Affine Term Structure Pricing with Bond Supply As Factors

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What This Paper Does: Extend Greenwood and Vayanos (2014)

- Greenwood and Vayanos (2014), "Bond Supply and Excess Bond Returns", *RFS*.
  - ATSM, Vayanos-Villa (2009).
  - The maturity structure $(s_t^{(1)}, \ldots, s_t^{(N)})$ is drive by a *single* factor.
- This paper:
  - The maturity structure unrestricted VAR.
  - IR of the yield curve to "local" shock to the maturity strucure. take a peek
- thus providing a modern formulation of Tobin (1969)'s **portfolio balance channel**.

The Portfolio Balance Channel

- Bernanke about LSAPII (his August 2010 Jackson Hole speech)

  "*I see the evidence as most favorable to the view that such purchases work primarily through the so-called* **portfolio balance channel**... *Specifically, the Fed's strategy [the operation twist] relies on the presumption that different financial assets are not perfect substitutes in investors' portfolios, so that changes in the net supply of an asset available to investors affect its yield and those of broadly similar assets.*"

- BOJ's Announcement about QQE (April 4, 2013). Has part a)-d). b) is

  b) An increase in JGB purchases and their maturity extension

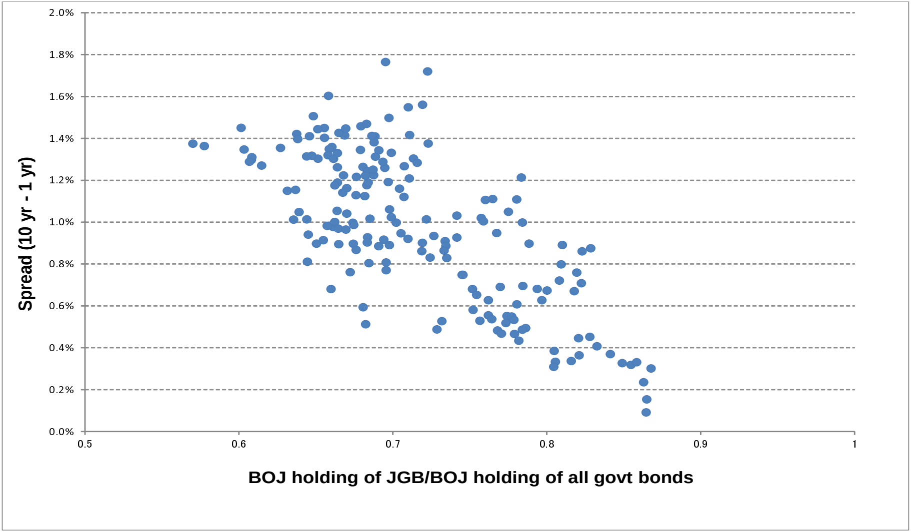
  "*With a view to encouraging a further decline in interest rates across the yield curve, the Bank will purchase JGBs... In addition, ... the average remaining maturity of the Bank's JGB purchases will be extended....*"

FYI: a) base targeting, c) ETF and J-REIT, d) Inflation exit condition is 2%.

The Irrelevance Theorem?

- A casual look at recent Japanese JGB yields.
- term premium $\equiv$ yield $-$ risk-neutral component (average of current and future short rates), i.e.,
  yield $=$ risk-neutral component $+$ term premium.
- Evidence in favor of the portfolio balance effect:
  - Gagnon, Raskin, Remache, and Sack (*Int'l J. of Central Banking*, March 2011): the Fed's LSAP lowered the term premia.
  - Joyce, Lasaosa, Stevens, and Tong (*IJCB*, September 2011): same for the U.K.

Effect of Maturity Structure on Spread: Apr 2001 - March 2016



Rest of Talk

- What is ATSM?
- Greenwood-Vayanos (2014)
- the extension
- IR

ATSM Default-Free Bond Pricing (well known)

- Notation:

$$P_t^{(n)} = \text{price of } n\text{-period bonds at } t, \quad y_t^{(n)} = \text{yield on } n\text{-period bonds} = -\frac{1}{n}\log P_t^{(n)}$$

- The model:

$$\text{(factor dynamics)} \quad \mathbf{f}_{t+1} = \mathbf{c} + \mathbf{\Phi}\mathbf{f}_t + \boldsymbol{\varepsilon}_{t+1}, \quad \boldsymbol{\varepsilon}_{t+1} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Omega}), \tag{1}$$

$$\text{(short rate equation)} \quad y_t^{(1)} = -\delta_0 - \boldsymbol{\delta}_1'\mathbf{f}_t, \quad \text{i.e.,} \quad P_t^{(1)} = \exp\left(\delta_0 + \boldsymbol{\delta}_1'\mathbf{f}_t\right), \tag{2}$$

$$\text{(no-arb condition)} \quad P_t^{(n)} = \mathrm{E}_t\left(M_{t+1} \cdot P_{t+1}^{(n-1)}\right), \quad \mathrm{E}_t(\cdot) \equiv \mathrm{E}(\cdot \mid \mathbf{f}_t).$$

$$\text{(pricing kernel/SDF)} \quad -\log(M_{t+1}) = y_t^{(1)} + \frac{1}{2}\boldsymbol{\lambda}_t'\mathbf{\Omega}\boldsymbol{\lambda}_t + \boldsymbol{\lambda}_t'\boldsymbol{\varepsilon}_{t+1}, \quad \boldsymbol{\lambda}_t \equiv \lambda_0 + \mathbf{\Lambda}_1\mathbf{f}_t,$$

- Can show: bond prices are **exponentially affine**, $P_t^{(n)} = \exp\left(\bar{a}_n + \bar{\mathbf{b}}_n'\mathbf{f}_t\right)$. The **recursion**:

$$\begin{cases} \bar{\mathbf{b}}_1 = \boldsymbol{\delta}_1, \ \bar{\mathbf{b}}_n' = \bar{\mathbf{b}}_{n-1}'\mathbf{\Phi}^{\mathbb{Q}} + \bar{\mathbf{b}}_1' \quad (n = 2, 3, ..., N), \\ \bar{a}_1 = \delta_0, \ \bar{a}_n = \bar{a}_{n-1} + \bar{\mathbf{b}}_{n-1}'\mathbf{c}^{\mathbb{Q}} + \frac{1}{2}\bar{\mathbf{b}}_{n-1}'\mathbf{\Omega}\bar{\mathbf{b}}_{n-1} + \bar{a}_1 \quad (n = 2, 3, ..., N), \end{cases} \tag{3}$$

with $\mathbf{\Phi}^{\mathbb{Q}} \equiv \mathbf{\Phi} - \mathbf{\Omega}\mathbf{\Lambda}_1$, $\mathbf{c}^{\mathbb{Q}} \equiv \mathbf{c} - \mathbf{\Omega}\boldsymbol{\lambda}_0$.

- So $y_t^{(n)} = -\frac{1}{n}\bar{a}_n - \frac{1}{n}\bar{\mathbf{b}}_n'\mathbf{f}_t$.

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Greenwood and Vayanos (2014): The Model

- Factor dynamics ((1) above): $\mathbf{f}_t = (f_{1t}, f_{2t})'$ in $\mathbf{f}_{t+1} = \mathbf{c} + \mathbf{\Phi}\mathbf{f}_t + \boldsymbol{\varepsilon}_{t+1}$. f_{2t} is the global shock to the maturity structure:

$$s_t^{(n)} = \xi_n + \theta_n f_{2t} \quad (n = 1, 2, ..., N) \;\text{ wtih }\; \sum_{n=1}^{N} s_t^{(n)} = 1.$$

- Short rate equation ((2) above): $f_{1t} = y_t^{(1)}$ (i.e., $\delta_0 = 0$, $\boldsymbol{\delta}_1 = (1, 0)'$ in $y_t^{(1)} = -\delta_0 - \boldsymbol{\delta}_1' \mathbf{f}_t$).

- Replace the no-arb condition by an explicit model of government bond market.
  - Arbitrageuers and the gov't (CB and Treasury).
  - The arbs' decision problem:

$$\max_{\{z_t^{(n)}\}_{n=1}^{N}} \left[\mathrm{E}_t(R_{t+1}) - \frac{\gamma}{2} \mathrm{Var}_t(R_{t+1}) \right] \qquad \text{subject to} \qquad \sum_{t=1}^{N} z_t^{(n)} = 1,$$

where

$$R_{t+1} \equiv \sum_{n=1}^{N} \frac{P_{t+1}^{(n-1)}}{P_t^{(n)}} z_t^{(n)}.$$

  - Bond market equilibrium: $s_t^{(n)} = z_t^{(n)}$, $n = 2, 3, ..., N$.

Greenwood and Vayanos (2014): The Recursion

- (ATSM recursion) $P_t^{(n)} = \exp\left(\bar{a}_n + \bar{\mathbf{b}}_n' \mathbf{f}_t\right)$. The **recursion**:

$$\begin{cases} \bar{\mathbf{b}}_1 = \delta_1, \ \bar{\mathbf{b}}_n' = \bar{\mathbf{b}}_{n-1}' \boldsymbol{\Phi}^{\mathbb{Q}} + \bar{\mathbf{b}}_1' \quad (n = 2, 3, ..., N), \\ \bar{a}_1 = \delta_0, \ \bar{a}_n = \bar{a}_{n-1} + \bar{\mathbf{b}}_{n-1}' \mathbf{c}^{\mathbb{Q}} + \frac{1}{2} \bar{\mathbf{b}}_{n-1}' \boldsymbol{\Omega} \bar{\mathbf{b}}_{n-1} + \bar{a}_1 \quad (n = 2, 3, ..., N), \end{cases} \tag{3}$$

  with $\boldsymbol{\Phi}^{\mathbb{Q}} \equiv \boldsymbol{\Phi} - \boldsymbol{\Omega}\boldsymbol{\Lambda}_1$, $\mathbf{c}^{\mathbb{Q}} \equiv \mathbf{c} - \boldsymbol{\Omega}\boldsymbol{\lambda}_0$.

- Can show: bond prices are exponentially affine. (3) with

$$\boldsymbol{\Phi}^{\mathbb{Q}} \equiv \underset{(2\times 2)}{\boldsymbol{\Phi}} - \underset{(2\times 2)}{\boldsymbol{\Omega}} \begin{bmatrix} \underset{(2\times 1)}{\mathbf{0}} & \gamma \underset{(2\times 1)}{\widetilde{\widetilde{\mathbf{b}}}} \end{bmatrix}, \quad \mathbf{c}^{\mathbb{Q}} \equiv \underset{(2\times 1)}{\mathbf{c}} - \underset{(2\times 1)}{\boldsymbol{\Omega}} \left(\gamma \underset{(2\times 1)}{\widetilde{\mathbf{b}}}\right),$$

$$\underset{(2\times 1)}{\widetilde{\mathbf{b}}} \equiv \bar{\mathbf{b}}_1 \xi_2 + \cdots + \bar{\mathbf{b}}_{N-1} \xi_N, \quad \underset{(2\times 1)}{\widetilde{\widetilde{\mathbf{b}}}} \equiv \bar{\mathbf{b}}_1 \theta_2 + \cdots + \bar{\mathbf{b}}_{N-1} \theta_N.$$

- No longer a recursion.

Sketch of Proof

- Under the conjecture of exponentially affine bond prices ($P_t^{(n)} = \exp\left(\bar{a}_n + \bar{\mathbf{b}}_n' \mathbf{f}_t\right)$),
  (a) derive arbs' FOCs under the conjecture, (b) impose $z_t^{(n)} = s_t^{(n)}$ $(n = 2, 3, ..., N)$.
- Part (a):

$$\underbrace{\underbrace{\mathrm{E}_t\left[\frac{P_{t+1}^{(n-1)}}{P_t^{(n)}}\right]}_{\text{gross holding-period return}} - \underbrace{\frac{1}{P_t^{(1)}}}_{\text{gross short-term interest rate}}}_{\text{risk premium on } n\text{-period bonds}} = \gamma \frac{1}{2} \frac{\partial \operatorname{Var}_t(R_{t+1})}{\partial z_t^{(n)}} \quad (n = 2, 3, ..., N).$$

$$\begin{aligned} \mathrm{E}_t\left[\frac{P_{t+1}^{(n-1)}}{P_t^{(n)}}\right] - \frac{1}{P_t^{(1)}} &\approx \mathrm{E}_t\left(\log P_{t+1}^{(n-1)}\right) - \log P_t^{(n)} + \frac{1}{2}\bar{\mathbf{b}}_{n-1}' \boldsymbol{\Omega} \bar{\mathbf{b}}_{n-1} - y_t^{(1)} \\ &= \bar{a}_{n-1} + \bar{\mathbf{b}}_{n-1}'(\mathbf{c} + \boldsymbol{\Phi}\mathbf{f}_t) - \bar{a}_n - \bar{\mathbf{b}}_n'\mathbf{f}_t + \frac{1}{2}\bar{\mathbf{b}}_{n-1}' \boldsymbol{\Omega} \bar{\mathbf{b}}_{n-1} + \bar{a}_1 + \bar{\mathbf{b}}_1'\mathbf{f}_t, \end{aligned}$$

$$\frac{1}{2} \frac{\partial \operatorname{Var}_t(R_{t+1})}{\partial z_t^{(n)}} \approx \bar{\mathbf{b}}_{n-1}' \boldsymbol{\Omega}\left(\bar{\mathbf{b}}_1 z_t^{(2)} + \cdots + \bar{\mathbf{b}}_{N-1} z_t^{(N)}\right).$$

- Part (b):
  (i) Set $z_t^{(n)} = s_t^{(n)}$ $(n = 2, 3, ..., N)$.
  (ii) Use $s_t^{(n)} = \xi_n + \theta_n f_{2t}$.

A Detour

- Reproducing,

$$\mathrm{E}_t\left[\frac{P_{t+1}^{(n-1)}}{P_t^{(n)}}\right] - \frac{1}{P_t^{(1)}} \approx \mathrm{E}_t\left(\log P_{t+1}^{(n-1)}\right) - \log P_t^{(n)} + \frac{1}{2}\overline{\mathbf{b}}'_{n-1}\boldsymbol{\Omega}\overline{\mathbf{b}}_{n-1} - y_t^{(1)} \equiv \mathrm{rp}_t^{(n)}.$$

- Routine to show:

$$y_t^{(n)} = \underbrace{\frac{1}{n}\sum_{i=0}^{n-1}\mathrm{E}_t\left(y_{t+i}^{(1)}\right)}_{\text{risk-neutral component}} + \underbrace{\frac{1}{n}\sum_{i=0}^{n-1}\mathrm{E}_t\left(\mathrm{rp}_{t+i}^{(n-i)}\right) + \frac{1}{n}A_n}_{\text{term premium}} \quad (n = 1, 2, ..., N),$$

where $A_n \equiv -\frac{1}{2}\sum_{i=0}^{n-1}\overline{\mathbf{b}}'_{n-i-1}\boldsymbol{\Omega}\overline{\mathbf{b}}_{n-i-1}$.

◂ IR

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The Model: Allowing for Local Supply Shocks

- $\underset{(N\times 1)}{\mathbf{f}_t} = \left(y_t^{(1)}, s_t^{(2)}, s_t^{(3)}, ..., s_t^{(N)}\right)'.$
- Go back to (b-i). Key observation:

$$\frac{1}{2}\frac{\partial \operatorname{Var}_t(R_{t+1})}{\partial z_t^{(n)}} = \underset{(1\times N)}{\overline{\mathbf{b}}'_{n-1}} \underset{(N\times N)}{\mathbf{\Omega}} \underbrace{\left(\overline{\mathbf{b}}_1 s_t^{(2)} + \cdots + \overline{\mathbf{b}}_{N-1} s_t^{(N)}\right)}_{(N\times 1)} \qquad \text{(by replacing } z_t^{(i)} \text{ by } s_t^{(i)}\text{)}$$

$$= \underset{(1\times N)}{\overline{\mathbf{b}}'_{n-1}} \underset{(N\times N)}{\mathbf{\Omega}} \underbrace{\begin{bmatrix} \overline{\mathbf{b}}_1 & \overline{\mathbf{b}}_2 & \cdots & \overline{\mathbf{b}}_{N-1} \end{bmatrix}}_{(N\times(N-1))} \underbrace{\begin{bmatrix} s_t^{(2)} \\ s_t^{(3)} \\ \vdots \\ s_t^{(N)} \end{bmatrix}}_{((N-1)\times 1)}$$

$$= \underset{(1\times N)}{\overline{\mathbf{b}}'_{n-1}} \underset{(N\times N)}{\mathbf{\Omega}} \underbrace{\begin{bmatrix} \overline{\mathbf{b}}_1 & \overline{\mathbf{b}}_2 & \cdots & \overline{\mathbf{b}}_{N-1} \end{bmatrix}}_{(N\times(N-1))} \underset{((N-1)\times N)}{\mathbf{S}} \underset{(N\times 1)}{\mathbf{f}_t}.$$

- We obtain (3) with

$$\underset{(N\times 1)}{\mathbf{c}^{\mathbb{Q}}} \equiv \mathbf{c}, \quad \underset{(N\times N)}{\mathbf{\Phi}^{\mathbb{Q}}} \equiv \mathbf{\Phi} - \underset{(N\times N)}{\mathbf{\Omega}} \underbrace{\begin{bmatrix} \gamma \underset{(N\times 1)}{\overline{\mathbf{b}}_1} & \gamma \underset{(N\times 1)}{\overline{\mathbf{b}}_2} & \cdots & \gamma \underset{(N\times 1)}{\overline{\mathbf{b}}_{N-1}} \end{bmatrix}}_{(N\times(N-1))} \underset{((N-1)\times N)}{\mathbf{S}}.$$

The QVE

- To reproduce the recursion for $\overline{\mathbf{b}}$:

$$\overline{\mathbf{b}}_1 = \boldsymbol{\delta}_1,\ \overline{\mathbf{b}}_n' = \overline{\mathbf{b}}_{n-1}' \boldsymbol{\Phi}^{\mathbb{Q}} + \overline{\mathbf{b}}_1' \quad (n = 2, 3, ..., N),$$

$$\underset{(N\times N)}{\boldsymbol{\Phi}^{\mathbb{Q}}} \equiv \boldsymbol{\Phi} - \underset{(N\times N)}{\boldsymbol{\Omega}} \underbrace{\begin{bmatrix} \gamma \underset{(N\times 1)}{\overline{\mathbf{b}}_1} & \gamma \underset{(N\times 1)}{\overline{\mathbf{b}}_2} & \cdots & \gamma \underset{(N\times 1)}{\overline{\mathbf{b}}_{N-1}} \end{bmatrix}}_{(N\times(N-1))} \underset{((N-1)\times N)}{\mathbf{S}}.$$

- Write this as a **quadratic vector equation (QVE)**

$$\underset{(N^2\times N^2)}{\mathbf{M}} \underset{(N^2\times 1)}{\overline{\mathbf{b}}} = \underset{(N^2\times 1)}{\mathbf{d}} - \gamma \underset{(N^2\times 1)}{\mathbf{g}(\overline{\mathbf{b}})},$$

where

$$\underset{(N^2\times 1)}{\overline{\mathbf{b}}} \equiv (\overline{\mathbf{b}}_1, ..., \overline{\mathbf{b}}_N) \text{ stacked}, \quad \underset{(N^2\times 1)}{\mathbf{d}} \equiv \underset{(N\times 1)}{\mathbf{1}} \otimes \underset{(N\times 1)}{\boldsymbol{\delta}_1},$$

The QVE (ctd.)

$$\underset{(N^2\times N^2)}{\mathbf{M}} \equiv \begin{bmatrix} \mathbf{I}_N & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ -\underset{(N\times N)}{\boldsymbol{\Phi}'} & \mathbf{I}_N & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & -\underset{(N\times N)}{\boldsymbol{\Phi}'} & \mathbf{I}_N & \cdots & \mathbf{0} \\ \vdots & & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & -\underset{(N\times N)}{\boldsymbol{\Phi}'} & \mathbf{I}_N \end{bmatrix},$$

$$\underset{(N^2\times 1)}{\mathbf{g}(\overline{\mathbf{b}})} \equiv \begin{bmatrix} \underset{(N\times 1)}{\mathbf{0}} \\ \underbrace{\underset{(N\times(N-1))}{\mathbf{S}'}\ \underset{((N-1)\times N)}{\overline{\mathbf{B}}'}\ \underset{(N\times N)}{\boldsymbol{\Omega}}\ \underset{(N\times 1)}{\overline{\mathbf{b}}_1}}_{(N\times 1)} \\ \vdots \\ \underbrace{\underset{(N\times(N-1))}{\mathbf{S}'}\ \underset{((N-1)\times N)}{\overline{\mathbf{B}}'}\ \underset{(N\times N)}{\boldsymbol{\Omega}}\ \underset{(N\times 1)}{\overline{\mathbf{b}}_{N-1}}}_{(N\times 1)} \end{bmatrix} = \underset{(N^2\times(N-1)^2)}{\mathbf{P}}\ \underbrace{\mathrm{vec}\left(\underbrace{\overline{\mathbf{B}}'\,\boldsymbol{\Omega}\,\overline{\mathbf{B}}}_{((N-1)\times(N-1))}\right)}_{((N-1)^2\times 1)},\quad \mathbf{P} \equiv \underbrace{\begin{bmatrix} \underset{(N\times(N-1)^2)}{\mathbf{0}} \\ \underbrace{\mathbf{I}_{N-1}\otimes\mathbf{S}'}_{((N-1)N\times(N-1)^2)} \end{bmatrix}}_{(N^2\times(N-1)^2)},$$

$$\underset{(N\times(N-1))}{\overline{\mathbf{B}}} \equiv \begin{bmatrix} \underset{(N\times 1)}{\overline{\mathbf{b}}_1} & \underset{(N\times 1)}{\overline{\mathbf{b}}_2} & \cdots & \underset{(N\times 1)}{\overline{\mathbf{b}}_{N-1}} \end{bmatrix}.$$

Solution Methods

- Recall the QVE is: $\mathbf{M}\overline{\mathbf{b}} = \mathbf{d} - \gamma \mathbf{g}(\overline{\mathbf{b}})$.
- Case: $\gamma = 0$. A solution exists and it is unique.

$$\overline{\mathbf{b}}^* \equiv \mathbf{M}^{-1}\mathbf{d}.$$

- Case: $\gamma > 0$. If a solution exists, there are generally more than one. Pick one that converges to $\overline{\mathbf{b}}^*$ as $\gamma \downarrow 0$ (as in Greenwood and Vayanos).
- This solution can be calculated by solving an appropriate differential equation (see next slide).
- Another option is the fixed-point algorithm:

$$\overline{\mathbf{b}}^{(k+1)} = \mathbf{M}^{-1}[\mathbf{d} - \gamma \mathbf{g}(\overline{\mathbf{b}}^{(k)})], \quad k = 0, 1, 2, ...$$

No theoretical reason for the fixed point to be the particular solution we picked. In the example below, they are the same.

Solution Methods (ctd.)

- Define **b** implicitly as a function of γ.

$$\mathbf{f}(\overline{\mathbf{b}}, \gamma) = \underset{(N^2 \times 1)}{\mathbf{0}}, \quad \mathbf{f}(\overline{\mathbf{b}}, \gamma) \equiv \mathbf{M}\overline{\mathbf{b}} - \mathbf{d} + \gamma\, \mathbf{g}(\overline{\mathbf{b}}).$$

By the implicit function theorem, there exists an interval U including 0 as an interior point and a vector-valued function of a single variable, $\overline{\mathbf{b}}(.)$: $U \to \mathbb{R}^{N^2}$, such that $\mathbf{f}(\overline{\mathbf{b}}(\widetilde{\gamma}), \widetilde{\gamma}) = \mathbf{0}$ for all $\widetilde{\gamma} \in U$ and its derivative $\overline{\mathbf{b}}'(.)$ is given by

$$\overline{\mathbf{b}}'(\widetilde{\gamma}) = -\left[\frac{\partial \mathbf{f}(\overline{\mathbf{b}}(\widetilde{\gamma}), \widetilde{\gamma})}{\partial \overline{\mathbf{b}}'}\right]^{-1} \frac{\partial \mathbf{f}(\overline{\mathbf{b}}(\widetilde{\gamma}), \widetilde{\gamma})}{\partial \gamma}$$

$$= -\underbrace{\left[\mathbf{M} + \widetilde{\gamma}\, \frac{\partial \mathbf{g}(\overline{\mathbf{b}}(\widetilde{\gamma}))}{\partial \overline{\mathbf{b}}'}\right]^{-1}}_{(N^2 \times N^2)} \underbrace{\mathbf{g}(\overline{\mathbf{b}}(\widetilde{\gamma}))}_{(N^2 \times 1)}.$$

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Specializing the Factor Dynamics of the Model

- $\underset{(N\times 1)}{\mathbf{f}_t} = \left(y_t^{(1)}, s_t^{(2)}, s_t^{(3)}, ..., s_t^{(N)}\right)'$.
- The model is tightly parameterized. The parameters are: γ, N, and the factor dynamics parameters $(\mathbf{c}, \mathbf{\Phi}, \mathbf{\Omega})$.
- The factor dynamics $(\mathbf{f}_{t+1} = \mathbf{c} + \mathbf{\Phi}\mathbf{f}_t + \boldsymbol{\varepsilon}_{t+1})$:

$$\text{(short rate)} \qquad y_t^{(1)} = c_1 + \rho y_{t-1}^{(1)} + \varepsilon_{1t}, \quad \varepsilon_{1t} \sim \mathcal{N}(0, \sigma_1^2),$$

$$\text{(maturity structue)} \qquad s_t^{(n)} = \begin{cases} c_n + \theta s_{t-1}^{(n+1)} + \varepsilon_{nt}, \quad \varepsilon_{nt} \sim \mathcal{N}(0, \sigma_n^2) & \text{if } n = 2, 3, ..., N-1, \\ c_N + \varepsilon_{Nt}, \quad \varepsilon_{Nt} \sim \mathcal{N}(0, \sigma_N^2) & \text{if } n = N, \end{cases}$$

  and $(\varepsilon_{1t},...,\varepsilon_{Nt})$ are uncorrelated.
- The IR function for the VAR factor dynamics: for $n = 2, 3, ..., N$,

$$\underbrace{\frac{\partial \mathbf{f}'_{t+j}}{\partial \varepsilon_{nt}}}_{(1\times N)} = \begin{cases} \left(0, 0, ..., 0, \underset{(n-j)}{\theta^j}, 0, ..., 0\right) & \text{for} \quad j = 0, 1, ..., n-2, \\ \underset{(1\times N)}{\mathbf{0}'} & \text{for} \quad j = n-1, n, ... \end{cases}$$

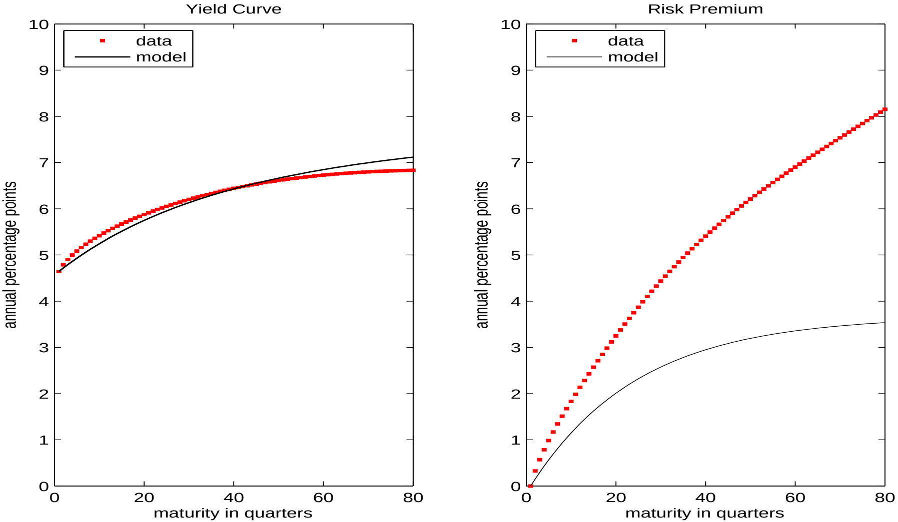
Calibration

- The unit inverval is a quarter. 20 years. So $N = 80$ quarters.
- Set $\gamma = 20$.
- The short rate process estimated on U.S. 3-month T-bill rate. Sample period is the Greenspan period. 1987:Q4-2007:Q4. This pins down (c_1, ρ, σ_1).
  - Zero-coupon yield data from Gurkaynak, Sack, and Wright (2007).
- Pick $c_n = (1-\theta)/N$ $(n = 2, 3, ..., N-1)$, $c_N = 1/N$ so that

$$\text{steady-state value of } \underset{(N\times 1)}{\mathbf{f}_t} = \Big(\frac{c_1}{1-\rho}, \underbrace{\frac{1}{N}, ..., \frac{1}{N}}_{N-1 \text{ elements}} \Big)'.$$

- $\sigma_n = \lambda/N$ $(n = 2, 3, ..., N)$ with $\lambda = 0.01$. Results insensitive to λ.
- $\theta = 0$ or $\theta = 1$.

Average Yield Curve and Risk Premium



Impulse Responses, $\theta = 1$, shock size is 1 percentage point