Globalization and Synchronization of Innovation Cycles

By

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Introduction and Preview of Main Results

Theoretical Motivation:

- How does globalization affect macro co-movements across countries?
- Most macroeconomists address this question by assuming that some *exogenous* processes drive productivity movements in each country.
- *But*, globalization can affect
  - productivity growth rates, as already shown by endogenous growth models
  - *synchronicity* of productivity fluctuations, as we show in an *endogenous cycles model*

Empirical Motivation:

- Countries that trade more with each other have more synchronized business cycles
  - Particularly among developed countries, also true among developing countries
  - Not so between developed and developing countries
- Standard International RBC has difficulty explaining this "***trade-comovement puzzle***."
  - Attempts to resolve it by appealing to vertical specialization met limited success
  - It would help if trade also synchronize TFP movements (as the evidence shows)
  - Our model might provide *one* theoretical argument (out of possibly many)

Intuition Behind Synchronization Effects of Globalization (via Innovation)

- Two structurally identical countries
- In each country, strategic complementarities in the *timing* of innovation among firms competing in the same market lead to a temporal clustering of innovation, and hence aggregate fluctuations.
- Without trade, fluctuations in the two countries are obviously *disconnected*.
- Trade integration makes firms based in different countries compete against each other and respond to an increasingly global (hence common) market environment.
- This leads to an alignment of innovation incentives, *synchronizing* innovation activities and hence productivity fluctuations across countries.

To capture this intuition, we propose and analyze a 2-country model of endogenous innovation cycles with *two* building blocks

- ✓ Judd (1985) for cycles
- ✓ Helpman & Krugman (1985) for trade

Judd (1985); Dynamic Dixit-Stiglitz monopolistic competitive model of innovation

- **Temporary Monopoly;** Innovators pay fixed cost to introduce a new (horizontally differentiated) variety, earn the monopoly profit for a limited time.
- Each variety sold initially at monopoly price; later at competitive price
- Impact of innovation; initially muted, full impact with a delay

➢ Past innovation discourages innovators more than contemporaneous innovation

➢ **Temporal clustering of innovation**, leading to aggregate fluctuations

Judd (1985; Sec.4); *also* ***Deneckere & Judd (1992; DJ for short)***

- *Discrete time* and *one period monopoly* for analytical tractability
- **1D state space** (the measure of competitive varieties inherited from past innovation determines how saturated the economy is)
- Unique equilibrium path obtained by iterating a **1D PWL noninvertible map**.
- Fluctuations for almost all initial conditions, whenever the unique steady state is unstable, converging either to a ***2-cycle*** or to a *chaotic attractor*
- They are driven neither by multiplicity *nor* self-fulfilling expectations.

Note: Discrete time is ***NOT*** responsible for fluctuations**. Judd (1985; Sec.3)** shows *in continuous time* that the economy alternates between the phases of active innovation and of no innovation along any equilibrium path for almost all initial conditions, if the duration of monopoly power is sufficiently *long* (but finite).

Deneckere-Judd (DJ) in a Nutshell

n_t: (Measure of) competitive varieties per labor inherited

Two Parameters: θ & δ

$\theta \equiv \left(1-\frac{1}{\sigma}\right)^{1-\sigma} \in (1,e)$, increasing in σ (the EoS)

Market share of a competitive variety
relative to a monopolistic variety

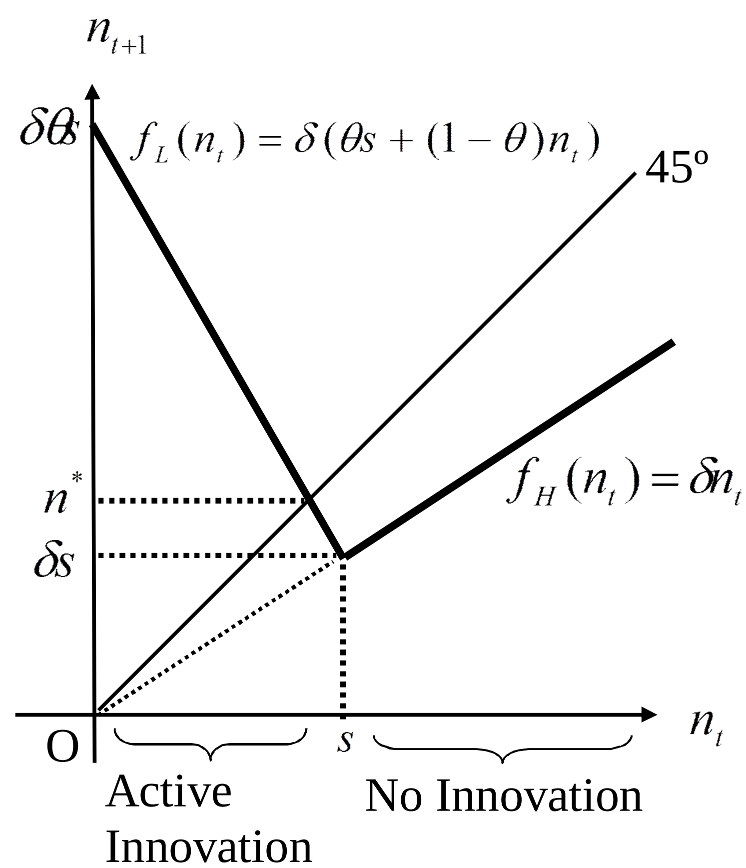
$\theta - 1 > 0$ measures the delayed impact of innovations.

$\delta \in (0,1)$, Survival rate of competitive varieties
due to *obsolescence* (or exogenous labor growth)

- Stable 2-cycle if $\delta^2(\theta-1) < 1 < \delta(\theta-1)$

Enough of innovations from one period ago
(but not from two periods ago) survive to discourage
current innovations.

- Chaotic attractor for $\delta^2(\theta-1) > 1$



Helpman & Krugman (1985; Ch.10):

Trade in horizontally differentiated (Dixit-Stiglitz) goods with *iceberg trade costs* between two structurally identical countries; only their sizes may be different.

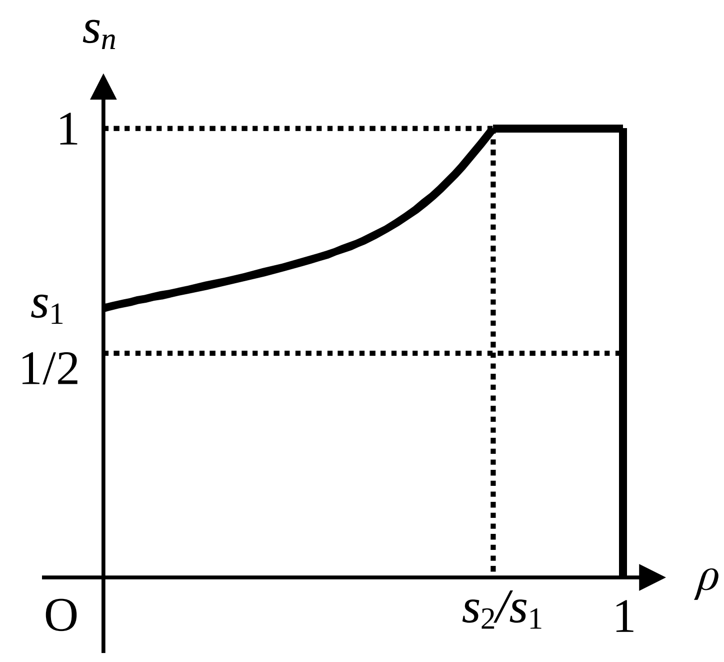
- **In autarky**, the number of firms based in each country is proportional to its size.
- **As trade costs fall,**
  - o Horizontally differentiated goods produced in the two countries mutually penetrate each other's home markets (Two-way flows of goods).
  - o Firm distribution becomes increasingly skewed toward the larger country *(Home Market Effect and its Magnification)*

Two Parameters: s_1 & ρ

$s_1 = 1 - s_2 \in [1/2, 1)$:
Bigger country's share in market size

$\rho \equiv (\tau)^{1-\sigma} \in [0,1)$: Degree of Globalization:
inversely related to the iceberg cost, $1 < \tau \le \infty$

s_n: Bigger country's share in firm distribution



Our Main Results: By combining DJ (1992) and HK (1985):

- **2D state space**: (Measures of competitive varieties in the two countries)

- Unique equilibrium path obtained by iterating a **2D-PWS, noninvertible map** with ***four parameters***: θ & δ & s_1 & ρ

  One unit of competitive varieties = θ (> 1) units of monopolistic varieties
  One unit of foreign varieties = ρ (< 1) unit of domestic varieties

- **In autarky,** the dynamics of the two are **decoupled.** Depending on the initial condition, they may converge to either synchronized or asynchronized fluctuations.

- **As trade costs fall,** they become more **synchronized** in the sense that:
  - *Basin of attraction* for synchronized 2-cycles ***expands.***
  - *Basin of attraction* for asynchronized 2-cycles ***shrinks*** and ***disappears***
  - Synchronization occurs ***faster*** with *unequal* country sizes
  - The larger country sets the tempo of global innovation cycles, with the smaller country adjusting its rhythm.

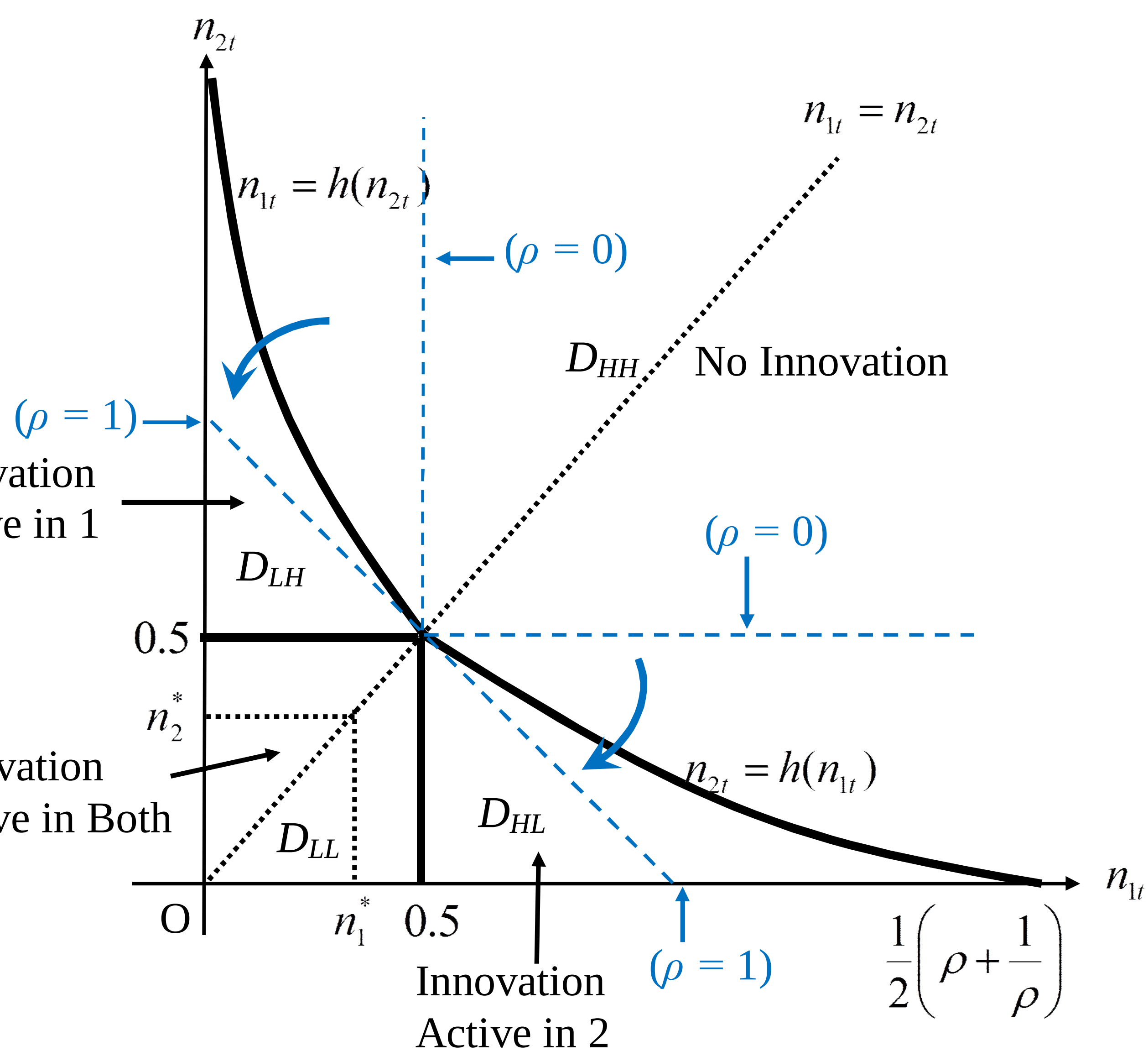
2D Dynamical System; $n_{t+1} = F(n_t)$ with $n_t \equiv (n_{1t}, n_{2t}) \in R_+^2$**;**
$(0 < \delta < 1;\ 1 < \theta < e;\ 0 \le \rho < 1;\ 1/2 \le s_1 < 1)$

$$
(17)\quad
\begin{aligned}
&n_{1t+1} = \delta\left(\theta s_1(\rho) + (1-\theta)n_{1t}\right) && \text{if } n_t \in D_{LL} \equiv \left\{(n_1, n_2) \in R_+^2 \,\middle|\, n_j \le s_j(\rho)\right\}\\
&n_{2t+1} = \delta\left(\theta s_2(\rho) + (1-\theta)n_{2t}\right) && \\
&n_{1t+1} = \delta n_{1t} && \text{if } n_t \in D_{HH} \equiv \left\{(n_1, n_2) \in R_+^2 \,\middle|\, n_j \ge h_j(n_k)\right\}\\
&n_{2t+1} = \delta n_{2t} && \\
&n_{1t+1} = \delta n_{1t} && \text{if } n_t \in D_{HL} \equiv \left\{(n_1, n_2) \in R_+^2 \,\middle|\, n_1 \ge s_1(\rho); n_2 \le h_2(n_1)\right\}\\
&n_{2t+1} = \delta\left(\theta h_2(n_{1t}) + (1-\theta)n_{2t}\right) && \\
&n_{1t+1} = \delta\left(\theta h_1(n_{2t}) + (1-\theta)n_{1t}\right) && \text{if } n_t \in D_{LH} \equiv \left\{(n_1, n_2) \in R_+^2 \,\middle|\, n_1 \le h_1(n_2); n_2 \ge s_2(\rho)\right\}\\
&n_{2t+1} = \delta n_{2t} &&
\end{aligned}
$$

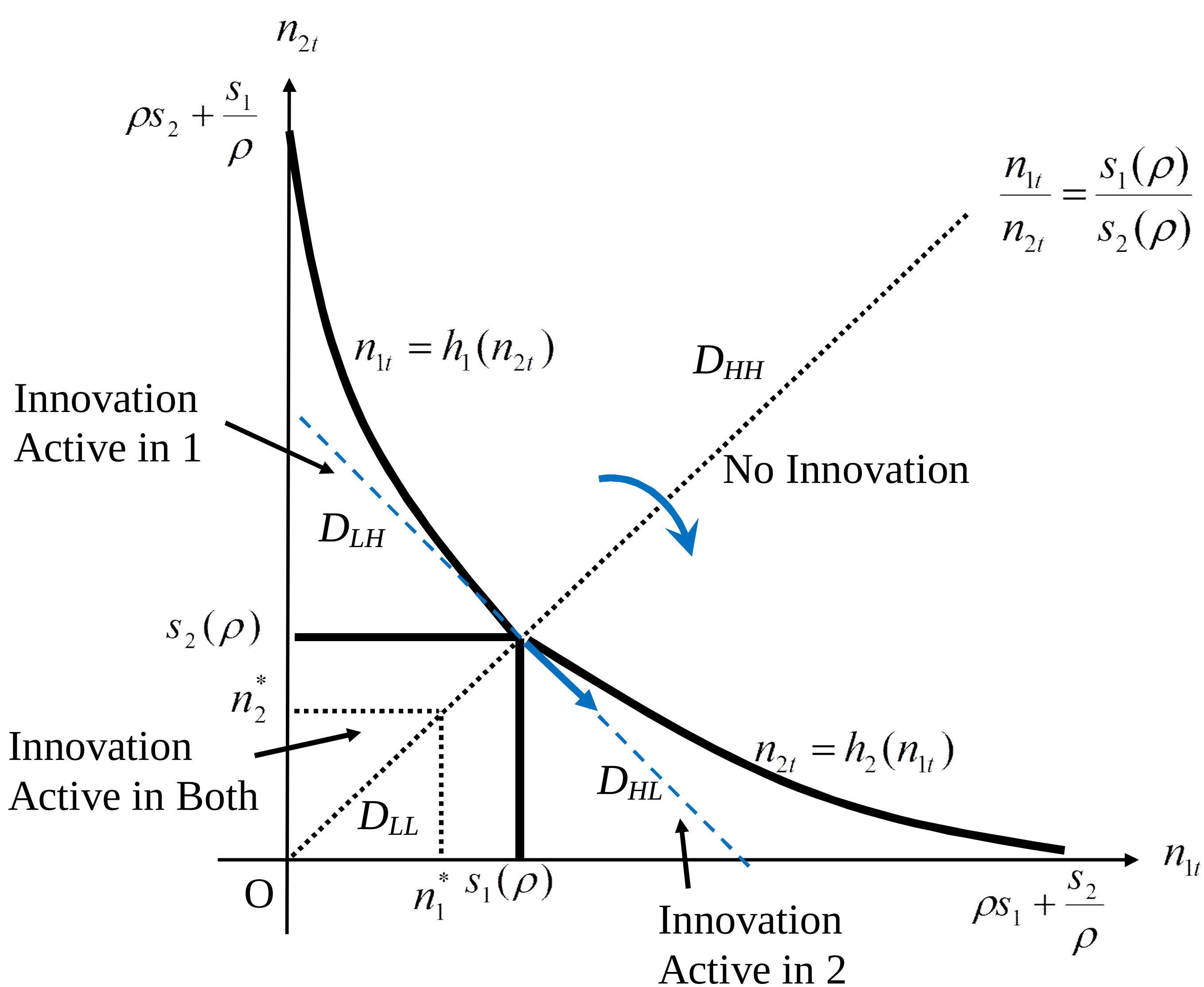
where $s_1(\rho) = 1 - s_2(\rho) = \min\left\{\dfrac{s_1 - \rho s_2}{1-\rho}, 1\right\}$, $\quad 0.5 \le s_1 = 1 - s_2 < 1;$

$h_j(n_k) > 0$ defined implicitly by $\dfrac{s_j}{h_j(n_k) + \rho n_k} + \dfrac{s_k}{h_j(n_k) + n_k/\rho} = 1.$

State Space & Four Domains for the Symmetric Case: $0 < \rho < s_2 / s_1 = 1$



State Space & Four Domains for the Asymmetric Case: $0 < \rho < s_2 / s_1 < 1$

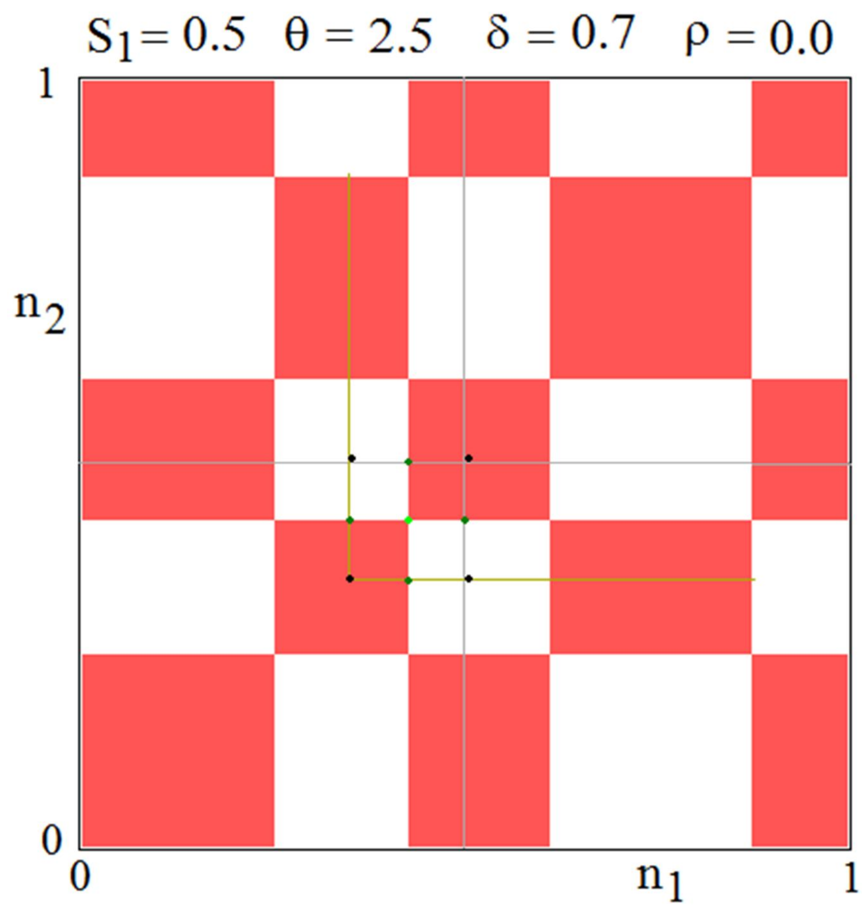


Synchronized vs. Asynchronized 2-Cycles in Autarky: $\rho = 0;\ \delta(\theta-1) > 1 > \delta^2(\theta-1)$,

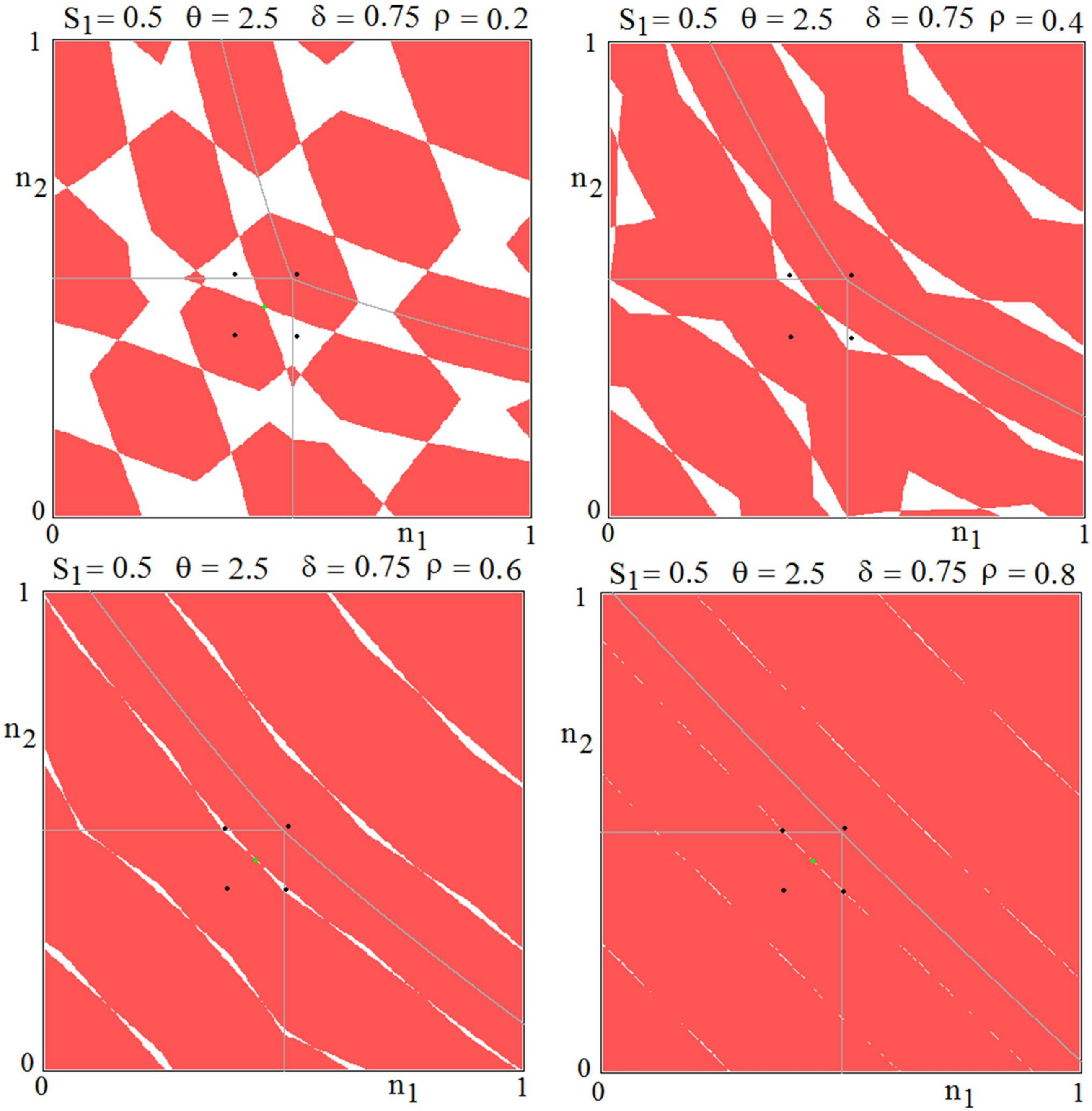
As a 2D-map, this system has

- **An unstable steady state;** (n_1^*, n_2^*)
- **A pair of stable 2-cycles**
  - **Synchronized;** $(n_{1L}^*, n_{2L}^*) \leftrightarrow (n_{1H}^*, n_{2H}^*)$, *Basin of Attraction* in red.
  - **Asynchronized;** $(n_{1L}^*, n_{2H}^*) \leftrightarrow (n_{1H}^*, n_{2L}^*)$, *Basin of Attraction* in white
- **A pair of saddle 2-cycles:**

$(n_{1L}^*, n_2^*) \leftrightarrow (n_{1H}^*, n_2^*)$; $(n_1^*, n_{2H}^*) \leftrightarrow (n_1^*, n_{2L}^*)$



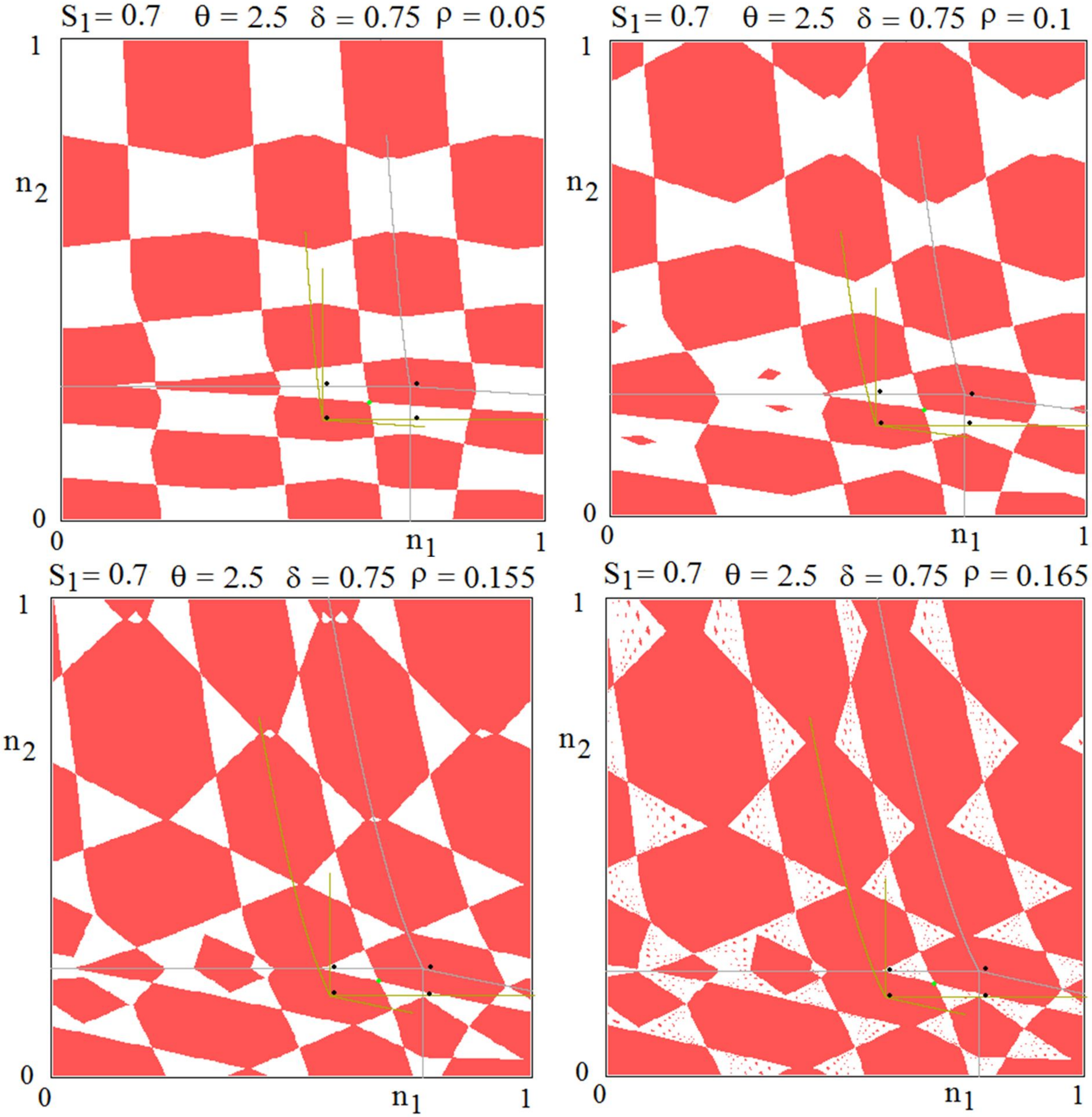
Symmetric Synchronized & Asynchronized 2-Cycles: $s_1 = 0.5$; $\theta = 2.5$; $\delta = 0.75$



Red (Sync. 2-cycle) becomes dominant.

Sym. Async. 2-cycle becomes
a node at ρ = .817867,
a saddle at ρ = .833323.

Asymmetric Synchronized & Asynchronized 2-Cycles $s_1 = 0.7$, $\theta = 2.5$; $\delta = 0.75$



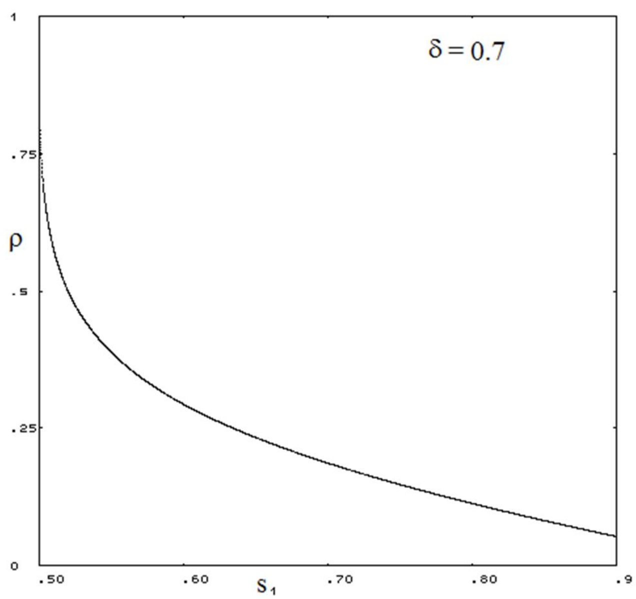
By ρ = .165, infinitely many Red islands appear inside White.

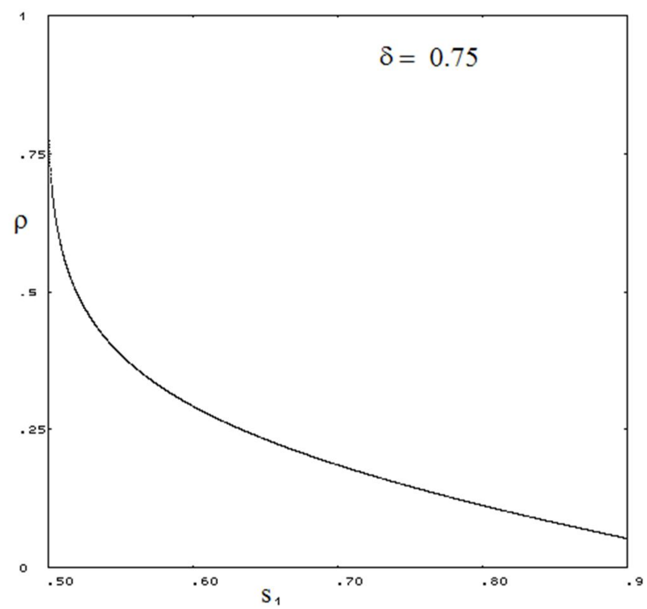
By ρ =.19, the stable asynchronized 2-cycle collides with its basin boundary and disappears, leaving **the Synchronized 2-cycle as the unique attractor**.

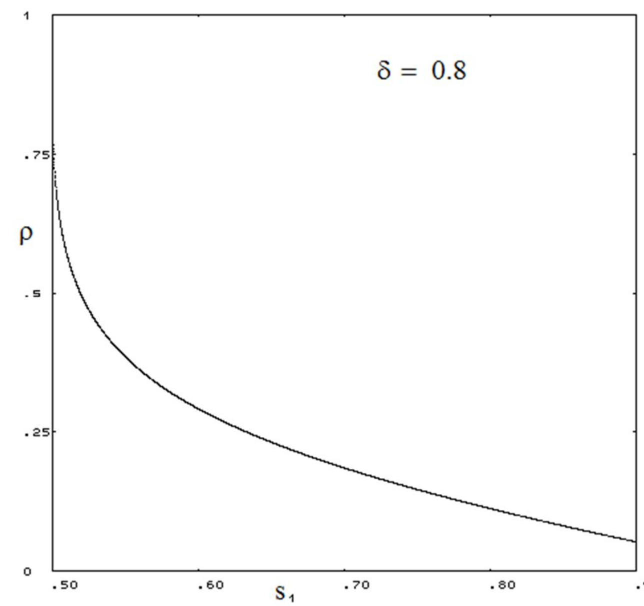
A Smaller Reduction in Trade Costs Synchronizes Innovation Cycles with Greater Country Size Asymmetry

Critical value of ρ_c at which the stable asynchronized 2-cycle disappears (a function of s_1)

- It declines very rapidly as s_1 increases from 0.5.
- It hardly changes with δ**.**

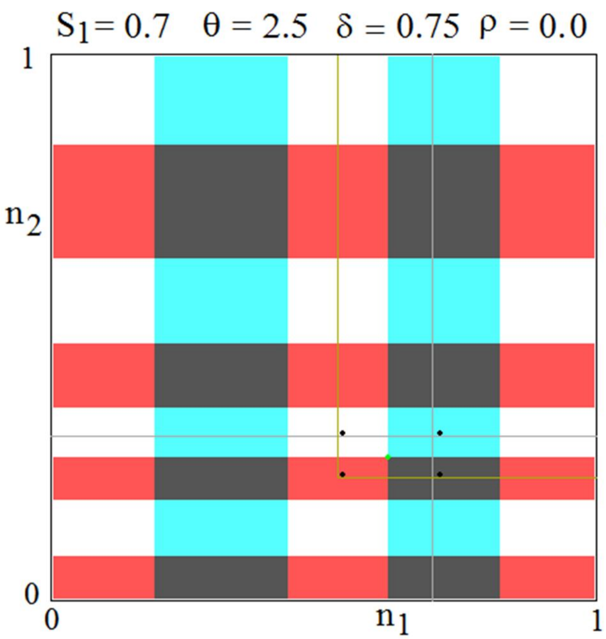


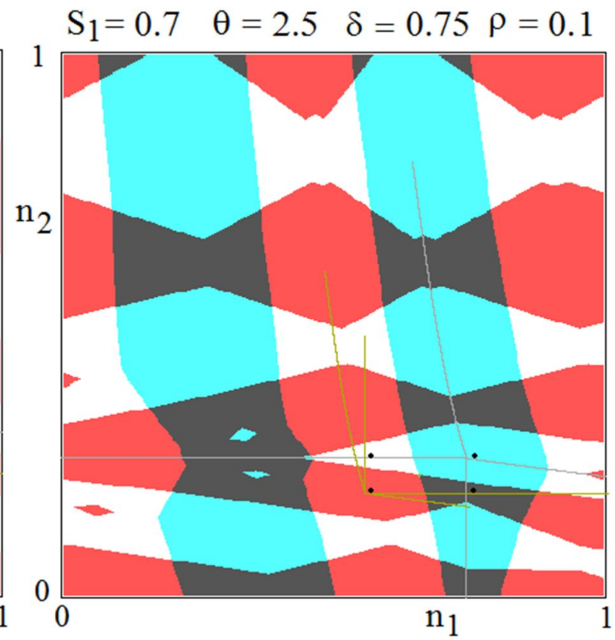


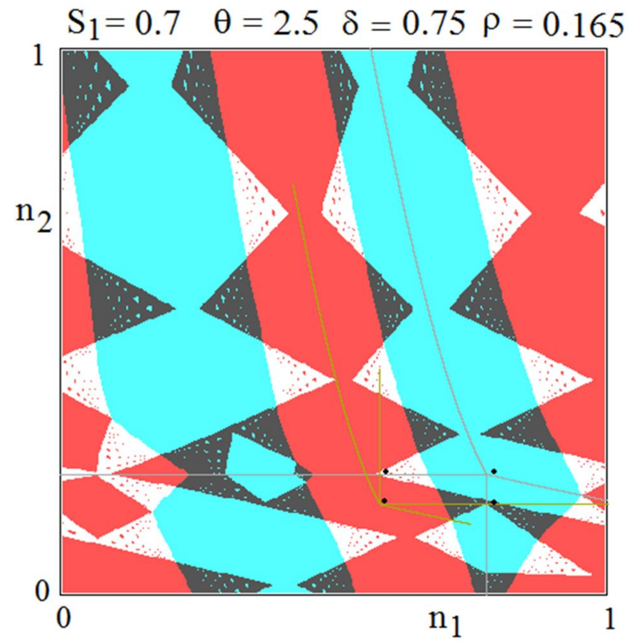


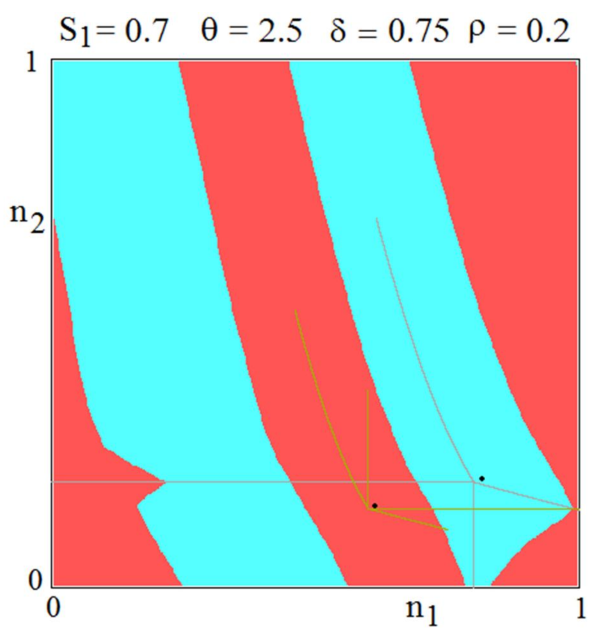
The Larger Country Dictates the Tempo of Global Innovation Cycles

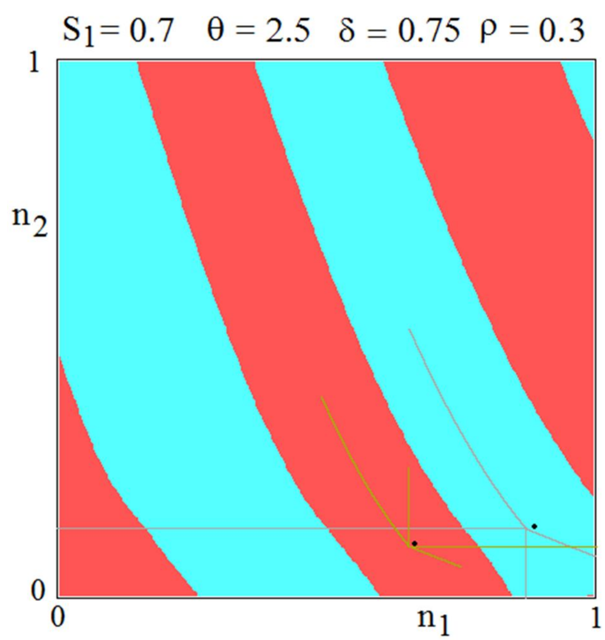
Four Basins of Attraction ($s_1 = 0.7$, $\theta = 2.5$, $\delta = 0.75$)











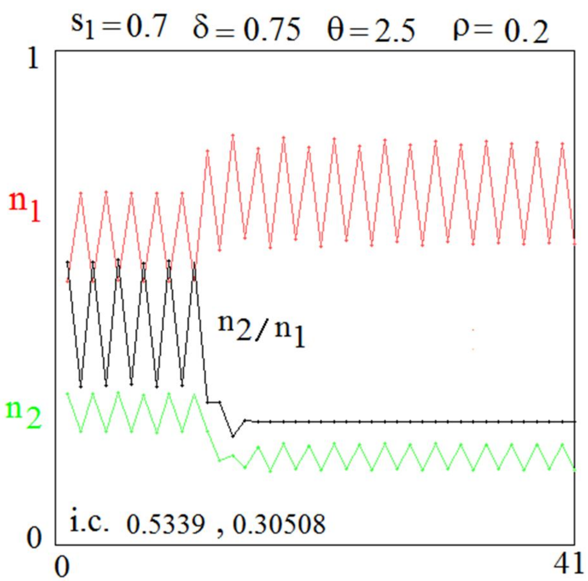
As ρ rises,
Red invades White, and
Azure invades Gray, and
vertical slips of Red and
Azure emerge.

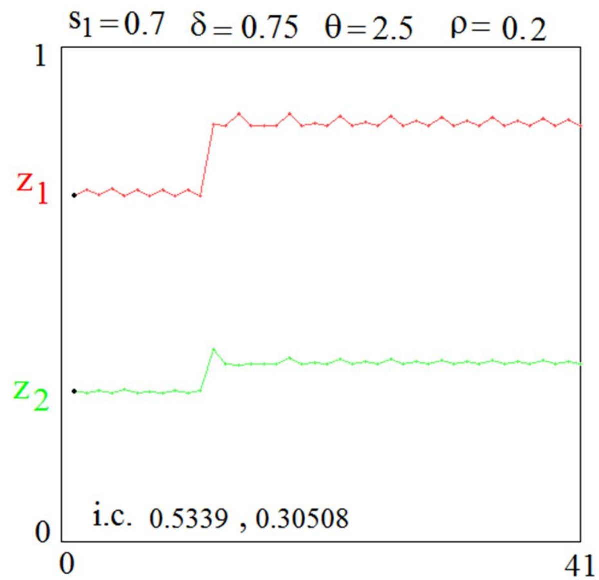
Three Effects of Globalization:

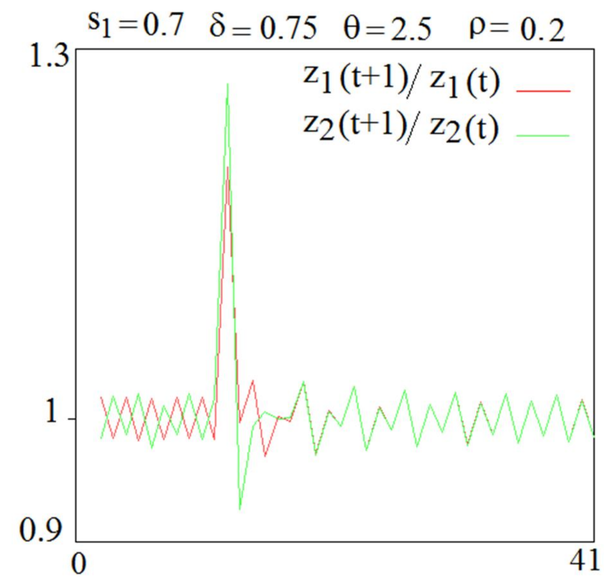
Home Market Effect

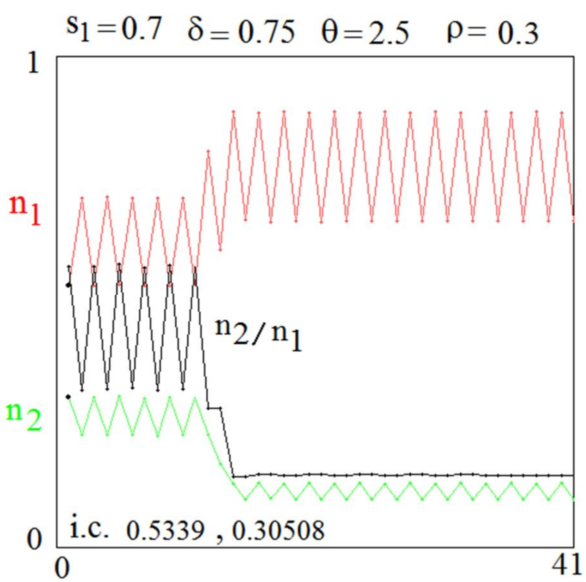
Productivity Gains

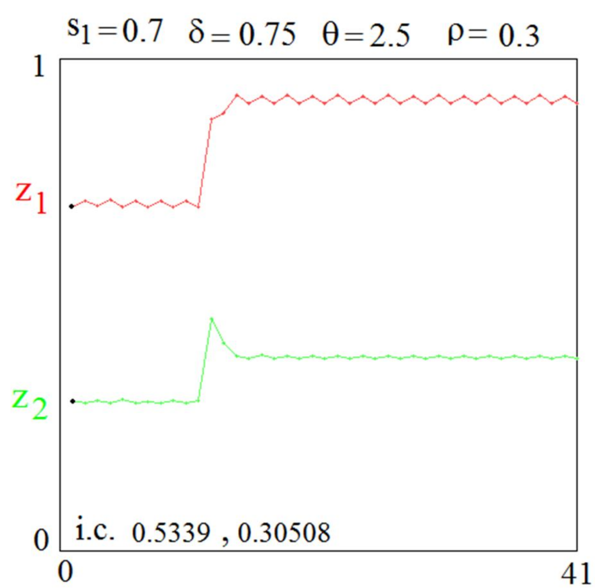
Synchronization

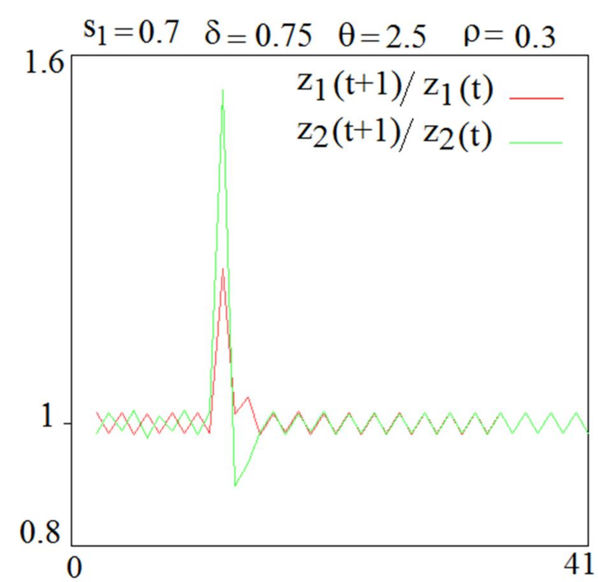












Related Work

***Intra-Industry Trade and Home Market Effects*:** Too numerous to cite:

Trade and Innovation-Driven Endogenous Growth: focus on Balanced Growth Path
Grossman & Helpman (1991); Rivera-Batiz & Romer (1991); Acemoglu & Zilibotti (2001); Acemoglu (2008; Ch.19); Acemoglu, Gancia & Zilibotti (2013) and many others

Endogenous Innovation Cycles: In addition to Judd (1985); Deneckere & Judd (1992), Shleifer (1986); Jovanovic & Rob (1990), Gale (1996), Evans, Honkaponja, and Romer (1998), Matsuyama (1999, 2001), Wälde (2002, 2005), Francois & Lloyd-Ellis (2003, 2008, 2009), Barmoulle & Saint-Paul (2010), Benhabib (2014), etc.

Matsuyama (1999, 2001) embed the DJ mechanism into an endogenous growth model with capital accumulation; See also Gardini, Sushko, and Naimzada (2008)

- Two engines of growth, capital accumulation and innovation, move *asynchronously*.
- They alternate, because fluctuations are driven by innovation and capital accumulation responds to it.

Globalization as a Coupling of Two Games of Strategic Complementarities:
Matsuyama, Kiyotaki, and Matsui (1993);
The bigger country's currency emerges as a Vehicle Currency of World Trade

Synchronization of Coupled Oscillators:

Natural Science: A Major Topic. Thousands of applications: Just to name a few,

- The Moon's rotation and revolution
- London Millennium Bridge

Economics: None? To the best of our knowledge, this is

- First 2-country, dynamic GE model of endogenous fluctuations
- One of the only two dynamic GE, whose equilibrium path is characterized by a system, which may be viewed as a coupling of two systems with self-sustained oscillations.

A companion piece, MSG, "Interdependent Innovation Cycles"

- o Two-sector, closed economy
- o Each sector produces a Dixit-Stiglitz composite, as in DJ
- o CES preferences over the two composites
- o Fluctuations in the two sectors are decoupled for Cobb-Douglas
- o Synchronized (asynchronized) if EoS increases (decreases) from one.

Organization

Section 1: **Introduction and Preview of Main Results**

Section 2: **Model**

Section 3: **Autarky and Decoupled Innovation Dynamics:**
- **3.1.** **1D-Analysis of the Skew Tent Map: *Revisiting Deneckere-Judd***
- **3.2.** **2D View of Autarky: Synchronized vs Asynchronized Cycles**

Section 4: **Globalization and Interdependent Innovation Dynamics: 2D Analysis**
- **4.1.** **A Brief Look at Steady State: *Reinterpreting Helpman-Krugman***
- **4.2.** **Synchronization Effects of Globalization: Symmetric Cases**
- **4.3.** **Synchronization Effects of Globalization: Asymmetric Cases**
- **4.4.** **Three Effects of Globalization: Some Trajectories**

Section 5: **Concluding Remarks**

Postscript: **Synchronization of Chaotic Fluctuations of Innovation: A First Look (Not in the paper)**

Model

A Two-Country Model of Endogenous Fluctuations of Innovation

Time: $t \in \{0,1,2,\ldots\}$

Two Countries: *j* or *k* = 1 or 2

Nontradeable Factor (Labor): supplied inelastically by $L_j = s_j L$ at w_{jt} $(s_1 + s_2 = 1)$
Two countries may differ only in L_j → 1/2 ≤ s_1 < 1, w. l.o.g.

Two Tradeable Intermediate Inputs Sectors: both produced with labor

- **Homogeneous Input (numeraire);** competitive, CRS (1-to-1), zero trade cost
  $w_{jt} \geq 1$; = 1 whenever country *j* produces the homogeneous input.

- **Differentiated Inputs:** $\Omega_t = \sum_j \Omega_{jt} = \sum_j (\Omega^c_{jt} + \Omega^m_{jt})$,

$\Omega_{jt} = \Omega^c_{jt} + \Omega^m_{jt}$: Set of differentiated inputs produced in j in equilibrium in period t.
Ω^c_{jt}: (Predetermined) set of competitively produced inputs in j.
Ω^m_{jt}: (Endogenous) set of new inputs introduced and produced in j; sold exclusively by their innovators for just one period.

Competitive Nontradeable Final (Consumption) Goods Sector:

$$Y_{kt} = C_{kt} = \left(\frac{X_{kt}^{o}}{1-\alpha}\right)^{1-\alpha}\left(\frac{X_{kt}}{\alpha}\right)^{\alpha}, \text{ where } \left[X_{kt}\right]^{1-\frac{1}{\sigma}} = \int_{\Omega_t}\left[x_{kt}(\nu)\right]^{1-\frac{1}{\sigma}} d\nu \quad (0 < \alpha < 1;\ \sigma > 1)$$

$$\Rightarrow x_{kt}(\nu) = \left(\frac{p_{kt}(\nu)}{P_{kt}}\right)^{-\sigma} X_{kt} = \left(\frac{p_{kt}(\nu)}{P_{kt}}\right)^{-\sigma} \frac{\alpha w_{kt} L_k}{P_{kt}} \quad (k = 1, 2);$$

where

$p_{kt}(\nu)$: price of variety ν in k;

P_{kt}: Price index for differentiated inputs in k, $\left[P_{kt}\right]^{1-\sigma} = \int_{\Omega_t}\left[p_{kt}(\nu)\right]^{1-\sigma} d\nu$

(Iceberg) trade costs: One unit of $\nu \in \Omega_{jt}$ in k needs $\tau_{jk} \geq 1$ units of $\nu \in \Omega_{jt}$

$$p_{kt}(\nu) = p_{jt}(\nu)\tau_{jk} \geq p_{jt}(\nu) \quad \text{for } \nu \in \Omega_{jt}$$

We assume $\tau_{11} = \tau_{22} = 1;\ \tau_{12} = \tau_{21} = \tau > 1$

Demand Curve for $\nu \in \Omega_{jt}$ **:** $D_t(\nu) = \sum_k \tau_{jk} x_{kt}(\nu) = \alpha A_{jt}(p_{jt}(\nu))^{-\sigma}$ where $A_{jt} \equiv \sum_k \frac{\rho_{jk} w_{kt} L_k}{(P_{kt})^{1-\sigma}}$

with $\rho_{11} = \rho_{22} = 1;\ \ \rho_{12} = \rho_{21} = \rho \equiv (\tau)^{1-\sigma} \in [0,1)$: **Degree of Globalization**

Market share (in value) of a foreign variety is $\rho < 1$ fraction of what it would be in the absence of trade cost.

Differentiated Inputs Pricing: ψ units of labor for producing one unit of each variety

$$p_{jt}(v) = \psi w_{jt} \equiv p^c_{jt}; \qquad y_{jt}(v) \equiv y^c_{jt} \qquad \text{for } v \in \Omega^c_{jt} \subset \Omega_{jt}$$

$$p_{jt}(v) = \frac{\psi w_{jt}}{1-1/\sigma} \equiv p^m_{jt}; \quad y_{jt}(v) \equiv y^m_{jt} \qquad \text{for } v \in \Omega^m_{jt} = \Omega_{jt} - \Omega^c_{jt}$$

$$\frac{p^c_{jt}}{p^m_{jt}} = 1 - \frac{1}{\sigma} < 1; \qquad \frac{y^c_{jt}}{y^m_{jt}} = \left(1-\frac{1}{\sigma}\right)^{-\sigma} > 1; \quad \frac{p^c_{jt}}{p^m_{jt}}\frac{y^c_{jt}}{y^m_{jt}} = \left(1-\frac{1}{\sigma}\right)^{1-\sigma} \equiv \theta \in (1, e).$$

θ: the market share (in value) of a competitive variety relative to a monopolistic variety; it is increasing in σ, but varies little with σ. Numerically, we set $\theta = 2.5$.

σ	$\to 1$	2	4	5	6	8	10	14	20	$\to \infty$
θ	$\to 1$	2	2.37	2.44	2.49	2.55	2.58	2.62	2.65	$\to e = 2.71828\ldots$

Price Indices: Let N^c_{jt} (N^m_{jt}) be the measure of Ω^c_{jt} (Ω^m_{jt})

$$\left(P_{kt}/\psi\right)^{1-\sigma} = M_{kt}\left(w_{kt}\right)^{1-\sigma} + \rho M_{jt}\left(w_{jt}\right)^{1-\sigma}, \text{ where } M_{jt} \equiv N^c_{jt} + \frac{N^m_{jt}}{\theta}.$$

$$= N^c_{kt} + N^m_{kt}/\theta + \rho\left(N^c_{jt} + N^m_{jt}/\theta\right), \text{ for } w_{1t} = w_{2t} = 1.$$

- One unit of competitive varieties = $\theta > 1$ units of monopolistic varieties
- One unit of foreign varieties = $\rho < 1$ units of domestic varieties

Introduction of New Varieties: Innovation cost per unit of variety, f

Complementarity Slackness Condition:

- *Non-Negativity Constraint on Innovation:* $N_{jt}^{m} = \theta(M_{jt} - N_{jt}^{c}) \geq 0;$
- *Free Entry (Zero Profit) Condition:* $\pi_{jt}^{m} \equiv p_{jt}^{m} y_{jt}^{m} - w_{jt}\left(\psi y_{jt}^{m} + f\right) \leq 0;$

$$\Rightarrow \frac{1}{\sigma}\left[\frac{\alpha L_j}{\theta(M_{jt} + \rho M_{kt})} + \frac{\alpha L_k}{\theta(M_{jt} + M_{kt}/\rho)}\right] \leq f \quad \text{for } w_{1t} = w_{2t} = 1$$

with $\theta(M_{jt} + \rho M_{kt}) = \theta N_{jt}^{c} + N_{jt}^{m} + \rho(\theta N_{kt}^{c} + N_{kt}^{m})$: ***Effective competition at home.***

$\theta(M_{jt} + M_{kt}/\rho) = \theta N_{jt}^{c} + N_{jt}^{m} + (\theta N_{kt}^{c} + N_{kt}^{m})/\rho$: ***Effective competition abroad.***

Obsolescence of Old Varieties:

$N_{jt+1}^{c} = \delta\left(N_{jt}^{c} + N_{jt}^{m}\right) = \delta\left(\theta M_{jt} + (1-\theta)N_{jt}^{c}\right)$, with $\delta \in (0,1)$, **the Survival Rate**

Additionally, labor supply in each country may grow at a common, constant factor, $L_{jt} = L_{j0}(G)^{t}$, with $G > 1$. Then, set $\delta/G < 1$, instead of $\delta < 1$.

Normalize as: $n_{jt} \equiv \dfrac{\theta\sigma f N^{c}_{jt}}{\alpha(L_1+L_2)}$; $i_{jt} \equiv \dfrac{\theta\sigma f N^{m}_{jt}}{\alpha(L_1+L_2)}$; $m_{jt} \equiv \dfrac{\theta\sigma f M_{jt}}{\alpha(L_1+L_2)} = n_{jt} + \dfrac{i_{jt}}{\theta}$

Pair of Complementarity Slackness Conditions:

(15) $$i_{1t} = \theta(m_{1t} - n_{1t}) \geq 0\ ; \qquad m_{1t} \geq h_1(m_{2t})$$
$$i_{2t} = \theta(m_{2t} - n_{2t}) \geq 0; \qquad m_{2t} \geq h_2(m_{1t})$$

where $h_j(m_k) > 0$ defined by $\dfrac{s_j}{h_j(m_k) + \rho m_k} + \dfrac{s_k}{h_j(m_k) + m_k/\rho} = 1$ with

$0 \leq \boldsymbol{\rho} < 1$ and $1 > \mathbf{s_1} \geq s_2 = 1 - \mathbf{s_1}$

Dynamics of Measures of Competitive Varieties:

(16) $$n_{1t+1} = \delta\big(\theta m_{1t} + (1-\theta)n_{1t}\big) \qquad (0 < \boldsymbol{\delta} < 1;\ 1 < \boldsymbol{\theta} < e)$$
$$n_{2t+1} = \delta\big(\theta m_{2t} + (1-\theta)n_{2t}\big)$$

By solving (15) for $m_t = (m_{1t}, m_{2t}) \in R^2_+$ as a function of $n_t = (n_{1t}, n_{2t}) \in R^2_+$ and then inserting it to (16),

2D Dynamical System; $n_{t+1} = F(n_t)$ with $n_t \equiv (n_{1t}, n_{2t}) \in R_+^2$**;**
(0 < δ < 1; 1 < θ < e; 0 ≤ ρ < 1; 1/2 ≤ s_1 < 1)

$$n_{1t+1} = \delta(\theta s_1(\rho) + (1-\theta)n_{1t}) \qquad \text{if } n_t \in D_{LL} \equiv \{(n_1, n_2) \in R_+^2 \mid n_j \le s_j(\rho)\}$$
$$n_{2t+1} = \delta(\theta s_2(\rho) + (1-\theta)n_{2t})$$

$$n_{1t+1} = \delta n_{1t} \qquad \text{if } n_t \in D_{HH} \equiv \{(n_1, n_2) \in R_+^2 \mid n_j \ge h_j(n_k)\}$$
$$n_{2t+1} = \delta n_{2t}$$

(17)

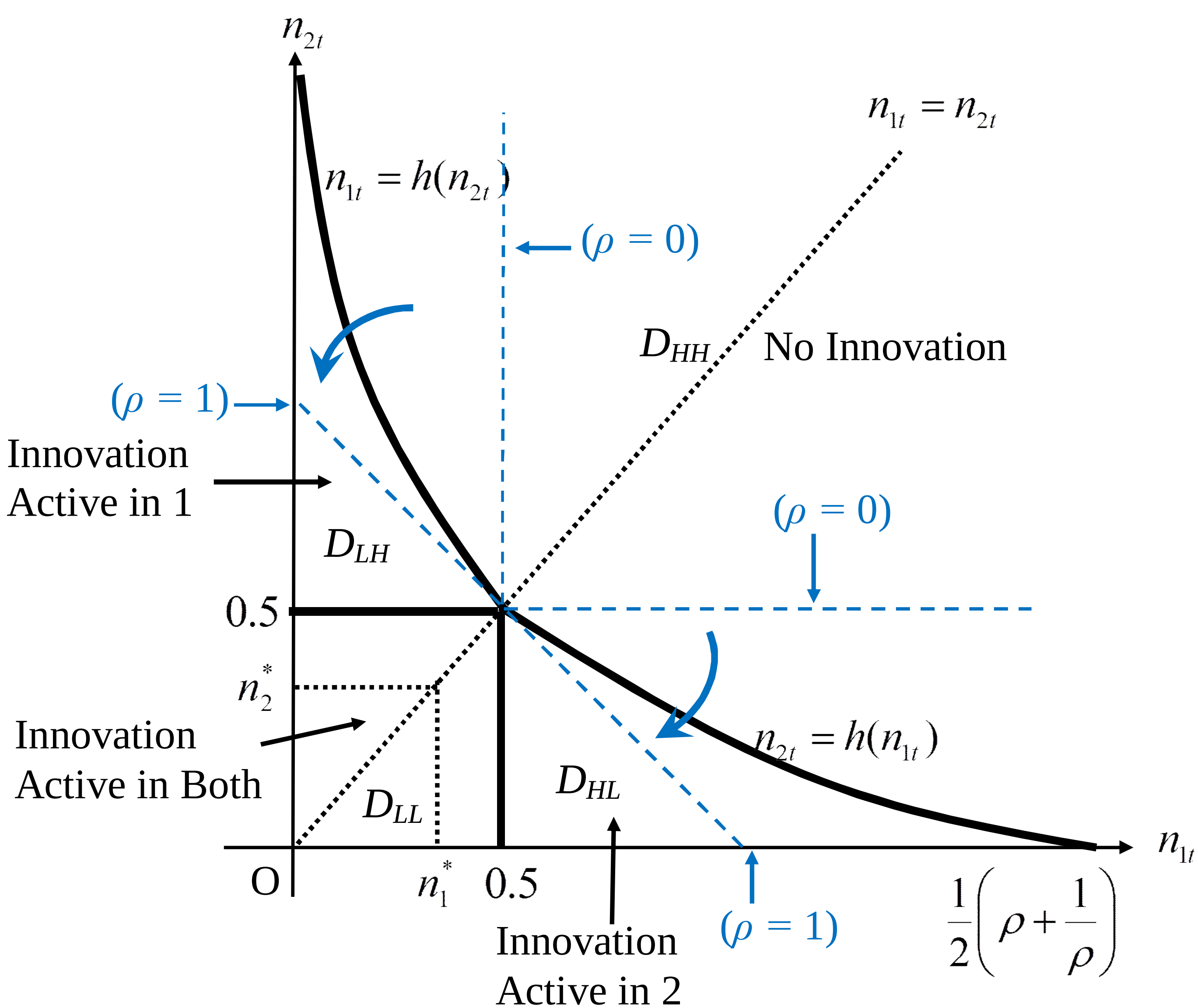
$$n_{1t+1} = \delta n_{1t} \qquad \text{if } n_t \in D_{HL} \equiv \{(n_1, n_2) \in R_+^2 \mid n_1 \ge s_1(\rho); n_2 \le h_2(n_1)\}$$
$$n_{2t+1} = \delta(\theta h_2(n_{1t}) + (1-\theta)n_{2t})$$

$$n_{1t+1} = \delta(\theta h_1(n_{2t}) + (1-\theta)n_{1t}) \qquad \text{if } n_t \in D_{LH} \equiv \{(n_1, n_2) \in R_+^2 \mid n_1 \le h_1(n_2); n_2 \ge s_2(\rho)\}$$
$$n_{2t+1} = \delta n_{2t}$$

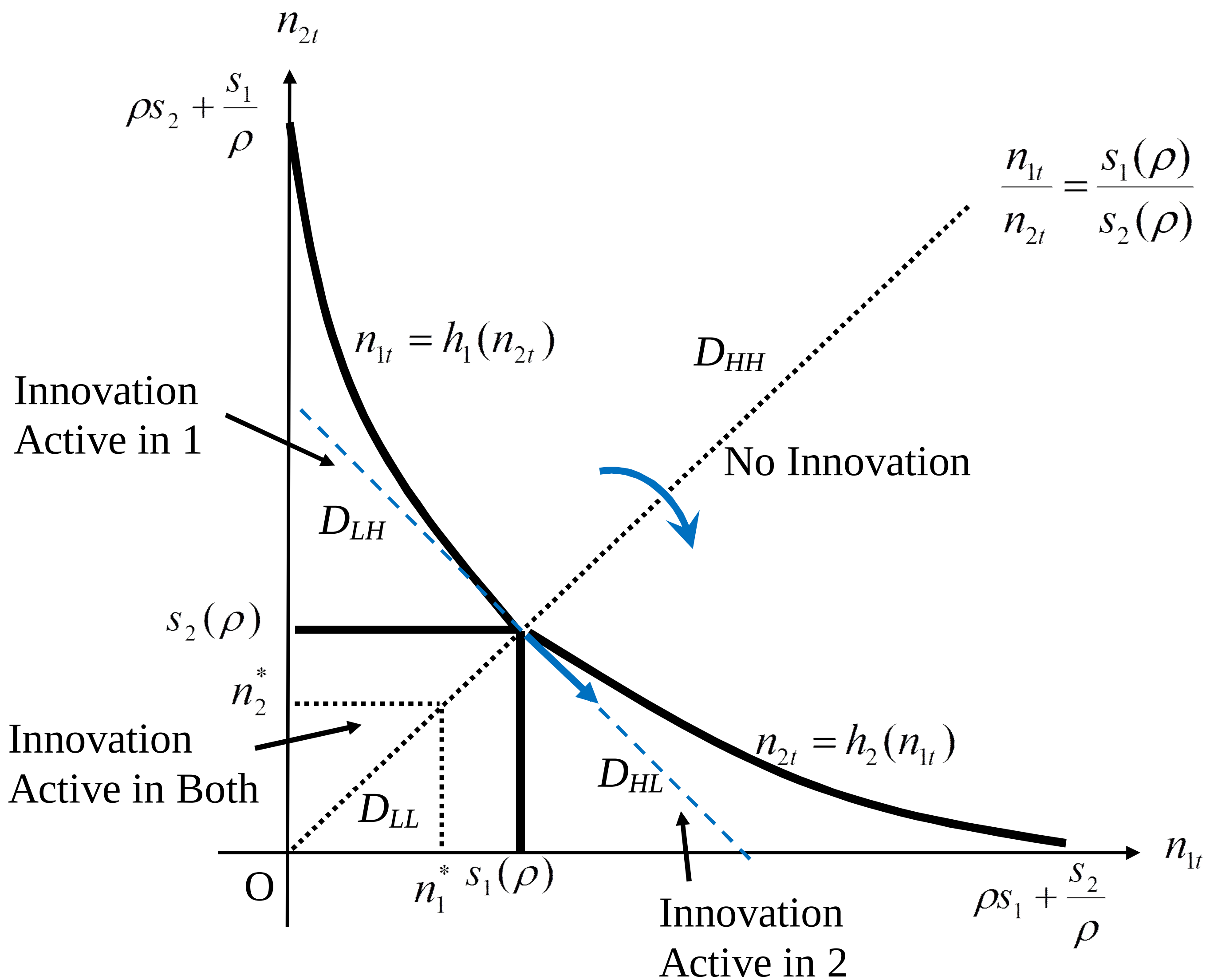
where $s_1(\rho) = 1 - s_2(\rho) = \min\left\{\frac{s_1 - \rho s_2}{1-\rho}, 1\right\}$, $\quad 0.5 \le s_1 = 1 - s_2 < 1;$

$h_j(n_k) > 0$ defined implicitly by $\dfrac{s_j}{h_j(n_k) + \rho n_k} + \dfrac{s_k}{h_j(n_k) + n_k/\rho} = 1.$

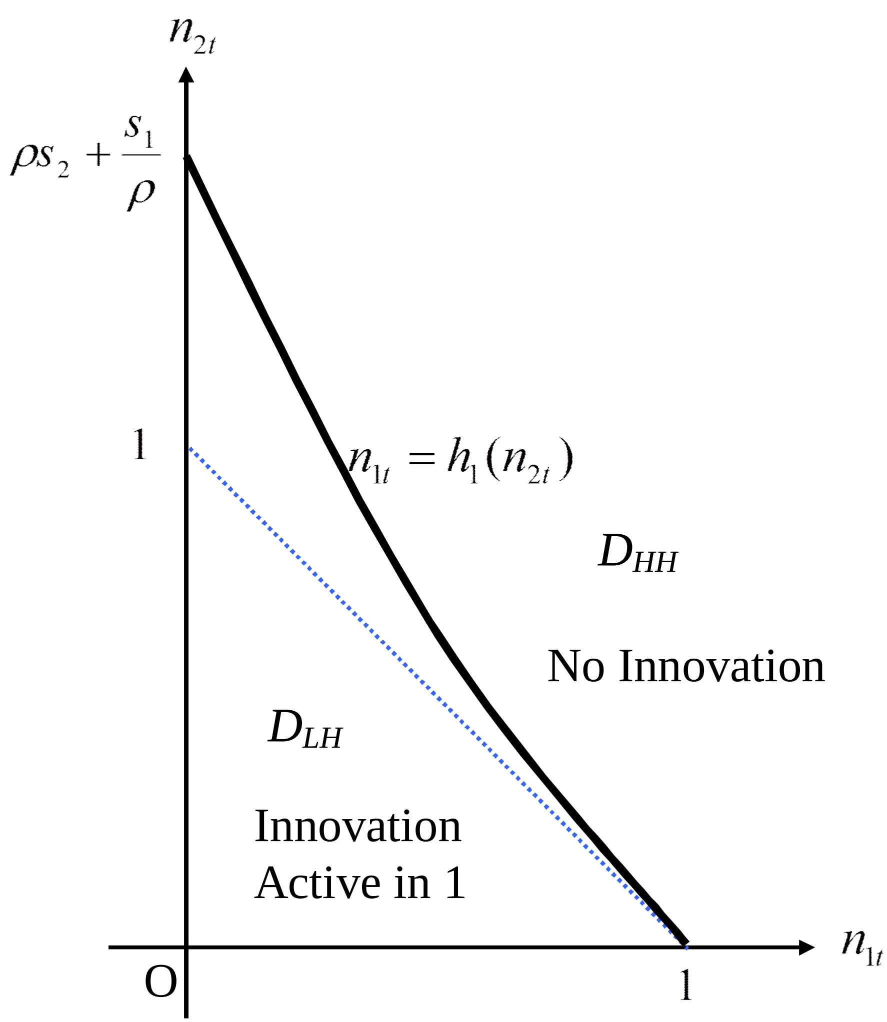
State Space & Four Domains for the Symmetric Case: $0 < \rho < s_2 / s_1 = 1$



State Space & Four Domains for the Asymmetric Case: $0 < \rho < s_2 / s_1 < 1$



State Space & Two Domains for the Asymmetric Case: $0 < s_2 / s_1 < \rho < 1$



Innovations: $i_{jt} = \theta(m_{jt} - n_{jt}) = (n_{jt+1} - \delta n_{jt})/\delta$.

(18)

$$i_{1t} = \theta(s_1(\rho) - n_{1t}); \qquad i_{2t} = \theta(s_2(\rho) - n_{2t}) \qquad \text{for } n_t \in D_{LL},$$

$$i_{1t} = 0 \qquad i_{2t} = 0 \qquad \text{for } n_t \in D_{HH},$$

$$i_{1t} = 0 \qquad i_{2t} = \theta(h_2(n_{1t}) - n_{2t}) \qquad \text{for } n_t \in D_{HL},$$

$$i_{1t} = \theta(h_1(n_{2t}) - n_{1t}) \qquad i_{2t} = 0 \qquad \text{for } n_t \in D_{LH}.$$

TFPs: $Z_{kt} \equiv \dfrac{Y_{kt}}{L_k}$, with $\log(Z_{kt}) = \omega_0 + \omega_1 \log(z_{kt})$

(19)

$$z_{1t} = (1+\rho)s_1 \qquad z_{2t} = (1+\rho)s_2 \qquad \text{for } n_t \in D_{LL}$$

$$z_{1t} = n_{1t} + \rho n_{2t} \qquad z_{2t} = \rho n_{1t} + n_{2t} \qquad \text{for } n_t \in D_{HH}$$

$$z_{1t} = n_{1t} + \rho h_2(n_{1t}) \qquad z_{2t} = \rho n_{1t} + h_2(n_{1t}) \qquad \text{for } n_t \in D_{HL}$$

$$z_{1t} = h_1(n_{2t}) + \rho n_{2t} \qquad z_{2t} = \rho h_1(n_{2t}) + n_{2t} \qquad \text{for } n_t \in D_{LH}$$

Autarky and Decoupled Innovation Dynamics

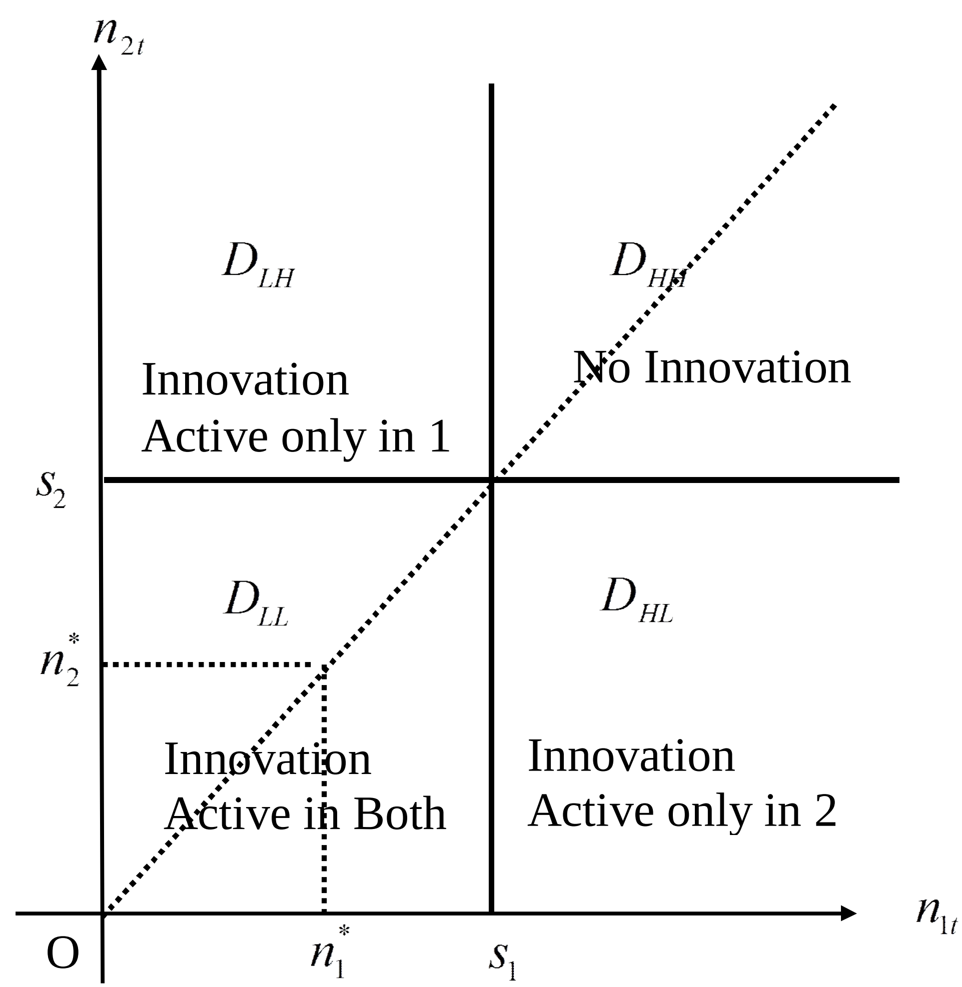
Autarky: For $\rho = 0$, $h_j(m_k) = s_j \Rightarrow m_{jt} = \max\{s_j, n_{jt}\}$.

The 2D system consists of two independent 1D systems:

$$n_{jt+1} = \delta\left(\theta \max\{s_j, n_{jt}\} + (1-\theta) n_{jt}\right).$$

TFP: $z_{jt} = \max\{s_j, n_{jt}\}$

Innovation: $i_{jt} = \theta \max\{s_j - n_{jt}, 0\}$



1D-Analysis of the Skew Tent Map: Revisiting DJ (country index omitted):

$$n_{t+1} = f(n_t) = \begin{cases} f_L(n_t) \equiv \delta(\theta s + (1-\theta)n_t) & \text{for } n_t \le s; \\ f_H(n_t) \equiv \delta n_t & \text{for } n_t \ge s. \end{cases} \qquad (0 < \delta < 1;\ 1 < \theta < e)$$

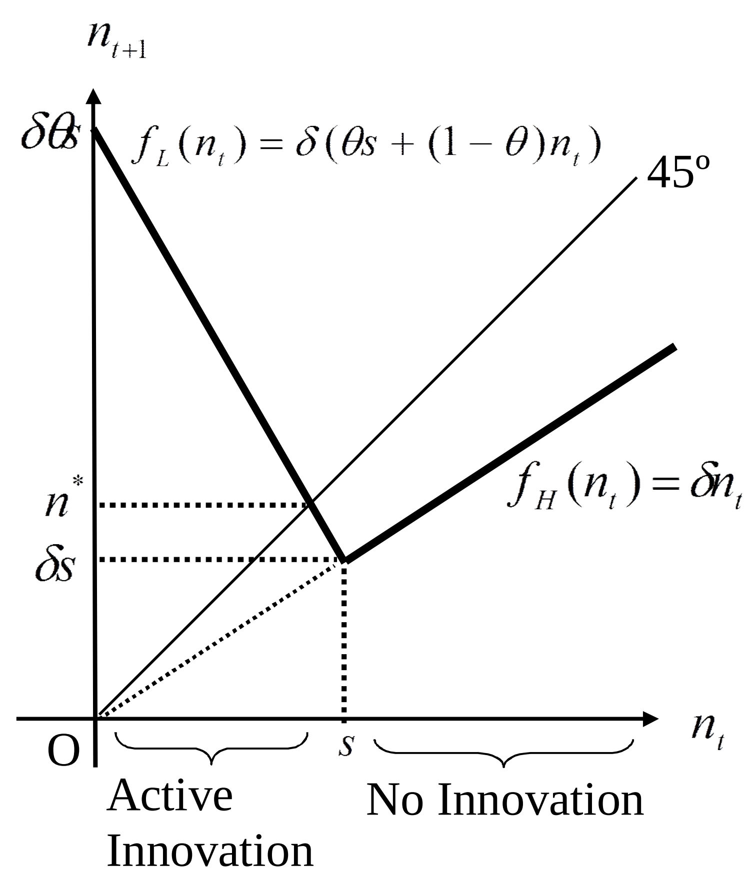
PWL noninvertible map with two branches:

***H*-branch;** upward-sloping, below the 45º line
With too many old goods ($n_t \ge s$), no innovation

- With $\delta < 1$, the map is contracting.

***L*-branch:** downward-sloping, cross the 45º line
W/o many old goods ($n_t < s$), active innovation

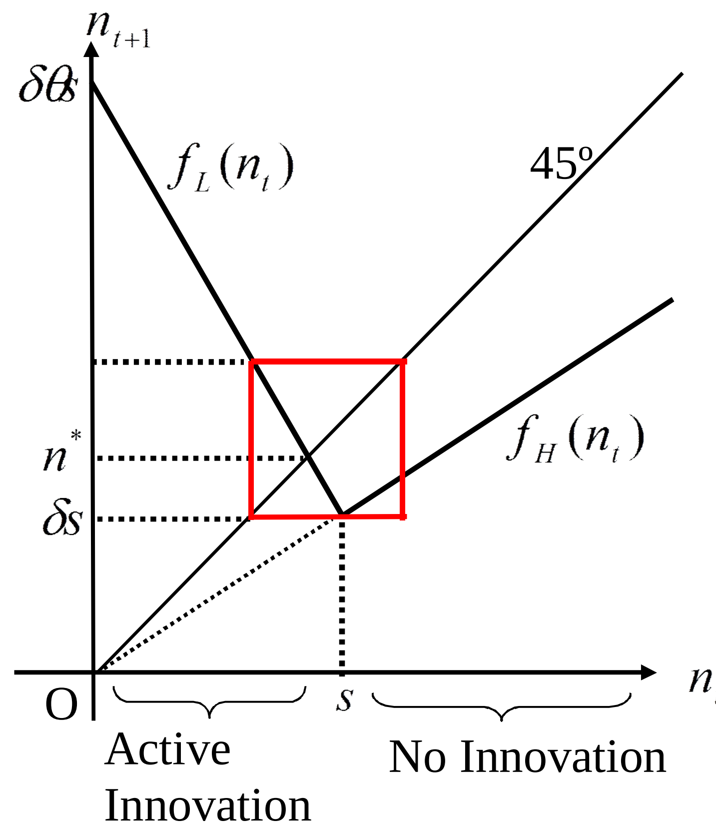
- With $\theta > 1$, downward sloping. (Unit measure of competitive varieties crowd out measure $\theta > 1$ of new varieties, hence the economy is left with fewer varieties in the next period.)
- This effect is stronger with higher σ (hence, θ).



The Skew Tent Map (Continued):

Unique Steady State in the *L*-branch,

$$n^* = \frac{\delta\theta s}{1+(\theta-1)\delta} < s \ ,$$



For $\delta(\theta-1)<1$, it is stable and globally attracting.

For $\delta(\theta-1)>1$

- Steady State is unstable.
- **Absorbing Interval**; $J=[\delta s, f_L(\delta s)]$,

indicated by the Red box.

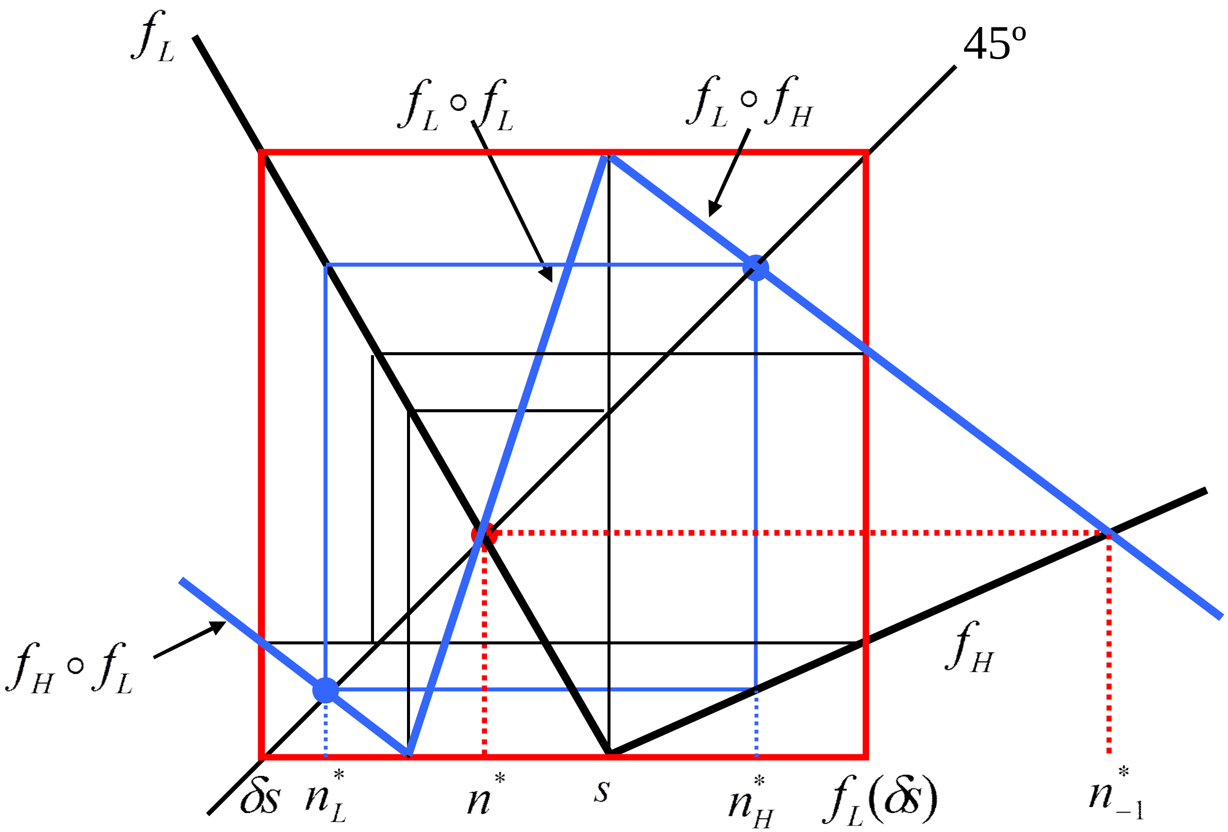
- **Unique 2-cycle,**

$$n_L^* = \frac{\delta^2\theta s}{1+(\theta-1)\delta^2} \leftrightarrow n_H^* = \frac{\delta\theta s}{1+(\theta-1)\delta^2},$$

For $\delta^2(\theta-1)<1$,
the 2-cycle is attracting from almost all initial conditions.

Illustrating the Stable 2-Cycle existing for $\delta^2(\theta-1)<1<\delta(\theta-1)$

Black graph: the map
Red dot: the steady state, unstable with $\delta(\theta-1)>1$; **Red box:** the absorbing interval
Blue graph: the 2nd-iterate of the map; **Blue dots**: the 2-cycle, stable with $\delta^2(\theta-1)<1$

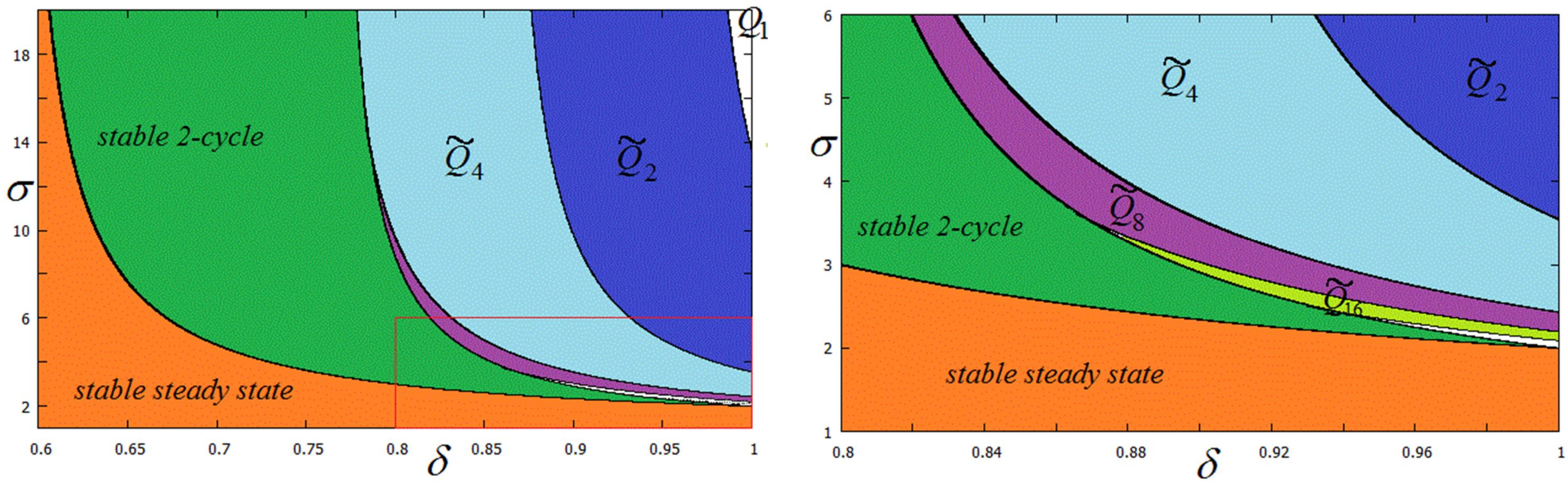


The Skew Tent Map (Continued):

For $\delta^2(\theta-1)>1$, 2-cycle is unstable → ***Robust Chaotic Attractor***

- Most examples of chaos in economics are **not** attracting.
- Most examples of chaotic attractors in economics are **not** robust.
- An **immediate** transition from 2-cycle to chaos

Bifurcation diagram in the (*σ*, *δ*)-plane and its magnification



Endogenous fluctuations with a higher *σ* (more competition across goods) and a higher *δ* (more past innovation survives to crowd out current innovation).

We focus on stable 2-cycle, $\delta^2(\theta-1)<1<\delta(\theta-1)$, or $0.666...<\delta<0.816...$ with $\theta=2.5$.

Synchronized vs. Asynchronized 2-Cycles in Autarky: $\rho = 0;\ \delta(\theta-1) > 1 > \delta^2(\theta-1)$,
Each component 1D-map has

- o **an *unstable* steady state,** $n_j^* = \dfrac{\theta\delta s_j}{1+(\theta-1)\delta}$
- o **a *stable* 2-cycle,** $n_{jL}^* = \dfrac{\delta^2\theta s_j}{1+(\theta-1)\delta^2} \leftrightarrow n_{jH}^* = \dfrac{\delta\theta s_j}{1+(\theta-1)\delta^2}$

As a 2D-map, this system has

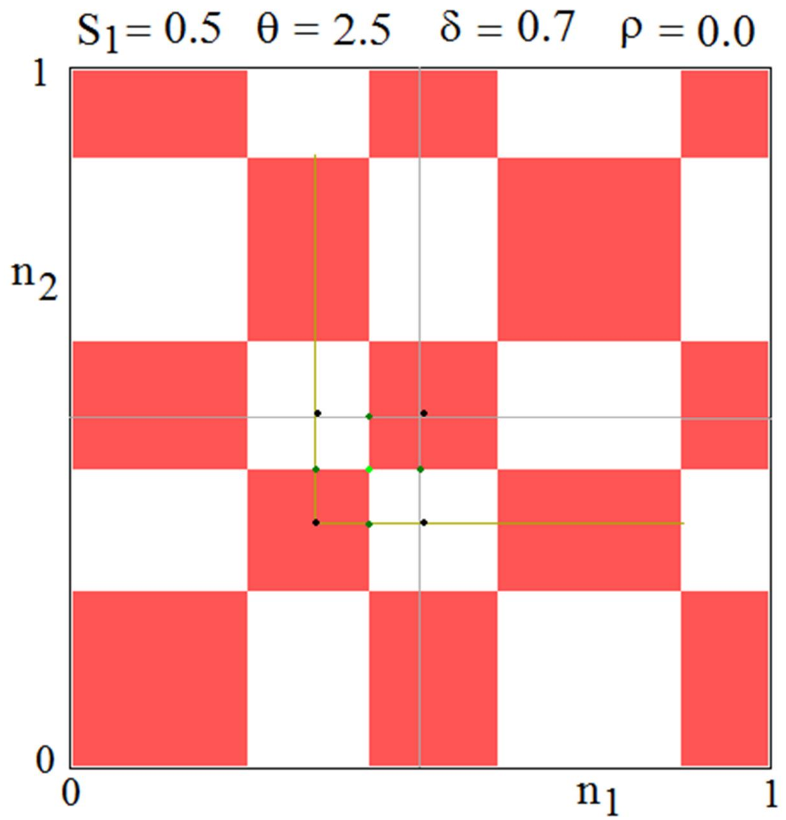
- **An unstable steady state;** (n_1^*, n_2^*) in light green
- **A pair of stable 2-cycles:**
  - o **Synchronized;** $(n_{1L}^*, n_{2L}^*) \leftrightarrow (n_{1H}^*, n_{2H}^*)$,
    Basin of Attraction in red.
  - o **Asynchronized;** $(n_{1L}^*, n_{2H}^*) \leftrightarrow (n_{1H}^*, n_{2L}^*)$,
    Basin of Attraction in white
- **A pair of saddle 2-cycles:**

$(n_{1L}^*, n_2^*) \leftrightarrow (n_{1H}^*, n_2^*)$; $(n_1^*, n_{2H}^*) \leftrightarrow (n_1^*, n_{2L}^*)$ in dark green

Notes:

✓ The two basins are NOT connected.

✓ The closure of the stable sets of the two saddles form the boundaries of the two basins.



Globalization and Interdependent Innovation Dynamics: 2D Analysis

A Brief Look at Unique Steady State: Reinterpreting HK

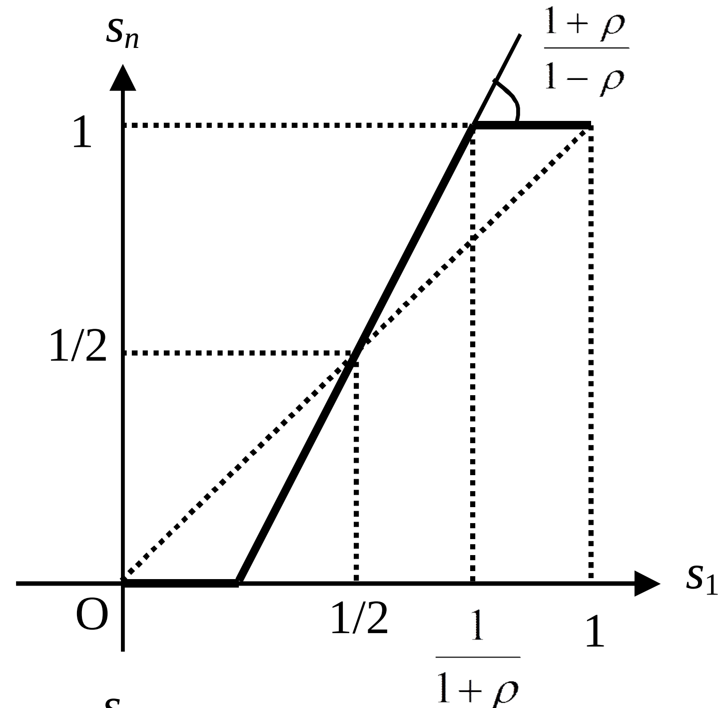
$$\left(n_1^*, n_2^*\right) = \frac{\delta\theta}{1+\delta(\theta-1)}\left(s_1(\rho), s_2(\rho)\right)$$

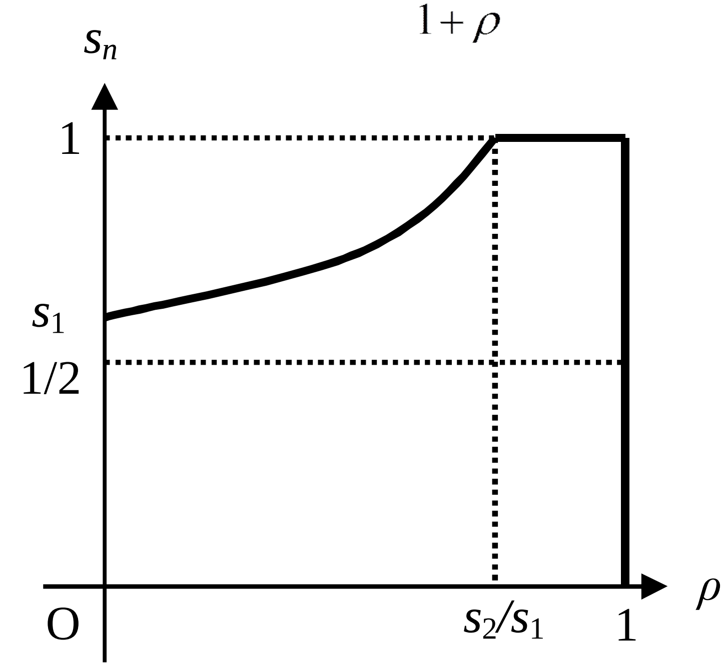
Share of Country 1:

$$s_n \equiv \frac{n_1^*}{n_1^*+n_2^*} = \frac{i_1^*}{i_1^*+i_2^*} = \frac{m_1^*}{m_1^*+m_2^*} = s_1(\rho)$$

- The slope is $(1+\rho)/(1-\rho) > 1$ in the interior. The larger country produces a disproportionately larger share of the differentiated inputs. (Home Market Effect)
- This effect is *larger* if the trade costs are *smaller* (i.e. with a *larger* ρ).
- This expression holds not only in SS but also along *Synchronized Fluctuations*.

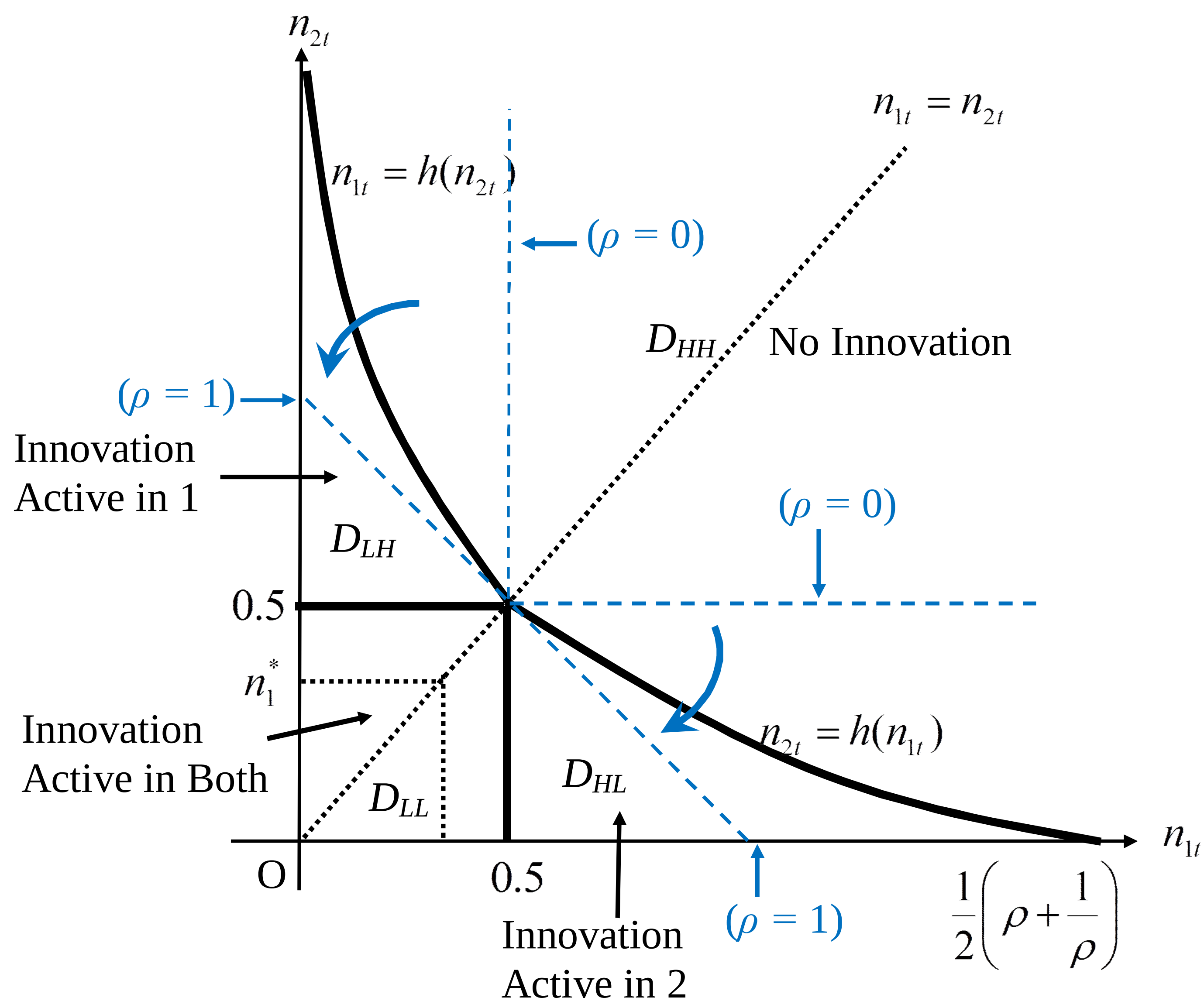
In what follows, we focus on the case where the unique steady state is unstable, $\delta(\theta-1) > 1$.





Synchronization Effects of Globalization: Symmetric Cases

State Space & Four Domains for the Symmetric Case: $0 < \rho < s_2 / s_1 = 1$



Symmetric Interdependent 2-Cycles, $s_1 = 0.5$, $\rho \in (0,1)$, $\delta(\theta-1) > 1 > \delta^2(\theta-1)$:

Each component 1D-map has:

- **an *unstable* steady state,** $n_j^* = n^* \equiv \dfrac{\theta\delta/2}{1+(\theta-1)\delta}$ &
- **a *stable* 2-cycle,** $n_{jL}^* = n_L^* \equiv \dfrac{\delta^2\theta/2}{1+(\theta-1)\delta^2} \leftrightarrow n_{jH}^* = n_H^* \equiv \dfrac{\delta\theta/2}{1+(\theta-1)\delta^2}$.

As a 2D-map,

- **Synchronized 2-cycle,** $(n_L^*, n_L^*) \in D_{LL} \leftrightarrow (n_H^*, n_H^*) \in D_{HH}$, is unaffected by $\rho \in (0,1)$.
- **Symmetric Asynchronized 2-cycle,** $(n_L^a, n_H^a) \in D_{LH} \leftrightarrow (n_H^a, n_L^a) \in D_{HL}$, depends on $\rho \in (0,1)$, no longer equal to $(n_L^*, n_H^*) \leftrightarrow (n_H^*, n_L^*)$. It exists for all $\rho \in (0,1)$; stable for $\rho \in (0, \rho_c)$ and **unstable** for $\rho \in (\rho_c, 1)$.

Furthermore, one could see numerically,

- For, $\rho \in (0, \rho_c)$, a higher ρ expands the basin of attraction for the synchronized 2-cycle, and reduces that for the asynchronized 2-cycle.

(Symmetric) Synchronized and Asynchronized 2-Cycles: A Comparison

Synchronized 2-cycle: $(n_L^*, n_L^*) \in D_{LL} \leftrightarrow (n_H^*, n_H^*) \in D_{HH}$, given by

$$n_H^* = \frac{1}{2\beta} > \frac{1}{2} > \delta n_H^* = n_L^* = \frac{\delta}{2\beta}$$

Asynchronized 2-cycle: $(n_L^a, n_H^a) \in D_{LH} \leftrightarrow (n_H^a, n_L^a) \in D_{HL}$, given by

$$n_H^a = \frac{1}{2}\left(\frac{1}{\beta+\rho} + \frac{\rho}{\rho\beta+1}\right) > \frac{1}{2} > h(n_H^a) = \beta n_H^a > \delta n_H^a = n_L^a = \frac{\delta}{2}\left(\frac{1}{\beta+\rho} + \frac{\rho}{\rho\beta+1}\right),$$

where $\beta \equiv \dfrac{1+\delta^2(\theta-1)}{\delta\theta} \in (\delta, 1)$ and $h(n) > 0$ solves $\dfrac{1}{h(n)+\rho n} + \dfrac{1}{h(n)+n/\rho} = 2$.

By inserting these expressions into (19),

$$z_H^* = \frac{1+\rho}{2\beta} > z_H^a = (1+\rho\beta)n_H^a;\ z_L^* = \frac{1+\rho}{2} > z_L^a = (\rho+\beta)n_H^a;\ \frac{z_H^*}{z_L^*} = \frac{1}{\beta} > \frac{z_H^a}{z_L^a} = \frac{1+\rho\beta}{\rho+\beta} > 1$$

Consumption is higher along the synchronized 2-cycle than the asynchronized 2-cycle, though it is more volatile.

Symmetric Asynchronized 2-Cycle: Local Stability

$$n_H^a = \frac{1}{2}\left(\frac{1}{\beta+\rho} + \frac{\rho}{\rho\beta+1}\right) > \frac{1}{2}; \qquad n_L^a = \frac{\delta}{2}\left(\frac{1}{\beta+\rho} + \frac{\rho}{\rho\beta+1}\right) = \delta n_H^a < \beta n_H^a = h(n_H^a),$$

where $\beta \equiv \dfrac{1+\delta^2(\theta-1)}{\delta\theta} \in (\delta,1)$ and $h(n) > 0$ solves $\dfrac{1}{h(n)+\rho n} + \dfrac{1}{h(n)+n/\rho} = 2$.

Jacobian at this 2-cycle:

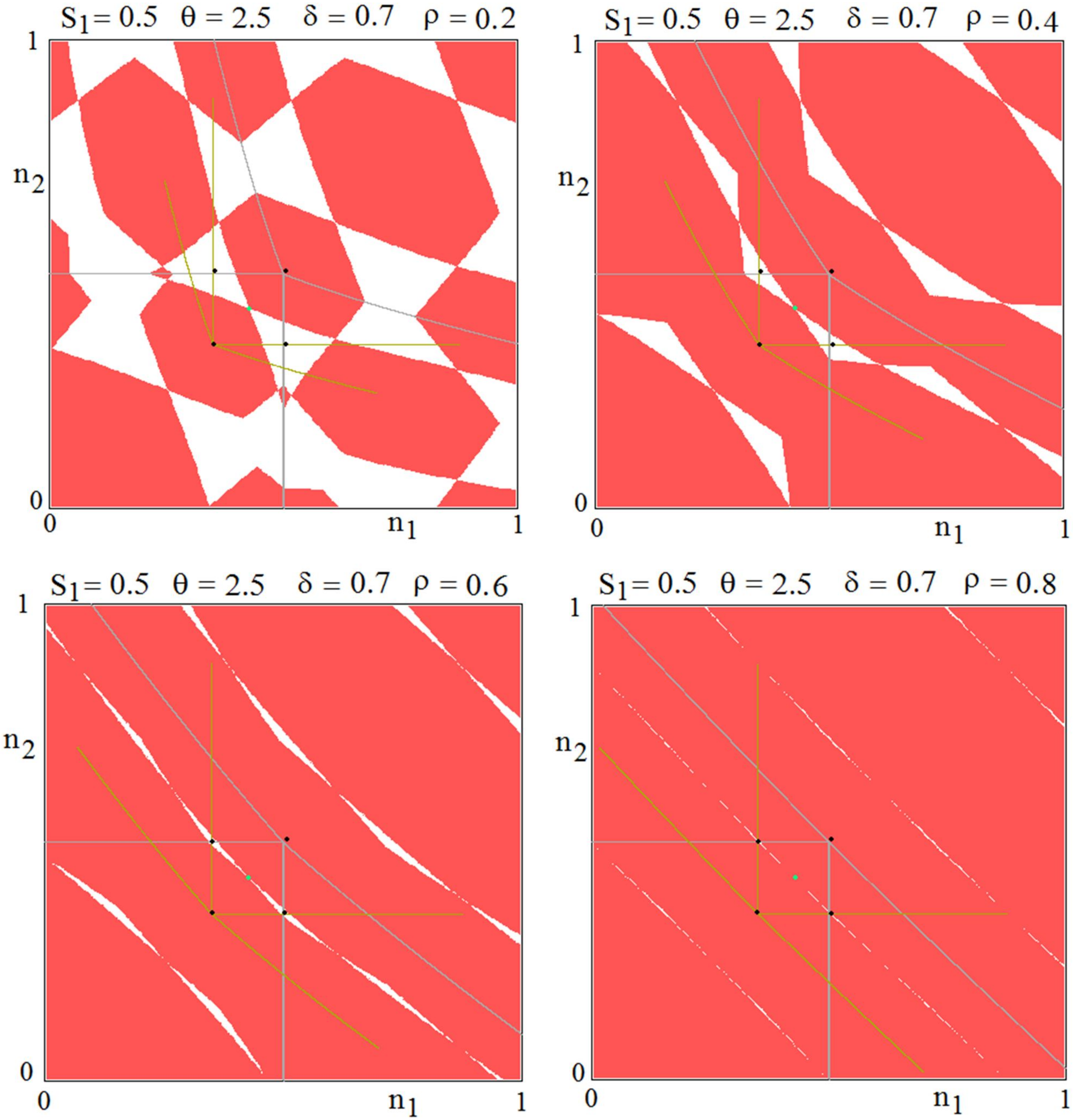
$$J = \delta^2 \begin{bmatrix} 1-\theta+\theta^2\gamma^2 & -(1-\theta)\theta\gamma \\ -\theta\gamma & 1-\theta \end{bmatrix},$$

where $\gamma \equiv -h'(n_H^a) = \dfrac{(\beta+1/\rho)^2\rho + (\beta+\rho)^2/\rho}{(\beta+1/\rho)^2 + (\beta+\rho)^2} \equiv \gamma(\rho)$ is continuous, increasing in ρ with $\gamma(0) = 0$ and $\gamma(1) = 1$.

Two Eigenvalues:

- Complex conjugated if $0 < \gamma(\rho) < 2\sqrt{\theta-1}/\theta$; **a stable focus,** as $Det(J) = \delta^4(1-\theta)^2 < 1$
- Real, both positive, less than one if $2\sqrt{\theta-1}/\theta < \gamma(\rho) < \beta$; **a stable node;**
- Real, both positive, one greater than one if $\beta < \gamma(\rho) < 1$; **an unstable saddle.**

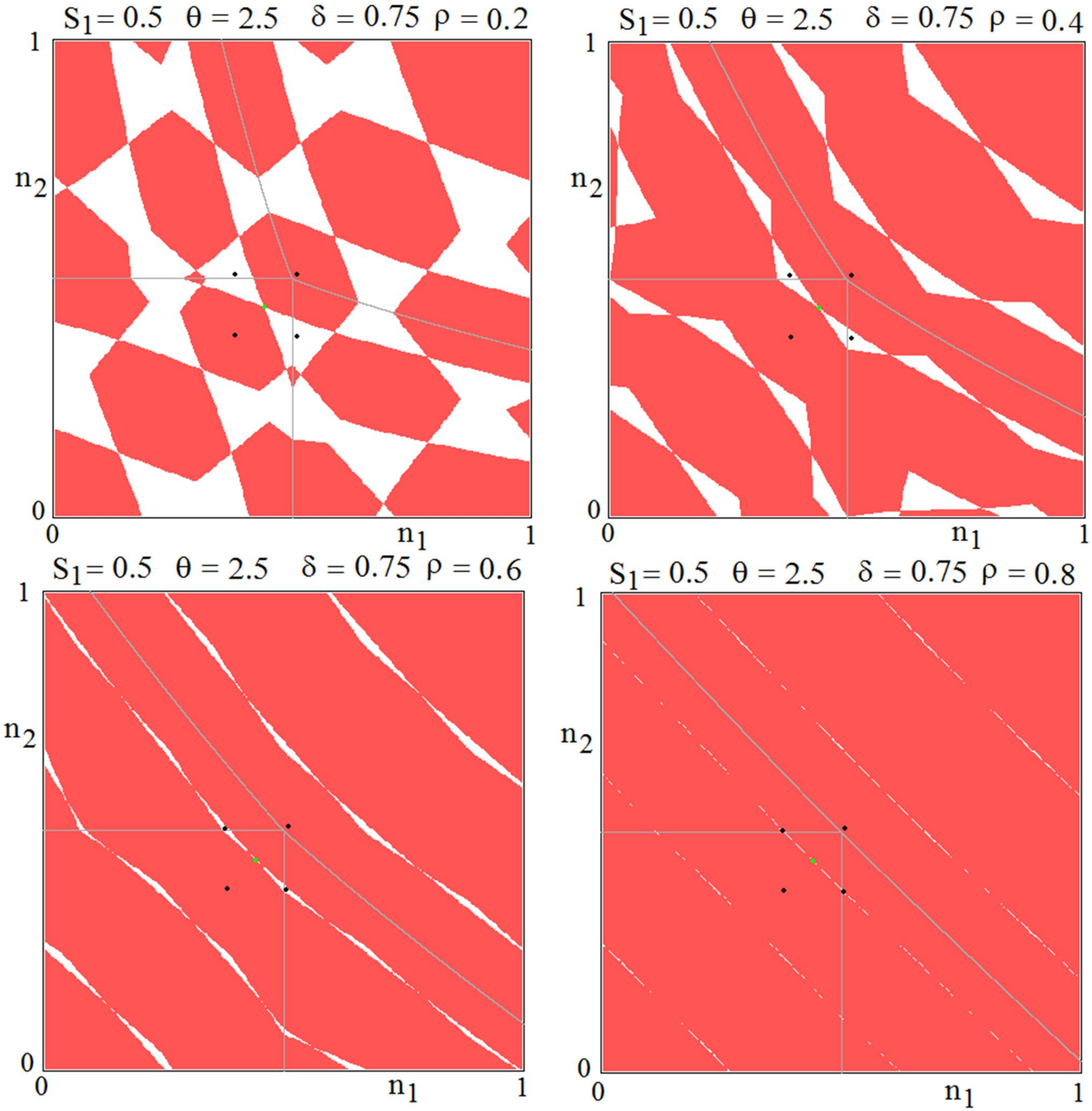
Symmetric Synchronized & Asynchronized 2-Cycles: $s_1 = 0.5$; $\theta = 2.5$; $\delta = 0.7$



Red (Sync 2-cycle) becomes dominant.

Sym. Async 2-cycle becomes
a node, at ρ = .817202;
a saddle at ρ = .877358.

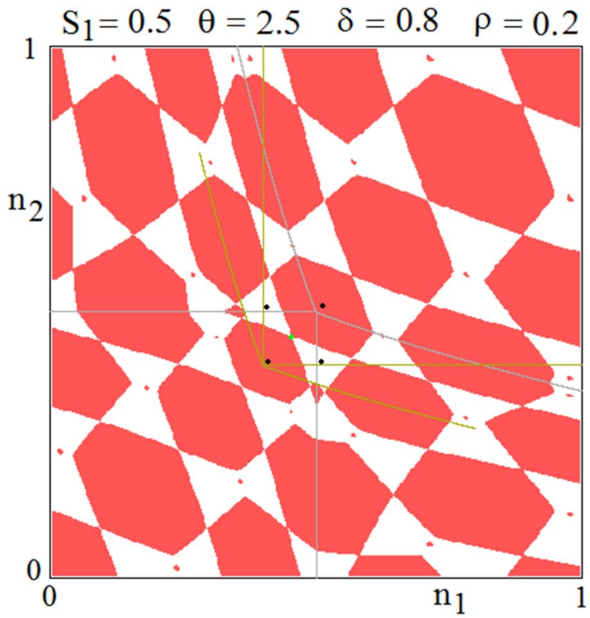
Symmetric Synchronized & Asynchronized 2-Cycles: $s_1 = 0.5$; $\theta = 2.5$; $\delta = 0.75$

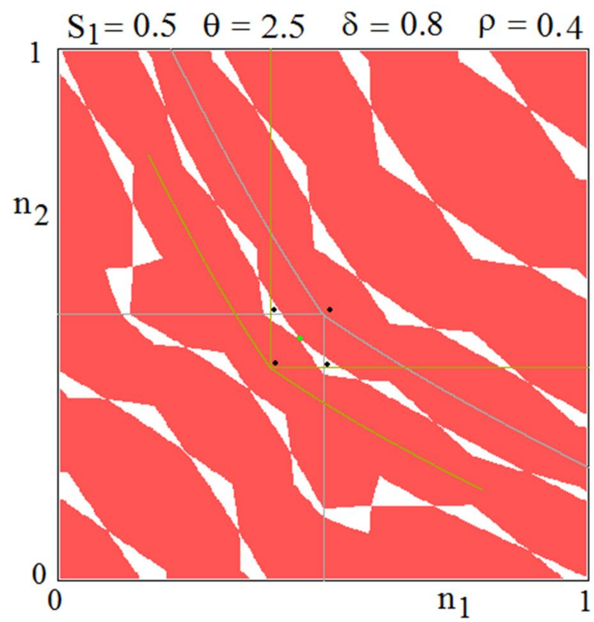


Red (Sync. 2-cycle) becomes dominant.

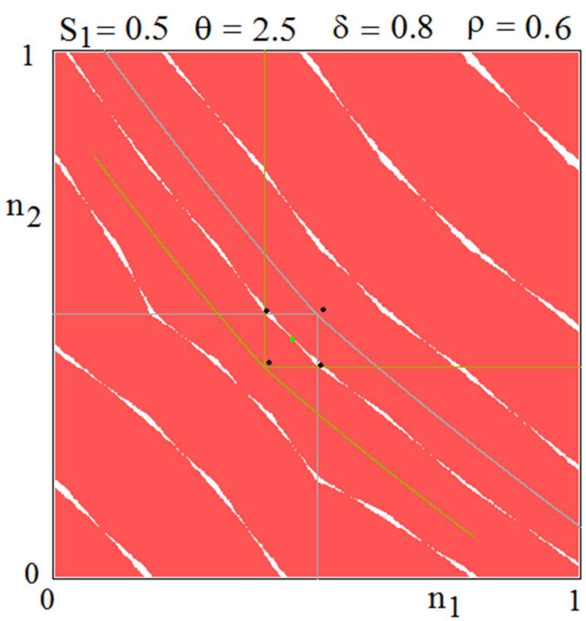
Sym. Async. 2-cycle becomes
a node at ρ = .817867,
a saddle at ρ = .833323.

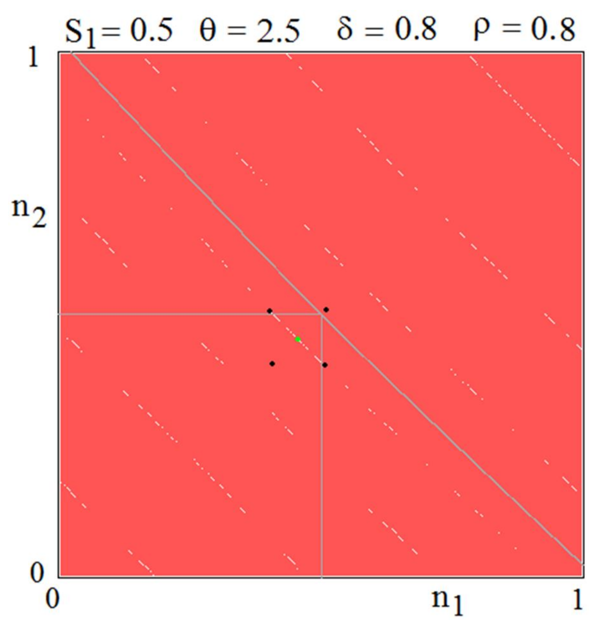
Symmetric Synchronized & Asynchronized 2-Cycles: $s_1 = 0.5$; $\theta = 2.5$; $\delta = 0.8$





Red (Sync. 2-cycle) becomes dominant.

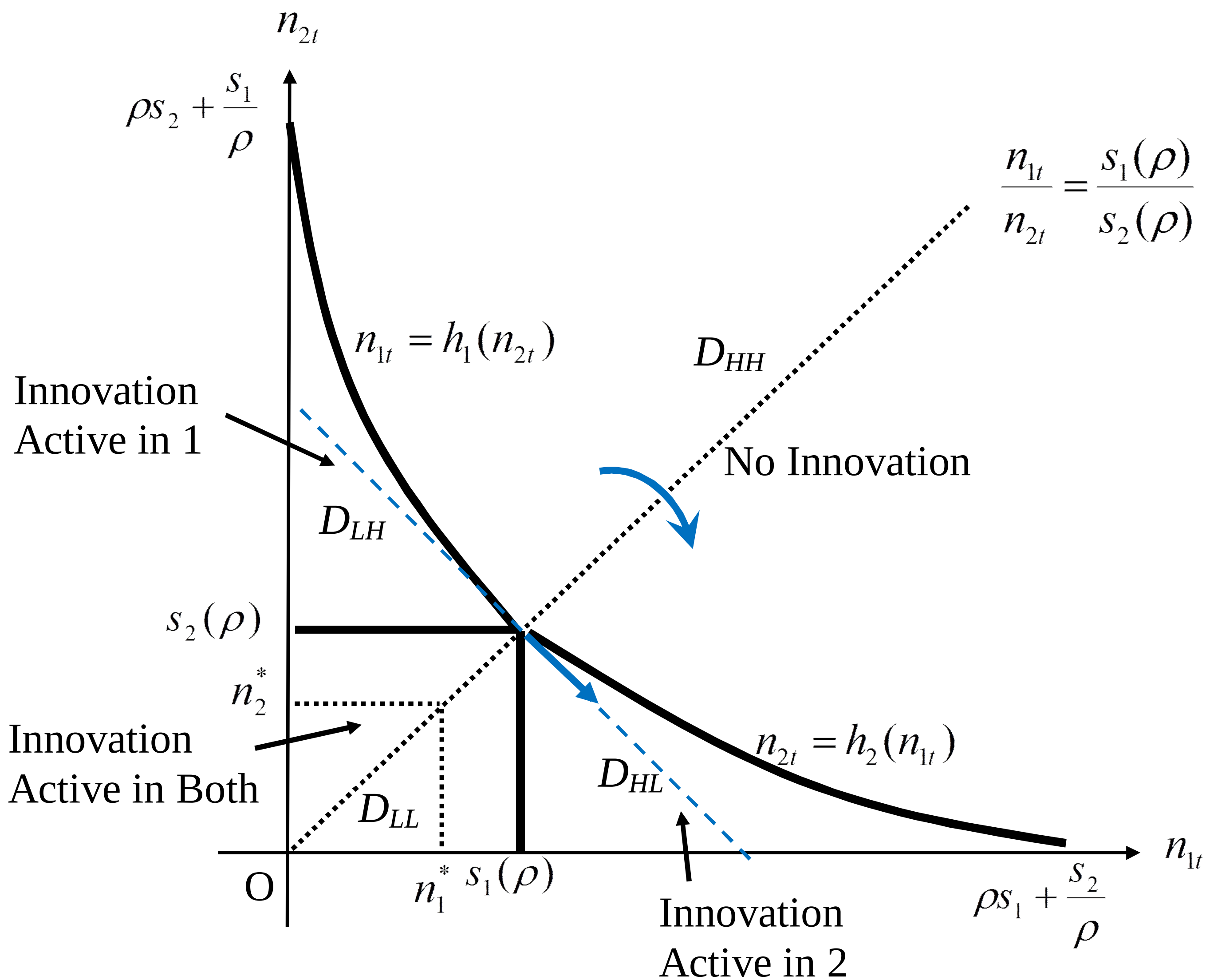




Sym. Async. 2-cycle becomes
a node at ρ = .81814;
a saddle at ρ = .818986.

Synchronization Effects of Globalization: Asymmetric Cases

State Space & Four Domains for the Asymmetric Case: $0 < \rho < s_2 / s_1 < 1$



Asymmetric Interdependent 2-Cycles, $s_1 > 0.5$, $\rho \in (0,1)$, $\delta(\theta - 1) > 1 > \delta^2(\theta - 1)$:

Each component 1D-map has:

- **an *unstable* steady state,** $n_j^* \equiv \dfrac{\theta\delta s_j(\rho)}{1+(\theta-1)\delta}$ &
- **a *stable* 2-cycle,** $n_{jL}^s \equiv \dfrac{\delta^2\theta s_j(\rho)}{1+(\theta-1)\delta^2} \leftrightarrow n_{jH}^s \equiv \dfrac{\delta\theta s_j(\rho)}{1+(\theta-1)\delta^2}$.

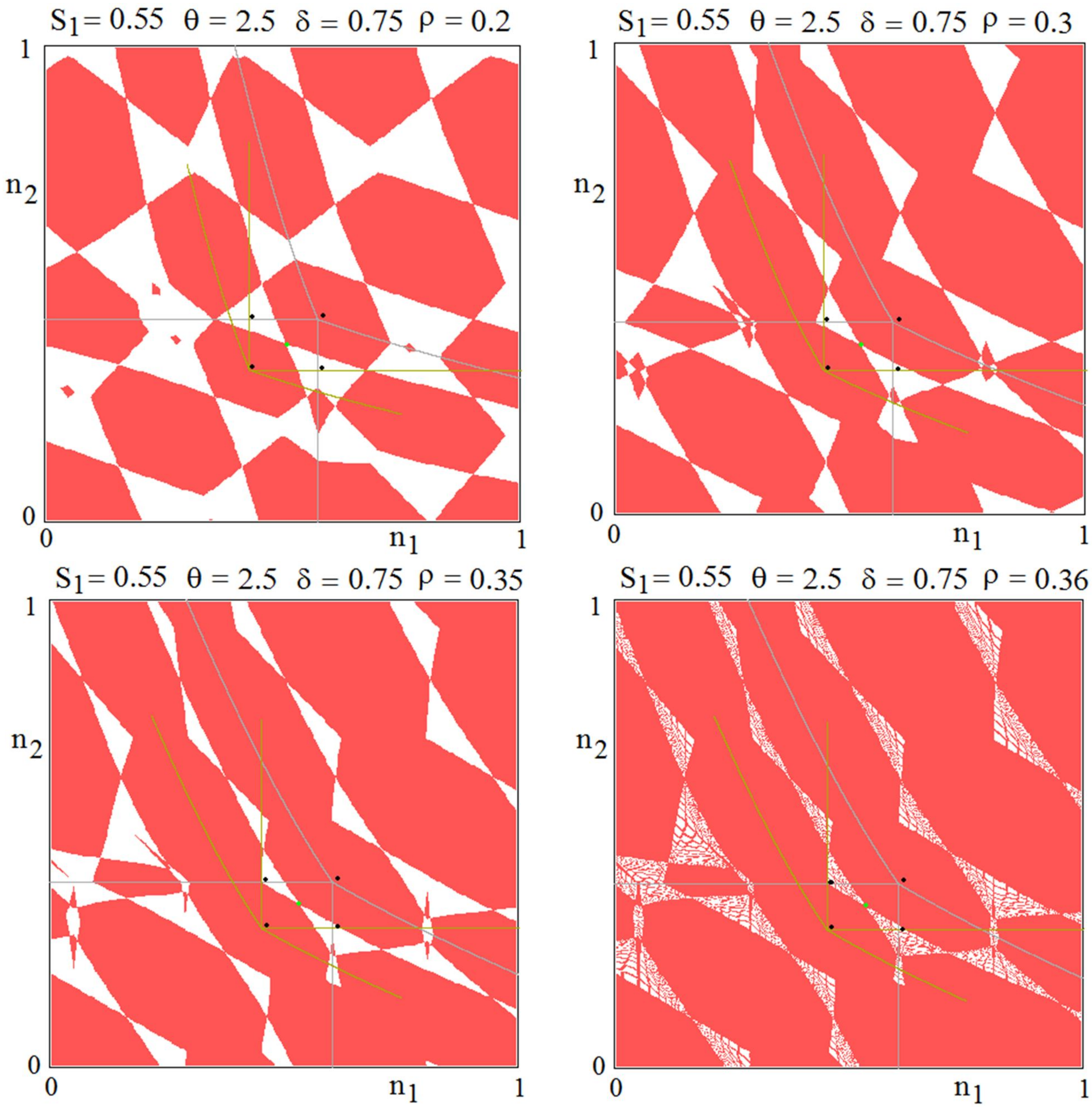
As a 2D-map,

- The two components are independent on D_{LL} and D_{HH}, which includes the ray, $n_1/n_2 = s_1(\rho)/s_2(\rho)$.
- **Stable Synchronized 2-cycle,** $(n_{1L}^s, n_{2L}^s) \in D_{LL} \leftrightarrow (n_{1H}^s, n_{2H}^s) \in D_{HH}$, exists for all $\rho \in (0,1)$
- **Stable Asynchronized 2-cycle,** $(n_{1L}^a, n_{2H}^a) \in D_{LH} \leftrightarrow (n_{1H}^a, n_{2L}^a) \in D_{HL}$, **disappears** for ρ sufficiently close to one.

Furthermore, even before the Stable Asynchronized 2-cycle disappears,

- a higher ρ expands the basin of attraction for the Synchronized 2-cycle, and reduces that for the Stable Asynchronized 2-cycle.
- This occurs more rapidly for a higher s_1.

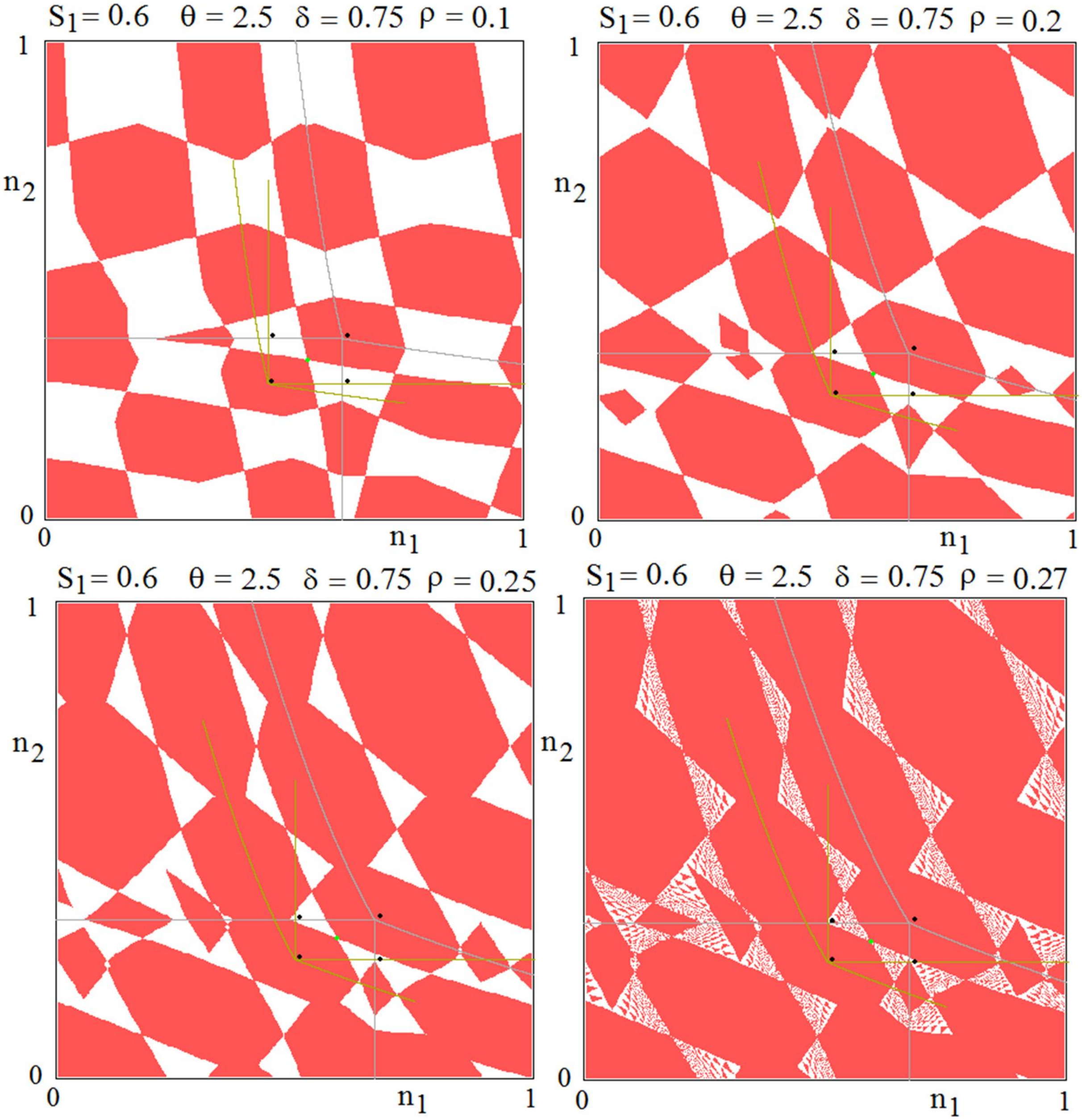
Asymmetric Synchronized & Asynchronized 2-Cycles $s_1 = 0.55$, $\theta = 2.5$; $\delta = 0.75$



By ρ = .36, infinitely many Red islands appear inside White.

By ρ = .39, the stable asynchronized 2-cycle collides with the basin boundary and disappears, leaving **the Synchronized 2-cycle as the unique attractor**.

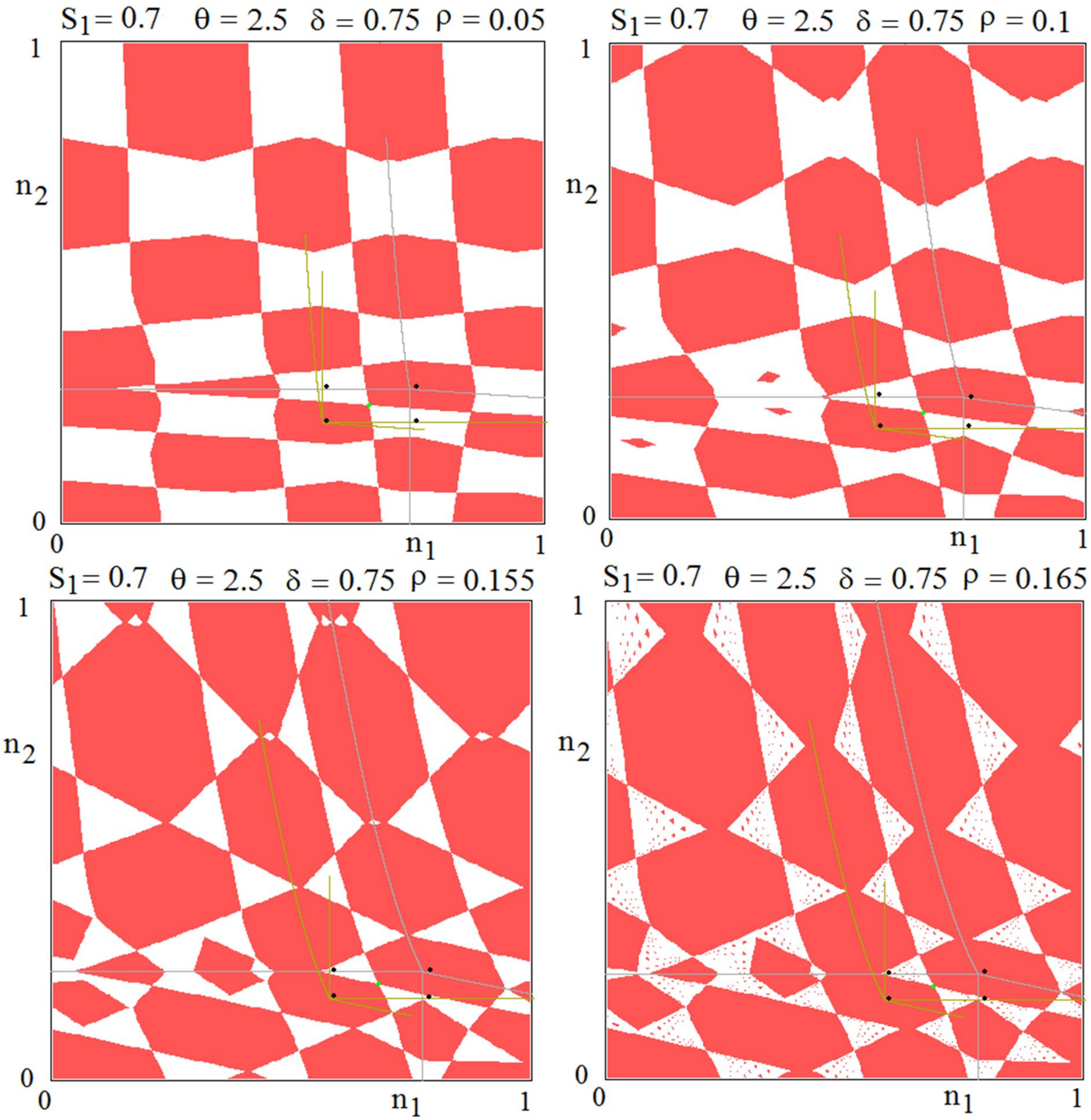
Asymmetric Synchronized & Asynchronized 2-Cycles $s_1 = 0.6$, $\theta = 2.5$; $\delta = 0.75$



By ρ = .27, infinitely many Red islands appear inside White region.

By ρ = .30, the stable asynchro. 2-cycle collides with its basin boundary and disappears, leaving **the Synchronized 2-cycle as the unique attractor**.

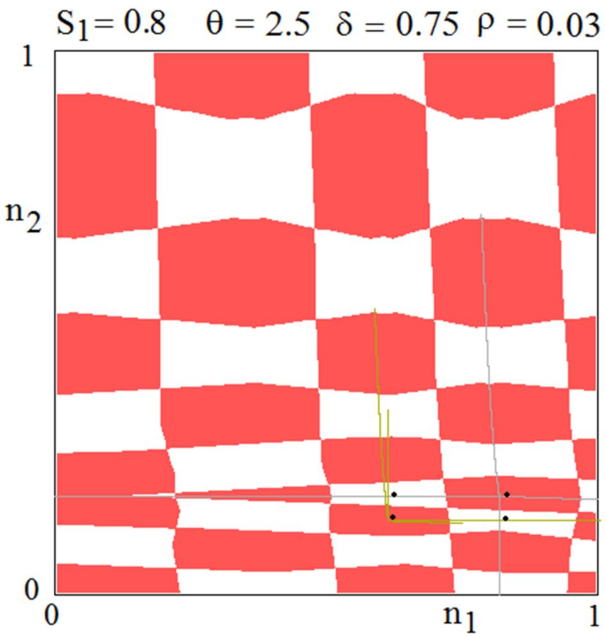
Asymmetric Synchronized & Asynchronized 2-Cycles $s_1 = 0.7$, $\theta = 2.5$; $\delta = 0.75$

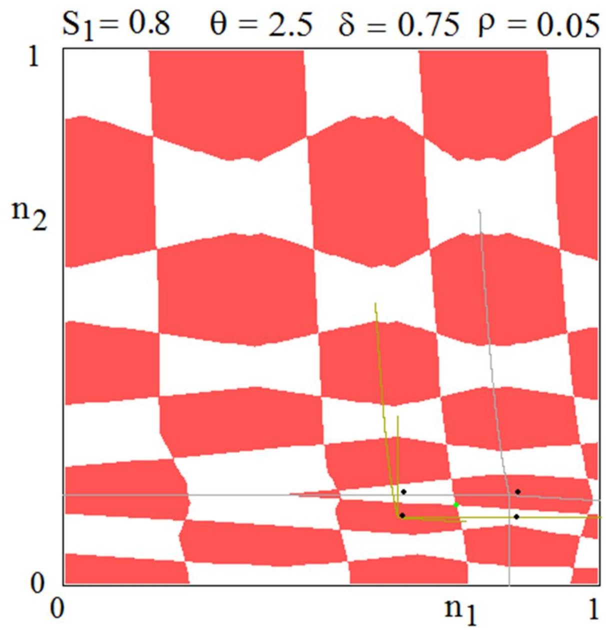


By ρ = .165, infinitely many Red islands appear inside White.

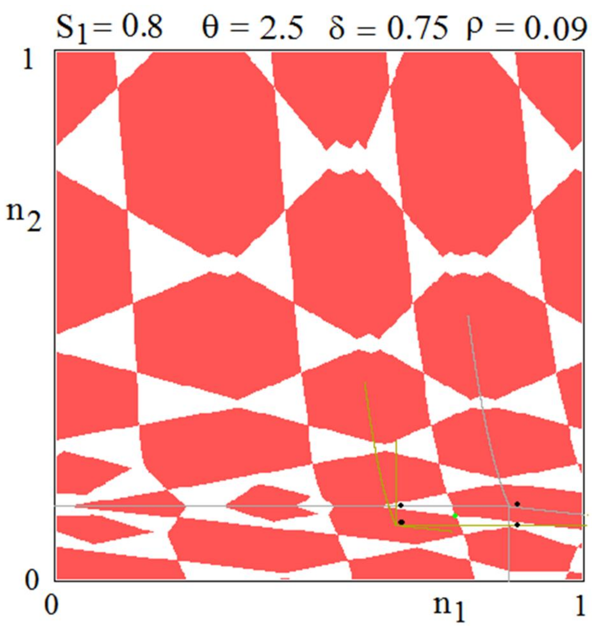
By ρ =.19, the stable asynchronized 2-cycle collides with its basin boundary and disappears, leaving **the Synchronized 2-cycle as the unique attractor**.

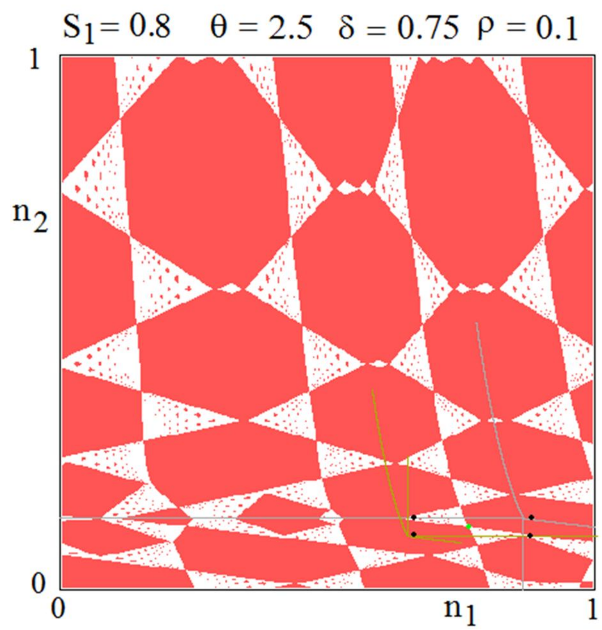
Asymmetric Synchronized & Asynchronized 2-Cycles: $s_1 = 0.8$, $\theta = 2.5$; $\delta = 0.75$





By ρ = .10, infinitely many Red islands appear inside White.





By ρ = .12, the stable asynchro. 2-cycle collides with its basin boundary and disappears, leaving **the Synch. 2-cycle as the unique attractor.**

Numerically, the stable asynchronized 2-cycle disappears at a lower value of ρ, with a higher value of s_1.

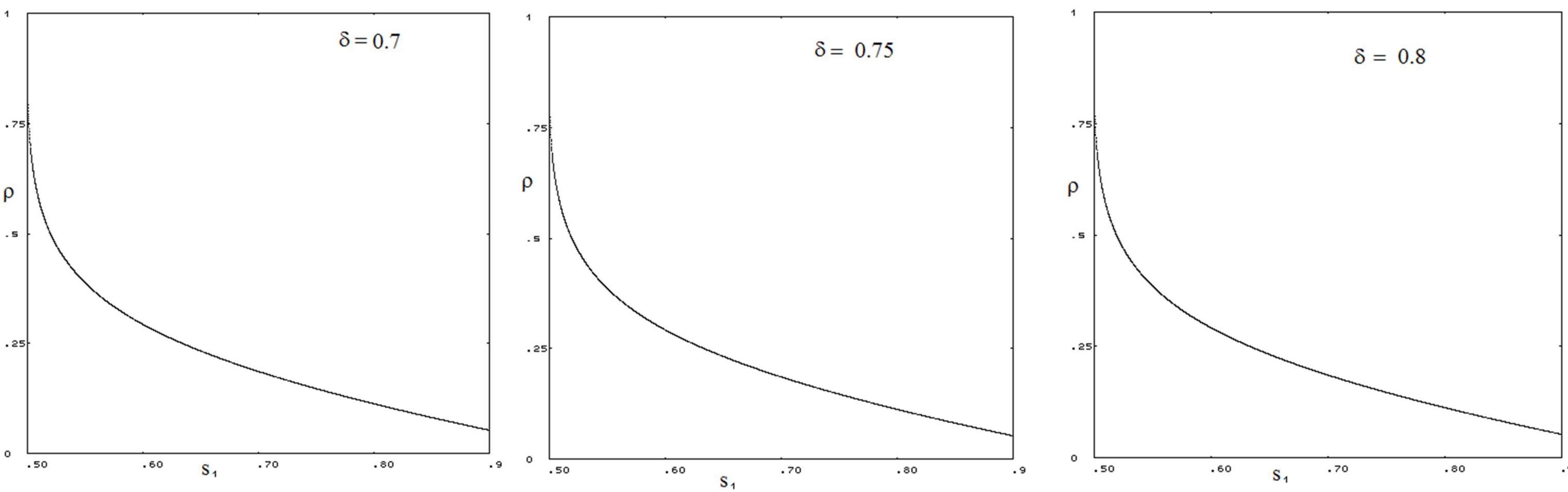
Estimating ρ_c , at which the stable asynchronized 2-cycle disappears ($\theta = 2.5$)

s_1	ρ_c			$\rho_{cc} = s_2/s_1$
	δ = 0.7	δ = 0.75	δ = 0.8	
0.5	0.8773	0.8333	0.8189	1
0.505	0.6416	0.6341	0.6310	0.9802
0.51	0.5749	0.5697	0.5676	0.9608
0.53	0.4513	0.4486	0.4475	0.8868
0.55	0.3871	0.3852	0.3845	0.8181
0.6	0.2929	0.2918	0.2913	0.6667
0.65	0.2325	0.2317	0.2314	0.5385
0.7	0.1860	0.1854	0.1851	0.4286
0.8	0.1126	0.1122	0.1120	0.2500
0.9	0.0525	0.0523	0.0522	0.1111

- Declines very rapidly as s_1 increases from 0.5.
- Hardly changes with δ.
- It happens long before the smaller country stops innovating.

Critical Value of ρ_c at which the Stable Asynchronized 2-cycle disappears (as a function of s_1)

- It declines very rapidly as s_1 increases from 0.5.
- It hardly changes with δ.



Which country sets the tempo of global innovation cycles?
Which country adjusts its rhythm to synchronize?

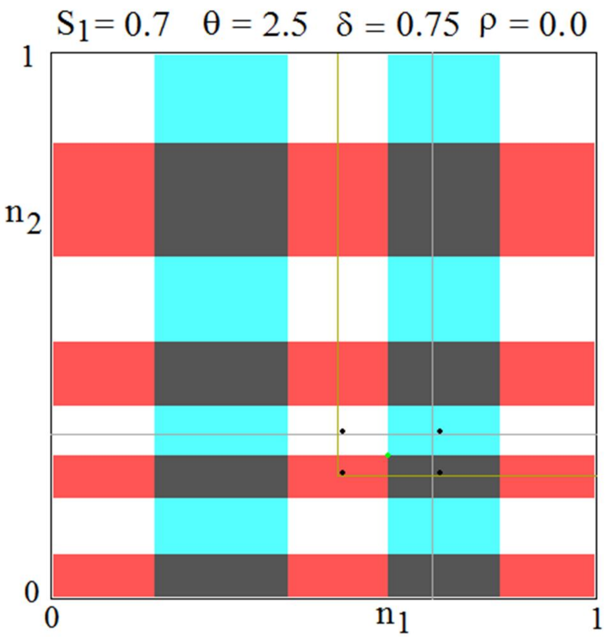
To answer this, we look at the *2nd* iterate of the map, $n_{t+2} = F \circ F(n_t) \equiv F^2(n_t)$, and the basins of attraction for its *four* stable steady states, which are the four points on the two stable 2-cycles.

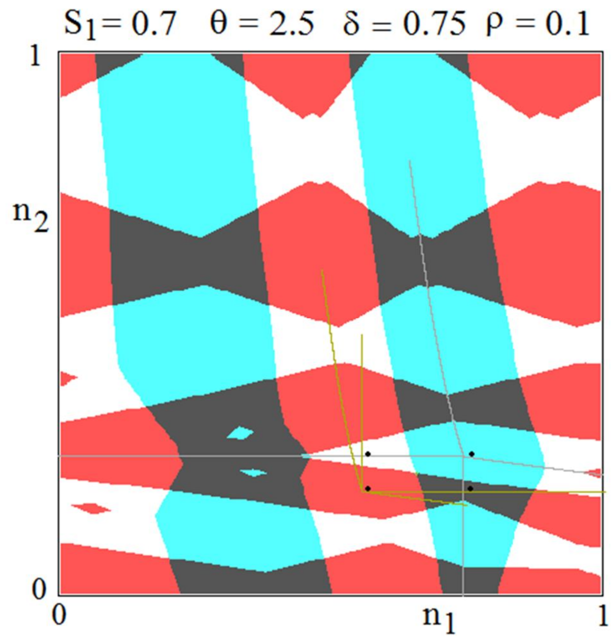
- **Red:** Basin of attraction for the steady state in D_{LL} (i.e., the synchronized 2-cycle along which it visits D_{LL} in even periods and D_{HH} in odd periods).
- **Azure:** Basin of attraction for the steady state in D_{HH} (i.e., the synchronized 2-cycle along which it visits D_{HH} in even periods and D_{LL} in odd periods).
- **White:** Basin of attraction for the steady state in D_{LH} (i.e., the asynchronized 2-cycle along which it visits D_{LH} in even periods and D_{HL} in odd periods).
- **Gray:** Basin of attraction for the steady state in D_{HL} (i.e., the asynchronized 2-cycle along which it visits D_{HL} in even periods and D_{LH} in odd periods).

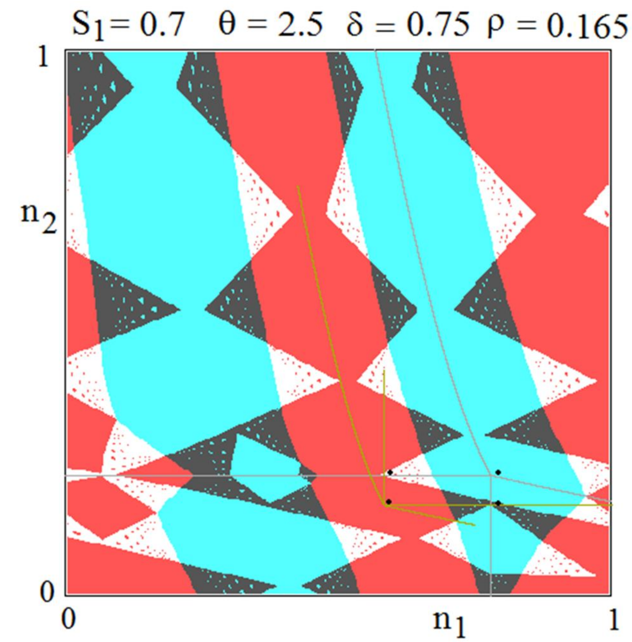
Numerically, we observe **Red invades White & Azure invades Gray**, and **vertical slips of Red and Azure** emerge, as ρ goes up.

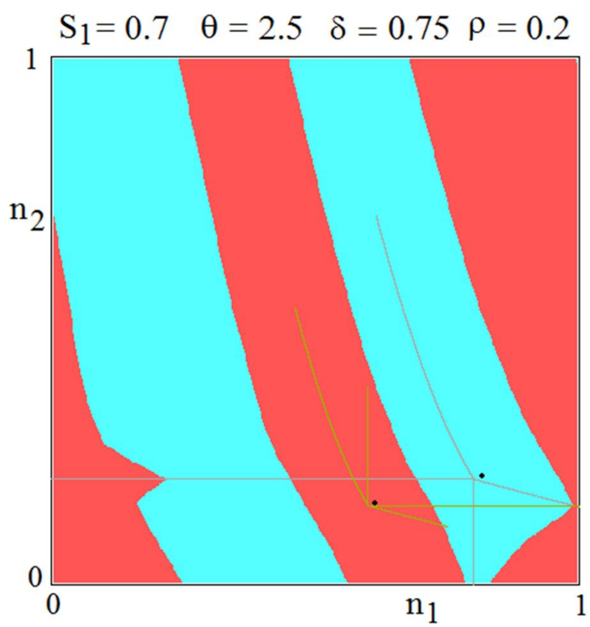
Thus, the bigger country sets the tempo of global innovation cycles and the smaller country adjusts its rhythm to the rhythm of the bigger country.

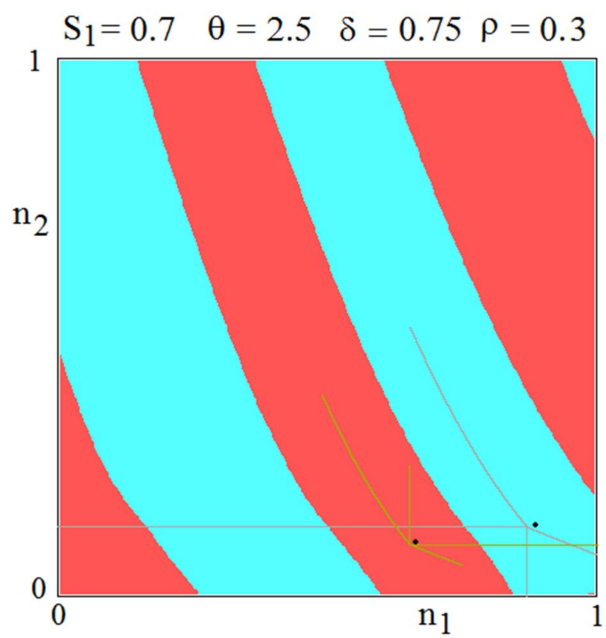
Four Basins of Attraction ($s_1 = 0.7$, $\theta = 2.5$, $\delta = 0.75$)







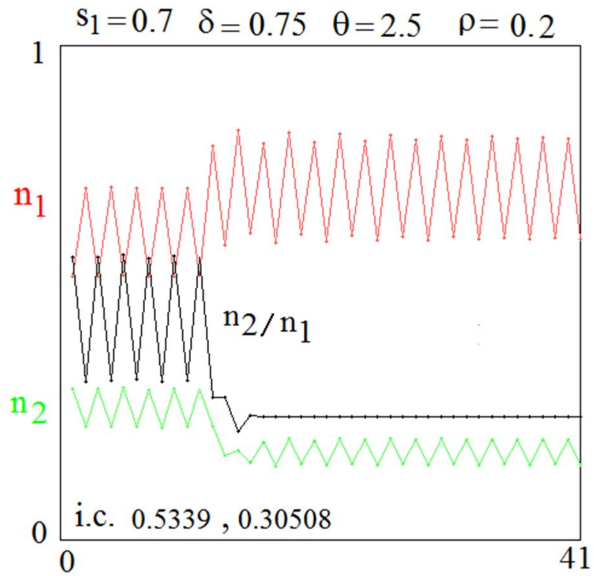


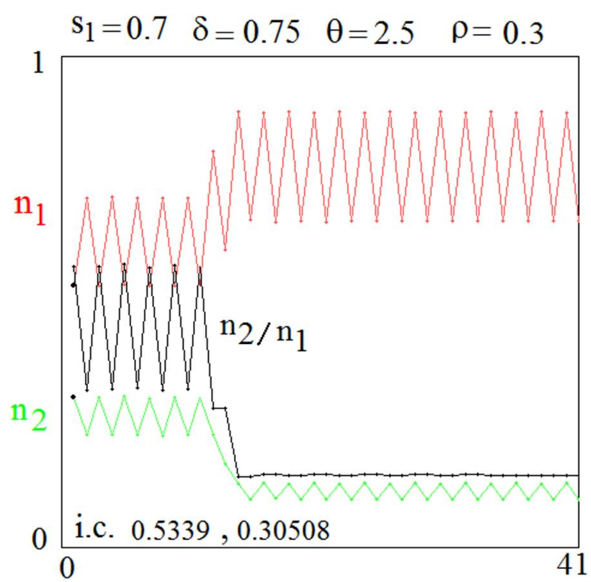


As ρ rises,
Red invades White, and
Azure invades Gray, and
vertical slips of Red and
Azure emerge.

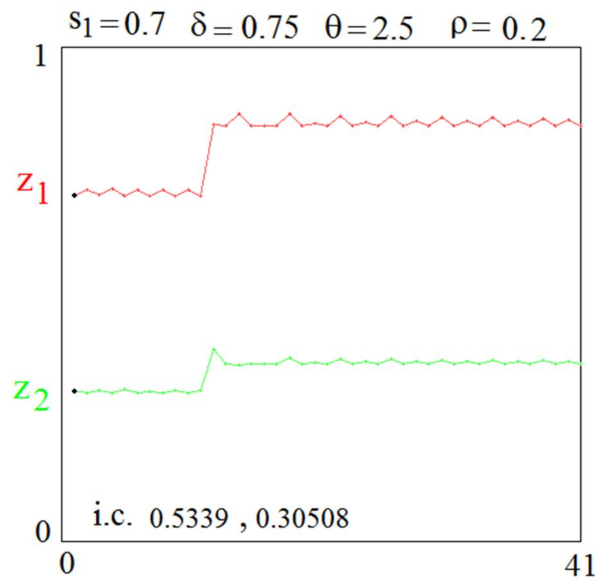
Three Effects of Globalization:

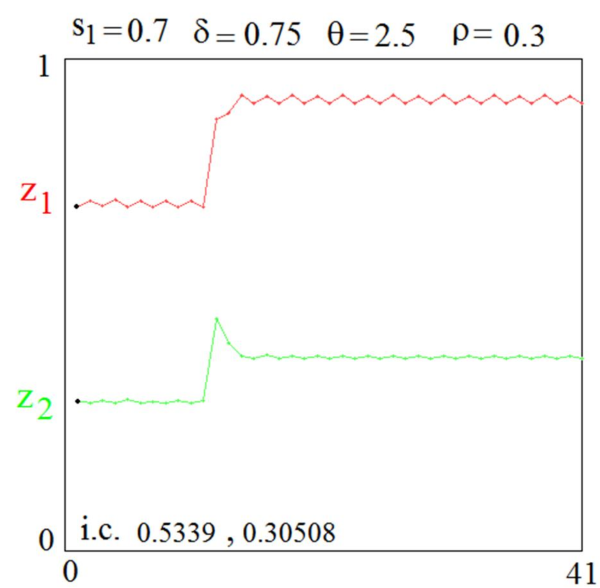
Home Market Effect



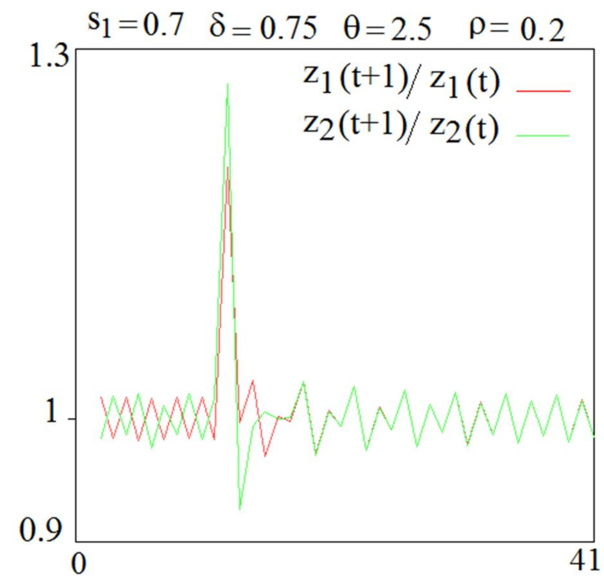


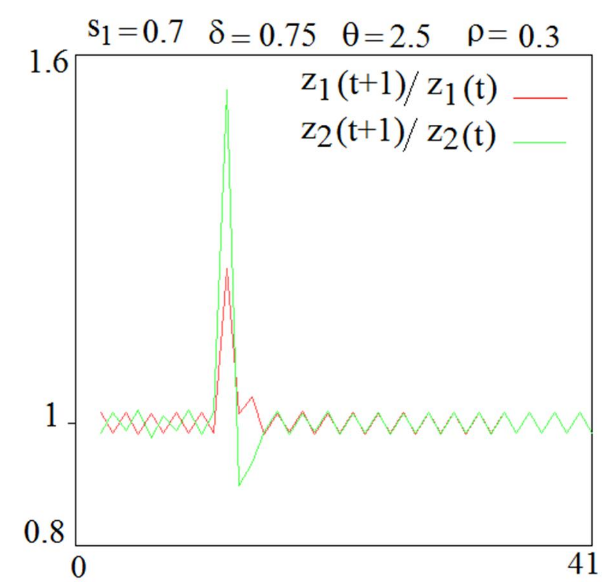
Productivity Gains





Synchronization





Concluding Remarks

Summary:

- 1st attempt to explain why globalization can cause synchronization of productivity fluctuations

- *Key Mechanism*: Globalization → Innovators from everywhere competing against each other in more integrated (hence common) market → Alignment of Incentives to Innovate → Synchronization

- Captured in a 2-country model of endogenous innovation cycles, built on DJ and HK
  - In autarky, innovation dynamics of the two countries are decoupled.
  - As trade cost falls and intra-industry trade rise, they become more synchronized.
  - Synchronization occurs faster with unequal country sizes.
  - The smaller country adjusts its rhythm to the rhythm of the bigger country.

- Adding endogenous sources of fluctuations might help to improve our understanding of “trade-co-movement puzzle.”

- Technical Contributions
  - 1^{st} two-country model of endogenous fluctuations
  - A New Model of Coupled Oscillators
  - Application of 2D noninvertible (PWS) discrete time dynamic system

Next Steps:

- **Synchronization of Chaotic Fluctuations:** *see the following pages.*

- **Different Models of Innovation Cycles:**
  - *My conjecture*: Globalization should cause synchronization as long as it causes innovators based in different countries to operate in a common market environment.
  - The assumption of structural similarity seems crucial.

What if two countries are structurally dissimilar?

- **Different Models of Trade:** For example,
  - What if the two countries are vertically specialized?; e.g., global supply chains
  - Two Industries: **Upstream & Downstream,** each produces DS composite as in DJ.
  - One country has comparative advantage in **U**; the other in **D**
  - *My conjecture*: Globalization leads to an asynchronization

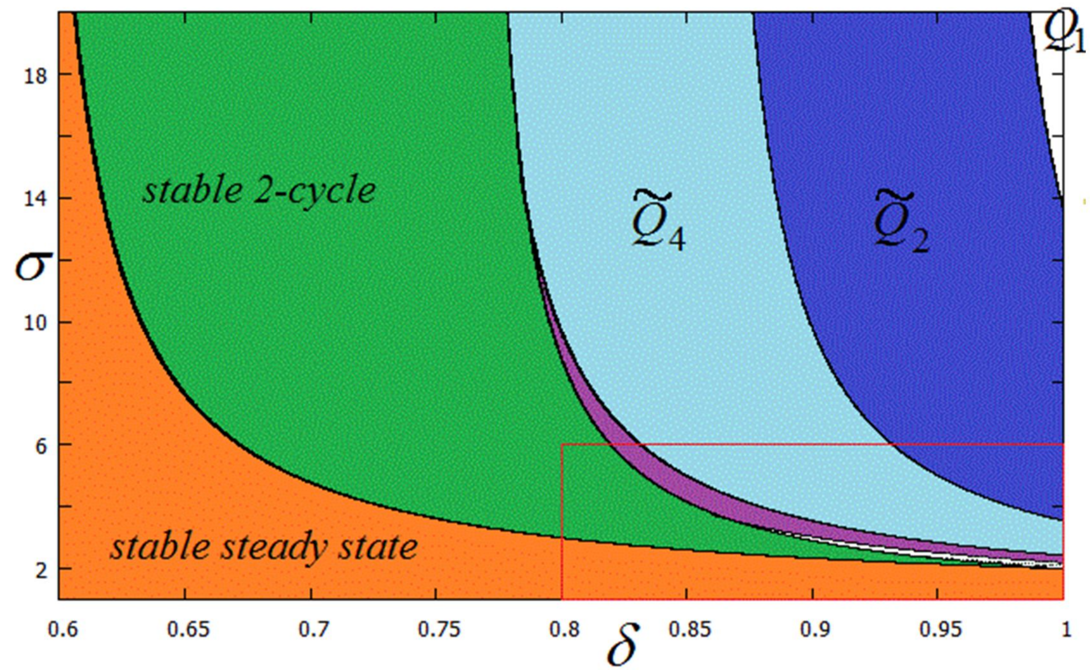
Empirically consistent, as the evidence for the synchronizing effect of trade is strong among developed countries, but *not so* btw developed and developing countries

- **"Globalization and Synchronization of Credit Cycles"**

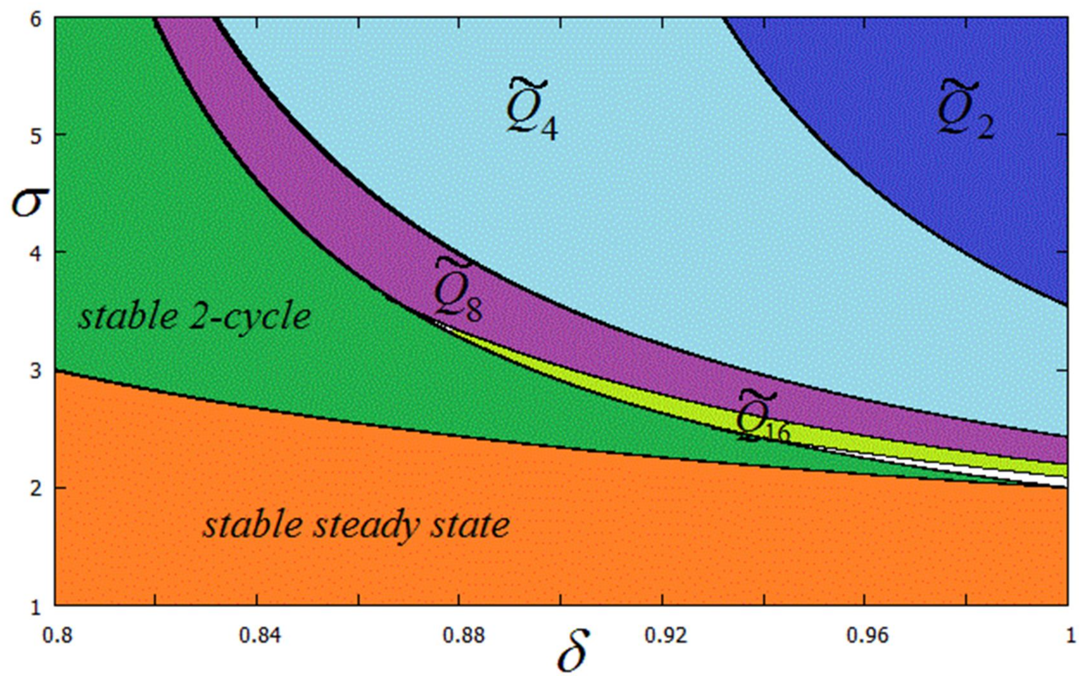
Postscript:
Interdependent Chaotic Fluctuations of Innovation: A First Look

Autarky Case:

Bifurcation diagram in the (*σ*, *δ*)-plane



And its magnification



$\tilde{Q}_{2^m}$: Robust chaotic attractor with 2^m-intervals (m = 0, 1, 2, …)

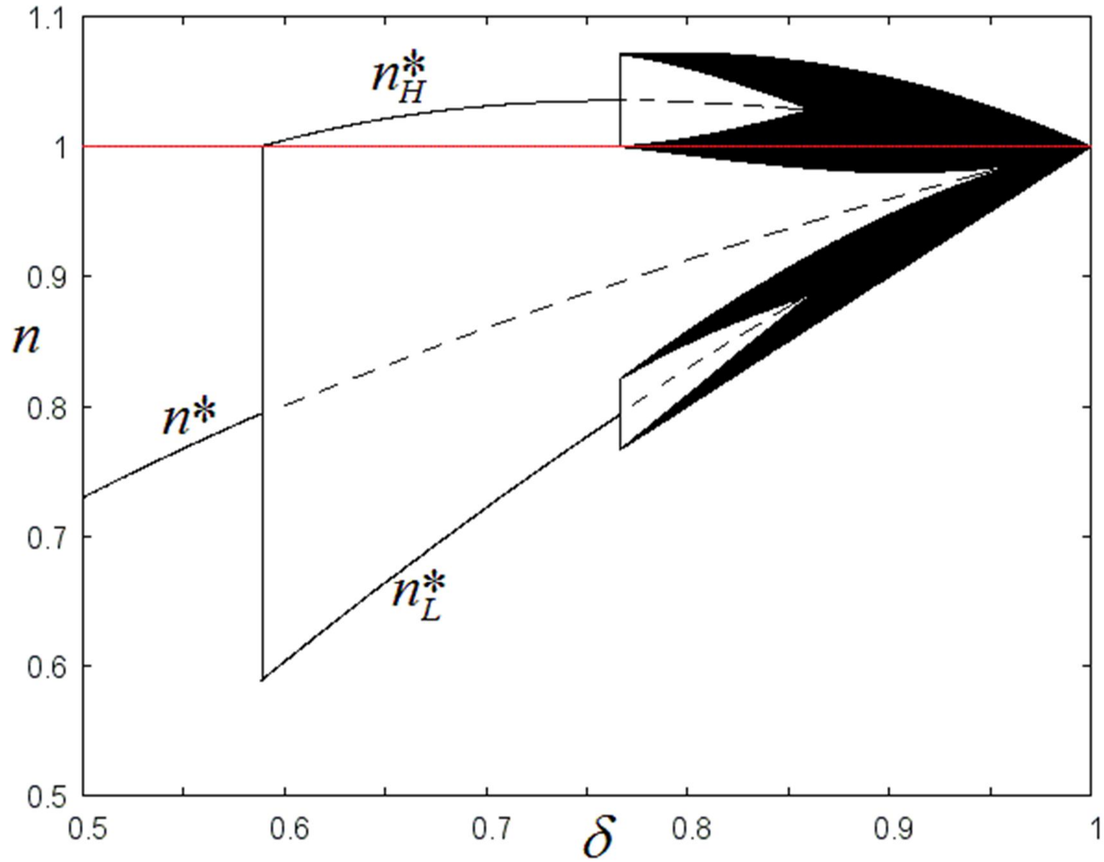
Effect of a higher δ

As we move along the ray, $(a,b)=(a,-ra)$, by increasing a from 0 to 1 for a given $r\in(1,e-1)$, it crosses DFB_1 at $(a,b)=(1/r,-1)$, the steady state becomes unstable to generate stable 2-cycle. As it crosses DFB_2 at $(a,b)=(1/\sqrt{r},-\sqrt{r})$, the system enters

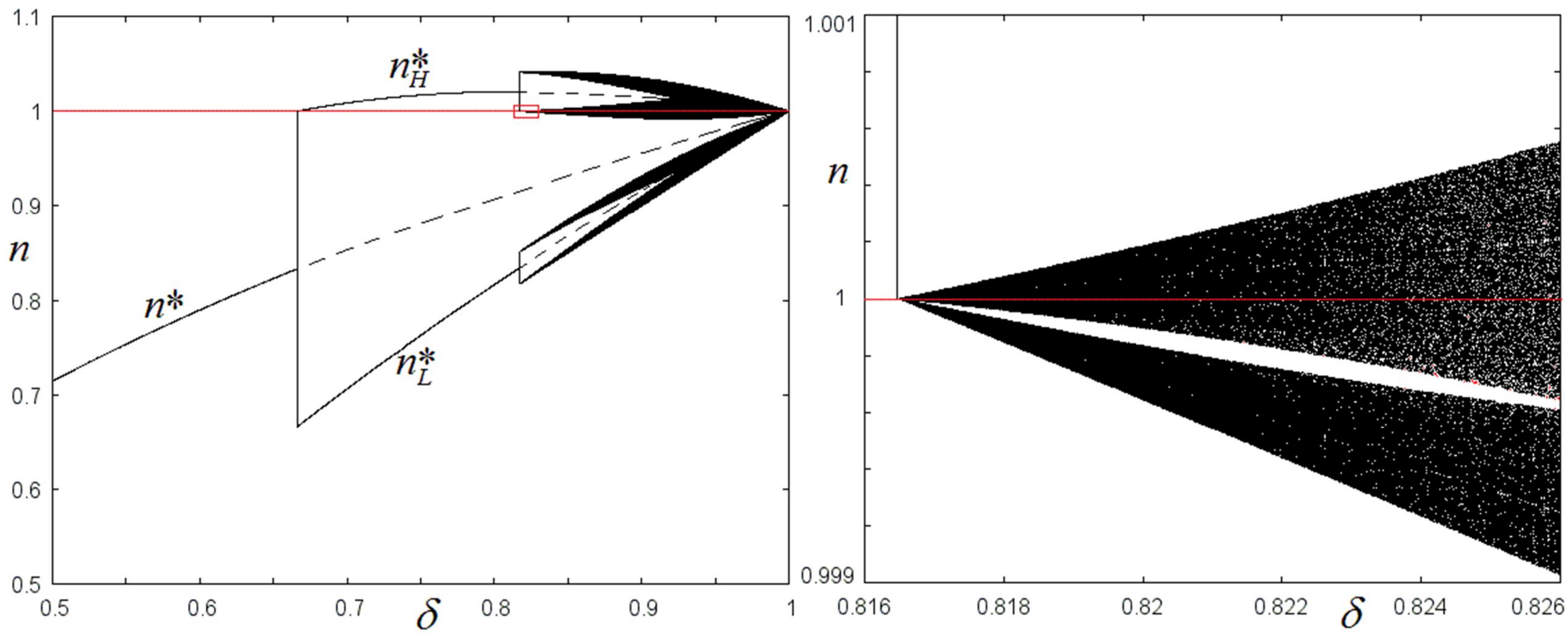
- $\tilde{Q}_4$ with 4-cyclical chaotic bands (4-CBs); then crosses H_2 to enter $\tilde{Q}_2$ with 2-CBs; then crosses H_1 to enter Q_1 with 1-CB, for $r\in(r_1,e-1)$, where r_1 = 1.618… is the root of $x^2-x-1=0$.

Thus, for $\sigma>\sigma_1$ = 13.64… ($\theta=2.7$ in the figure).

Along the 4-cyclical chaotic attractor, the trajectory visits the 4 intervals such that $n_1<n_3<n_4<n_2$.



$\theta = 2.5$ below.



- $\tilde{Q}_8$ with 8-CBs; then crosses H_4 to enter $\tilde{Q}_4$ with 4-CBs; then crosses H_2 to enter $\tilde{Q}_2$ with 2-CBs and reaches (a,b) = $(1,-r)$, without crossing H_1, for $r \in (r_2, r_1)$, where r_2 = 1.3247… is the root of $x^3 - x - 1 = 0$. (Or, σ_2 = 3.548… < σ < σ_1 = 13.64…)

and so on.

As $r = \theta - 1 \rightarrow 1$ ($\sigma \rightarrow 2$), the system enters $\tilde{Q}_{2^m}$, the region of 2^m-CBs (m = 2, 3, …) after crossing DFB_2 at $(a,b) = (1/\sqrt{r}, -\sqrt{r})$.

Independent 4-cyclical chaos in Autarky ($\rho = 0$)

With $\delta^2(\theta - 1) > 1$, each component 1D-map has

- *an unstable steady state,* $n_j^* = \dfrac{\theta\delta s_j}{1+(\theta-1)\delta}$
- *an unstable 2-cycle,* $n_{jL}^* = \dfrac{\delta^2\theta s_j}{1+(\theta-1)\delta^2} \leftrightarrow n_{jH}^* = \dfrac{\delta\theta s_j}{1+(\theta-1)\delta^2}$

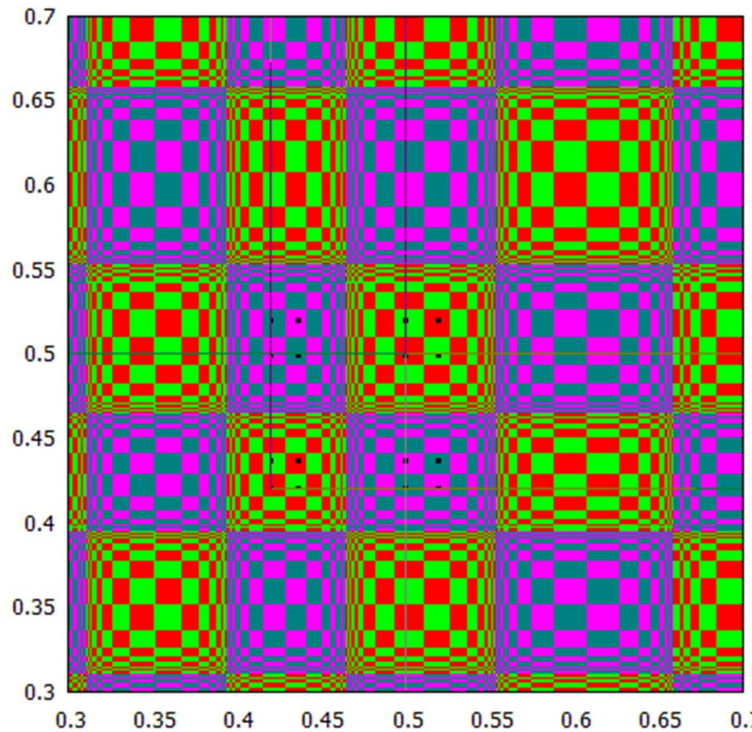
For $\theta = 2.5$, $\delta = 0.84$, each 1d map has **4-cyclical chaotic bands** as the unique attractor, in which the orbit visits the 4 intervals such that $n_1 < n_3 < n_4 < n_2$.

Autarky ($\rho = 0$),

These 1d maps are decoupled, and hence, the 2D-map has 4 distinct 4-cyclical chaotic attractors, shown in 16 rectangles.

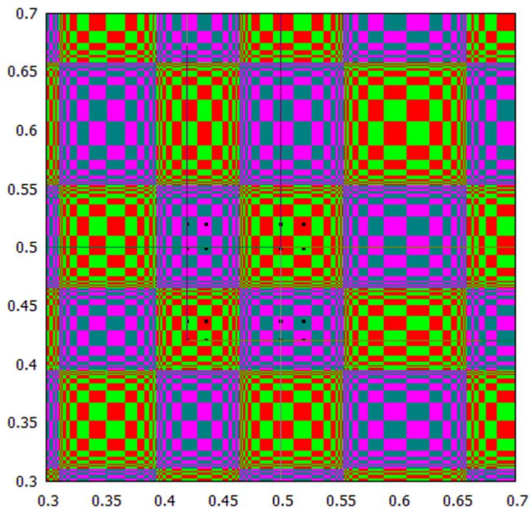
- Red: $(1,1) \to (2,2) \to (3,3) \to (4,4)$
- Yellow Green $(1,3) \to (2,4) \to (3,1) \to (4,2)$
- Pink: $(1,4) \to (2,1) \to (3,2) \to (4,3)$
- Dark Green: $(1,2) \to (2,3) \to (3,4) \to (4,1)$

Starting in Red or Yellow Green (Pink or Dark Green), the two variables become *positively* (*negatively*) correlated asymptotically.

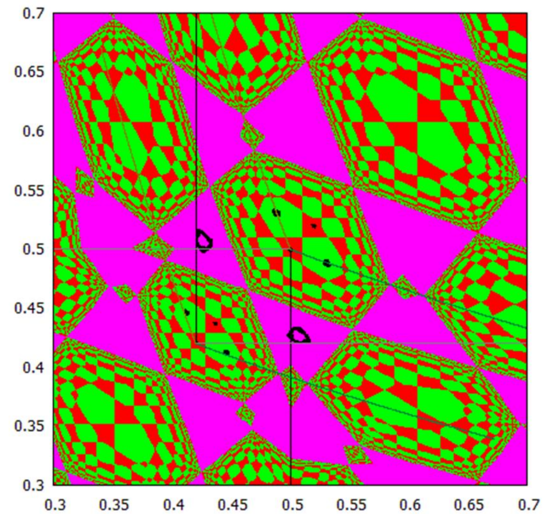


Symmetric Interdependent 4-cyclical Chaos: $s_1 = s_2 = 0.5$; $\theta = 2.5$ and $\delta = 0.84$

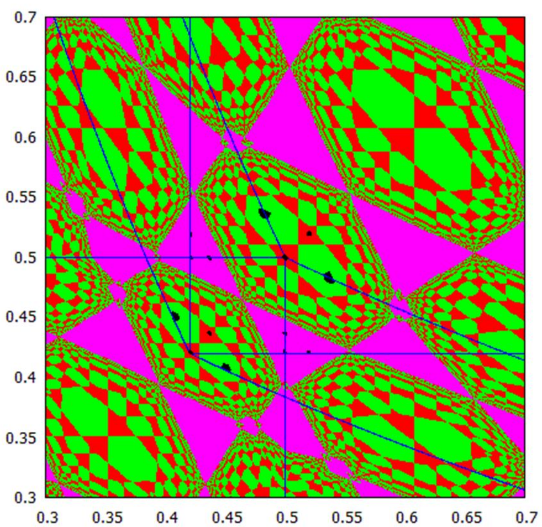
$\rho = 0$



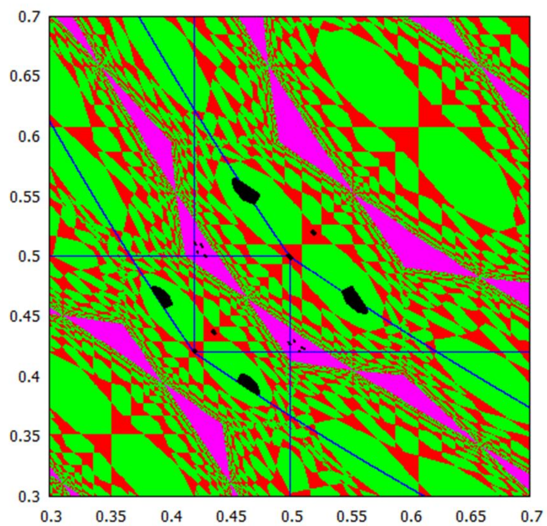
$\rho = 0.2$



$\rho = 0.26$

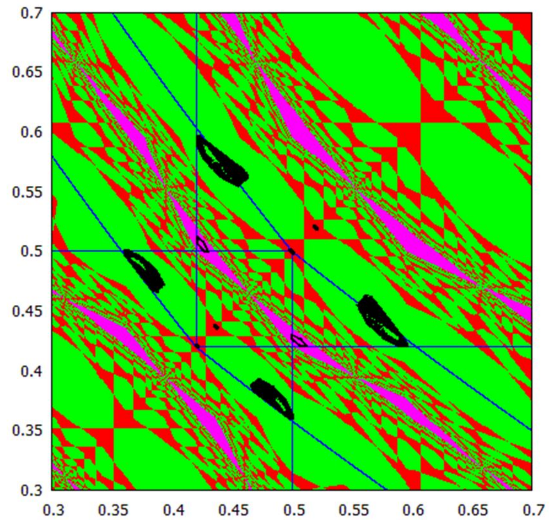


$\rho = 0.4$

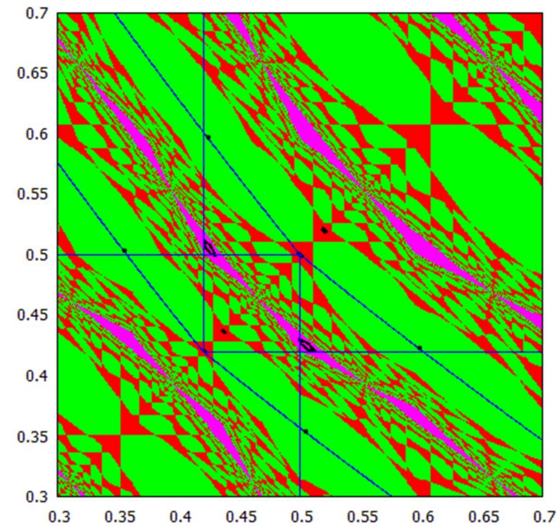


By $\rho = 0.2$, the two attractors for Pink and Dark Green merge into a 2-piece annular attractor (whose basin is shown in Pink). By $\rho = 0.26$, the attractor for Pink becomes a 6-piece. By $\rho = 0.4$, it becomes a 8-piece. By $\rho = 0.5$, it becomes a 2-piece annular again. The attractor for Yellow green also becomes a 4-piece annular.

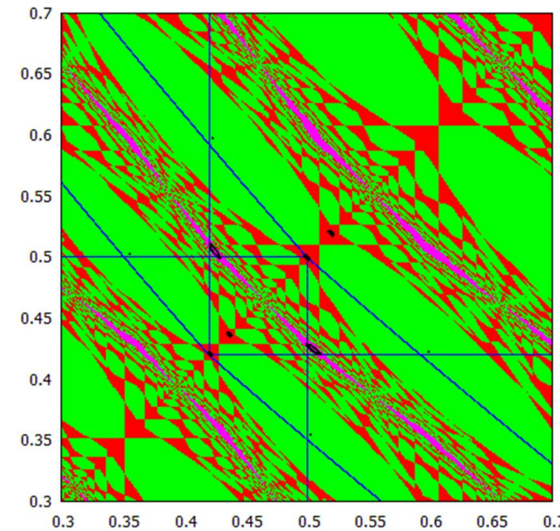
$\rho = 0.5$



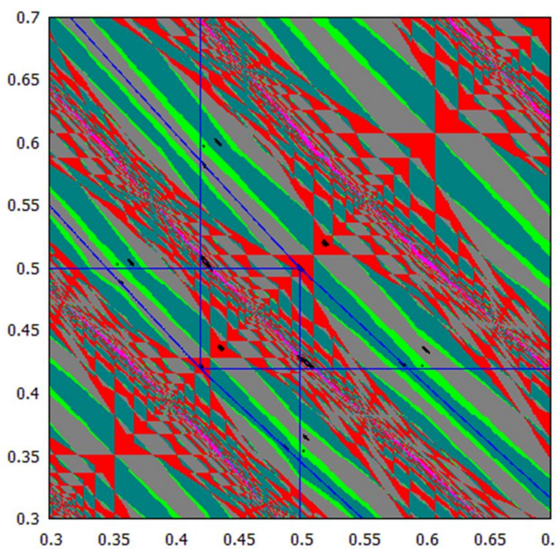
$\rho = 0.516$



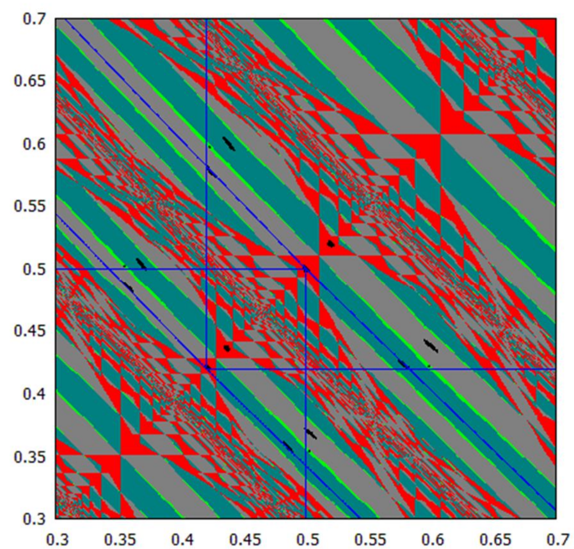
$\rho = 0.6$



$\rho = 0.7$



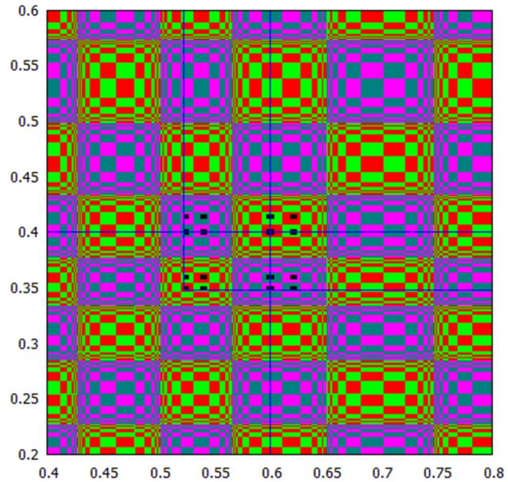
$\rho = 0.8$



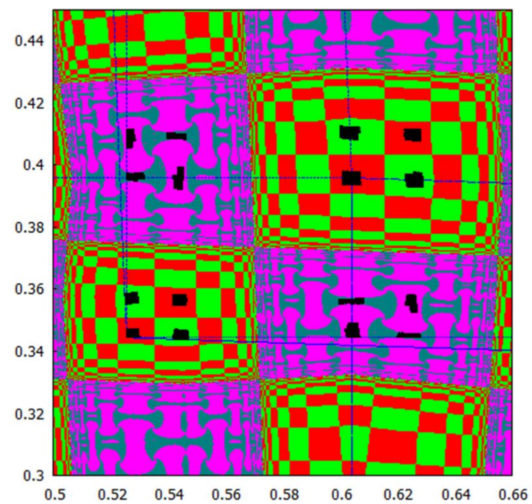
By ρ = 0.516, a pair of 4-cycles (one attracting and one saddle) appear in Yellow Green (and the former chaotic attractor becomes a repellor). By ρ = 0.7, two new 4-piece chaotic attractors (Blue and Gray) emerge inside the former Yellow Green. Pink barely exists. By ρ = 0.8, Pink disappears, as its attractor becomes a repellor. The remaining basins have fractal structures.

Asymmetric Interdependent 4-cyclical chaos: $s_1 = 0.6$; $\theta = 2.5$; $\delta = 0.87$

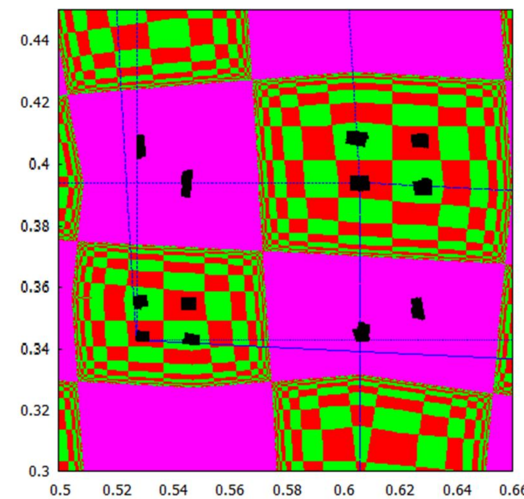
$\rho = 0$



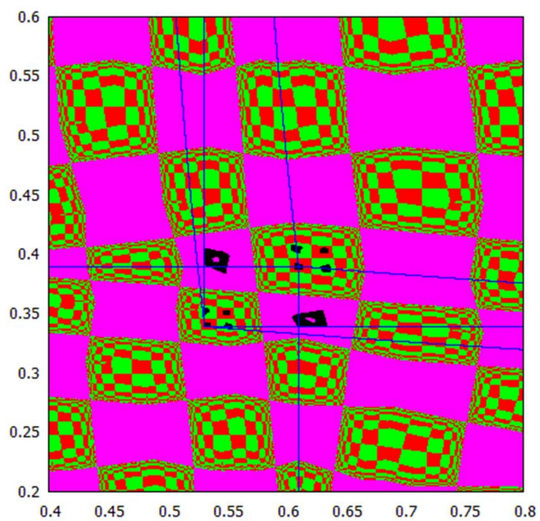
$\rho = 0.02$



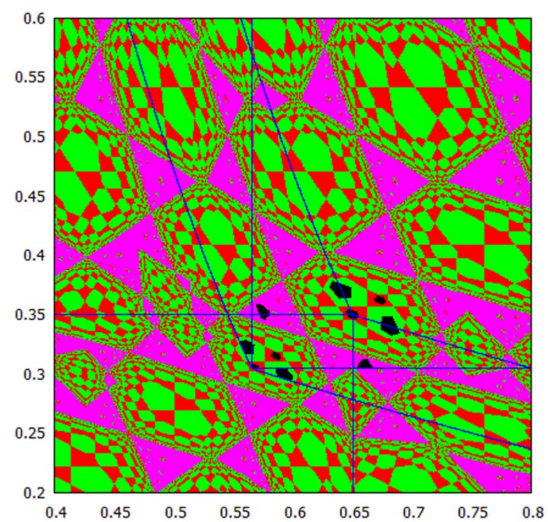
$\rho = 0.03$



$\rho = 0.05$

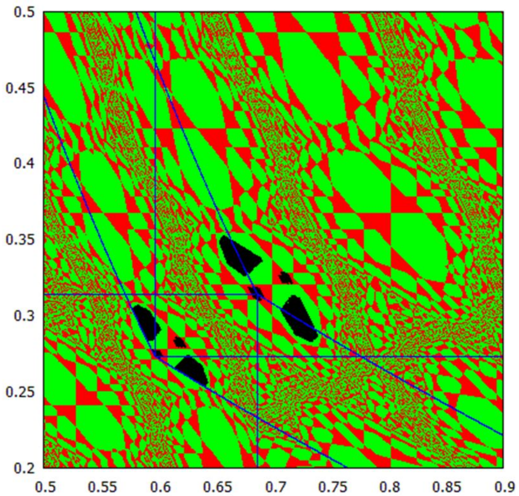


$\rho = 0.2$

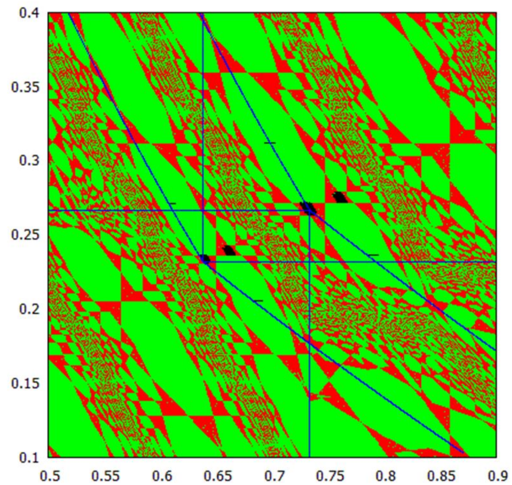


By ρ= .02, the Asynchro. 2-cycle is a focus. Then, one of the two Asynchro. 4-piece chaotic attractors collides with the borders and becomes a chaotic repellor and Dark Green is absorbed by Pink (ρ= .03). Then, 2nd Asynchro. 4-piece chaotic attractor collides with this repellor, leading to a 2-piece annular chaotic attractor (ρ= .05). By ρ= .2, the yellow green islands appear in the Pink region.

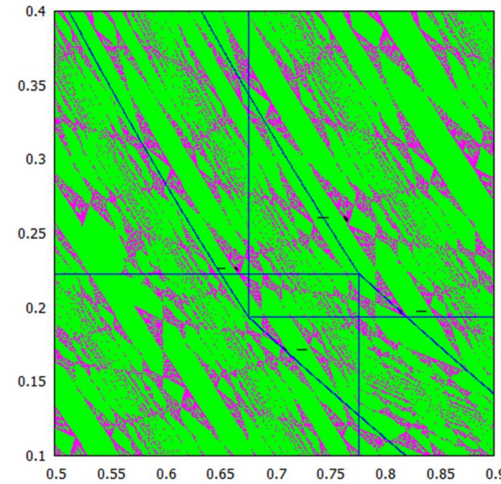
$\rho = 0.3$



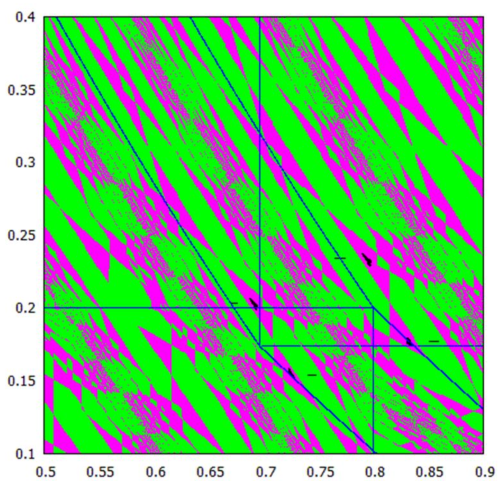
$\rho = 0.4$



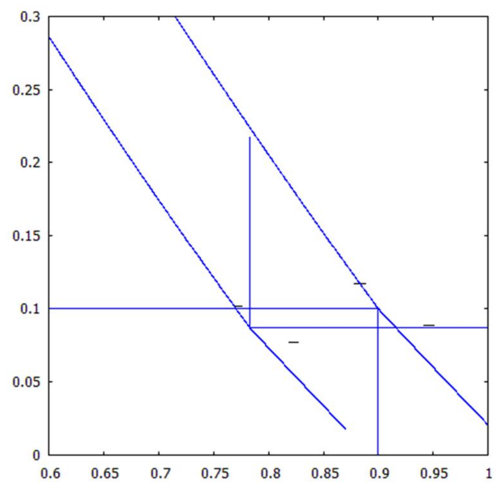
$\rho = 0.47$



$\rho = 0.5$



$\rho = 0.6$



By ρ= .3, the attractor in Pink collides with its basin boundary, and becomes a repellor; leading to Red and Yellow green basins with a fractal structure. By ρ= .4, the attractor in yellow green are horizontal lines. (a 4-cycle for the 2nd variable.) Then, the Red disappears, **leaving only one color, Yellow Green.** Then, an attracting 4-cycle (with Pink basin) appears, which becomes repelling, leading to a 4-piece chaotic attractor by ρ = .47. At ρ= .5, the Pink basin of the 4-piece chaotic attractor increases but this attractor collides with the boundary and becomes a repellor, **leaving Yellow Green the only color again** by ρ= .6.