D) \geq 0)$ imply that $\tilde{k}(s,D) \in \Gamma^{(n+1)}(s,D)$, which implies that $\tilde{k}(s,D) \leq k(s,D)$, a contradiction. Thus, it must be the case that $\tilde{k}(s,D) \leq k(s,D)$. *(End of proof of Claim 3).*

Let $(k,b) = (k(s,D), b(s,D))$ and $(\tilde{k},\tilde{b}) = (\tilde{k}(s,D), \tilde{b}(s,D))$. Then, Claim 3 implies that there exist a non-negative integer m and a non-negative real number ε such that

$$F(s,\tilde{k}) - R\tilde{k} = F(s,k) - Rk - \varepsilon,$$
$$\tilde{b} = b - m\beta\delta.$$

Thus,

$$\begin{aligned} \tilde{V}^{(n+1)}(s,D) &= F(s,\tilde{k}) - R\tilde{k} - \tilde{b} + \beta \mathbb{E} V^{(n)}(s_{+1}, \beta^{-1}(D - \tilde{b})), \\ &= F(s,k) - Rk - \varepsilon - b + m\beta\delta + \beta \mathbb{E} V^{(n)}(s_{+1}, \beta^{-1}(D - b) + m\delta), \\ &= -\varepsilon + F(s,k) - Rk - b + \beta \mathbb{E}[m\delta + V^{(n)}(s_{+1}, \beta^{-1}(D - b) + m\delta) \\ &\leq -\varepsilon + F(s,k) - Rk - b + \beta \mathbb{E} V^{(n)}(s_{+1}, \beta^{-1}(D - b)) \\ &= -\varepsilon + V^{(n+1)}(s,D) \leq V^{(n+1)}(s,D), \end{aligned}$$

where the first inequality is from Assumption (iv'). Note that Assumption (iv') applies, since $\beta^{-1}(D - \tilde{b}) < D^{(n)}(s)$ because $D + \delta < D^{(n+1)}(s)$. (31) implies that $V^{(n+1)}(s, D+\delta) =$

$$-\delta + \tilde{V}^{(n+1)}(s,D) \leq -\delta + V^{(n+1)}(s,D).$$

Proof of (v). For $D > \bar{D}^{(n+1)}(s)$, it is the case that $V^{(n+1)}(s,D) = G^{npl}(s)$ as proven at part (iii). Next, we consider the case where $D \leq \bar{D}^{(n+1)}(s)$. For a fixed (s,D), Assumption (v') implies that $\Gamma^{(n+1)}(s,D) \subset \Gamma^{(n)}(s,D)$ and $\Lambda^{(n+1)}(s,D) \subset \Lambda^{(n)}(s,D)$. The following claim holds.

Claim 4. The variables for $(n+1)-$th problem satisfy $b^{(n+1)}(s,D) \leq b^{(n)}(s,D)$ and $k^{(n+1)}(s,D) \leq k^{(n)}(s,D)$.

(Proof of Claim 4). Since $\Gamma^{(n+1)}(s,D) \subset \Gamma^{(n)}(s,D)$, Claim 2 implies that $b^{(n+1)}(s,D) \leq b^{(n)}(s,D)$. Next, we prove $k^{(n+1)}(s,D) \leq k^{(n)}(s,D)$. Denote by $(C^{(n)})$ and $(C^{(n+1)})$ the following conditions:

$$(C^{(n)}) \qquad \begin{cases} F(s,k) - Rk - b + \beta \mathbb{E} V^{(n-1)}(s_{+1}, \beta^{-1}(D-b)) \geq G(s,k), \\ F(s,k) - Rk - b \geq 0, \end{cases}$$

$$(C^{(n+1)}) \qquad \begin{cases} F(s,k) - Rk - b + \beta \mathbb{E} V^{(n)}(s_{+1}, \beta^{-1}(D-b)) \geq G(s,k), \\ F(s,k) - Rk - b \geq 0, \end{cases}$$

- Case 1: Suppose that $b^{(n+1)} = b^{(n)}$
  In this case, $k^{(n+1)} \leq k^{(n)}$ should hold because $(C^{(n+1)})$ is (weakly) tighter than $(C^{(n)})$ for $b = b^{(n+1)} = b^{(n)}$.

- Case 2: Suppose that $b^{(n+1)} < b^{(n)}$.
  In this case, we first prove that the following condition holds:

  $$0 \leq F(s,k^{(n+1)}(s,D)) - Rk^{(n+1)}(s,D) - b^{(n+1)}(s,D) < \delta(s,k^{(n+1)}(s,D)) + \beta\delta, \tag{34}$$

  where $\delta(s,k^{(n+1)}(s,D))$ is defined by $\delta(s,k^{(n+1)}(s,D)) \equiv F(s,k^{(n+1)}(s,D)) - Rk^{(n+1)}(s,D) - F(s,k_-^{(n+1)}(s,D)) + Rk_-^{(n+1)}(s,D)$, where $k_-^{(n+1)}(s,D)$ is defined by $f(s,k_-^{(n+1)}(s,D)) - f(s,k^{(n+1)}(s,D)) = \beta\delta$. Thus, $k_-^{(n+1)}(s,D)$ is the value of k, which is smaller than and adjacent to $k^{(n+1)}(s,D)$. The condition (34) is proven by contradiction.[7] Then, as $b^{(n)}(s,D) \geq b^{(n+1)}(s,D) + \beta\delta$, the condition (34) implies that

  $$F(s,k_-^{(n+1)}(s,D)) - Rk_-^{(n+1)}(s,D) - b^{(n)}(s,D) < 0,$$

  which implies that $k^{(n)}(s,D) > k_-^{(n+1)}(s,D)$, which means $k^{(n)}(s,D) \geq k^{(n+1)}(s,D)$.

[7] Suppose that $F(s,k^{(n+1)}(s,D)) - Rk^{(n+1)}(s,D) - b^{(n+1)}(s,D) \geq \delta(s,k^{(n+1)}(s,D)) + \beta\delta$. Then, $k = k_-^{(n+1)}(s,D)$ and $b = b^{(n+1)}(s,D) + \beta\delta$ satisfies $(C^{(n+1)})$, as follows. First, the limited liability $(F(s,k) - Rk - b \geq 0)$ is obviously satisfied. Second, since $F(s,k^{(n+1)}(s,D)) - Rk^{(n+1)}(s,D) - G(s,k^{(n+1)}(s,D)) = F(s,k_-^{(n+1)}(s,D)) - Rk_-^{(n+1)}(s,D) - G(s,k_-^{(n+1)}(s,D)) - \beta\delta$ and $V^{(n)}(s_{+1},\beta(D - b^{(n+1)}(s,D))) \leq V^{(n)}(s_{+1},\beta(D - b^{(n+1)}(s,D) - \beta\delta))$, the enforcement constraint is satisfied for $k = k_-^{(n+1)}(s,D)$ and $b = b^{(n+1)}(s,D) + \beta\delta$. Thus, they are in $\Gamma^{(n+1)}(s,D)$. Then, the solution to $(n+1)$-th problem should be $b^{(n+1)}(s,D) + \beta\delta$, instead of $b^{(n+1)}(s,D)$. This is a contradiction.

(End of proof of Claim 4).

Let $(k, b) = (k^{(n)}(s, D), b^{(n)}(s, D))$ and $(\tilde{k}, \tilde{b}) = (k^{(n+1)}(s, D), b^{(n+1)}(s, D))$. The above claim implies that there exists a non-negative integer m and a non-negative real number ε such that $F(s, \tilde{k}) - R\tilde{k} = F(s, k) - Rk - \varepsilon$ and $\tilde{b} = b - m\beta\delta$. Thus,

$$\begin{aligned}
V^{(n+1)}(s, D) &= F(s, \tilde{k}) - R\tilde{k} - \tilde{b} + \beta\mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - \tilde{b})), \\
&\leq F(s, k) - Rk - \varepsilon - b + m\beta\delta + \beta\mathbb{E}V^{(n-1)}(s_{+1}, \beta^{-1}(D - b) + m\delta), \\
&= -\varepsilon + F(s, k) - Rk - b + \beta\mathbb{E}[m\delta + V^{(n-1)}(s_{+1}, \beta^{-1}(D - b) + m\delta)] \\
&\leq -\varepsilon + F(s, k) - Rk - b + \beta\mathbb{E}V^{(n-1)}(s_{+1}, \beta^{-1}(D - b)) \\
&= -\varepsilon + V^{(n)}(s, D) \leq V^{(n)}(s, D),
\end{aligned}$$

where the first inequality is from Assumption (v') and the second inequality is from Assumption (iv'). Note that Assumption (iv') applies since $D \leq \bar{D}^{(n+1)}(s)$, which implies that $\beta^{-1}(D - b) \leq \bar{D}^{(n)}(s) \leq \bar{D}^{(n-1)}(s)$. The fact that $k^{(n+1)}(s, D) \geq k^{npl}(s)$ and the enforcement constraint $[V^{(n+1)}(s, D) \geq G(s, k^{(n+1)}(s, D))]$ directly imply that

$$V^{(n+1)}(s, D) \geq G^{npl}(s).$$

Proof of (vi). First, we prove $\bar{D}^{(n+1)}(s) \leq \bar{D}^{(n)}(s)$ by contradiction. Suppose that $\exists s,\ \bar{D}^{(n+1)}(s) > \bar{D}^{(n)}(s)$. Then, we can pick D such that $\bar{D}^{(n)}(s) < D \leq \bar{D}^{(n+1)}(s)$, which satisfies

$$\begin{aligned}
D_{+1}^{(n+1)}(s, D) &= \beta^{-1}[D - b^{(n+1)}(s, D)] < \bar{D}^{(n)}(s_H) \leq \bar{D}^{(n-1)}(s_H), \\
D_{+1}^{(n)}(s, D) &= \beta^{-1}[D - b^{(n)}(s, D)] \geq \bar{D}^{(n-1)}(s_H).
\end{aligned}$$

These inequalities imply $b^{(n+1)}(s, D) > b^{(n)}(s, D)$, while $b^{(n+1)}(s, D)$ is feasible in (n)-th problem:

$$b^{(n+1)}(s, D) \in \Gamma^{(n+1)}(s, D) \subset \Gamma^{(n)}(s, D).$$

Therefore, $b^{(n)}(s, D)$ and $b^{(n)}(s, D) + \beta\delta$ are both feasible in (n)-th problem. Assumption (i') implies

$$\begin{aligned}
d^{(n)}(s, D) &= b^{(n)}(s, D) + \beta\mathbb{E}d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s, D)) \\
&\leq b^{(n)}(s, D) + \beta\mathbb{E}[\delta + d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s, D) - \delta)] \\
&= b^{(n)}(s, D) + \beta\delta + \beta\mathbb{E}d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s, D) - \delta).
\end{aligned}$$

If $d^{(n)}(s, D) < b^{(n)}(s, D) + \beta\delta + \beta\mathbb{E}d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s, D) - \delta)$, then $b^{(n)} + \beta\delta$ should be the solution to the (n)-th problem. This is a contradiction because $b^{(n)}(s, D)$ is the solution. If $d^{(n)}(s, D) = b^{(n)}(s, D) + \beta\delta + \beta\mathbb{E}d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s, D) - \delta)$, then the fact that $d^{(n)}(s, D) = d^{npl}(s)$ and $b^{(n)}(s, D) = b^{npl}(s)$ for $D > \bar{D}^{(n)}(s)$, and $d^{npl}(s) = b^{npl}(s) + \beta\mathbb{E}d^{npl}(s_{+1})$ imply that

$$\mathbb{E}d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s, D) - \delta) < \mathbb{E}d^{npl}(s_{+1}),$$

which, in turn, implies that $\exists s_{+1},\ d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s,D)-\delta) < d^{npl}(s_{+1})$. On the other hand, $D > \bar{D}^{(n)}(s) > d^{npl}(s_H)$ implies that $D \geq d^{npl}(s_H) + 2\delta$, which, in turn, implies that $D_{+1}^{(n)}(s,D) - \delta \geq D - \delta > d^{npl}(s_H)$. Then, Assumption (ii') implies that $d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s,D)-\delta) \geq d^{npl}(s_{+1})$. Thus, we have demonstrated that $\exists s_{+1}$, such that $d^{npl}(s_{+1}) \leq d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s,D)-\delta) < d^{npl}(s_{+1})$, which is a contradiction. Therefore, it cannot be the case that $\exists s,\ \bar{D}^{(n+1)}(s) > \bar{D}^{(n)}(s)$.

G Proof of Lemma 12

Claim 2 implies that $b(s,D) = \lim_{n\to\infty} b^{(n)}(s,D)$ satisfies $b(s,D) \geq \delta$ for $D < D_{\max}(s)$. For $D \geq D_{\max}(s)$, Lemmas 10 and 11 imply $b(s,D) = b^{npl}(s) \geq \delta$. Therefore, $b(s,D) \geq \delta$ for all (s,D).

Lemmas 10 and 11 imply that $V(s,D) = \lim_{n\to\infty} V^{(n)}(s,D)$ and $D_{\max}(s) = \lim_{n\to\infty} \bar{D}^{(n)}(s)$ satisfy that $V(s,D+\delta) \leq V(s,D) - \delta$ for $D < D_{\max}(s)$.