Population Aging, Government Policy and the Postwar Japanese Economy

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Abstract

This paper analyses the Postwar Japanese economy within a parsimonious neoclassical growth framework in order to quantitatively investigate the impacts of the demographic transition in Japan. We find that the decline in the employment rate and the increase in payroll tax due to the increase in the fraction of the population aged above 65 years old significantly reduced output relative to its potential over the 1975-2014 period. On the other hand, the decline in population growth and the increase in government consumption due to the rise in demand for health services had a positive effect on output.

1 Introduction

A key feature of the postwar Japanese economy is the rapid population aging. The share of population above 65 years old among the population above 15 years old has increased from 8% in 1955 to 30% in 2014 which is currently

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the highest among all economies in the world. In this paper, we construct a parsimonious neoclassical growth model to quantitatively assess the impact of population aging and related policies over the 1975-2014 period.

The basis of our model is a representative agent neoclassical growth model with exogenous productivity growth. Hayashi and Prescott (2002) and Chen, Imrohoroglu and Imrohoroglu (2006) show that total factor productivity growth is important in accounting for the postwar Japanese economic performance. In order to assess the impact of population aging, we modified the representative agent model such that a representative household consists of young and old adults. Population aging defined as the increase in the fraction of old adults among total adult population is treated as exogenous. We assume that young and old adults have fixed employment rates where that of old adults is lower so that population aging directly affects the aggregate employment rate. This directly reduces labor input. In addition, we incorporate government fiscal policies that are related to population aging such as government consumption, capital income tax and labor income tax.

There are several related literature on population aging in Japan. Chen, Imrohoroglu and Imrohoroglu (2007) and Braun, Ikeda and Joines (2009) use over-lapping generations model to study the effects of the demographic transition on the Japanese savings rate over the 1960-2000 period. They find that total factor productivity plays a more important role than demographic transitions in accounting for the fluctuation in the savings rate. Yamada (2012) further introduces idiosyncratic labor income shocks in the over-lapping generation model in order to analyze the inter and intra-generational distribution of earning and consumption in Japan over time.

In addition to the growth path of output, we also discuss the effects of various shocks on the hours worked per worker which has declined since the 1990s. Otsu (2009) shows that productivity growth and subsistence consumption can explain the decline in hours worked during the rapid growth period but not the decline in hours during the 1990s. Hayashi and Prescott (2002) and Yamada (2012) argue that the government policy to reduce the workweek from 5.5 days per week to 5 days per week over the 1988 to 1993 period are important for the decline in hours worked. We explicitly model the workweek shortening policy as a decline in maximum available hours over the workweek and find that it is quantitatively much more important than other labor discouraging shocks such as the increase in labor income tax and the decline in total factor productivity.

The remainder of the paper is as follows. In section 2 we will discuss the postwar Japanese macroeconomy. In section 3 we describe the model. In section 4 we explain the quantitative method and the simulation results. Section 5 concludes the paper.

2 The Postwar Japanese Macroeconomy

In this section, we present macroeconomic data that characterizes the postwar Japanese economy over the 1955-2014 period. We focus on the demographic transition and the evolution of GDP per adult, its expenditure components, production factors, technology measures, government policy variables.

2.1 GDP

Figure 1 plots the log real GDP per adult deflated by the consumption price deflator. The GDP and consumption price data are from the ESRI SNA database. The figure clearly shows that output grew rapidly during the 1960s to early 1970s also known as the rapid growth period in Japan. The economy was hit by the oil shock in 1974 and average economic growth slowed down thereafter. The economy experienced a boom known as the "bubble economy" during the late 1980s followed by a long-lasting stagnation from 1991 known as the "lost decade". Finally, during the last decade the Great Recession in 2009 and the East Japan earthquake in 2011.

Table 1 lists the average GDP per adult and its growth rate over the 1955-1974, 1975-1991 and 1992-2014 period. The average GDP per adult roughly doubled over the first two subperiods. This is a result of a high average per adult output growth rate over the 1955-1974 period at 7.1%. The average growth rate over the 1975-1991 period is 2.4% and there has been practically no growth over the 1992-2014 period.

Table 1. Economic Growth

	GDP per Adult in 2000 yen	GDP per Adult Growth (%)
1955 – 1974	1,871,062	7.1
1975 – 1991	3,637,000	2.4
1992 – 2014	4,537,831	0.0

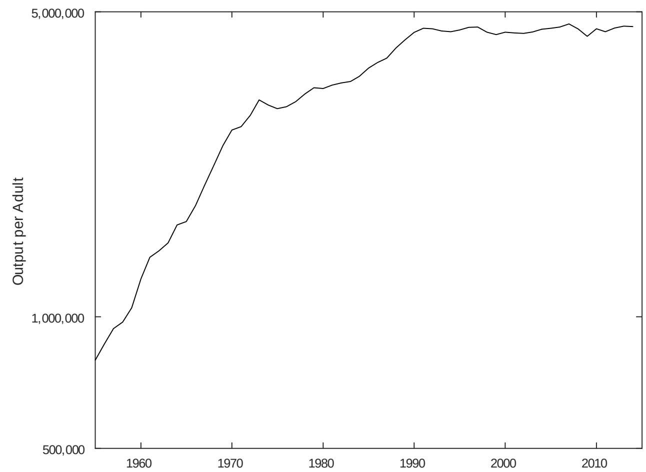

Figure 1: Real GDP per adult

2.2 Demographics

The demographic transition, namely population aging, has taken place more rapidly in Japan than any other country in the world. For population shares we consider two age groups: "Young" population defined as the population aged 15 years old to 64 years old and the "Old" population defined as the population of those above 65 years old. The population data are from the Labor Force Survey for 1973-2014 extrapolated backwards using the census data for 1955-1972.

Figure 2 plots the population of the two groups over the 1955-2014 period. Both groups are growing until during the 1990s where the Young age group starts to shrink. This is the result of the decline in the fertility rate which has fallen below the reproductive rate. On the other hand, the Old age group continues to grow which reflects the ongoing extension of life expectancy. Both of these forces contribute to the increase in the share of old population.

Table 2 reports the average demographic statistics during the 1955-1974, 1975-1991, and 1992-2014 periods. The share of old population rapidly increases from 8.7% to 12.6% and to 22.5% over the periods. The growth rate of adult population over the same periods are 1.8%, 1.2% and 0.3% respectively. In terms of age groups, the growth rate of the young population over the three subperiods are 1.6%, 0.8% and -0.4% while that of the old

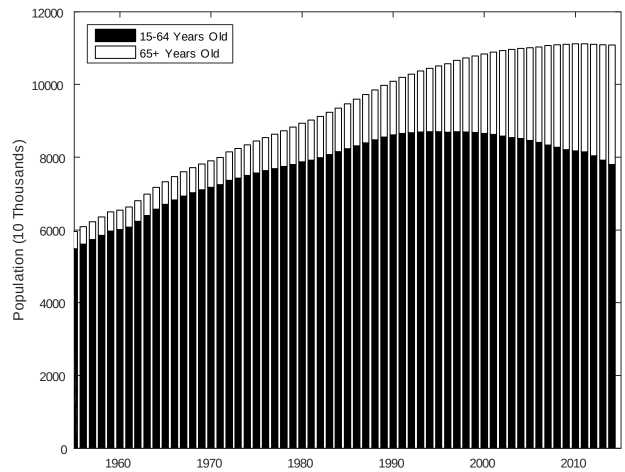

Figure 2: Population Share 1955-2014

population over the same periods are 3.0%, 3.5% and 3.2%. Therefore, the decline in the adult population growth rate is driven by the decline in fertility while the growth rate of the old population remains high due to extended life expectancy.

Table 2. Demographic Transition

	Old Share	Adult Growth	Young Growth	Old Growth
$1955 - 1974$	8.7	1.8	1.6	3.0
$1975 - 1991$	12.6	1.2	0.8	3.5
$1992 - 2014$	22.2	0.3	-0.4	3.3

2.3 Expenditure

Table 3 reports the real per adult growth rates and GDP shares of each GDP expenditure component over the 1955-1974, 1975-1991 and 1992-2014 periods. The GDP expenditure component data are from the ESRI SNA database. Each component is deflated by the consumption price indicator.

The first panel shows the growth rates of each GDP expenditure component over the three subperiods. Consumption initially grows slower than GDP during the rapid growth period but then slightly exceeds the pace of output growth in the following subperiods. Investment initially grows faster than output but the slow-down of growth after the oil-shock is much more severe than that of output and consumption. Surprisingly, in the final subperiod during the lost decades investment has been shrinking by 1.7% per year. The growth rates of government expenditure has been similar to consumption over the first two periods whereas it is growing rapidly during the final subperiod.

The second panel shows the GDP share of expenditure components. Since consumption has been growing faster than output in the later periods, its GDP share is increasing from 53.4% to 54.2% and 57.5% over the three subperiods. Since investment has been growing much slower than output in the later periods, the GDP share of investment has been falling from 33.8% to 30.5% and to 24.1% over the three subperiods. Since the government consumption is growing much more rapidly than output in the last subperiod, its GDP share has increased over the three subperiods from 11.7% to 13.9% and to 17.6%. The main driver of the increase in government consumption is the increase in health care services due to population aging. The share of health care service on total government consumption grew from 23.2% in 1980 to 34.5% in 2014.[1] The average GDP share of the trade balance over the three subperiods are 1.1%, 1.4% and 0.7% respectively which means that Japan was in trade surplus on average throughout the entire period.

Table 3. GDP Expenditure Components

a. Growth Rates of Expenditure Components (%)				
	Cons.	*Inv.*	*Gov.*	*T.B.*
1955 – 1974	6.1	9.5	6.5	-
1975 – 1991	2.6	1.5	2.5	-
1992 – 2014	0.6	−1.7	1.9	-
b. Expenditure Share of GDP (%)				
	Cons.	*Inv.*	*Gov.*	*T.B.*
1955 – 1974	53.4	33.8	11.7	1.1
1975 – 1991	54.2	30.5	13.9	1.4
1992 – 2014	57.5	24.1	17.6	0.7

[1] The treatment of government expenditure on health services changed dramatically from SNA68 to SNA93 so we cannot extrapolate the series to the past beyond 1980.

2.4 Production Factors

Table 4 presents the evolution of production factors, capital stock, employment and hours worked per worker over time. Capital stock is defined as per adult net capital stock at the beginning of the year deflated by the consumption price deflator. The sources are ESRI SNA93 dataset for 1981-2014 extrapolated backwards using the ESRI SNA68 dataset for 1970-1980 and Hayashi and Prescott (2002) for 1956-1969. The data source for hours worked after 1968 is the non-agricultural working hours data from the Labor Force Survey while for the years before that we extrapolate using the hours worked per employee data from the Monthly Labor Statistics of the Ministry of Health, Labor and Welfare and the hours worked per total employment data from the Total Economy Database of the Conference Board. The data for employment is from the Labor Force Survey.

The first panel presents the growth rates of each production factor. Capital stock per adult rapidly grew during the initial period and slowed down after the oil shock as output does. Hours worked per worker also declines especially during the final period coinciding with the timing when the government introduced the workweek shortening policy. Employment per adult has been declining especially during the first and last period.

The second panel presents the levels of each production factors. The first column shows the capital to output ratio which represents capital deepening. Since capital stock has been growing much faster than output, the average capital output ratio has doubled from 1.12 to 2.23 over the three subperiods. The hours worked per worker declined by more than 10 percent over the 1975-1991 period to the 1992-2014 period.[2]

[2] The employment rate of the young group has been rising from 68.0% in 1968 to 73.3% in 2015. Two major reasons of this trend is the increase in female participation and the extended retirement age. In contrast, the employment rate of the old group has been declining from 33.2% in 1968 to 21.7% in 2015. The main reason of this trend is the extended life expectancy where people live longer after retirement today compared to before.

Table 4. Production Factors

a. Growth Rates of Production Factors (%)			
	Cap. per Adult	*Hours per Worker*	*Emp. per Adult*
1955 – 1974	7.0	0.1	−0.5
1975 – 1991	3.9	−0.2	−0.0
1992 – 2014	0.9	−0.6	−0.4

b. Levels of Production Factors			
	Cap. – Output Ratio	*Weekly Hours*	*Emp.*
1955 – 1974	1.12	47.9	65.8
1975 – 1991	1.80	46.1	61.7
1992 – 2014	2.23	41.0	58.9

In order to illustrate the effect of population aging on the employment rate, we compute the average employment rate for each age group and project the employment rate by changing the share of each age group according to data while keeping the employment rates constant at 0.68 for the young and 0.24 for the old, which are the average over the 1968 to 2015 period. Figure 3 plots the projected series against the data. The projected employment to adult population matches the data quite well which shows that population aging alone can account for the decline of the employment rate over time especially during the 1992-2014 period.

2.5 Productivity

In this paper, we consider Total Factor Productivity as an important source of the postwar Japanese growth. We assume a standard Cobb-Douglas production function of capital and labor and define TFP as

$$Y_t = A_t K_t^{\theta} \left(E_t H_t\right)^{1-\theta}$$

where Y_t, K_t, E_t, H_t stand for aggregate output, capital, employment and hours worked per worker and θ is the capital share. The capital share $\theta = 0.404$ is calibrated to match the income data following the method described in Cooley and Prescott (1995) as described below.

Table 5 presents the growth rates of TFP over the three subperiods. This shows that TFP rapidly grew during the rapid growth period and practically

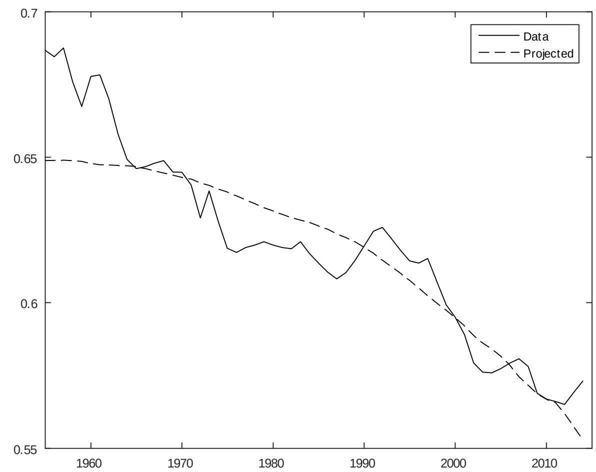

Figure 3: Aging Effect on Employment Share

experienced no growth during the lost decades. We also compute TFP using the projected employment presented above instead of the actual employment data. The second column shows that the assumption of constant employment shares of the young and old has a negligible effect on the measurement of TFP.

Table 5. Technological Progress (%)

	TFP	
	Data	Projected
1955 – 1974	4.5	4.3
1975 – 1991	1.0	1.1
1992 – 2014	0.2	0.3

2.6 Government Policy

Table 6 presents the tax rates on labor income and capital income computed by McDaniel (2009). The first column shows the capital income tax rate over the three subperiods. Capital income tax initially rises as income rises during the rapid growth period and bubble economy and then declines during the lost decades. The second column shows that the labor income tax rate steadily rise throughout the three subperiods. The main driver of this is the increase in the social security tax rate due to population aging and the rise in the dependency ratio.

The government also introduced several labor market restrictions after the oil shock. In the late 1980s the government set a target to reduce the average annual working hours to 1800 hours per worker and the 5 day workweek came into operation. In 1992 the government introduced the Act for Enforcement of Work Time Shortening which promoted shortening labor hours through firm subsidies. In 1994 the regular weekly working hours was officially reduced from 48 hours to 40 hours by the Labor Standards Act which reinforced the transition to the 5 day work week.

Table 6. Government Policy

	Capital	*Labor Tax(%)*	
	Tax(%)	*Total*	*Soc. Sec.*
1955 – 1974	18.1	9.6	5.7
1975 – 1991	24.2	17.7	11.3
1992 – 2014	20.7	23.3	17.3

3 Model

The model consists of a representative household of young and aged adults, a representative firm and the government. The head of the household decides the optimal resource allocation for the whole family. The firm hires labor and capital from the household. The government taxes the households' labor and capital income to finance its expenditure.

3.1 Household

3.1.1 Individual Preference

The period utility of each member of the household i depends on consumption c and leisure l :

$$u_i = \psi \ln c_{i,t} + (1-\psi) \ln l_{i,t}.$$

Since there are only two types of household members, $i = y, o$ denotes young and old adults. We assume that the utility from leisure l is derived from the time allocated to activities during the workweek l^{ww} and the activities during the weekend l^{we}.

First, consider the case for a worker. For simplicity, we assume a separable utility function over time allocated to each leisure activity:

$$\ln l_{i,t} = \phi \ln(l_{i,t}^{ww}) + (1-\phi)\ln(l_{i,t}^{we}).$$

The time allocated to each type of activities are defined as

$$\begin{aligned} l_{i,t}^{ww} &= (\overline{\omega} - \omega_{i,t}) \times workweek_t, \\ l_{i,t}^{we} &= \overline{\omega} \times (7 - workweek_t) \end{aligned}$$

where $\overline{\omega}$ and ω_t stand for the maximum available hours to work per day and actual hours worked per day. The workweek can exogenously change over time due to government policy. We define maximum hours per week and hours worked per week as

$$\begin{aligned} \overline{h_t} &= \overline{\omega} \times workweek_t, \\ h_{i,t} &= \omega_{i,t} \times workweek_t. \end{aligned}$$

Therefore, the period preference function is

$$u_i = \psi \ln c_{i,t} + (1-\psi)\,\phi \ln\left(\overline{h_t} - h_{i,t}\right) + (1-\psi)\,\phi \ln\left(7\overline{\omega} - \overline{h_t}\right).$$

Due to homogeneity, the scaling of the preference weight parameters do not affect the maximization problem. Therefore, we rewrite the utility function of each member of the worker as

$$u_i = \Psi \ln c_{i,t} + (1-\Psi)\ln\left(\overline{h_t} - h_{i,t}\right) + \Phi_t$$

where

$$\Phi_t = (1-\Psi)\ln\left(7\overline{\omega} - \overline{h_t}\right).$$

The case of a non-worker is simply that the hours worked ω_t and thus h_t is equal to zero.

We assume that the employment rate is fixed at π so that on average the utility of each individual is

$$u_i = \Psi \ln c_{i,t} + \pi_i(1-\Psi)\ln\left(\overline{h_t} - h_{i,t}\right) + \Omega_t$$

where

$$\Omega_t = (1-\pi_i)(1-\Psi)\ln \overline{h_t} + \Phi_t.$$

Since Ω_t only includes exogenous variables, it will not affect the maximization problem given the separable utility function.

3.1.2 Household Optimization

In this model, we assume that the head of the household solves the resource allocation problem of the family which consists of young and old adults. For simplicity, we assume that the size of each family is equal to 1 and define the population share of young adults as η. The average representative family utility is

$$\begin{aligned} u(c,h) &= \eta_t \left[\Psi \ln c_{y,t} + \pi_y (1-\Psi) \ln \left(\overline{h_t} - h_{y,t}\right) + \Omega_{y,t}\right] \\ &\quad + (1-\eta_t) \left[\Psi \ln c_{o,t} + \pi_o (1-\Psi) \ln \left(\overline{h_t} - h_{o,t}\right) + \Omega_{o,t}\right] \end{aligned} \tag{1}$$

where the subscripts y, o stand for the young and old family members.

It turns out that given the separability of the utility function, both the optimal consumption level and working hours are identical across the young and the old so that $c_{y,t} = c_{o,t} = c_t$ and $h_{y,t} = h_{o,t} = h_t$. In addition, the separable term Ω_t only includes exogenous variables so that it does not affect the optimization problem. Therefore, we can imagine that the head of the household is maximizing the following family utility function:

$$U = \max \sum_t \beta^t \left[\Psi \ln c_t + e_t (1-\Psi) \ln \left(\overline{h_t} - h_t\right)\right], \tag{2}$$

where

$$e_t = \eta_t \pi_y + (1-\eta_t) \pi_o$$

stands for the employment rate. Since we assume constant employment shares for each age group and population aging is exogenous, the employment rate changes over time exogenously.

The household faces the following budget constraint

$$c_t + i_t = (1-\tau_{l,t}) w_t h_t e_t + (1-\tau_{k,t}) r_t k_t + \zeta_t, \tag{3}$$

where i_t is investment, w_t is the after tax wage rate, r_t is the rental rate on capital, k_t is the capital stock per adult, $\tau_{l,t}$ and $\tau_{k,t}$ are labor and capital income tax rates and ζ_t is a lump sum transfer from the government.

In this model, population growth is interpreted as an increase in the number of households. We assume that existing households will support the new households by sharing their capital. Therefore, the capital stock of a representative household evolves over time according to a capital law of motion

$$(1+n_t) k_{t+1} = i_t + (1-\delta) k_t, \tag{4}$$

where n_t is the population growth rate.

3.2 Firm

The representative firm will produce a single good by combining capital and labor according to the following Cobb-Douglas production function:

$$Y_t = A_t K_t^{\theta} \left(h_t e_t N_t\right)^{1-\theta},$$

where Y_t is total output, A_t is the total factor productivity and N_t is the number of families in the economy.

The firm maximizes profits

$$\pi_t N_t = Y_t - w_t h_t e_t N_t - r_t K_t,$$

by choosing the optimal labor and capital level where π_t is the profit per family

$$\pi_t = y_t - w_t h_t e_t - r_t k_t.$$

3.3 Government

The government purchases goods and services for exogenous reasons and pays for this through labor income tax. They rebate all excess revenue to the household through lump sum transfer. Therefore, the government budget constraint is

$$G_t = \tau_{l,t} w_t h_t e_t N_t + \tau_{k,t} r_t K_t - \zeta_t N_t. \tag{5}$$

For simplicity, we assume that the government decides the amount of expenditure as a fraction of current output so that

$$G_t = g_t Y_t.$$

The government budget constraint together with the household budget constraint and firm profits, we get the resource constraint

$$(1 - g_t) y_t = c_t + i_t \tag{6}$$

3.4 Equilibrium

The deterministic competitive equilibrium is a set of quantities and prices

$$\{y_t, c_t, i_t, h_t, k_{t+1}, \zeta_t, w_t, r_t, e_t, n_t, \tau_{l,t}, \tau_{k,t}, g_t, A_t\}_{t=0}^{T}$$

such that;

1. The household optimizes given the series of $\{w_t, r_t, e_t, n_t, \tau_{l,t}, \tau_{k,t}, \zeta_t\}_{t=0}^{T}$ and k_0
2. The firm optimizes given $\{w_t, r_t, A_t\}$ each period
3. The government budget constraint (5) holds
4. The resource constraint (6) holds

4 Quantitative Analysis

In order to analyze the quantitative impacts of population aging and government policy we calibrate the model parameters to the Japanese data and solve the model numerically using the shooting algorithm.

4.1 Solution Method

The model leads to the following equilibrium conditions.

$$\frac{\Psi}{c_t} = \mu_t \tag{7a}$$

$$\frac{1-\Psi}{\overline{h_t} - h_t} = \mu_t(1-\tau_{l,t})w_t \tag{7b}$$

$$(1+n_t)\mu_t = \beta\mu_{t+1}\left\{(1-\tau_{k,t+1})r_{t+1} + 1 - \delta\right\} \tag{7c}$$

$$r_t = \theta\frac{y_t}{k_t} \tag{7d}$$

$$w_t = (1-\theta)\frac{y_t}{h_t e_t} \tag{7e}$$

$$(1+n_t)k_{t+1} = i_t + (1-\delta)k_t, \tag{7f}$$

$$y_t = A_t k_t^{\theta}\left(h_t e_t\right)^{1-\theta} \tag{7g}$$

$$(1-g_t)y_t = c_t + i_t \tag{7h}$$

For each t, there are 8 equations 8 endogenous variables, $\{k_{t+1}, \mu_t, h_t, y_t, c_t, i_t, r_t, w_t\}$ as well as $\left\{k_1, \mu_{T+1}\right\}$, where note that k_t is the capital at the beginning of period t. Hence, if there are T time periods, we have $8T$ equations and $8T+2$ endogenous variables. Adding the initial condition for k_1 and a terminal condition (TVC), we can regard the equilibrium condition as a system of equations with $8T+2$ equations and $8T+2$ variables. Our equilibrium is defined as $\{k_{t+1}, \mu_t, h_t, y_t, c_t, i_t, r_t, w_t\}_{t=1}^{T}$ as well as $\left\{k_1, \mu_{T+1}\right\}$ that satisfies the above equilibrium equations and initial and terminal conditions, given exogenous processes.

The most straightforward strategy to solve this model is to solve a system of $8T + 2$ equations for $8T + 2$ variables by using a numerical solver (e.g., "fsolve" of Matlab). This is easy to programme, but it has a lot of equations because T tends to be large, it requires very good initial guess and it does not exploit some specific features of this system of equitations, which we discuss shortly.

Instead, we can use the method called the "shooting algorithm", which numerically solves the system of ordinary differential (difference) equations with boundary conditions. In our case, we conduct the computation as follows. Suppose that we know k_1 (the initial condition), and the terminal condition is given by $k_{T+1} = \overline{k}$ (a certain value exogenously specified). Then, pick a certain value for μ_0.[3] Given $\{k_1, \mu_0\}$, we have 8 equations and 8 remaining variables for time 1; $\{k_2, \mu_1, h_1, y_1, c_1, i_1, r_1, w_1\}$, which means we can solve time $t = 1$ equations for $t = 1$ variables.[4] Now, having $\{k_2, \mu_1\}$ on hand, we can solve for time $t = 2$ equations for $\{k_3, \mu_2, h_2, y_2, c_2, i_2, r_2, w_2\}$. We repeat this until $t = T$. Of course, in general, k_{T+1} obtained in this way does not match to the terminal value $\overline{k}$. Hence, we try different values of μ_0 until k_{T+1} gets close enough to $\overline{k}$. For each trial of μ_0, k_t evolves to k_{T+1} and we keep trying different μ_0 "shooting" at target value $\overline{k}$.

There are a couple of practical issues. First, it is often not clear what kind of terminal condition we should use. In this paper we compute a sort of steady state value of k_{T+1}, assuming that total factor productivity grows at the same rate as that in period T and the other exogenous variables stay at the values in T forever. Alternatively, we could have imposed a terminal condition such as $\mu_{T+1} = \overline{\mu}$. Algebraically speaking, any condition would work as long as we can reduce one degree of freedom, but it should depend on our economic intuition; i.e., the terminal condition itself is part of the model. Moreover, the quantitative result might depend on the choice of the terminal condition.[5]

[3] Note that μ_0 is not included in our solution.

[4] One of the caveats of solver type algorithms is that it fails to exploit this property; that is, time t equations have only time t variables and time $t+1$ variables, but not, say, $t+2$ variables.

[5] The choice of the terminal condition affects the solution near the terminal date T, while it has relatively little impact for the rest. This is exactly what the Turnpike theorem

Second, whether using a numerical solver or the shooting algorithm, we solve the same system of equations and hence theoretically the results should be the same. In actual computation, while the shooting algorithm is stable, it tends to accumulate numerical errors toward the terminal date. In this regard, we could say a numerical solver is more accurate. However, practically speaking, the maximum Euler equation error of the shooting algorithm is negligible. We employ both algorithms but it is rather to ensure error-proof results than numerical accuracy. We first run the shooting algorithm and then use its results as the initial guess for a standard non-linear equation solver algorithm.

4.2 Calibration

In order to carry out the numerical simulation, we need to pin down the structural parameters of the model.

The capital income share θ is calibrated following Cooley and Prescott (1995) using income data from ESRI SNA data base over the 1955-2007 period. In specific, we compute the average of

$$\theta_t = \frac{OS + DEP}{Y - (MI + IBT - SUB)}$$

where OS, DEP, MI, IBT and SUB stand for operating surplus, capital depreciation, mixed income, indirect business tax and subsidies respectively.

Then we select the capital depreciation rate δ, subjected discount factor β and the preference weight Ψ to match capital stock, consumption and hours worked in the final period to their data levels given the data of all exogenous variables.[6] The parameter values are listed in Table 7. The parameter levels are consistent with literature.

Table 7. Parameter Values

θ	Capital Income Share	0.404
β	Subjective Discount Factor	0.956
δ	Capital Depreciation Rate	0.088
Ψ	Preference Weight	0.494

suggest. Therefore, this sort of solution method best fits to the models that study the transition from one steady state to the other.

[6] The production function and resource constraint guarantee that output and investment are also equal to the data level in the final period.

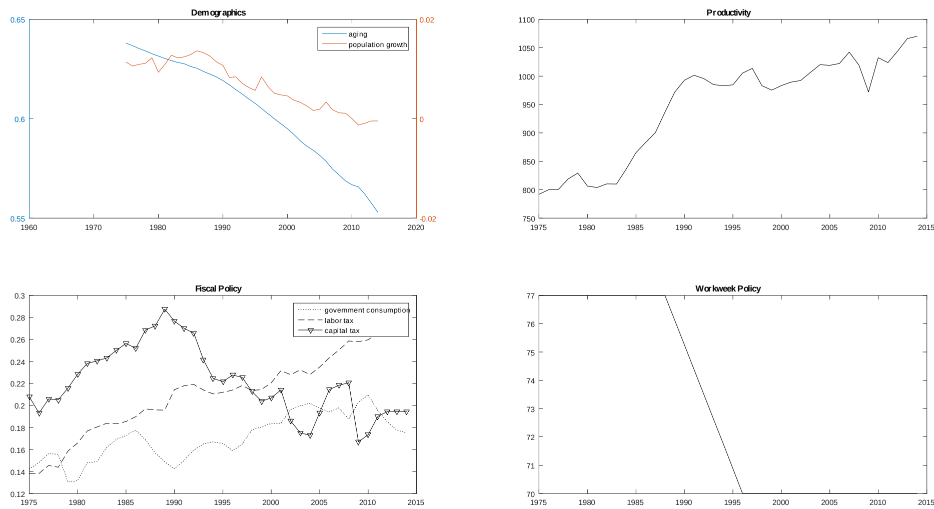

Figure 4: Exogenous Variables

4.3 Benchmark Simulations

We first run a simulation with the benchmark model which incorporates the effects of the changes in all exogenous variables: population aging, population growth, productivity growth, labor income tax, capital income tax, government expenditure and workweek reduction. Figure 4 shows all the exogenous variables we use over the 1975-2014 period.[7]

Figure 5 presents the benchmark results. The simulated output replicates the data well. However, the simulated output level is lower during the earlier years. This is due to both the under-prediction of hours worked and investment. Finally, simulated consumption is extremely close to the data.

4.4 Counterfactual Analysis

Next we run simulations with counterfactual models turning off the fluctuation of one exogenous variable at a time. For each simulation we reset the terminal condition and set the final period capital as that implied by each

[7]We do not have data for capital and labor income tax for 2013 and 2014 so we assume that they stay constant at their 2012 level.

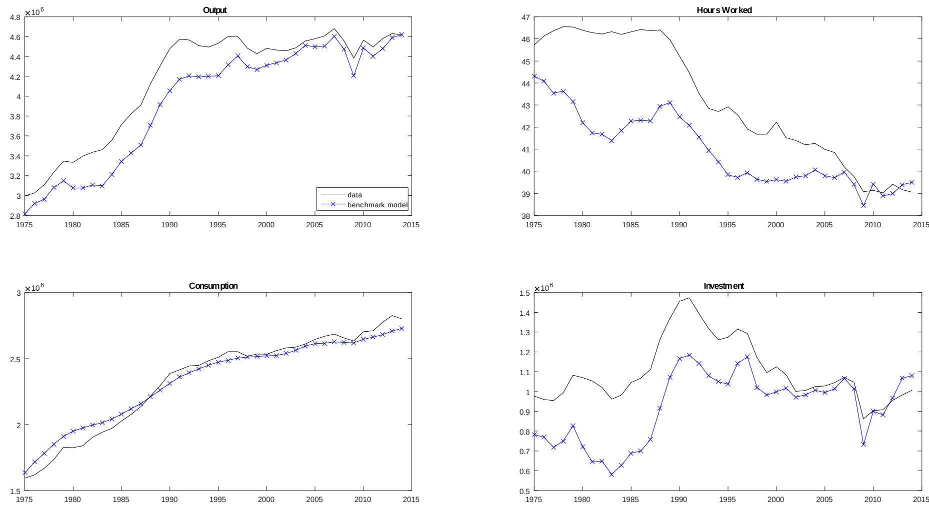

Figure 5: Simulated Variables: Benchmark

counterfactual model. The difference between the benchmark and the counterfactual simulation represents the effect of the selected exogenous variable.

Figure 6 presents the result from simulations from the model shutting down demographic effects. The model without population aging has higher output than the benchmark model implying that population aging reduced output due to the decline in employment. Interestingly, population aging has a positive effect on hours worked per worker as the firms want substitute employment by hours. However, overall total hours worked declines. This reduces the marginal product of capital and hence investment. The reduction of labor and capital stock results in lower output. Consequently consumption is lower than the benchmark.

Population growth in our model works like capital depreciation rate as the aggregate capital stock has to be spread out among more families each period, which is known as the capital dilution effect. The model with constant population growth has lower output than the benchmark model implying that the decline in the population growth rate increased output due to a decrease in capital dilution.

Figure 7 shows the simulation results for the model with constant productivity. The dominant force of output change in this model is the transition

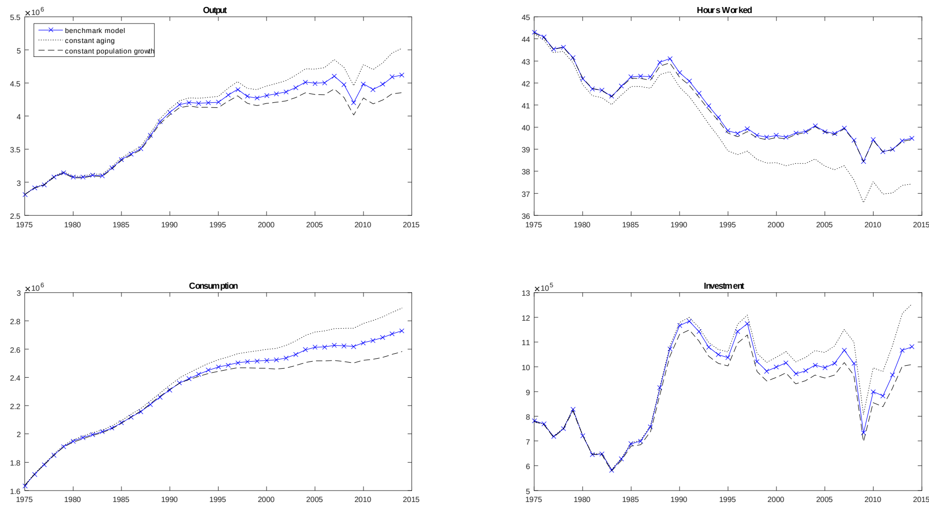

Figure 6: Simulated Variables: No Demographic Transition

from the initial state to the final steady state. Since the initial capital stock k_1 is lower than the terminal capital stock k_{T+1} the marginal product of capital is initially high. This causes rapid growth in initial periods and a slow down as the economy converges to its steady state. Consumption follows the time path of output. Investment is much less volatile than the data which implies that productivity shocks are important in accounting for the fluctuation in investment. Finally, productivity has little effect on the time path of hours worked..

Figure 8 shows the simulation results for the model with constant fiscal policy variables. The model with constant government expenditure has lower output than the benchmark model which implies that the increase in government expenditure increased output due to higher aggregate demand. Consumption and leisure both decrease due to a negative income effect. The resulting increase in hours worked increases the marginal product of capital and leads to higher investment.

The model with constant labor income tax has higher output than the benchmark model which implies that the increase in labor income tax reduced output. The main reason is the depressing effect of labor income tax on hours worked per worker. Consumption and investment are also depressed due to

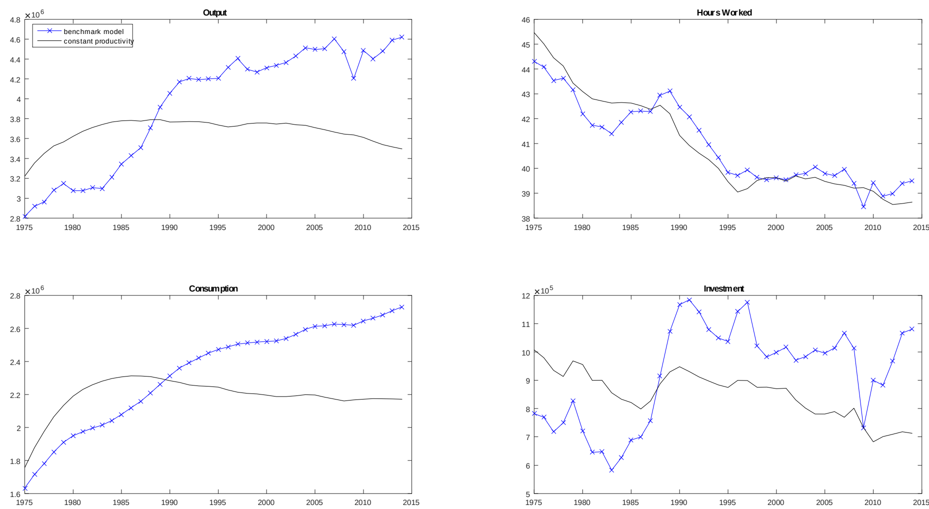

Figure 7: Simulated Variables: Constant Productivity

the reduced income.

The model with constant capital income tax has higher output than the benchmark model which implies that the increase in capital income tax reduced output. The main reason is the depressing effect on capital accumulation. Investment is particularly affected during the 1980s when the tax rate increased dramatically and less so onwards as it declined. The rise in capital income tax also depresses hours worked as the decline in capital accumulation decreases the marginal product of labor. Consumption is depressed due to the reduced income.

Figure 9 shows the simulation result for the model with constant workweek. The results show that if the workweek remained constant output would have been higher than the benchmark model which implies that the reduction in workweek reduced output. In fact, all of the decline in hours worked during the 1990s can be attributed to the workweek reduction policy. Consumption and investment is also depressed by this policy due to the decline in income.

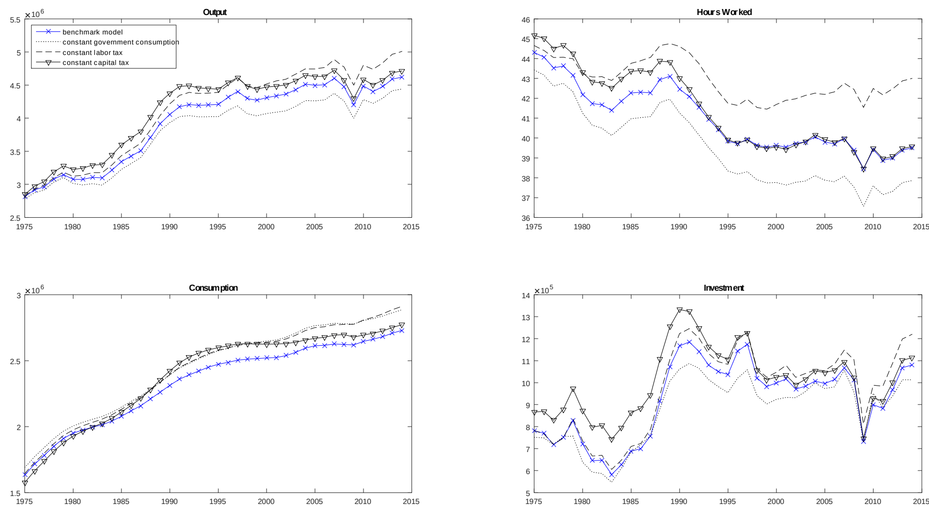

Figure 8: Simulated Variables: Constant Fiscal Policy

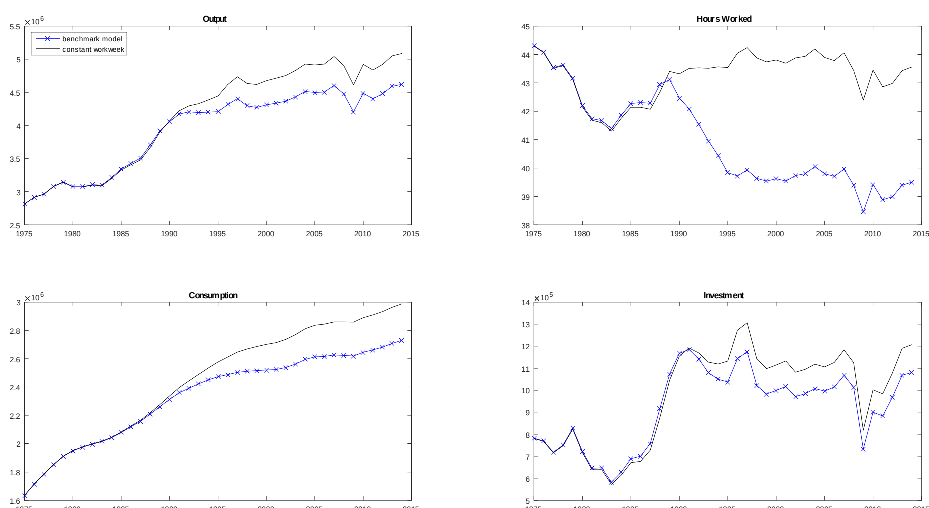

Figure 9: Simulated Variables: Constant Workweek

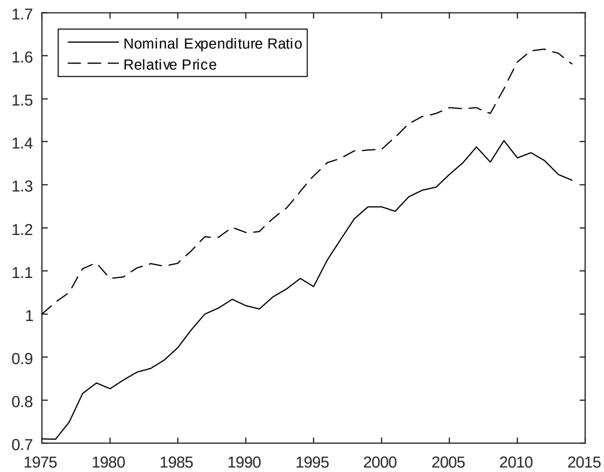

Figure 10: Structural Change Data

5 Population Aging and Structural Transformation

Up to now we have treated the increase in government expenditure as exogenous. In this section, we explicitly model the rise in government expenditure in relationship with population aging and structural transformation.

Figure 10 plots two key observations of structural transformation in Japan over the 1975-2015 period. The first observation is the rise in the service sector relative to the goods sector. The solid line plots the nominal household expenditure on services relative to that on goods. The ratio has been constantly growing until the recent Great Recession. The dashed line plots the price of services relative to that of goods. The relative price of services has been rising, which implies a slower productivity growth rate in the service sector than that in the goods sector.

5.1 Household's Problem

Imagine that consumption consists of consumption expenditure of goods and services. Then, we can modify the definition of consumption of the young and old adults in (1) as follows

$$
\begin{aligned}
c_{y,t} &= \left(\omega_y^{\frac{1}{\varepsilon}} c_{yg,t}^{\frac{\varepsilon-1}{\varepsilon}} + (1-\omega_y)^{\frac{1}{\varepsilon}} c_{ys,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}}, \\
c_{o,t} &= \left(\omega_o^{\frac{1}{\varepsilon}} c_{og,t}^{\frac{\varepsilon-1}{\varepsilon}} + (1-\omega_o)^{\frac{1}{\varepsilon}} c_{os,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}},
\end{aligned}
$$

where $c_{ij,t}$ stands for the consumption of age group i for consumption type j. We assume that the relative demand for services is higher for aged adults compared to that of young adults so that $\omega_y > \omega_o$.

The budget constraint (3) becomes

$$
\begin{aligned}
& \eta_t(c_{yg,t} + (1-s_y)p_t c_{ys,t}) + (1-\eta_t)(c_{og,t} + (1-s_o)p_t c_{os,t}) + i_t \\
= & \ (1-\tau_{l,t})\, w_t h_t e_t + (1-\tau_{k,t})\, r_t k_t + \zeta_t,
\end{aligned}
$$

where everything is denominated at the price of goods and p_t is the price of services relative to goods. We assume that consumption goods and investment goods are identical manufactured goods. We also assume that the government subsidizes the purchase of services by the old. We can consider this as the government payment of health care services.

Household optimality implies

$$
\begin{aligned}
\frac{c_{yg,t}}{c_{ys,t}} &= \frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon}, \\
\frac{c_{og,t}}{c_{os,t}} &= \frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon}, \\
\frac{c_{yg,t}}{c_{og,t}} &= \frac{\omega_y}{\omega_o}\left(\frac{c_{y,t}}{c_{o,t}}\right)^{1-\varepsilon}, \\
\frac{c_{ys,t}}{c_{os,t}} &= \frac{1-\omega_y}{1-\omega_o}\left(\frac{1-s_o}{1-s_y}\right)^{\varepsilon}\left(\frac{c_{y,t}}{c_{o,t}}\right)^{1-\varepsilon}.
\end{aligned}
$$

5.2 Firm's Problem

The production structure also changes such that two different types of products are produced from capital and labor:

$$
\begin{aligned}
y_{g,t} &= A_{g,t} k_{g,t}^{\theta} \left(h_{g,t} e_{g,t}\right)^{1-\theta}, \\
y_{s,t} &= A_{s,t} k_{s,t}^{\theta} \left(h_{s,t} e_{s,t}\right)^{1-\theta}.
\end{aligned}
$$

If we assume that labor and capital are fully mobile across sectors, the firms' optimality conditions are

$$\begin{aligned} w_t &= (1-\theta)\frac{y_{g,t}}{h_{g,t}e_{g,t}} = (1-\theta)\frac{p_t y_{s,t}}{h_{s,t}e_{s,t}}, \\ r_t &= \theta\frac{y_{g,t}}{k_{g,t}} = \theta\frac{p_t y_{s,t}}{k_{s,t}}. \end{aligned}$$

Therefore, we can show that the growth in relative prices are uniquely determined by the relative productivity growths:

$$p_t = \frac{A_{g,t}}{A_{s,t}}.$$

The trend in relative prices shown in Figure 10 implies a higher productivity growth rate in goods relative to that in services.

5.3 Government

The government expenditure consists of the subsidies for consumer services and unrelated expenditures $\widetilde{G_t}$. The government budget constraint can be modified to

$$\begin{aligned} G_t &= S_t + \widetilde{G_t} \\ &= \tau_{l,t} w_t h_t e_t N_t + \tau_{k,t} r_t K_t - \zeta_t N_t. \end{aligned}$$

where $S_t = \eta_t s_y p_t c_{ys,t} + (1-\eta_t) s_o p_t c_{os,t}$ is the total consumer service subsidies.

5.4 Quantitative Exercise

The key variables we consider are the nominal expenditure ratio of the service sector to the goods sector and government expenditure on consumer services. From the model, we can derive the nominal expenditure ratio as

$$\frac{p_t c_{s,t}}{c_{g,t}} = \frac{\left(\eta_t \frac{1-s_o}{1-s_y} \frac{\frac{\omega_o}{1-\omega_o}((1-s_o)p_t)^{\varepsilon-1}+1}{\frac{\omega_y}{1-\omega_y}((1-s_y)p_t)^{\varepsilon-1}+1} + 1 - \eta_t\right)}{\left(\eta_t \frac{1+\frac{1-\omega_o}{\omega_o}((1-s_o)p_t)^{1-\varepsilon}}{1+\frac{1-\omega_y}{\omega_y}((1-s_y)p_t)^{1-\varepsilon}} + 1 - \eta_t\right)} \frac{p_t}{\left(\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon}\right)}.$$

we also derive government expenditure on consumer services relative to total consumption expenditure as

$$\frac{S_t}{C_t} = \eta_t \frac{s_y}{1-s_y} \frac{1}{1+\frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon-1}} + (1-\eta_t)\frac{s_o}{1-s_o}\frac{1}{1+\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon-1}}.$$

We further define the share of government subsidies in total government consumption as

$$\phi_t = \frac{S_t}{C_t} \times \frac{C_t}{G_t}$$

.

We can compute these variables from the model taking the relative price, population aging and the private consumption to government consumption ratio as given. The chosen parameter levels are listed in Table 8. An important parameter is the consumption elasticity parameter ε. Since the price of the services are rising, they must be strong complements to increase relative to goods. We set this parameter to 0.3 to ensure that structural change takes place in our model. We further assume that the old has a greater preference weight on services than the young. Given the elasticity parameter, we set the preference weights for young and old so that the nominal ratio of services to goods roughly match the data. Finally, we set the subsidy rate such that the old is subsidized 25 percent and the young is subsidized 10 percent of their service consumption. This is set so that the government subsidy to total government consumption ratio roughly matches the government health expenditure to total government consumption over the 1980-2000 period.

Table 8. Parameter Values II

ε	Consumption Elasticity	0.3
ω_y	Preference Weight Young	0.55
ω_o	Preference Weight Old	0.2
s_y	Subsidy Rate Young	0.1
s_o	Subsidy Rate Old	0.25

Figure 11 shows the results. We report the nominal share of services to goods consumption and the share of subsidies in total government consumption. The first panel shows that nominal share computed from the model roughly matches the trend in the data. The second panel shows that the resulting subsidies paid by the government is increasing in terms of the share among total government consumption. We further consider a counterfactual

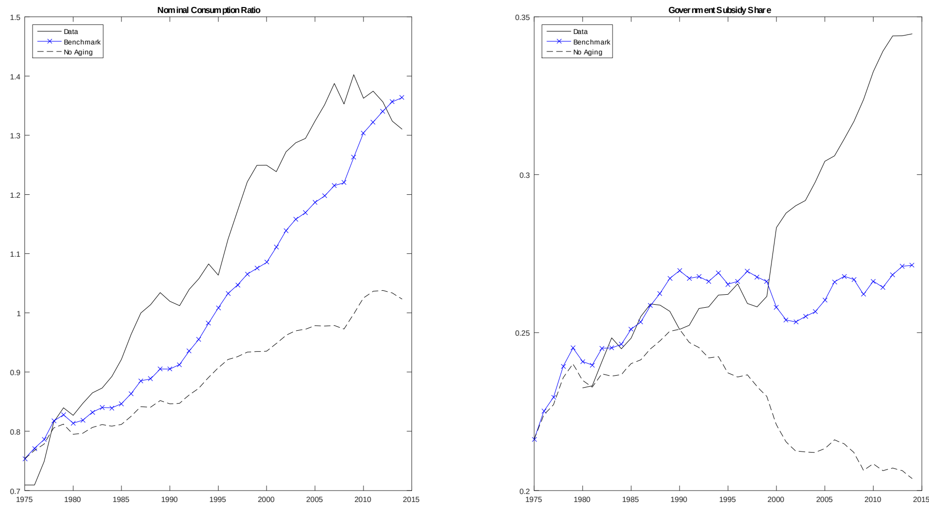

Figure 11: Structural Change Simulation

in which population aging did not take place. The dashed lines in both panels show the variables with no population aging. The results shows that if population aging did not take place, the nominal share of services relative to goods would be more than 30 percent below data in 2014. While we could not replicate the dramatic rise in the share of government health care expenditure among total government consumption after the 2000 in our model, we find that without population aging, the government share of health care expenditure would have been around 20 percent instead of 27 percent in the benchmark simulation.

6 Conclusion

In this paper we constructed a dynamic general equilibrium model to quantitative analyze the impacts of demographics, productivity and government policy on the Japanese economy during the 1975-2014 period. We find that total factor productivity is necessary to account for the growth, population aging, the increase in social security contribution and the workweek shortening policy significantly dampened economic growth. We further show that population aging can account for a significant portion of the structural transformation from goods to services and the increase in government consumption.

In order to simplify the computation, we have made several assumption in the model. First, we assume a representative household with only young and old people. This is as if we assume a overlapping generation model with complete capital markets and full altruistic bequests. It is interesting to investigate whether our results hold in an overlapping generation setting as in Chen, Imrohoroglu and Imrohoroglu (2007) and Braun, Ikeda and Joines (2009). Second, we assumed log utility which led to equal consumption levels across age groups. Allowing more general utility functions can enable us to investigate issues such as inter-generational inequality. Third, we took the employment rates of each group as a constant. The data shows that the employment rate of the young is slightly increasing reflecting the increase in female labor market participation. On the other hand, the employment rate of the old has been declining over time reflecting the increase in the share of the old-old who are incapable to work. Therefore, incorporating the changes in employment rates in each group should increase the output dampening effect of population aging. Finally, we have been silent about the relationship between population aging and productivity growth. One possibility is that the increase in health care expenditures crowded out government resources from projects such as education and innovation which could have led to productivity growth. While these are all interesting extensions, we will leave these for future research as they are beyond the scope of this paper.

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A Detailed Derivation of the Structural Transformation Model

$$\begin{aligned} c_{y,t} &= \left(\omega_y^{\frac{1}{\varepsilon}} c_{yg,t}^{\frac{\varepsilon-1}{\varepsilon}} + (1-\omega_y)^{\frac{1}{\varepsilon}} c_{ys,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}}, \\ c_{o,t} &= \left(\omega_o^{\frac{1}{\varepsilon}} c_{og,t}^{\frac{\varepsilon-1}{\varepsilon}} + (1-\omega_o)^{\frac{1}{\varepsilon}} c_{os,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}}, \end{aligned}$$

The budget constraint (3) becomes

$$\begin{aligned} & \eta_t(c_{yg,t} + p_t(1-s_y)c_{ys,t}) + (1-\eta_t)(c_{og,t} + (1-s_o)p_t c_{os,t}) + i_t \\ = \ & (1-\tau_{l,t})\, w_t h_t e_t + (1-\tau_{k,t})\, r_t k_t + \zeta_t, \end{aligned}$$

where everything is denominated at the price of goods and p_t is the price of services relative to goods. We assume that consumption goods and investment goods are identical goods. Household optimality implies

$$
\begin{aligned}
\Psi \frac{\omega_y^{\frac{1}{\varepsilon}} c_{yg,t}^{-\frac{1}{\varepsilon}}}{\left(\omega_y^{\frac{1}{\varepsilon}} c_{yg,t}^{\frac{\varepsilon-1}{\varepsilon}} + (1-\omega_y)^{\frac{1}{\varepsilon}} c_{ys,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)} &= \Psi \omega_y^{\frac{1}{\varepsilon}} c_{yg,t}^{-\frac{1}{\varepsilon}} c_{y,t}^{\frac{1-\varepsilon}{\varepsilon}} = \lambda_t, \\
\Psi \frac{(1-\omega_y)^{\frac{1}{\varepsilon}} c_{ys,t}^{-\frac{1}{\varepsilon}}}{\left(\omega_y^{\frac{1}{\varepsilon}} c_{yg,t}^{\frac{\varepsilon-1}{\varepsilon}} + (1-\omega_y)^{\frac{1}{\varepsilon}} c_{ys,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)} &= \Psi (1-\omega_y)^{\frac{1}{\varepsilon}} c_{ys,t}^{-\frac{1}{\varepsilon}} c_{y,t}^{\frac{1-\varepsilon}{\varepsilon}} = \lambda_t (1-s_y) p_t, \\
\Psi \frac{\omega_o^{\frac{1}{\varepsilon}} c_{og,t}^{-\frac{1}{\varepsilon}}}{\left(\omega_o^{\frac{1}{\varepsilon}} c_{og,t}^{\frac{\varepsilon-1}{\varepsilon}} + (1-\omega_o)^{\frac{1}{\varepsilon}} c_{os,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)} &= \Psi \omega_o^{\frac{1}{\varepsilon}} c_{og,t}^{-\frac{1}{\varepsilon}} c_{o,t}^{\frac{1-\varepsilon}{\varepsilon}} = \lambda_t, \\
\Psi \frac{(1-\omega_o)^{\frac{1}{\varepsilon}} c_{os,t}^{-\frac{1}{\varepsilon}}}{\left(\omega_o^{\frac{1}{\varepsilon}} c_{og,t}^{\frac{\varepsilon-1}{\varepsilon}} + (1-\omega_o)^{\frac{1}{\varepsilon}} c_{os,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)} &= \Psi (1-\omega_o)^{\frac{1}{\varepsilon}} c_{os,t}^{-\frac{1}{\varepsilon}} c_{o,t}^{\frac{1-\varepsilon}{\varepsilon}} = \lambda_t (1-s_o) p_t,
\end{aligned}
$$

so that

$$
\begin{aligned}
\frac{c_{yg,t}}{c_{ys,t}} &= \frac{\omega_y}{1-\omega_y} \left((1-s_y)p_t\right)^{\varepsilon}, \\
\frac{c_{og,t}}{c_{os,t}} &= \frac{\omega_o}{1-\omega_o} \left((1-s_o)p_t\right)^{\varepsilon}, \\
\frac{c_{yg,t}}{c_{og,t}} &= \frac{\omega_y}{\omega_o} \left(\frac{c_{y,t}}{c_{o,t}}\right)^{1-\varepsilon}, \\
\frac{c_{ys,t}}{c_{os,t}} &= \frac{1-\omega_y}{1-\omega_o} \left(\frac{1-s_o}{1-s_y}\right)^{\varepsilon} \left(\frac{c_{y,t}}{c_{o,t}}\right)^{1-\varepsilon},
\end{aligned}
$$

where

$$\begin{aligned}
c_{y,t} &= \left(\omega_y^{\frac{1}{\varepsilon}} c_{yg,t}^{\frac{\varepsilon-1}{\varepsilon}} + (1-\omega_y)^{\frac{1}{\varepsilon}} c_{ys,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}} \\
&= \left(\frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon-1}+1\right)^{\frac{\varepsilon}{\varepsilon-1}} (1-\omega_y)^{\frac{1}{\varepsilon-1}} c_{ys,t} \\
&= \left(1+\frac{1-\omega_y}{\omega_y}\left((1-s_y)p_t\right)^{1-\varepsilon}\right)^{\frac{\varepsilon}{\varepsilon-1}} \omega_y{}^{\frac{1}{\varepsilon-1}} c_{yg,t}, \\
c_{o,t} &= \left(\omega_o^{\frac{1}{\varepsilon}} c_{og,t}^{\frac{\varepsilon-1}{\varepsilon}} + (1-\omega_o)^{\frac{1}{\varepsilon}} c_{os,t}^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}} \\
&= \left(\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon-1}+1\right)^{\frac{\varepsilon}{\varepsilon-1}} (1-\omega_o)^{\frac{1}{\varepsilon-1}} c_{os,t} \\
&= \left(1+\frac{1-\omega_o}{\omega_o}\left((1-s_o)p_t\right)^{1-\varepsilon}\right)^{\frac{\varepsilon}{\varepsilon-1}} \omega_o{}^{\frac{1}{\varepsilon-1}} c_{og,t}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{c_{y,t}}{c_{o,t}} &= \frac{\left(1+\frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon-1}\right)^{\frac{\varepsilon}{\varepsilon-1}} (1-\omega_y)^{\frac{1}{\varepsilon-1}}}{\left(1+\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon-1}\right)^{\frac{\varepsilon}{\varepsilon-1}} (1-\omega_o)^{\frac{1}{\varepsilon-1}}} \frac{c_{ys,t}}{c_{os,t}}, \\
&= \frac{\left(1+\frac{1-\omega_y}{\omega_y}\left((1-s_y)p_t\right)^{1-\varepsilon}\right)^{\frac{\varepsilon}{\varepsilon-1}} \omega_y{}^{\frac{1}{\varepsilon-1}}}{\left(1+\frac{1-\omega_o}{\omega_o}\left((1-s_o)p_t\right)^{1-\varepsilon}\right)^{\frac{\varepsilon}{\varepsilon-1}} \omega_o{}^{\frac{1}{\varepsilon-1}}} \frac{c_{yg,t}}{c_{og,t}}. \\
\frac{c_{yg,t}}{c_{og,t}} &= \frac{\omega_y}{\omega_o}\left(\frac{\left(1+\frac{1-\omega_y}{\omega_y}\left((1-s_y)p_t\right)^{1-\varepsilon}\right)^{\frac{\varepsilon}{\varepsilon-1}} \omega_y{}^{\frac{1}{\varepsilon-1}}}{\left(1+\frac{1-\omega_o}{\omega_o}\left((1-s_o)p_t\right)^{1-\varepsilon}\right)^{\frac{\varepsilon}{\varepsilon-1}} \omega_o{}^{\frac{1}{\varepsilon-1}}} \frac{c_{yg,t}}{c_{og,t}}\right)^{1-\varepsilon} \\
&= \frac{1+\frac{1-\omega_o}{\omega_o}\left((1-s_o)p_t\right)^{1-\varepsilon}}{1+\frac{1-\omega_y}{\omega_y}\left((1-s_y)p_t\right)^{1-\varepsilon}} \\
\frac{c_{ys,t}}{c_{os,t}} &= \frac{1-\omega_y}{1-\omega_o}\left(\frac{1-s_o}{1-s_y}\right)^{\varepsilon}\left(\frac{\left(1+\frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon-1}\right)^{\frac{\varepsilon}{\varepsilon-1}} (1-\omega_y)^{\frac{1}{\varepsilon-1}}}{\left(1+\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon-1}\right)^{\frac{\varepsilon}{\varepsilon-1}} (1-\omega_o)^{\frac{1}{\varepsilon-1}}} \frac{c_{ys,t}}{c_{os,t}}\right)^{1-\varepsilon}, \\
&= \frac{1-s_o}{1-s_y}\frac{1+\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon-1}}{1+\frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon-1}}.
\end{aligned}$$

Then the consumption expenditure of the household for each type of product is

$$
\begin{aligned}
c_{s,t} &= \eta_t(1-s_y)c_{ys,t} + (1-\eta_t)(1-s_o)c_{os,t}, \\
&= \left(\eta_t \frac{1+\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon-1}}{1+\frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon-1}} + 1 - \eta_t\right)(1-s_o)c_{os,t}, \\
c_{g,t} &= \eta_t c_{yg,t} + (1-\eta_t)c_{og,t}, \\
&= \left(\eta_t \frac{1+\frac{1-\omega_o}{\omega_o}\left((1-s_o)p_t\right)^{1-\varepsilon}}{1+\frac{1-\omega_y}{\omega_y}\left((1-s_y)p_t\right)^{1-\varepsilon}} + 1 - \eta_t\right)c_{og,t},
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{p_t c_{s,t}}{c_{g,t}} &= \frac{\left(\eta_t \frac{1-s_o}{1-s_y}\frac{1+\frac{\omega_o}{1-\omega_o}((1-s_o)p_t)^{\varepsilon-1}}{1+\frac{\omega_y}{1-\omega_y}((1-s_y)p_t)^{\varepsilon-1}} + 1 - \eta_t\right)}{\left(\eta_t \frac{1+\frac{1-\omega_o}{\omega_o}((1-s_o)p_t)^{1-\varepsilon}}{1+\frac{1-\omega_y}{\omega_y}((1-s_y)p_t)^{1-\varepsilon}} + 1 - \eta_t\right)} \frac{(1-s_o)p_t c_{os,t}}{c_{og,t}}, \\
&= \frac{\left(\eta_t \frac{1-s_o}{1-s_y}\frac{1+\frac{\omega_o}{1-\omega_o}((1-s_o)p_t)^{\varepsilon-1}}{1+\frac{\omega_y}{1-\omega_y}((1-s_y)p_t)^{\varepsilon-1}} + 1 - \eta_t\right)}{\left(\eta_t \frac{1+\frac{1-\omega_o}{\omega_o}((1-s_o)p_t)^{1-\varepsilon}}{1+\frac{1-\omega_y}{\omega_y}((1-s_y)p_t)^{1-\varepsilon}} + 1 - \eta_t\right)} \frac{(1-s_o)p_t}{\left(\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon}\right)}.
\end{aligned}
$$

Next, total consumption expenditure is

$$
\begin{aligned}
C_t &= \eta_t(c_{yg,t} + (1-s_y)p_t c_{ys,t}) + (1-\eta_t)(c_{og,t} + (1-s_o)p_t c_{os,t}) \\
&= .\eta_t\left(\frac{c_{yg,t}}{c_{ys,t}} + (1-s_y)p_t\right)\frac{c_{ys,t}}{c_{os,t}}c_{os,t} + (1-\eta_t)\left(\frac{c_{og,t}}{c_{os,t}} + (1-s_o)p_t\right)c_{os,t} \\
&= \eta_t\left(\frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon} + p_t(1-s_y)\right)\frac{1-s_o}{1-s_y}\frac{1+\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon-1}}{1+\frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon-1}}c_{os,t} \\
&\quad +(1-\eta_t)\left(\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon} + (1-s_o)p_t\right)c_{os,t} \\
&= \eta_t\left(1+\frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon-1}\right)(1-s_y)p_t\frac{1-s_o}{1-s_y}\frac{1+\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon-1}}{1+\frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon-1}}c_{os,t} \\
&\quad +(1-\eta_t)\left(1+\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon-1}\right)(1-s_o)p_t c_{os,t} \\
&= \left(1+\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon-1}\right)(1-s_o)p_t c_{os,t}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{S_t}{C_t} &= \frac{\eta_t s_y p_t c_{ys,t} + (1-\eta_t) s_o p_t c_{os,t}}{\left(1 + \frac{\omega_o}{1-\omega_o} \left((1-s_o)p_t\right)^{\varepsilon-1}\right)(1-s_o) p_t c_{os,t}} \\
&= \frac{\left(\eta_t s_y \frac{c_{ys,t}}{c_{os,t}} + (1-\eta_t) s_o\right) p_t c_{os,t}}{\left(1 + \frac{\omega_o}{1-\omega_o} \left((1-s_o)p_t\right)^{\varepsilon-1}\right)(1-s_o) p_t c_{os,t}} \\
&= \frac{\left(\eta_t s_y \frac{1-s_o}{1-s_y} \frac{1+\frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon-1}}{1+\frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon-1}} + (1-\eta_t) s_o\right)}{\left(1 + \frac{\omega_o}{1-\omega_o} \left((1-s_o)p_t\right)^{\varepsilon-1}\right)(1-s_o)} \\
&= \eta_t \frac{s_y}{1-s_y} \frac{1}{1 + \frac{\omega_y}{1-\omega_y}\left((1-s_y)p_t\right)^{\varepsilon-1}} + (1-\eta_t) \frac{s_o}{1-s_o} \frac{1}{1 + \frac{\omega_o}{1-\omega_o}\left((1-s_o)p_t\right)^{\varepsilon-1}}
\end{aligned}
$$