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Banking Crises and Central Bank Digital Currency in a Monetary Economy*

Tarishi Matsuoka† Makoto Watanabe‡

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Abstract

This paper examines the role of Central Bank Digital Currency (CBDC) in a monetary model in which fundamental-based bank runs arise endogenously. We demonstrate that introducing a CBDC designed to replicate the properties of cash displaces physical cash and, when offered at a sufficiently attractive rate, can increase the likelihood of a bank run. In contrast, when the CBDC is designed to resemble bank deposits, cash, CBDC, and deposits can coexist as media of exchange, and a high CBDC rate can eliminate the risk of runs. We further characterize the optimal CBDC policy within this framework.

Keywords: Monetary Equilibrium, Bank Run, CBDC

JEL Classification Number: E42, E58, G21

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1 Introduction

Many central banks have initiated research and development efforts concerning the potential issuance of a central bank digital currency (CBDC) that would be widely used by retail firms and households as a medium of exchange. A central concern in these discussions is the impact of introducing a CBDC on financial stability. Since a CBDC could serve as a safe alternative to bank deposits, it may increase the risk of bank runs. The collapses of Silicon Valley Bank and Signature Bank underscore how digital bank runs can pose significant threats to financial systems.

This paper studies the economic role and implications of a CBDC, with a particular focus on its effects on financial stability and welfare. We develop a monetary model in which fundamental-based bank runs arise endogenously. As in Williamson (2012, 2022b), means of payment are not perfectly substitutable in our baseline framework. Agents use both cash and bank deposits to purchase goods in decentralized transactions. We assume that cash protects privacy, whereas bank deposits do not. Due to idiosyncratic uncertainty over which payment method—cash or deposits—will be required in a given transaction, banks endogenously emerge as liquidity insurers. Banks invest depositors' funds in a portfolio of cash and real assets and offer deposit contracts allowing individuals to transact with either cash or deposits backed by real assets. When the return on real assets (i.e., economic fundamentals) is sufficiently low, the transaction value of deposits falls below that of cash. In such cases, agents who hold deposits choose to convert them into cash. Thus, bank runs in our model are driven by adverse fundamentals, consistent with Allen and Gale (1998).

We analyze the introduction of two types of CBDCs in this setting: a cash-like CBDC and a deposit-like CBDC. A cash-like CBDC is a universal digital currency that, like physical cash, protects privacy and is widely accepted in all transactions, regardless of whether privacy is needed. It therefore competes directly with cash in decentralized transactions. In contrast, a deposit-like CBDC is designed for online, remote, or high-volume transactions and provides less privacy than physical cash. Consequently, it does not compete with cash in transactions where privacy is valued. Both types of CBDC carry an interest rate—a key distinction from physical cash.

The introduction of a cash-like CBDC crowds out physical cash in our model, as it is a perfect substitute but pays interest, unlike cash. In this sense, the cash-like CBDC resembles a retail CBDC. We show that a higher CBDC interest rate raises the transaction value of CBDC and increases the probability of a bank run, as deposit users are more likely to convert deposits into CBDC when fundamentals are weak. Therefore, issuing a cash-like CBDC can reduce financial stability. Nonetheless, we also find that a higher CBDC rate improves welfare, highlighting a trade-off between financial stability and welfare. In this case, a modified version of the Friedman rule is optimal and can eliminate the possibility of a run.

By contrast, introducing a deposit-like CBDC improves financial stability. Because it is an imperfect substitute for cash, cash, CBDC, and deposits can coexist in equilibrium. The CBDC can serve as collateral backing deposits and as a buffer against low returns on real assets. This reduces the likelihood that the transaction value of deposits falls below that of cash, thereby diminishing the incentive for agents to run. While a deposit-like CBDC enhances stability, the welfare effect of a higher CBDC rate is ambiguous. It reduces the amount of cash reserves and thus lowers the consumption of agents relying on cash. In this case, the optimal policy is to follow the traditional Friedman rule with a (net) zero CBDC rate, minimizing the cost of holding both cash and CBDC and ensuring that agents consume efficient quantities.

Unlike Allen and Gale (1998) and related work, bank runs are inefficient in our model. In their framework, deposit contracts are real and incomplete, and bank runs can improve risk sharing by restoring some contractual contingencies. In contrast, in our model, bank runs disrupt retail payments, reducing depositors' consumption and welfare despite improved risk sharing. Furthermore, the panic-based bank run of Diamond and Dybvig (1983) does not occur here, as real assets cannot be liquidated at an intermediate stage. Hence, bank runs in our model are purely fundamental-driven and always inefficient.

Related Literature. There is a rapidly growing literature on central bank digital currencies (CBDCs). Structured overviews of recent research are provided by Ahnert et al. (2022), Auer et al. (2022), and Chapman et al. (2023). Our study contributes to three main strands of this literature.

The first strand focuses on the introduction of CBDC and the supply of liquidity services, including Barrdear and Kumhof (2022), Brunnermeier and Niepelt (2019), Davoodalhosseini (2021), Andolfatto (2021), Chiu et al. (2022), Williamson (2022b), and Sanches and Keister (2023). In contrast, our paper emphasizes the implications of CBDC issuance for financial stability.

The second strand examines the impact of CBDC on financial stability, with contributions from Fernández-Villaverde et al. (2020, 2021), Kim and Kwon (2022), Williamson (2022a), Monnet and Keister (2022), and Ahnert et al. (2023). Our approach differs from these studies in several respects. First, we focus on fundamental-based bank runs, as opposed to the panic-based runs examined in some of these works. Second, we compare two distinct CBDC designs and their differential implications for financial stability and welfare. Third, our model incorporates microfoundations for the exchange process of money, a feature absent in some existing models.

The third strand draws on the New Monetarist approach to banking, following Lagos and Wright (2005) and Rocheteau and Wright (2005). This literature includes Andolfatto et al. (2019), Berentsen et al. (2007), Ferraris and Watanabe (2008, 2011), Bencivenga and Camera (2011), Williamson (2012, 2016), Gu et al. (2013, 2019), Sanches (2018), and Matsuoka and Watanabe (2019). Building on the theoretical frameworks developed in these studies, we explore the implications of CBDC issuance for financial stability.

Finally, our model shares key features with Jiang (2008), who analyzes the effects of inflation on fundamental-based bank runs in an overlapping generations model with random relocation. In contrast, our paper adopts a modern monetary framework to study the introduction of CBDC, rather than a traditional one.

This paper proceeds as follows. Section 2 presents the baseline model with cash and bank deposits. Section 3 derives the monetary equilibrium under a cash-like CBDC, and Section 4 analyzes the case of a deposit-like CBDC. Section 5 concludes. All proofs are provided in the Appendix.

2 Model

2.1 Environment

Time is discrete and continues forever. Each period is divided into two subperiods: day and night. A market opens in each subperiod. The day market is a centralized settlement market (CM), which is frictionless. In the CM, agents can produce divisible goods, which are numeraire and referred to as CM goods, trade their goods for assets, and settle their debts from the previous period. The night market is a decentralized goods market (DM) involving bilateral random matching and bargaining. There are two types of $[0, 1]$ continuum of infinitely-lived agents: sellers and buyers. Sellers have production technologies in the DM, which allow them to produce perishable and divisible goods, referred to as DM goods. Buyers do not have such production technologies in the DM but want to consume the DM goods. Agents discount future payoffs at $\beta \in (0, 1)$ across periods, but there is no discounting between the two subperiods.

The instantaneous utility functions for buyers and sellers are $x - h + u(q^b)$ and $x - h - q^s$, respectively, where x denotes the amount of the CM good consumed in a period, h denotes the daytime hours of work in a period, q^b denotes the amount of the DM good consumed by a buyer in a period, and q^s denotes the amount of the DM good produced by a seller in a period where we assume a constant marginal production cost normalized to unity. The utility function $u(q)$ is strictly increasing, strictly concave, and twice continuously differentiable, with $u(0) = 0$ and $u'(0) = \infty$. Let q^* denote the efficient quantity, which solves $u'(q^*) = 1$. For analytical tractability, we assume that the coefficient of relative risk aversion is constant, $\xi \equiv -\frac{qu''(q)}{u'(q)} \in (0, 1)$.

There is physical money (cash), which is perfectly divisible, storable, and recognizable. Let $\phi \geq 0$ denote a money price in terms of the CM good in a period. The total supply of cash is denoted by $M > 0$. In addition to physical money, there is an interest-bearing digital asset called central bank digital currency (CBDC). A CBDC is an electronic form of deposit account provided by the central bank and bears a (gross) nominal interest rate of $i^e \geq 1$, where

"e" stands for electronic money. The total supply of the CBDC is $E > 0$. Both monies are injected or withdrawn in a lump-sum fashion in the CM.

In addition to the central bank's liabilities, there is a one-period Lucas tree, which is also perfectly divisible and storable. One unit of the real asset generates a dividend payoff (e.g., fruit) equal to $\kappa \geq 0$ units of the CM good at the beginning of the CM and depreciates fully after producing the fruit. Let φ denote the asset's price in terms of the CM good. The total supply of the tree, denoted by $A > 0$, is fixed and constant over time. Given the quasi-linear preference, there is no loss of generality in assuming that the sellers are endowed with a set of trees at the beginning of each CM.

We assume that the dividend κ is a nonnegative random variable (due to the weather conditions). It is publicly observable and identically distributed over time. Let F be the cumulative distribution function, which is smooth and strictly increasing on $[0, \infty)$, where f represents the associated density function. F is common knowledge. We define $\mathbb{E}(\kappa) \equiv \int_0^\infty \kappa f(\kappa) d\kappa$ as the expected value of κ and assume $\beta\mathbb{E}(\kappa) > 1$, which ensures a positive amount of the asset holding in equilibrium.

2.2 DM

At night, buyers and sellers can trade the DM goods bilaterally. At the beginning of the night, sellers find their counterparts, while buyers learn whether their counterparties want privacy in their trades. In the DM, there is a fraction $\alpha \in (0, 1)$ of sellers who require their trades to be private and only accept cash and a fraction $1 - \alpha$ of sellers who do not care about privacy and accept both cash and deposits (claims on a bank) as a means of payment. The design of a CBDC for privacy will determine a buyer's means of payment in the DM. If a transaction using CBDC is designed to keep privacy, a buyer uses a CBDC in both types of transactions, while if not, a buyer must use cash only in a privacy transaction. We assume that claims to trees cannot be used as a medium of exchange in the DM because these claims can be counterfeited perfectly at zero cost (see Lester et al., 2012), and the lack of commitment prevents buyers from using credit (issuing personal IOU) in the DM. For simplicity, when a buyer meets a seller, the buyer makes a take-it-or-leave-it offer in exchange for goods.

The privacy shock plays a similar role as a "liquidity preference shock" in the Diamond and Dybvig (1983) model, which renders banks a role for pooling risks and providing a better combination of asset returns and liquidity.

2.3 Government

Letting T denote a lump-sum real transfer from the government to buyers in the CM, the consolidated government budget constraint is

$$T + \phi(M_- + i^e E_-) = \phi(M + E),$$

where " $-$ " stands for the previous period. Let π denote the (gross) rate of growth of nominal government liabilities; that is,

$$M_+ + E_+ = \pi(M + E),$$

with $\pi \geq \beta$, where " $+$ " stands for next period. In a stationary monetary equilibrium, the rate of inflation is $\pi \equiv \frac{\phi}{\phi_+}$.[1] We assume that $\pi \geq \beta i^e$. In what follows, we treat π and i^e as key policy parameters.

Fig 1 summarizes the timing of events in the model. At the end of a CM, buyers form private banks, and the banks collect funds from their depositors (buyers) and make a portfolio (z, e, a), where z denotes real cash balances, e denotes real CBDC balances, and a denotes the amount of the tree. These assets altogether back the issuance of deposits. At the beginning of the DM, the fundamentals (including the dividend) become known, and the buyers learn about their types of meetings. Then, buyers who want cash can withdraw it from their banks and use it for their consumption in the DM. The buyers who trade using money in a privacy exchange, referred to as *cash buyers*, consume $q^c \geq 0$, while the buyers who trade using deposits in a non-privacy exchange, referred to as *deposit buyers*, consume $q^d \geq 0$. Finally, at the beginning

[1] Most central banks consider the issuance of CBDC, which can be exchanged for cash one-to-one. For this practical reason and analytical simplicity, we consider the case where the cash and CBDC growth rates are the same.

of the next CM, the banks distribute their remaining wealth among their depositors equally and dissolve.

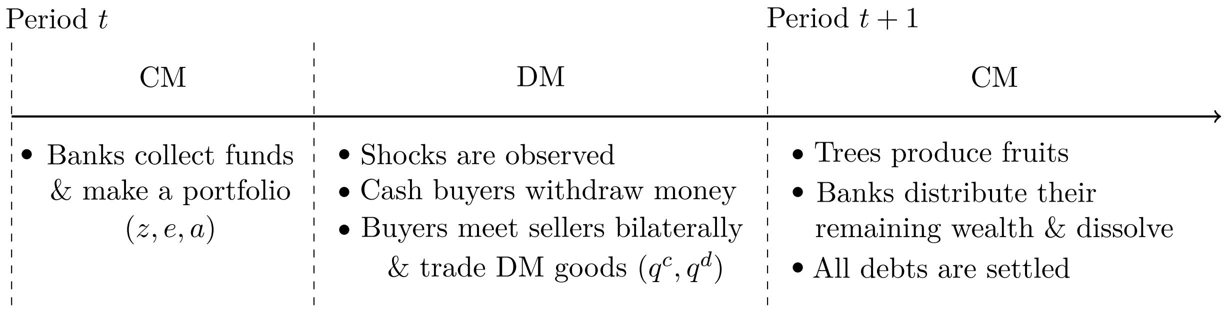

Fig 1: Timing of Events

2.4 Welfare

In a steady state, the welfare measure $\mathcal{W}$ is

$$\mathcal{W} = \int_0^\infty \left[\alpha \left\{u(q^c(\kappa)) - q^c(\kappa)\right\} + (1-\alpha)\left\{u(q^d(\kappa)) - q^d(\kappa)\right\} + \kappa A\right] f(\kappa) d\kappa, \tag{1}$$

which is the weighted sum of expected surpluses in the DM and the return on the real asset in the CM. Note that $\mathcal{W}$ depends mainly on the amount of the DM goods consumed because the utilities from consuming and producing the CM good add up to zero, except for κA. The first-best consumption quantities in the DM satisfy $q^c(\kappa) = q^d(\kappa) = q^* \equiv u^{-1\prime}(1)$ for all κ.

3 Equilibrium with Cash-Like CBDC

Central banks can generally design CBDCs that can be used in targeted transactions. This paper considers two types of CBDC: a *cash-like* CBDC and a *deposit-like* CBDC. We consider each in turn.

A cash-like CBDC is designed to keep privacy and be recognizable to all sellers in our model economy. This CBDC design may minimize the fees and user costs, impose no holding limits, and enable offline use with a high-security level. If $i^e > 1$, no bank will hold cash reserves, so that cash will be replaced with a CBDC completely, that is, $z = 0 < e$. Banks are indifferent between cash and CBDC if $i^e = 1$, which is equivalent to a model without CBDC.

At the beginning of the night, banks decide how much CBDC to allocate to each buyer. After the buyers learn their meeting type, the banks choose a payment schedule given their holdings of CBDC reserves, $e \geq 0$, and the asset, $a \geq 0$, selected in the previous CM. The payment can be contingent on the realized aggregate state. We assume competitive banks with free entry so that each bank maximizes the expected value of its representative depositor (buyer). Without loss of generality, we assume that a bank's remaining reserves and wealth are distributed uniformly among depositors after the DM closes.

For a given value of κ and a given portfolio (e, a), a bank's problem in the DM can be written as

$$\max_{q^c, q^d \geq 0} \; \alpha u(q^c) + (1-\alpha)u(q^d) + \left[\beta\left(\frac{i^e e}{\pi} + \kappa a\right) - \alpha q^c - (1-\alpha)q^d\right],$$

subject to

$$\alpha q^c \leq \frac{\beta i^e e}{\pi}, \tag{2}$$

$$\alpha q^c + (1-\alpha)q^d \leq \beta\left(\frac{i^e e}{\pi} + \kappa a\right). \tag{3}$$

The first two terms in the objective function represent the expected utility of a buyer in the DM, and the third term represents the bank's remaining wealth distributed among buyers in the next CM. Constraint (2) states that a bank's CBDC reserves must finance consumption for buyers who trade in a privacy transaction: a unit of CBDC stocked in the previous CM becomes i^e (where i^e is a gross interest rate) and is worth $\frac{\beta i^e}{\pi}$ in real term (where β is a discount factor and π is an inflation rate). Constraint (3) is the usual balance sheet constraint. If the inequality is strict, there are remaining assets distributed in the next period.

At this point, we can define the following.

Definition 1 *A (partial) bank run occurs if*

1. *(some) deposit buyers dash for bank money;*

2. some bank money is actually allocated to those deposit buyers;

3. all bank resources are used up for liquidity services.

According to our definition (which is consistent with the one provided by a series of papers by Allen and Gale), a partial bank run in our model describes a situation where the realized return of assets is short of total liquidity demand by deposit buyers. This induces some deposit buyers who can trade without money (CBDC) to choose to withdraw money (CBDC). Since all bank resources are used up for liquidity services, such a money withdrawal by deposit buyers reduces the amount of money allocated to cash buyers. We should note that this definition allows banks to operate even after a run, i.e. even with a run, bank deposits can still be used as a payment, and they circulate in the economy.

In our model, a run occurs only if (2) is slack and (3) is binding. A run is impossible when (2) is binding; otherwise, there is no extra money that can be allocated to deposit buyers, which violates item 2 of Definition 1. When a bank uses up all the resources for liquidity services, (3) must be binding. In our model, a run occurs if and only if (2) is slack and (3) binds – as we will see shortly below, such an occurrence depends on the realized value of κ.

The first-order conditions are:

$$u'(q^c) - 1 = \mu_c + \mu_d, \tag{4}$$

$$u'(q^d) - 1 = \mu_d, \tag{5}$$

with complementary slackness conditions, where $\mu_c \geq 0$ and $\mu_d \geq 0$ are the Lagrange multipliers of (2) and (3), respectively.

Lemma 1 *The bank's optimal repayment is given by*

$$q^c = q^d = \beta\left(\frac{i^e e}{\pi} + \kappa a\right)$$

if $\kappa \leq \kappa_c$ *and*

$$q^c = \min\left\{\frac{\beta i^e e}{\alpha\pi}, q^*\right\} \quad \textit{and} \quad q^d = \min\left\{\frac{\beta\kappa a}{1-\alpha}, q^*\right\}$$

if $\kappa > \kappa_c$*, where the critical value is* $\kappa_c \equiv \min\left\{\frac{(1-\alpha)i^e e}{\alpha\pi a}, \frac{q^* - \frac{\beta i^e e}{\pi}}{\beta a}\right\}$*. Further, a bank run occurs if and only if* $\kappa \in [0, \kappa_c)$*.*

When $\kappa \leq \kappa_c$, the available resources are so scarce that (3) is binding, $\mu_d > 0$, leading to a disruption in retail payments and a low level of DM consumption. On the other hand, (2) is slack, $\mu_c = 0$, because the aggregate consumption of cash buyers αq^c is low enough. Given $\mu_c = 0$ and $\mu_d > 0$, the first order conditions (4) and (5) imply $q_c = q_d < q^*$, i.e. everyone receives the pro-rata share $\beta(\frac{i^e e}{\pi} + \kappa a)$. This situation describes a bank run because there is a positive measure of deposit buyers forced to run and withdraw CBDC. To see this, let $\lambda(\kappa) \in [0, 1-\alpha]$ denote the fraction of such deposit-buyers. Noting that all the available resources are used up (since (3) is binding), the fraction $\alpha + \lambda(\kappa)$ of buyers who demand CBDC and the fraction $1 - \alpha - \lambda(\kappa)$ of deposit buyers who do not demand CBDC (i.e., who wait and use only deposit) must have the same individual consumption level:

$$(q^c =)\ \frac{\frac{\beta i^e e}{\pi}}{\alpha + \lambda(\kappa)} = \frac{\beta \kappa a}{1 - \alpha - \lambda(\kappa)}\ (= q^d).$$

This leads to

$$\lambda(\kappa) = \frac{(1-\alpha)\frac{i^e e}{\pi} - \alpha\kappa a}{\frac{i^e e}{\pi} + \kappa a},$$

which is strictly positive and decreasing in $\kappa \in [0, \kappa_c)$. Note that $\lambda(0) = 1-\alpha$ and $\lambda\left(\frac{(1-\alpha)i^e e}{\alpha\pi a}\right) = 0$. The former situation describes the full bank run because all the deposit buyers run for money and bank deposits have no value.

Therefore, a run occurs if and only if $\kappa < \kappa_c$, i.e. (2) is slack but (3) binds. In our model, a run occurs when the dividend turns out to be so low that the DM value of deposits is insufficient to cover the needs of deposit buyers.

The critical value of the dividend under which a bank run occurs, κ_c, depends on the DM value of CBDC reserves. When $\frac{\beta i^e e}{\alpha\pi} < q^*$, it is derived based on the slack (2) and the critical value is given by the just binding (2), which is $\kappa_c = \frac{(1-\alpha)i^e e}{\alpha\pi a} \geq 0$. In this case, a bank run is likely to occur when the ratio of the DM value of CBDC reserves, given by $\frac{\beta i^e e}{\pi}$, to the DM value of asset dividend, given by $\beta\kappa a$, is relatively high. Intuitively, deposit buyers are likely to want to dash for money when $\beta\kappa a$ is relatively low and more money is allocated to them when $\frac{\beta i^e e}{\pi}$ is relatively high (see items 1 and 2 of Definition 1). When $\frac{\beta i^e e}{\alpha\pi} \geq q^*$, (2) is not the determinant of the critical value. Rather, simply (3) and $q^c = q^d \leq q^*$ dictates the determination of the critical value, $\kappa_c = \frac{q^* - \frac{\beta i^e e}{\pi}}{\beta a}$ because for high values of κ, bank assets are abundant enough to cover even the first best level (and so there is no need to run) and a run can occur only when bank assets are low enough which happens for low values of κ (see items 1 and 3 of Definition 1). Therefore, this time, unlike the previous case, a run is less likely to occur when $\frac{\beta i^e e}{\pi}$ is relatively higher, because it contributes to increasing the bank's total resource.

In sum, a higher CBDC reserve value leads to a higher likelihood of a bank run (because CBDC becomes better able to satisfy the needs of deposit–buyers) when it is relatively low but to a lower likelihood of a bank run (because a higher CBDC reserve value compensates for low realizations of asset dividend) when it is relatively high.

When the dividend increases to the region $\kappa > \kappa_c$, (2) can become binding, $\mu_d > 0$, as well. Then, all the CBDC reserves are allocated to cash buyers, leading to $q^c = \frac{\beta i^e e}{\alpha\pi}$, while deposit buyers only use deposits as a payment instrument, leading to $q^d = \frac{\beta\kappa a}{1-\alpha}$. In an extreme situation where κ becomes large enough, that is, $\kappa \geq \bar{\kappa}_c$, the DM value of deposits is sufficiently high to cover the liquidity needs of deposit buyers. This can either make (3) slack, leading to $q^d = q^*$, or even make both (2) and (3) slack, leading to $q^c = q^d = q^*$. Fig. 2 illustrates this result.

Given the above result, we now derive the optimal portfolio. Since a portfolio has to be selected before knowing the realization of asset dividends, the portfolio choice problem of a bank is:

$$\max_{e,a\geq 0} \; -e - \varphi a + \int_0^{[\kappa_c]^+} u\left(\beta\left(\frac{i^e e}{\pi} + \kappa a\right)\right) f(\kappa) d\kappa \\ + \int_{[\kappa_c]^+}^{\infty} \left[\alpha u\left(q^c\right) + (1-\alpha)u(q^d) + \beta\left(\frac{i^e e}{\pi} + \kappa a\right) - \alpha q^c - (1-\alpha) q^d\right] f(\kappa) d\kappa,$$

where $[\kappa_c]^+ \equiv \max\{0, \kappa_c\}$, $\kappa_c = \kappa_c(e,a)$, $q^c = q^c(e)$ and $q^d = q^d(a)$ are given in Lemma 1. In

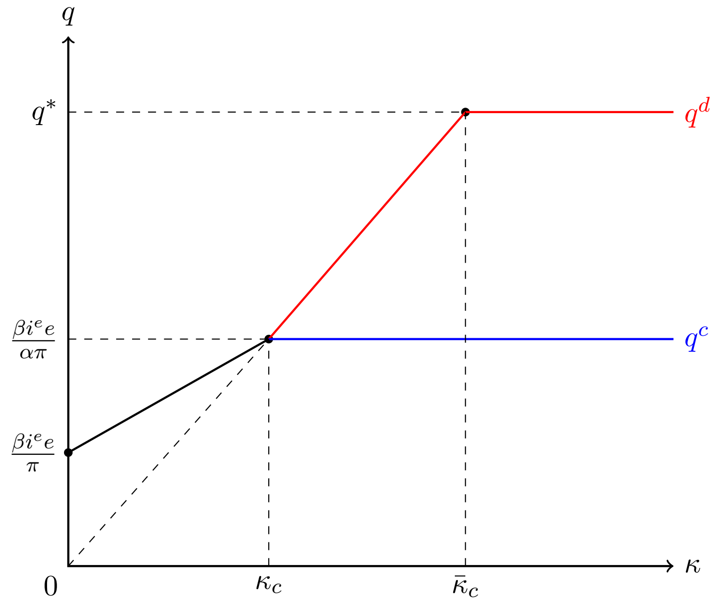

Fig 2: Consumption with a Cash-Like CBDC in the case of $\frac{\beta i^e e}{\alpha\pi} < q^*$

the second integral, we consider the possibility that not all the resources are used in the DM, in which case the remaining wealth is consumed in the next CM.

The first-order conditions are:

$$\frac{\pi - \beta i^e}{\beta i^e} = \int_0^\infty \left[u'(q^c) - 1\right] f(\kappa) d\kappa, \tag{6}$$

$$\frac{\varphi}{\beta} - \mathbb{E}(\kappa) = \int_0^\infty \kappa \left[u'(q^d) - 1\right] f(\kappa) d\kappa. \tag{7}$$

The left-hand sides of (6) and (7) are the marginal costs of holding CBDC and the asset, respectively, while the right-hand sides are the liquidity premium of CBDC and the asset, respectively. Of course, the liquidity premium depends on the dividend value κ. For instance, if the solution satisfies $\frac{\beta i^e e}{\alpha\pi} < q^*$ (which is indeed the case when π is high, see the proof of Proposition 1), then as shown in Lemma 1, $q^c = q^d = \beta\left(\frac{i^e e}{\pi} + \kappa a\right)$ for $\kappa \leq [\kappa_c]^+ = \kappa_c = \frac{(1-\alpha)i^e e}{\alpha\pi a} \geq 0$, $q^c = \frac{\beta i^e e}{\alpha\pi}$ for $\kappa > \kappa_c$, $q^d = \frac{\beta\kappa a}{1-\alpha}$ for $\kappa \in (\kappa_c, \bar{\kappa}_c)$ where $\bar{\kappa}_c \equiv \frac{(1-\alpha)q^*}{\beta a} > 0$ and $q^d = q^*$ for $\kappa \geq \bar{\kappa}_c$. Hence, the liquidity premium of CBDC (i.e., the R.H.S. of (6)) in this case becomes

$$\int_0^{\kappa_c} \left[u'\left(\beta\left(\frac{i^e e}{\pi} + \kappa a\right)\right) - 1\right] f(\kappa) d\kappa + \left[u'\left(\frac{\beta i^e e}{\alpha\pi}\right) - 1\right] (1 - F(\kappa_c)),$$

and the liquidity premium of asset (i.e., the R.H.S. of (7)) becomes

$$\int_0^{\kappa_c} \kappa \left[u'\left(\beta \left(\frac{i^e e}{\pi} + \kappa a \right) \right) - 1 \right] f(\kappa) d\kappa + \int_{\kappa_c}^{\bar{\kappa}_c} \kappa \left[u'\left(\frac{\beta \kappa a}{1-\alpha} \right) - 1 \right] f(\kappa) d\kappa.$$

Similarly, if the solution satisfies $\frac{\beta i^e e}{\alpha \pi} \geq q^*$ (which is indeed the case when π is low), then as shown in Lemma 1, $q^c = q^d = \beta \left(\frac{i^e e}{\pi} + \kappa a \right)$ for $\kappa \leq [\kappa_c]^+$ and $q^c = q^d = q^*$ for $\kappa > [\kappa_c]^+$. In this case, the liquidity premium of CBDC and the asset in this case become

$$\int_0^{[\kappa_c]^+} \left[u'\left(\beta \left(\frac{i^e e}{\pi} + \kappa a \right) \right) - 1 \right] f(\kappa) d\kappa \quad \text{and} \quad \int_0^{[\kappa_c]^+} \kappa \left[u'\left(\beta \left(\frac{i^e e}{\pi} + \kappa a \right) \right) - 1 \right] f(\kappa) d\kappa,$$

respectively.

A stationary monetary equilibrium is characterized by a pair (e, φ), satisfying the first-order conditions (6) and (7), and the market clearing conditions, $e = \phi E$ and $a = A$. The following proposition establishes the existence and uniqueness of equilibrium with a cash-like CBDC.

Proposition 1 *A stationary monetary equilibrium with a cash-like CBDC exists and is unique, satisfying $\frac{\beta i^e e}{\alpha \pi} < q^*$ for $\pi > \pi^*(i^e)$ and $\frac{\beta i^e e}{\alpha \pi} \geq q^*$ for $\pi \leq \pi^*(i^e)$, some $\pi^*(i^e) \in (\beta i^e, \infty)$. Further, a bank run occurs with positive probability, $F(\kappa_c) > 0$, for all $\pi \in (\beta i^e, \infty)$. The equilibrium allocation is efficient under the modified Friedman rule, $\pi = \beta i^e$.*

Just like cash reserves in the standard monetary equilibrium, the CBDC reserves e and its DM value $\frac{\beta i^e e}{\alpha \pi}$ are relatively low (high) for relatively high (low) rates of inflation π (see below). If an inflation rate is above $\pi^*(i^e)$, the CBDC reserves are scarce (i.e., (2) is binding), so that a cash buyer gets low DM consumption described in Fig 2. On the other hand, if an inflation rate is below $\pi^*(i^e)$, (2) is slack and both types of buyers consume the efficient quantity in the DM if $\kappa \geq \kappa_c$. Otherwise, a bank's assets are scarce (i.e., (3) is binding), and both types of buyers get the same low DM consumption, implying a bank run. The optimal policy is the modified Friedman rule, which sets the cost of CBDC holding (i.e., L.H.S. of (6)) to zero. At the optimal policy, κ_c must be zero, implying that there is no bank run.

Lemma 2 *A higher inflation rate π reduces the CBDC reserves e and increases the asset price φ, while a higher CBDC rate i^e increases e and reduces φ.*

An increase in the inflation rate π decreases the CBDC reserves e and increases the demand for the asset because it increases the cost of holding CBDC and the relative merit of holding the real asset, leading to a higher asset price φ. An increase in the CBDC rate i^e implies a higher rate of return on CBDC, and so, due to substitution, it increases the demand for CBDC and reduces the demand for the asset, leading to a lower asset price.

Now, consider the effects of policy changes on financial stability.

Proposition 2 *The probability of a bank run $F(\kappa_c)$ decreases with the CBDC rate i^e for $\pi \leq \pi^*(i^e)$ and increases with i^e for $\pi > \pi^*(i^e)$. Welfare increases with i^e for all $\pi \in (\beta i^e, \infty)$*

A higher CBDC rate leads to a larger amount of CBDC reserves, which increases the threshold for a bank run $\kappa_c = \frac{(1-\alpha)i^e e}{\alpha \pi A}$ for $\pi \geq \pi^*(i^e)$ and decreases $[\kappa_c]^+ = \max\left\{\frac{q^* - \frac{\beta i^e e}{\pi}}{\beta a}, 0\right\}$ for $\pi < \pi^*(i^e)$. The former is because, with more CBDC reserves available, more deposit-buyers are induced to run and convert their deposit to CBDC when the CBDC reserve value is relatively low. In contrast, the latter is because more CBDC reserves available make up for low realizations of asset dividends when the CBDC reserve value is relatively high. The main policy implication of this section is that the introduction of a cash-like CBDC crowds out cash and improves welfare, but accompanies an increase in financial instability when inflation is high.

4 Equilibrium with Deposit-Like CBDC

We next consider a CBDC that can not be used in privacy DM transactions. The CBDC design may permit less privacy protection, impose holding limits, and/or enable only online use in retail payments. Another design may allow only financial institutions to use it, like central bank digital reserves, in wholesale payments rather than retail payments. We refer to this type of CBDC as a deposit-like CBDC. We assume that cash and CBDC are imperfect substitutes, while CBDC and deposits are perfect substitutes. So, a bank chooses a portfolio (z, e, a) consisting of three types of assets, altogether backing the issuance of deposits in the first place.

As before, we solve the problem backward. For each realized value κ, dropping the constant terms, a bank's maximization problem in the DM can be written as

$$\max_{q^c,q^d\geq 0} \ \alpha u(q^c) + (1-\alpha)u(q^d) + \left[\beta\left(\frac{z+i^e e}{\pi} + \kappa a\right) - \alpha q^c - (1-\alpha)q^d\right],$$

subject to

$$\alpha q^c \leq \frac{\beta z}{\pi}, \tag{8}$$

$$\alpha q^c + (1-\alpha)q^d \leq \beta\left(\frac{z+i^e e}{\pi} + \kappa a\right). \tag{9}$$

Constraint (8) states that a bank's cash reserves must finance consumption for buyers who trade in a privacy DM transaction. That is, CBDC no longer provides a liquidity service in this transaction. In this sense, bank money in items 1 and 2 of Definition 1 is cash rather than CBDC, so a bank run is a traditional run away from deposits to cash in this case. Constraint (9) is the balance sheet constraint when a bank holds three types of assets.

The first order conditions take the same form as in (4) and (5), with complementary slackness conditions, except that this time the Lagrange multipliers, μ_c and μ_d, are associated with constraints (8) and (9), respectively.

We obtain the following results:

Lemma 3 *The bank's optimal repayment is given by*

$$q^c = q^d = \beta\left(\frac{z+i^e e}{\pi} + \kappa a\right)$$

if $\kappa \leq \kappa_d$ *and*

$$q^c = \min\left\{\frac{\beta z}{\alpha\pi}, q^*\right\} \quad \textit{and} \quad q^d = \min\left\{\frac{\frac{\beta i^e e}{\pi} + \beta\kappa a}{1-\alpha}, q^*\right\}$$

if $\kappa > \kappa_d$*, where the critical value is* $\kappa_d = \min\left\{\frac{(1-\alpha)z-\alpha i^e e}{\alpha\pi a}, \frac{q^* - \frac{\beta(z+i^e e)}{\pi}}{\beta a}\right\}$*. Further, a bank run occurs if and only if* $\kappa \in [0, \kappa_d)$*.*

A similar interpretation applies as before (see Figs 3 and 4). A bank run occurs if and only if the realized dividend value is low enough, $\kappa \in [0, \kappa_d)$, where (8) is slack but (9) is binding.

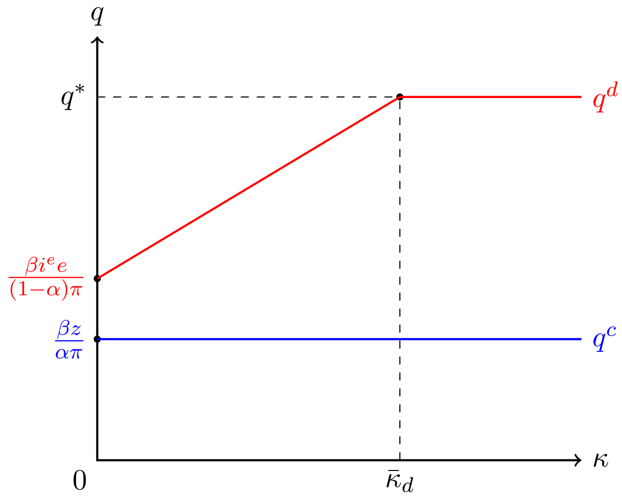

Fig 3: No Bank Runs ($\kappa_d < 0$)

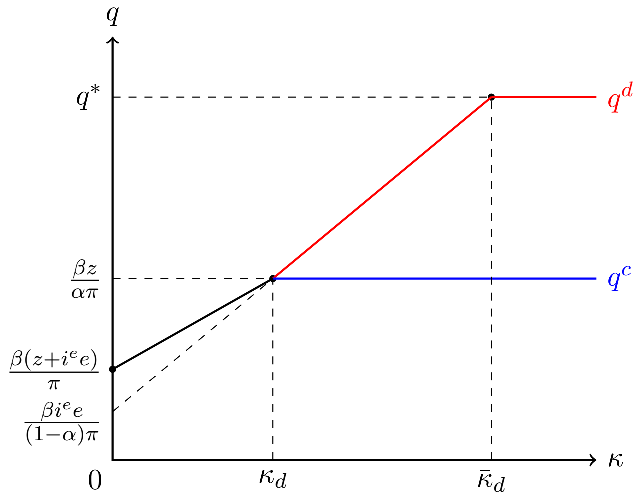

Fig 4: Bank Runs ($\kappa_d > 0$)

One significant difference is that everything else being equal, the critical value of a bank run κ_d with deposit-like CBDC takes a lower value when the DM value of CBDC is higher, irrespective of either $\frac{\beta z}{\alpha\pi} < q^*$ or $\frac{\beta z}{\alpha\pi} \geq q^*$. This contrasts with the one with the cash-like CBDC, which can take a higher value when the DM value of CBDC is higher. This occurs because a bank run is a run away from deposits to cash, and an increase in the DM value of CBDC improves the balance sheet by compensating for low realizations of asset dividends. Another difference is that a run may not occur with probability one (see Fig 3), i.e., $\kappa_d < 0$, because a deposit buyer does not have an incentive to withdraw cash, which is never happened before.

Given the above result, we now derive the optimal portfolio. The procedure is pretty similar to before. The portfolio choice problem of a bank is:

$$\max_{z,e,a\geq 0} -(z+e) - \varphi a + \int_0^{[\kappa_d]^+} u\left(\beta\left(\frac{z+i^e e}{\pi}+\kappa a\right)\right) f(\kappa)d\kappa \\ + \int_{[\kappa_d]^+}^{\infty}\left[\alpha u\left(q^c\right) + (1-\alpha)u(q^d) + \beta\left(\frac{z+i^e e}{\pi}+\kappa a\right) - \alpha q^c - (1-\alpha)q^d\right] f(\kappa)d\kappa,$$

where $[\kappa_d]^+ \equiv \max\{0, \kappa_d\}$, $\kappa_d = \kappa_d(e,a)$, $q^c = q^c(z)$, and $q^d = q^d(e,a)$ are given in Lemma 3.

The first order conditions with respect to z and e are

$$\frac{\pi-\beta}{\beta} = \int_0^\infty \left[u'(q^c)-1\right] f(\kappa)d\kappa, \tag{10}$$

$$\frac{\pi-\beta i^e}{\beta i^e} \geq \int_0^\infty \left[u'(q^d)-1\right] f(\kappa)d\kappa, \tag{11}$$

respectively. In (11), the inequality reflects the possibility of $e=0$ as a solution (see below). The first order condition with respect to a takes the same form as in (7). The liquidity premium (i.e., the R.H.S. of the first-order condition) is characterized as follows. If the solution satisfies $\frac{\beta z}{\alpha\pi} < q^*$ (which is indeed the case when π is high, see the proof of Proposition 3), then as shown in Lemma 3, $q^c = q^d = \beta\left(\frac{z+i^e e}{\pi} + \kappa a\right)$ for $\kappa \leq \kappa_d = \frac{(1-\alpha)z-\alpha i^e e}{\alpha\pi a}$, $q^c = \frac{\beta z}{\alpha\pi}$ for $\kappa > \kappa_d$, $q^d = \frac{\frac{\beta i^e e}{\pi}+\beta\kappa a}{1-\alpha}$ for $\kappa \in (\kappa_d, \bar{\kappa}_d)$ where $\bar{\kappa}_d \equiv \frac{(1-\alpha)q^* - \frac{\beta i^e e}{\pi}}{\beta a}$ and $q^d = q^*$ for $\kappa \geq \bar{\kappa}_d$. Hence, the liquidity premium of cash (i.e., the R.H.S. of (10)) in this case becomes

$$\int_0^{[\kappa_d]^+} \left[u'\left(\beta\left(\frac{z+i^e e}{\pi} + \kappa a\right)\right) - 1\right] f(\kappa)d\kappa + \left[u'\left(\frac{\beta z}{\alpha\pi}\right) - 1\right](1-F(\kappa_d)),$$

the liquidity premium of CBDC (i.e., the R.H.S. of (11)) becomes

$$\int_0^{[\kappa_d]^+} \left[u'\left(\beta\left(\frac{z+i^e e}{\pi} + \kappa a\right)\right) - 1\right] f(\kappa)d\kappa + \int_{[\kappa_d]^+}^{[\bar{\kappa}_d]^+} \left[u'\left(\frac{\beta(\frac{i^e e}{\pi}+\kappa a)}{1-\alpha}\right) - 1\right] f(\kappa)d\kappa,$$

and the liquidity premium of asset (i.e., the R.H.S. of (7)) becomes

$$\int_0^{[\kappa_d]^+} \kappa\left[u'\left(\beta\left(\frac{z+i^e e}{\pi} + \kappa a\right)\right) - 1\right] f(\kappa)d\kappa + \int_{[\kappa_d]^+}^{[\bar{\kappa}_d]^+} \kappa\left[u'\left(\frac{\beta(\frac{i^e e}{\pi}+\kappa a)}{1-\alpha}\right) - 1\right] f(\kappa)d\kappa.$$

Similarly, if the solution satisfies $\frac{\beta z}{\alpha\pi} \geq q^*$ (which is indeed the case when π is low), then as shown in Lemma 3, $q^c = q^d = \beta\left(\frac{z+i^e e}{\pi} + \kappa a\right)$ for $\kappa \leq \kappa_d = \frac{q^* - \frac{\beta(z+i^e e)}{\pi}}{\beta a}$ and $q^c = q^d = q^*$ for $\kappa > \kappa_d$. In this case, the liquidity premium of cash and CBDC becomes

$$\int_0^{[\kappa_d]^+} \left[u'\left(\beta\left(\frac{z+i^e e}{\pi} + \kappa a\right)\right) - 1\right] f(\kappa)d\kappa,$$

and the liquidity premium of the asset becomes

$$\int_0^{[\kappa_d]^+} \kappa\left[u'\left(\beta\left(\frac{z+i^e e}{\pi} + \kappa a\right)\right) - 1\right] f(\kappa)d\kappa,$$

respectively.

A stationary monetary equilibrium is characterized by a pair (z, e, φ), satisfying the first-order conditions (7), (10) and (11), and the market clearing conditions, $z = \phi M$, $e = \phi E$ and $a = A$. The following proposition establishes the existence and uniqueness of equilibrium with a deposit-like CBDC.

Proposition 3 *A stationary monetary equilibrium with a deposit-like CBDC exists and is unique for all $i^e \in [1, \frac{\pi}{\beta}]$ for the following parameter configurations, characterized by critical values $\underline{i}^e(\pi), \bar{i}^e(\pi) \in (1, \frac{\pi}{\beta})$, $\pi^*(1) \in (\beta, \infty)$:*

- *For $i^e \in [\bar{i}^e(\pi), \frac{\pi}{\beta})$ and all $\pi \in (\beta i^e, \infty)$, a bank run can never happen, $[\kappa_d]^+ = 0$, and banks hold a positive amount of CBDC, $e > 0$;*
- *For $i^e \in (\underline{i}^e(\pi), \bar{i}^e(\pi))$ and all $\pi \in (\beta i^e, \infty)$, a bank run occurs with a positive probability, $\kappa_d > 0$, and banks hold a positive amount of CBDC, $e > 0$;*
- *For $i^e \in (1, \underline{i}^e(\pi)]$ and $\pi \in (\pi^*(1), \infty)$, a bank run occurs with a positive probability, $\kappa_d > 0$, and banks do not hold CBDC, $e = 0$;*
- *For $i^e = 1$ and $\pi \in (\beta, \pi^*(1)]$, a bank run occurs with a positive probability, $\kappa_d > 0$, and cash and CBDC are perfect substitute, $z + e > 0$.*

The monetary equilibrium allocation is efficient when $\pi \to \beta$ and $i^e = 1$.

We have four cases with a deposit-like CBDC. For $i^e \in [\underline{i}^e(\pi), \frac{\pi}{\beta})$, a bank has a positive amount of CBDC reserves because the CBDC rate is high enough. If the rate is sufficiently high, i.e., $i^e \in [\bar{i}^e(\pi), \frac{\pi}{\beta})$, a bank has enough CBDC reserves to prevent a bank run. In this case, since the CBDC reserves act as a buffer that offsets bad fundamentals (low values of κ), an increase in the DM value of CBDC improves the balance sheet by compensating for low realizations of asset dividends.[2] This stands in sharp contrast to the case with cash-like CBDC. For $i^e \in (\underline{i}^e(\pi), \bar{i}^e(\pi))$, a bank still has a positive amount of CBDC reserves, but it is

[2]This mechanism is shared in many papers on fiat money as a means of self-insurance against income risk. See, for example, Bewley (1980) and Kitagawa (1994, 2001).

not sufficient to eliminate runs. In this case, a bank run occurs with a positive probability, which is relatively small.

For $i^e \in (1, \underline{i}^e(\pi)]$ and $\pi \in (\pi^*(1), \infty)$, a bank does not hold CBDC reserves because CBDC yields a low return and is dominated by the other two assets. Clearly, the CBDC rate no longer affects the equilibrium allocation and welfare. Finally, for $i^e = 1$ and $\pi \in (\beta, \pi^*(1)]$, cash and CBDC are perfect substitutes, and the monetary reserves $z + e$ are determined uniquely, but z and e individually are not. Notice that the last two cases are equivalent to ones without CBDC, and there are no equilibria when $i^e \in (1, \underline{i}^e(\pi)]$ and $\pi \in (\beta, \pi^*(1)]$.

The optimal policy is different from the one in the case of cash-like CBDC. That is, the combination of the Friedman rule ($\pi \to \beta$) and the net zero CBDC rate ($i^e = 1$) is optimal because both the costs of cash and CBDC holdings (i.e., L.H.S. of (10) and (11)) must be zero at the same time to achieve the first best.

The important message from the analysis in this section is that introducing a deposit-like CBDC with a high interest rate improves financial stability; if the CBDC rate is high enough, a fundamental-based bank run is eliminated.

Consider the consequences of CBDC policies when $\pi > \pi^*(1)$. Suppose that a bank holds three types of assets, i.e., $i^e \in (\underline{i}^e(\pi), \frac{\pi}{\beta}]$. Then, we have:

Lemma 4 (Comparative Statics) *Suppose that ξ is sufficiently small. Inflation reduces both cash and CBDC reserves and increases the asset price. In addition, an increase in the CBDC rate increases CBDC reserves and decreases cash reserves and the asset price.*

The intuitions of this lemma are straightforward and similar to those of Lemma 2. Because an increase in the inflation rate increases the cost of holding cash and CBDC reserves, it decreases a bank's demand for these assets and increases the demand for the real asset, leading to a high asset price. On the other hand, an increase in the CBDC rate increases the relative benefit of holding CBDC reserves compared to cash and real assets, so a bank increases CBDC reserves by reducing other asset holdings.

Finally, the main implications of the CBDC policy for financial stability will be considered. Given the results of Lemma 4, we obtain the following result.

Proposition 4 *Suppose that $\pi > \pi^*(1)$. If a bank run is possible, i.e., $i^e \in (\underline{i}^e(\pi), \bar{i}^e(\pi))$, a high CBDC rate decreases the probability of a run, but its effect on welfare is ambiguous. If a bank run is not possible, i.e., $i^e \in [\bar{i}^e(\pi), \frac{\pi}{\beta}]$, a high CBDC rate increases welfare.*

In a situation where cash and CBDC coexist and a bank run is possible, a higher CBDC rate leads to increased consumption inequality between cash buyers and deposit buyers because it reduces the DM value of cash but improves the DM value of deposits. In addition, a higher CBDC rate also increases the CBDC reserves that act as a buffer to absorb the shock, resulting in a low probability of a bank run. The model with a deposit-like CBDC can exhibit a trade-off between financial stability and welfare for $i^e \in (\underline{i}^e(\pi), \bar{i}^e(\pi))$.

On the other hand, in a situation where cash and CBDC coexist but a bank run is not possible, cash and CBDC reserves are dichotomized, so cash reserves do not depend on the CBDC rate. Then, a high CBDC rate increases the value of CBDC and welfare, like in the case of cash-like CBDC.

5 Conclusion

This paper has examined the effects of introducing two types of central bank digital currencies (CBDCs) on financial stability within a New Monetarist framework. In our model, a cash-like CBDC crowds out physical cash, and a high CBDC interest rate can increase the probability of a bank run while simultaneously improving welfare. This generates a trade-off between financial stability and welfare, for which a modified version of the Friedman rule is optimal. In contrast, introducing a deposit-like CBDC does not fully displace cash and lowers the likelihood of a run. Notably, a sufficiently high CBDC rate can eliminate bank runs entirely. However, a higher CBDC rate does not necessarily raise welfare, as it reduces the amount of cash reserves. In this case, the optimal policy combines the traditional Friedman rule with a (net) zero CBDC interest rate, thereby eliminating consumption inequality between cash users and deposit users.

The main message of this paper is that CBDC design choices have distinct implications for financial stability, economic welfare, and optimal monetary policy. Our findings suggest that the design features of CBDCs must be carefully considered when contemplating their introduction into the financial system.

The analysis can be extended in several directions, two of which are particularly promising. First, in our model, the deposit contract is contingent on the realized signal of economic fundamentals, whereas, in reality, deposit contracts typically promise a fixed nominal payment regardless of the state of nature. This could be addressed by modeling a bank that offers depositors fixed nominal returns across all states. Second, an important extension would be to study the optimal design of a lender of last resort that issues CBDC during banking crises. Because CBDC withdrawals provide the central bank with real-time information about emerging crises—unlike cash—policy responses can be faster and more targeted (Keister and Monnet, 2022). We leave these extensions for future research.

Appendix

Proof of Lemma 1

We derive the solution for $q^c, q^d \in (0, q^*]$ and $\mu_c, \mu_d \geq 0$ that satisfies the first order conditions, (4) and (5), and the constraints (2) and (3).

⊙ **Case 1: $\mu_c = 0$ and $\mu_d > 0$:** From (4) and (5), combined with (3), we must have

$$q^c = q^d = \beta\left(\frac{i^e e}{\pi} + \kappa a\right) < q^*.$$

Since $q^c = q^d \leq \min\{\frac{\beta i^e e}{\alpha\pi}, q^*\}$ by (2), there are two possible cases. If $\frac{\beta i^e e}{\alpha\pi} < q^*$, then constraint (2) implies that

$$q^c = q^d = \beta\left(\frac{i^e e}{\pi} + \kappa a\right) \leq \frac{\beta i^e e}{\alpha\pi} \iff \kappa \leq \frac{(1-\alpha)i^e e}{\alpha\pi a}.$$

If $\frac{\beta i^e e}{\alpha\pi} \geq q^*$, then $q^c = q^d < q^*$ implies that

$$q^c = q^d = \beta\left(\frac{i^e e}{\pi} + \kappa a\right) < q^* \iff \kappa < \frac{q^* - \frac{\beta i^e e}{\pi}}{\beta a}.$$

Defining

$$\kappa_c \equiv \min\left\{\frac{(1-\alpha)i^e e}{\alpha\pi a}, \frac{q^* - \frac{\beta i^e e}{\pi}}{\beta a}\right\},$$

we see that $\kappa_c = \frac{(1-\alpha)i^e e}{\alpha\pi a}$ if $\frac{\beta i^e e}{\alpha\pi} < q^*$, and $\kappa_c = \frac{q^* - \frac{\beta i^e e}{\pi}}{\beta a}$ if $\frac{\beta i^e e}{\alpha\pi} \geq q^*$. In this case, we can say that a bank run occurs (see Definition 1).

⊙ **Case 2: $\mu_c > 0$ and $\mu_d > 0$:** In this case, both the constraints (2) and (3) are binding, so that

$$q^c = \frac{\beta i^e e}{\alpha\pi} \quad \text{and} \quad q^d = \frac{\beta\kappa a}{1-\alpha}.$$

By (4) and (5), we must have $q^c < q^d < q^*$. Observe that $q^c < q^d < q^*$ if and only if

$$\kappa_c = \frac{(1-\alpha)i^e e}{\alpha\pi a} < \kappa < \frac{(1-\alpha)q^*}{\beta a} \equiv \bar{\kappa}_c,$$

where it holds that $\kappa_c = \frac{(1-\alpha)i^e e}{\alpha\pi a} < \bar{\kappa}_c = \frac{(1-\alpha)q^*}{\beta a}$ (since $\frac{\beta i^e e}{\alpha\pi} < q^*$).

⊙ **Case 3: $\mu_c > 0$ and $\mu_d = 0$:** By (4) and (5), we must have $q^c < q^d = q^*$. Also, by (2) and $\mu_c > 0$, we have

$$q^c = \frac{\beta i^e e}{\alpha\pi}.$$

Combined with these results, (3) and $\mu_d = 0$ lead to

$$q^d = q^* \leq \frac{\beta\kappa a}{1-\alpha} \quad \Longleftrightarrow \quad \kappa \geq \bar{\kappa}_c,$$

where again it holds that $\kappa_c < \bar{\kappa}_c$ (since $\frac{\beta i^e e}{\alpha\pi} < q^*$).

⊙ **Case 4:** $\mu_c = \mu_d = 0$**:** From (4) and (5), we must have $q^c = q^d = q^*$. Then, both the constraints (2) and (3) must be satisfied, i.e., we should have

$$q^* \leq \frac{\beta i^e e}{\alpha\pi} \quad \text{and} \quad q^* \leq \beta\left(\frac{i^e e}{\pi} + \kappa a\right),$$

which implies that

$$\kappa \geq \kappa_c = \frac{q^* - \frac{\beta i^e e}{\pi}}{\beta a}.$$

The above findings can be summarized as follows. If $\frac{\beta i^e e}{\alpha\pi} < q^*$ then $\kappa_c = \frac{(1-\alpha)i^e e}{\alpha\pi a}$, and

$$q^c = q^d = \beta\left(\frac{i^e e}{\pi} + \kappa a\right) < q^*$$

for $\kappa \leq \kappa_c$ (which corresponds to Case 1),

$$q^c = \frac{\beta i^e e}{\alpha\pi} < q^d = \frac{\beta\kappa a}{1-\alpha}$$

for $\kappa \in (\kappa_c, \bar{\kappa}_c)$, where $\bar{\kappa}_c \equiv \frac{(1-\alpha)q^*}{\beta a}$ (which corresponds to Case 2), and

$$q^c = \frac{\beta i^e e}{\alpha\pi} < q^d = q^*$$

for $\kappa \geq \bar{\kappa}_c$ (which corresponds to Case 3).

If $\frac{\beta i^e e}{\alpha\pi} \geq q^*$, then $\kappa_c = \max \frac{q^* - \frac{\beta i^e e}{\pi}}{\beta a}$ and

$$q^c = q^d = \beta\left(\frac{i^e e}{\pi} + \kappa a\right) < q^*$$

for $\kappa \leq \kappa_c$ (which corresponds to Case 1) and

$$q^c = q^d = q^*$$

for $\kappa \geq \kappa_c$ (which corresponds to Case 4). The above covers all the possible cases, and a bank run occurs if and only if $\kappa < \kappa_c$, which completes the proof of this lemma. ■

Proof of Proposition 1

The proof proceeds with the following steps. In Step 1, we show that given $\frac{\beta i^e e}{\alpha\pi} < q^*$, a unique equilibrium solution exists if $\pi > \pi^*$, some $\pi^* \in (\beta i^e, \infty)$. In Step 2, we show that given $\frac{\beta i^e e}{\alpha\pi} \geq q^*$, a unique equilibrium solution exists if $\pi \leq \pi^*$. In Step 3, we show that $\frac{\beta i^e e}{\alpha\pi} < q^*$ occurs if and only if $\pi > \pi^*$ and $\frac{\beta i^e e}{\alpha\pi} \geq q^*$ occurs if and only if $\pi \leq \pi^*$. Hence, since the critical value π^* is unique, Steps 1–3 show the existence and uniqueness of a steady-state monetary equilibrium for all $\pi \in (\beta i^e, \infty)$. Given the established monetary equilibrium with $e > 0$ and $\frac{\beta i^e e}{\pi} < q^*$ for all $\pi > \beta i^e$, we have a positive probability of bank run, $F(\kappa_c) > 0$ for all $\pi \in (\beta i^e, \infty)$, where $\kappa_c = \min\left\{\frac{(1-\alpha)i^e e}{\alpha\pi a}, \frac{q^* - \frac{\beta i^e e}{\pi}}{\beta a}\right\}$. The last claim on the modified Friedman rule will become clear immediately from the following analysis. This completes the proof of Proposition 1.

<u>Step 1. The Case of $\frac{\beta i^e e}{\alpha\pi} < q^*$:</u>

As shown in the main text, the first order conditions, (6) and (7), together with the asset market clearing condition, $a = A$, imply that a stationary monetary equilibrium for $\frac{\beta i^e e}{\alpha\pi} < q^*$ is characterized by the pair (e, φ), satisfying

$$\frac{\pi - \beta i^e}{\beta i^e} = \int_0^{\kappa_c} \left\{u'\left(\beta\left(\frac{i^e e}{\pi} + \kappa A\right)\right) - 1\right\} f(\kappa)d\kappa + \left[u'\left(\frac{\beta i^e e}{\alpha\pi}\right) - 1\right](1 - F(\kappa_c)) \equiv \Phi(e), \quad \text{(A.1)}$$

$$\frac{\varphi}{\beta} - \mathbb{E}(\kappa) = \int_0^{\kappa_c} \kappa\left\{u'\left(\beta\left(\frac{i^e e}{\pi} + \kappa A\right)\right) - 1\right\} f(\kappa)d\kappa + \int_{\kappa_c}^{\bar{\kappa}_c} \kappa\left\{u'\left(\frac{\beta\kappa A}{1-\alpha}\right) - 1\right\} f(\kappa)d\kappa \equiv \Psi(e), \quad \text{(A.2)}$$

where, as shown in the proof of Lemma 1, $\kappa_c = \frac{(1-\alpha)i^e e}{\alpha\pi A}$ (since $\frac{\beta i^e e}{\alpha\pi} < q^*$) and $\bar{\kappa}_c = \frac{(1-\alpha)q^*}{\beta A}$.

Since (A.1) is independent of φ, the equilibrium value of $e > 0$ is determined by (A.1) and, given the determined value of e, (A.2) determines $\varphi > 0$.

To find a solution of $e > 0$, observe that

$$\frac{\partial\Phi(e)}{\partial e} = \frac{\beta i^e}{\pi}\left[\int_0^{\kappa_c} u''\left(\beta\left(\frac{i^e e}{\pi} + \kappa A\right)\right) f(\kappa)d\kappa + \frac{1}{\alpha}u''\left(\frac{\beta i^e e}{\alpha\pi}\right)(1 - F(\kappa_c))\right] < 0,$$

$\Phi(0) = \infty$, and $\Phi\left(\frac{\alpha\pi q^*}{\beta i^e}\right) = \int_0^{\kappa_c}\{u'(\alpha q^* + \beta\kappa A) - 1\} f(\kappa)d\kappa > 0$, where $\kappa_c = \frac{(1-\alpha)q^*}{\beta A}$. Therefore, there exists a unique solution $e \in (0, \frac{\alpha\pi q^*}{\beta i^e})$ that solves (A.1) if

$$\frac{\pi - \beta i^e}{\beta i^e} > \Phi\left(\frac{\alpha\pi q^*}{\beta i^e}\right),$$

or

$$\pi > \pi^*(i^e) \equiv \beta i^e\left[1 + \int_0^{\frac{(1-\alpha)q^*}{\beta A}} \left\{u'(\alpha q^* + \beta\kappa A) - 1\right\} f(\kappa)d\kappa\right] > \beta i^e.$$

Given this solution, a solution $\varphi \in \left(\beta\mathbb{E}(\kappa) + \beta\Psi\left(\frac{\alpha\pi q^*}{\beta i^e}\right), \beta\mathbb{E}(\kappa) + \beta\Psi(0)\right)$ is pinned down uniquely by (A.2) for all $\pi \in (\pi^*(i^e), \infty)$ and $i^e \in (1, \infty)$, because $\Psi(e)$ is a monotone decreasing function of $e \in (0, \frac{\alpha\pi q^*}{\beta i^e})$.

Step 2. The Case of $\frac{\beta i^e e}{\alpha\pi} \geq q^*$:

Similarly as before, with the market clearing condition, $a = A$, a stationary monetary equilibrium for $\frac{\beta i^e e}{\alpha\pi} \leq q^*$ is characterized by the pair (e, φ), satisfying

$$\frac{\pi - \beta i^e}{\beta i^e} = \int_0^{[\kappa_c]^+} \left\{u'\left(\beta\left(\frac{i^e e}{\pi} + \kappa A\right)\right) - 1\right\} f(\kappa)d\kappa \equiv \tilde{\Phi}(e), \tag{A.3}$$

$$\frac{\varphi}{\beta} - \mathbb{E}(\kappa) = \int_0^{[\kappa_c]^+} \kappa\left\{u'\left(\beta\left(\frac{i^e e}{\pi} + \kappa A\right)\right) - 1\right\} f(\kappa)d\kappa \equiv \tilde{\Psi}(e), \tag{A.4}$$

where, as shown in the proof of Lemma 1, $\kappa_c = \frac{q^* - \frac{\beta i^e e}{\pi}}{\beta A}$ (since $\frac{\beta i^e e}{\alpha\pi} \geq q^*$) and $[\kappa_c]^+ = \max\{\kappa_c, 0\}$.

Since $\frac{\partial\tilde{\Phi}(e)}{\partial e} < 0$, $\tilde{\Phi}\left(\frac{\alpha\pi q^*}{\beta i^e}\right) = \int_0^{\kappa_c}\{u'(\alpha q^* + \beta\kappa A) - 1\} f(\kappa)d\kappa = \Phi\left(\frac{\alpha\pi q^*}{\beta i^e}\right) > 0$ and $\tilde{\Phi}\left(\frac{\pi q^*}{\beta i^e}\right) = 0$, there exists a unique solution $e \in [\frac{\alpha\pi q^*}{\beta i^e}, \frac{\pi q^*}{\beta i^e})$ that solves (A.3) if

$$\frac{\pi - \beta i^e}{\beta i^e} \leq \Phi\left(\frac{\alpha\pi q^*}{\beta i^e}\right),$$

or $\pi \leq \pi^*(i^e)$ where $\pi^*(i^e) \in (\beta i^e, \infty)$ is defined above.

Given this solution, a solution $\varphi \in \left(\beta\mathbb{E}(\kappa), \beta\mathbb{E}(\kappa) + \beta\tilde{\Psi}\left(\frac{\alpha\pi q^*}{\beta i^e}\right)\right)$ is pinned down uniquely by (A.4) for all $\pi \in (\beta i^e, \pi^*(i^e)]$ because $\tilde{\Psi}(e)$ is a monotone decreasing function of $e \in [\frac{\alpha\pi q^*}{\beta i^e}, \frac{\pi q^*}{\beta i^e})$.

In the limit as $\pi \to \beta i^e$, we must have $\kappa_c \to 0$, implying that, as shown in Lemma 1, it holds that $q^c = q^d = q^*$ for all $\kappa > \kappa_c \to 0$, i.e. the first best is achieved.

Step 3. $\frac{\beta i^e e}{\alpha\pi} < q^*$ occurs if and only if $\pi > \pi^*$ and $\frac{\beta i^e e}{\alpha\pi} \geq q^*$ occurs if and only if $\pi \leq \pi^*$:

Step 1 shows that e is determined by (A.1). We see that $e \to 0$ as $\pi \to \infty$ and so $\frac{\beta i^e e}{\alpha\pi} \to 0 < q^*$ as $\pi \to \infty$. Similarly, Step 2 shows that e is determined by (A.3). We see that $e \to \frac{\pi q^*}{\beta i^e}$ (which implies $\kappa_c \to 0$) as $\pi \to \beta i^e$ and so $\frac{\beta i^e e}{\alpha\pi} \to \frac{q^*}{\alpha} > q^*$ as $\pi \to \beta i^e$. Hence, we must be in the region $\frac{\beta i^e e}{\alpha\pi} < q^*$ when π is high and $\frac{\beta i^e e}{\alpha\pi} \geq q^*$ when π is low.

To prove the claim, it is therefore sufficient to show that e is monotone decreasing in π. For $\frac{\beta i^e e}{\alpha\pi} < q^*$, the implicit differentiation of (A.1) yields

$$\frac{\partial e}{\partial\pi} = \frac{\frac{1}{\beta i^e} - \frac{\partial\Phi}{\partial\pi}}{\frac{\partial\Phi}{\partial e}}$$

where $\frac{\partial \Phi}{\partial e} < 0$. Differentiation yields:

$$\frac{\partial \Phi}{\partial \pi} = -\frac{\beta i^e e}{\pi^2}\left[\int_0^{\kappa_c} u''\left(\beta\left(\frac{i^e e}{\pi}+\kappa A\right)\right) f(\kappa)d\kappa + \frac{1-F(\kappa_c)}{\alpha}u''\left(\frac{\beta i^e e}{\alpha\pi}\right)\right] > 0.$$

Observe from (A.1) that

$$\frac{1}{\beta i^e} = \frac{1}{\pi} + \frac{\Phi}{\pi}.$$

Applying this, we can compute that

$$\frac{1}{\beta i^e} - \frac{\partial \Phi}{\partial \pi} = \int_0^{\kappa_c} \frac{1}{\pi}\left(1 - \frac{\xi \frac{i^e e}{\pi}}{\frac{i^e e}{\pi}+\kappa A}\right) u'\left(\beta\left(\frac{i^e e}{\pi}+\kappa A\right)\right) f(\kappa)d\kappa + \frac{1-\xi}{\pi}u'\left(\frac{\beta i^e e}{\alpha\pi}\right)(1-F(\kappa_c)) > 0$$

where $\xi = -\frac{u''(q)q}{u'(q)} \in (0,1)$. This shows $\frac{\partial e}{\partial \pi} < 0$.

Similarly, for $\frac{\beta i^e e}{\alpha\pi} \geq q^*$, we can compute from (A.3) that

$$\frac{1}{\beta i^e} - \frac{\partial \tilde{\Phi}}{\partial \pi} = \int_0^{[\kappa_c]^+} \frac{1}{\pi}\left(1 - \frac{\xi \frac{i^e e}{\pi}}{\frac{i^e e}{\pi}+\kappa A}\right) u'\left(\beta\left(\frac{i^e e}{\pi}+\kappa A\right)\right) f(\kappa)d\kappa + \frac{1-F(\kappa_c)}{\pi} > 0,$$

which shows $\frac{\partial e}{\partial \pi} < 0$. ■

Proof of Lemma 2

⊙ The effects of π on e and φ: In Step 3 of the proof of Proposition 1, we have already shown that $\frac{\partial e}{\partial \pi} < 0$ for all $\pi \in (\beta i^e, \infty)$.

For $\pi > \pi^*(i^e)$, differentiation of (A.2) yields $\frac{\partial \varphi}{\partial \pi} = \beta\left(\frac{\partial \Psi}{\partial \pi} + \frac{\partial \Psi}{\partial e}\frac{\partial e}{\partial \pi}\right) > 0$ since $\frac{\partial \Psi}{\partial \pi} > 0$ and $\frac{\partial \Psi}{\partial e} < 0$. For $\pi \leq \pi^*(i^e)$, differentiation of (A.4) yields $\frac{\partial \varphi}{\partial \pi} = \beta\left(\frac{\partial \tilde{\Psi}}{\partial \pi} + \frac{\partial \tilde{\Psi}}{\partial e}\frac{\partial e}{\partial \pi}\right) > 0$ since $\frac{\partial \tilde{\Psi}}{\partial \pi} > 0$ and $\frac{\partial \tilde{\Psi}}{\partial e} < 0$. Therefore, $\frac{\partial \varphi}{\partial \pi} > 0$ for all $\pi \in (\beta i^e, \infty)$.

⊙ The effects of i^e on e and φ: For $\pi > \pi^*(i^e)$, the implicit differentiation of (A.1) yields

$$\frac{\partial e}{\partial i^e} = -\frac{\frac{\pi}{\beta (i^e)^2} + \frac{\partial \Phi}{\partial i^e}}{\frac{\partial \Phi}{\partial e}}.$$

where $\frac{\partial \Phi}{\partial e} < 0$. Differentiation yields:

$$\frac{\partial \Phi}{\partial i^e} = \frac{\beta e}{\pi}\left[\int_0^{\kappa_c} u''\left(\beta\left(\frac{i^e e}{\pi}+\kappa A\right)\right) f(\kappa)d\kappa + \frac{1}{\alpha}u''\left(\frac{\beta i^e e}{\alpha\pi}\right)(1-F(\kappa_c))\right] < 0.$$

Observe from (A.1) that

$$\frac{\pi}{\beta (i^e)^2} = \frac{1}{i^e} + \frac{\Phi}{i^e}.$$

Applying this, we can compute that

$$\frac{\pi}{\beta(i^e)^2}+\frac{\partial\Phi}{\partial i^e}=\int_0^{\kappa_c}\frac{1}{i^e}\left(1-\frac{\xi\frac{i^e e}{\pi}}{\frac{i^e e}{\pi}+\kappa A}\right)u'\left(\beta\left(\frac{i^e e}{\pi}+\kappa A\right)\right)f(\kappa)d\kappa+\frac{1-\xi}{i^e}u'\left(\frac{\beta i^e e}{\alpha\pi}\right)(1-F(\kappa_c))>0$$

where $\xi=-\frac{u''(q)q}{u'(q)}\in(0,1)$. This shows $\frac{\partial e}{\partial i^e}>0$.

Similarly, for $\pi\leq\pi^*(i^e)$, we can compute from (A.3) that

$$\frac{\pi}{\beta(i^e)^2}+\frac{\partial\tilde{\Phi}}{\partial i^e}=\int_0^{[\kappa_c]^+}\frac{1}{i^e}\left(1-\frac{\xi\frac{i^e e}{\pi}}{\frac{i^e e}{\pi}+\kappa A}\right)u'\left(\beta\left(\frac{i^e e}{\pi}+\kappa A\right)\right)f(\kappa)d\kappa+\frac{1-F(\kappa_c)}{i^e}>0,$$

which shows $\frac{\partial e}{\partial i^e}>0$. Therefore, $\frac{\partial e}{\partial i^e}>0$ for all $\pi\in(\beta i^e,\infty)$

As for the effect on φ, for $\pi>\pi^*$, differentiation of (A.2) yields $\frac{\partial\varphi}{\partial i^e}=\beta\left(\frac{\partial\Psi}{\partial i^e}+\frac{\partial\Psi}{\partial e}\frac{\partial e}{\partial i^e}\right)<0$ since $\frac{\partial\Psi}{\partial i^e}<0$ and $\frac{\partial\Psi}{\partial e}<0$. For $\pi\leq\pi^*$, differentiation of (A.4) yields $\frac{\partial\varphi}{\partial i^e}=\beta\left(\frac{\partial\tilde{\Psi}}{\partial i^e}+\frac{\partial\tilde{\Psi}}{\partial e}\frac{\partial e}{\partial i^e}\right)<0$ since $\frac{\partial\tilde{\Psi}}{\partial i^e}<0$ and $\frac{\partial\tilde{\Psi}}{\partial e}<0$. Therefore, $\frac{\partial\varphi}{\partial i^e}<0$ for all $\pi\in(\beta i^e,\infty)$. ■

Proof of Proposition 2

The first part of this proposition is immediate from Lemma 2, $\frac{\partial e}{\partial i^e}>0$.

To prove the second part, observe that welfare with a cash-like CBDC is given by

$$\begin{aligned}\mathcal{W}=&\int_0^{\kappa_c}\left\{u\left(\beta\left(\frac{i^e e}{\pi}+\kappa A\right)\right)-\beta\left(\frac{i^e e}{\pi}+\kappa A\right)\right\}f(\kappa)d\kappa\\&+\int_{\kappa_c}^{\bar{\kappa}_c}\left[\alpha\left\{u\left(\frac{\beta i^e e}{\alpha\pi}\right)-\frac{\beta i^e e}{\alpha\pi}\right\}+(1-\alpha)\left\{u\left(\frac{\beta\kappa A}{1-\alpha}\right)-\frac{\beta\kappa A}{1-\alpha}\right\}\right]f(\kappa)d\kappa\\&+\int_{\bar{\kappa}_c}^{\infty}\left[\alpha\left\{u\left(\frac{\beta i^e e}{\alpha\pi}\right)-\frac{\beta i^e e}{\alpha\pi}\right\}+(1-\alpha)\{u(q^*)-q^*\}\right]f(\kappa)d\kappa+\mathbb{E}(\kappa)A,\end{aligned}$$

for $\pi>\pi^*(i^e)$, where $\kappa_c=\frac{(1-\alpha)i^e e}{\alpha\pi A}$ and $\bar{\kappa}_c=\frac{(1-\alpha)q^*}{\beta A}$, and

$$\mathcal{W}=\int_0^{[\kappa_c]^+}\left\{u\left(\beta\left(\frac{i^e e}{\pi}+\kappa A\right)\right)-\beta\left(\frac{i^e e}{\pi}+\kappa A\right)\right\}f(\kappa)d\kappa+\int_{[\kappa_c]^+}^{\infty}\{u(q^*)-q^*\}f(\kappa)d\kappa+\mathbb{E}(\kappa)A,$$

for $\pi\leq\pi^*(i^e)$, where $[\kappa_c]^+=\max\left\{\frac{q^*-\frac{\beta i^e e}{\pi}}{\beta A},0\right\}$. We obtain

$$\frac{\partial\mathcal{W}}{\partial i^e}=\frac{\beta}{\pi}\left(e+i^e\frac{\partial e}{\partial i^e}\right)\left[\int_0^{\kappa_c}\left\{u'\left(\beta\left(\frac{i^e e}{\pi}+\kappa A\right)\right)-1\right\}f(\kappa)d\kappa+\int_{\kappa_c}^{\infty}\left\{u'\left(\frac{\beta i^e e}{\alpha\pi}\right)-1\right\}f(\kappa)d\kappa\right]>0,$$

for $\pi>\pi^*(i^e)$, and

$$\frac{\partial\mathcal{W}}{\partial i^e}=\frac{\beta}{\pi}\left(e+i^e\frac{\partial e}{\partial i^e}\right)\int_0^{[\kappa_c]^+}\left\{u'\left(\beta\left(\frac{i^e e}{\pi}+\kappa A\right)\right)-1\right\}f(\kappa)d\kappa>0,$$

for $\pi\leq\pi^*(i^e)$. This completes the proof of this proposition. ■

Proof of Lemma 3

We derive the solution for $q^c, q^d \in (0, q^*]$ and $\mu_c, \mu_d \geq 0$ that satisfies the first order conditions, (4) and (5), and constraints (8) and (9).

$\odot$ **Case 1: $\mu_c = 0$ and $\mu_d > 0$:** From (4) and (5), combined with (9), we must have

$$q^c = q^d = \beta\left(\frac{z + i^e e}{\pi} + \kappa a\right) < q^*.$$

Since $q^c = q^d \leq \min\{\frac{\beta z}{\alpha\pi}, q^*\}$ by (8), there are two possible cases. If $\frac{\beta z}{\alpha\pi} < q^*$, then constraint (8) implies that

$$q^c = q^d = \beta\left(\frac{z + i^e e}{\pi} + \kappa a\right) \leq \frac{\beta z}{\alpha\pi} \iff \kappa \leq \frac{(1-\alpha)z - \alpha i^e e}{\alpha\pi a}.$$

If $q^* \leq \frac{\beta z}{\alpha\pi}$, then $q^c = q^d < q^*$ implies that

$$q^c = q^d = \beta\left(\frac{z + i^e e}{\pi} + \kappa a\right) < q^* \iff \kappa < \frac{q^* - \frac{\beta(z+i^e e)}{\pi}}{\beta a}.$$

Defining

$$\kappa_d \equiv \min\left\{\frac{(1-\alpha)z - \alpha i^e e}{\alpha\pi a}, \frac{q^* - \frac{\beta(z+i^e e)}{\pi}}{\beta a}\right\},$$

we see that $\kappa_d = \frac{(1-\alpha)z-\alpha i^e e}{\alpha\pi a}$ if $\frac{\beta z}{\alpha\pi} < q^*$, and $\kappa_d = \frac{q^* - \frac{\beta(z+i^e e)}{\pi}}{\beta a}$ if $\frac{\beta z}{\alpha\pi} \geq q^*$. According to our definition of a bank run, there are bank runs in this case.

$\odot$ **Case 2: $\mu_c > 0$ and $\mu_d > 0$:** In this case, both the constraints (8) and (9) are binding, so that

$$q^c = \frac{\beta z}{\alpha\pi} \quad \text{and} \quad q^d = \frac{\frac{\beta i^e e}{\pi} + \beta\kappa a}{1-\alpha}.$$

By (4) and (5), we must have $q^c < q^d < q^*$. Observe that $q^c < q^d < q^*$ if and only if

$$\kappa_d = \frac{(1-\alpha)z - \alpha i^e e}{\alpha\pi a} < \kappa < \frac{(1-\alpha)q^* - \frac{\beta i^e e}{\pi}}{\beta a} \equiv \bar{\kappa}_d,$$

where it holds that $\kappa_d = \frac{(1-\alpha)z-\alpha i^e e}{\alpha\pi a} < \bar{\kappa}_d = \frac{(1-\alpha)q^* - \frac{\beta i^e e}{\pi}}{\beta a}$ (since $\frac{\beta z}{\alpha\pi} < q^*$).

$\odot$ **Case 3: $\mu_c > 0$ and $\mu_d = 0$:** By (4) and (5), we must have $q^c < q^d = q^*$. Also, by (8) and $\mu_c > 0$, we have

$$q^c = \frac{\beta z}{\alpha\pi}.$$

Combined with these results, (9) and $\mu_d = 0$ lead to

$$q^d = q^* \leq \frac{\frac{\beta i^e e}{\pi} + \beta\kappa a}{1-\alpha} \quad \Longleftrightarrow \quad \kappa \geq \bar{\kappa}_d$$

where again it holds that $\kappa_d < \bar{\kappa}_d$ (since $\frac{\beta z}{\alpha\pi} < q^*$).

$\odot$ **Case 4:** $\mu_c = \mu_d = 0$**:** From (4) and (5), we have $q^c = q^d = q^*$. Then, both the constraints (8) and (9) must be satisfied, i.e., we should have

$$q^* \leq \frac{\beta z}{\alpha\pi} \quad \text{and} \quad q^* \leq \beta\left(\frac{z + i^e e}{\pi} + \kappa a\right),$$

which implies

$$\kappa \geq \kappa_d = \frac{q^* - \frac{\beta(z+i^e e)}{\pi}}{\beta a}.$$

The above findings can be summarized as follows. If $\frac{\beta z}{\alpha\pi} < q^*$ then $\kappa_d = \frac{(1-\alpha)z - \alpha i^e e}{\alpha\pi a}$, and

$$q^c = q^d = \beta\left(\frac{z + i^e e}{\pi} + \kappa a\right) < q^*$$

for $\kappa \leq \kappa_d$ (which corresponds to Case 1),

$$q^c = \frac{\beta z}{\alpha\pi} < q^d = \frac{\frac{\beta i^e e}{\pi} + \beta\kappa a}{1-\alpha}$$

for $\kappa \in (\kappa_d, \bar{\kappa}_d)$, where $\bar{\kappa}_d \equiv \frac{(1-\alpha)q^* - \frac{\beta i^e e}{\pi}}{\beta a}$ (which corresponds to Case 2), and

$$q^c = \frac{\beta z}{\alpha\pi} < q^d = q^*$$

for $\kappa \geq \bar{\kappa}_d$ (which corresponds to Case 3).

If $\frac{\beta z}{\alpha\pi} \geq q^*$, then $\kappa_d = \frac{q^* - \frac{\beta(z+i^e e)}{\pi}}{\beta a}$, and

$$q^c = q^d = \beta\left(\frac{z + i^e e}{\pi} + \kappa a\right) < q^*$$

for $\kappa < \kappa_d$ (which corresponds to Case 1) and

$$q^c = q^d = q^*$$

for $\kappa \geq \kappa_d$ (which corresponds to Case 4). The above covers all the possible cases, and a bank run occurs if and only if $\kappa < \kappa_d$, which completes the proof of this lemma. ■

Proof of Proposition 3

In the main text, we have shown that the first order conditions, (7), (10), and (11), together with the market clearing conditions, $z = \phi M$, $e = \phi E$, and $a = A$, are necessary and sufficient conditions for a stationary monetary equilibrium. In this proof, we show that a unique solution exists to these conditions. We first examine $\frac{\beta z}{\alpha \pi} < q^*$ and $e > 0$, where the pair (z, e) is determined by

$$\frac{\pi - \beta}{\beta} = \int_0^{[\kappa_d]^+} \left\{ u'\left(\beta\left(\frac{z + i^e e}{\pi} + \kappa A\right)\right) - 1 \right\} f(\kappa) d\kappa + \left\{ u'\left(\frac{\beta z}{\pi \alpha}\right) - 1 \right\} (1 - F(\kappa_d)) \equiv \Phi(z, e), \tag{A.5}$$

$$\frac{\pi - \beta i^e}{\beta i^e} = \int_0^{[\kappa_d]^+} \left\{ u'\left(\beta\left(\frac{z + i^e e}{\pi} + \kappa A\right)\right) - 1 \right\} f(\kappa) d\kappa + \int_{[\kappa_d]^+}^{[\bar{\kappa}_d]^+} \left\{ u'\left(\frac{\beta(\frac{i^e e}{\pi} + \kappa A)}{1 - \alpha}\right) - 1 \right\} f(\kappa) d\kappa \equiv X(z, e), \tag{A.6}$$

where $\kappa_d = \frac{(1-\alpha)z - \alpha i^e e}{\alpha \pi A} \geq 0$ ($[\kappa_d]^+ = 0$ for $e \geq \frac{(1-\alpha)z}{\alpha i^e}$) and $\bar{\kappa}_d = \frac{(1-\alpha)q^* - \frac{\beta i^e e}{\pi}}{\beta A} > \kappa_d$ ($[\bar{\kappa}_d]^+ = 0$ for $e \geq \frac{(1-\alpha)q^* \pi}{\beta i^e}$). There are two cases. In Case 1, we examine $[\kappa_d]^+ = 0$ (i.e., $e \geq \frac{(1-\alpha)z}{\alpha i^e}$), and show that a unique solution exists if and only if $i^e \in [\bar{i}^e, \frac{\pi}{\beta})$, with some $\bar{i}^e \in (1, \frac{\pi}{\beta})$. In Case 2, we examine $\kappa_d > 0$ (i.e., $e < \frac{(1-\alpha)z}{\alpha i^e}$) and show that a unique solution exists if and only if $i^e \in (\underline{i}^e, \bar{i}^e)$, with some $\underline{i}^e \in (1, \bar{i}^e)$. Then, we examine in Case 3,

$$\frac{\pi - \beta i^e}{\beta i^e} > X(z, 0), \tag{A.7}$$

and show that a unique solution exists for $\frac{\beta z}{\alpha \pi} < q^*$ and $e = 0$ if and only if $i^e \in (1, \underline{i}^e]$ and $\pi > \pi^*(1)$ (see the proof of Proposition 1). In all of these cases, once e and z are determined jointly by (A.5) and (A.6), $\phi \geq \beta \mathbb{E}(\kappa)$ is uniquely pinned down by

$$\frac{\varphi}{\beta} - \mathbb{E}(\kappa) = \int_0^{[\kappa_d]^+} \kappa \left\{ u'\left(\beta\left(\frac{z + i^e e}{\pi} + \kappa A\right)\right) - 1 \right\} f(\kappa) d\kappa + \int_{[\kappa_d]^+}^{[\bar{\kappa}_d]^+} \kappa \left\{ u'\left(\frac{\beta(\frac{i^e e}{\pi} + \kappa A)}{1 - \alpha}\right) - 1 \right\} f(\kappa) d\kappa \equiv \Psi(z, e). \tag{A.8}$$

Finally, in Case 4, we show that the case $\frac{\beta z}{\alpha \pi} \geq q^*$ is essentially the same as in Step 2 in the proof of Proposition 1 and a unique equilibrium solution exists if and only if $\pi \leq \pi^*(1)$ and $i^e = 1$.

Note in the above analysis, we have taken $\frac{\beta z}{\alpha \pi} < q^*$ as given to establish a solution for $i^e \in (1, \frac{\pi}{\beta})$, and $\frac{\beta z}{\alpha \pi} \geq q^*$ as given to a solution for $i^e = 1$. The above analysis shows this is indeed the case. For $i^e \in [\bar{i}^e(\pi), \frac{\pi}{\beta}]$, Step 1 shows that $z = \underline{z}$, satisfying $\frac{\beta \underline{z}}{\alpha \pi} = u^{-1\prime}(\frac{\pi}{\beta}) < q^* \equiv u^{-1\prime}(1)$ for all $\pi > \beta$. Further, as shown in the proof of Lemma 4 (see below), z is monotone decreasing in $i^e \in (\underline{i}^e(\pi), \bar{i}^e(\pi))$, and so Step 2 shows $\frac{\beta z}{\alpha \pi} < q^*$ for $i^e \in (\underline{i}^e(\pi), \bar{i}^e(\pi))$ as well, as long as $\pi > \pi^*(1)$. Step 3 shows that $\frac{\beta z}{\alpha \pi}$ is independent of $i^e \in [1, \underline{i}^e(\pi)]$ (because $e = 0$), and $\frac{\beta \bar{z}_1}{\alpha \pi} < q^*$ if and only if $\pi > \pi^*(1)$. In sum, given Lemma 4, Step 1 – 3 indeed show that we must have $\frac{\beta z}{\alpha\pi} < q^*$ when $i^e \in [1, \frac{\pi}{\beta})$ and $\pi > \pi^*(1)$. Simultaneously, they show that there is no equilibrium solution that satisfies $\frac{\beta z}{\alpha\pi} < q^*$ when $\pi \leq \pi^*(1)$. Therefore, Step 4 shows that we must have $\frac{\beta z}{\alpha\pi} \geq q^*$ when $i^e = 1$ and $\pi \leq \pi^*(1)$. This establishes the existence and uniqueness of equilibrium for all $i^e \in [1, \frac{\pi}{\beta})$ and completes the proof of this proposition.

Case 1. $\frac{\beta z}{\alpha\pi} < q^*$**,** $\kappa_d = 0$**, and** $e > 0$**:**

Note that for $[\kappa_d]^+ = 0$ (i.e. $e \geq \frac{(1-\alpha)z}{\alpha i^e}$) to be part of a solution, we must have $\bar{\kappa}_d > 0$ (i.e $e < \frac{(1-\alpha)q^*\pi}{\beta i^e}$) in (A.6) with $i^e < \frac{\pi}{\beta}$. When $\kappa_d = 0$, we have

$$\Phi(z, e) = u'\left(\frac{\beta z}{\pi\alpha}\right) - 1,$$

and (A.5) has a unique solution

$$z = \underline{z} \equiv \frac{\alpha\pi}{\beta} u^{-1\prime}\left(\frac{\pi}{\beta}\right).$$

Similarly, when $\kappa_d = 0$, we have

$$X(\underline{z}, e) = \int_0^{\bar{\kappa}_d} \left\{ u'\left(\frac{\beta(\frac{i^e e}{\pi} + \kappa A)}{1-\alpha}\right) - 1 \right\} f(\kappa) d\kappa,$$

which is decreasing in e. Since $X\left(\underline{z}, \frac{(1-\alpha)\pi q^*}{\beta i^e}\right) = 0$ (where $[\bar{\kappa}_d]^+ = 0$ holds), a unique solution $e \in \left[\frac{(1-\alpha)\underline{z}}{\alpha i^e}, \frac{(1-\alpha)\pi q^*}{\beta i^e}\right)$ exists if and only if

$$\frac{\pi - \beta i^e}{\beta i^e} \leq X\left(\underline{z}, \frac{(1-\alpha)\underline{z}}{\alpha i^e}\right) = \int_0^{\bar{\kappa}_d} \left\{ u'\left(u^{-1\prime}\left(\frac{\pi}{\beta}\right) + \frac{\beta\kappa A}{1-\alpha}\right) - 1 \right\} f(\kappa) d\kappa,$$

(where $[\kappa_d]^+ = 0$ and $\bar{\kappa}_d = \frac{(1-\alpha)\{q^* - u^{-1\prime}(\frac{\pi}{\beta})\}}{\beta A}$ hold) or equivalently,

$$i^e \geq \bar{i}^e(\pi) \equiv \frac{\frac{\pi}{\beta}}{1 + \int_0^{\bar{\kappa}_d} \left\{ u'\left(u^{-1\prime}\left(\frac{\pi}{\beta}\right) + \frac{\beta\kappa A}{1-\alpha}\right) - 1 \right\} f(\kappa) d\kappa} \in \left(1, \frac{\pi}{\beta}\right),$$

because $\bar{i}^e(\pi)$ is strictly increasing in π and satisfies $\bar{i}^e(\pi) \to 1$ as $\pi \to \beta$.

In sum, for $\frac{\beta z}{\alpha\pi} < q^*$ and $i^e \in [\bar{i}^e(\pi), \frac{\pi}{\beta})$, a unique equilibrium solution exists with $e \in [\frac{(1-\alpha)\underline{z}}{\alpha i^e}, \frac{(1-\alpha)\pi q^*}{\beta i^e})$, $z = \underline{z}$ and $[\kappa_d]^+ = 0$.

Case 2. $\frac{\beta z}{\alpha\pi} < q^*$**,** $\kappa_d > 0$**, and** $e > 0$**:**

When $\kappa_d > 0$, i.e. $e < \frac{(1-\alpha)\underline{z}}{\alpha i^e}$, an equilibrium pair (z, e) is determined jointly by (A.5) and (A.6). Let $z = \phi(e)$ and $z = \chi(e)$ be defined by (A.5) and (A.6), respectively. The proof proceeds by showing the following steps:

- Step 1. $\phi'(e) > \chi'(e)$;
- Step 2. $\underline{e} < \phi^{-1}(\underline{z})$;
- Step 3. $\bar{z}_1 < \bar{z}_2$.

Step 1-3 altogether imply that the two curves, $z = \phi(e)$ and $z = \chi(e)$, must have a unique intersection (see Figure 5).

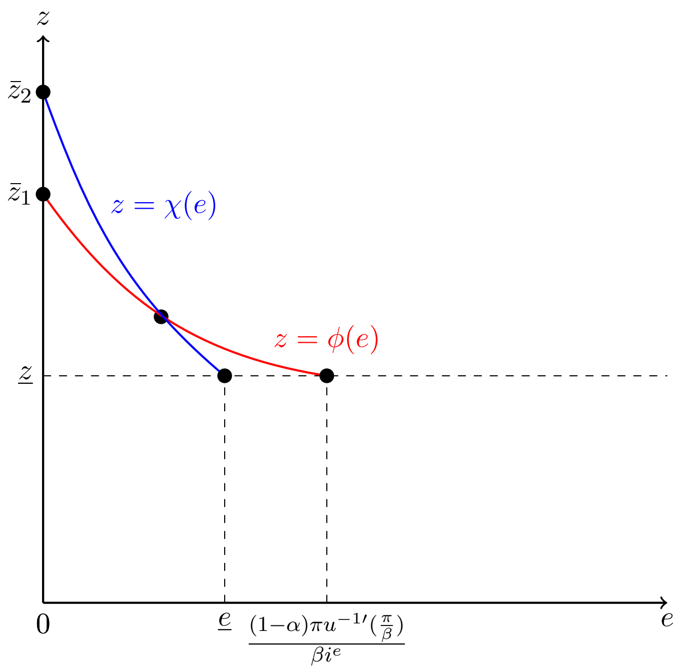

Fig 5: Existence of a Monetary Equilibrium with a Deposit-Like CBDC for $i^e \in [1, \bar{i}^e(\pi))$ with $\pi > \beta i^e$

Proof of Step 1.

The total derivatives of the identities, $\frac{\pi-\beta}{\beta} = \Phi(\phi(e), e)$ and $\frac{\pi-\beta i^e}{\beta i^e} = X(\chi(e), e)$, imply

$$\phi'(e) = -\frac{\Phi_e}{\Phi_z} < 0 \quad \text{and} \quad \chi'(e) = -\frac{X_e}{X_z} < 0,$$

where $\Phi_z \equiv \frac{\partial \Phi}{\partial z} < 0$, $\Phi_e \equiv \frac{\partial \Phi}{\partial e} < 0$, $X_z \equiv \frac{\partial X}{\partial z} < 0$ and $X_e \equiv \frac{\partial X}{\partial e} < 0$. Further,

$$\phi'(e) - \chi'(e) = \frac{X_e \Phi_z - \Phi_e X_z}{X_z \Phi_z} > 0,$$

since $X_z\Phi_z > 0$ and

$$X_e\Phi_z - \Phi_e X_z = i^e\left(\frac{\beta}{\pi}\right)^2\left[\left\{\int_0^{\kappa_d} u''\left(\beta\left(\frac{z+i^e e}{\pi}+\kappa A\right)\right)f(\kappa)d\kappa\right\}\left\{\int_{\kappa_d}^{\infty} u''\left(\frac{\beta z}{\pi\alpha}\right)\frac{f(\kappa)}{\alpha}d\kappa\right\}\right.$$
$$\left.+\left\{\int_{\kappa_d}^{\bar{\kappa}_d} u''\left(\frac{\beta(\frac{i^e e}{\pi}+\kappa A)}{1-\alpha}\right)\frac{f(\kappa)}{1-\alpha}d\kappa\right\}\left\{\int_0^{\kappa_d} u''\left(\beta\left(\frac{z+i^e e}{\pi}+\kappa A\right)\right)f(\kappa)d\kappa + \int_{\kappa_d}^{\infty} u''\left(\frac{\beta z}{\pi\alpha}\right)\frac{f(\kappa)}{\alpha}d\kappa\right\}\right] > 0. \tag{A.9}$$

Proof of Step 2.

Note that $z = \phi\left(\frac{(1-\alpha)\underline{z}}{\alpha i^e}\right) = \underline{z}(=\alpha(\frac{\pi}{\beta})u^{-1\prime}(\frac{\pi}{\beta}))$ (where $\kappa_d = 0$ holds). Let $\underline{e}$ be defined by

$$z = \underline{z} = \chi\left(\underline{e}\right).$$

Then, since $X(\underline{z}, e)$ is strictly decreasing in e, we have

$$\phi^{-1}(\underline{z}) = \frac{(1-\alpha)\pi u^{-1\prime}(\frac{\pi}{\beta})}{\beta i^e} > \underline{e}$$

if and only if

$$\frac{\pi - \beta i^e}{\beta i^e} > \int_0^{\bar{\kappa}_d}\left\{u'\left(\frac{(1-\alpha)u^{-1\prime}(\frac{\pi}{\beta})+\beta\kappa A}{1-\alpha}\right)-1\right\}f(\kappa)d\kappa,$$

or $i^e < \bar{i}^e(\pi)$.

Proof of Step 3.

Denote $\bar{z}_1 = \phi(0)$ and $\bar{z}_2 = \chi(0)$. Since $\Phi(\bar{z}_1, 0)$ does not depend on i^e, $\bar{z}_1$ does not depend on i^e either. On the other hand, the implicit differentiation of $\frac{\pi-\beta i^e}{\beta i^e} = X(\bar{z}_2, 0)$ yields

$$\frac{\partial \bar{z}_2}{\partial i^e} = -\frac{\pi}{\beta(i^e)^2 X_z} > 0,$$

where $X_z \equiv \frac{\partial X}{\partial z} < 0$.

We will show that $\bar{z}_1 > \bar{z}_2$ when $i^e = 1$. Assume that $i^e = 1$. Then,

$$\Gamma(\bar{z}_1, \bar{z}_2) \equiv \Phi(\bar{z}_1, 0) - X(\bar{z}_2, 0) = 0.$$

Consider

$$\Gamma(\bar{z}_1, \bar{z}_1) = \left\{u'\left(\frac{\beta\bar{z}_1}{\pi\alpha}\right)-1\right\}(1-F(\kappa_d)) - \int_{\kappa_d}^{\bar{\kappa}_d}\left\{u'\left(\frac{\beta\kappa A}{1-\alpha}\right)-1\right\}f(\kappa)d\kappa,$$

where $\kappa_d = \frac{(1-\alpha)\bar{z}_1}{\alpha\pi A}$ and $\bar{\kappa}_d = \frac{(1-\alpha)q^*}{\beta A}$. Observe that $\Gamma(\bar{z}_1, \bar{z}_1)$ is strictly decreasing in $\bar{z}_1$, and satisfies $\Gamma\left(\bar{z}_1, \bar{z}_1\right) \to 0$ as $\bar{z}_1 \to \frac{\alpha\pi q^*}{\beta}$. Therefore, $\Gamma(\bar{z}_1, \bar{z}_1) > 0$ for all $\bar{z}_1 \in \left(\underline{z}, \frac{\alpha\pi q^*}{\beta}\right)$. Since $\Gamma(\bar{z}_1, \bar{z}_2) = 0$, this implies that we must have $\bar{z}_1 > \bar{z}_2$.

Given the above, we can say that $\bar{z}_1 < \bar{z}_2$ if and only if $\frac{\pi-\beta i^e}{\beta i^e}(= X(\bar{z}_2,0)) < X(\bar{z}_1,0)$, which can be written as

$$i^e > \underline{i}^e(\pi) \equiv \frac{\frac{\pi}{\beta}}{1+\int_0^{\kappa_d}\left\{u'\left(\beta\left(\frac{\bar{z}_1}{\pi}+\kappa A\right)\right)-1\right\}f(\kappa)d\kappa+\int_{\kappa_d}^{\bar{\kappa}_d}\left\{u'\left(\frac{\beta\kappa A}{1-\alpha}\right)-1\right\}f(\kappa)d\kappa} \in (1,\bar{i}^e(\pi)),$$

where $\kappa_d = \frac{(1-\alpha)\bar{z}_1}{\alpha\pi A}$ and $\bar{\kappa}_d = \frac{(1-\alpha)q^*}{\beta A}$, because $u'(\beta\left(\frac{\bar{z}_1}{\pi}+\kappa A\right)) > u'(\frac{\beta\kappa A}{1-\alpha}) > u'(u^{-1\prime}(\frac{\pi}{\beta})+\frac{\beta\kappa A}{1-\alpha})$ for $\kappa < \bar{\kappa}_d$. Therefore, $\bar{z}_1 \geq \bar{z}_2$ if $i^e \leq \underline{i}^e$ and $\bar{z}_1 < \bar{z}_2$ if $i^e > \underline{i}^e$.

In sum, for $\frac{\beta z}{\alpha\pi} < q^*$ and $i^e \in (\underline{i}^e(\pi),\bar{i}^e(\pi))$, a unique equilibrium solution exists with $e \in (0,\frac{(1-\alpha)\underline{z}}{\alpha i^e})$, $z \in (\underline{z},\frac{\alpha\pi q^*}{\beta})$ and $\kappa_d > 0$.

Case 3. $\frac{\beta z}{\alpha\pi} < q^*$**,** $\kappa_d > 0$**, and** $e = 0$**:**

The above analysis shows that when $i^e \leq \underline{i}^e$, there is no equilibrium solution with $e > 0$. The only possibility with $\frac{\beta z}{\alpha\pi} < q^*$ is when (A.6) is replaced by (A.7) i.e., $e = 0$, where $\bar{z}_1$ is a solution to

$$\frac{\pi-\beta}{\beta} = \Phi(\bar{z}_1,0)$$

(see Step 3 of Case 2). We know from Step 1 in the proof of Proposition 1 (just apply $i^e = 1$ and replace e with z in (A.1)) that there exists a unique solution $\bar{z}_1 \in (0,\frac{\alpha\pi q^*}{\beta})$ if and only if $\pi > \pi^*(1) \in (\beta,\infty)$. Given $z = \bar{z}_1$, $\varphi \geq \beta\mathbb{E}(\kappa)$ is uniquely pinned down by

$$\varphi = \beta\mathbb{E}(\kappa)+\beta\left[\int_0^{\kappa_d}\kappa\left\{u'\left(\beta\left(\frac{\bar{z}_1}{\pi}+\kappa A\right)\right)-1\right\}f(\kappa)d\kappa+\int_{\kappa_d}^{\bar{\kappa}_d}\kappa\left\{u'\left(\frac{\beta\kappa A}{1-\alpha}\right)-1\right\}f(\kappa)d\kappa\right],$$

where $\kappa_d = \frac{(1-\alpha)\bar{z}_1}{\alpha\pi A}$ and $\bar{\kappa}_d = \frac{(1-\alpha)q^*}{\beta A}$.

In sum, for $\frac{\beta z}{\alpha\pi} < q^*$ and $i^e \in [1,\underline{i}^e(\pi)]$, a unique equilibrium solution exists with $e = 0$, $z = \underline{z}_1 \in (0,\frac{\alpha\pi q^*}{\beta})$ and $\kappa_d > 0$ if and only if $\pi > \pi^*(1) \in (\beta,\infty)$.

Case 4. $\frac{\beta z}{\alpha\pi} \geq q^*$**:**

When $\frac{\beta z}{\alpha\pi} \geq q^*$, the first-order conditions together with the market-clearing conditions yield $\frac{\pi-\beta}{\beta} = \Phi(z,e)$ and $\frac{\pi-\beta i^e}{\beta i^e} = X(z,e)$ where

$$\Phi(z,e) = X(z,e) = \int_0^{\bar{\kappa}_d}\left\{u'\left(\beta\left(\frac{z+i^e e}{\pi}+\kappa A\right)\right)-1\right\}f(\kappa)d\kappa$$

where $\bar{\kappa}_d = \frac{q^*-\frac{\beta(z+i^e e)}{\pi}}{\beta a} \geq 0$. This implies that we must have $i^e = 1$, and $z+e$ is determined uniquely, whereas z and e individually are not. Indeed, we know from Step 2 in the proof of

Proposition 1 (just apply $i^e = 1$ and set $e = z + e$ in (A.3)) that a unique equilibrium exists with $z + e \in [\frac{\alpha\pi q^*}{\beta}, \frac{\pi q^*}{\beta})$ if and only if $\pi \leq \pi^*(1)$. Given this solution, $\varphi \geq \beta\mathbb{E}(\kappa)$ is uniquely pinned down by

$$\frac{\varphi}{\beta} - \mathbb{E}(\kappa) = \int_0^{\bar{\kappa}_d} \kappa \left\{ u'\left(\beta\left(\frac{z + i^e e}{\pi} + \kappa A\right)\right) - 1 \right\} f(\kappa) d\kappa.$$

In the limit as $\pi \to \beta$, we have $z + e \to q^*$, leading to $\varphi \to \beta\mathbb{E}(\kappa)$ and $\bar{\kappa}_d \to 0$, the first best.

In sum, for $\frac{\beta z}{\alpha\pi} \geq q^*$ and $i^e = 1$, a unique equilibrium solution exists with $z+e \in [\frac{\alpha\pi q^*}{\beta}, \frac{\pi q^*}{\beta})$ and $\kappa_d > 0$ if and only if $\pi \leq \pi^*(1)$. ■

Proof of Lemma 4

Consider the case of $\underline{i}^e(\pi) < i^e \leq \frac{\pi}{\beta}$ and $\pi > \pi^*(1)$. Differentiating the first-order conditions, (A.5), (A.6), and (A.8), with respect to π, we obtain

$$\begin{pmatrix} \Phi_z & \Phi_e & 0 \\ X_z & X_e & 0 \\ \Psi_z & \Psi_e & -\frac{1}{\beta} \end{pmatrix} \begin{pmatrix} \frac{\partial z}{\partial \pi} \\ \frac{\partial e}{\partial \pi} \\ \frac{\partial \varphi}{\partial \pi} \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta} - \Phi_\pi \\ \frac{1}{\beta i^e} - X_\pi \\ -\Psi_\pi \end{pmatrix}$$

Letting

$$\Lambda^d \equiv \begin{pmatrix} \Phi_z & \Phi_e & 0 \\ X_z & X_e & 0 \\ \Psi_z & \Psi_e & -\frac{1}{\beta} \end{pmatrix},$$

we have

$$\det(\Lambda^d) = \frac{1}{\beta}(\Phi_e X_z - \Phi_z X_e) < 0,$$

since (A.9).

We will make use of the following derivatives:

$$\frac{\partial\Phi(z,e)}{\partial\pi} = -\frac{\beta}{\pi^2}\left[\int_0^{\kappa_d}(z+i^e e)u''\left(\beta\left(\frac{z+i^e e}{\pi}+\kappa A\right)\right)f(\kappa)d\kappa + \int_{\kappa_d}^{\infty}\frac{z}{\alpha}u''\left(\frac{\beta z}{\pi\alpha}\right)f(\kappa)d\kappa\right] > 0, \tag{A.10}$$

$$\frac{\partial\Phi(z,e)}{\partial i^e} = \frac{\beta e}{\pi}\int_0^{\kappa_d}u''\left(\beta\left(\frac{z+i^e e}{\pi}+\kappa A\right)\right)f(\kappa)d\kappa \le 0, \tag{A.11}$$

$$\frac{\partial X(z,e)}{\partial\pi} = -\frac{\beta}{\pi^2}\left[\int_0^{\kappa_d}(z+i^e e)u''\left(\beta\left(\frac{z+i^e e}{\pi}+\kappa A\right)\right)f(\kappa)d\kappa + \int_{\kappa_d}^{\bar\kappa_d}\frac{i^e e}{1-\alpha}u''\left(\frac{\beta(\frac{i^e e}{\pi}+\kappa A)}{1-\alpha}\right)f(\kappa)d\kappa\right] > 0, \tag{A.12}$$

$$\frac{\partial X(z,e)}{\partial i^e} = \frac{\beta e}{\pi}\left[\int_0^{\kappa_d}u''\left(\beta\left(\frac{z+i^e e}{\pi}+\kappa A\right)\right)f(\kappa)d\kappa + \int_{\kappa_d}^{\bar\kappa_d}\frac{1}{1-\alpha}u''\left(\frac{\beta(\frac{i^e e}{\pi}+\kappa A)}{1-\alpha}\right)f(\kappa)d\kappa\right] < 0, \tag{A.13}$$

$$\frac{\partial\Psi(z,e)}{\partial\pi} = -\frac{\beta}{\pi^2}\left[\int_0^{\kappa_d}\kappa(z+i^e e)u''\left(\beta\left(\frac{z+i^e e}{\pi}+\kappa A\right)\right)f(\kappa)d\kappa + \int_{\kappa_d}^{\bar\kappa_d}\frac{\kappa i^e e}{1-\alpha}u''\left(\frac{\beta(\frac{i^e e}{\pi}+\kappa A)}{1-\alpha}\right)f(\kappa)d\kappa\right] > 0, \tag{A.14}$$

$$\frac{\partial\Psi(z,e)}{\partial i^e} = \frac{\beta e}{\pi}\left[\int_0^{\kappa_d}\kappa u''\left(\beta\left(\frac{z+i^e e}{\pi}+\kappa A\right)\right)f(\kappa)d\kappa + \int_{\kappa_d}^{\bar\kappa_d}\frac{\kappa}{1-\alpha}u''\left(\frac{\beta(\frac{i^e e}{\pi}+\kappa A)}{1-\alpha}\right)f(\kappa)d\kappa\right] < 0. \tag{A.15}$$

Further, combining the FOCs and the above result yields

$$\frac{1}{\beta}-\Phi_\pi = \frac{1}{\pi}\left[\int_0^{\kappa_d}\left\{1-\frac{\xi(\frac{z+i^e e}{\pi})}{\frac{z+i^e e}{\pi}+\kappa A}\right\}u'\left(\beta\left(\frac{z+i^e e}{\pi}+\kappa A\right)\right)f(\kappa)d\kappa + \int_{\kappa_d}^{\infty}(1-\xi)u'\left(\frac{\beta z}{\pi\alpha}\right)f(\kappa)d\kappa\right] > 0,$$

$$\begin{aligned}\frac{1}{\beta i^e}-X_\pi = &\int_0^{\kappa_d}\frac{1}{\pi}\left\{1-\frac{\xi(\frac{z+i^e e}{\pi})}{\frac{z+i^e e}{\pi}+\kappa A}\right\}u'\left(\beta\left(\frac{z+i^e e}{\pi}+\kappa A\right)\right)f(\kappa)d\kappa \\ &+\int_{\kappa_d}^{\bar\kappa_d}\frac{1}{\pi}\left\{1-\frac{\xi(\frac{i^e e}{\pi})}{\frac{i^e e}{\pi}+\kappa A}\right\}u'\left(\frac{\beta(\frac{i^e e}{\pi}+\kappa A)}{1-\alpha}\right)f(\kappa)d\kappa > 0,\end{aligned}$$

where $\Phi_\pi \equiv \frac{\partial\Phi}{\partial\pi} > 0$ and $X_\pi \equiv \frac{\partial X}{\partial\pi} > 0$, since $\xi < 1$.

⊙ The effects of π on z: Using Cramer's rule, we have

$$\frac{\partial z}{\partial\pi} = \frac{\det(\Lambda^d_{1\pi})}{\det(\Lambda^d)},$$

where

$$\Lambda^d_{1\pi} \equiv \begin{pmatrix}\frac{1}{\beta}-\Phi_\pi & \Phi_e & 0\\ \frac{1}{\beta i^e}-X_\pi & X_e & 0\\ -\Psi_\pi & \Psi_e & -\frac{1}{\beta}\end{pmatrix},$$

$$\det(\Lambda^d_{1\pi}) = \frac{1}{\beta}\left[\left(\frac{1}{\beta i^e}-X_\pi\right)\Phi_e - \left(\frac{1}{\beta}-\Phi_\pi\right)X_e\right].$$

Note that if $\kappa_d \le 0$, then $\Phi_e = 0$, implying $\det(\Lambda^d_{1\pi}) = -\frac{1}{\beta}\left(\frac{1}{\beta}-\Phi_\pi\right)X_e > 0$ and $\frac{\partial z}{\partial\pi} < 0$. If

$\kappa_d > 0$,

$$\left(\frac{1}{\beta i^e} - X_\pi\right)\Phi_e - \left(\frac{1}{\beta} - \Phi_\pi\right)X_e$$
$$= \frac{\xi}{\pi}\left[\int_0^{\kappa_d}\left\{1 - \frac{\xi(\frac{z+i^e e}{\pi})}{\frac{z+i^e e}{\pi} + \kappa A}\right\}u'\left(\beta\left(\frac{z+i^e e}{\pi} + \kappa A\right)\right)f(\kappa)d\kappa\right]\left[\int_{\kappa_d}^{\bar{\kappa}_d}\frac{\frac{i^e}{\pi}}{\frac{i^e e}{\pi} + \kappa A}u'\left(\frac{\beta(\frac{i^e e}{\pi} + \kappa A)}{1-\alpha}\right)f(\kappa)d\kappa\right]$$
$$+ \frac{\xi}{\pi}\left[\int_{\kappa_d}^{\infty}(1-\xi)u'\left(\frac{\beta z}{\pi\alpha}\right)f(\kappa)d\kappa\right]\left[\int_{\kappa_d}^{\bar{\kappa}_d}\frac{\frac{i^e}{\pi}}{\frac{i^e e}{\pi} + \kappa A}u'\left(\frac{\beta(\frac{i^e e}{\pi} + \kappa A)}{1-\alpha}\right)f(\kappa)d\kappa\right]$$
$$+ \frac{\xi}{\pi}\left[\int_0^{\kappa_d}\frac{\frac{i^e}{\pi}}{\frac{z+i^e e}{\pi} + \kappa A}u'\left(\beta\left(\frac{z+i^e e}{\pi} + \kappa A\right)\right)f(\kappa)d\kappa\right]$$
$$\times\left[\int_{\bar{\kappa}_d}^{\infty}(1-\xi)u'\left(\frac{\beta z}{\pi\alpha}\right)f(\kappa)d\kappa + \int_{\kappa_d}^{\bar{\kappa}_d}\left\{(1-\xi)u'\left(\frac{\beta z}{\pi\alpha}\right) - \left(1 - \frac{\xi(\frac{i^e e}{\pi})}{\frac{i^e e}{\pi} + \kappa A}\right)u'\left(\frac{\beta(\frac{i^e e}{\pi} + \kappa A)}{1-\alpha}\right)\right\}f(\kappa)d\kappa\right] > 0.$$

Note that since $u'\left(\frac{\beta z}{\pi\alpha}\right) > u'\left(\frac{\beta(\frac{i^e e}{\pi}+\kappa A)}{1-\alpha}\right)$ for $\kappa > \kappa_d$, we have

$$(1-\xi)u'\left(\frac{\beta z}{\pi\alpha}\right) > \left(1 - \frac{\xi(\frac{i^e e}{\pi})}{\frac{i^e e}{\pi} + \kappa A}\right)u'\left(\frac{\beta(\frac{i^e e}{\pi} + \kappa A)}{1-\alpha}\right)$$

if $\xi < \xi^* \equiv \dfrac{u'(\frac{\beta z}{\pi\alpha}) - u'\left(\frac{\beta(\frac{i^e e}{\pi}+\kappa A)}{1-\alpha}\right)}{u'(\frac{\beta z}{\pi\alpha}) - \frac{\frac{i^e e}{\pi}}{\frac{i^e e}{\pi}+\kappa A}u'\left(\frac{\beta(\frac{i^e e}{\pi}+\kappa A)}{1-\alpha}\right)}$. Thus, we obtain

$$\frac{\partial z}{\partial \pi} < 0,$$

if ξ is sufficiently small.

$\odot$ The effects of π on e: Using Cramer's rule, we have

$$\frac{\partial e}{\partial \pi} = \frac{\det(\Lambda^d_{2\pi})}{\det(\Lambda^d)},$$

where

$$\Lambda^d_{2\pi} \equiv \begin{pmatrix} \Phi_z & \frac{1}{\beta} - \Phi_\pi & 0 \\ X_z & \frac{1}{\beta i^e} - X_\pi & 0 \\ \Psi_z & -\Psi_\pi & -\frac{1}{\beta} \end{pmatrix},$$
$$\det(\Lambda^d_{2\pi}) = \frac{1}{\beta}\left[X_z\left(\frac{1}{\beta} - \Phi_\pi\right) - \Phi_z\left(\frac{1}{\beta i^e} - X_\pi\right)\right].$$

If $\kappa_d \leq 0$, then $X_z = 0$, implying that $\det(\Lambda^d_{2\pi}) = -\frac{1}{\beta}\Phi_z\left(\frac{1}{\beta i^e} - X_\pi\right) > 0$ and $\frac{\partial e}{\partial \pi} < 0$. If

$\kappa_d > 0$, we get

$$
\begin{aligned}
&X_z\left(\frac{1}{\beta}-\Phi_\pi\right)-\Phi_z\left(\frac{1}{\beta i^e}-X_\pi\right)\\
&=\frac{\xi}{\pi^2}\left[\int_{\kappa_d}^{\infty}u'\left(\frac{\beta z}{\pi\alpha}\right)f(\kappa)d\kappa\right]\left[\int_0^{\kappa_d}\left\{\frac{\pi}{z}-\frac{1+\xi\frac{i^e e}{z}}{\frac{z+i^e e}{\pi}+\kappa A}\right\}u'\left(\beta\left(\frac{z+i^e e}{\pi}+\kappa A\right)\right)f(\kappa)d\kappa\right],\\
&+\frac{\xi}{\pi^2}\left[\int_{\kappa_d}^{\bar{\kappa}_d}\left\{1-\frac{\xi(\frac{i^e e}{\pi})}{\frac{i^e e}{\pi}+\kappa A}\right\}u'\left(\frac{\beta(\frac{i^e e}{\pi}+\kappa A)}{1-\alpha}\right)f(\kappa)d\kappa\right],\\
&\times\left[\int_0^{\kappa_d}\frac{1}{\frac{z+i^e e}{\pi}+\kappa A}u'\left(\beta\left(\frac{z+i^e e}{\pi}+\kappa A\right)\right)f(\kappa)d\kappa+\int_{\kappa_d}^{\infty}\frac{\pi}{z}u'\left(\frac{\beta z}{\pi\alpha}\right)f(\kappa)d\kappa\right]>0,
\end{aligned}
$$

since $\frac{\pi}{z}-\frac{1+\xi\frac{i^e e}{z}}{\frac{z+i^e e}{\pi}+\kappa A}>0 \iff \xi<1+\frac{\pi\kappa A}{i^e e}$. This result implies $\det(\Lambda^d_{2\pi})>0$, which leads to

$$\frac{\partial e}{\partial \pi}=\frac{\det(\Lambda^d_{2\pi})}{\det(\Lambda^d)}<0,$$

for any $\xi\in(0,1)$.

$\odot$ The effects of π on φ: Using Cramer's rule, we have

$$\frac{\partial \varphi}{\partial \pi}=\frac{\det(\Lambda^d_{3\pi})}{\det(\Lambda^d)},$$

where

$$\Lambda^d_{3\pi}\equiv\begin{pmatrix}\Phi_z & \Phi_e & \frac{1}{\beta}-\Phi_\pi\\ X_z & X_e & \frac{1}{\beta i^e}-X_\pi\\ \Psi_z & \Psi_e & -\Psi_\pi\end{pmatrix}.$$

Since $(\frac{1}{\beta}-\Phi_\pi)X_z-(\frac{1}{\beta i^e}-X_\pi)\Phi_z>0$, $\Psi_e<0$, $(\frac{1}{\beta i^e}-X_\pi)\Phi_e-(\frac{1}{\beta}-\Phi_\pi)X_e>0$, $\Psi_z<0$, $\Psi_\pi>0$, and $\Phi_zX_e-\Phi_eX_z>0$, we obtain

$$
\begin{aligned}
\det(\Lambda^d_{3\pi})=&\left\{\left(\frac{1}{\beta}-\Phi_\pi\right)X_z-\left(\frac{1}{\beta i^e}-X_\pi\right)\Phi_z\right\}\Psi_e+\left\{\left(\frac{1}{\beta i^e}-X_\pi\right)\Phi_e-\left(\frac{1}{\beta}-\Phi_\pi\right)X_e\right\}\Psi_z\\
&-\Psi_\pi(\Phi_zX_e-\Phi_eX_z)<0,
\end{aligned}
$$

which leads to

$$\frac{\partial \varphi}{\partial \pi}=\frac{\det(\Lambda^d_{3\pi})}{\det(\Lambda^d)}>0.$$

Similarly, differentiating the first-order conditions, (A.5), (A.6), and (A.8), with respect to i^e, we obtain

$$\begin{pmatrix}\Phi_z & \Phi_e & 0\\ X_z & X_e & 0\\ \Psi_z & \Psi_e & -\frac{1}{\beta}\end{pmatrix}\begin{pmatrix}\frac{\partial z}{\partial i^e}\\ \frac{\partial e}{\partial i^e}\\ \frac{\partial \varphi}{\partial i^e}\end{pmatrix}=\begin{pmatrix}-\Phi_{i^e}\\ -\frac{\pi}{\beta(i^e)^2}-X_{i^e}\\ -\Psi_{i^e}\end{pmatrix}$$

where

$$
\begin{aligned}
&-\frac{\pi}{\beta(i^e)^2}-X_{i^e}=\frac{1}{i^e}\left(-\frac{\pi}{\beta i^e}-i^eX_{i^e}\right)\\
&=-\frac{1}{i^e}\left[\{1-F(\bar{\kappa}_d)\}+\int_0^{\kappa_d}\left\{1-\frac{\frac{\xi i^e e}{\pi}}{\frac{z+i^e e}{\pi}+\kappa A}\right\}u'\left(\beta\left(\frac{z+i^e e}{\pi}+\kappa A\right)\right)f(\kappa)d\kappa\right.\\
&\left.+\int_{\kappa_d}^{\bar{\kappa}_d}\left\{1-\frac{\frac{\xi i^e e}{\pi}}{\frac{i^e e}{\pi}+\kappa A}\right\}u'\left(\frac{\beta(\frac{i^e e}{\pi}+\kappa A)}{1-\alpha}\right)f(\kappa)d\kappa\right]<0.
\end{aligned}
$$

$\odot$ The effects of i^e on z: Using Cramer's rule, we have

$$
\frac{\partial z}{\partial i^e}=\frac{\det(\Lambda^d_{1i^e})}{\det(\Lambda^d)},
$$

where

$$
\Lambda^d_{1i^e}\equiv\begin{pmatrix}-\Phi_{i^e} & \Phi_e & 0\\ -\frac{\pi}{\beta(i^e)^2}-X_{i^e} & X_e & 0\\ -\Psi_{i^e} & \Psi_e & -\frac{1}{\beta}\end{pmatrix},
$$

$$
\det(\Lambda^d_{1i^e})=\frac{1}{\beta}\left[\Phi_{i^e}X_e-\left(\frac{\pi}{\beta(i^e)^2}+X_{i^e}\right)\Phi_e\right]
$$

If $\kappa_d\le 0$, then we have $\Phi_e=\Phi_{i^e}=0$, implying $\det(\Lambda^d_{1i^e})=0$. In this case, a change in i^e does not affect z. If $\kappa_d>0$, we have

$$
\begin{aligned}
&\Phi_{i^e}X_e-\left(\frac{\pi}{\beta(i^e)^2}+X_{i^e}\right)\Phi_e\\
&=-\frac{\beta}{\pi}\left[\int_0^{\kappa_d}u''\left(\beta\left(\frac{z+i^e e}{\pi}+\kappa A\right)\right)f(\kappa)d\kappa\right]\\
&\times\left[\{1-F(\bar{\kappa}_d)\}+\int_0^{\kappa_d}u'\left(\beta\left(\frac{z+i^e e}{\pi}+\kappa A\right)\right)f(\kappa)d\kappa+\int_{\kappa_d}^{\bar{\kappa}_d}u'\left(\frac{\beta(\frac{i^e e}{\pi}+\kappa A)}{1-\alpha}\right)f(\kappa)d\kappa\right]>0.
\end{aligned}
$$

Thus, we obtain $\det(\Lambda^d_{1i^e})>0$ and

$$
\frac{\partial z}{\partial i^e}<0.
$$

$\odot$ The effects of i^e on e: Using Cramer's rule, we have

$$
\frac{\partial e}{\partial i^e}=\frac{\det(\Lambda^d_{2i^e})}{\det(\Lambda^d)},
$$

where

$$
\Lambda^d_{2i^e}\equiv\begin{pmatrix}\Phi_z & -\Phi_{i^e} & 0\\ X_z & -\frac{\pi}{\beta(i^e)^2}-X_{i^e} & 0\\ \Psi_z & -\Psi_{i^e} & -\frac{1}{\beta}\end{pmatrix}.
$$

Since $-\Phi_{i^e} \geq 0$, $X_z \leq 0$, $\frac{\pi}{\beta(i^e)^2} + X_{i^e} > 0$, and $\Phi_z < 0$, then we have

$$\det(\Lambda^d_{2i^e}) = \frac{1}{\beta}\left[-\Phi_{i^e} X_z + \left(\frac{\pi}{\beta(i^e)^2} + X_{i^e}\right)\Phi_z\right] < 0,$$

which implies

$$\frac{\partial e}{\partial i^e} = \frac{\det(\Lambda^d_{2i^e})}{\det(\Lambda^d)} > 0.$$

$\odot$ The effects of i^e on φ: Using Cramer's rule, we have

$$\frac{\partial \varphi}{\partial i^e} = \frac{\det(\Lambda^d_{3i^e})}{\det(\Lambda^d)},$$

where

$$\Lambda^d_{3i^e} \equiv \begin{pmatrix} \Phi_z & \Phi_e & -\Phi_{i^e} \\ X_z & X_e & -\frac{\pi}{\beta(i^e)^2} - X_{i^e} \\ \Psi_z & \Psi_e & -\Psi_{i^e} \end{pmatrix}.$$

Note that

$$\begin{aligned}\Psi_z X_e - \Psi_e X_z = \frac{\beta^2 i^e}{\pi^2}\Bigg[&\left\{\int_0^{\kappa_d} \kappa u''\left(\beta\left(\frac{z+i^e e}{\pi} + \kappa A\right)\right) f(\kappa)d\kappa\right\}\left\{\int_{\kappa_d}^{\bar{\kappa}_d} u''\left(\frac{\beta(\frac{i^e e}{\pi} + \kappa A)}{1-\alpha}\right)\frac{f(\kappa)}{1-\alpha}d\kappa\right\} \\ &- \left\{\int_0^{\kappa_d} u''\left(\beta\left(\frac{z+i^e e}{\pi} + \kappa A\right)\right) f(\kappa)d\kappa\right\}\left\{\int_{\kappa_d}^{\bar{\kappa}_d} \kappa u''\left(\frac{\beta(\frac{i^e e}{\pi} + \kappa A)}{1-\alpha}\right)\frac{f(\kappa)}{1-\alpha}d\kappa\right\}\Bigg].\end{aligned} \tag{A.16}$$

Applying the Simpson's rule to the right-hand side of (A.16) yields:

$$\begin{aligned}&\left\{\int_0^{\kappa_d} \kappa u''\left(\beta\left(\frac{z+i^e e}{\pi} + \kappa A\right)\right) f(\kappa)d\kappa\right\}\left\{\int_{\kappa_d}^{\bar{\kappa}_d} u''\left(\frac{\beta(\frac{i^e e}{\pi} + \kappa A)}{1-\alpha}\right)\frac{f(\kappa)}{1-\alpha}d\kappa\right\} \\ &- \left\{\int_0^{\kappa_d} u''\left(\beta\left(\frac{z+i^e e}{\pi} + \kappa A\right)\right) f(\kappa)d\kappa\right\}\left\{\int_{\kappa_d}^{\bar{\kappa}_d} \kappa u''\left(\frac{\beta(\frac{i^e e}{\pi} + \kappa A)}{1-\alpha}\right)\frac{f(\kappa)}{1-\alpha}d\kappa\right\} \\ &\approx \left\{\frac{(\kappa_d)^2}{6}(B+2C)\right\}\left\{\frac{\bar{\kappa}_d - \kappa_d}{6}(E+4F+G)\right\} - \left\{\frac{\kappa_d}{6}(B+4C+D)\right\}\left\{\frac{\bar{\kappa}_d - \kappa_d}{6}(\bar{\kappa}_d E + 2(\bar{\kappa}_d + \kappa_d)F + \kappa_d G)\right\} \\ &= \frac{\kappa_d(\bar{\kappa}_d - \kappa_d)}{36}[-2(\bar{\kappa}_d - \kappa_d)(B+2C)F - (2C+D)(\bar{\kappa}_d E + 2(\bar{\kappa}_d + \kappa_d)F + \kappa_d G)] < 0,\end{aligned}$$

where $B = u''\left(\beta\left(\frac{z+i^e e}{\pi} + \kappa_d A\right)\right) f(\kappa_d) < 0$, $C = u''\left(\beta\left(\frac{z+i^e e}{\pi} + \frac{\kappa_d}{2}A\right)\right) f\left(\frac{\kappa_d}{2}\right) < 0$, $D = u''(\frac{\beta(z+i^e e)}{\pi})f(0) < 0$, $E = u''\left(\frac{\beta(\frac{i^e e}{\pi} + \bar{\kappa}_d A)}{1-\alpha}\right)\frac{f(\bar{\kappa}_d)}{1-\alpha} < 0$, $F = u''\left(\frac{\beta(\frac{i^e e}{\pi} + \frac{\kappa_d + \bar{\kappa}_d}{2}A)}{1-\alpha}\right) f\left(\frac{\kappa_d + \bar{\kappa}_d}{2}\right) < 0$, and $G = u''\left(\frac{\beta(\frac{i^e e}{\pi} + \kappa_d A)}{1-\alpha}\right)\frac{f(\kappa_d)}{1-\alpha} < 0$. Then, we obtain

$$\Psi_z X_e - \Psi_e X_z < 0.$$

Note also that

$$\Psi_e\Phi_z - \Psi_z\Phi_e = \frac{\beta^2 i^e}{\pi^2}\left[\int_0^{\kappa_d} \kappa u''\left(\beta\left(\frac{z+i^e e}{\pi}+\kappa A\right)\right) f(\kappa)d\kappa\right]\left[\int_{\kappa_d}^{\infty} u''\left(\frac{\beta z}{\pi\alpha}\right)\frac{f(\kappa)}{\alpha}d\kappa\right]$$
$$+\frac{\beta^2 i^e}{\pi^2}\left[\int_{\kappa_d}^{\bar{\kappa}_d} \kappa u''\left(\frac{\beta(\frac{i^e e}{\pi}+\kappa A)}{1-\alpha}\right)\frac{f(\kappa)}{1-\alpha}d\kappa\right]$$
$$\times\left[\int_0^{\kappa_d} u''\left(\beta\left(\frac{z+i^e e}{\pi}+\kappa A\right)\right) f(\kappa)d\kappa + \int_{\kappa_d}^{\infty} u''\left(\frac{\beta z}{\pi\alpha}\right)\frac{f(\kappa)}{\alpha}d\kappa\right] > 0. \tag{A.17}$$

Since $\Phi_{i^e} \leq 0$, $\Psi_z X_e - \Psi_e X_z < 0$ (from (5)), $\frac{\pi}{\beta(i^e)^2} + X_{i^e} > 0$, $\Psi_e\Phi_z - \Psi_z\Phi_e > 0$ (from (A.17)), $\Psi_{i^e} < 0$, and $\Phi_e X_z - \Phi_z X_e < 0$ (from (A.9)), we obtain

$$\det(\Lambda^d_{3i^e}) = \Phi_{i^e}(\Psi_z X_e - \Psi_e X_z) + \left(\frac{\pi}{\beta(i^e)^2} + X_{i^e}\right)(\Psi_e\Phi_z - \Psi_z\Phi_e) + \Psi_{i^e}(\Phi_e X_z - \Phi_z X_e) > 0$$

which leads to

$$\frac{\partial\varphi}{\partial i^e} = \frac{\det(\Lambda^d_{3i^e})}{\det(\Lambda^d)} < 0,$$

since $\det(\Lambda^d) < 0$. ■

Proof of Proposition 4

Consider first the case of $\pi > \pi^*(1)$ and $i^e \in (\underline{i}^e, \bar{i}^e)$. Differentiating the probability of a bank run, $F(\kappa_d)$, with respect to i^e, we obtain

$$\frac{\partial F(\kappa_d)}{\partial i^e} = f(\kappa_d) \times \frac{(1-\alpha)\frac{\partial z}{\partial i^e} - \alpha(e + i^e\frac{\partial e}{\partial i^e})}{\alpha\pi A} < 0,$$

since $\frac{\partial z}{\partial i^e} < 0$ and $\frac{\partial e}{\partial i^e} > 0$, which proves the first part of this proposition.

Next, we obtain the welfare as follows:

$$\mathcal{W}_1^d \equiv \int_0^{[\bar{\kappa}_d]^+}\left[\alpha\left\{u\left(\frac{\beta z}{\alpha\pi}\right) - \frac{\beta z}{\alpha\pi}\right\} + (1-\alpha)\left\{u\left(\frac{\beta(\frac{i^e e}{\pi}+\kappa A)}{1-\alpha}\right) - \frac{\beta(\frac{i^e e}{\pi}+\kappa A)}{1-\alpha}\right\}\right] f(\kappa)d\kappa$$
$$+\int_{[\bar{\kappa}_d]^+}^{\infty}\left[\alpha\left\{u\left(\frac{\beta z}{\alpha\pi}\right) - \frac{\beta z}{\alpha\pi}\right\} + (1-\alpha)\{u(q^*) - q^*\}\right] f(\kappa)d\kappa + \mathbb{E}(\kappa)A,$$

for $i^e \in [\bar{i}^e, \frac{\pi}{\beta}]$, and

$$\mathcal{W}_2^d \equiv \int_0^{[\kappa_d]^+}\left[u\left(\beta\left(\frac{z+i^e e}{\pi}+\kappa A\right)\right) - \beta\left(\frac{z+i^e e}{\pi}+\kappa A\right)\right] f(\kappa)d\kappa$$
$$+\int_{[\kappa_d]^+}^{[\bar{\kappa}_d]^+}\left[\alpha\left\{u\left(\frac{\beta z}{\alpha\pi}\right) - \frac{\beta z}{\alpha\pi}\right\} + (1-\alpha)\left\{u\left(\frac{\beta(\frac{i^e e}{\pi}+\kappa A)}{1-\alpha}\right) - \frac{\beta(\frac{i^e e}{\pi}+\kappa A)}{1-\alpha}\right\}\right] f(\kappa)d\kappa$$
$$+\int_{[\bar{\kappa}_d]^+}^{\infty}\left[\alpha\left\{u\left(\frac{\beta z}{\alpha\pi}\right) - \frac{\beta z}{\alpha\pi}\right\} + (1-\alpha)\{u(q^*) - q^*\}\right] f(\kappa)d\kappa + \mathbb{E}(\kappa)A,$$

for $i^e \in (\underline{i}^e, \bar{i}^e)$. Differentiating these welfare measures with respect to i^e, we obtain

$$\frac{\partial \mathcal{W}_1^d}{\partial i^e} = \frac{\beta}{\pi} \int_0^{[\bar{\kappa}_d]^+} \left(e + i^e \frac{\partial e}{\partial i^e}\right) \left\{ u' \left(\frac{\beta(\frac{i^e e}{\pi} + \kappa A)}{1-\alpha} \right) - 1 \right\} f(\kappa) d\kappa > 0,$$

for $i^e \in [\bar{i}^e, \frac{\pi}{\beta}]$, since $\frac{\partial e}{\partial i^e} > 0$. Recall that z does not depend on i^e in this case since $z = \underline{z}$ (see the proof of Proposition 3). Similarly, we have

$$\begin{aligned}
\frac{\partial \mathcal{W}_2^d}{\partial i^e} &= \frac{\beta}{\pi} \Bigg[\int_0^{[\kappa_d]^+} \left(\frac{\partial z}{\partial i^e} + e + i^e \frac{\partial e}{\partial i^e} \right) \left\{ u' \left(\beta \left(\frac{z + i^e e}{\pi} + \kappa A \right) \right) - 1 \right\} f(\kappa) d\kappa \\
&+ \int_{[\kappa_d]^+}^{[\bar{\kappa}_d]^+} \left(e + i^e \frac{\partial e}{\partial i^e} \right) \left\{ u' \left(\frac{\beta(\frac{i^e e}{\pi} + \kappa A)}{1-\alpha} \right) - 1 \right\} f(\kappa) d\kappa + \int_{[\kappa_d]^+}^{\infty} \frac{\partial z}{\partial i^e} \left\{ u' \left(\frac{\beta z}{\alpha \pi} \right) - 1 \right\} f(\kappa) d\kappa \Bigg], \\
&= \frac{\beta}{\pi} \left[\frac{\pi - \beta}{\beta} \frac{\partial z}{\partial i^e} + \frac{\pi - \beta i^e}{\beta i^e} \left(e + i^e \frac{\partial e}{\partial i^e} \right) \right],
\end{aligned}$$

for $i^e \in (\underline{i}^e, \bar{i}^e)$, where the last equality uses the first-order conditions. Since $\frac{\partial z}{\partial i^e} < 0$ and $\frac{\partial e}{\partial i^e} > 0$, the effect of an increase in i^e on welfare $\mathcal{W}_2^d$ is ambiguous, which proves the second part of this proposition. ■

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