Does the New Keynesian Model Have a Uniqueness Problem?*

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Abstract

This paper addresses whether non-uniqueness of equilibrium is a substantive problem for policy analysis in New-Keynesian (NK) models. There would be a substantive problem if there were no compelling way to select among different equilibria that give different answers to critical policy questions. In fact there is: stability-under-learning. We focus our analysis on the efficacy of fiscal policy when the economy is in the ZLB. We study a fully non-linear NK model with Calvo-pricing frictions and argue that the model has a unique stable-under-learning rational expectations equilibrium. In that equilibrium,

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the implications of the model for fiscal policy inherit all of the key properties of linearized NK models.

1. Introduction

New Keynesian (NK) models have been enormously influential in terms of their policy implications[1]. The models' implications for fiscal policy are particularly striking when the zero lower bound (ZLB) on the nominal rate of interest is binding.[2] Eggertsson and Woodford (2003) (EW) and Eggertsson (2004) develop an elegant and transparent framework for studying fiscal policy in the NK model at the ZLB.

The key results that emerge from the literature can be summarized as follows[3]. First, when the ZLB binds, the fall in output is potentially very large. Second, the output multiplier associated with government consumption is larger when the ZLB binds than when it does not bind. Third, the more flexible are prices and the longer is the expected duration of the ZLB is longer, the larger is the drop in output and the larger is the government consumption multiplier.

These controversial results are based on literature that uses a linearized version of the NK model, which has a unique solution. In fact, the non-linear NK models have multiple equilibria, even if one restricts attention, as did EW, to minimum state variable ZLB equilibria. As stressed by Mertens and Ravn (2015), policy prescriptions can vary a great deal across those equilibria. At some ZLB equilibria, the government consumption multiplier is small or even negative. In others, it is very large. So, in principle, non-uniqueness of equilibria poses an enormous challenge for

[1]For a classic exposition of the NK model see Woodford (2003.)

[2]It is widely understood that zero is not the critical lower bound. What is critical is that some lower bound on the interest becomes binding on monetary policy.

[3]see, for example, EW, Eggertsson (2011) and Christiano, Eichenbaum and Rebelo (2011) (CER),

policy analysis based on NK models.

This paper addresses a simple question: is non-uniqueness of equilibria a *substantive* problem for policy analysis in NK models? There would be a substantive problem if there were no compelling way to select among different equilibria that give different answers to critical policy questions. To be concrete we focus our analysis on the impact of changes in government consumption when the economy is in the ZLB.

Our argument starts from the presumption that the assumption of rational expectations is *obviously* wrong. But it can be a useful modeling strategy for thinking about a world where the strong assumptions associated with rational expectations aren't literally satisfied.[4] In the spirit of the literature summarized by Evans and Honkapohja (2001), we adopt the following selection criterion for rational expectations equilibria (REE). Suppose agents make a 'small' error in forming expectations about variables relative to their values in a particular REE. Would the economy converge to a REE, if agents form expectations using simple learning rules? If yes, then we say the REE is stable-under-learning, or for short, learnable. From this perspective, stability-under-learning is a necessary condition for an REE and the associated policy implications to be empirically interesting. REE equilibria that aren't learnable are best treated as mathematical curiosities.

We apply this stable-under-learning criterion to a standard fully non-linear NK model with Calvo pricing frictions. Working with this model poses two interesting challenges. First, unlike linearized NK models of the type considered by EW, the

[4] Indeed that is how Lucas viewed it: "... the model describe above 'assumes' that agents know a great deal about the structure of the economy and perform some non-routine computations. It is in order to ask, then: will an economy with agents armed with 'sensible' rules-of-thumb, revising these rules from time to time so as to claim observed rents, tend as time passes to behave as described..." Lucas (1978)

ZLB REE can't be characterized by a set of numbers. Because there is an endogenous state variable (past price dispersion), the ZLB REE is a set of functions. Second, we must think about how agents might learn about these functions.

Our basic results can be summarized as follows. First, consistent with Mertens and Ravn (2015) we find that there are multiple REE, including sunspot equilibria. When we consider fundamental shocks that trigger ZLB episodes, we find two minimum state variable ZLB equilibria. These equilibria converge to different inflation rates if the ZLB episode lasts forever. Second, like Mertens and Ravn (2015), we find that impact of government consumption can be very different in the different ZLB equilibria. For example, there exist both sunspot and minimum state ZLB REE in which the government consumption multiplier is actually negative. Third, we argue that there exists a unique interior ZLB equilibrium in the non-linear Calvo model that is stable-under-learning. Fourth, and most importantly, the controversial predictions of the linearized NK model about fiscal policy in the ZLB, including the large size of the government consumption multiplier at the ZLB are satisfied at the unique learnable ZLB REE. That equilibrium is the one that converges to a relatively low ZLB deflation rate. Based on this analysis we conclude that the Calvo model does not have a substantive uniqueness problem, as least for the analysis of fiscal policy in the ZLB.

Many authors have used non-linear versions of the Rotemberg (1982) model of nominal price rigidities to proxy for the Calvo model. In the Rotemberg model the representative firm faces a quadratic cost of adjusting nominal prices. It is well known that linear approximations to the Calvo and the Rotemberg models give rise to the same set of equations whose solution defines an REE. In contrast, non-linear versions of the model are potentially very different. As it turns out some of the non-linear properties of the Rotemberg model are very sensitive to the details of how one formulates adjustment costs for prices. Specifically, we show that the number of rational expectations ZLB equilibria and their stability properties depend on whether and exactly how one scales adjustment costs for growth. Remarkably, we still always find that there exists a unique ZLB REE that is stable-under-learning. Moreover, all of the predictions of the log-linear NK for the impact of fiscal policy in the ZLB hold at that equilibrium. Indeed, for our benchmark parametrizations, the value of the government consumption multiplier in the linear and non-linear model are remarkably similar.

As a by-product of our analysis, we use our non-linear model to assess the robustness of policy implications about fiscal policy at the ZLB that have been derived using log linear approximations to the NK model. We find that linear approximations work quite well for assessing the size of the government spending multiplier and the drop in GDP that occurs in the ZLB. Evidence that the quality of linear approximations is poor rests on examples where output deviates by more than roughly 20 percent from its steady state, cases where no one would expect linear approximations to work well. There is one interesting difference between the linear and non-linear models. It is well know that for some parameters values, the multiplier in the linear model shoots off to infinity, say as the expected length of the ZLB episode becomes large or prices become very flexible (see for example CER (2011)). For the same parameter values, these extreme results manifest themselves in a different way in the non-linear Calvo model: a ZLB REE simply ceases to exist.

The Great Recession was a very unusual event. So the learning equilibrium underlying our stability calculations are of interest as a way of modeling how agents behaved in the wake of a shock that pushes the economy into a prolonged ZLB episode. So we analyze the impact of an increase in government consumption along the learning equilibrium that converges to the stable ZLB REE. Our findings here can be summarized as follows. First, the learning equilibrium is unique. Second, the size of the multiplier is large in the learning equilibrium. The latter finding is different than results reported in Mertens and Ravn (2015). As it turns out the main reason for the difference in our results is that despite their backwards looking learning rule, Mertens and Ravn change agents expectations about future consumption and inflation when they change government consumption. We do not.

The remainder of this paper is organized as follows. In section 2 we discuss multiplicity and learnability in the context of a standard flexible price model. We do so in order to define learnability in a very simple environment and contrast it with the notion of stability of a REE employed by Benhabib, Schmidt-Gorhe and Uribe (2001). In section three we analyze ZLB REE in a nonlinear Calvo model. We also assess the quality of linear approximations to the Calvo model in this section. Section four contains our main results regarding stability-under-learning of different ZLB REE. In section five we discuss learning equilibrium. Section six contains our analysis of the non-linear Rotemberg model. Concluding remarks are contained in section seven.

2. Fiscal Policy in the ZLB

In this section we derive the implications of the NK model for the effects of changes in government purchases when the ZLB in binding. We conduct our analysis in a non-linear version of the NK model in which firms face Calvo price-setting frictions. Authors like Christiano and Eichenbaum (2012) and Braun, Boneva, and Waki (2015) interpret the price frictions in their nonlinear analysis of the NK model as stemming from Rotemberg (1982) type adjustment costs. This interpretation is interesting because it implies the same linearized equations that EW study. The advantage of adopting Rotemberg adjustment costs is analytic simplicity. In contrast to the Rotemberg approach, the Calvo approach implies the existence an endogenous state variable (past price dispersion). However, as we show in Section 5, there are some important pitfalls associated with using the Rotemberg model that arise from its sensitivity to how the costs of adjusting prices is formulated.

2.1. Model Economy

A representative household maximizes

$$E_0 \sum_{t=0}^{\infty} d_t \left[\log\left(C_t\right) - \frac{\chi}{2} h_t^2 \right]$$

where C_t denotes consumption, h_t denotes hours work, and

$$d_t = \prod_{j=0}^{t} \left(\frac{1}{1 + r_{j-1}} \right).$$

As in EW, we assume that r_t can take on two values: r and r^ℓ, where $r^\ell < 0$. The stochastic process for r_t is given by

$$\Pr\left[r_{t+1} = r^\ell | r_t = r^\ell\right] = p, \ \ \Pr\left[r_{t+1} = r | r_t = r^\ell\right] = 1 - p, \ \ \Pr\left[r_{t+1} = r^\ell | r_t = r\right] = 0. \tag{2.1}$$

We assume that r_t is known at time t. The household faces the budget constraint

$$P_t C_t + B_t \leq \left(1 + R_{t-1}\right) B_{t-1} + W_t h_t + \Pi_t.$$

Here P_t is the price of the consumption good, B_t denotes the quantity of risk-free bonds that the household owns, R_{t-1} is the gross nominal interest rate paid on bonds held from period $t-1$ to period t, W_t is the nominal wage, and Π_t represents lump-sum profits net of lump-sum government taxes. The two first order necessary conditions associated with an interior solution to the household's problem are:

$$\chi h_t C_t = \frac{W_t}{P_t} \tag{2.2}$$

$$\frac{1}{1+R_t} = \frac{1}{1+r_t} E_t \frac{P_t C_t}{P_{t+1} C_{t+1}}. \tag{2.3}$$

A final homogeneous good, Y_t, is produced by competitive and identical firms using the technology:

$$Y_t = \left[\int_0^1 (Y_{j,t})^{\frac{\varepsilon}{\varepsilon-1}} \, dj \right]^{\frac{\varepsilon-1}{\varepsilon}}, \tag{2.4}$$

where $\varepsilon > 1$. The representative firm chooses inputs, $Y_{j,t}$, to maximize profits:

$$P_t Y_t - \int_0^1 P_{j,t} Y_{j,t} dj,$$

subject to the production function (2.4). The firm's first order condition for the j^{th} input is:

$$Y_{j,t} = (P_t / P_{j,t})^{-\varepsilon} \, Y_t. \tag{2.5}$$

The j^{th} input good in (2.4) is produced by firm j who is a monopolist in the product market and is competitive in factor markets. Monopolist j has the production function:

$$Y_{j,t} = h_{j,t}. \tag{2.6}$$

Here $h_{j,t}$ is the quantity of labor used by the j^{th} monopolist. The monopolist maximizes

$$E_t \sum_{k=0}^{\infty} \beta^k \lambda_{t+k} \left((1+\upsilon) P_{j,t} - P_{t+k} s_{t+k} \right) Y_{j,t+k} \tag{2.7}$$

The j^{th} monopolist sets its price, $P_{j,t}$, subject to the demand curve, (2.5), and the following Calvo sticky price friction (2.8):

$$P_{j,t} = \begin{cases} P_{j,t-1} & \text{with probability } \theta \\ \tilde{P}_{j,t} & \text{with probability } 1-\theta \end{cases} . \tag{2.8}$$

Here $\tilde{P}_{j,t}$ is the price chosen by the monopolist j in the event that he can re-optimize his price. The variable υ is a subsidy designed to remove steady state distortions stemming from monopoly power. The monopolist satisfies whatever demand occurs at its posted price. The real marginal cost facing each monopolist is given by:

$$s_t \equiv \frac{W_t}{P_t} = \chi h_t C_t. \tag{2.9}$$

Since all monpolists face the same problem, $\tilde{P}_{j,t}$ is independent of j and we denote its value by $\tilde{P}_t$. The first order condition of monopolist j can be written as

$$\tilde{p}_t = \frac{\tilde{P}_t}{P_{t-1}} = \pi_t \frac{K_t}{F_t}$$

where

$$K_t = \frac{Y_t}{C_t} s_t + \theta \frac{1}{1+r_t} E_t \pi_{t+1}^{\varepsilon} K_{t+1}$$

and

$$F_t = \frac{Y_t}{C_t} + \theta \frac{1}{1+r_t} E_t \pi_{t+1}^{\varepsilon-1} F_{t+1}.$$

Here π_t denotes the gross rate of inflation.

It is well known that aggregate output can be written as[5]

$$Y_t = p_t^* h_t \tag{2.10}$$

where p_t^* is a measure of price dispersion, which evolves according to

$$p_t^* = \left[(1-\theta) \left[\frac{1-\theta\pi_t^{\varepsilon-1}}{1-\theta} \right]^{\frac{-\varepsilon}{1-\varepsilon}} + \theta\pi_t^{\varepsilon} (p_{t-1}^*)^{-1} \right]^{-1}.$$

The aggregate resource constraint is given by

$$C_t + G_t \leq Y_t. \tag{2.11}$$

In equilibrium, this constraint is satisfied as an equality because households and government go to the boundary of their budget constraints. Government consumption is an exogenous process discussed below.

Monetary policy rule is given by

$$R_t = \max \{1, 1 + r + \alpha (\pi_t - 1)\} \tag{2.12}$$

The max operator reflects the ZLB constraint on nominal interest rates and α is assumed to be larger than $1 + r$. As in BSGU, the latter assumption guarantees the existence of two steady states.

We assume that the economy begins in steadt state. At time 0, there is a shock to agents' discount rate so that $r = r^{\ell}$. We consider two scenarios. In the first, the government does not respond to the discount–rate shock. In the second, G_t increases by one percent of steady state output as long as $r_t = r^{\ell}$.

[5] See for example Woodford (2003).

2.2. Solving the Non-Linear Calvo Model

Other than the exogenous discount factor shock, the price dispersion term, p^*_{t-1} is the only state variable in our system. It is convenient to collect the equilibrium conditions of the model:

$$\begin{aligned}
p^*_t &= \left[(1-\theta) \left[\frac{1-\theta \pi_t^{\varepsilon-1}}{1-\theta} \right]^{\frac{\varepsilon}{\varepsilon-1}} + \theta \pi_t^{\varepsilon} (p^*_{t-1})^{-1} \right]^{-1} \\
\frac{1}{Y_t - G_t} &= \frac{1}{1+r_t} \max\left(1, 1+r+\alpha\left(\pi_t - 1\right)\right) E_t \frac{1}{Y_{t+1} - G_{t+1}} \frac{1}{\pi_{t+1}} \\
F_t &= \frac{Y_t}{Y_t - G_t} + \theta \frac{1}{1+r_t} E_t \pi_{t+1}^{\varepsilon-1} F_{t+1} \\
F_t \left[\frac{1-\theta \pi_t^{\varepsilon-1}}{1-\theta} \right]^{\frac{1}{1-\varepsilon}} &= \chi \frac{Y_t^2}{p^*_t} + \theta \frac{1}{1+r_t} E_t \pi_{t+1}^{\varepsilon} F_{t+1} \left[\frac{1-\theta \pi_{t+1}^{\varepsilon-1}}{1-\theta} \right]^{\frac{1}{1-\varepsilon}}
\end{aligned} \tag{2.13}$$

A solution to the model is a set of functions $Y(p^*_{t-1}, r_t)$, $\pi(p^*_{t-1}, r_t)$, $F(p^*_{t-1}, r_t)$, $p^*(p^*_{t-1}, r_t)$ which satisfy the four equilibrium conditions (2.13).

We solve for the equilibrium in two stages. In the first stage, we solve for the equilibrium functions that obtain when $r_t = r$, i.e. after the economy has exited the ZLB. As in Bizer and Judd (1989) we begin with a conjectured set of equilibrium functions, $\tilde{Y}(p^*_{t-1}, r), \tilde{\pi}(p^*_{t-1}, r), \tilde{F}(p^*_{t-1}, r), \tilde{p}^*\left(p^*_{t-1}, r\right)$, for the time $t+1$ variables that appear in (2.13). The equilibrium conditions give us a mapping

$$\begin{aligned}
&\left[Y\left(p^*_{t-1}, r\right), \quad \pi\left(p^*_{t-1}, r\right), \quad F\left(p^*_{t-1}, r\right), \quad p^*\left(p^*_{t-1}, r\right) \right] \\
= \;& T\left[\tilde{Y}(p^*_t, r), \tilde{\pi}(p^*_t, r), \tilde{F}(p^*_t, r), \tilde{p}^*\left(p^*_t, r\right) \right].
\end{aligned}$$

In a rational expectations equilibrium

$$\begin{aligned}&\left[Y\left(p_{t-1}^{*},r\right),\quad \pi\left(p_{t-1}^{*},r\right),\quad F\left(p_{t-1}^{*},r\right),\quad p^{*}\left(p_{t-1}^{*},r\right)\right]\\ =\;& T\left[Y\left(p_{t}^{*},r\right),\quad \pi\left(p_{t}^{*},r\right),\quad F\left(p_{t}^{*},r\right),\quad p^{*}\left(p_{t}^{*},r\right)\right].\end{aligned}$$

We approximate these functions using finite elements methods on a grid defined over p_{t-1}^{*} (see the Appendix for details). Given a value of p_{t-1}^{*}, and the conjectured set of equilibrium functions, (2.13) reduces to a systems of four equations in four unknowns, Y_t, π_t, F_t and p_t^{*}. We solve these equations for all of the values of p_{t-1}^{*} in the grid. In this way we construct a function from the state variable, p_{t-1}^{*} to the equilibrium quantities. If the resulting functions are the same as the conjectured equilibrium functions, then we have found an equilibrium. If they aren't the same, then we use the newly computed functions as conjectured equilibrium functions and repeat the process until the approximating functions converge.

In the second stage, we solve for the equilibrium functions that obtain when r_t is equal to r_ℓ. It is convenient to define

$$\begin{aligned}Y_\ell(p_{t-1}^{*}) &= Y(p_{t-1}^{*},r_\ell),\ p_\ell^{*}(p_{t-1}^{*})=p^{*}(p_{t-1}^{*},r_\ell),\\ F_\ell(p_{t-1}^{*}) &= F(p_{t-1}^{*},r_\ell),\ \pi_\ell(p_{t-1}^{*})=\pi(p_{t-1}^{*},r_\ell).\end{aligned}$$

In the ZLB, we can write (2.13) as

$$p_\ell^{*}(p_{t-1}^{*})=\left[(1-\theta)\left[\frac{1-\theta\pi_\ell(p_{t-1}^{*})^{\varepsilon-1}}{1-\theta}\right]^{\frac{\varepsilon}{\varepsilon-1}}+\theta\frac{\pi_\ell(p_{t-1}^{*})^{\varepsilon}}{p_{t-1}^{*}}\right]^{-1}$$

$$\frac{1}{Y_\ell(p^*_{t-1}) - G_\ell} = \frac{1}{1+r^l} \max\left(1 + r + \alpha\left(\pi_\ell(p^*_{t-1}) - 1\right), 1\right) \left[p \frac{1}{Y_\ell(p^*_t) - G_\ell} \frac{1}{\pi_\ell(p^*_t)} \right. \tag{2.14}$$

$$\left. + (1-p) \frac{1}{Y(p^*_t) - G} \frac{1}{\pi(p^*_t)} \right]$$

$$F_\ell(p^*_{t-1}) = \frac{Y_\ell(p^*_{t-1})}{Y_\ell(p^*_{t-1}) - G_\ell} + \theta \frac{1}{1+r^l} \left[p\pi_\ell(p^*_t)^{\varepsilon-1} F_\ell(p^*_t) + p\pi(p^*_t)^{\varepsilon-1} F(p^*_t) \right]$$

$$\begin{aligned} F_\ell(p^*_{t-1}) \left[\frac{1 - \theta \pi_\ell(p^*_{t-1})^{\varepsilon-1}}{1-\theta} \right]^{\frac{1}{1-\varepsilon}} = & \chi Y_\ell(p^*_{t-1}) (Y_\ell(p^*_{t-1}) / p^*_\ell(p^*_{t-1})) \\ & + \theta \frac{1}{1+r^l} p \pi_\ell(p^*_t)^{-\varepsilon} F_\ell(p^*_t) \left[\frac{1 - \theta \pi_\ell(p^*_t)^{\varepsilon-1}}{1-\theta} \right]^{\frac{1}{1-\varepsilon}} \\ & + \theta \frac{1}{1+r^l} (1-p) \pi(p^*_t)^{-\varepsilon} F(p^*_t) \left[\frac{1 - \theta \pi(p^*_t)^{\varepsilon-1}}{1-\theta} \right]^{\frac{1}{1-\varepsilon}} \end{aligned}$$

We solve for the equilibrium functions $Y_\ell(p^*_{t-1}), \pi_\ell(p^*_{t-1})$, $F_\ell(p^*_{t-1})$ and $p^*_\ell(p^*_{t-1})$ using the same algorithm used in the first stage. Note that our solution recovers minimum state variable equilibria. We comment on sunspot equilibria.

Define a steady-state ZLB equilibrium as the equilibrium prices and quantities of the economy if $r_t = r_\ell$ in the limit as t goes to infinity. It is easy to verify that if a steady state equilibrium exists, then p^*_t converges to $\hat{p}$. Then (2.14) collapses to a system of four equations in four unknowns, $\pi_\ell(\hat{p})$, $Y_\ell(\hat{p})$, $p^*_\ell(\hat{p})$,and $F_\ell(\hat{p})$.We compute the steady-state ZLB equilibrium as follows. Conjecture a guess for $\pi_\ell(\hat{p})$. Then calculate the implied value of $\hat{p}$ from the first equation of (2.14), calculate $C_\ell(\hat{p})$ from the second equation of (2.14) and compute $F_\ell(\hat{p})$ from the third equation of (2.14). Then check if the final equation of (2.14) holds with equality. If it holds, $\pi_\ell(\hat{p})$ is a steady-state ZLB equilibrium value of inflation. If it doesn't hold, search for another $\pi_\ell(\hat{p})$.

Employing the previous algorithm, we reduce the equations defining an interior steady-state ZLB equilibrium into one equation one unknown

$$f(\pi_\ell) = 0. \tag{2.15}$$

In a slight abuse of notation we have dropped the explicit dependence of π_ℓ on $\hat{p}$. If this condition doesn't hold, then π_ℓ can't be the steady-state ZLB equilibrium value of inflation. So, a necessary condition for the ZLB equilibrium to be unique is that there is a unique solution to (2.15).

In our experiments we use following baseline parameterization of the model:

$$\begin{aligned} \varepsilon &= 7.0,\ \beta = 0.99,\ \alpha = 2.0,\ p = 0.75, \\ r^{\ell} &= -0.02/4,\ \theta = 0.85,\ \eta_g = 0.2. \end{aligned} \tag{2.16}$$

Steady state output is normalized to 1 by setting $\chi = 1.25$. The appendix contains a sensitivity analysis of all our key results to perturbations of the benchmark parameter values.

2.3. Baseline RE Equilibria Results

Recall that the equation defining an interior steady-state ZLB equilibrium is given by (2.15). Figure 2.1, displays $f(\pi_\ell)$ as a function of π_ℓ. The solid line is calculated assuming that G is equal to its steady state value, 0.20. Note that $f(\pi_\ell)$ has an inverted U shape. It follows that there are either two interior steady-state ZLB equilibria or none. Given our assumed parameter values, there are in act two such equilibria. We refer to them as the high and low inflation steady state ZLB equilibria.

Figure 2.1: Steady State ZLB Equilibrium Function

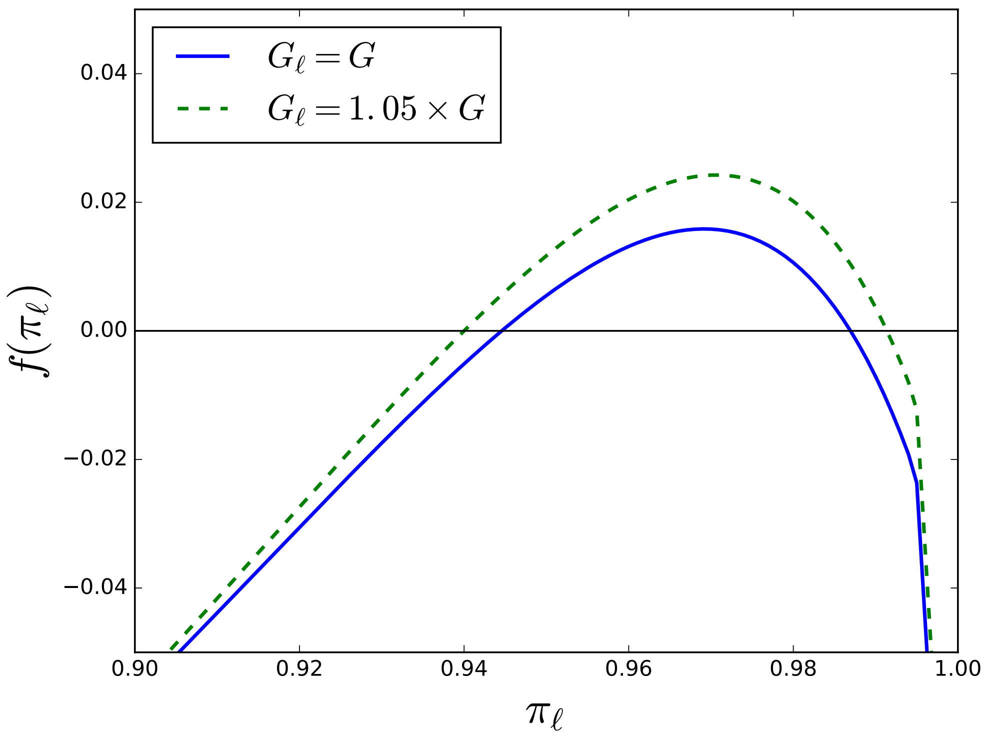
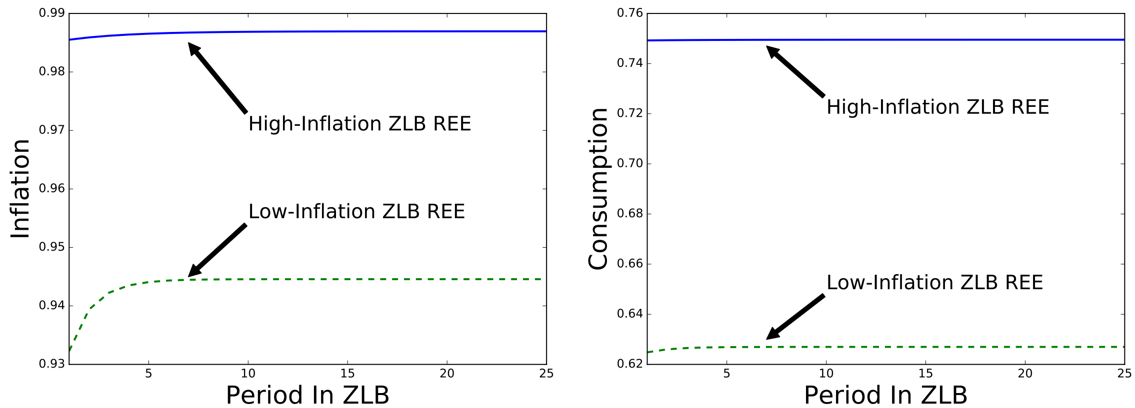


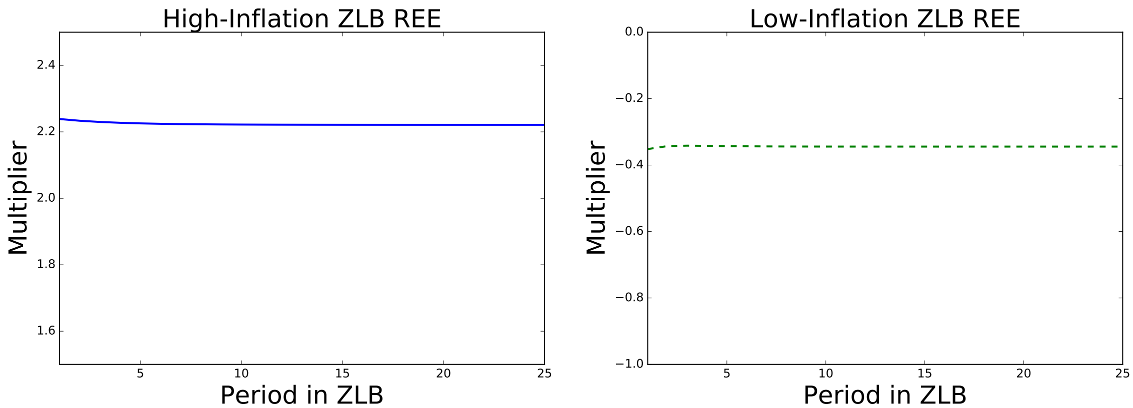
Figure 2.2: RE Equilibrium Paths In ZLB



In practice we find that the number of ZLB equilibria coincides with the number of steady state ZLB equilibria. To be clear, this is a numerical result, not a theorem.

The dotted lines of the panels of Figure 2.2 displays the dynamic response of inflation and consumption, respectively, to the discount rate shock as the economy converges to the high and low inflation steady-state ZLB equilibrium. We refer to these paths as the high and low inflation ZLB equilibria, respectively. A number of features are worth noting. First, along the high inflation ZLB equilibrium path, quarterly inflation and consumption drop in the impact period of the shock by 1.5 and 6.35 percentage points, respectively. After about 5 quarters these declines stabilize at 1.3 and 6.3 percentage points, respectively. Second, along the low inflation ZLB equilibrium path, quarterly inflation and consumption drop in the impact period of the shock by 7.25 and 23.5 percentage points, respectively. After about 5 quarters these declines stabilize at 6.0 and 23.3 percentage points, respectively. Third, the dynamics induced by the evolving state variable p_t^* are larger in magnitude for the low inflation ZLB equilibrium. But event for that case, the system effectively converges

Figure 2.3: RE Multiplier In ZLB



after one year.

To derive values for the government spending multiplier we assume that $G^{\ell} = 1.05 \times G^{h}$, i.e. when the economy is in the ZLB, G rises by 1 percentage of steady state output. We define the multiplier in the first period to be

$$\frac{G^{\ell}}{(C^{\ell}(p_{t-1}^{*}) + G^{\ell})} \frac{\Delta\left(C^{\ell}(p_{t-1}^{*}) + G^{\ell}\right)}{\Delta G^{\ell}}.$$

We compute this ratio assuming that if the economy is in the high inflation (low inflation) ZLB equilibrium for a low value of G, it is in the high inflation (low inflation) ZLB equilibrium for the high value of G.[6] The two panels of Figure 2.3 display the multiplier in the high inflation and low inflation ZLB equilibrium as a function of time. Notice that the multiplier in the high inflation ZLB equilibrium is large, exceeding two over the time period displayed. In contrast, the multiplier is actually *negative* in the low inflation ZLB equilibrium. This change in sign is

[6]This assumption is non-trivial because one can easily construct examples in which G serves as a sunspot inducing a switch from one equilibrium to the other. As in Mertens and Raven (2015), we abstract from this issue.

a dramatic illustration of the basic result in Mertens and Raven (2011) where the multiplier is much lower in the analog to our low inflation ZLB equilibrium. To understand why the sign of the multiplier depends on which equilibrium we are in, note an increase in G^{ℓ} shifts f upwards (see Figure 2.1). This shift implies that the effect of an increase in G depends on which equilibrium we focus on.

The size of the multiplier in the high-inflation ZLB equilibrium increases as p rises or θ falls, i.e. as the expected duration of the ZLB rises or as prices become more flexible. These results are consistent with the intuition in CER (2011) and EW. In contrast, the size of the multiplier associated with the low-inflation ZLB equilibrium become more negative as the multiplier increases as p rises or θ falls.

2.3.1. Comparisons to linearized version of the model

Table 2.1 summarizes our results regarding the impact of changes in G for the non-linear and linear versions of the Calvo model. We report the response of inflation, output and the multiplier in the impact period of a shock to the discount rate accompanied by a rise in G. Notice that the equilibrium behavior of the linearized model is similar to that of the non-linear model in the high-inflation ZLB equilibrium. For example, the impact multiplier in the linear model is 1.63 while it is 2.24 in the high-inflation ZLB equilibrium. While the magnitudes of the two multipliers are different, both deserve the adjective, 'large'. The initial percent drop in GDP in the linear and high-inflation ZLB equilibrium model is 2.18% and 2.84%,respectively. Again, while the numbers are different, the decline in output is large in both cases. In stark contrast, the properties of the non-linear model in the low-inflation ZLB equilibrium are very different than those of the linear model. For example the impact multiplier is -0.35% and the initial drop in GDP is 17.87%.

Table 2.1: Comparing the Linear and Non-linear Models

Model	Output	Inflation	Multiplier
Linear	-2.18	-0.0066	1.63
Nonlinear, high-inflation	-2.84	-0.0093	2.24
Nonlinear, low-inflation	-17.87	-0.0734	-0.35

The multiplier in the linear model is inversely related

$$\Delta = (1-p)(1-\theta p) - p(2-\eta_g)\frac{(1-\theta)(1-\beta\theta)}{\theta}.$$

It is evident that the multiplier is strictly increasing in p and θ. See CER (2015) for the intuition underlying this result. As noted above, a similar result obtains for the high-inflation ZLB equilibria of the non linear model. There is one interesting difference between the linear and non- linear models. Carlstrom, Fuerst and Paustian (2014) prove that the linear model does not have an interior equilibrium when Δ is negative. Before Δ turns negative, the multiplier can be *arbitrarily* large. We found that increases in p and declines in θ which reduce the value of Δ,have the effect of shifting the $f(\pi_\ell)$ function down. At some point $f(\pi_\ell)$ is not equal to zero for any π_ℓ, i.e. an interior equilibrium non longer exists. So non-existence leads to an effective bound on the multiplier in the non-linear model. In practice we found that the upper and lower bounds associated with high and low inflation ZLB equilibria were 4.3 and -2.5 percent respectively.

To summarize, the basic qualitative results reported in CER using a log-linear approximation obtain when we consider the nonlinear solution as long as we focus attention on the high-inflation ZLB equilibrium.

2.4. Sunspot Equilibria

In the analysis above, we assumed that the ZLB becomes a binding because of a shock to the household's discount rate. We now consider a scenario in which the ZLB binds because of a non-fundamental shock. This case is the one considered by Mertens and Ravn (2015). Suppose that at $t = 0$, before any agent has made a decision, the economy is in the high-inflation steady state equilibrium. Each firms observes a sunspot. Conditional on the sunspot firms can either believe that other firms behave as in they did in the high inflation steady state or they will set their prices sufficiently low to make the ZLB bind. With probability p firms continue to hold this belief. With probability $(1 - p)$, firms believe that other firms will set their prices sufficiently high to make the ZLB non-binding and behave as they did in high inflation steady state. The latter belief is an absorbing state.

Figure 2.4 displays the $f(\pi_\ell)$ function for the case under consideration. Notice that there are two steady ZLB equilibria corresponding a low and high inflation rate, respectively.

As stressed in Mertens and Ravn (2015), the sunspot equilibrium can be characterized as a situation in which the shock driving the economy into a binding ZLB is a loss in confidence. The basic intuition is as follows. Suppose that agents anticipate deflation, creating the perception that the real interest rate is high. Households respond to the high real interest rate by reducing expenditures, thus drive the economy into a recession. The lower level of output leads to a fall in real wages and marginal cost. The latter effect leads to sustained downward pressure on the price level because of price-setting frictions. So the initial fear of deflation is self-fulfilling. Mertens and Ravn (2015) propose this non-fundamental 'loss of confidence' shock as an alternative to a fundamental shock that drives the economy into the ZLB.

Figure 2.4: Steady State Sunspot Equilibrium Function

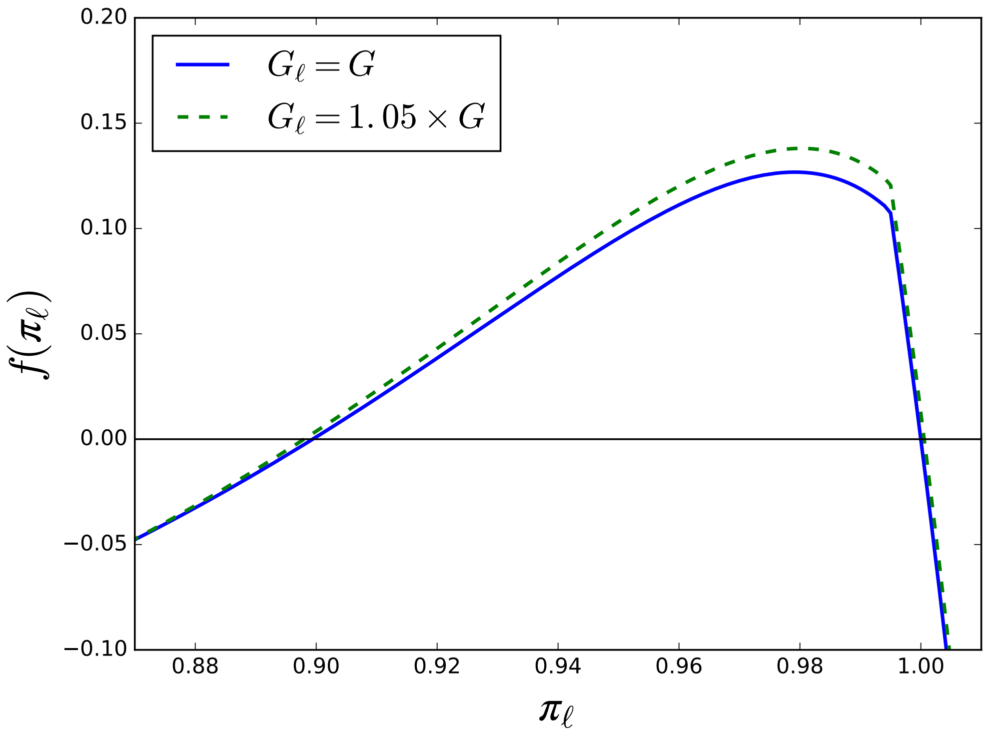
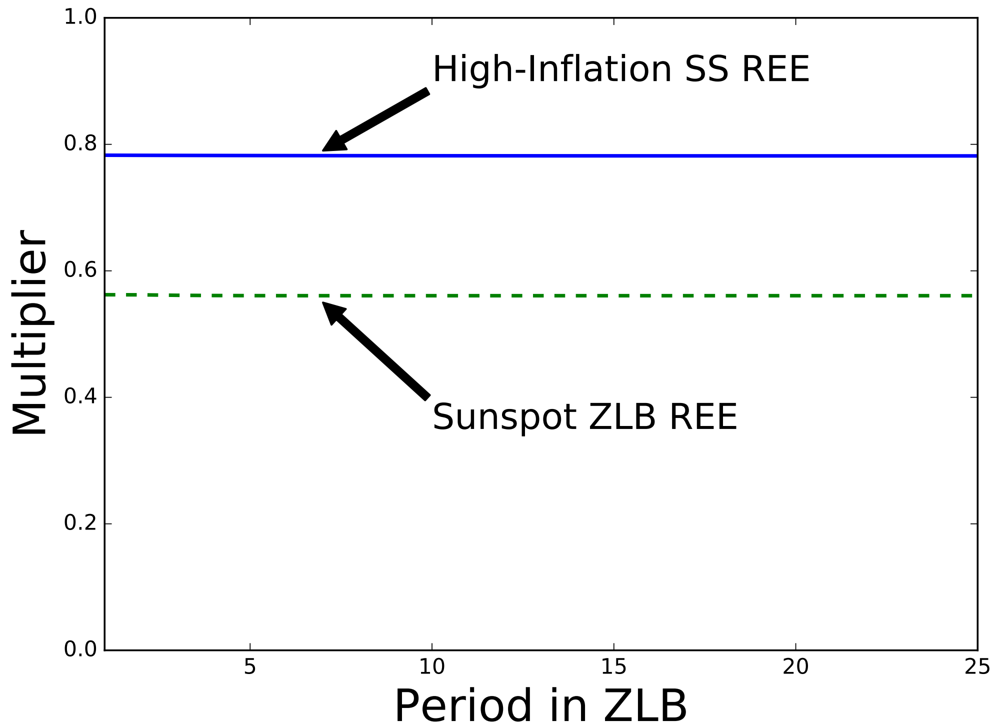


Figure 2.5: Sunspot Multipliers



Now consider the effect of a rise in G when the sunspot occurs. Depending on beliefs, the economy will either be in the high or low inflation equilibrium. Note that the ZLB is only binding in the low inflation equilibrium. We compute the multiplier assuming that the economy is initially in the high inflation steady state and agents think the economy will go back there when the ZLB ends. Figure 2.5 displays the multiplier as a function of time for the case where the economy is in the ZLB and the case where the economy remains in the high inflation steady state equilibrium after the sunspot is operative. As it turns out, the multiplier at the ZLB can be larger or smaller than in the steady state, depending on parameter values. But the robust result is that the multiplier is quite small: (0.56) in the ZLB and (0.79) at the steady state. The steady state multiplier is small when the ZLB is not binding because an increase in government spending leads to inflation. Monetary policy responds by rasing the real interest which crowds private consumption. In the other equilibrium, the real interest is also high because the ZLB binds and there is deflation. So again consumption falls, leading to a relatively small multiplier.

3. Stability Under Learning at the ZLB

In this section we investigate stability under learning of the high and low inflation ZLB equilibria. In order to determine what happens when agents don't have rational expectations, we must make assumptions about how their beliefs evolve over time.

3.1. The benchmark case

In the rational expectations version of the Calvo model, intermediate good firms choose their price level, $\tilde{P}_{j,t}$, based in part on the value of the aggregate price level, P_t. But, the later is a function of firms' collective price decisions. So firms cannot actually observe P_t when they choose $\tilde{P}_{j,t}$. The standard assumption is that these firms form a 'belief' about P_t when they make their decision. In a rational expectations equilibrium that belief is correct. In a world where firms don't necessarily have rational expectations it is not natural to assume that firms actually see P_t at the time they choose$\tilde{P}_{j,t}$. Note that if they don't see P_t they also don't know what the demand for their output (C_t) will be.

At time t firms make their decisions given the state variable p^*_{t-1} and views about the equilibrium functions for current and future values of P_t and C_t.We assume that firms believe they are in a stationary environment, i.e. firms think that the equilibrium functions won't change over time. Denote by $x^{e,f}_\ell(p^*_{t-1}, t-1)$ the typical firm's belief, formed using information up to time $t-1$, about the equilibrium function for x_ℓ. The only argument of the function is the state variable p^*_{t-1}. While the firm knows the actual value of p^*_{t-1}, we must attribute to it beliefs about the entire equilibrium function for x_ℓ. The reason is that the firm's first order conditions involve objects like $x^{e,f}_\ell(p^*_{t-1}, t-1)$ and $x^{e,f}_\ell(p^*_{t+j}, t-1)$ for $j \geq 0$. The fact that all these functions have time $t-1$ as an argument summarizes the standard assumption that firms think they are in a stationary environment, i.e. they don't expect that their beliefs about the functions will change in the future (see Evans and Honkapohja (2003)).

Given new information, firms' beliefs evolve over time according to

$$x^{e,f}_\ell(p^*_t, t) = \omega x_\ell(p^*_{t-1}, t-1) + (1-\omega) x^{e,f}_\ell(p^*_{t-1}, t-1). \tag{3.1}$$

For $\omega > 0$, this formulation assumes that at time t, agents know the time $t-1$ equilibrium function for x_ℓ. This assumption is clearly heroic. So we also investigate what happens when firms just assume that the value of the variables that they have to forecast are equal to their current value. (See the appendix for details). As an aside, it is worth noting that in Rotemberg model, discussed in Section 5, there are no state variables in the ZLB. So we can replace (3.1) with the assumption that agents' expectations about the values of future variables evolve according to a simple constant gain algorithm.

When households make their time t consumption decisions, firms' actions have already determined the aggregate price level. Given this information, the households can compute the time t equilibrium function for inflation.[7] Denote by $\pi_\ell^{e,h}(p_{t-1}^*, t)$ households' belief, at time t, about the equilibrium function for inflation, π_ℓ. Households think they are in a stationary environment, i.e. they don't expect that their beliefs about this function will change in the future Given new information, households beliefs evolve according to

$$\pi_\ell^{e,h}(p_t^*, t+1) = \omega \pi_\ell(p_{t-1}^*, t) + (1-\omega)\pi_\ell^{e,h}(p_{t-1}^*, t). \tag{3.2}$$

The first-order condition of the firm when $r_t = r_\ell$ can be written as

$$\frac{\tilde{P}_{\ell,t}}{P_{t-1}} = \frac{P_t}{P_{t-1}} \frac{K_{\ell,t}^{e,f}}{F_{\ell,t}^{e,f}} \tag{3.3}$$

where

$$K_{\ell,t}^{e,f} = \chi \frac{(Y_{\ell,t}^{e,f})^2}{p_t^*} + \theta \frac{1}{1+r_\ell} (\pi_{\ell,t+1}^{e,f})^{\varepsilon} \left[p K_{\ell,t+1}^{e,f} + (1-p) K_{n,t+1}^{e,f} \right] \tag{3.4}$$

and

$$F_{\ell,t}^{e,f} = \frac{Y_{\ell,t}^{e,f}}{C_{\ell,t}^{e,f}} + \theta \frac{1}{1+r_\ell} (\pi_{\ell,t+1}^{e,f})^{\varepsilon-1} \left[p F_{\ell,t+1}^{e,f} + (1-p) F_{n,t+1}^{e,f} \right]. \tag{3.5}$$

[7] They can do so under the further heroic assumption that they can solve the problem that the firm just solved.

Here, $F^{e,f}_{\ell,t}$, $K^{e,f}_{\ell,t}$, $Y^{e,f}_{\ell,t}$, $C^{e,f}_{\ell,t}$, and $\pi^{e,f}_{\ell,t}$ denote firms' beliefs about F_t, K_t, Y_t, C_t, and π_t when $r_t = r_\ell$. Similarly, $F^{e,f}_{n,t}$, $K^{e,f}_{n,t}$, $Y^{e,f}_{n,t}$, $C^{e,f}_{n,t}$, and $\pi^{e,f}_{n,t}$ denote firms' beliefs about F_t, K_t, Y_t, C_t, and π_t when $r_t = r$. Here, the superscript, 'e', indicates the typical firms' belief about the value of the corresponding variable. Note that the beliefs $F^{e,f}_{\ell,t}$, $K^{e,f}_{\ell,t}$, and $Y^{e,f}_{\ell,t}$ are derived from the beliefs $C^{e,f}_{\ell,t}$, and $\pi^{e,f}_{\ell,t}$ which evolve according to (3.1).

The first-order conditions of the household when $r_t = r_\ell$ can be written

$$\chi C_{\ell,t} h_{\ell,t} = \frac{W_{\ell,t}}{P_{\ell,t}}, \tag{3.6}$$

and

$$\frac{1}{C_{\ell,t}} = \frac{1}{1+r_\ell} \max\left\{1, 1+r+\alpha(\pi_{\ell,t}-1)\right\} \left[\frac{p}{C^{e,h}_{\ell,t+1}\pi^{e,h}_{\ell,t+1}} + \frac{1-p}{C^{e,h}_{n,t+1}\pi^{e,h}_{n,t+1}}\right]. \tag{3.7}$$

Here $C_{\ell,t}$, $h_{\ell,t}$, $R_{\ell,t}$, and $\frac{W_{\ell,t}}{P_{\ell,t}}$ are the time t realized values of consumption, labor supply, the nominal interest rate, and the real wage. Household beliefs about future inflation evolve according to (3.2).

Definition 1. *A learning ZLB equilibrium is a sequence of functions $\pi_{\ell,t}(\cdot)$, $C_{\ell,t}(\cdot)$, $h_{\ell,t}(\cdot)$, $\frac{W_{\ell,t}}{P_t}(\cdot)$, $\frac{\tilde{P}_t}{P_{t-1}}(\cdot)$, and $R_{\ell,t}(\cdot)$ that satisfy the resource constraint, the monetary policy rule, and the household and firm optimality conditions for all t, given an initial set of beliefs $\pi^{e,f}_{\ell}(\cdot,0)$, $C^{e,f}_{\ell}(\cdot,0)$, and $\pi^{e,h}_{\ell}(\cdot,0)$ that evolve according to (3.1) and (3.2).*

Here we have assumed that households and firms know the equilibrium functions when $r_t = r$, i.e. they have rational expectations about the economy when it's not in the ZLB.

Our selection criterion for a rational expectations equilibrium (REE) is based on the following notion of stability.

Definition 2. *Suppose that in a neighborhood of a ZLB REE, either $x_{\ell}^{e,f}(\cdot, t-1)$ is not equal to $x_{\ell}(\cdot)$ or $\pi_{\ell}^{e,h}(\cdot, t)$ is not equal to $\pi_{\ell}(\cdot)$. Here $\pi_{\ell}(\cdot)$ and $x_{\ell}(\cdot)$ are the ZLB REE functions for x_{ℓ} and π_{ℓ}. A ZLB REE is said to be stable-under learning if a learning equilibrium converges back to the ZLB REE.*

If the economy stays in the ZLB forever, it will converge to a steady state ZLB REE. The learning equilibrium must also approach the same steady state if the initial ZLB REE is stable under learning. This fact is very useful because it allows us to eliminate *all* of the ZLB REE equilibria that lead to the low inflation ZLB steady state REE as not being stable-under-learning.

We now establish this numerically and provide the underlying intuition. To this end, suppose that a firm incorrectly believes that the steady state inflation rate is $\pi_{\ell}^{e,f}$ and that the economy is in the corresponding steady state. Also, assume that p_{t-1}^{*} is consistent with this belief. Since the belief $\pi_{\ell}^{e,f}$ is not a rational expectations belief, $f(\pi_{\ell}^{e,f})$ is not equal to zero. Note that there is an equivalence between the belief $\pi_{\ell}^{e,f}$ and the value of $\tilde{p}_{\ell}^{e,f}$ that will be chosen by firms who can update their price. So we use the function $f(\pi_{\ell}^{e,f})$ to define a new function $\tilde{f}\,(\tilde{p}_{\ell}^{e})$ that must be equal to zero at a steady state ZLB REE.

Combining (3.3)-(3.5), using the aggregate resource (2.11) and the household Euler equation (3.7) we represent the first order condition of the firm under consideration as

$$\tilde{F}\left(\tilde{p}_{\ell}, \tilde{p}_{\ell}^{e,f}\right) = 0. \tag{3.8}$$

Define the best-response function

$$\tilde{p}_\ell = g(\tilde{p}_\ell^{e,f}). \tag{3.9a}$$

This function has the property,

$$\tilde{F}\left(g(\tilde{p}_\ell^{e,f}), \tilde{p}_\ell^{e,f}\right) = 0, \tag{3.10}$$

i.e. for arbitrary $\tilde{p}_\ell^{e,f}$, equation (3.8) is satisfied. In a steady state ZLB REE

$$\tilde{p}_\ell = \tilde{p}_\ell^{e,f}. \tag{3.11}$$

The typical firm's belief about the current aggregate inflation, is given by

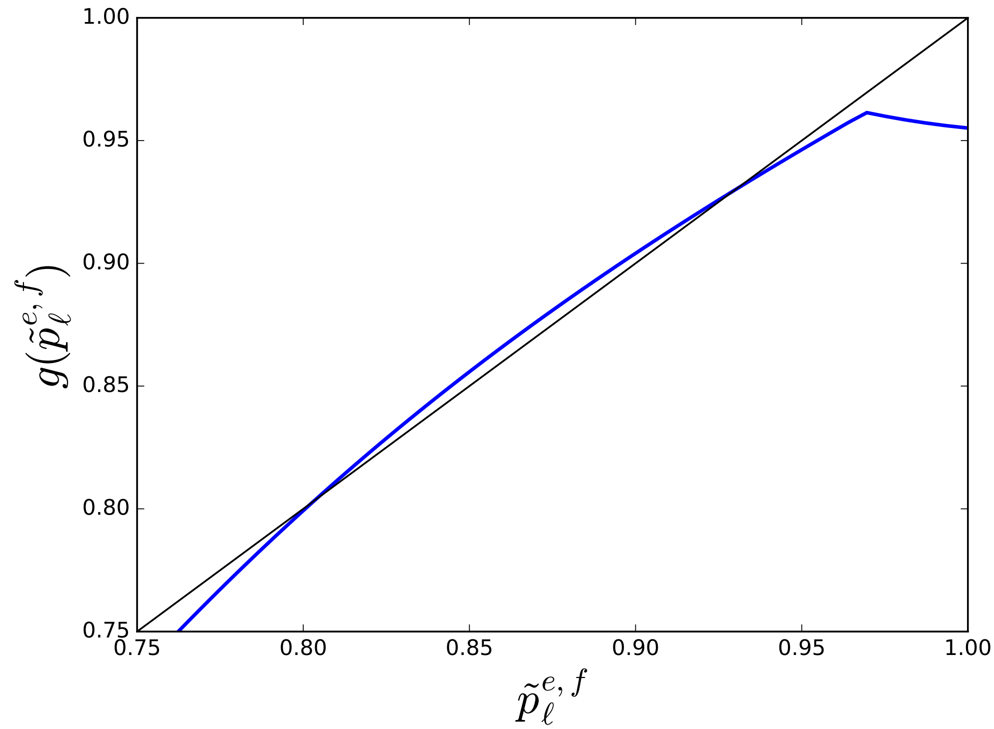
$$\pi_\ell^{e,f} = \left(\theta + (1-\theta)(\tilde{p}_\ell^{e,f})^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}. \tag{3.12}$$

In an REE,

$$\pi_\ell = \pi_\ell^{e,f} \tag{3.13}$$

Figure 3.1 plots the typical firm's best response function (3.9a), i.e. $\tilde{p}_\ell$ as a function of $\tilde{p}_\ell^{e,f}$. The two steady state ZLB REE equilibria correspond to the two points where the best response function intersects the 45 degree line. Notice that given any belief, $\tilde{p}_\ell^{e,f}$, between the steady state REE ZLB beliefs, the best response $g(\tilde{p}_\ell^{e,f})$ is greater than $\tilde{p}_\ell^{e,f}$. It follows that realized inflation will exceed beliefs about inflation. So the learning equilibrium will move towards the high inflation ZLB REE steady state. Now consider any belief, $\tilde{p}_\ell^{e,f}$, that exceeds its value in the high inflation

Figure 3.1: Best Response Function



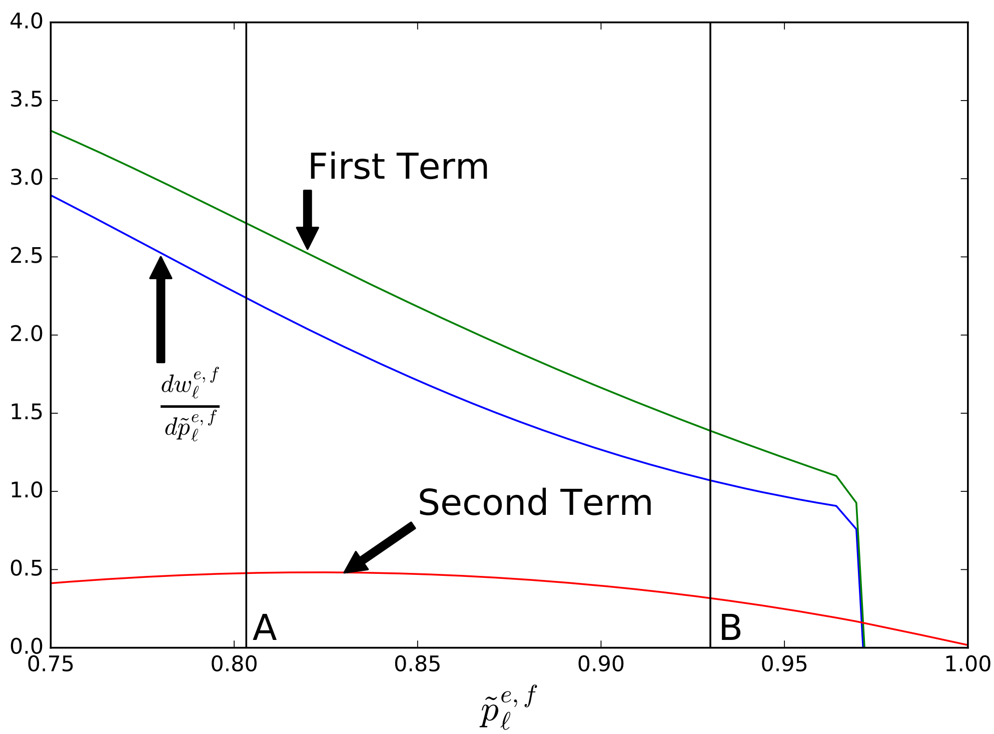
steady state REE. Here the best response function $g(\tilde{p}_\ell^{e,f})$ is less than $\tilde{p}_\ell^{e,f}$. So realized inflation will be lower than beliefs about inflation and the learning equilibrium will move towards the high inflation ZLB REE steady state. Finally, consider any belief, $\tilde{p}_\ell^{e,f}$, that is less than its value in the low inflation ZLB REE steady state. Here the best response function $g(\tilde{p}_\ell^{e,f})$ is less than $\tilde{p}_\ell^{e,f}$. It follows that realized inflation will be lower than beliefs about inflation. So the learning equilibrium will move away from the low inflation RE steady state.

The basic intuition for the previous results is as follows. Consider a firm whose expectations $\tilde{p}_\ell^{e,f}$ aren't equal to an REE value. Associated with $\tilde{p}_\ell^{e,f}$ is an expectation about aggregate consumption and the wage rate. In the ZLB, a low value of $\tilde{p}_\ell^{e,f}$ implies a low expected value of inflation and a high value of the real interest rate. From the household's Euler equation (2.14), we see that a high real rate means that aggregate consumption will be low. The production function and the household's first-order condition imply that aggregate employment and the real wage will also be low. It follows that a low value of $\tilde{p}_\ell^{e,f}$ is associated with a low expected value of marginal cost. Since the firm's price is an increasing function of marginal cost, a low value of $\tilde{p}_\ell^{e,f}$ will be associated with a low value of $\tilde{p}_\ell$. This result is shows up in Figure 3.1 since the best response function is an increasing function of $\tilde{p}_\ell$ when the ZLB binds.

We now show that a small change in $\tilde{p}_\ell^{e,f}$ is associated with a smaller movement in marginal cost when we start from the high inflation ZLB REE (point B in Figure 3.2) than when we start from the low inflation ZLB REE (point A in Figure 3.2). As it turns out the basic force driving the result is that the consumption response to changes in inflation are much larger when inflation is low (near point A) than when inflation is high (near point B).

Consider the change in the expected real wage associated with a change in $\tilde{p}_\ell^{e,f}$:

Figure 3.2: Expected Changes in Marginal Cost



$$\frac{dw_t^{e,f}}{d\tilde{p}_\ell^{e,f}} = \frac{d\left(\chi \frac{(G_\ell^{e,f}+C_\ell^{e,f})C_\ell^{e,f}}{p_\ell^*}\right)}{d\tilde{p}_\ell^{e,f}} = \chi\left(\frac{G_\ell^{e,f}+2C_\ell^{e,f}}{p_\ell^*}\right)\frac{dC_\ell^{e,f}}{d\pi_\ell^{e,f}}\frac{d\pi_\ell^{e,f}}{d\tilde{p}_\ell^{e,f}} - \chi\left(\frac{(G_\ell^{e,f}+C_\ell^{e,f})C_\ell^{e,f}}{(p_\ell^*)^2}\right)\frac{dp_\ell^*}{d\tilde{p}_\ell^{e,f}}. \tag{3.14}$$

Figure 3.2 displays the behavior of this derivative as a function of $\tilde{p}_\ell^{e,f}$. As it turns out, the key determinant of this derivative is the first group of terms on the right hand side of the equation. That term captures the effect of $\tilde{p}_\ell^{e,f}$ on aggregate consumption, hours worked, and the real wage that operate through the real interest rate. To analyze this effect, we re-write the household's Euler Equation when the ZLB binds as

$$1 = \frac{1}{1+r_\ell}\left[\frac{p}{\pi_\ell} + \frac{(1-p)C_\ell}{C(p_\ell^*)\pi(p_\ell^*)}\right]. \tag{3.15}$$

When π_ℓ is near 1, there is a negative relationship between C_ℓ and π_ℓ that is roughly linear. For values of π_ℓ that are relatively far from 1, C_ℓ is more sensitive to changes in π_ℓ. This increased sensitivity reflects the convexity of the term p/π_ℓ which appears in the household's Euler equation. Since $\frac{d\pi_\ell^{e,f}}{d\tilde{p}_\ell^{e,f}}$ is roughly a constant, this convexity implies that the derivative of wages with respect to $\tilde{p}_\ell^{e,f}$ is much larger at point A than at point B.

Next, consider the second term in 3.14 that involves $\frac{dp_\ell^*}{d\tilde{p}_\ell^{e,f}}$. This term captures the impact of changes in p_ℓ^* on marginal cost induced by a change in $\tilde{p}_\ell^{e,f}$. Equation (2.10) implies that the amount of labor required to produce a given Y_t depends negatively on p_t^*. At the steady state ZLB REE , an increase in $\tilde{p}_\ell^{e,f}$ leads a rise in inflation and a higher value of p_ℓ^*. So less labor is needed to produce the same amount of output. Other things equal this effect induces a decline in hours worked, the real wage rate and marginal cost.

Figure 3.2 displays $\frac{dp_\ell^*}{d\tilde{p}_\ell^{e,f}}$ as a function $\tilde{p}_\ell^{e,f}$.[8] Note that at point B, p_ℓ^* is near one and $\frac{dp_\ell^*}{d\tilde{p}_\ell^{e,f}}$ is near zero. In contrast, at point A, p_ℓ^* is roughly 0.9 and $\frac{dp_\ell^*}{d\tilde{p}_\ell^{e,f}}$ is greater than zero. So other things equal, a firm contemplating a decrease in $\tilde{p}_\ell^{e,f}$ thinks that marginal costs are falling more if its initial expectations are near point A rather than point B. But this effect is small relative to the impact of the first term in 3.14.

Critically, $\frac{dw_t^{e,f}}{d\tilde{p}_\ell^{e,f}}$ is bigger at point A than at point B. So at point A a firm will increases its price by *more* than it would at point B. Critically, at point A the firm increases its price by more than one-for-one with an increase in $\tilde{p}_\ell$. In sharp contrast, a firm at point B will increase its price by *less* than one-for-one with expected increase in $\tilde{p}_\ell$. This result is precisely why the low-inflation steady state ZLB REE is not stable under learning and the high-inflation steady state ZLB REE is stable under learning.

The previous discussion focused on the limiting point of the ZLB REE. To be stable-under-learning, the functions defining a learning equilibrium must converge point wise to the functions defining a ZLB REE for every possible value of p_t^*. The previous discussion establishes that any ZLB REE that converges to the low inflation ZLB REE steady state does not satisfy this condition, and is therefor not stable-under-learning. It does not establish that a ZLB REE equilibrium that converges to the high inflation steady state ZLB REE is stable-under-learning. We now establish, numerically, the stability-under-learning of such an equilibrium.

Recall that we parameterize the ZLB REE functions with a finite number of parameters, z_t. The learning algorithm specified above defines a mapping from the current values of those parameters to the values that they take in the subsequent period

$$z_{t+1} = s\left(z_t\right). \tag{3.16}$$

[8] We compute this term using the fact that $p_t^* = \left[(1-\theta)\tilde{p}_t + \theta\pi_t^{\varepsilon}(p_{t-1}^*)^{-1}\right]^{-1}$.

Define

$$S(\tilde{z}) = \left[\frac{ds_i(z)}{dz_j} \right] |_{\tilde{z}}, \tag{3.17}$$

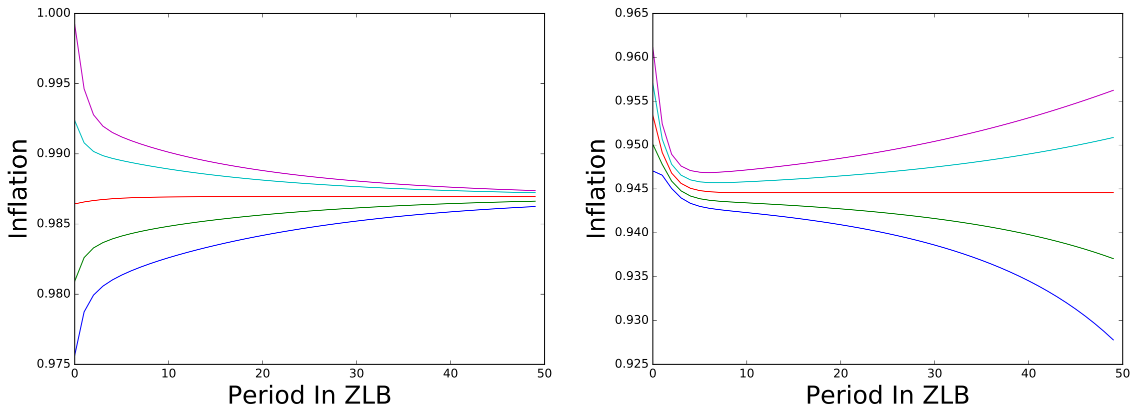
for all $i, j < N$ where N is the number of parameters. When we evaluate S for the parameters of the high inflation ZLB REE, we find that the maximum eigenvalue is less than one in absolute value. Consider a learning equilibrium. The previous result establishes that, locally, beliefs in the neighborhood of the high inflation ZLB REE will converge to REE beliefs. By contrast, when we evaluate S for the parameters of the low inflation ZLB REE, we find that the maximum eigenvalue is greater than one in absolute value. This result implies that beliefs in the neighborhood of the low-inflation ZLB REE will diverge from the REE beliefs.

To illustrate the process of convergence and divergence, suppose that at time -1 the economy is in the high inflation steady state ZLB REE where $p^*_{-1} = p_\ell$. Then at time 0, for reasons unexplained, (i) $x^{e,f}_\ell(p^*_{-1}, -1) = x_\ell(p^*_{-1}) + \bar{x}_\ell$,where $\bar{x}_\ell$ is a positive constant, and (ii) all agents think that if the ZLB ends, the economy will be in a REE that converges to the high inflation steady state.[9] For simplicity we assume that the parameter ω in (3.1) and (3.2) is equal to one.

The first panel of Figure 3.3 displays the evolution of realized inflation for $\bar{x}_\ell$= (-.02,-.01, 0, 0.01, 0.02). The red line corresponding to $\bar{x}_\ell = 0.0$ is the inflation rate in the high inflation steady state ZLB REE. Regardless of the value of $\bar{x}_\ell$, inflation converges to the high inflation steady state ZLB REE. The second panel is the analog to the first, where we begin from the low inflation steady state ZLB REE. Here inflation diverges from that equilibrium in the learning equilibrium. For positive values of $\bar{x}_\ell$, inflation converges to the high inflation steady state ZLB RE equilibrium. Interestingly for $\bar{x}_\ell < 0$, there does not exist an interior ZLB learning

[9] We obtain virtually identical results regardless of whether $\bar{x}_\ell$ is applied to firms' beliefs about only inflation, only consumption or both.

Figure 3.3: Learning Equilibria Near Steady State ZLB

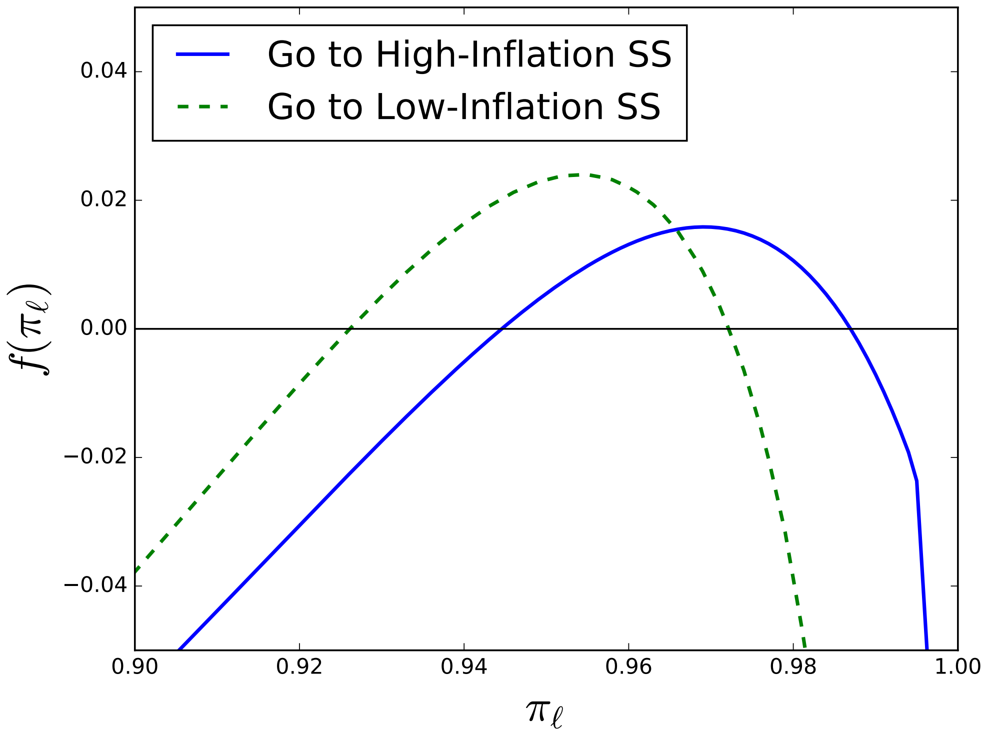


equilibrium.

Until now we supposed that agents believe that once the ZLB is not binding, the economy will go to an REE that converges to the high inflation steady state. It's natural to ask what happens if agents believe that the economy will converge to the low inflation steady state. Figure 3.4 is the analog to Figure 2.1 for this alternative assumption. Notice that the curve is shifted to the left, meaning that there are two steady state ZLB REE, and their inflation rates are lower than under our previous assumptions. The reason that the curve is shifted to the left is that agents expect a lower rate of inflation after the ZLB is over. This effect means that the real interest in the ZLB is higher which leads to lower consumption.

It is still the case that REE which converge to the low inflation steady state are not stable-under-learning while those which converge to the high inflation steady state are stable-under-learning. So regardless of which assumption we make about agents beliefs about the post ZLB period the high inflation ZLB RE equilibrium is stable-under-learning and the low inflation ZLB RE equilibrium is not.

Figure 3.4: Steady State ZLB Equilibrium Function, Alternative SS Expectations



We conclude by noting, that there may be multiple RE ZLB equilibria that converge to the high inflation steady state ZLB equilibrium. But as a practical matter we could find not any of those equilibria. As it turns out, this potential ambiguity is resolved once we redo the analysis using consider the Rotemberg model.

4. Fiscal Policy in the Learning Equilibrium

In this section we analyze the value of government spending multipliers in the learning equilibrium. The effect of learning dynamics on fiscal policy is interesting because the Great Recession was such an unusual event and the rational expectations assumption is of questionable validity.

4.1. Fiscal policy under benchmark learning scheme

We initially assume that agents think that when the ZLB episode is over, the economy reverts to the REE that converges to the high inflation steady state. Later we assess the robustness of our results to this assumption.

Assume that the economy begins in the steady state of the high inflation REE. At time 0, $p^*_{-1} = 1$, r falls to r_ℓ and evolves according to (2.1). Firms and households beliefs about equilibrium functions evolve according to (3.1) and (3.2).

Figure 4.1 displays the paths of consumption, inflation, and the government spending multiplier in the learning equilibrium (the blue lines), as well as the high-inflation ZLB REE paths (the green lines). The paths for inflation and consumption are computed holding government consumption at its steady state value (0.20). Notice that consumption and inflation converge to the high inflation steady state ZLB REE from above. The reason is that agents initially have expectations about inflation and consumption that are higher than warranted after the shock to r. Expectations

Figure 4.1: Learning Equilibrium, Starting from Steady State

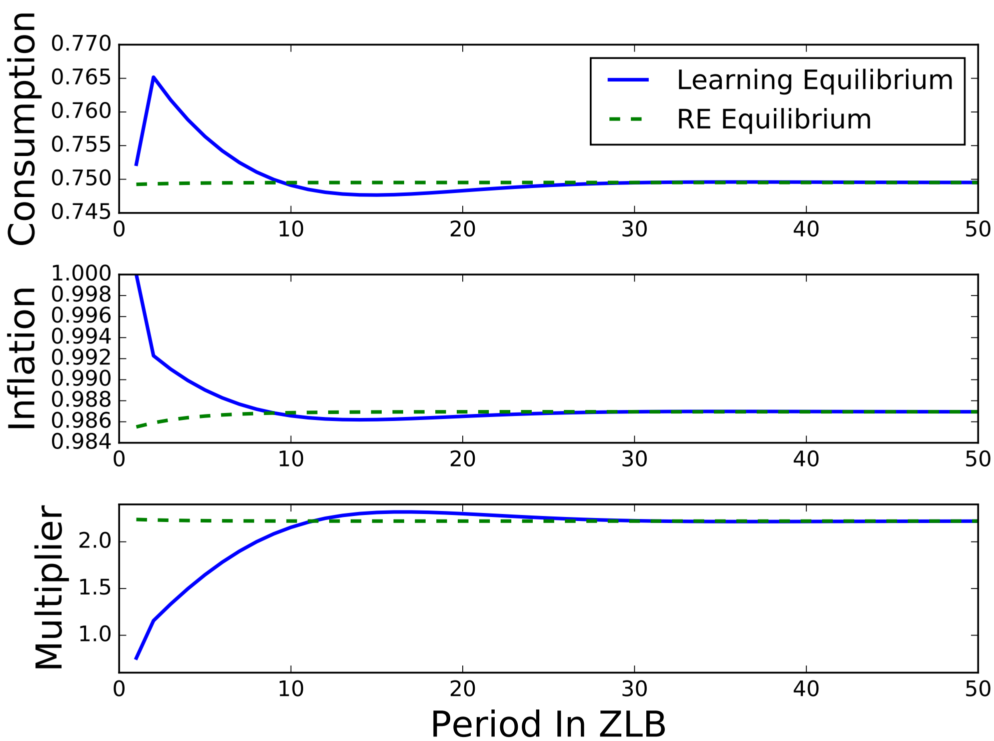
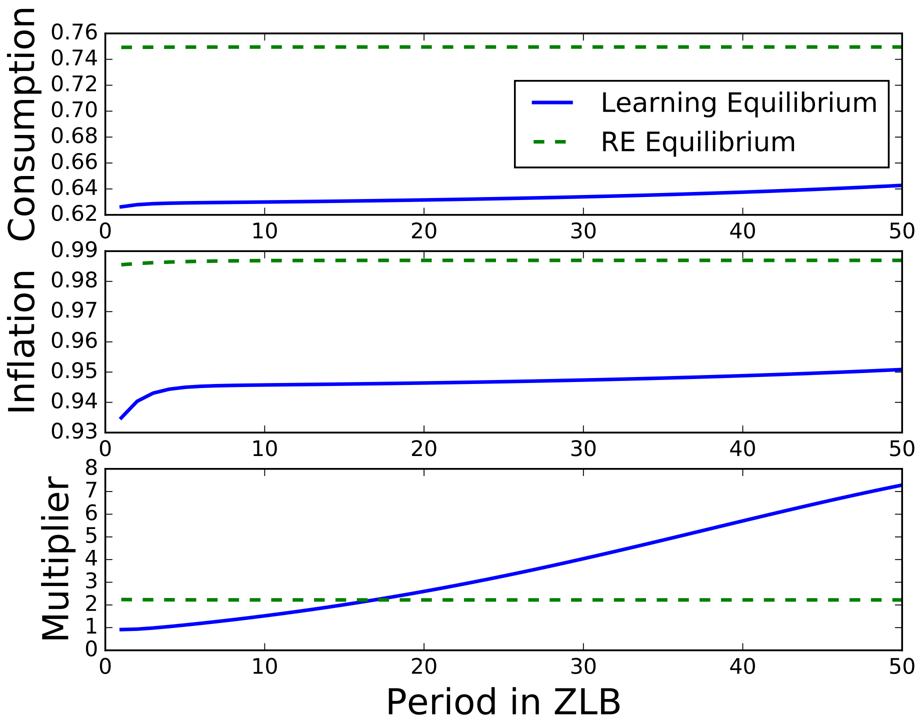


Figure 4.2: Learning Equilibrium, Starting Near Low-Inflation RE ZLB Equilibrium



about higher future inflation and consumption spur demand in the present. As expectations adjust downward in response to the ongoing binding ZLB, inflation and consumption decline further. The value of the multiplier is initially low because the ZLB isn't binding in the first few periods. Once the ZLB starts to bind, the multiplier quickly rises above 1.

Now imagine that after the shock to r_t, firms and household have beliefs near the low-inflation ZLB REE. Figure 4.2 displays the paths of consumption, inflation, and the government spending multiplier in the learning equilibrium (the blue lines), as well as the high-inflation ZLB REE paths (the green lines). As before, the paths for inflation and consumption are computed holding government consumption at its steady state value (0.20). Notice that consumption and inflation converge to the high inflation steady state ZLB REE from below. The reason is that agents initially expect future inflation and consumption to be low. These expectations current consumption and inflation. The multiplier has an initial value of about 1 and then rises. The reason the multiplier rises is that the fiscal expansion helps quickly move expectations toward those associated with the high-inflation ZLB REE. Notably, the multiplier continues to rise for some time. After many periods, the multiplier eventually approaches the high-inflation RE ZLB steady-state equilibrium value.

We conclude that in both scenarios the multiplier is large and eventually to its value in the ZLB REE.

4.2. Reconciling with Mertens and Ravn (2015)

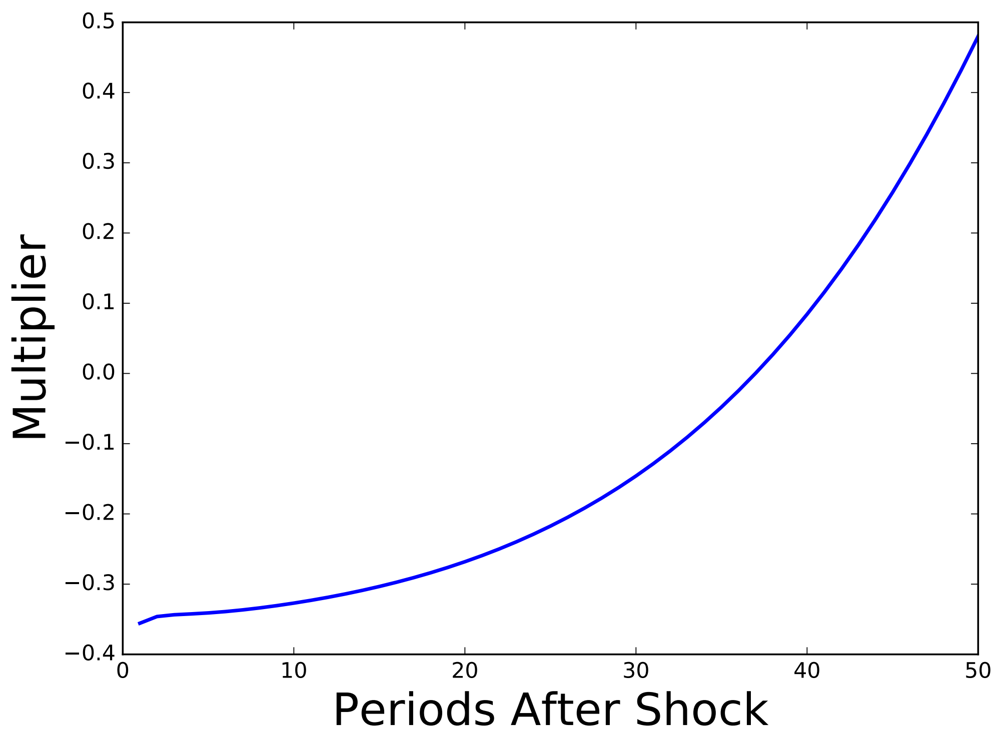
Mertens and Ravn (2015) report that the fiscal multiplier is small when they analyze a learning equilibrium near the low inflation steady state ZLB REE. This result contrasts sharply with our result that the multiplier is very large when we begin near the same equilibrium. There are four differences our analysis and theirs'. First, they work with a linearized Calvo model when they study the learning equilibrium,. Second, they assume that firms who choose prices at time t, see the time t aggregate price level when they choose prices. Third, they model household learning behavior about future consumption as in Evans and Honkapohja (2001). In contrast we suppose that households believe that the function mapping the state p^*_{t-1} to the household consumption decision is the same in the subsequent period. Fourth, the experiment that underlies their multiplier calculation is subtly but very significantly different than ours. When we calculate the multiplier we initially consider an economy in which agents initial expectations about inflation differ by ϵ_π from the low inflation steady state ZLB REE. We then consider a separate economy with shocks that set r to r_ℓ and a shock to G. Agents' expectations about inflation and consumption start in the same place for the two economies. We then use the difference in output between the two economies to calculate the multiplier. Mertens and Ravn (2015) proceed in the same way with one crucial difference.When G increases, the rate of inflation in the steady state low inflation ZLB REE falls by ϵ'_π. When Mertens and Ravn raise G in the learning equilibrium, they also decrease agents' expectations about inflation by ϵ'_π. As discussed above, this fall in inflation expections would in and of itself reduce output in the ZLB.

In the Appendix we show that first three differences between our analysis and Mertens and Ravn (2015) do not have a large impact on the mutliplierIn contrast the fourth difference is very important. Figure 4.3 displays the multiplier as a function of time if we adopt the assumption of Mertens and Ravn (2015) about how expectations about inflation change when G increases. Notice that we obtain a *negative* multiplier that persists for roughly 10 years. This results reflects that, in this example, the change in expectations is quantitatively much more important than the increase in G. From our perspective, the Mertens and Ravn experiment confounds the effects of two shocks.

5. The Rotemberg Model

A number of authors have studied the behavior of the economy in the ZLB interpreting the price frictions in the EW analysis as stemming from adjustment costs as proposed by Rotemberg (1982). A prominent example in this literature is Braun, Boneva and Waki (2015) who study the accuracy of linear approximations to the

Figure 4.3: Multiplier with Shock to Expectations



model.[10] This model is interesting because it implies the same linearized equations that EW study. In this section we highlight an important potential shortcoming of using Rotemberg adjustment costs when studying multiplicity and learnability issues.

With once exception, the Rotemberg model is identical to the Calvo model discussed above. The exception is that instead of (2.7) - (2.8) we assume that the monopolist who produces the j^{th} good has the following objective:

$$E_t \sum_{k=0}^{\infty} \beta^k \lambda_{t+k}[(1+\nu)\frac{P_{j,t+k}}{P_{t+k}} Y_{j,t+k} - s_{t+k} Y_{j,t+k} - \Phi_{t+k}\left(\frac{P_{j,t+k}}{P_{j,t+k-1}} - 1\right)^2]. \tag{5.1}$$

The variable Φ_t denotes a potentially state dependent function that scales the firm's costs of adjusting prices. In the classic Rotemberg model,

$$\Phi_t = \phi \tag{5.2}$$

To accommodate growth, Christiano and Eichenbaum (2012) assume

$$\Phi_t = \frac{\phi}{2}(C_t + G_t). \tag{5.3}$$

In contrast, authors like Braun et. al. (2015) and Gust, Herbst, Lopez-Salido and Smith (2015), assume

$$\Phi_t = \frac{\phi}{2} Y_t. \tag{5.4}$$

As it turns out, existence and learnability of equilibria in the Rotemberg model depend on exactly which specification of Φ_t one adopt.

[10] Braun, Boneva, and Waki (2015) paper was first written in 2012. As best as we can tell, it is the first paper to analyze the accuracy of the linearized EW model of the ZLB relative to the underlying nonlinear model.

It is well known that an interior minimum state variable equilibrium for all three versions of the Rotemberg model is a set of eight numbers:

$$\pi, C, R, h, \pi_\ell, C_\ell, R_\ell, h_\ell,$$

that, when $r_t = r_\ell$, satisfy:

$$\begin{aligned}
R_\ell &= \max\left\{1, \frac{1}{\beta} + \alpha\left(\pi_\ell - 1\right)\right\} && (5.5)\\
\frac{1}{R_\ell} &= \frac{1}{1+r_\ell}\left[p\frac{C_\ell}{\pi_\ell C_\ell} + (1-p)\frac{C_\ell}{\pi C}\right] && (5.6)\\
h_\ell &= C_\ell + G_\ell + \Phi_\ell\left(\pi_\ell - 1\right)^2 && (5.7)\\
\left(\pi_\ell - 1\right)\pi_\ell &= \frac{1}{2\Phi_\ell}\varepsilon\left(\chi h_\ell C_\ell - 1\right)\left[C_\ell + G_\ell + \Phi_\ell\left(\pi_\ell - 1\right)^2\right] && (5.8)\\
&\quad + \frac{1}{1+r_\ell}\left[p\left(\pi_\ell - 1\right)\pi_\ell + (1-p)\left(\pi - 1\right)\pi\frac{C_\ell}{C}\frac{\Phi}{\Phi_\ell}\right]
\end{aligned}$$

Subscript ℓ denotes the value of a variable when $r_t = r_\ell$ and no subscript denotes the value of a variable after $r_t = r$.[11]

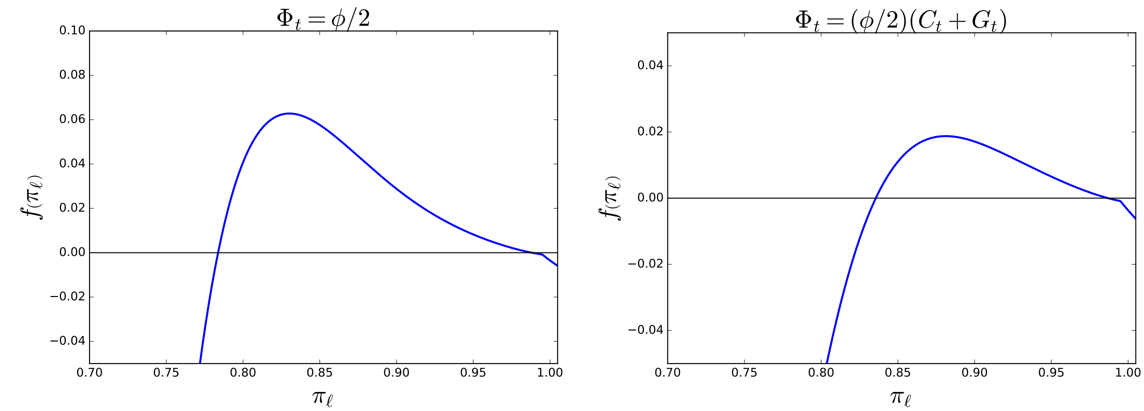
The equations defining a RE equilibrium collapse into one equation in one unknown, π_ℓ,

$$f(\pi_\ell) = 0. \tag{5.9}$$

This equation is analogous (2.15) in the Calvo model. The key difference is that the latter is an equation that determines the steady state ZLB REE. Since there is no state variable in the Rotemberg model, (5.9) determines the ZLB equilibrium values so long as $r_t = r_\ell$.

[11] We formally derive these equations in the appendix and describe the way we solve for an equilibrium.

Figure 5.1: $f(\pi_\ell)$ in Rotemberg Model



The two panels of Figure 5.1 plot $f(\pi_\ell)$ for Φ_t given by (5.2) and (5.3). In all cases we use the benchmark parameters given in (2.16). The parameter ϕ is chosen so that the log-linearized model implies the same system of equations implied by the log-linearized Calvo model, respectively.[12] The domain of admissible values of π_ℓ is restricted by the conditions that $C_\ell > 0$ and $Y_\ell > 0$.

Two features of the figures are worth noting. First, the plots of $f(\pi_\ell)$ are very similar when Φ_t is given by (5.2) or (5.3). Second, there are two ZLB equilibria, both of which feature deflation. Note that the curve looks very similar to the analogous curve that determines the two steady state ZLB REE in the non-linear Calvo model.

The two panels of Figure 5.1 display $f(\pi_\ell)$ for different specifications for Φ_t. The first and second panel correspond to the case where Φ_t is given by (5.2) and (5.3), respectively. Both specification give rise to $f(\pi_\ell)$ functions that are similar to each other and to (2.15). The level of inflation in that equilibrium is similar to the level of inflation in the stable - under-learning ZLB REE of the Calvo model.

[12] Given our normalization that steady state output is one, this requirement implies that ϕ satisfies $(\varepsilon - 1)\,\phi = \frac{(1-\theta)(1-\beta\theta)}{\theta}$.

Figure 5.2: $f(\pi_\ell)$ in Rotemberg Model

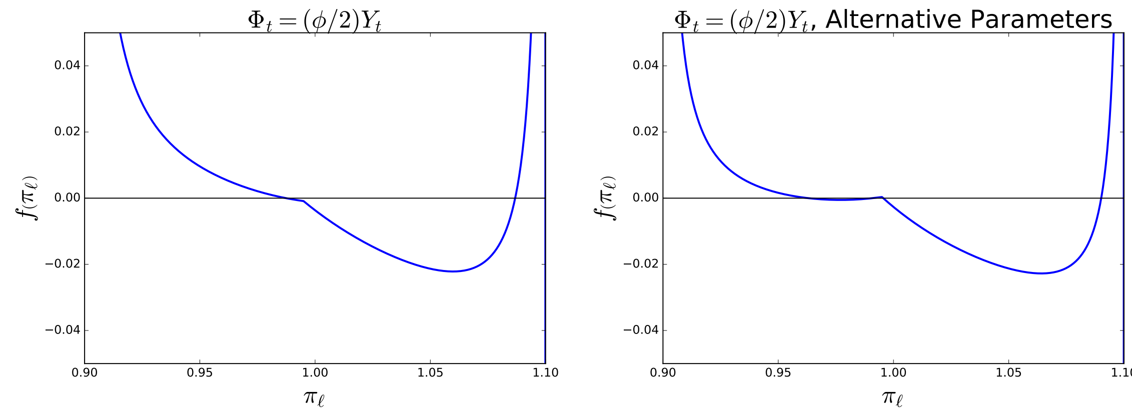


Figure (5.2) displays $f(\pi_\ell)$ for the case where Φ_t is given by (5.4) and our benchmark parameter values. Notice that there are two equilibria when $r_t = r_\ell$. In one case the ZLB binds and that equilibrium is stable under learning. In the other case the ZLB doesn't bind. As it turns out with this specification of Φ_t it is possible to generate more exotic equilibria. The second panel of Figure 5.2 is the analog to the first except that model's parameters are given by:

$$
\begin{aligned}
\varepsilon &= 7.0,\ \beta = 0.99,\ \alpha = 2.0,\ p = 0.83, \\
r^\ell &= -0.0001,\ \phi = 200,\ \eta_g = 0.2, g_\ell = 0.23
\end{aligned}
$$

Strikingly when $r_t = r_\ell$ there are now two equilibria where the ZLB binds and two equilibria where the ZLB isn't binding. This example is consistent with results in Braun et. al. (2015). Note that in both panels of Figure 5.2 $f(\pi_\ell)$ has asymptotes, at quarterly rates of deflation and inflation of 10%. At these rates of inflation, the costs of adjustment consume all of output so that consumption can no longer be non-negative.

It is easy to characterize which ZLB equilibria are stable under learning for the Rotemberg model. Going from left to right in the plots, whenever $f(\pi_\ell)$ crosses from above, the equilibrium is stable under learning. From figure 5.1, we see that when Φ_t is given by (5.2) or (5.3) there is a unique equilibrium that is stable under learning. That equilibrium is the one with less deflation. When Φ_t is given by (5.4) and we work with the benchmark parameter values there is only one ZLB equilibrium and it is stable under learning. Notably, the non-ZLB equilibrium is not stable when adjustment costs are given by (5.4). Even with two ZLB equilibria, as in the second panel of 5.2, there is only one that is stable under learning. Interestingly, that equilibrium is the one that has more deflation. So our key conclusion from Calvo for the Rotemberg model: there is a unique ZLB REE that is stable under learning.

Unlike the Calvo model, the multiplier in the ZLB for the Rotemberg model is constant. Table 5.1 summarizes the values of the multiplier for the equilibria in Figures 5.1 and 5.2 that are stable. Notice that these multipliers are remarkably similar to each other and to the multiplier in the linear Calvo model (1.63). Viewed as a whole our results strongly support the view that once we focus on learnable equilibria, the implications of the NK model for multipliers in the ZLB are very robust: the multiplier is large and increasing the more binding is the ZLB.[13]

6. Conclusion

In this paper we analyze whether the non-uniqueness of equilibria in NK models poses a substantive challenge to the key conclusions in the literature about the efficacy of

[13] When adjustment costs are scaled by (5.2) or (5.3), we are able to find sunspot equilibria similar to the equilibria studied by Mertens and Ravn (2015). However, when adjustment costs are scaled by (5.4), there is no such ZLB equilibrium under our benchmark parameterization. Instead, the sunspot equilibrium exhibits high inflation. Again, we find that the sunspot equilibrium is not stable under learning.

Table 5.1: Multipliers in the Rotemberg Model

Adj. Cost	Stable Equilibrium	Unstable Equilibrium
$\Phi_t = \frac{\phi}{2}$	1.56	0.98
$\Phi_t = \frac{\phi}{2}(C_t + G_t)$	1.70	0.36
$\Phi_t = \frac{\phi}{2}Y_t$	1.65	1.07

fiscal policy in ZLB episodes. We argue that it does not. This conclusion rests on our view that if an REE is not stable-under learning, then it is simply too fragile to be taken seriously as a description of the data. We make our argument using particular models of learning. While we have explored alternative learning mechanisms, it is certainly possible that there exist alternative learning models for which our results do not go through. Still we believe our results are very supportive of the view that the key properties of linearized NK models regarding the impact of changes in government consumption in the ZLB are robust and should be taken seriously.

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