Screening and Adverse Selection in Frictional Markets*

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Abstract

We develop a tractable framework for analyzing adverse selection economies with imperfect competition. In our environment, uninformed buyers offer a general menu of screening contracts to privately informed sellers. Some sellers receive offers from multiple buyers while others receive offers from only one buyer, as in Burdett and Judd (1983). This specification allows us to smoothly vary the degree of competition, nesting monopsony and perfect competition à la Rothschild and Stiglitz (1976) as special cases. We show that the unique symmetric mixed-strategy equilibrium exhibits a *strict rank-preserving* property, in that different types of sellers have an identical ranking over the various menus offered in equilibrium. These menus can be all separating, all pooling, or a mixture of both, depending on the distribution of types and the degree of competition in the market. This calls into question the practice of using the incidence of separating contracts as evidence of adverse selection without controlling for market structure. We examine the relationship between *ex-ante* welfare and the degree of competition, and show that in some cases an interior level of frictions maximizes welfare, while in other cases competition is unambiguously bad for welfare. Finally, we study the effects of various policy interventions — such as disclosure and non-discrimination requirements — and show that our model generates new, and perhaps counter-intuitive insights.

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1 Introduction

Many important markets suffer from *adverse selection*, including markets for insurance, loans, and even certain financial securities. In these markets, a common way to ameliorate the effects of asymmetric information is by designing nonlinear contracts that *screen* (or separate) different types. As a result, a large literature — both theoretical and empirical — has developed in order to study prices, allocations, and welfare in adverse selection markets with nonlinear pricing schedules.

However, perhaps surprisingly, this literature has focused primarily on the two extreme cases of either perfectly competitive or monopolistic market structures.[1] This focus has certainly not been motivated by empirical considerations, as many insurance, credit, and even financial markets are characterized by some degree of *imperfect competition*. Instead, it seems that an important reason for focusing on these two special cases has been a shortage of tractable models that can accommodate the analysis of the more general case.[2]

As a result, in contrast to the extreme cases of perfect competition and monopoly, much less is known about the continuum of cases in between. Does an equilibrium always exist? Is it unique? What is the nature of the menu of contracts that are offered in equilibrium? Do they pool or screen different types? Do some buyers end up predominantly trading with one type of seller, or do buyers typically trade equally often with different types? Finally, and perhaps most importantly, how does welfare respond to an increase in competition? Should policymakers necessarily be promoting competition and transparency in markets with adverse selection?

In this paper, we incorporate adverse selection, screening, and imperfect competition into a simple, parsimonious framework, and then we use this framework to answer all of the questions posed above. The basic building blocks of our model are completely standard: sellers are endowed with a perfectly divisible good, which is either of low or high quality, and this is the seller's private information; buyers offer a seller a menu of contracts, using price-quantity pairs to potentially screen high- and low-quality sellers; and sellers can accept at most one

[1] We discuss several notable exceptions in the literature review below.

[2] As Einav et al. (2010b) remark in a recent survey, relative to the progress made along other dimensions in the insurance literature, "there has been much less progress on empirical models of insurance market competition, or empirical models of insurance contracting that incorporate realistic market frictions. One challenge is to develop an appropriate conceptual framework."

contract, i.e., contracts are exclusive. Given these assumptions, if all sellers received offers from multiple buyers, our environment is equivalent to the perfectly competitive case studied in, e.g., Rothschild and Stiglitz (1976), Rosenthal and Weiss (1984), and Dasgupta and Maskin (1986). Alternatively, if all sellers receive only a single offer, our environment is equivalent to the monopsonist case studied in, e.g., Mirrlees (1971), Stiglitz (1977), and Maskin and Riley (1984).

Our main point of departure from the existing literature is that we assume buyers are uncertain about whether they are competing with other buyers. In particular, in the tradition of Burdett and Judd (1983), we assume that each buyer will be competing with one other buyer with probability π, and he will be a monopsonist with probability $1-\pi$. In this way, we can smoothly vary the degree of competition in the market between the two extremes discussed above.

Now, even though the environment is fairly simple, the characterization of equilibrium is potentially very complicated. This is because, in almost all regions of the parameter space, buyers optimally choose to mix across menus according to a non-generate distribution function.[3] Since each menu is comprised of two price-quantity pairs (one for each type), this implies that the key equilibrium object is a probability distribution over four-dimensional offers. A priori, there is no obvious reason to believe that these offers take any particular form; equilibria at the limits of $\pi = 0$ and $\pi = 1$, alone, can be separating or pooling, and can involve trade with only one type or full trade with both types.

Despite these complications, we develop techniques that enable us to provide a complete characterization of the equilibrium set. We first show that any contract can be summarized by the indirect utility it offers the type of seller it is intended for; this reduces the dimensionality of each menu from four to two. Then, we establish a key property of every equilibria: we show that any two menus that are offered in equilibrium are ranked in exactly the same way by both low and high type sellers. In other words, a buyer's offer for high- and low-quality sellers fall in the exact same percentile of the marginal distributions of equilibrium offers. This property of equilibria, which we call "strictly rank preserving," simplifies the characterization even more, as the marginal distribution of offers for high type sellers can be expressed as a

[3] Mixing is to be expected for at least two reasons. First, this is a robust feature of nearly all models in which buyers are both monopsonists and Bertrand competitors with some probability, even without adverse selection or non-linear contracts. Second, even in perfectly competitive markets, it is well known that pure strategy equilibria may not exist in an environment with both adverse selection and non-linear contracts.

strictly monotonic transformation of the marginal distribution of offers for low type sellers.

Using these results, we characterize the unique equilibrium for any value of π and any fraction of high- and low-quality sellers. We then exploit this characterization to explore the implications—both positive and normative—of imperfect competition in markets suffering from adverse selection.

We show that, in contrast with many papers in the literature, whether buyers offer sellers separating or pooling menus depends on the underlying distribution of types in the market. However, the nature of equilibrium contracts does not depend on this distribution alone, but also on the market structure: separating menus are more prevalent when markets are more competitive, while pooling menus emerge when markets are more frictional. These results suggest that observing separating contracts in a market is not necessarily a sign of severe adverse selection; identifying the severity of information frictions requires knowledge of the prevailing trading frictions.

Turning to the model's normative implications, we show that ex ante welfare is inverse U-shaped in π when adverse selection is severe, i.e., when μ_h is sufficiently small. Therefore, in this region of the parameter space, there is an *interior* level of market frictions that maximizes surplus from trade. When adverse selection is mild, on the other hand, ex ante welfare is monotonic and decreasing in π, so that competition unambiguously hinders the process of realizing gains from trade.

Finally, we demonstrate that explicitly modeling competition and allowing for general contracts yields novel and interesting policy implications. Specifically, we analyze the effect of making additional information about sellers' types available to buyers on *ex-ante* welfare. This is a very topical question in the context of recent developments, both due to policy and/or technological changes, in a number of insurance and financial markets. Our main finding is that the desirability of additional information depends both on the distribution of types and the degree of competition. In particular, when adverse selection is relatively mild to begin with or when the market is very competitive, additional information is detrimental to welfare. The opposite is true when adverse selection and trading frictions are relatively severe. Thus, evaluating the implications of these policies requires knowledge of the distribution of types in the population as well as the extent of frictions.

Literature Review Our paper contributes to a vast body of work on adverse selection. Our focus on contracts as screening devices puts us in the tradition of Rothschild and Stiglitz (1976) as opposed to the branch of the literature which restricts attention to single price contracts, following Akerlof (1970). Both approaches have been used extensively in empirical work on markets for financial assets, loans and insurance, ranging from tests of adverse selection to effects of policy interventions.[4]

The main novelty of our analysis - and our primary contribution - is a tractable and flexible specification of imperfect competition without restrictions on contracts. The literature, on the other hand, has generally stayed within the perfectly competitive paradigm. There are a few notable exceptions. In an important paper, Guerrieri et al. (2010) study a environment with search frictions, where principals post contracts and match bilaterally with agents who direct their search efforts towards specific contracts. The departure from the perfectly competitive benchmark comes from the bilateral matching technology - which implies that, depending on the relative measure of principals and agents, respectively offering and searching for a specific contract, a subset of agents may be rationed, i.e. fail to trade. Under a plausible restriction on off-equilibrium beliefs, the resulting equilibrium is unique. We view our approach to modeling competition as distinct but complementary to the competitive search paradigm. Both approaches present explicit models of trading and allow principals to offer general contracts. There are, however, a few important differences. First, we obtain a unique equilibrium without additional assumptions or refinements. Second, we find that, depending on parameters, equilibrium contracts can be full pooling, separation or a combination of both (the Guerrieri et al. (2010) equilibrium, on the other hand, always features separation). In this sense, our approach has the potential to speak to a richer set of observed outcomes. Finally, our specification allows us to vary the degree of competition in a simple and intuitive way, nesting the well-known limit cases of monopsony and perfect competition as special cases.

The other approach to modeling imperfect competition with selection effects is product differentiation. Two recent examples are Benabou and Tirole (2014) and Mahoney and Weyl (2014). Identical contracts offered from different principals are valued differently by agents, due to an orthogonal attribute - 'distance' in a Hotelling interpretation, 'taste' in a random utility, discrete choice framework. This additional dimension of heterogeneity is the source of market power.

[4] For example, Ivashina (2009) and Chiappori and Salanie (2000) test for adverse selection, while Einav et al. (2012) and Einav et al. (2010a) examine effects of policy interventions

Changes in competition are induced by varying the importance of this alternative attribute, i.e. by altering preferences. We take a different approach to modeling (and varying) competition, which holds constant preferences and therefore, the potential social surplus. It is also worth noting that we arrive at very different conclusions about the desirability of competition compared to the aforementioned papers. In Benabou and Tirole (2014), a trade-off from increased competition arises not due to adverse selection *per se*, but from the need to provide incentives to allocate effort between multiple, imperfectly observable or contractible tasks. Without the issues raised by multi-tasking, competition improves welfare even in the presence of asymmetric information. In Mahoney and Weyl (2014), where attention is restricted to single-price contracts, welfare always increases with competition under adverse selection.

Our formalization of imperfect competition draws from the literature on search frictions and in particular, from the seminal work of Burdett and Judd (1983). In a paper contemporaneous to this one, Garrett et al. (2014) also introduce similar frictions into an environment with private information and screening contracts. Importantly, however, they restrict attention to private values. In other words, the private information of the agents is about their own payoffs and not that of the principals. This has significant Under these conditions, screening only serves as a tool for monopsony rent extraction by the principals. Competition reduces (and ultimately, eliminates) these rents and along with them, incentives to screen. Thus, when the asymmetric information is about private values, screening disappears under perfect competition. In contrast, with common values, screening plays a central role in mitigating adverse selection problem. As a result, it disappears only when that problem is sufficiently mild - increased competition serves to strengthen incentives to separate.

2 Model

Environment. We consider a market populated by a measure of sellers and a measure of buyers. Each seller is endowed with a single unit of a perfectly divisible good. A fraction $\mu_l \in (0, 1)$ of sellers posses a low (l) quality good, while the remaining fraction $\mu_h = 1 - \mu_l$ possess a high (h) quality good. Buyers and sellers derive utility v_i and c_i, respectively, from consuming each unit of a quality $i \in \{l, h\}$ good. We assume that

$$v_i > c_i \quad \text{for } i \in \{l, h\}, \tag{1}$$

so that there are gains from trading both high and low quality goods.

There are two types of frictions in the market. First, there is *asymmetric information*: sellers observe the quality of the good they possess while buyers do not, though the probability μ_i that a randomly selected good is quality $i \in \{l, h\}$ is common knowledge. In order to generate the standard "lemons problem," we focus on the case in which

$$v_l < c_h. \tag{2}$$

The second type of friction is a *search* friction: the buyers in our model post offers, but sellers only sample a finite number of these offers. In particular, we assume that each seller samples one offer with probability $1 - \pi$ and two offers with probability π. Throughout the paper, we refer to sellers with one offer as "captive," while we refer to those with two offers as "non-captive" sellers.[5]

We allow buyers to post menus that specify different price-quantity pairs. By the revelation principle, we can restrict attention to menus with two pairs, $\{(x_l, t_l), (x_h, t_h)\} \in \left([0, 1] \times \mathbb{R}_+\right)^2$, that specify a quantity $x_i \leqslant 1$ of the good to be sold in exchange for a transfer t_i from the buyer to the seller, given that the seller reports owning a quality $i \in \{l, h\}$ good. Importantly, we assume that *contracts are exclusive*: if a seller samples two buyers' offers, he can only accept the offer of one buyer. Throughout the paper, we refer to a quantity-transfer pair (x, t) as a "contract," while we refer to a pair of contracts $\{(x_l, t_l), (x_h, t_h)\}$ as a "menu."

Payoffs. A seller who owns a quality i good and accepts a contract $(x_{i'}, t_{i'})$ receives a payoff

$$t_{i'} + (1 - x_{i'})c_i,$$

while a buyer who acquires a quality i good at terms $(x_{i'}, t_{i'})$ receives a payoff

$$-t_{i'} + x_{i'}v_i.$$

Meanwhile, a seller with a quality i good who does not trade receives a payoff c_i, while a buyer who does not trade receives zero payoff.

[5] This way of modeling search frictions follows from a long tradition, starting with Butters (1977), Varian (1980), and Burdett and Judd (1983).

Strategies and Definition of Equilibrium. Let $\mathbf{z}_i = (x_i, t_i)$ denote the contract that is intended for a seller of type $i \in \{l, h\}$, and let $\mathbf{z} = (\mathbf{z}_l, \mathbf{z}_h)$. A buyer's strategy, then, is a distribution across menus, $\Phi \in \Delta\big(([0,1] \times \mathbb{R}_+)^2\big)$.

A seller's strategy is much simpler: given the available menus, a seller should choose the menu with the contract that maximizes her payoffs, or mix between menus if she is indifferent. Of course, conditional on a menu, the seller chooses the contract which maximizes her payoffs. In what follows, we will take the seller's optimal behavior as given.

A symmetric equilibrium is thus a distribution $\Phi^\star(\mathbf{z})$ such that:

1. *Incentive compatibility*: for almost all $\mathbf{z} = \{(x_l, t_l), (x_h, t_h)\}$ in the support of $\Phi^\star(\mathbf{z})$,

$$t_l + c_l(1 - x_l) \geqslant t_h + c_l(1 - x_h) \tag{3}$$

$$t_h + c_h(1 - x_h) \geqslant t_l + c_h(1 - x_l). \tag{4}$$

2. *Buyer's optimize*: for almost all $\mathbf{z} = \{(x_l, t_l), (x_h, t_h)\}$ in the support of $\Phi^\star(\mathbf{z})$,

$$\mathbf{z} \in \arg\max_{\mathbf{z}} \sum_{i \in \{l,h\}} \mu_i (v_i x_i - t_i) \left[1 - \pi + \pi \int_{\mathbf{z}'} \chi_i(\mathbf{z}, \mathbf{z}') \Phi^\star(d\mathbf{z}')\right], \tag{5}$$

where

$$\chi_i(\mathbf{z}, \mathbf{z}') = \left\{\begin{array}{c} 0 \\ \frac{1}{2} \\ 1 \end{array}\right\} \quad \text{if} \quad t_i + c_i(1 - x_i) \left\{\begin{array}{c} < \\ = \\ > \end{array}\right\} t_i' + c_i(1 - x_i'). \tag{6}$$

The function χ_i represents the seller's optimal choice between two menus. Note that we have assumed that if the seller is indifferent between menus then she chooses among menus with equal probability. Within a given menu, we have assumed that sellers do not randomize; for any incentive compatible contract, sellers choose the contract intended for their type, as in most of the mechanism design literature (see Myerson (1979), Dasgupta et al. (1979), for examples).

It is worth noting that the sellers in our model are heterogenous along two dimensions: the quality of the good that they are selling, and both the number and the type of alternative menus available to them. Moreover, *both* the quality of their good and their available alternatives are privately known. One might argue that buyers should try to screen the second dimension as well as the first, i.e., offer contracts that explicitly depend on whether a seller is captive, along with the details of the alternative menu when the seller is not captive. However, captive and non-captive sellers receive the same payoff from accepting any contract, and hence contracts that explicitly depend on the number or type of outside menus are irrelevant. That is, buyers cannot attain better outcomes if they are permitted to offer multiple menus intended for sellers that have the same quality good but differ with respect to the alternative menus that are available. We have implicitly used this result in our setup above. Note that when contracts are non-exclusive and sellers can trade with multiple buyers, this result no longer holds;.

3 Properties of Equilibria

Characterizing the equilibrium described above—a distribution over four-dimensional menus that maximize buyers' payoffs while preserving incentive compatibility—is a daunting task. In this section, we establish a series of results that reduce the dimensionality of the equilibrium characterization.

First, we show that each menu offered by a buyer can be summarized by the indirect utilities that it delivers to each type of seller, so that equilibrium strategies can in fact be defined by a joint distribution over two-dimensional menus. Then, we establish that the marginal distributions of offers intended for each type of seller are well-behaved, i.e., that they have fully connected support and no mass points.

Finally, we establish that there is a very precise link between the two contracts offered by any buyer, which imposes even more structure on the joint distribution of offers. In particular, we show that any two menus that are offered in equilibrium are ranked in exactly the same way by both low and high type sellers; that is, one menu is strictly preferred by a low type seller if and only if it is also preferred by a high type seller. This property of equilibria, which we call "strictly rank preserving," simplifies the characterization even more, as the marginal distribution of offers for high type sellers can be expressed as a strictly monotonic transformation of the marginal distribution of offers for low type sellers. This property of equilibria also has important implications regarding the correlation of offers and terms of trade across different types of sellers; we explore these predictions in more detail in Section 5, when we explore the positive implications that emerge from our model in greater detail.

3.1 Utility Representation

As a first step, we establish two results that imply any menu can be summarized by two numbers, (u_l, u_h), where

$$u_i = t_i + c_i(1 - x_i) \tag{7}$$

denotes the utility received by a type $i \in \{l, h\}$ seller from accepting a contract $\mathbf{z}_i$.

Lemma 1. *In any equilibrium, for almost all* $\mathbf{z}$ *in the support of* $\Phi^\star$*, it must be that* $x_l = 1$ *and* $t_l = t_h + c_l(1 - x_h)$.

In words, Lemma 1 states that all equilibrium menus require that low quality sellers trade their entire endowment, and that the incentive compatibility constraint always binds for low quality sellers. This is reminiscent of the "no-distortion-at-the-top" result in the taxation literature, or that of full-insurance for the high-risk agents in Rothschild and Stiglitz (1976).

Corollary 2. *In equilibrium, any menu of contracts* $\{(x_l, t_l), (x_h, t_h)\} \in \left([0,1] \times \mathbb{R}_+\right)^2$ *can be summarized by a pair* (u_l, u_h) *with* $x_l = 1$, $t_l = u_l$,

$$x_h = 1 - \frac{u_h - u_l}{c_h - c_l}, \text{ and} \tag{8}$$

$$t_h = \frac{u_l c_h - u_h c_l}{c_h - c_l}. \tag{9}$$

Since $0 \leqslant x_h \leqslant 1$, note that the pair (u_l, u_h) must satisfy

$$c_h - c_l \geqslant u_h - u_l \geqslant 0 \tag{10}$$

in order to satisfy feasibility. Note that when $u_h = u_l$, Corollary 2 implies that $x_h = 1$ and $t_h = t_l$.

3.2 Recasting the Buyer's Problem and Equilibrium

Buyer's Problem. Given the results above, we can recast the problem of a representative buyer as choosing a menu of indirect utilities, (u_l, u_h), taking as given the distribution of indirect utilities offered by other buyers. For any menu (u_l, u_h), buyers must infer the probability that the menu will be accepted by a type $i \in \{l, h\}$ seller. In order to calculate these probabilities, let us define the marginal distributions

$$\begin{aligned} F_l(u_l) &= \int_{\mathbf{z}'_l} \mathbf{1}\left[t'_l + c_l\left(1 - x'_l\right) \leqslant u_l\right] \Phi\left(d\mathbf{z}'_l\right) \\ F_h(u_h) &= \int_{\mathbf{z}'_h} \mathbf{1}\left[t'_h + c_h\left(1 - x'_h\right) \leqslant u\right] \Phi\left(d\mathbf{z}'_h\right). \end{aligned}$$

$F_l(u_l)$ and $F_h(u_h)$ are the probability distributions of indirect utilities arising from each buyer's mixed strategy. When these distributions are continuous and have no mass points, the probability that a contract intended for a type i seller is accepted is simply $1 - \pi + \pi F_i(u_i)$, i.e., the probability that the seller is captive plus the probability that he is non-captive but receives an offer less than u_i. However, if $F_i(\cdot)$ has a mass point at u_i, then the fraction of non-captive sellers of type i attracted to a contract with value u_i is given by $\tilde{F}_i(u_i) = \frac{1}{2}F_i^-(u_i) + \frac{1}{2}F_i(u_i)$, where $F_i^-(u_i) = \lim_{u \nearrow u_i} F_i(u)$ is the left limit of F_i at u_i.[6] Given $\tilde{F}_i(\cdot)$, each buyer solves

$$\max_{u_l \geqslant c_l,\, u_h \geqslant c_h} \quad \mu_l\left(1 - \pi + \pi\tilde{F}_l(u_l)\right)\Pi_l(u_h, u_l) + \mu_h\left(1 - \pi + \pi\tilde{F}_h(u_h)\right)\Pi_h(u_h, u_l) \tag{11}$$

$$\text{s. t.} \quad c_h - c_l \geqslant u_h - u_l \geqslant 0, \tag{12}$$

with

$$\Pi_l(u_h, u_l) \equiv \nu_l x_l - t_l = \nu_l - u_l \tag{13}$$

$$\Pi_h(u_h, u_l) \equiv \nu_h x_h - t_h = \nu_h - u_h\frac{\nu_h - c_l}{c_h - c_l} + u_l\frac{\nu_h - c_h}{c_h - c_l}. \tag{14}$$

In words, $\Pi_i(u_h, u_l)$ is the buyer's payoff conditional on the offer u_i being accepted by a type i seller. We refer to the objective in (11) as $\Pi(u_h, u_l)$.

Equilibrium. Using the optimization problem described above, we can redefine the equilibrium in terms of the distributions of indirect utilities. In particular, for each u_l, let

$$\begin{aligned} U_h(u_l) &= \arg\max_{u'_h \geqslant c_h} \Pi\left(u'_h, u_l\right) \\ &\quad \text{s. t.} c_h - c_l \geqslant u'_h - u_l \geqslant 0. \end{aligned}$$

[6]Since F_i is a distribution function it is right continuous and its left limits exists everywhere (it is cádlág) and it has countable points of discontinuity . We make use of these properties repeatedly throughout the paper.

The equilibrium can then be described by the marginal distributions $\{F_i(u_i)\}_{i\in\{l,h\}}$ together with the requirement that a joint distribution function must exist. In other words, a probability measure μ over the set of feasible (u_l, u_h)'s must exist such that, for each $u_l > u_l'$ and $u_h > u_h'$

$$\begin{aligned} 1 &= \mu\left(\{(\hat{u}_l, \hat{u}_h); \hat{u}_h \in U_h(\hat{u}_l)\}, \hat{u}_l \in [c_l, v_h]\right) \\ F_l^-(u_l) - F_l\left(u_l'\right) &= \mu\left(\left\{(\hat{u}_l, \hat{u}_h); \hat{u}_h \in U_h(\hat{u}_l), \hat{u}_l \in \left(u_l', u_l\right)\right\}\right), \quad (15) \\ F_h^-(u_h) - F_h\left(u_h'\right) &= \mu\left(\left\{(\hat{u}_l, \hat{u}_h); \hat{u}_h \in U_h(\hat{u}_l), \hat{u}_h \in \left(u_h', u_h\right)\right\}\right). \quad (16) \end{aligned}$$

Note that this definition of equilibrium imposes two different requirements. The first is that buyers behave optimally: for each u_l, the joint probability measure puts a positive weight only on $u_h \in U_h(u_l)$. The second is aggregate consistency: the fact that F_l and F_h are marginal distributions associated with a joint measure of menus.

3.3 Basic Properties of Equilibrium Distributions

In this section, we establish that, in equilibrium, the distributions $F_l(u_l)$ and $F_h(u_h)$ are continuous and have connected support, i.e., there are neither mass points nor gaps in either distribution.

Proposition 3. *The marginal distributions F_l and F_h have connected support. They are also continuous, with the possible exception of a mass point at the lower bound of their support.*

Much like Burdett and Judd (1983), the basic idea behind the proof of Proposition 3 is to come up with deviations that rule out having gaps and mass points in the distribution. The difficulty in our model is that payoffs are interdependent, e.g., a change in the utility offered to low type sellers changes the contract—and hence the profits—that a buyer receives from high type sellers. It is possible, however, to prove these claims sequentially, first for the distribution F_h and then for the distribution F_l. We sketch the proofs here, and present the formal arguments in the Appendix.

To see that F_h has connected support, suppose towards a contradiction that F_h is constant on some interval, and consider an equilibrium menu $(\overline{u}_l, \overline{u}_h)$ with $\overline{u}_h$ equal to the upper bound of this interval. We show that offering a menu $(\overline{u}_l, u_h)$, with $u_h < \overline{u}_h$ and $F_h(u_h) = F_h(\overline{u}_h)$, is a profitable deviation because the offer u_h attracts the same fraction of high quality sellers but makes more profit per trade. Such a deviation is feasible, however, only if $\overline{u}_h > \overline{u}_l$. If instead $\bar{u}_h = \bar{u}_l$, we show that a deviation to a menu $(u_l, \bar{u}_h)$, with $u_l < \bar{u}_l$, must be a profitable deviation.

To see that F_h has no mass points, suppose towards a contradiction that it does. If $\Pi_h(u_h, u_l)$ is strictly positive for such a value of u_h, then a small increase in u_h will increase profits by attracting a mass of high quality sellers. If, instead, $Pi_h(u_h, u_l)$ is not positive, then we can establish several facts. First, since overall profits must be strictly positive for any $\pi \in (0, 1)$, then $\Pi_l(u_l, u_h) > 0$ when $\Pi_h(u_l, u_h) \leqslant 0$. Second, it must be that $u_h = c_h$, as otherwise it would be profitable to decrease u_h by a small amount and *shed* a positive mass of high type sellers. Finally, if $u_h = c_h$ and $\Pi_h(u_l, u_h) \leqslant 0$, it must be that $x_h = 0$—that is, the buyer will not actually trade with high type sellers—in which case it must be that $u_l = c_l$. Therefore, if there is a mass point at $u_h = c_h$, there must also be a mass point at $u_l = c_l$. However, if this is true, then a small increase in u_l is a profitable deviation, which completes the contradiction.

Having shown that F_h is continuous and strictly increasing, we then apply an inductive argument to prove that F_l has connected support and is continuous, with a possible exception at the lower bound of the support. An important step in the induction argument, which we use more generally, is to show that the objective function $\Pi(u_h, u_l)$ is strictly supermodular. We state this here as a lemma.

Lemma 4. *Suppose F_h has connected support and is continuous over its support. Then the profit function is strictly supermodular so that*

$$\Pi(u_{h1}, u_{l1}) + \Pi(u_{h2}, u_{l2}) \geqslant \Pi(u_{h1}, u_{l2}) + \Pi(u_{h2}, u_{l1}), \forall u_{i1} \geqslant u_{i2}, i \in \{l, h\}$$

with strict inequality when $u_{i1} > u_{i2}$.

The supermodularity of the buyer's profit function reflects a basic complementarity between the indirect utilities offered to low and high quality sellers. In particular, an increase in the indirect utility offered to low quality sellers relaxes their incentive constraint and allows the buyer to increase the quantity traded with high quality sellers, thus increasing u_h.

An important implication of Lemma 4 is that the correspondence $U_h(u_l)$ is weakly increasing. We use this property to construct deviations similar to those described above in order to rule out gaps and mass points in the distribution F_l at everywhere but, perhaps, the lower bound of the support. However, in Section 4, we show that mass points only occur at the lower bound in a knife-edge case; generically, the marginal distribution F_l has connected support and no mass points everywhere in its support. We defer the details of the proof to Appendix A.

3.4 Strict Rank Preserving

In this section, we establish that the mapping between a buyer's optimal offer to low and high quality sellers, $U_h(u_l)$, is a well-defined, strictly increasing function. As a result, every equilibria has the property that menus are *strictly rank preserving*—that is, low and high types share the same ranking over the set of contracts offered in equilibrium—with the possible exception of the knife-edge case discussed above. To formalize this result, the following definition is helpful.

Definition 5. *For any subset $\mathcal{U}_l$ of $\mathrm{Supp}\,(F_l)$, an equilibrium is **strictly rank-preserving over** $\mathcal{U}_l$ if the correspondence $U_h\,(u_l)$ is a strictly increasing function of u_l for all $u_l \in \mathcal{U}_l$. An equilibrium is **strictly rank-preserving** if it is strictly rank-preserving over $\mathrm{Supp}\,(F_l)$.*

Equivalently, an equilibrium is strictly rank-preserving when, for any two points in the equilibrium support (u_l, u_h) and $\left(u_l', u_h'\right)$, $u_l > u_l'$ if and only if $u_h > u_h'$. Given this terminology, we can now establish one of our key results.

Theorem 6. *Let $\underline{u}_l = \min \mathrm{Supp}\,(F_l)$. Then all equilibria are strictly rank-preserving over the set $\mathrm{Supp}\,(F_l) \setminus \{\underline{u}_l\}$.*

The proof of Theorem 6, which can be found in the Appendix, makes use of the key facts established above; namely, that the distributions $F_l\,(\cdot)$ and $F_h\,(\cdot)$ are strictly increasing and continuous, and that $\Pi(u_l, u_h)$ is supermodular, which implies that $U_h\,(u_l)$ is a weakly increasing correspondence. In words, if there exists a $u_l > \underline{u}_l$ and $u_h' > u_h$ such that $u_h, u_h' \in U_h(u_l)$, then the supermodularity of $\Pi(u_l, u_h)$ implies that $[u_h, u_h'] \subset U_h(u_l)$. Since $F_h(\cdot)$ has connected support, this implies that $F_l(\cdot)$ must have a mass point at u_l, which contradicts Proposition 3. A similar argument rules out the possibility that there exist u_h and $u_l' > u_l$ that are offered in equilibrium with $U_h(u_l') = U_h(u_l) = u_h$. Hence, $U_h(u_l)$ must be a strictly increasing function for all $u_l > \underline{u}_l$.

Moreover, when $F_l(\cdot)$ is continuous everywhere—that is, when there is no mass point at the lower bound of the support—then every menu offered in equilibrium is accepted by exactly the same fraction of low and high quality non-captive sellers. We state this result in the following Corollary to Theorem 6.

Corollary 7. *If* F_l *and* F_h *are continuous, then* $F_h(U_h(u_l)) = F_l(u_l)$.

Taken together, theorem 6 and Corollary 7 simplify the search for equilibrium, which we undertake in the next Section. Specifically, if an equilibrium exists in which the marginal distributions F_l and F_h are continuous, then the equilibrium can be described compactly by the marginal F_l along with the policy function $U_h(u_l)$.

4 Construction of Equilibrium

In this section, we use the properties established above to help construct equilibria. Then we show that the equilibrium we construct is unique. In this sense, we characterize the entire set of equilibrium outcomes in our model.

To fix ideas, it is useful to discuss the well-known extreme cases of $\pi = 0$ and $\pi = 1$, i.e., when sellers face a monopsonist and when markets are perfectly competitive, respectively. Using these two extreme cases, we identify a key parameter that governs the structure of the equilibrium set for the general case of $\pi \in (0, 1)$.

4.1 Monopsony and Perfect Competition

We start by describing the equilibrium outcome when sellers face a monopsonist and perfectly competitive markets.

Monopsony. When each seller meets with at most one buyer, i.e., $\pi = 0$, the buyers choose a pair (u_l, u_h) to maximize

$$\mu_l(v_l - u_l) + \mu_h \left[v_h - u_h \frac{v_h - c_l}{c_h - c_l} + u_l \frac{v_h - c_h}{c_h - c_l} \right],$$

subject to the constraint described in (12). The solution to this problem, summarized in Lemma 8 below, is standard and hence we omit the proof.

Lemma 8. *Let*

$$\phi = 1 - \frac{\mu_h}{\mu_l} \left(\frac{v_h - c_h}{c_h - c_l} \right). \tag{17}$$

When $\pi = 0$*, the equilibrium is as follows:*

(i) if $\phi > 0$*, then* $u_l = c_l$ *with* $x_l = 1$ *and* $u_h = c_h$ *with* $x_h = 0$*;*

(ii) if $\phi < 0$, *then* $u_l = u_h = c_h$ *with* $x_l = x_h = 1$.

(iii) if $\phi = 0$, *then* $u_l \in [c_l, c_h]$ *with* $x_l = 1$ *and* $u_h = c_h$ *with* $x_h = 0$.

Intuitively, the buyer optimally chooses not to trade with the type h seller when $\phi \geqslant 0$, or

$$\mu_l \geqslant \mu_h \left(\frac{\nu_h - c_h}{c_h - c_l} \right). \tag{18}$$

The left hand side of (18) is the marginal benefit of lowering $u_l = t_l$: with probability μ_l, the buyer pays a marginal unit less in exchange for the low quality good. The right hand side of (18) is the cost of a small decrease in u_l: since reducing u_l tightens the incentive compatibility constraint, the buyer must trade a smaller amount with high quality sellers. Since the objective is linear, $x_h = 0$ when (18) is satisfied. On the other hand, when (18) is violated, the buyer pools high and low quality sellers, offering c_h in exchange for one unit of either good.

Given this interpretation, the parameter ϕ can be thought of as capturing the buyer's net loss from a marginal increase in the utility offered to the low quality seller, taking into account the fact that such an increase relaxes the incentive constraint and leads to higher profits from trading with the high quality seller. This loss is positive when the fraction of high quality sellers, μ_h, is small and/or the gains from trading with high quality sellers, $\nu_h - c_h$, is relatively small.

Note that an important feature of the monopsony allocations is that the buyer makes non-negative profits on each type when $\phi > 0$. However, when $\phi < 0$, the monopolist subsidizes the low quality sellers in order to be able to trade with high quality sellers, i.e., allocations involve *cross-subsidization*. As we will show, this pattern extends to the set of equilibria in our general model with $\pi \in (0, 1)$.

Bertrand Competition. When markets are perfectly competitive, i.e., when $\pi = 1$, our setup becomes the same as that in Riley (1975), Riley (1979) and Rosenthal and Weiss (1984), and similar to that of Rothschild and Stiglitz (1976). As is typical in adverse selection economies, a pure strategy equilibrium in our model must satisfy separation ($u_h > u_l$) and buyers must earn zero profits on each type of seller ($u_l = \nu_l$ and $\Pi_h(u_h, \nu_l) = 0$). Indeed, when μ_h is sufficiently small, so that $\phi \geqslant 0$, the unique equilibrium is the least-cost separating contract: low type sellers earn $u_l = \nu_l$ and high type sellers receive ν_h per unit sold on a fraction of their endowment, which is sufficiently small to ensure that incentive compatibility holds.

However, when μ_h is larger, so that $\phi < 0$, one can construct a pooling menu which yields buyers strictly positive payoffs, and hence dominates the pure strategy equilibrium described above. In this case, no pure strategy equilibrium exists, and thus we turn our attention to equilibria in mixed strategies, as in Dasgupta and Maskin (1986) and Rosenthal and Weiss (1984).

In the equilibrium that emerges, buyers mix across menus (u_l, u_h) in which the payoff from trading with low types is negative, the payoff from trading with high types is positive, but the two payoffs offset each other exactly, so that buyers ultimately break even. This is true across each contract, independent of the marginal distributions. In order to ensure that this is an equilibrium, however, the marginal distribution $F_l(\cdot)$ must be constructed to ensure that pure strategy deviations (either pooling or cream-skimming) attract disproportionately many low quality sellers and therefore are not profitable. The following lemma summarizes this equilibrium in the Bertrand game; the proof is in the Appendix.

Lemma 9. *When $\pi = 1$, the unique equilibrium of the game is as follows:*
(i) if $\phi \geqslant 0$, then $u_l = v_l$ with $x_l = 1$ and $u_h = \frac{v_h(c_h - c_l) + v_l(v_h - c_h)}{v_h - c_l}$ with $x_h = \frac{v_l - c_l}{v_h - c_l}$;
(ii) if $\phi < 0$, then symmetric equilibrium is described by the distribution

$$F_l\left(u_l\right) = \left(\frac{u_l - v_l}{\mu_h\left(v_h - v_l\right)}\right)^{-\phi} \tag{19}$$

for value of the low type, with $\text{Supp}\left(F_l\right) = [v_l, \bar{v}]$ where $\bar{v} = \mu_h v_h + \mu_l v_l$. Furthermore, $U_h\left(u_l\right)$ satisfies

$$\mu_h \Pi_h\left(U_h\left(u_l\right), u_l\right) + \mu_l \Pi_l\left(U_h\left(u_l\right), u_l\right) = 0. \tag{20}$$

Note, again, that the equilibrium features cross-subsidization when $\phi < 0$, but not when $\phi \geqslant 0$.

4.2 Imperfect Competition

We are now ready to analyze the general model with imperfect competition, i.e., when $\pi \in (0, 1)$. As in the two cases studied above, the sign of the parameter ϕ is a key determinant of the structure of the equilibrium outcome. Hence, we consider the three relevant cases—$\phi > 0$, $\phi < 0$, and $\phi = 0$—sequentially.

4.2.1 No Cross-subsidizing Equilibria ($\phi > 0$)

We proceed by first constructing an equilibrium when $\phi > 0$, and then establishing in Section 4.3 that this equilibrium is unique. We construct the equilibrium distribution F_l as the solution to the following differential equation

$$\frac{\pi f_l(u_l)}{1-\pi+\pi F_l(u_l)}(\nu_l - u_l) = \phi \tag{21}$$

along with the boundary condition $\min \operatorname{Supp}(F_l) = c_l$. The policy function, $U_h(u_l)$, satisfies

$$(1-\pi+\pi F_l(u_l))\left[\mu_h \Pi_h(U_h(u_l), u_l) + \mu_l(\nu_l - u_l)\right] = \mu_l(1-\pi)(\nu_l - c_l). \tag{22}$$

To understand this equilibrium, note that the differential equation (21) describes the first order condition for a buyer with respect to u_l when facing the distribution $F_l(u_l)$. In this sense, (21) describes necessary conditions for an equilibrium because it ensures local deviations away from equilibrium menus are not profitable. Implicitly, this first order condition requires three assumptions. First, that there is not a mass point at the lower bound of the support of $F_l(u_l)$. Second, that for each menu (u_l, u_h), the utility for the high quality seller is strictly greater than that for the low quality seller; we call such an equilibrium a *separating equilibrium*. Lastly, that the implied quantity traded by the high quality seller is interior in all trades, i.e., $0 < x_h = (u_h - u_l)/(c_h - c_l) < 1$, except possibly at the boundary of the support of F_l. We call an equilibrium that satisfies this latter property an *interior equilibrium*.

Equation (21) is familiar from basic production theory and resembles optimality condition of a monopolist facing a demand function. For example, in equation (21), the first term on the left-hand side can be interpreted as the elasticity of demand—it represents the percentage change in the fraction of low quality sellers attracted to the contract from a one percent change in the utility offered to the low type seller. Since the second term is profits per trade with low quality sellers, the left-hand side captures the marginal benefit to the buyer of increasing u_l. On the right-hand side of (21), ϕ represents the marginal cost of increasing the utility of the low quality seller, taking into account the fact that increasing u_l relaxes the incentive constraint. The fact that ϕ incorporates this additional benefit of increasing u_l implies that $\phi < 1$, i.e., an increase in the utility of the low quality seller decreases profits by less than 1.

The boundary condition requires that the lowest utility offered to the low quality seller is equal to c_l. From (22), and using the fact that $F_l(c_l) = 0$, we find $U_h(c_l) = c_h$. That is, the worst menu offered in such an equilibrium coincides with the monopsony outcome. Loosely speaking, if the worst equilibrium menu offers more utility to high quality sellers than c_h, then a buyer offering this worst menu could increase profit by decreasing u_h—his payoff from trading with high type sellers would increase without changing the payoffs from trading with low types. On the other hand, if the worst menu offers more utility to low quality sellers than c_l, the buyer could profit by decreasing u_l and u_h—the gains associated with trading from low types would exceed the losses associated from trading with high types precisely because $\phi > 0$.

The final step in our construction of the equilibrium is to determine the policy function, $U_h(u_l)$. Since the worst menu offered in equilibrium is the monopsony contract and all menus offered must earn equal profits, $U_h(u_l)$ is determined by the equal profit condition, equation (22). The right-hand side of (22) define monopsony profits. The left-hand side of (22) defines profits earned from the menu $(u_l, U_h(u_l))$ given the distribution F_l. The following proposition asserts that our construction of F_l and $U_h(u_l)$ from (21) and (22) constitutes an equilibrium.

Proposition 10. *If $\phi > 0$, there exists an interior, separating equilibrium with continuous distributions (F_l, F_h). The equilibrium distribution F_l is characterized by the differential equation (21); the distribution F_h satisfies $F_h(U_h(u_l)) = F_l(u_l)$ with $U_h(u_l)$ determined by (22).*

As discussed above, the differential equation (21) ensures that all local deviations by a buyer from an equilibrium menu are unprofitable. To complete the proof that F_l and $U_h(u_l)$ are an equilibrium, we need only ensure that no global deviations are profitable; we establish that this is true in Appendix A. Notice also from (21) that since $\phi > 0$, our equilibrium satisfies $v_l > u_l$ for all menus in equilibrium. That is, buyers earn strictly positive profits from trading with low quality sellers. It is straightforward to show that buyers also earn strictly positive profits from trading with high quality sellers. Therefore, the equilibrium we have constructed features no cross-subsidization.

4.2.2 Cross-subsidizing Equilibria ($\phi < 0$)

We now turn our attention to the case of $\phi < 0$. As with monopsony and perfect competition, cross-subsidization occurs in the region with $\phi < 0$. Notice that under Bertrand competition cross-subsidization is achieved with separating menus. In contrast, under monopsony cross-subsidization is achieved through a pooling menu. It is then natural to expect equilibrium menus to feature pooling when $\phi < 0$ and markets are imperfectly competitive. We show that in fact this is the case when the number of high quality sellers is high enough. We proceed by constructing an equilibrium when $\phi < 0$,and then establishing in Section 4.3 that this equilibrium is unique.

Our equilibrium construction relies on finding a threshold, $\hat{u}_l$, in the support of the distribution F_l. For utility levels in the support of F_l, equilibrium menus feature pooling with $U_h(u_l) = u_l$; above the threshold $\hat{u}_l$, equilibrium menus feature separation with $U_h(u_l) > u_l$. We find the threshold $\hat{u}_l$ as follows. Let $\bar{\Pi}(\hat{u}_l)$ be defined as

$$\bar{\Pi}(\hat{u}_l) = (1-\pi)\left(\mu_h(\nu_h - c_h) + \mu_l(\nu_l - \min\{c_h, \hat{u}_l\})\right). \tag{23}$$

and let $H(\hat{u}_l)$ satisfy

$$(1-\pi+\pi H(\hat{u}_l))(\mu_h\nu_h + \mu_l\nu_l - u_l) = \bar{\Pi}(\hat{u}_l). \tag{24}$$

Finally, define $\bar{u}_l(\hat{u}_l)$ as

$$\bar{u}_l(\hat{u}_l) = \nu_l + (\hat{u}_l - \nu_l)\left[1-\pi+\pi H(\hat{u}_l)\right]^{\frac{1}{\phi}}. \tag{25}$$

Then, the threshold, $\hat{u}_l$ is the value satisfying

$$\mu_h\nu_h + \mu_l\nu_l - \bar{u}_l(\hat{u}_l) = \bar{\Pi}(\hat{u}_l). \tag{26}$$

If the solution satisfies $\hat{u}_l \in (c_h, \bar{u}_l(\hat{u}_l))$, then, we let the equilibrium distribution F_l be given by

$$(1-\pi+\pi F_l(u_l))(\mu_h\nu_h + \mu_l\nu_l - u_l) = \bar{\Pi}(\hat{u}_l) \tag{27}$$

for $u_l \leqslant \hat{u}_l$. For such values of u_l, we set $U_h(u_l) = u_l$ so that menus in this region feature pooling. For values of $u_l \geqslant \hat{u}_l$, F_l satisfies (21) together with the boundary condition

$$1-\pi+\pi F_l(\hat{u}_l) = \frac{\bar{\Pi}(\hat{u}_l)}{(\mu_h\nu_h + \mu_l\nu_l - \hat{u}_l)}$$

and with $U_h(u_l)$ determined by the equal profit condition

$$(1-\pi+\pi F_l(u_l))\left[\mu_h\Pi_h(U_h(u_l), u_l) + \mu_l(\nu_l - u_l)\right] = \bar{\Pi}(\hat{u}_l). \tag{28}$$

In this region of the support of F_l, $U_h(u_l) > u_l$ so the menus are separating. The support of the distribution F_l is given by $[c_h, \bar{u}_l(\hat{u}_l)]$.

If the solution to (26) satisfies $\hat{u}_l \leqslant c_h$, then F_l satisfies (21) with the boundary condition $\min \operatorname{Supp}(F_l) = \hat{u}_l$ and $U_h(u_l)$ is determined by (28). If the solution to (26) satisfies $\hat{u}_l \geqslant \bar{u}_l(\hat{u}_l)$, then F_l satisfies (27), $\operatorname{Supp}(F_l) = [c_h, \hat{u}_l]$,and $U_h(u_l) = u_l$ for all $u_l \in \operatorname{Supp}(F_l)$.

Relative to our construction when $\phi > 0$, it is immediate that our construction when $\phi < 0$ is more complicated. Ultimately, this computation relies on finding a solution to the non-linear equation (26). Note that it is possible for the threshold that solves (26) may be interior to the support of F_l or at either end of the support. Thus, the constructed equilibrium may be separating for almost all menus, pooling for all menus, or pooling for some menus and separating for others. We discuss below parameter values such that each of these cases are possible. Importantly, our equilibrium provides a straightforward partition of the support of F_l into pooling and separating region which simplifies the characterization greatly. Note also that the equation (26) may have multiple solutions; in this case, we characterize the equilibrium using the lowest solution $\hat{u}_l$.

To understand the equilibria we have constructed, assume that the threshold $\hat{u}_l$ satisfies $\hat{u}_l \in (c_h, \bar{u}_l(\hat{u}_l))$, and consider first the pooling region $u_l \in [\underline{u}_l, \widehat{u}_l)$. Since buyers must be indifferent between offering any menu in the pooling region, each of these menus must earn the same profits in spite of the fact that better utility levels attract a greater number of sellers. Hence, equation (27) determines the distribution $F_l(u_l)$ in this interval. In addition, profits must satisfy (23) since the worst contract is a pooling menu with a utility offer of $u_l = c_h$. The object, $H(\hat{u}_l)$, determined in (24) specifies the distribution F_l at the upper bound of the pooling region.

Next consider the separating region. As in Section 4.2.1, local deviations must be unprofitable which requires F_l to satisfy (21) and that buyers earn equal profits at all equilibrium menus determines $U_h(u_l)$ in this region according to (28) as a function of $F_l(u_l)$. To solve the differential equation (21) we must specify a boundary condition. Continuity of the distribution F_l at the threshold, $\hat{u}_l$, yields the necessary condition given by equation (24). The solution to the differential equation is given by

$$1 - \pi + \pi F_l(u_l) = (1 - \pi + \pi F_l(\widehat{u}_l)) \left(\frac{\widehat{u}_l - v_l}{u_l - v_l} \right)^{\phi}. \tag{29}$$

Having solved the differential equation exactly, we then determine the upper bound of the support of F_l by solving for $F_l(\overline{u}_l) = 1$ using equation (29). In a slight abuse of notation, we denote this upper bound $\overline{u}_l(\widehat{u}_l)$, which is given by equation (25).

The last step in the construction is to ensure that at the upper bound of the support of F_l, the pooling menu $(\overline{u}_l(\widehat{u}_l), \overline{u}_l(\widehat{u}_l))$ yields profits exactly equal to the profits at the lower bound $\bar{\Pi}(\hat{u}_l)$ according to (28). By construction, then, we have imposed that the best menu offered in equilibrium is a pooling menu. When $\phi < 0$, it is straightforward to show that any equilibrium features this property. If the best menu offered in equilibrium were a separating menu, then a buyer offering this menu purchases strictly fewer than 1 unit of goods from each high quality seller; an increase in the utility offered to low quality sellers necessarily improves the buyer's profits by increasing the amount purchased from high quality sellers with no impact on the number of sellers the buyer attracts. These results lead us to the non-linear equation (26) which we solve to determine the threshold, $\hat{u}_l$.

To prove that our constructed distribution F_l along with the policy function $U_h(u_l)$ constitute an equilibrium we must prove that there are no profitable local deviations in the pooling region and that there are no global deviations. The following proposition asserts that our construction of F_l and $U_h(u_l)$ using the threshold $\hat{u}_l$ yield an equilibrium and provides a full characterization of the threshold $\hat{u}_l$ with respect to the support of F_l.

Proposition 11. *If $\phi < 0$, there exists a threshold equilibrium with continuous distributions (F_l, F_h) with F_l defined by (26) for $u_l < \hat{u}_l$ and (28) for $u_l \geqslant \hat{u}_l$, $U_h(u_l) = u_l$ for $u_l < \hat{u}_l$, $U_h(u_l)$ given by (28) for $u_l \geqslant \hat{u}_l$ and $F_h(U_h(u_l)) = F_l(u_l)$. There exists two cutoffs $\phi_2 < \phi_1 < 0$ such that:*

1. *When $\phi_2 < \phi < \phi_1$, the threshold $\hat{u}_l$ is interior to the support of F_l,*
2. *When $\phi_1 \leqslant \phi$, the threshold $\hat{u}_l$ satisfies $\hat{u}_l = \min \operatorname{Supp}(F_l)$,*
3. *When $\phi \leqslant \phi_2$, the threshold $\hat{u}_l$ satisfies $\hat{u}_l = \max \operatorname{Supp}(F_l)$.*

Proposition 11 describes three types of equilibria: semi-separating equilibria when $\phi_2 < \phi < \phi_1$, separating when $\phi_1 \leqslant \phi$, and pooling when $\phi \leqslant \phi_2$. The cutoffs, ϕ_1 and ϕ_2 represent necessary conditions for existence of an equilibrium with only separating menus ($\phi \geqslant \phi_1$) or with only pooling menus ($\phi \leqslant \phi_2$).

Consider first the cutoff ϕ_1. An equilibrium with only separating menus has $U_h(u_l) > u_l$ for almost all u_l in the support of F_l. In particular, at the lower bound of the support of F_l, in this case given by the threshold $\hat{u}_l$, the equilibrium must satisfy $\hat{u}_l \leqslant c_h$. In the Appendix, we show that this condition is satisfied for the constructed $\hat{u}_l$ if and only if

$$c_h \geqslant \nu_l + \frac{\pi(1-\mu_l)\left(\nu_h - \nu_l\right)}{(1-\pi)\left[(1-\pi)^{\frac{1-\phi}{\phi}} - 1\right]}. \tag{30}$$

The condition (30) yields an lower bound on ϕ, which we refer to as ϕ_1, such that a separating equilibrium exists. Note also that a lower bound on ϕ is equivalent to an upper bound on μ_h.

Next consider the cutoff ϕ_2. An equilibrium with only pooling menus has $\bar{\Pi}\left(\hat{u}_l\right) = (1-\pi)\left(\mu_h\nu_h + \mu_l\nu_l - c_h\right)$ and F_l determined by (27). To verify this distribution F_l yields an equilibrium, we must ensure that the standard cream-skimming deviation is not profitable relative to any pooling menu (u_l, u_l) with u_l in the support of F_l. Such a cream-skimming deviation, locally, is of the form $(u_l - \varepsilon, u_l)$. The decrease in u_l can be attractive for two reasons: first, it decreases the loss in a trade with a low quality seller; and second, it decreases the number of non-captive low quality sellers that the buyer attracts, inducing a more favorable mix of non-captive high and low quality sellers. Since the best pooling menu offers the largest subsidy to low quality sellers, one can that show a buyer's incentive to cream-skim are strongest at the most generous menu offered in equilibrium, which occurs at the threshold $\hat{u}_l$. Thus, if cream-skimming is not profitable at the pooling menu offering utility $\widehat{u}_l$, then it is not profitable at any pooling menu offering utility less than $\widehat{u}_l$. As we show in the Appendix, cream-skimming is not profitable at the pooling menu offering utility $\widehat{u}_l$ if, and only if,

$$1 - \pi \geqslant \frac{\mu_h\nu_h + \mu_l\nu_l - \nu_l}{(1-\phi)(\mu_h\nu_h + \mu_l\nu_l - c_h)}. \tag{31}$$

The condition (31) yields an upper bound on ϕ, which we refer to as ϕ_2, such that a pooling equilibrium exists. One can prove that $\phi_2 < \phi_1$ so that if a pooling equilibrium exists, then a separating equilibrium does not and vice versa.

When ϕ lies between these cutoffs, $\phi_2 < \phi < \phi_1$, one can prove that the threshold $\hat{u}_l$ is interior to the support of the constructed F_l. To confirm that the candidate threshold equilibrium is indeed an equilibrium, we must prove that buyers have no profitable deviations. By construction, buyers have no incentive to deviate locally to alternative menus in the separating region. Hence, we need only check deviations in the pooling interval. As in the pooling equilibrium, it suffices to check that cream-skimming is not profitable at the pooling menu offering utility $\hat{u}_l$.

In the Appendix, we show that when $\phi_2 < \phi < \phi_1$, such cream-skimming deviations are not profitable at $\hat{u}_l$.

4.2.3 Equilibria with Mass Points ($\phi = 0$)

When $\phi \neq 0$, we have described the construction of equilibria assuming that the distributions F_l and F_h do not feature mass points. Here, we construct an equilibrium $\phi = 0$ that does feature mass points. In particular, we construct an equilibrium in which the distribution F_l assigns full mass to $u_l = v_l$. The distribution F_h is determined by the equation

$$(1 - \pi + \pi F_h(u_h))\, \mu_h \Pi_h(u_h, v_l) = (1 - \pi)\, \mu_l (v_l - c_l) \tag{32}$$

with $\text{Supp}(F_h) = \left[c_h, c_h + \pi \frac{(v_l - c_l)(v_h - c_h)}{v_h - c_l}\right]$.

To understand this equilibrium, note that (32) is determined to ensure that buyers have no incentive to deviate by changing only u_h. Next, consider the incentives of a buyer to deviate by perturbing u_l away from v_l. An marginal increase in u_l leads to a marginal increase in profits earned per high quality seller attracted. However, the number of non-captive low quality sellers increases discretely. A marginal decrease in u_l is also unprofitable. Such a deviation increases profits earned per captive low-quality seller but decrease profits earned per high-quality seller. It is straightforward to show that when $\phi = 0$, the losses from this deviation are larger than the gains. The next proposition asserts that the equilibrium we have constructed is indeed an equilibrium.

Proposition 12. *When $\phi = 0$, the pair of distribution functions - F_l degenerate at v_l and F_h satisfying (32) - constitute an equilibrium.*

4.3 Uniqueness

In this section, we establish uniqueness of the equilibrium we have constructed. We proceed by discussing the key results needed to ensure uniqueness for each ϕ.

Consider first the separating equilibrium we construct when $\phi > 0$. We demonstrate uniqueness of the separating equilibrium by ruling out the two relevant alternatives. To be more precise, we first show that there are no equilibria with mass points when $\phi > 0$. Then we establish that there are no equilibria that feature pooling, i.e., there are no equilibria in which

$U_h(u_l) = u_l$ for some $u_l \in \text{Supp}(F_l)$.

Next, consider the threshold equilibrium we construct when $\phi < 0$. We demonstrate uniqueness of the threshold equilibrium as follows: first, we show that any equilibrium features pooling at the upper bound of the support of F_l; second, we prove that any equilibrium features at most one interval of pooling followed by at most one interval of separation; third, we prove that the equilibria characterized in Proposition 11 are mutually exclusive so that equilibria without mass points are unique; fourth, we prove no equilibrium features mass points demonstrating uniqueness of the equilibrium characterized in Proposition 11.

Finally, when $\phi = 0$, it is straightforward to prove that in any equilibrium, F_l must be degenerate at $u_l = \nu_l$. We summarize these results in the following Theorem.

Theorem 13. *For any (π, ϕ), the applicable equilibrium constructed in Section 4.2.1, 4.2.2, or 4.2.3 is unique.*

5 Discussion

In this section, we explore the implications of the equilibrium characterization provided above. We begin with positive implications, paying particular attention to how the nature of the contracts that are offered in equilibrium are affected by both the severity of the adverse selection problem and the degree of competition. We show that, in contrast with many papers in the literature, whether buyers offer sellers separating or pooling menus depends on the underlying distribution of types in the market. Moreover, this decision depends systematically on the underlying market structure: separating menus are more prevalent when markets are more competitive, while pooling menus emerge when markets are more frictional. Hence, observing separating contracts in a market is not necessarily a sign of severe adverse selection; identifying the severity of information frictions requires knowledge of the prevailing trading frictions.

We then explore some of the normative implications, fleshing out the affect of competition on ex ante welfare in different regions of the parameter space. We show that, when adverse selection is severe, ex ante welfare is inverse U-shaped in π, so that there is an *interior* level of frictions that maximizes surplus from trade. When adverse selection is mild, on the other hand, ex ante welfare is monotonic and decreasing in π, so that competition unambiguously hinders the process of realizing gains from trade.

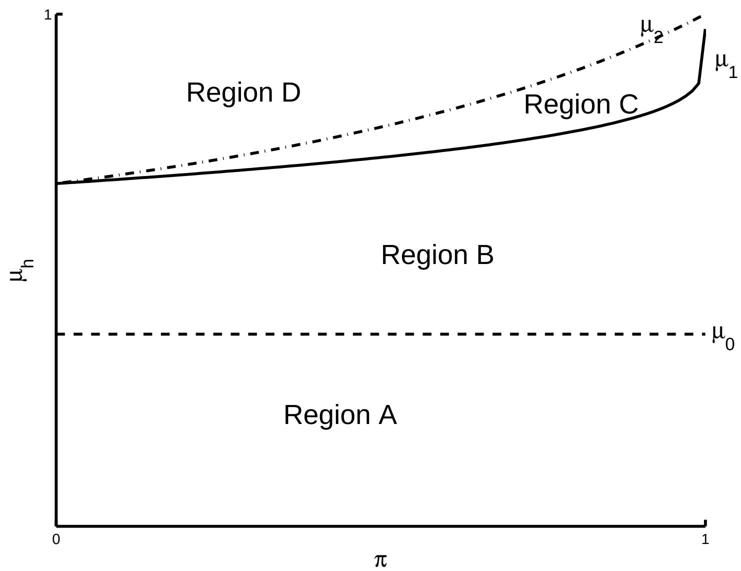

Figure 1: Equilibrium thresholds. Region A: no cross-subsidization, separating. Region B: cross-subsidization, separating. Region C: cross-subsidization, pooling at low u_l, separating at high u_l. Region D: cross-subsidization, pooling.

Finally, we demonstrate that explicitly modeling competition and allowing for general contracts yields novel and interesting policy implications. Specifically, we analyze the effect of making additional information about sellers' types available to buyers on *ex-ante* welfare. This is a very topical question in the context of recent developments, both due to policy and/or technological changes, in a number of insurance and financial markets. Our main finding is that the desirability of additional information depends both on the distribution of types and the degree of competition. In particular, when adverse selection is relatively mild to begin with or when the market is very competitive, additional information is detrimental to welfare. The opposite is true when adverse selection and trading frictions are relatively severe. Thus, evaluating the implications of these policies requires knowledge of the distribution of types in the population as well as the extent of frictions.

5.1 Equilibrium Contracts, Adverse Selection, and Competition

Figure 1 depicts the various types of equilibria that arise for different values of μ_h and π, which represent the fraction of high quality sellers and the degree of competition in the market, respectively. The figure shows that for each value of $\pi \in (0, 1)$, there are three critical cutoffs: μ_0, μ_1, and μ_2. When the adverse selection problem is relatively severe — that is, when $\mu_h < \mu_0$

— buyers optimally choose to offer separating contracts (region A). Moreover, these contracts do not feature cross subsidization; all contracts offered in equilibrium yield a positive expected payoff from both low and high type sellers. As μ_h increases, however, the incentive to trade larger quantities with high type sellers increases, as such sellers are more abundant. Indeed, for all $\mu_h > \mu_0$ (regions B,C and D), all equilibrium contracts feature cross-subsidization, as low quality sellers earn information rents that relax incentive constraints and enable buyers to trade larger quantities with high quality sellers.

In region B, i.e. when $\mu_h \in (\mu_0, \mu_1)$, all equilibrium menus are composed of separating contracts. In this region, even though a monopsonist would like to pool low and high quality sellers, such an offer cannot be sustained. The reason, as we discussed above, is that *cream-skimming* is still optimal in this region: a buyer could profitably deviate by offering less utility to low types, while offering the same utility to high types by trading a smaller quantity at a higher price. Since μ_h is still relatively small in this region, the benefits of trading with fewer low types (and hence suffering fewer losses) outweigh the cost of extracting less rents from high types.

As μ_h continues to increase, however, the benefits of cream-skimming deviations subside while the costs grow. As a result, in region C, when $\mu_h > \mu_1$, at least some pooling menus can be sustained in equilibrium. In particular, whenever some pooling menus are offered, they will be the menus that deliver low levels of utility to sellers. Intuitively, those buyers whose offers fall in the lower portion of the distribution of indirect utilities earn most of their profits from captive sellers; the benefits of altering the mix of low and high quality non-captive sellers they attract by deviating to a cream-skimming menu are small relative to the costs of extracting fewer rents from the mass of captive, high quality sellers. Conversely, those buyers offering indirect utilities at the top of the distribution trade with a large fraction of non-captive sellers, and hence the cream skimming deviation may still be profitable. Thus, when $\mu_h \in (\mu_1, \mu_2)$, menus that offer high indirect utilities remain separating menus. However, when $\mu_h > \mu_2$ (region D), cream-skimming is not profitable anywhere in the distribution of equilibrium offers, and all buyers offer pooling menus. Overall, comparative statics with respect to μ_h reveal that the underlying distribution of asset quality affects whether menus feature cross-subsidization *and* whether they feature pooling or separating contracts. Interestingly, this is not the case in related models like Guerrieri et al. (2010).

Figure 1 also shows that pooling is more prevalent when markets are less competitive. In particular, when μ_h is sufficiently high, the equilibrium features at least some (if not all) pooling menus. However, for similar reasons to those discussed above, the incentive to deviate to a cream-skimming menu grows as π increases and a larger fraction of profits are generated from trades with non-captive sellers. When π gets sufficiently close to 1, in fact, no pooling menus can be sustained. Note that these last results imply that the relationship between the contracts that are traded in a market and the severity of the adverse selection problem can ultimately depend heavily on the degree of competition in that market. In particular, the observation that separating contracts are observed in market A and not in market B does *not* necessarily imply that adverse selection is more severe in market A than B.

5.2 Ex-ante Welfare and Competition with Adverse Selection

We now turn our attention to the relationship between ex ante welfare and competition, as captured by π, for varying degrees of adverse selection, as captured by μ_h. To start, note that ex ante welfare is given by

$$\begin{aligned} \mu_l v_l \quad & + (1-\pi) \int \mu_h \left(v_h x_h \left(u_l\right) + c_h \left(1 - x_h \left(u_l\right)\right)\right) dF_l \left(u_l\right) \\ & + \pi \int \mu_h \left(v_h x_h \left(u_l\right) + c_h \left(1 - x_h \left(u_l\right)\right)\right) d \left(F_l \left(u_l\right)^2 \right) \end{aligned} \tag{33}$$

where

$$x_h \left(u_l\right) = 1 - \frac{U_h \left(u_l\right) - u_l}{c_h - c_l}. \tag{34}$$

The first term represents the total surplus from trade between low quality sellers and buyers; since $x_l = 1$ with probability 1, all low quality trades generate a surplus of v_l, the buyers' valuation. The second term in (33) captures the total surplus created from trades between buyers and captive high-quality sellers; a captive high-quality seller trades a quantity $x_h(u_l)$, where u_l is drawn from $F_l(u_l)$. Likewise, the last term in (33) captures the total surplus created from trades between buyers and non-captive high-quality sellers; since non-captive sellers choose the maximum indirect utility among the two offers they receive, they trade $x_h(u_l)$ with u_l drawn from $F_l(u_l)^2$.

Using this definition of ex ante welfare, we show that ex ante welfare is inverse U-shaped in π when adverse selection is severe; that is, there is an *interior* level of frictions that maximize the surplus generated from trade. On the other hand, when adverse selection is mild, ex ante welfare is strictly decreasing in π; that is, more competition is unambiguously bad for welfare. The following proposition summarizes.

Proposition 14. *When $\mu_h < \mu_0$, there exists a $\pi \in (0,1)$ that maximizes ex ante welfare. On the other hand, when $\mu_h > \mu_0$, ex ante welfare is maximized at $\pi = 0$.*

Two key features of our equilibrium can account for the first result. First, when π is sufficiently large, $x_h(u_l)$ is inverse U-shaped in u_l. Second, as π increases, $F_l(u_l)$ increases in the sense of first order stochastic dominance; that is, F_l shifts to the right, putting more weight on offers with lower values of x_h. This drives down the total quantity traded with high quality sellers, and thus ex ante welfare.

But why is $x_h(u_l)$ decreasing for high values of u_l when π is sufficiently close to 1? Intuitively, when $\phi > 0$, buyers compete for low quality sellers more intensely than they compete for high quality sellers, since the former are relatively more abundant. Thus, offers to low quality sellers are clustered very tightly at the top of F_l; demand in this region is very elastic and, in order to satisfy the equal profits condition, prices offered to low quality sellers have to rise relatively slowly.[7] Competition for high quality sellers, on the other hand, is less fierce. As a result, offers to high quality sellers are more dispersed at the top of F_h; demand in this region is more inelastic, and hence prices offered to high quality sellers have to rise very quickly. In order to achieve this increase, the quantity traded with high quality sellers has to fall.

To see the mathematics behind this result, note that (8) implies that $x_h(u_l) = x_h(U_h(u_l))$ is decreasing if and only if $U_h'(u_l) > 1$. Moreover, from (22), one can derive

$$U_h'(u_l) = \left[\frac{\phi(c_h - c_l)}{v_h - c_l}\right] \times \frac{\Pi_h(U_h(u_l), u_l)}{v_l - u_l}. \tag{35}$$

Therefore, as $\pi \to 1$, $U_h'(u_l) > 1$ if u_l converges to v_l faster than $\Pi_h(u_l, U_h(u_l))$ converges to zero. This is exactly the case when $\phi > 0$, and the incentive to outbid other buyers to trade with non-captive, low quality sellers is strong relative to the incentive to outbid other buyers for non-captive, high quality sellers.

The second result — that ex ante welfare is maximized at $\pi = 0$ when $\phi < 0$ — is more straightforward. Since a monopsonist offers a pooling contract in this region of the parameter

[7] Indeed, from (21), one can see that as π tends to 1, $\lim_{u_l \to v_l} f_l(u_l) = \infty$.

space, all gains from trade are exhausted. By introducing competition, one simply increases a buyer's incentive to deviate to a cream-skimming menu. In order to make such a deviation unprofitable, equilibrium contracts offer high quality sellers a higher price but a lower quantity to trade, and this causes a decline in ex ante welfare.

5.3 Information, Disclosure, Discrimination, and Welfare

In this section, we show that explicitly modeling imperfect competition in an environment with both adverse selection and general contracts reveals new and interesting implications regarding the impact of certain policy choices and technological innovations. In other words, several normative insights that have been derived under a particular market structure (e.g., perfect competition) or with *ad-hoc* restrictions on contracts may not be particularly robust.

We demonstrate this insight by analyzing what happens when buyers have better information about sellers. Consider an environment in which the probability that a particular seller is high quality, μ_h', is a random variable drawn from a distribution $G(\mu_h')$. In the baseline case, i.e. without additional information, G is degenerate. We then ask: what is the effect of a mean-preserving spread of G ? As we discuss below, this is a natural way to model the effect of additional information — as a result policy-induced or other changes — in a variety of important markets.

1. **Non-discrimination in insurance markets.** Under the Affordable Care Act, insurance providers are not allowed to discriminate based on certain observable characteristics (e.g. health factors). In other words, participants with known differences in risk must be offered the same menus (of benefits, premiums etc.). In the context of our model, we can interpret these differences in risk as differences in μ_h across subgroups of the population. The distribution of group-specific μ_h is a mean-preserving spread of the population G. Then, under the non-discrimination policy, insurance companies would offer contracts based on the pooled (or population) μ_h. In the absence of such a policy, i.e. if companies were free to offer different terms to different subgroups, the set of contracts offered in equilibrium to each subgroup would be based on their μ_h.

2. **Credit scores in consumer loan markets.** The development of standardized scoring systems to assess borrower creditworthiness is widely believed to have played an important

role in the expansion of US consumer credit markets over the last three decades[8]. Again, interpreting these credit scores as signals about borrower types, it is easy to see that they induce a mean-preserving spread of G.

3. **Transparency in financial markets.** An important debate in financial regulation relates to disclosure of trades conducted in decentralized or over-the-counter settings. Many securities in the US are now subject to rules requiring public dissemination of trade information. For example, all transactions in corporate bonds and asset-backed securities are now required to be reported within 15 minutes through a centralized reporting system called Trade Reporting and Compliance Engine (TRACE). This is intended to induce "price discovery and improved bargaining power of previously uninformed participants", and in turn " a decrease in bond price dispersion and an increase in trading activity" (Levitt (1999)). To map this into our mean-preserving spread experiment, suppose that the fraction of high quality assets is state-dependent. Disclosure of past trades essentially generates a signal about the state of the world (i.e. about μ_h) and thus, induces a mean-preserving spread of the unconditional distribution.

To start, let us define welfare explicitly in terms of μ_h, so that

$$
\begin{aligned}
W(\mu_h) &= (1-\mu_h)\,v_l + \mu_h v_h X_h(\mu_h) + \mu_h c_h\,(1 - X_h(\mu_h)) \\
\text{where} \quad X_h(\mu_h) &\equiv \int x_h(u)\, dR(u, \mu_h) \\
\text{and} \quad R(u, \mu_h) &= (1-\pi)\, F_l(u, \mu_h) + \pi F_l^2(u, \mu_h)\,.
\end{aligned}
$$

Note that, for expositional simplicity, we have retained the dependence of allocations and the distribution on μ_h, while suppressing the dependence on π. The effect of a mean-preserving thus depends on the curvature of $W(\mu_h)$. If W is convex (concave) in μ_h over the relevant region, a mean-preserving spread increases (reduces) welfare.

It is important to mention that we are interested in local mean-preserving spreads. Additional information of sufficiently high quality always improves welfare. To see this, consider the extreme case of a perfectly informative signal. This eliminates the adverse selection problem and achieves first-best welfare, irrespective of the degree of competition - clearly, not a very interesting (or realistic) experiment. Therefore, in what follows, we restrict attention to 'small'

[8]See Chatterjee et al. (2011) and Einav et al. (2013)

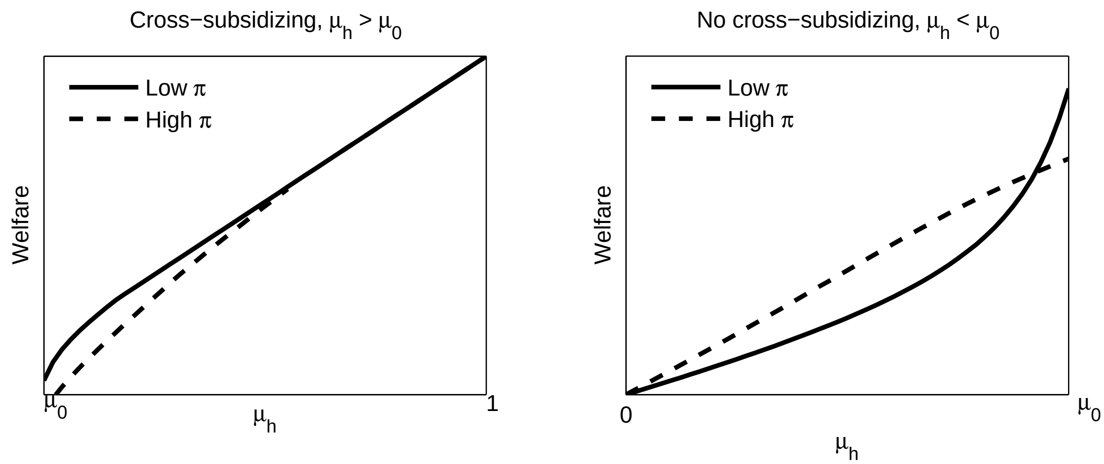

Figure 2: Welfare as a function of μ_h

spreads.

Before considering the general case, it is worthwhile to examine the limiting cases of monopsony and Bertrand competition. Recall from Lemma 8, that under monopsony, $x_h = 0$ when $\mu_h < \bar{\mu}_0$, so $W(\mu_h) = (1-\mu_h)v_l + \mu_h c_h = v_l + \mu_h(c_h - v_l)$. When $\mu_h \geqslant \bar{\mu}_0$, the monopsonist pools so $x_h = 1$, so we have $W(\mu_h) = v_l + \mu_h(v_h - v_l)$. In other words, under monopsony, W consists of two piece-wise linear segments, with a discontinuity at $\mu_h = \bar{\mu}_0$. Directly, a mean-preserving spread is welfare-neutral unless such spreading spans the discontinuity. At the other extreme, when $\pi = 1$, W is also linear in the region where a pure strategy equilibrium exists (i.e. which $\mu_h < \bar{\mu}_0$). This is because x_h is pinned down entirely by the incentive constraint and independent of μ_h. Again, within this region, the policy experiments have no effect on welfare. The analysis of the case with mixed strategies is more involved and closely related to the general case of $\pi \in (0,1)$ to which we turn next.

Our results for $\pi \in (0,1)$ are depicted in Figure 2, which shows the effect of mean preserving spreads for the two cases of cross-subsidizing and no-cross subsidizing equilibria. Each panel plots the W over an interval of μ_h, for two different values of π. The first panel considers an interval entirely in the cross-subsidizing region, i.e. when $\mu_h > \bar{\mu}_0$. In this region, W is concave, both for high and low π, implying that mean-preserving spreads are detrimental. To see why, first note that higher μ_h shifts the distributions of u_l and u_h to the right. This is intuitive - as the probability of high quality assets increases, the adverse selection becomes less severe, so both seller types receive higher utilities. This also results in higher x_h, increasing overall welfare. However, this change in the distribution becomes more gradual as μ_h rises closer to the pooling region. In other words, a reduction in μ_h has a bigger effect on the distribution - and consequently on expected trade X_h and welfare - than an increase of similar magnitude.

The opposite happens in the no-cross subsidization region, provided π is sufficiently low. In this region, W is convex, so mean-preserving spreads *increase* welfare. With π is high, however, there is an additional effect. Recall from Section 5.2 that x_h is decreasing in u_l when π is high. This moderates the direct gains from a rightward shift in the distribution of utilities induced by higher μ_h. The combined effect of these two forces results in a region where W is concave, making mean-preserving spreads undesirable. Thus, in the no cross subsidization region, the curvature of W varies with π so mean-preserving spreads can be good or bad for welfare[9].

Thus, the concavity of W depends in a rich way both on the type of equilibrium (cross-subsidizing vs no cross subsidization) as well as on π. Broadly speaking, our results suggest that providing additional information to buyers is less likely to be desirable when the adverse selection problem is relatively mild to begin with (in the sense that μ_h is high) or when the market is very competitive (i.e. when π is high). Therefore, in order to assess even the qualitative effect of policies like the ones described at the beginning of the section, one needs to first accurately measure both the degree of competition and the severity of adverse selection.

[9]Numerical calculations, however, suggest that the concavity of W is relatively mild and occurs only over a narrow region of the parameter space - in other words, W appears to be mostly convex in the no cross subsidization region.

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Appendix

A Proofs

A.1 Proof of Lemma 1

Both results are similar to existing results (see, for example, Dasgupta and Maskin (1986)), and thus we keep the exposition brief. To establish the first result, consider a contract $\mathbf{z} = (\mathbf{z}_l, \mathbf{z}_h)$ with $x_l < 1$ and $t_l \in \mathbb{R}_+$ yielding a low quality seller utility u_l. Now, consider a deviation $\mathbf{z}' = (\mathbf{z}_l', \mathbf{z}_h)$ with $x_l' = x_l + \epsilon$ for $\epsilon \in (0, 1 - x_l]$ and $t_l' = t_l + \epsilon c_l$. Note that $u_l' = u_l$, so that $\mathbf{z}_l$ and $\mathbf{z}_l'$ are accepted with the same probability, but

$$x_l v_l - t_l < x_l v_l - t_l + \epsilon(v_l - c_l) = x_l' v_l - t_l',$$

so that $\mathbf{z}_l'$ earns the buyer a higher payoff when it's accepted.

Second, to establish that the incentive compatibility constraint is always binding for low quality sellers, consider a contract $\mathbf{z} = (\mathbf{z}_l, \mathbf{z}_h)$ with $t_l > t_h + c_l(1 - x_h)$. Now, consider a deviation $\mathbf{z}' = (\mathbf{z}_l, \mathbf{z}_h')$ with $x_h' = x_h + \epsilon$ and $t_h' = t_h + \epsilon c_h$ for $\epsilon \in \left(0, \frac{t_l - t_h - c_l(1 - x_h)}{c_h - c_l}\right]$, which is a non-empty interval by assumption. The upper bound on ϵ ensures that the incentive compatibility constraint on type l sellers is not violated. In addition, note that $u_h' = u_h$, so that $\mathbf{z}_h$ and $\mathbf{z}_h'$ are accepted with the same probability, but

$$x_h v_h - t_h < x_h v_h - t_h + \epsilon(v_h - c_h) = x_h' v_h - t_h',$$

so that $\mathbf{z}_h'$ earns the buyer a higher payoff when it's accepted. ■

A.2 Proof of Proposition 3

We prove this result in a through a sequence of three lemmas.

Lemma 15. *The marginal distribution F_h is continuous and strictly increasing.*

Proof. Suppose not. Then there exists a contract (u_l, u_h) in the support of the equilibrium distribution such that either:

1. F_h has a *flat*, i.e., $\exists\ \delta > 0$ such that $F_h(u_h) - F_h(u_h - \delta) = 0$.
2. F_h has a *mass point*, i.e., $\exists\ \delta > 0$ such that $F_h(u_h) - F_h(u_h') > \delta$ for all $u_h' < u_h$.

We start by showing that F_h cannot have a flat.
There are two relevant cases.

1. **Case 1:** $u_h > u_l$.

   Consider the menu $(u_h - \varepsilon, u_l)$ for some $0 < \varepsilon < \delta$, which implies $F_h(u_h - \varepsilon) = F_h(u_h)$. This menu generates strictly higher profits than (u_h, u_l) since,

$$\Pi(u_h - \varepsilon, u_l) - \Pi(u_h, u_l) = \mu_h (1 - \pi + \pi F_h(u_h)) \frac{v_h - c_l}{c_h - c_l} \varepsilon \quad > \quad 0.$$

which is a contradiction.

2. **Case 2:** $u_h = u_l = u$.
Suppose that F_h has a flat. We prove a contradiction by working with three different deviations. Note first that if F_l has a flat in an interval of the form $(u-\epsilon', u)$, then a small equal reduction in both values of u_l and u_h increases profits. Therefore, it must be that F_l does not have a flat in a neighborhood below u. Furthermore, since (u, u) is offered in equilibrium a small reduction in both u_h and u_l should not raise profits. In other words, we must have

$$\mu_h(1-\pi+\pi F_h(u)) - \underline{D}^{-}\hat{\Pi}_l(u)\mu_l \leqslant 0 \tag{36}$$

where $\hat{\Pi}_l\left(u_l\right) = \left(1-\pi+\pi F\left(u_l\right)\right)\left(q_l - u_l\right)$ and $\underline{D}^{-}$ is a notion of derivative defined as

$$\underline{D}^{-}\hat{\Pi}_l(u) \quad = \quad \liminf_{x \nearrow u} \frac{\hat{\Pi}_l(x) - \hat{\Pi}_l(u)}{x-u}$$

Similarly, a small reduction in u_l should not raise the profits either and therefore, we must have

$$-\mu_h(1-\pi+\pi F_h(u))\frac{\nu_h - c_h}{c_h - c_l} - \underline{D}^{-}\hat{\Pi}_l(u)\mu_l \leqslant 0 \tag{37}$$

Since the left hand side of (37) is strictly lower than that of (36), we must have that

$$-\mu_h(1-\pi+\pi F_h(u))\frac{\nu_h - c_h}{c_h - c_l} - \underline{D}^{-}\hat{\Pi}_l(u)\mu_l < 0 \tag{38}$$

or

$$0 < \mu_h(1-\pi+\pi F_h(u))\frac{\nu_h - c_h}{c_h - c_l} + \underline{D}^{-}\hat{\Pi}_l(u)\mu_l \tag{39}$$

Note that since F_l does not have a flat in a neighborhood below u, there must exists a sequence $\{u_l^k\}_{k=0}^{\infty}$ where $u_l^k < u$ and

$$u_l^k \in \mathrm{Supp}(F_l), \lim_{k\to\infty} u_l^k = u$$

Since u_l^k is in the support of F_l, there must exist u_h^k's such that profits evaluated at (u_l^k, u_h^k) is equal to the equilibrium profits, $\overline{\Pi}$. Note that we must have $u_l^k \leqslant u_h^k$. Furthermore, since F_h is flat in the interval $[u-\delta, u]$, it must be that for k large enough $u_h^k \geqslant u$. Therefore, for k large enough, $u_l^k < u \leqslant u_h^k$. We can use the definition of the derivative $\underline{D}^{-}$, (39) implies that

$$0 < \mu_h(1-\pi+\pi F_h(u))\frac{\nu_h - c_h}{c_h - c_l} + \frac{\hat{\Pi}_l(u) - \hat{\Pi}_l(u_l^k)}{u - u_l^k}\mu_l$$

which implies that

$$\begin{aligned} 0 &< \mu_h(1-\pi+\pi F_h(u))\frac{\nu_h-c_h}{c_h-c_l}\left(u-u_l^k\right)+\left(\hat{\Pi}_l(u)-\hat{\Pi}_l(u_l^k)\right)\mu_l \\ &\leqslant \mu_h(1-\pi+\pi F_h(u_h^k))\frac{\nu_h-c_h}{c_h-c_l}\left(u-u_l^k\right)+\left(\hat{\Pi}_l(u)-\hat{\Pi}_l(u_l^k)\right)\mu_l \end{aligned}$$

Using the fact that $\mu_l\hat{\Pi}_l(u_l^k)+\mu_h(1-\pi+\pi F_h(u_h^k))\Pi_h(u_l^k,u_h^k)=\overline{\Pi}$, and adding $\overline{\Pi}$ to the above inequality, we have

$$\begin{aligned} \overline{\Pi} &< \overline{\Pi}+\mu_h(1-\pi+\pi F_h(u_h^k))\frac{\nu_h-c_h}{c_h-c_l}\left(u-u_l^k\right)+\left(\hat{\Pi}_l(u)-\hat{\Pi}_l(u_l^k)\right)\mu_l \\ &= \mu_h(1-\pi+\pi F_h(u_h^k))\left[\frac{\nu_h-c_h}{c_h-c_l}\left(u-u_l^k\right)+\Pi_h(u_l^k,u_h^k)\right]+\hat{\Pi}_l(u)\mu_l \\ &= \mu_h(1-\pi+\pi F_h(u_h^k))\Pi_h(u,u_h^k)+\hat{\Pi}_l(u)\mu_l \end{aligned}$$

This implies that the menu represented by (u,u_h^k) is a profitable deviation which is a contradiction. Therefore, in this case, F_l cannot have a flat.

Next, we show that F_h cannot have a mass point. Suppose that u_h exists such that $F_h(u_h)>\lim_{u\nearrow u_h}F_h(u)$. Note that $u_l\leqslant u_h$ must exists so that profits from the menu (u_l,u_h) is equal to equilibrium profits $\overline{\Pi}$. There are two possibilities:

1. $\Pi_h(u_l,u_h)>0$: In this case a small increase in u_h leads to a discontinuous increase in the probability of trading with high quality sellers while the profits per trade with high quality sellers change continuously. Thus a small enough increase in u_h is a profitable deviation.

2. $\Pi_h(u_l,u_h)\leqslant 0$: In this case, it must be that profits from the low quality sellers are non-negative (since equilibrium profits must be non-negative) and therefore, $u_l\leqslant \nu_l<c_h\geqslant u_h$. Therefore, if $u_h>c_h$, a downward deviation in u_h is profitable since it is feasible and it leads to higher profits from trading with high quality sellers. Therefore, we must have $u_h=c_h$. In this case, non-positivity of the profits from high quality sellers imply that

$$\Pi(u_l,c_h)=\nu_h-c_h\frac{\nu_h-c_l}{c_h-c_l}+\frac{\nu_h-c_h}{c_h-c_l}u_l=\frac{\nu_h-c_h}{c_h-c_l}(u_l-c_l)\leqslant 0$$

Thus, we must have that $u_l=c_l$. This implies that $c_h\in U_h(c_l)$. Using the consistency requirement for the equilibrium, we must have that

$$\begin{aligned} F_h(c_h)-F_h^-(c_h) &= \mu\left(\{(c_l,c_h)\}\right)>0 \\ F_l(c_l)-F_l^-(c_l) &= \mu\left(\{(c_l,u_h),u_h\in U_h(c_l)\}\right)\geqslant\mu\left(\{(c_l,c_h)\}\right)>0 \end{aligned}$$

Thus F_l also has a mass point at c_l. Now, consider the equilibrium menu (c_l,c_h) and consider the deviation to $(c_l+\varepsilon,c_h)$ for small and positive ε. This deviation leads to a discontinuous increase in the probability of trade with low quality sellers and thus increases the profits for low quality sellers. Furthermore, since the function Π_h is increasing in u_l, this deviation also raises profits from the high quality sellers and thus is a profitable deviation. This establishes the contradiction.

Therefore, F_h cannot have a mass point.

■

Lemma 16. *The profit function* $\Pi(u_l, u_h)$ *is strictly super-modular.*

Proof. Suppose $u_{l2} > u_{l1}$ and $u_{h2} > u_{h1}$. Then, letting $\xi_1 \equiv \frac{v_h - c_h}{c_h - c_l} > 0$ and $\xi_2 \equiv \frac{v_h - c_l}{c_h - c_l} > 0$,

$$\begin{aligned}
& \Pi(u_{l1}, u_{h2}) - \Pi(u_{l1}, u_{h1}) \\
= \; & \mu_h \{[1 - \pi + \pi F_h(u_{h2})]\Pi_h(u_{l1}, u_{h2}) - [1 - \pi + \pi F_h(u_{h1})]\Pi_h(u_{l1}, u_{h1})\} \\
= \; & \mu_h \{[1 - \pi + \pi F_h(u_{h2})]\,[v_h + \xi_1 u_{l1} - \xi_2 u_{h2}] - [1 - \pi + \pi F_h(u_{h1})]\,[v_h + \xi_1 u_{l1} - \xi_2 u_{h1}]\} \\
< \; & \mu_h \{[1 - \pi + \pi F_h(u_{h2})]\,[v_h + \xi_1 u_{l2} - \xi_2 u_{h2}] - [1 - \pi + \pi F_h(u_{h1})]\,[v_h + \xi_1 u_{l2} - \xi_2 u_{h1}]\} \\
= \; & \Pi(u_{l2}, u_{h2}) - \Pi(u_{l2}, u_{h1}),
\end{aligned}$$

where the inequality follows from the fact that F_h is strictly increasing, and hence

$$\pi \xi_1 (u_{l2} - u_{l1})[F_h(u_{h2}) - F_h(u_{h1})] > 0.$$

■

Lemma 17. *The correspondence* $U_h(u_l)$ *is weakly increasing. That is, if* $u_{l2} > u_{l1}$ *then* $u_{h2} \geqslant u_{h1}$ *for all* $u_{h1} \in U_h(u_{l1})$ *and* $u_{h2} \in U_h(u_{l2})$.

Proof. Suppose by way of contradiction that $u_{l2} > u_{l1}$ but $u_{h2} < u_{h1}$ for some $u_{h1} \in U_h(u_{l1})$ and $u_{h2} \in U_h(u_{l2})$. Since $u_{l1} < u_{l2} \leqslant u_{h2} < u_{h1} \leqslant u_{l1} + v_h - v_l < u_{l2} + v_h - v_l$, all pairs $\{(u_{li}, u_{hj})\}_{i,j \in \{1,2\}}$ satisfy the constraint set. Then, since $u_{hi} \in U_h(u_{li})$ for $i \in \{1, 2\}$, it must be that $\Pi(u_{l1}, u_{h1}) \geqslant \Pi(u_{l1}, u_{h2})$ and $\Pi(u_{l2}, u_{h2}) \geqslant \Pi(u_{l2}, u_{h1})$, so that

$$\Pi(u_{l1}, u_{h1}) + \Pi(u_{l2}, u_{h2}) \geqslant \Pi(u_{l1}, u_{h2}) + \Pi(u_{l2}, u_{h1}). \tag{40}$$

However, since $u_{l1} < u_{l2}$ and $u_{h2} < u_{h1}$, the strict supermodularity of the profit function implies

$$\Pi(u_{l1}, u_{h1}) + \Pi(u_{l2}, u_{h2}) < \Pi(u_{l1}, u_{h2}) + \Pi(u_{l2}, u_{h1}),$$

which contradicts (40). ■

Lemma 18. *The marginal distribution* F_l *is strictly increasing and continuous (except possibly at its lower bound.)*

Proof. To show that F_l is strictly increasing, we show that it cannot have flats. Suppose not. That is suppose $u_{l1} < u_{l2}$ exists such that $F_l(u_{l1}) = F_l(u_{l2}) = F$ for some F and all $u_l \in (u_{l1}, u_{l2})$, $F_l(u_l) = F$. Let $u_{h1} = \sup U_h(u_{l1})$ and $u_{h2} = \inf U_h(u_{l2})$. Since $u_{l2} \geqslant u_{l1}$, Lemma 17 implies $u_{h2} \geqslant u_{h1}$.

Next, we argue that $u_{h2} = u_{h1}$. To see this, suppose instead that $u_{h2} > u_{h1}$. By Lemma 15, since F_h has connected support, for any sub-interval $[u, u'] \subset [u_{h1}, u_{h2}]$, $F_h(u') - F_h(u) > 0$. This means that there exists a positive measure of $u_h \in (u_{h1}, u_{h2})$, such that for each u_h there exists u_l with $u_h \in U_h(u_l)$. By our initial contradiction assumption, we must have $u_l < u_{l1}$ or $u_l > u_{l2}$. If $u_l < u_{l1}$ then the fact that $u_h > u_{h1}$ contradicts the weak rank preserving property. Similarly, if $u_l > u_{l2}$, then the fact that $u_h < u_{h2}$ contradicts Lemma 17. Hence, it must be that $u_{h2} = u_{h1} = \hat{u}_h$.

Since F_h has connected support and no mass, $\Pi(u_l, u_h)$ is continuous in u_h and therefore, $\hat{u}_h = \max U_h(u_{l1})$ and $\hat{u}_h = \min U_h(u_{l2})$. That is $\Pi(u_{l1}, \hat{u}_h) = \Pi(u_{l2}, \hat{u}_h) = \bar{\Pi}$ where $\bar{\Pi}$ is equilibrium profits.

Note that

$$\forall u_l \in [u_{l1}, u_{l2}], \tilde{\Pi}(u_l, u_h) = (1 - \pi + \pi F)\, \mu_l (q_l - u_l) + \mu_h (1 - \pi + \pi F_h(u_h))\, \Pi_h(u_l, u_h) \quad (41)$$

which is a linear function in u_l. Hence, if $\frac{\partial}{\partial u_l}\tilde{\Pi}(u_l, \hat{u}_h) \neq 0$, we would be able to construct profitable deviations (either $\tilde{\Pi}(u_{l1} + \varepsilon, \hat{u}_h) > \overline{\Pi}$ or $\tilde{\Pi}(u_{l2} - \varepsilon, \hat{u}_h) > \overline{\Pi}$). This means for all $u_l \in [u_{l1}, u_{l2}]$, $\frac{\partial}{\partial u_l}\tilde{\Pi}(u_l, \hat{u}_h) = 0$; that is, profits must be constant in this interval so that $\tilde{\Pi}(u_l, \hat{u}_h) = \overline{\Pi}$ $\forall u_l \in [u_{l1}, u_{l2}]$.

Furthermore, note that the profit function in (41) is differentiable with respect to u_h. Since this derivative is linear with respect to u_l, there must exist a value of $u_l \in (u_{l1}, u_{l2})$ so that $\frac{\partial}{\partial u_h}\tilde{\Pi}(ul, \hat{u}_h) \neq 0$. Therefore when $\frac{\partial}{\partial u_h}\tilde{\Pi}(ul, \hat{u}_h) > 0$, a deviation of the form $(u_l, \hat{u}_h + \varepsilon)$ with $\varepsilon > 0$ small is profitable and vice versa. This establishes the contradiction and hence F_l must not have a flat.

Next, we show that F_l is continuous except possibly at its lower bound. That is, suppose that $u_l \in Supp(F_l)$ exists such that $u_l > \min Supp(F_l)$ and $F_l(u_l) > F_l^-(u_l)$. There are two possible cases:

1. $u_l \neq v_l$: Since F_h has no mass point, it must be that $U_h(u_l)$ is an interval when F_l puts a mass point on u_l. This implies that there are u_h's in $U_h(u_l)$ that are interior in the sense that $u_h > u_l$. Now consider a deviation to $u_l + \varepsilon$ for a small and positive ε. This increases the number of low types by a large amount and although the profits per low types decline the increase in the number of low types is more than compensating and therefore is a profitable deviation for small enough ε. This is a valid deviation since u_h is interior. A similar opposite deviation rules out $u_l > v_l$.

2. $u_l = v_l$: In this case, we show that $F_l^-(v_l) = 0$ which implies that $u_l = v_l$ is at the lower bound of the support and hence a contradiction. Suppose that $F_l^-(v_l) > 0$ and that $\phi \geqslant 0$. Note that when F_l puts mass on v_l it must be that $U_h(v_l)$ is a non-degenerate interval. Hence for a value of $u_h \in U_h(v_l)$ that is above its min value, the change in profits from offering $(v_l - \epsilon, u_h)$ for a small value of ϵ is approximately given by

$$-\mu_h(1 - \pi + \pi F_h(u_h))\frac{v_h - c_h}{c_h - c_l}\epsilon + \mu_l(1 - \pi + \pi F_l^-(v_l))\epsilon$$

Since $u_h > \min U_h(v_l)$, $F_h(u_h) > F_l^-(v_l)$. Therefore, since $\phi > 0$, the above expression is positive and this is a profitable deviation.

Now suppose that $\phi < 0$. Then a similar logic as above establishes that a deviation of the form $(v_l + \epsilon, \max U_h(v_l))$ is profitable. This establishes the contradiction and completes the proof.

■

A.3 Proof of Proposition 10

Separating Allocation. We first show that the equilibrium allocations constructed in (21) and (22) are indeed separating. Note that solution to the differential equation in (22) together with boundary condition $F_l(c_l) = 0$, must satisfy

$$1 - \pi + \pi F_l(u_l) = (1-\pi)(v_l - c_l)^{\phi}(v_l - u_l)^{-\phi} \tag{42}$$

Therefore, from (22), $U_h(u_l)$ must satisfy

$$\mu_h \Pi_h(u_l, U_h(u_l)) + \mu_l(v_l - u_l) = \mu_l(v_l - c_l)^{1-\phi}(v_l - u_l)^{\phi}$$

or

$$U_h(u_l) = \frac{\mu_h v_h + \mu_l v_l - \mu_l \phi u_l - \mu_l(v_l - c_l)^{1-\phi}(v_l - u_l)^{\phi}}{\mu_h \frac{v_h - c_l}{c_h - c_l}}$$

For this to be a separating allocation, we must show that

$$u_l + c_h - c_l \geqslant U_h(u_l) > u_l, \forall u_l \in \mathrm{Supp}(F_l) \tag{43}$$

where

$$\mathrm{Supp}(F_l) = \left[c_l, v_l - (1-\pi)^{\frac{1}{\phi}}(v_l - c_l)\right]$$

The right hand side inequality is satisfied if and only if

$$\mu_h v_h + \mu_l v_l > \mu_l (v_l - c_l)^{1-\phi}(v_l - u_l)^{\phi} + u_l \tag{44}$$

Let the function $H(u_l) =$ be defined as the right hand side of the above inequality. Then $H(u_l)$ is a concave function:

$$\begin{aligned} H'(u_l) &= -\phi\mu_l(v_l - c_l)^{1-\phi}(v_l - u_l)^{\phi-1} + 1 \\ H''(u_l) &= \phi(\phi-1)\mu_l(v_l - c_l)^{1-\phi}(v_l - u_l)^{\phi-2} < 0 \end{aligned}$$

where the last inequality follows from $\phi < 1$. Note further that, since $\phi < 1$, $H'(v_l) = -\infty$ and $H'(c_l) = 1 - \phi\mu_l > 0$. Therefore, to show (44), it is sufficient to show that it is satisfied where $H(u_l)$ is maximized in the interval $[c_l, v_l]$. The function $H(u_l)$ is maximized at u_l^* given by

$$\phi\mu_l(v_l - c_l)^{1-\phi}(v_l - u_l^*)^{\phi-1} = 1 \to u_l^* = v_l - (\phi\mu_l)^{\frac{1}{1-\phi}}(v_l - c_l)$$

Therefore

$$\begin{aligned} H(u_l^*) &= \frac{1}{\phi}(v_l - u_l^*) + u_l^* \\ &= \frac{(\phi\mu_l)^{\frac{1}{1-\phi}}(v_l - c_l)}{\phi} + v_l - (\phi\mu_l)^{\frac{1}{1-\phi}}(v_l - c_l) \\ &= v_l + (v_l - c_l)\,\mu_l^{\frac{1}{1-\phi}}\,\phi^{\frac{\phi}{1-\phi}}\,[1-\phi] \end{aligned}$$

Note that since $c_h \geqslant v_l$, we must have the following

$$(\phi\mu_l)^{\frac{\phi}{1-\phi}} < 1 \leqslant \frac{(c_h - c_l)(v_h - v_l)}{(v_l - c_l)(v_h - c_h)}$$

The above inequality implies that

$$(v_l - c_l)(\phi\mu_l)^{\frac{\phi}{1-\phi}} \frac{v_h - c_h}{c_h - c_l} < (v_h - v_l) \rightarrow (v_l - c_l)\mu_l(\phi\mu_l)^{\frac{\phi}{1-\phi}} \frac{\mu_h}{\mu_l}\frac{v_h - c_h}{c_h - c_l} < \mu_h(v_h - v_l)$$

Hence,

$$(v_l - c_l)\mu_l(\phi\mu_l)^{\frac{\phi}{1-\phi}}(1-\phi) < \mu_h(v_h - v_l)$$

and

$$v_l + (v_l - c_l)\mu_l(\phi\mu_l)^{\frac{\phi}{1-\phi}}(1-\phi) < \mu_h(v_h - v_l) + v_l$$

or

$$\max_{u_l e[c_l, v_l]} H(u_l) < \mu_h v_h + \mu_l v_l$$

which in turn implies that $U_h(u_l) > u_l$ for all $u_l \in \text{Supp}(F_l)$.

Next, we show the left inequality in (43). For this to hold, we must have

$$\frac{\mu_h v_h + \mu_l v_l - \mu_l \phi u_l - \mu_l (v_l - c_l)^{1-\phi}(v_l - u_l)^{\phi}}{\mu_h \frac{v_h - c_l}{c_h - c_l}} \leqslant u_l + c_h - c_l$$

or equivalently

$$\mu_h c_l + \mu_l v_l \leqslant u_l + \mu_l (v_l - c_l)^{1-\phi}(v_l - u_l)^{\phi}, \forall u_l \in \text{Supp}(F_l) \subset [c_l, v_l] \tag{45}$$

Since, the right hand side of (45) is a concave function, it takes its minimum values at the extremes of the interval $[v_l, c_l]$. These values are given by v_l and $\mu_l v_l + \mu_h c_l$ both of which are at least as large the left side of (45). Hence, (45) must be satisfied for all $u_l \in [v_l, c_l]$.

Global Deviations. Note that our conditions (21) and (22) imply that local deviations with respect to u_h and u_l are not profitable. It, thus, remains to show that for all (u'_h, u'_l)

$$\Pi(u'_h, u'_l) = \mu_l(1 - \pi + \pi F_l(u'_l))(v_l - u'_l) + \mu_h(1 - \pi + \pi F_h(u'_h))\Pi_h(u'_h, u'_l) \leqslant \mu_l(1-\pi)(v_l - c_l)$$

We consider two cases:

1. $u'_h > \max \text{Supp}(F_h) = \bar{u}_h$: In this case, $1 - \pi + \pi F_h(u'_h) = 1$. If $u'_l > \max \text{Supp}(F_l) = \bar{u}_l$, then the profits from this menu are given by

$$\mu_l(v_l - u'_l) + \mu_h \Pi_h(u'_h, u'_l)$$

Since $\phi > 0$, the above function is decreasing in (u'_h, u'_l) and therefore

$$\mu_l(v_l - u'_l) + \mu_h \Pi_h(u'_h, u'_l) < \mu_l(v_l - \bar{u}_l) + \mu_h \Pi_h(\bar{u}_h, \bar{u}_l) = \mu_l(1-\pi)(v_l - c_l)$$

When $u_l' \leqslant \max \text{Supp}(F_l)$, the partial derivative of the above profit function with respect to u_l' is given by

$$-\mu_l\left(1-\pi+\pi F_l\left(u_l'\right)\right)+\mu_l\pi f_l\left(u_l'\right)\left(\nu_l-u_l'\right)+\mu_h\frac{\nu_h-c_h}{c_h-c_l} \geqslant$$
$$-\mu_l\left(1-\pi+\pi F_l\left(u_l'\right)\right)+\mu_l\pi f_l\left(u_l'\right)\left(\nu_l-u_l'\right)+\mu_h\left(1-\pi+\pi F_l\left(u_l'\right)\right)\frac{\nu_h-c_h}{c_h-c_l}=0$$

Thus for a given value of u_h', we must have

$$\Pi\left(u_h',u_l'\right) \leqslant \Pi\left(u_h',\bar{u}_l\right) < \Pi\left(\bar{u}_h,\bar{u}_l\right)=\mu_l\left(1-\pi\right)\left(\nu_l-c_l\right)$$

This proves the claim.

2. $u_h' \in [c_h, \bar{u}_h]$. In this case, we must have there exists $\tilde{u}_l$ such that $u_h' = U_h(\tilde{u}_l)$ and thus $F_h\left(u_h'\right) = F_l(\tilde{u}_l)$. We can thus write the profit function as

$$\mu_l\left(1-\pi+\pi F_l\left(u_l'\right)\right)\left(\nu_l-u_l'\right)+\mu_h\left(1-\pi+\pi F_l\left(\tilde{u}_l\right)\right)\Pi_h\left(u_h',u_l'\right) \tag{46}$$

We show that the above function is maximized at $u_l' = \tilde{u}_l$. This proves our claim.

To show this, we show that the function is strictly concave for values of $u_l' \in \text{Supp}(F_l)$ and decreasing for values of $u_l' > \bar{u}_l$. When $u_l' \in \text{Supp}(F_l)$, since Π_h is linear in u_l', the second derivative of (46) with respect to u_l' is given by

$$\frac{\partial^2}{\partial\left(u_l'\right)^2}\mu_l\left(1-\pi+\pi F_l\left(u_l'\right)\right)\left(\nu_l-u_l'\right)$$

Using (42), we can rewrite the above as

$$\begin{aligned}\frac{\partial^2}{\partial\left(u_l'\right)^2}\mu_l\left(1-\pi+\pi F_l\left(u_l'\right)\right)\left(\nu_l-u_l'\right) &= \frac{\partial^2}{\partial\left(u_l'\right)^2}\mu_l\left(1-\pi\right)\left(\nu_l-c_l\right)^{\phi}\left(\nu_l-u_l'\right)^{1-\phi} \\ &= \left(\phi-1\right)\phi\mu_l\left(1-\pi\right)\left(\nu_l-c_l\right)^{\phi}\left(\nu_l-u_l'\right)^{-1-\phi}<0\end{aligned}$$

This implies that (46) is strictly concave in u_l' for values of $u_l' \in \text{Supp}(F_l)$. When $u_l' > \bar{u}_l$, $1-\pi+\pi F_l\left(u_l'\right)=1$ and thus the value in (46) is given by

$$\mu_l\left(\nu_l-u_l'\right)+\mu_h\left(1-\pi+\pi F_l\left(\tilde{u}_l\right)\right)\Pi_h\left(u_h',u_l'\right)$$

The derivative of this function with respect to u_l' is given by

$$-\mu_l+\mu_h\left(1-\pi+\pi F_l\left(\tilde{u}_l\right)\right)\frac{\nu_h-c_h}{c_h-c_l} < -\mu_l+\mu_h\frac{\nu_h-c_h}{c_h-c_l}=-\mu_l\phi<0$$

Therefore, (46) is maximized at a value of u_l' which equates the partial derivative of (46) with zero. This value must satisfy

$$-\mu_l\left(1-\pi+\pi F_l\left(u_l'\right)\right)+\mu_l\pi f_l\left(u_l'\right)\left(v_l-u_l'\right)+\mu_h\left(1-\pi+\pi F_l\left(\tilde{u}_l\right)\right)\frac{v_h-c_h}{c_h-c_l}=0$$

Note that since (46) is strictly concave, at most one u_l' exists that satisfies the above. The differential equation (21) implies that $u_l'=\tilde{u}_l$ is a solution to the above equation. This implies that (46) must be maximized at $u_l'=\tilde{u}_l$. This concludes the proof.

■

A.4 Proof of Proposition 11

We first show that the thresholds, ϕ_1 and ϕ_2 defined as the values of ϕ that satisfy (30) and (31) represent lower and upper bounds on ϕ respectively and that $\phi_2<\phi_1<0$. We state these results as Lemma 19. Next, we show for each case described in the proposition, that there exist no profitable local or global deviations.

Lemma 19. *Let ϕ_1 and ϕ_2 satisfy (30) and (31) respectively. Then (30) is satisfied for all $\phi_1\leqslant\phi\leqslant 0$ and (31) is satisfied for all $\phi\leqslant\phi_2$. Moreover, $\phi_2<\phi_1<0$.*

Proof. First, note that equation (30) which determines the threshold ϕ_1 can be re-written as

$$(1-\pi)^{\frac{1-\phi}{\phi}}\geqslant\frac{\pi}{1-\pi}\frac{v_h-v_l}{c_h-v_l}\mu_h+1, \tag{47}$$

or, after taking logs and substituting for ϕ,

$$\frac{\mu_h\left(v_h-c_h\right)}{c_h-c_l-\mu_h\left(v_h-c_l\right)}\log\left(1-\pi\right)-\log(\mu_h\pi\left(v_h-v_l\right)+\left(1-\pi\right)\left(c_h-v_l\right))-\log\left[\left(1-\pi\right)\left(c_h-v_l\right)\right]\geqslant 0. \tag{48}$$

We show that the left-hand side of (48) is a decreasing function of μ_h which is strictly satisfied when μ_h is such that $\phi=0$ and is weakly violated when $\mu_h=1$. Hence, there is a unique threshold μ_1 (and implied threshold ϕ_1) such that for all $\mu_h\leqslant\mu_1$ and $\phi<0$, the separating condition (30) is satisfied. Differentiating the left-hand side of (48) with respect to μ_h we obtain

$$\log\left(1-\pi\right)\frac{\left(v_h-c_h\right)\left(c_h-c_l\right)}{\left[c_h-c_l-\mu_h\left(v_h-c_l\right)\right]^2}-\frac{\pi\left(v_h-v_l\right)}{\mu_h\pi\left(v_h-v_l\right)+\left(1-\pi\right)\left(c_h-v_l\right)}$$

which is negative for all $\pi\leqslant 1$. Next, as $\phi\to 0$ from below, it is immediate that (47) is satisfied since the left-hand side tends to infinity. As $\mu_h\to 1$, the term $\left(1-\phi\right)/\phi\to -1$ and so (47) tends to the requirement that

$$1\geqslant\pi\frac{v_h-v_l}{c_h-v_l}+\left(1-\pi\right)$$

which is violated since $c_h<v_h$.

Next, consider equation (31) which determines the threshold ϕ_2. Substituting for ϕ and straightforward algebra shows this equation is equivalent to

$$\mu_h\left(v_h-v_l\right)\left[1+\left(1-\pi\right)\frac{v_h-c_h}{c_h-c_l}\right]\geqslant v_h-v_l+\left(c_h-v_l\right)\left(1-\pi\right)\frac{v_h-c_h}{c_h-c_l}. \tag{49}$$

Clearly, (49) represents a lower bound on μ_h, or, equivalently, an upper bound on ϕ. Note that this equation is necessarily satisfied at $\mu_h = 1$. It is immediate that when μ_h is such that $\phi = 0$, equation (31) is violated since $c_h > \nu_l$.

We now prove that if $\phi \leqslant \phi_2$, then $\phi < \phi_1$ to establish that $\phi_2 < \phi_1$. Suppose $\phi \leqslant \phi_2$ and let $\bar{\nu} = \mu_h \nu_h + \mu_l \nu_l$ so that

$$1 - \pi \geqslant \frac{\bar{\nu} - \nu_l}{(1-\phi)(\bar{\nu} - c_h)}. \tag{50}$$

Note that (50) implies

$$1 - \phi \geqslant \frac{\bar{\nu} - \nu_l}{(\bar{\nu} - c_h)(1-\pi)} > \frac{\bar{\nu} - \nu_l}{\bar{\nu} - c_h}$$

and so

$$-\phi > \frac{c_h - \nu_l}{\bar{\nu} - c_h}.$$

We will show that (30) is violated. First, we re-arrange (30) as

$$(1-\pi)\left[(1-\pi)^{\frac{1-\phi}{\phi}} - 1\right](c_h - \nu_l) - \pi\mu_h(\nu_h - \nu_l) \geqslant 0$$

or, using straightforward algebra, as

$$(1-\pi)(\bar{\nu} - c_h) + (1-\pi)^{\frac{1}{\phi}}(c_h - \nu_l) \geqslant \bar{\nu} - \nu_l.$$

We show that if (50) holds, then

$$(1-\pi)(\bar{\nu} - c_h) + (1-\pi)^{\frac{1}{\phi}}(c_h - \nu_l) < \bar{\nu} - \nu_l.$$

Towards this end, define a function

$$H(\pi) = (1-\pi)(\bar{\nu} - c_h) + (1-\pi)^{\frac{1}{\phi}}(c_h - \nu_l).$$

We argue that

$$H\left(1 - \frac{\bar{\nu} - \nu_l}{(1-\phi)(\bar{\nu} - c_h)}\right) < \bar{\nu} - \nu_l \tag{51}$$

and that this result implies $H(\pi) < \bar{\nu} - \nu_l$. Note that using the expression for $H(\pi)$, we have

$$H\left(1 - \frac{\bar{\nu} - \nu_l}{(1-\phi)(\bar{\nu} - c_h)}\right) = \frac{\bar{\nu} - \nu_l}{(1-\phi)(\bar{\nu} - c_h)}(\bar{\nu} - c_h) + \left(\frac{\bar{\nu} - \nu_l}{(1-\phi)(\bar{\nu} - c_h)}\right)^{\frac{1}{\phi}}(c_h - \nu_l). \tag{52}$$

Straightforward algebra can be applied to (52) to show that (51) holds if and only if

$$\left(\frac{\bar{\nu} - \nu_l}{(1-\phi)(\bar{\nu} - c_h)}\right)^{\frac{1-\phi}{\phi}} < -\phi\frac{\bar{\nu} - c_h}{c_h - \nu_l}.$$

Raising both sides of the last inequality by ϕ, we obtain (51) holds if and only if

$$\left(\frac{\bar{v}-v_l}{(1-\phi)\left(\bar{v}-c_h\right)}\right)^{1-\phi} > \left(-\phi\frac{\bar{v}-c_h}{c_h-v_l}\right)^{\phi}$$

or

$$\left(\frac{c_h-v_l}{\bar{v}-c_h}\right)^{\phi}\left(\frac{\bar{v}-v_l}{\bar{v}-c_h}\right)^{1-\phi} > (-\phi)^{\phi}(1-\phi)^{1-\phi}. \tag{53}$$

Let $B(x) = x^{\phi}(1+x)^{1-\phi}$ and since $(\bar{v}-v_l)/(\bar{v}-c_h) = 1+(c_h-v_l)/(\bar{v}-c_h)$, (53) can be written as

$$B\left(\frac{c_h-v_l}{\bar{v}-c_h}\right) > B(-\phi).$$

Observe that

$$B'(x) = B(x)\left(\frac{\phi}{x}+\frac{1-\phi}{1+x}\right) = B(x)\frac{\phi+x}{x(1+x))}$$

so that for $0 < x < -\phi$, $B'(x) < 0$. Since $-\phi > (c_h-v_l)/(\bar{v}-c_h)$ and $B(x)$ is strictly decreasing, we have proved (53) and, therefore, (51) are satisfied.

Finally, note that the function $H(x)$ is convex since

$$\begin{aligned} H'(\pi) &= -(\bar{v}-c_h)-\frac{1}{\phi}(1-\pi)^{\frac{1}{\phi}-1}(c_h-v_l) \\ H''(\pi) &= \frac{1}{\phi}\left(\frac{1}{\phi}-1\right)(1-\pi)^{\frac{1}{\phi}-2}(c_h-v_l) > 0. \end{aligned}$$

Additionally, $H(0) = \bar{v}-v_l$, $H'(0) \leqslant 0$ when $-\phi \geqslant (c_h-v_l)/(\bar{v}-c_h)$ and $\lim_{\pi\to 1} H(\pi) = \infty$. Thus, there is a unique value $\pi^s > 0$ such that for all $\pi < \pi^s$, $H(\pi) \leqslant \bar{v}-v_l$. Since (51) holds, and since $\pi \leqslant 1-(\bar{v}-v_l)/((1-\phi)(\bar{v}-c_h))$, we must have $H(\pi) < \bar{v}-v_l$. This proves $\phi < \phi_1$ as desired. ■

We now prove that in each region ($\phi \leqslant \phi_2, \phi_2 < \phi < \phi_1, \phi_1 \leqslant \phi < 0$), there are no local or global deviations for buyers which earn strictly positive profits.

Case 1: Only Separating Menus. Here, it suffices to rule out global deviations since the distribution $F_l(u_l)$ is chosen to ensure no local deviations are profitable. To rule out global deviations a proof similar to that of Proposition 10 can be used. Here, we show that for a given value of u_h', the profit function is strictly concave in u_l' and therefore it must be maximized at $u_l' = U_h^{-1}(u_h')$ – since at this value its derivative is set to zero.

The profits are given by

$$\mu_l\left(1-\pi+\pi F_l\left(u_l'\right)\right)\left(v_l-u_l'\right)+\mu_h\left(1-\pi+\pi F_h\left(u_h'\right)\right)\Pi_h\left(u_h',u_l'\right)$$

Since Π_h is linear in u_l', the second derivative of the above function is equal to the second derivative of the profits for the low quality sellers. Using differential equation (21), we know that $\left(1-\pi+\pi F_l\left(u_l'\right)\right)=\kappa\left(u_l'-\nu_l\right)^{-\phi}$ for some constant κ. Therefore, we have

$$\begin{aligned}\frac{\partial^2}{\partial\left(u_l'\right)^2}\mu_l\left(1-\pi+\pi F_l\left(u_l'\right)\right)\left(\nu_l-u_l'\right) &= -\mu_l\kappa\frac{\partial^2}{\partial\left(u_l'\right)^2}\left(u_l'-\nu_l\right)^{1-\phi}\\ &= -\mu_l\kappa\left(1-\phi\right)\left(-\phi\right)\left(u_l'-\nu_l\right)^{-1-\phi}<0\end{aligned}$$

This completes the proof.

Case 2: Only Pooling Menus. We first prove that no local deviations in the pooling equilibrium yield strictly positive profits. Below we demonstrate global deviations are also unprofitable. Recall that in a pooling equilibrium, the distributions $F_l\left(u_l\right)$ satisfies

$$\left(1-\pi+\pi F_l(u_l)\right)\left[\bar{\nu}-u_l\right]=\left(1-\pi\right)\left(\bar{\nu}-c_h\right) \tag{54}$$

where $\bar{\nu}=\mu_h\nu_h+\mu_l\nu_l$, $U_h(u_l)=u_l$, $F_h(u_l)=F_l(u_l)$, and $\mathrm{Supp}(F_l)=\left[c_h,\bar{\nu}-\left(1-\pi\right)\left(\bar{\nu}-c_h\right)\right].$ Fix any utility, u_l, interior to the support of F_l and consider a perturbation that increases u_h by ε ($\hat{u}_h=u_l+\varepsilon$) holding u_l fixed. Profits from such a deviation satisfy

$$\begin{aligned}&\mu_l\left(1-\pi+\pi F_l(u_l)\right)\left(\nu_l-u_l\right)+\mu_h\left(1-\pi+\pi F_l\left(u_l+\varepsilon\right)\right)\Pi_h\left(u_l+\varepsilon,u_l\right)\\ =\ &\mu_l\left(1-\pi+\pi F_l(u_l)\right)\left(\nu_l-u_l\right)+\mu_h\left(1-\pi+\pi F_l\left(u_l+\varepsilon\right)\right)\left[\nu_h-u_l-\varepsilon\frac{\nu_h-c_l}{c_h-c_l}\right].\end{aligned}$$

If local deviations are unprofitable, this function must be maximized at $\varepsilon=0$, so that F_l must satisfy

$$\mu_h\pi f_l\left(u_l\right)\left[\nu_h-u_l\right]-\mu_h\left(1-\pi+\pi F_l\left(u_l\right)\right)\frac{\nu_h-c_l}{c_h-c_l}\leqslant 0.$$

Totally differentiating (54) yields the following relationship between F_l and f_l,

$$\pi f_l(u_l)\left(\bar{\nu}-u_l\right)=\left(1-\pi+\pi F_l(u_l)\right) \tag{55}$$

so that for local deviations to be unprofitable, we require

$$\mu_h\pi f_l\left(u_l\right)\left[\nu_h-u_l\right]-\mu_h\pi f_l(u_l)\left(\bar{\nu}-u_l\right)\frac{\nu_h-c_l}{c_h-c_l}\leqslant 0.$$

Since F_l is continuous in our constructed equilibrium, we may simplify this condition using straightforward algebra as

$$u_l\left(\nu_h-c_h\right)\leqslant\bar{\nu}\left(\nu_h-c_l\right)-\nu_h\left(c_h-c_l\right).$$

Consequently, we see that it suffices to check that this deviation is unprofitable at $\max\mathrm{Supp}(F_l)$. Using $u_l=\bar{\nu}-\left(1-\pi\right)\left(\bar{\nu}-c_h\right)$, simple algebraic manipulations show that this local deviation is unprofitable as long as

$$\frac{\bar{\nu}-\nu_l}{\left(1-\phi\right)\left(\bar{\nu}-c_h\right)}\leqslant 1-\pi. \tag{56}$$

In order to rule out global deviations, we show that for any value of $u_h'\in\mathrm{Supp}\left(F_l\right)$, when

$u'_l \leqslant u'_h$, the profit function is increasing in u'_l. This implies that profits are maximized at $u'_l = u'_h$ and therefore there are no profitable deviation. To see this, note that profits are given by

$$\mu_l \left(1 - \pi + \pi F_l(u'_l)\right) \left(v_l - u'_l\right) + \mu_h \left(1 - \pi + \pi F_l \left(u'_h\right)\right) \Pi_h \left(u'_h, u'_l\right)$$

The derivative of this function with respect to u'_l is given by

$$\begin{aligned}
&\mu_l \pi f_l \left(u'_l\right) \left(v_l - u'_l\right) - \mu_l \left(1 - \pi + \pi F_l(u'_l)\right) + \mu_h \left(1 - \pi + \pi F_l \left(u'_h\right)\right) \frac{v_h - c_h}{c_h - c_l} \geqslant \\
&\mu_l \pi f_l \left(u'_l\right) \left(v_l - u'_l\right) - \mu_l \left(1 - \pi + \pi F_l(u'_l)\right) + \mu_h \left(1 - \pi + \pi F_l \left(u'_l\right)\right) \frac{v_h - c_h}{c_h - c_l} = \\
&\mu_l \pi f_l \left(u'_l\right) \left(v_l - u'_l\right) - \mu_l \phi \left(1 - \pi + \pi F_l(u'_l)\right) \qquad (57)
\end{aligned}$$

Using (55), we can write the above as

$$\begin{aligned}
&\mu_l \frac{v_l - u'_l}{\bar{v} - u'_l} (1 - \pi + \pi F_l(u_l)) - \mu_l \phi \left(1 - \pi + \pi F_l(u'_l)\right) = \\
&\mu_l (1 - \pi + \pi F_l(u'_l)) \left[\frac{v_l - u'_l}{\bar{v} - u'_l} - \phi\right] = \mu_l (1 - \pi + \pi F_l(u'_l)) \left[1 + \frac{v_l - \bar{v}}{\bar{v} - u'_l} - \phi\right]
\end{aligned}$$

Since $u'_l \leqslant u'_h \leqslant \max \mathrm{Supp}\,(F_l)$, the expression in the bracket takes its minimum value at $u'_l = \max \mathrm{Supp}\,(F_l)$ and we have

$$1 + \frac{v_l - \bar{v}}{\bar{v} - u'_l} - \phi \geqslant 1 + \frac{v_l - \bar{v}}{(1 - \pi)\left(\bar{v} - c_h\right)} - \phi \geqslant 0$$

where the second inequality follows from (56). This implies that the expression in (57) is positive and concludes the proof.

Case 3: Equilibrium Features Pooling and Separating Menus. As in Case 1 with only separation, the distribution F_l for $u_l \in [\hat{u}_l, \max \mathrm{Supp}(F_l)]$ is chosen to ensure local deviations are not profitable. Thus, we need only prove that local deviations are not profitable in the pooling region and that no global deviations are profitable. As in Case 2 with only pooling menus, it suffices to ensure that at the upper bound of the pooling region, $\hat{u}_l$, no local deviations are profitable, or

$$\hat{u}_l \left(v_h - c_h\right) \leqslant \bar{v} \left(v_h - c_l\right) - v_h \left(c_h - c_l\right). \qquad (58)$$

To prove that (58) holds, first note that since $\phi_2 < \phi < \phi_1$, we have $c_h < \hat{u}_l < \bar{u}\left(\hat{u}_l\right)$. To see this, guess that $\hat{u}_l > c_h$ (we will verify it later), in which case $\hat{u}_l$ satisfies

$$\bar{v} - \left\{ v_l + \left(\hat{u}_l - v_l\right) \left[(1 - \pi) \frac{\bar{v} - c_h}{\bar{v} - \hat{u}_l}\right]^{\frac{1}{\phi}} \right\} - (1 - \pi)\left(\bar{v} - c_h\right) = 0. \qquad (59)$$

Let $H(u)$ denote the left-hand side of (59). Observe that one solution to $H(u) = 0$ is given by

$$\bar{u} = \bar{v} - (1 - \pi)\left(\bar{v} - c_h\right).$$

This solution occurs when $\bar{u}(\hat{u}_l) = \hat{u}_l$. For $\hat{u}_l < \bar{u}$, we have $\bar{u}_l(\hat{u}_l) > \bar{u}_l$. We now show that when $\phi_2 < \phi < \phi_1$, then there is another solution with $\hat{u}_l \in (c_h, \bar{u})$. We show this by proving that $H'(\bar{u}) > 0$, $H(\bar{u}) = 0$ $H(c_h) > 0$ and $H''(u) \geqslant 0$. Then, there exists a unique threshold, $\hat{u}_l$ such that $H(u) = 0$. Note that

$$H'(u) = -\left[(1-\pi)\frac{\bar{v}-c_h}{\bar{v}-u}\right]^{\frac{1}{\phi}} - (u - v_l)\frac{1}{\phi}\left[(1-\pi)\frac{\bar{v}-c_h}{\bar{v}-u}\right]^{\frac{1}{\phi}-1}(1-\pi)(\bar{v}-c_h)(\bar{v}-u)^{-2}.$$

Algebraic computations, available on request, then demonstrate that $H''(u) \geqslant 0$. Next, observe that by construction, $H(\bar{u}) = 0$ and

$$H'(\bar{u}) = -1 - \frac{1}{\phi}\frac{\bar{u}-v_l}{\bar{v}-\bar{u}}.$$

Substituting for $\bar{u}$ and re-arranging terms, we find that

$$H'(\bar{u}) = \frac{1-\phi}{\phi}\left[1 - \frac{\bar{v}-v_l}{(1-\pi)(1-\phi)(\bar{v}-c_h)}\right].$$

When $\phi > \phi_2$, the term in brackets is negative so that $H'(\bar{u}) > 0$.

Finally, note that $H(c_h)$ satisfies

$$\begin{aligned} H(c_h) &= \bar{v} - v_l - (c_h - v_l)(1-\pi)^{\frac{1}{\phi}} - (1-\pi)(\bar{v}-c_h) \\ &= \pi(\bar{v}-v_l) - (c_h - v_l)(1-\pi)^{\frac{1}{\phi}} + (1-\pi)(c_h - v_l) \\ &= \pi(\bar{v}-v_l) - (c_h - v_l)\left[(1-\pi)^{\frac{1}{\phi}} - (1-\pi)\right] \\ &= \frac{1}{(1-\pi)^{\frac{1}{\phi}} - (1-\pi)}\left[v_l + \frac{\pi(\bar{v}-v_l)}{(1-\pi)^{\frac{1}{\phi}} - (1-\pi)} - c_h\right]. \end{aligned}$$

The fact that $\phi < \phi_1 < 0$ implies that the term in brackets is strictly positive and the leading fraction is also positive so that $H(c_h) > 0$.

We now prove that (58) is satisfied at $\hat{u}_l$. Algebra, available upon request shows that (58) may be written as

$$\hat{u}_l \leqslant \frac{-\phi}{1-\phi}\bar{v} + \frac{1}{1-\phi}v_l.$$

We demonstrate that this condition is verified by proving that $H\left(\frac{-\phi}{1-\phi}\bar{v} + \frac{1}{1-\phi}v_l\right) \leqslant 0$. To see this, note that

$$\begin{aligned}
& H\left(\frac{-\phi}{1-\phi}\bar{v}+\frac{1}{1-\phi}v_l\right) \\
= \;& \bar{v}-v_l-\left(\frac{-\phi}{1-\phi}\bar{v}+\frac{1}{1-\phi}v_l-v_l\right)\left[(1-\pi)\,\frac{\bar{v}-c_h}{\bar{v}-\frac{-\phi}{1-\phi}\bar{v}-\frac{1}{1-\phi}v_l}\right]^{\frac{1}{\phi}} \\
& -(1-\pi)\,(\bar{v}-c_h) \\
= \;& \bar{v}-v_l+\frac{\phi}{1-\phi}\,(\bar{v}-v_l)\left[\frac{(1-\pi)\,(1-\phi)\,(\bar{v}-c_h)}{\bar{v}-v_l}\right]^{\frac{1}{\phi}}-(1-\pi)\,(\bar{v}-c_h) \\
= \;& (\bar{v}-v_l)\left[\frac{\bar{v}-v_l-(1-\pi)\,(\bar{v}-c_h)}{\bar{v}-v_l}+\phi\frac{(1-\phi)^{\frac{1}{\phi}-1}\,(1-\pi)^{\frac{1}{\phi}}\,(\bar{v}-c_h)^{\frac{1}{\phi}}}{(\bar{v}-v_l)^{\frac{1}{\phi}}}\right]. \qquad (60)
\end{aligned}$$

We now show that the term in brackets on the right side of (60) is negative. To simplify notation, define $\xi=(1-\pi)\,(\bar{v}-c_h)\,/\,(\bar{v}-v_l)$. Since $\phi>\phi_2$, we have that $\xi<1/(1-\phi)$. As well, the term in brackets can be written compactly as

$$1-\xi+\phi\,(1-\phi)^{\frac{1}{\phi}-1}\,\xi^{\frac{1}{\phi}}.$$

Let $G(\xi)=1-\xi+\phi\,(1-\phi)^{\frac{1}{\phi}-1}\,\xi^{\frac{1}{\phi}}$ and observe that for $\xi\leqslant 1/(1-\phi)$, we have

$$G'(\xi)=-1+[(1-\phi)\,\xi]^{\frac{1}{\phi}-1}\geqslant 0$$

so that for low values of ξ, $G(\xi)$ is an increasing function. Finally, we see that $G(1/(1-\phi))=0$ so that $G(\xi)\leqslant G(1/(1-\phi))\leqslant 0$ which ensures the term in brackets in (60) is indeed negative as desired.

To rule out global deviations, one can use the arguments provided in the two cases above (when equilibrium is fully separating or pooling) in each region of the $\mathrm{Supp}\,(F_h)$. The argument is the exact replication of the above arguments and is thus omitted. ■

A.5 Proof of Proposition 12

We show that there are no profitable deviations. In other words,

$$\forall\,(u_h',u_l'):\mu_h\,(1-\pi+\pi F_l\,(u_l'))\,\Pi_h\,(u_h',u_l')+\mu_l\,(1-\pi+\pi F_l\,(u_l'))\,(v_l-u_l')\leqslant(1-\pi)\,\mu_l\,(v_l-c_l)$$

We consider two cases:

1. $u_h'>\max\mathrm{Supp}\,(F_h)=\bar{u}_h$: In this case, when $u_l'>v_l$, the profit function is given by

$$\mu_h\Pi_h\,(u_h',u_l')+\mu_l\,(v_l-u_l')$$

   Since $\phi=0$, the above function is invariant to changes in u_l' and is strictly decreasing in u_l'. Therefore, its value must be less than its value evaluated at $(\bar{u}_h,v_l)$ which gives the equilibrium profits. When, $u_l'\leqslant v_l$, the profits are given by $\mu_h\Pi_h\,(u_h',u_l')$ which is decreasing in u'_h and therefore

$$\mu_h \Pi_h \left(u'_h, u'_l\right) + \mu_l (1-\pi) \left(v_l - u'_l\right) < \mu_h \Pi_h \left(\bar{u}_h, u'_l\right) + \mu_l (1-\pi) \left(v_l - u'_l\right)$$

Note that the right hand side of the above inequality is a linear function of u'_l whose derivative is given by

$$\begin{aligned} \mu_h \frac{v_h - c_h}{v_l - c_l} - \mu_l (1-\pi) &= \mu_h \frac{v_h - c_h}{c_h - c_l} - \mu_l + \mu_l \pi \\ &= -\mu_l \phi + \mu_l \pi = \mu_l \pi > 0. \end{aligned}$$

Therefore, we must have that

$$\mu_h \Pi_h \left(\bar{u}_h, u'_l\right) + \mu_l (1-\pi) \left(v_l - u'_l\right) \leqslant \mu_h \Pi_h \left(\bar{u}_h, v_l\right) = (1-\pi) \mu_l \left(v_l - c_l\right)$$

where the last equality follows from (32).

2. $u'_h \in [c_h, \bar{u}_h]$. In this case, when $u'_l > v_l$, profits are given by

$$\begin{aligned} \mu_h \left(1 - \pi + \pi F_l \left(u'_h\right)\right) \Pi_h \left(u'_h, u'_l\right) + \mu_l \left(v_l - u'_l\right) &\leqslant \mu_h \left(1 - \pi + \pi F_l \left(u'_h\right)\right) \Pi_h \left(u'_h, v_l\right) \\ &= (1-\pi) \mu_l \left(v_l - c_l\right) \end{aligned}$$

where the inequality is satisfied since $u'_l > v_l$ and the last equality follows from (32). When $u'_l \leqslant v_l$, profits are given by

$$\mu_h \left(1 - \pi + \pi F_l \left(u'_h\right)\right) \Pi_h \left(u'_h, u'_l\right) + \mu_l (1-\pi) \left(v_l - u'_l\right)$$

The above function is linear in u'_l and its derivative is given by

$$\begin{aligned} \mu_h \left(1 - \pi + \pi F_l \left(u'_h\right)\right) \frac{v_h - c_h}{c_h - c_l} - \mu_l (1-\pi) &= (1-\pi) \left(\mu_h \frac{v_h - c_h}{c_h - c_l} - \mu_l\right) + \pi F_l \left(u'_h\right) \frac{v_h - c_h}{c_h - c_l} \\ &= \pi F_l \left(u'_h\right) \frac{v_h - c_h}{c_h - c_l} \geqslant 0 \end{aligned}$$

Therefore it is maximized at $u'_l = v_l$. This completes the proof.

■

A.6 Proof of Theorem 13

We prove uniqueness of the equilibrium first for $\phi > 0$ and then for $\phi < 0$.

Case 1: $\phi > 0$. As discussed in the text, we need to prove that no equilibrium features a mass point in F_l and that no equilibrium features pooling ($U_h(u_l) = u_l$) on any interval in the support of F_l. We state these results and proofs as independent lemmas below. These results imply that any equilibrium is separating and has continuous distribution F_l; since there is a unique solution to the differential equation (21) with the boundary condition $\min \mathrm{Supp}(F_l) = c_l$, we have then shown that the equilibrium described in Proposition 10 is unique.

Lemma 20. *If $\phi > 0$, then F_l has no mass points.*

Proof. We have already shown that if F_l has a mass point, then it must occur at the lower bound of the support of F_l which satisfies $\min \mathrm{Supp}(F_l) = c_l$. Suppose by way of contradiction that F_l has a mass point at c_l. Since F_h has no mass point, $U_h(u_l)$ must be a correspondence at $u_l = c_l$. Hence, there exists a $u_h \in U_h(u_l)$ such that $u_h > u_l = c_l$. Consider a deviation by a buyer to the menu $(c_l + \varepsilon, u_h)$. Such a deviation induces a jump in the number of non-captive low quality sellers attracted while decreasing the profits earned per low quality seller marginally. Hence, this deviation yields strictly positive profits, implying a contradiction. ■

Lemma 21. *If $\phi > 0$, in any equilibrium, there is no interval $[u_1, u_2] \in \mathrm{Supp}\, F_l$ such that $U_h(u_l) = u_l$ for $u_l \in [u_1, u_2]$. That is, there is no interval which features pooling menus.*

Proof. Suppose by way of contradiction that a pooling interval exists. We will show that this necessarily implies that the separating equilibrium we construct in Proposition 10 must not exist.

Towards this end, we first prove that there can be at most one interval of pooling and that the pooling thresholds u_1 and u_2 are uniquely determined. First, note that since the worst equilibrium menu is the menu (c_l, c_h) which satisfies $U_h(c_l) > c_l$, the equilibrium near the lower bound of the support must be separating. Let $[u_1, u_2]$ denote the lowest interval of pooling and assume $u_2 < \max \mathrm{Supp}(F_l)$. We show that for all $u_l > u_2$, $U_h(u_l) > u_l$. Since the equilibrium is locally pooling below u_2, $\lim_{u_l \nearrow u_2} U_h'(u_l) = 1$. IF $u_2 < \max \mathrm{Supp}(F_l)$, the equilibrium must be locally separating above u_2 so that $U_h(u_l)$ satisfies

$$(1 - \pi + \pi F_l(u_l)) \left[\mu_l (v_l - u_l) + \mu_h \Pi_h(u_l, U_h(u_l)\right] = (1 - \pi)(v_l - c_l)$$

with $F_l(u_l)$ satisfying the differential equation

$$\frac{\pi f_l(u_l)}{1 - \pi + \pi F_l(u_l)} = \frac{\phi}{v_l - u_l}. \tag{61}$$

Twice differentiating the equal profit condition with respect to u_l, we obtain

$$U_h''(u_l) = \phi(1 - \phi) \frac{(1 - \pi)(v_l - c_l)(v_h - v_l)}{(1 - \pi + \pi F_l(u_l))(v_l - u_l)^2 \mu_h (v_h - c_l)} > 0$$

where the inequality holds when $\phi > 0$ (recall $\phi \leqslant 1$) so that U_h is convex. This means that for all $u_l > u_2$, $U_h(u_l) > u_l$. We conclude that there can be at most one region of pooling menus.

For the pooling region to exist, cream-skimming deviations must be unprofitable at the upper threshold of the pooling interval, u_2 so that

$$u_2 \leqslant \bar{v} + (\bar{v} - v_h) \frac{c_h - c_l}{v_h - c_h} \equiv v_{pool}.$$

The lower threshold is determined by imposing the boundary condition $U_h(u_1) = u_1$ on the differential equation (61). Solving the differential equation, we find u_1 satisfies

$$\bar{v} = u_1 + \mu_l (v_l - c_l)^{1-\phi} (v_l - u_1)^{\phi}. \tag{62}$$

We now analyze solutions to the non-linear equation (62). Let

$$H(u) = u + \mu_l (v_l - c_l)^{1-\phi} (v_l - u)^{\phi}$$

so that

$$H'(u) = 1 - \phi\mu_l (v_l - c_l)^{1-\phi} (v_l - u)^{\phi-1}$$

Straightforward calculations show that $H(u)$ is a concave function when $\phi \geqslant 0$ and $u \leqslant v_l$. Since $H'(v_l) < 0$, we have that $\arg\max_u H(u) < v_l$. Since the pooling region exists, by assumption, we must have $\max_u H(u) \geqslant \bar{v}$. Since $U_h(u_l) \to u_l$ as $u_l \to u_1$, we must have $U_h'(u_l) \leqslant 1$ as $u_l \to u_1$. Or, using the equal profit condition again

$$-\mu_l\phi - \mu_h \frac{v_h - c_l}{c_h - c_l} U_h'(u_l) = H'(u) - 1. \tag{63}$$

Using (63), one can show that requiring $U_h'(u_1) \leqslant 1$ is equivalent to requiring $H'(u) \geqslant 0$. Hence, u_1 is uniquely determined as the lowest solution to $H(u) = \bar{v}$. Extensive algebraic manipulations, available upon request, can be used to show that $H'(v_{pool}) < 0$ so that by concavity of $H(u)$, we know that $u_1 \leqslant v_{pool}$. A similar argument, applied to the upper bound implies that the upper threshold is also uniquely determined if $u_2 < \mathrm{Supp}(F_l)$.

Now, note that the equilibrium we describe in Proposition 10 satisfies $\max \mathrm{Supp}(F_l) = v_l - (1-\pi)^{1/\phi} (v_l - c_l)$. If the pooling interval exists, it is immediate that $u_1 \leqslant v_l - (1-\pi)^{1/\phi} (v_l - c_l)$. (If $u_1 > v_l - (1-\pi)^{1/\phi} (v_l - c_l)$, then $F(u_1) > 1$). If the threshold u_1 satisfies

$$u_1 < v_l - (1-\pi)^{1/\phi} (v_l - c_l), \tag{64}$$

then the equilibrium with only separating menus described in Proposition 10 does not exist, which is a contradiction. If such an equilibrium exists when u_1 satisfies the inequality (64), then for some values $u_l \in \left[u_1, v_l - (1-\pi)^{1/\phi} (v_l - c_l)\right]$, we have $U_h(u_l) = u_l$ violating the separating feature of the equilibrium. ∎

Case 2: $\phi < 0$**.** To prove the equilibrium characterized in Proposition 11 is unique, we first prove that in any equilibrium with $\phi < 0$, if $\bar{u} = \max \mathrm{Supp}(F_l)$, then $U_h(\bar{u}) = \bar{u}$ so that the best menu in equilibrium is a pooling menu. We then demonstrate that F_l has no mass points when $\phi < 0$. Next, We then prove that if the equilibrium has a pooling region, the region begins at the lower bound of the support of F_l or ends at the upper bound of F_l. Additionally, if the equilibrium features a separating region, this region must end at the upper bound of the support of F_l. These results imply that any equilibrium must take one of the three forms described in Proposition 11: only separating, only pooling, or semi-separating. Finally, we show that the necessary conditions for each type of equilibrium to exist are mutually exclusive so that at most one type of equilibrium exists for each region of the parameter space, ensuring our equilibrium is unique for all $\phi < 0$. We prove these results in the sequence of following lemmas.

Lemma 22. *If $\phi < 0$, then F_l has no mass points.*

Proof. Suppose by way of contradiction that F_l has a mass point at the lower bound of the support of F_l. Then it must be that $\min \mathrm{Supp}(F_l) = v_l$. Exactly as in Lemma 20, if F_l has a mass point at some $u_l < v_l$, then a deviation to a menu $(u_l \pm \varepsilon, u_h)$ must be feasible for some $u_h \in U_h(u_l)$ and earn strictly positive profits.

Next note that if F_l has a mass point at v_l, then we must have $F_l^+(v_l) = 1$. If instead $F_l^+(v_l) = m < 1$, then, since $U_h(v_l)$ is an interval, there must be an interval of u_l such that for $u_l \in (v_l, v_l + \varepsilon)$ for $\varepsilon > 0$ the equilibrium menus are separating, or $U_h(u_l) > u_l$. In this interval of $\mathrm{Supp}(F_l)$, since local deviations must be unprofitable, F_l must satisfy

$$1 - \pi + \pi F_l(u_l) = C\,(u_l - v_l)^{-\phi}$$

for some positive constant C. But then,

$$\lim_{u_l \searrow v_l} 1 - \pi + \pi F_l(u_l) = \lim_{u_l \searrow v_l} C(u_l - v_l)^{-\phi} = 0$$

which contradicts the fact that

$$\lim_{u_l \searrow v_l} 1 - \pi + \pi F_l(u_l) = 1 - \pi + \pi F_l\,(v_l) = 1 - \pi(1 - m) > 0.$$

Hence, $F_l^+(v_l) = 1$.

As a consequence, we have that $\mathrm{Supp}(F_l) = \{v_l\}$ while $\max \mathrm{Supp}(F_h) > \underline{u}_h \geqslant v_l = \max \mathrm{Supp}(F_l)$ (this last fact follows since F_h is continuous). Let $\bar{u}_h = \max \mathrm{Supp}(F_h)$ and consider the profits a buyer would obtain by following a deviation to a menu $(v_l + \epsilon, \bar{u}_h)$. Profits at the original contract satisfy

$$\mu_h \left(v_h - \bar{u}_h \frac{v_h - c_l}{c_h - c_l} + v_l \frac{v_h - c_h}{c_h - c_l} \right)$$

while the deviation contract yields profits equal to

$$\begin{aligned} &\mu_h \left(v_h - \bar{u}_h \frac{v_h - c_l}{c_h - c_l} + (v_l + \epsilon) \frac{v_h - c_h}{c_h - c_l} \right) + \mu_l \left(v_l - (v_l + \epsilon) \right) \\ &= \mu_h \left(v_h - \bar{u}_h \frac{v_h - c_l}{c_h - c_l} + v_l \frac{v_h - c_h}{c_h - c_l} \right) - \mu_l \epsilon + \mu_h \frac{v_h - c_h}{c_h - c_l} \epsilon. \end{aligned}$$

Since the increment to profits earned from this deviation is given by

$$-\mu_l \epsilon + \mu_h \frac{v_h - c_h}{c_h - c_l} \epsilon = -\mu_l \phi \epsilon$$

and since $\phi < 0$, this deviation earns strictly greater profits than the equilibrium menu $(v_l, \bar{u}_h)$, which yields the necessary contradiction. ■

Lemma 23. *If $\phi < 0$, then the best equilibrium menu is a pooling menu.*

Proof. Let $\bar{u} = \max \mathrm{Supp}(F_l)$ and suppose for contradiction that $U_h(\bar{u}) > \bar{u}$. Consider a deviation menu with $(u_l, u_h) = (\bar{u} + \varepsilon, U_h(\bar{u}))$. Since $U_h\,(\bar{u}) > \bar{u}$, this menu is incentive compatible and has $F_l(u_l) = F_l\,(u_h) = 1$. This menu increases the buyer's profits relative to the menu $(\bar{u}, U_h(\bar{u}))$ by the amount

$$-\mu_l \varepsilon + \mu_h \frac{v_h - c_h}{c_h - c_l} = -\mu_l \phi \varepsilon > 0$$

where the inequality follows from $\phi < 0$. This profitable deviation yields the necessary contradiction so that we must have $U_h(\bar{u}) = \bar{u}$. ■

Lemma 24. *If* $\phi < 0$ *and an equilibrium features* $[u_1, u_2] \subseteq \text{Supp}(F_l)$ *such that* $U_h(u_l) = u_l$ *for* $u_l \in [u_1, u_2]$, *then either* $u_1 = \min \text{Supp}(F_l)$ *or* $u_2 = \max \text{Supp}(F_l)$.

Proof. Suppose for contradiction that an interval of pooling exists and $u_1 > \min \text{Supp}(F_l)$ and $u_2 < \max \text{Supp}(F_l)$. Then there must exist intervals sufficiently close to and below u_1 and above u_2 respectively in which the equilibrium menus feature separation. Consequently, we must have $\lim_{u_l \nearrow u_1} U_h'(u_l) \leqslant 1$ and $\lim_{u_l \searrow u_2} U_h'(u_l) \geqslant 1$. In any region with $U_h(u_l) > u_l$, the distribution F_l must also satisfy

$$\frac{\pi f_l(u_l)}{1 - \pi + \pi F_l(u_l)} = \frac{-\phi}{u_l - v_l}$$

to rule out profitable local deviations and U_h must satisfy

$$\bar{v} - \mu_l \phi u_l - \mu_h \frac{v_h - c_l}{c_h - c_l} U_h(u_l) = \bar{\Pi} \left(1 - \pi + \pi F_l(u_l)\right)^{-1}$$

to ensure all menus earn the equilibrium level of profits which we denote with $\bar{\Pi}$.

Using these features of the conjectured equilibrium, we have that in the separating regions, $U_h'(u_l)$ satisfies

$$-\mu_l \phi - (1 - \mu_l \phi) U_h'(u_l) = \frac{\bar{\Pi}}{1 - \pi + \pi F_l(u_l)} \frac{\phi}{u_l - v_l}.$$

Moreover, the second derivative of U_h satisfies

$$-(1 - \mu_l \phi) U_h''(u_l) = \frac{\bar{\Pi} \pi f_l(u_l)}{[1 - \pi + \pi F_l(u_l)]^2} \frac{\phi}{u_l - v_l} + \frac{\bar{\Pi}}{1 - \pi + \pi F_l(u_l)} \frac{-\phi}{[u_l - v_l]^2}$$

which implies $U_h''(u_l) \leqslant 0$ since $\phi < 0$. However, the pooling region requires $U_h'^{+}(u_2) \geqslant 1 \geqslant U_h'^{-}(u_1)$ which contradicts the concavity of U_h given that $u_1 < u_2$. Hence, either $u_1 = \min \text{Supp}(F_l)$ or $u_2 = \max \text{Supp}(F_l)$. ■

Lemma 25. *If* $\phi < 0$ *and an equilibrium features* $[u_1, u_2] \subseteq \text{Supp}(F_l)$ *such that* $U_h(u_l) > u_l$ *for* $u_l \in (u_1, u_2)$, *then* $u_2 = \max \text{Supp}(F_l)$.

Proof. Suppose such a separating interval exists with $\min \text{Supp}(F_l) < u_2 < \max \text{Supp}(F_l)$. Then there exists a pooling interval $[u_2, \bar{u}]$ for some $\bar{u}$. Since $u_2 > \min \text{Supp}(F_l)$, Lemma 24 implies that $\bar{u} = \max \text{Supp}(F_l)$. Since the conjectured equilibrium features separation in $[u_1, u_2]$ with $U_h(u_l) \to u_l$ as $u_l \to u_2$, we must have $U_h'^{-}(u_2) \leqslant 1$ or

$$\frac{1}{1 - \mu_l \phi} \left[-\mu_l \phi + \frac{\bar{\Pi}}{1 - \pi + \pi F_l(u_2)} \frac{-\phi}{u_2 - v_l}\right] \leqslant 1$$

which can be re-written as

$$-\phi \bar{\Pi} \leqslant [1 - \pi + \pi F_l(u_2)] (u_2 - v_l).$$

Since $u_2 < \bar{u}$, $F(u_2) < 1$ so that

$$-\phi\bar{\Pi} < u_2 - v_l. \tag{65}$$

Moreover, since the best menu is pooling with utility $\bar{u}$, equilibrium profits satisfy

$$\bar{\Pi} = \bar{v} - \bar{u}$$

equation (65) implies

$$v_l - \phi(\bar{v} - \bar{u}) < u_2 < \bar{u}$$

or

$$v_l - \phi\bar{v} < (1-\phi)\bar{u} \tag{66}$$

Since the equilibrium features pooling in the interval $[u_2, \bar{u}]$, we must have cream-skimming is not a profitable deviation, in particular from the pooling menu $(\bar{u}, \bar{u})$, or

$$\bar{u} < \bar{v}\frac{v_h - c_l}{v_h - c_h} - v_h\frac{c_h - c_l}{v_h - c_h}. \tag{67}$$

Conditions (66) and (67) then require

$$\frac{v_l - \phi\bar{v}}{1-\phi} < \bar{v}\frac{v_h - c_l}{v_h - c_h} - v_h\frac{c_h - c_l}{v_h - c_h}.$$

By substituting for ϕ and using straightforward algebraic manipulations we show that

$$\frac{v_l - \phi\bar{v}}{1-\phi} = \bar{v}\frac{v_h - c_l}{v_h - c_h} - v_h\frac{c_h - c_l}{v_h - c_h}$$

which yields the needed contradiction. ■

Since the only possible equilibria when $\phi < 0$, then, are fully separating (except at the upper bound of the support of F_l), fully pooling, or semi-separating, we need only prove that only one of these equilibria may exist for any value of ϕ. We have already shown in the proof of Proposition 11 that $\phi_2 < \phi_1 < 0$. Recall that a necessary condition for a fully pooling equilibrium is that $\phi \leqslant \phi_2$. Hence, there is no fully pooling equilibrium when $\phi > \phi_2$. Similarly, a necessary condition for a fully separating equilibrium is that $\phi \geqslant \phi_1$ so that when $\phi < \phi_1$, no fully separating equilibrium exists. This means that in the interval $\phi_2 < \phi < \phi_1$, the only possible equilibrium is a semi-separating equilibrium. Moreover, the threshold in the semi-separating equilibrium is interior to the support of F_l only if ϕ lies between ϕ_2 and ϕ_1. Hence, at most one of these types of equilibria may exist for any value of $\phi < 0$, proving that the equilibrium described in Proposition 13 is unique.

A.7 Proof of Proposition 14

When $\phi < 0$, it is immediate that welfare is (weakly) maximized when $\pi = 0$. To prove that welfare is maximized for $\pi \in (0,1)$ when $\phi > 0$ we prove that our measure of welfare is decreasing in π at $\pi = 1$. Since welfare associated with the endpoint $\pi = 1$ is strictly larger than at the other extreme, when $\pi = 0$, this finding necessarily implies that welfare is maximized at an interior π.

To show that welfare is decreasing at $\pi = 1$, we first demonstrate that the volume of loans sold by high-quality sellers is decreasing in the level of indirect utility offered to low-quality sellers when this level is sufficiently close to v_l. We state this result as the following Lemma. Below, we use this finding to prove that welfare is decreasing in π at $\pi = 1$.

Lemma 26. *As u_l tends to v_l, the volume of loans sold by high-quality sellers at the associated u_h is decreasing.*

Proof. First, note that for any equilibrium menu, $(u_l, U_h(u_l))$, the associated volume of loans sold by high quality sellers satisfies

$$x(u_l) = 1 - \frac{U_h(u_l) - u_l}{c_h - c_l}.$$

Thus, $x(u_l)$ is decreasing if and only if $U_h'(u_l) > 1$. We determine $U_h'(u_l)$ using properties of the equilibrium, namely equations (21) and (22). Specifically, if we differentiate (22) with respect to u_l, we find

$$\mu_h \frac{v_h - c_l}{c_h - c_l} U_h'(u_l) = -\mu_l \phi + \frac{\mu_l (1-\pi)(v_l - c_l)}{1 - \pi + \pi F_l(u_l)} \frac{\pi f_l(u_l)}{1 - \pi + \pi F_l(u_l)}.$$

Using (21), we can simplify this expression as

$$\mu_h \frac{v_h - c_l}{c_h - c_l} U_h'(u_l) = -\mu_l \phi + \frac{\mu_l (1-\pi)(v_l - c_l)}{1 - \pi + \pi F_l(u_l)} \frac{\phi}{v_l - u_l}.$$

Using (22) again to replace $\mu_l(1-\pi)(v_l - c_l) / [1 - \pi + \pi F_l(u_l)]$ yields equation (35) in the text, which we replicate here.

$$\begin{aligned} \mu_h \frac{v_h - c_l}{c_h - c_l} U_h'(u_l) &= -\mu_l \phi + \frac{\phi\left[\mu_h \Pi_h(U_h(u_l), u_l) + \mu_l (v_l - u_l)\right]}{v_l - u_l} \\ \Rightarrow \quad U_h'(u_l) &= \left[\frac{\phi (c_h - c_l)}{v_h - c_l}\right] \times \frac{\Pi_h(U_h(u_l), u_l)}{v_l - u_l}. \end{aligned}$$

We show that as $u_l \to v_l$, $U_h'(u_l) \to \infty$. To see this, note that

$$\frac{\Pi_h(U_h(u_l), u_l)}{v_l - u_l} = \frac{\mu_l}{\mu_h} \left[\frac{(1-\pi)(v_l - c_l)}{v_l - u_l} \frac{1}{1 - \pi + \pi F_l(u_l)} - 1\right].$$

When $\phi > 0$, the equilibrium distribution (that is, the solution to the differential equation) satisfies

$$1 - \pi + \pi F_l(u_l) = (1-\pi) \left(\frac{v_l - c_l}{v_l - u_l}\right)^{\phi}$$

and hence

$$\frac{\Pi_h(U_h(u_l), u_l)}{v_l - u_l} = \frac{\mu_l}{\mu_h} \left[\left(\frac{v_l - c_l}{v_l - u_l}\right)^{1-\phi} - 1\right].$$

Since $\phi > 0$, as $u_l \to v_l$, we see that this term tends to infinity. ∎

Next, we show that welfare is a decreasing function of π at $\pi = 1$.

Note ex-ante welfare is given by $\mu_l(v_l - c_l) + (v_h - c_h)$ times the expected trade by the high quality sellers which is given by

$$\begin{aligned}
& (1-\pi)\mu_h \int_{c_l}^{\bar{u}_l} x_h(u_l) f(u_l)\, du_l + \pi\mu_h \int_{c_l}^{\bar{u}_l} 2x_h(u_l) f(u_l) F(u_l)\, du_l \\
= \quad & \mu_h \int_{c_l}^{\bar{u}_l} x_h(u_l) f(u_l) \left[1 - \pi + 2\pi F(u_l)\right] du_l \\
= \quad & \mu_h \int_{c_l}^{\bar{u}_l} \left(\frac{\mu_l (v_l - c_l)^{1-\phi} (v_l - u_l)^{\phi} + u_l - \mu_h c_l - \mu_l v_l}{\mu_h (v_h - c_l)} \right) \left(\frac{1-\pi}{\pi} \phi (v_l - c_l)^{\phi} (v_l - u_l)^{-1-\phi} \right) \\
& \qquad \left[(1-\pi) + 2(1-\pi) \left(\frac{v_l - c_l}{v_l - u_l} \right)^{\phi} - 2(1-\pi) \right] du_l \\
= \quad & \phi\mu_h \frac{(1-\pi)^2}{\pi} \int_{c_l}^{\bar{u}_l} \frac{\mu_l (v_l - c_l)^{1-\phi} (v_l - u_l)^{\phi} + u_l - \mu_h c_l - \mu_l v_l}{\mu_h (v_h - c_l)} (v_l - c_l)^{\phi} (v_l - u_l)^{-1-\phi} \\
& \qquad \left[2 \left(\frac{v_l - c_l}{v_l - u_l} \right)^{\phi} - 1 \right] du_l \\
= \quad & \phi \frac{(v_l - c_l)^{\phi}}{v_h - c_l} \frac{(1-\pi)^2}{\pi} \int_{c_l}^{\bar{u}_l} \left(\mu_l (v_l - c_l)^{1-\phi} (v_l - u_l)^{\phi} + u_l - \mu_h c_l - \mu_l v_l \right) (v_l - u_l)^{-1-\phi} \\
& \qquad \left[2 \left(\frac{v_l - c_l}{v_l - u_l} \right)^{\phi} - 1 \right] du_l
\end{aligned}$$

Expected trade by the high type is $\phi \frac{(v_l - c_l)^{\phi}}{v_h - c_l}$ times

$$\begin{aligned}
& \frac{(1-\pi)^2}{\pi} \int_{c_l}^{\bar{u}_l} \left(\mu_l (v_l - c_l)^{1-\phi} (v_l - u_l)^{-1} + (u_l - \mu_h c_l - \mu_l v_l)(v_l - u_l)^{-1-\phi} \right) \\
& \qquad \left[2 \left(\frac{v_l - c_l}{v_l - u_l} \right)^{\phi} - 1 \right] du_l \\
= \quad & \frac{(1-\pi)^2}{\pi} 2(v_l - c_l) \left(\int_{c_l}^{\bar{u}_l} \mu_l (v_l - u_l)^{-1-\phi} + (u_l - \mu_h c_l - \mu_l v_l)(v_l - u_l)^{-1-2\phi} (v_l - c_l)^{\phi-1} du_l \right) \\
& - \frac{(1-\pi)^2}{\pi} \left(\int_{c_l}^{\bar{u}_l} \left(\mu_l (v_l - c_l)^{1-\phi} (v_l - u_l)^{-1} + (u_l - \mu_h c_l - \mu_l v_l)(v_l - u_l)^{-1-\phi} \right) du_l \right) \\
= \quad & \frac{(1-\pi)^2}{\pi} 2(v_l - c_l) \left(\int_{c_l}^{\bar{u}_l} \mu_l (v_l - u_l)^{-1-\phi} + (u_l - v_l + \mu_h (v_l - c_l))(v_l - u_l)^{-1-2\phi} (v_l - c_l)^{\phi-1} du_l \right) \\
& - \frac{(1-\pi)^2}{\pi} \left(\int_{c_l}^{\bar{u}_l} \left(\mu_l (v_l - c_l)^{1-\phi} (v_l - u_l)^{-1} + (u_l - v_l + \mu_h (v_l - c_l))(v_l - u_l)^{-1-\phi} \right) du_l \right)
\end{aligned}$$

The indefinite integral in the first term:

$$\begin{aligned}
&\int \left[\mu_l \left(\nu_l - u_l\right)^{-1-\phi} + \left(u_l - \nu_l + \mu_h \left(\nu_l - c_l\right)\right) \left(\nu_l - u_l\right)^{-1-2\phi} \left(\nu_l - c_l\right)^{\phi-1}\right] du_l \\
= \;& -\int \left[\mu_l z^{-1-\phi} + \left(-z + \mu_h \left(\nu_l - c_l\right)\right) z^{-1-2\phi} \left(\nu_l - c_l\right)^{\phi-1}\right] dz \\
= \;& -\left[-\mu_l \frac{z^{-\phi}}{\phi} - \left(\nu_l - c_l\right)^{\phi-1} \frac{z^{1-2\phi}}{1-2\phi} - \mu_h \left(\nu_l - c_l\right) \left(\nu_l - c_l\right)^{\phi-1} \frac{z^{-2\phi}}{2\phi}\right] \\
= \;& \mu_l \frac{z^{-\phi}}{\phi} + \left(\nu_l - c_l\right)^{\phi-1} \frac{z^{1-2\phi}}{1-2\phi} + \mu_h \left(\nu_l - c_l\right)^{\phi} \frac{z^{-2\phi}}{2\phi} \\
= \;& \mu_l \frac{\left(\nu_l - u_l\right)^{-\phi}}{\phi} + \left(\nu_l - c_l\right)^{\phi-1} \frac{\left(\nu_l - u_l\right)^{1-2\phi}}{1-2\phi} + \mu_h \left(\nu_l - c_l\right)^{\phi} \frac{\left(\nu_l - u_l\right)^{-2\phi}}{2\phi}
\end{aligned}$$

and in the second term

$$\begin{aligned}
= \;& \int \left(\mu_l \left(\nu_l - c_l\right)^{1-\phi} \left(\nu_l - u_l\right)^{-1} + \left(u_l - \nu_l + \mu_h \left(\nu_l - c_l\right)\right) \left(\nu_l - u_l\right)^{-1-\phi}\right) du_l \\
= \;& -\int \left(\mu_l \left(\nu_l - c_l\right)^{1-\phi} z^{-1} + \left(-z + \mu_h \left(\nu_l - c_l\right)\right) z^{-1-\phi}\right) dz \\
= \;& -\left[\mu_l \left(\nu_l - c_l\right)^{1-\phi} \ln z - \frac{z^{1-\phi}}{1-\phi} - \mu_h \left(\nu_l - c_l\right) \frac{z^{-\phi}}{\phi}\right] \\
= \;& -\mu_l \left(\nu_l - c_l\right)^{1-\phi} \ln z + \frac{z^{1-\phi}}{1-\phi} + \mu_h \left(\nu_l - c_l\right) \frac{z^{-\phi}}{\phi} \\
= \;& -\mu_l \left(\nu_l - c_l\right)^{1-\phi} \ln \left(\nu_l - u_l\right) + \frac{\left(\nu_l - u_l\right)^{1-\phi}}{1-\phi} + \mu_h \left(\nu_l - c_l\right) \frac{\left(\nu_l - u_l\right)^{-\phi}}{\phi}
\end{aligned}$$

Using the above two integrals and extensive algebra implies that expected value of trade is given by

$$\begin{aligned}
= \;& \frac{1}{\pi} \left(\nu_l - c_l\right)^{1-\phi} 2 \left[\mu_l \frac{\left(1-\pi\right) \pi}{\phi} + \frac{\left(1-\pi\right)^{\frac{1}{\phi}} - \left(1-\pi\right)^2}{1-2\phi} + \mu_h \frac{1 - \left(1-\pi\right)^2}{2\phi}\right] \\
& - \frac{1}{\pi} \left(\nu_l - c_l\right)^{1-\phi} \left[-\mu_l \frac{\left(1-\pi\right)^2}{\phi} \ln \left(1-\pi\right) + \frac{\left(1-\pi\right)^{\frac{1+\phi}{\phi}} - \left(1-\pi\right)^2}{1-\phi} + \mu_h \frac{\left(1-\pi\right) \pi}{\phi}\right]
\end{aligned}$$

Using the above formula, the derivative of expected value of trade by the high quality seller w.r.t π is given by

$$
\begin{aligned}
&-\frac{1}{\pi^2}\left(v_l-c_l\right)^{1-\phi} 2\left[\mu_l \frac{(1-\pi)\pi}{\phi}+\frac{(1-\pi)^{\frac{1}{\phi}}-(1-\pi)^2}{1-2\phi}+\mu_h \frac{1-(1-\pi)^2}{2\phi}\right] \\
&+\frac{1}{\pi}\left(v_l-c_l\right)^{1-\phi} 2\left[\mu_l \frac{1-2\pi}{\phi}+\frac{-\frac{1}{\phi}(1-\pi)^{\frac{1}{\phi}-1}+2(1-\pi)}{1-2\phi}+\mu_h \frac{2(1-\pi)}{2\phi}\right] \\
&+\frac{1}{\pi^2}\left(v_l-c_l\right)^{1-\phi}\left[-\mu_l \frac{(1-\pi)^2}{\phi} \ln(1-\pi)+\frac{(1-\pi)^{\frac{1+\phi}{\phi}}-(1-\pi)^2}{1-\phi}+\mu_h \frac{(1-\pi)\pi}{\phi}\right] \\
&-\frac{1}{\pi}\left(v_l-c_l\right)^{1-\phi}\left[\mu_l \frac{2(1-\pi)}{\phi} \ln(1-\pi)+\mu_l \frac{(1-\pi)}{\phi}-\frac{\frac{1+\phi}{\phi}(1-\pi)^{\frac{1}{\phi}}-2(1-\pi)}{1-\phi}+\mu_h \frac{1-2\pi}{\phi}\right]
\end{aligned}
$$

Evaluated at $\pi=1$, this becomes

$$
\begin{aligned}
& -\left(v_l-c_l\right)^{1-\phi} \frac{\mu_h}{\phi}+\left(v_l-c_l\right)^{1-\phi}\left[\frac{-2\mu_l}{\phi}\right]-\left(v_l-c_l\right)^{1-\phi}\left[\frac{-\mu_h}{\phi}\right] \\
= & \frac{\left(v_l-c_l\right)^{1-\phi}}{\phi}\left[-\mu_h-2\mu_l+\mu_h\right]=-2\mu_l \frac{\left(v_l-c_l\right)^{1-\phi}}{\phi} \\
< & 0
\end{aligned}
$$