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Unbalanced Growth, Elasticity of Substitution, and Land Overvaluation

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Abstract

We study the long-run behavior of land prices when land plays the dual role of factor of production and store of value. In modern economies where technological progress is faster in non-land sectors, when the elasticity of substitution in production exceeds 1 at high input levels (which always holds if non-land factors do not fully depreciate), unbalanced growth occurs and land becomes overvalued on the long-run trend relative to the fundamental value defined by the present value of land rents. Around the trend, land prices exhibit recurrent stochastic fluctuations, with expansions and contractions in the size of land overvaluation.

Keywords: asset price, elasticity of substitution, land, unbalanced growth.

JEL codes: D53, G12, O41.

1 Introduction

As economies develop and per capita incomes rise, the importance of land as a factor of production diminishes.[1] This is partly because people face bio-

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[1]Echevarria (1997, Figure 2) documents that the GDP share of agriculture is lower in countries with higher per capita incomes. She also notes that the employment share

logical constraints regarding the amount of food they can consume (where land produces agricultural products) or the amount of leisure time they can spend (where land produces amenities like tennis courts and national parks). Although people living in modern capitalistic societies have tremendously benefited from technological progress over the past decades such as the development of computers, Internet, smartphones, and electric vehicles, introspection suggests that our dining and outdoor experiences—the quality of "land-intensive products"—have not changed much. On the other hand, land also has an important role as a scarce means of savings and has significant value as a financial asset.[2]

Land has a few characteristics that make it suitable as a store of value compared to other means of savings such as gold or cryptocurrency. First, unlike cryptocurrency, land has an intrinsic value because it can be used as a factor of production in agriculture, construction, housing, and leisure. Second, unlike gold (which is chemically homogeneous), each land parcel is immobile and unique and hence property rights are well-defined, which makes it difficult to steal. Third, relative to durable goods such as vehicles, land is more durable as it cannot be destroyed absent natural disasters, sea level rise, and pollution.

This paper theoretically studies the long-run behavior of land prices in modern economies where the importance of land as a factor of production diminishes, yet, land remains to play a significant role as a store of value. In a plausible economic model with land and aggregate risk, we establish a theorem—Land Overvaluation Theorem—showing the tight link between unbalanced productivity growth, elasticity of substitution between production factors, and overvaluation of land, meaning that the equilibrium land price exceeds its fundamental value defined by the present value of land rents.

To illustrate the key mechanism of how land overvaluation emerges on the long-run trend, we first present two example models. Throughout this

of agriculture decreases with incomes, both across countries and time. Acemoglu (2009, p. 698, Figure 20.1) documents that the employment share of agriculture in U.S. has declined from about 80% to below 5% over the past 200 years.

[2] According to OECD (2022, Figure 2.1), among 29 OECD countries, real estate (owner-occupied housing and secondary real estate) comprises more than 50% of household wealth in 27 countries. See also https://www.oecd.org/housing/policy-toolkit/data-dashboard/wealth-distribution/.

paper, we employ a standard two-period overlapping generations (OLG) model with land, where land plays the dual role of factor of production and store of value. The two-period OLG model is the simplest model with heterogeneous agents—there are just the young and the old. It illustrates speculative behaviour of heterogeneous agents trading assets with each other—individuals buying an asset largely on the basis of beliefs (here assumed to be rational) of what they can sell it for. The basis of heterogeneity is that the young buy land from the old, in anticipation of receiving rents and selling land to the next generation before exiting the market.

In the first example, we consider a two-sector growth economy where one ("tech") sector uses skilled labor (human capital) as the primary input for production such as technology, finance, and information and communication, while the other ("land" sector) uses unskilled labor and land as the primary inputs such as agriculture and construction. Specifically, the production function is linear in the tech sector and Cobb-Douglas in the land sector. There may or may not be labor mobility between different sectors and we examine both cases. Importantly and realistically, productivity growth rates are different across the two sectors. Unlike standard models with balanced growth, we suppose that the tech sector will eventually grow faster, exhibiting unbalanced growth. In this setting, land prices will grow, pulled by the savings motive of the young and the productivity growth in the tech sector. On the other hand, land rents will not grow as fast because productivity growth is lower in the land sector. This implies that the land price will eventually exceed the present value of land rents (its fundamental value), generating land overvaluation, and a backward induction argument shows that land is always overvalued.

The second example is a one-sector growth economy. To illustrate the role of the elasticity of substitution, we employ a standard constant elasticity of substitution (CES) production function where labor and land are used as inputs, with factor-augmenting technological progress. As in the first example, we assume that labor productivity grows faster than land productivity, which generates land overvaluation. The intuition is as follows. Along the equilibrium path, the land price increases together with wages, whose growth rate will be the same as labor productivity growth. On the other hand, the growth rate of land rents will be suppressed if the elasticity of substitution between land and labor exceeds 1, in which case the price-rent ratio will rise and land will be overvalued. To be precise, the ratio increases over time depending on whether the elasticity of substitution exceeds 1 at high input levels, not necessarily globally.

Motivated by these examples, we consider an abstract stochastic overlapping generations model with land and establish the Land Overvaluation Theorem. We identify economic conditions under which land overvaluation will necessarily emerge in equilibrium. Let us denote the labor and land productivities at time t by A_{Ht} and A_{Xt}, respectively. Let us also denote by σ (a lower bound of) the elasticity of substitution between land and labor at sufficiently high input levels and assume $\sigma > 1$. The main result of this paper, Theorem 1, shows that if

$$\mathrm{E}_0 \sum_{t=0}^{\infty} (A_{Ht}/A_{Xt})^{1/\sigma-1} < \infty,$$

then land is overvalued in equilibrium. Noting that $\sigma > 1$ and hence $1/\sigma - 1 < 0$, this land overvaluation condition holds whenever labor productivity A_{Ht} grows faster than land productivity A_{Xt} in the long run, i.e., unbalanced growth occurs. The intuition is the same as that for the two examples just described, which are special cases of this theorem. In Section 3.2, we justify our assumption of $\sigma > 1$ in several ways based on both empirical and theoretical grounds. To the best of our knowledge, this theorem is the first that proves land overvaluation in an economy with aggregate uncertainty.

There are three important implications to be drawn from our Land Overvaluation Theorem. First, our analysis illustrates the key mechanism of how land overvaluation emerges, where unbalanced growth and elasticity of substitution play a crucial role. To our knowledge, the link between the elasticity of substitution and asset overvaluation is new.[3] Second, unlike the usual perspective on land overvaluation (sometimes called land bub-

[3]The importance of the *intertemporal* elasticity of substitution in macro-finance models is well known (Bansal and Yaron, 2004; Pohl, Schmedders, and Wilms, 2018). The analogy here is only superficial because (i) the relevant elasticity of substitution in our model is between production factors in the production function, not between consumption in different periods in the utility function, and (ii) macro-finance models typically assume outright that the asset price equals its fundamental value.

bles) as short-run phenomena with boom-bust cycles, our analysis shows land overvaluation on the long-run trend in the process of economic development. In reality, as economies develop, structural transformation occurs from the land-intensive agricultural economy to the labor- or knowledge-intensive economy. During this transition, while the importance of land as a factor of production would diminish, as long as land remains important as a store of value, land necessarily becomes overvalued. Third, our theorem in an economy with aggregate risk also provides a new insight on short-term fluctuations that deviate from the long-run trend. When productivities of the economy swing up and down, the land price also fluctuates. In standard asset pricing models, these valuations and fluctuations always reflect fundamentals. In contrast, our analysis shows that land is always overvalued, associated with expansions and contractions in the size of overvaluation that may appear to be the emergence and collapse of large land bubbles. Our model provides a theoretical foundation for recurrent stochastic bubbles.

1.1 Related literature

As in McCallum (1987), Hansen and Prescott (2002), Mountford (2004), and Stiglitz (2015), we employ a standard two-period OLG model with land where land plays the dual role of factor of production and store of value. Our paper is different because we focus on asset pricing, unbalanced growth, and land overvaluation.

As in Lucas (1978) and the large subsequent literature, we study asset pricing in an economy with aggregate uncertainty. In this literature, it is well known that there is a fundamental difficulty in generating asset overvaluation (sometimes called asset bubbles) in dividend-paying assets including the Lucas (1978) tree model, even if dividends are slightly positive.[4] Since land in our paper is used as a factor of production yielding positive rents, land may be interpreted as a variant of the Lucas tree. In this sense, our paper identifies conditions under which the tree is necessarily overvalued as the unique equilibrium outcome.[5] Perhaps because of

[4] See, for instance, Santos and Woodford (1997, Theorem 3.3) for details.

[5] Of course, it is well known since Samuelson (1958) and Bewley (1980) that for zero-dividend assets like fiat money, overvaluation may occur, in which case there usually

this difficulty, progress in macro-finance models that describe realistic asset overvaluation in stocks, land, and housing has been slow. Our paper contributes towards this direction.

Concerning unbalanced growth, Baumol (1967) points out the implications for economic development when different sectors have different productivity growth rates. Hansen and Prescott (2002) consider a two-sector OLG model with uneven productivity growth rates across the capital-intensive (Solow) sector and the land-intensive (Malthus) sector and argue that land becomes unimportant as a factor of production as the economy develops. Acemoglu and Guerrieri (2008) show in a two-sector general equilibrium model that differences in factor proportions across different sectors combined with capital deepening leads to unbalanced growth. The elasticity of substitution between the two sectors play a key role for growth dynamics. Matsuyama (1992), Buera and Kaboski (2012), Boppart (2014), and Fujiwara and Matsuyama (2022) use non-homothetic preferences to generate unbalanced growth. A crucial difference between our work and this literature is that we show the tight theoretical link between unbalanced growth, elasticity of substitution, and land overvaluation, while the literature abstracts from asset pricing.

Concerning land overvaluation, several papers such as Fostel and Geanakoplos (2012, 2016) argue that land or housing can be overvalued under incomplete markets and financial innovation because they serve as collateral that can be seized upon default by borrowers. Our model is different because markets are complete and frictionless. In the model of Kocherlakota (2013), which builds on Kocherlakota (1992), land is intrinsically useless but may have a positive value (hence be overvalued) because it is a scarce means of savings under a low interest rate environment. However, our model has substantial differences. First, in Kocherlakota (2013), land is intrinsically useless, which leads to equilibrium indeterminacy including equilibria in which land is worthless. In contrast, in our model, land is a productive asset used as an input and hence necessarily has a positive price, with the size of overvaluation fluctuating with productivities in the unique

exist a continuum of monetary equilibria. Our paper is about asset overvaluation in models of a Lucas tree type asset with positive dividends, not about pure bubbles with no dividends, which are fundamentally different.

equilibrium. Second, and more importantly, we highlight the importance of unbalanced growth and elasticity of substitution for generating land overvaluation, whose relevance is obscured in Kocherlakota (2013) because land rents are zero and he focuses on the steady state.

2 An example two-sector OLG model

To clearly present the tight connection between unbalanced growth and land overvaluation, we start the discussion with a simple two-sector overlapping generations (OLG) model that admits a unique equilibrium in closed-form.

2.1 Model

Time is discrete, runs forever, and is indexed by $t = 0, 1, \dots$.

Preferences At each time t, a constant mass H of agents are born, who live for two periods and derive utility

$$(1-\beta)\log y_t + \beta \log z_{t+1} \tag{2.1}$$

from consumption (y_t, z_{t+1}) when young and old. Each period, the young are endowed with one unit of labor, while the old are not. Masses H_1 and $H_2 = H - H_1$ of agents are skilled and unskilled.

At $t = 0$, there is a mass H of initial old agents who only care about their consumption z_0. The initial old is endowed with a unit supply of land, which is durable and non-reproducible.

Technologies There are two production sectors denoted by $j = 1, 2$. Sector 1 is a knowledge-intensive industry such that skilled labor is the primary input for production, such as technology, finance, and information and communication. Sector 2 is a land-intensive industry such that both unskilled labor and land are inputs for production, such as agriculture and construction. The time t production function of sector j is $F_{jt}(H, X)$, where H, X denote the labor and land inputs. For simplicity, we suppose that technologies in sectors 1 and 2 are linear and Cobb-Douglas, respectively:

$$F_{1t}(H,X) = A_{1t}H, \tag{2.2a}$$

$$F_{2t}(H,X) = A_{2t}H^{\alpha}X^{1-\alpha}, \tag{2.2b}$$

where $A_{jt} > 0$ denotes the total factor productivity in sector j at time t and $\alpha \in (0,1)$ is the labor share of sector 2.

Equilibrium As usual, the competitive equilibrium is characterized by utility maximization, profit maximization, and market clearing. Without loss of generality, assume that each sector has one representative firm. Firm j chooses labor and land inputs H, X to maximize the profit

$$F_{jt}(H,X) - w_{jt}H - r_tX, \tag{2.3}$$

where w_{jt} denotes the wage in sector j. Note that because the two sectors employ different types of labor (skilled and unskilled), the wages differ. Noting that labor supply is exogenous at (H_1, H_2) and land supply is 1, using the functional form of the production functions (2.2), profit maximization implies the wages and rent

$$w_{1t} = A_{1t}, \tag{2.4a}$$

$$w_{2t} = \alpha A_{2t}H_2^{\alpha-1}, \tag{2.4b}$$

$$r_t = (1-\alpha)A_{2t}H_2^{\alpha}. \tag{2.4c}$$

Define the aggregate labor income by

$$w_t := w_{1t}H_1 + w_{2t}H_2 = A_{1t}H_1 + \alpha A_{2t}H_2^{\alpha}. \tag{2.5}$$

Because agents have identical homothetic preferences, demand aggregation holds and the aggregate consumption of the young and the old (y_t, z_{t+1}) and land holdings x_t maximize utility (2.1) subject to the budget constraints

$$\text{Young:} \quad y_t + P_tx_t = w_t, \tag{2.6a}$$

$$\text{Old:} \quad z_{t+1} = (P_{t+1} + r_{t+1})x_t, \tag{2.6b}$$

where P_t is the land price and we choose the consumption good as the numéraire. Clearly, the two budget constraints (2.6) can be combined into one as

$$y_t + \frac{1}{R_t} z_{t+1} = w_t, \tag{2.7}$$

where $R_t := (P_{t+1} + r_{t+1})/P_t$ denotes the gross risk-free rate between time t and $t+1$. Applying the familiar Cobb-Douglas formula to the combined budget constraint (2.7), the consumption of the young is $y_t = (1-\beta)w_t$. Using the budget constraint of the young (2.6a) and the land market clearing condition $x_t = 1$ (the old exit the economy so the young must hold the entire land), we obtain the land price

$$P_t = P_t x_t = \beta w_t = \beta(A_{1t}H_1 + \alpha A_{2t}H_2^\alpha). \tag{2.8}$$

Therefore we obtain the following proposition.

Proposition 1. *There exists a unique equilibrium, which is characterized by* (2.4)*,* (2.5)*,* (2.8)*, and* $y_t = (1-\beta)w_t$*.*

2.2 Unbalanced growth and land overvaluation

We now study conditions under which land is overvalued. For simplicity, suppose that productivity growth is exponential, so $A_{jt} = G_j^t$ for some $G_j > 0$. Using (2.4c) and (2.8), both the land price and rent grow exponentially:

$$\begin{aligned} P_t &= \beta(G_1^t H_1 + \alpha G_2^t H_2^\alpha), \\ r_t &= (1-\alpha)G_2^t H_2^\alpha. \end{aligned}$$

Therefore if $G_1 > G_2$, then the land price grows at a faster rate than the rent and the dividend yield decreases, which suggests that land is overvalued.

We make this argument more formal. Define the gross risk-free rate between time $t-1$ and t by the return on land

$$R_{t-1} = \frac{P_t + r_t}{P_{t-1}} = \frac{\beta G_1^t H_1 + (\beta\alpha + 1 - \alpha)G_2^t H_2^\alpha}{\beta G_1^{t-1} H_1 + \beta\alpha G_2^{t-1} H_2^\alpha}. \tag{2.9}$$

Define the date-0 price of consumption delivered at time t (the price of a zero-coupon bond with face value 1 and maturity t) by $q_t = 1/\prod_{s=0}^{t-1} R_s$, with the normalization $q_0 = 1$. The fundamental value of land at time t is defined by the present value of rents

$$V_t := \frac{1}{q_t}\sum_{s=1}^{\infty} q_{t+s} r_{t+s}. \tag{2.10}$$

We say that land is *overvalued* if $P_t > V_t$. We obtain the following proposition.

Proposition 2. *Land is overvalued ($P_t > V_t$) if $G_1 > G_2$.*

Proof. See Corollary 2 below. □

The intuition for Proposition 2 is as follows. In this economy, land serves as a store of value as well as a factor of production. Because agents have labor income only when young, there is a strong savings motive for retirement, which pushes up the land price and hence $P_t \sim G_1^t$. When $G_1 > G_2$, the gross risk-free rate R_{t-1} in (2.9) converges to G_1 as $t \to \infty$. Consequently, we have the order of magnitude $q_{t+s}/q_t \sim G_1^{-s}$ and $r_{t+s} \sim G_2^{t+s}$, so a straightforward calculation yields $V_t \sim G_2^t$. Therefore $P_t > V_t$ for large enough t, and a backward induction argument shows $P_t > V_t$ for all t.

2.3 Two variants

The previous example is arguably highly stylized as labor supply in both sectors is exogenous and one of the production functions is linear. This section presents two other variants that give rise to land overvaluation.

Example 1. This example is a simplified version of the model of Hansen and Prescott (2002), where we abstract from capital. The production functions are the same as in (2.2), but labor is homogeneous (with aggregate supply normalized to 1) and mobile between the two sectors. Letting w_t be the (common) wage and H_{jt} be the labor demand in sector j, profit maximization implies

$$A_{1t} = w_t = \alpha A_{2t} H_{2t}^{\alpha-1} \iff H_{2t} = (\alpha A_{2t}/A_{1t})^{\frac{1}{1-\alpha}}, \tag{2.11}$$

where we assume $A_{1t} > \alpha A_{2t}$ to guarantee an interior solution. Using (2.4c), the land rent is

$$r_t = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}(A_{2t}/A_{1t}^{\alpha})^{\frac{1}{1-\alpha}}.$$

Therefore if $A_{jt} = G_j^t$, then the dividend yield on land is

$$\frac{r_t}{P_t} = \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{\beta}(G_2/G_1)^{\frac{t}{1-\alpha}},$$

which geometrically decays to 0 if $G_1 > G_2$. By Corollary 3 below, land is overvalued.

Example 2. There is only one sector with a constant elasticity of substitution (CES) aggregate production function

$$F_t(H,X) = \left(\alpha(A_{Ht}H)^{1-\rho} + (1-\alpha)(A_{Xt}X)^{1-\rho}\right)^{\frac{1}{1-\rho}}, \tag{2.12}$$

where $\alpha \in (0,1)$ is a parameter, $\sigma = 1/\rho$ is the elasticity of substitution between labor and land, and (A_{Ht}, A_{Xt}) are factor-augmenting productivities. Without loss of generality, normalize the labor and land supply as $(H,X) = (1,1)$. Then a straightforward calculation yields the rent-wage ratio

$$\frac{r_t}{w_t} = \frac{1-\alpha}{\alpha}(A_{Xt}/A_{Ht})^{1-\rho}.$$

As before, the land price satisfies $P_t = \beta w_t$, so the dividend yield on land is

$$\frac{r_t}{P_t} = \frac{1-\alpha}{\beta\alpha}(A_{Xt}/A_{Ht})^{1-1/\sigma} = \frac{1-\alpha}{\beta\alpha}(G_X/G_H)^{(1-1/\sigma)t},$$

where the last equality assumes $(A_{Ht}, A_{Xt}) = (G_H^t, G_X^t)$. Therefore if the elasticity of substitution σ between land and labor exceeds 1 and $G_H > G_X$, then the dividend yield geometrically decays to 0. By Corollary 4 below, land is overvalued.

In what follows, following Baumol (1967), we refer to a situation with uneven productivity growth between different sectors or different production factors as "unbalanced growth". When the economy features multiple sectors as in reality, there is no reason to expect equal growth rates across sectors. The slightest introspection suggests that it would be a miracle if the rate of technological progress were the same in 19th century trains and (horse-drawn) carriages, 20th century computers and calculators, or early 21st century electric vehicle batteries and internal combustion engines. Although models with unbalanced growth are not so common in economics, unbalanced growth is a natural and general feature in the process of economic development.

3 Substitution elasticity and land overvaluation

The examples in Section 2 suggest that unbalanced growth and land overvaluation may be closely related. This section confirms this conjecture in a general setting and highlights the role of the elasticity of substitution between land and other production factors.

3.1 Model

We consider a stochastic two-period overlapping generations model. Uncertainty is resolved according to a filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$ over a probability space $(\Omega, \mathcal{F}, P)$. We denote conditional expectations by $\mathrm{E}_t[\cdot] = \mathrm{E}[\cdot \mid \mathcal{F}_t]$.

Preferences Agents born at time t have the Cobb-Douglas utility

$$(1-\beta)\log y_t + \beta\,\mathrm{E}_t[\log z_{t+1}], \tag{3.1}$$

where $\beta \in (0,1)$. There are two factors of production, labor and land (denoted by H, X), both of which are in unit supply. As in Section 2, only the young are endowed with labor, and the initial old is endowed with land, which is durable and non-reproducible.

Technologies Without loss of generality, we only specify the aggregate production function, as it is well known that if each sector or firm is competitive and markets are frictionless, profit maximization at the individual and aggregate level are equivalent. (See Corollary 3.) Below, we say that a production function $F(H,X)$ is *neoclassical* if $F : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ is homogeneous of degree 1, concave, continuously differentiable, and satisfies $F_H, F_X > 0$.

Assumption 1. *The time t aggregate production function takes the form*

$$F_t(H,X) = F(A_{Ht}H, A_{Xt}X),$$

where F is a neoclassical production function and $A_{Ht}, A_{Xt} > 0$ are $\mathcal{F}_t$-measurable factor-augmenting productivities.

Equilibrium The definition of a competitive equilibrium is standard.

Definition 1. A competitive equilibrium consists of adapted processes of prices $\{(P_t, r_t, w_t)\}_{t=0}^{\infty}$, allocations $\{(x_t, y_t, z_t)\}_{t=0}^{\infty}$, and factor inputs $\{(H_t, X_t)\}_{t=0}^{\infty}$ such that,

(i) (Utility maximization) (x_t, y_t, z_{t+1}) maximizes utility (3.1) subject to the budget constraints (2.6),

(ii) (Profit maximization) (H_t, X_t) maximizes the profit $F_t(H_t, X_t) - w_t H_t - r_t X_t$,

(iii) (Market clearing) $H_t = 1$, $X_t = 1 = x_t$, and $y_t + z_t = F_t(H_t, X_t)$.

Note that the market clearing condition $x_t = 1$ follows because the old exit the economy and the young must buy the entire land. Due to log utility, the existence and uniqueness of equilibrium are immediate.

Proposition 3. *If Assumption 1 holds, then the economy has a unique equilibrium, which is characterized by the following equations.*

Wage:	$w_t = F_H(A_{Ht}, A_{Xt})A_{Ht},$	(3.2a)
Rent:	$r_t = F_X(A_{Ht}, A_{Xt})A_{Xt},$	(3.2b)
Land price:	$P_t = \beta w_t,$	(3.2c)
Young consumption:	$y_t = (1-\beta)w_t,$	(3.2d)
Old consumption:	$z_t = \beta w_t + r_t.$	(3.2e)

3.2 Elasticity of substitution

As Example 2 suggests, the elasticity of substitution plays a crucial role in generating land overvaluation. Recall that the elasticity of substitution σ between production factors is defined by the percentage change in relative factor inputs with respect to the percentage change in relative factor prices

$$\sigma = -\frac{\partial \log(H/X)}{\partial \log(w/r)}, \tag{3.3}$$

where the derivative is taken along the production possibility frontier $F(H, X) =$ constant. A mathematically more convenient way to define the elasticity of substitution is the following. Let $h = \log(H/X)$ be the log relative inputs. Then noting that $w = F_H$ and $r = F_X$, (3.3) can be rewritten as

$$\rho(H, X) := \frac{1}{\sigma(H, X)} = -\frac{\partial \log(F_H/F_X)}{\partial h}, \tag{3.4}$$

where we set $(H, X) = (Xe^h, X)$ to compute the derivative and substitute $h = \log(H/X)$. The following lemma provides an explicit formula for the elasticity of substitution of a neoclassical production function.

Lemma 3.1. *Let F be a neoclassical production function. Then its elasticity of substitution $\sigma_F(H, X)$ satisfies*

$$\sigma_F = \frac{F_H F_X}{F F_{HX}}. \tag{3.5}$$

To derive asset pricing implications, we restrict the elasticity of substitution as follows.

Assumption 2. *The elasticity of substitution of the neoclassical production function F exceeds 1 at high input levels:*

$$\liminf_{H \to \infty} \sigma_F(H, 1) > \sigma > 1.$$

We justify the economic relevance of Assumption 2 in several ways.

The first justification is empirical. Epple, Gordon, and Sieg (2010) find that the elasticity of substitution between land and non-land factors for producing housing service is 1.16 for residential properties and 1.39 for commercial properties in Allegheny County, Pennsylvania. Ahlfeldt and McMillen (2014) argue that the estimation approach of Epple, Gordon, and Sieg (2010) is less susceptible to measurement error than the old estimates, which are likely biased downwards. They find that the elasticity of substitution is around 1.25 for Chicago and Berlin.

The second justification is the pathological behavior of interest rates with $\sigma < 1$. To see this, suppose $\sigma < 1$ in Example 2. Using (2.9) and (3.2), we can bound the gross risk-free rate from below as

$$\begin{aligned} R_{t-1} &= \frac{\beta w_t + r_t}{\beta w_{t-1}} \geq \frac{r_t}{\beta w_{t-1}} \\ &= \frac{1-\alpha}{\alpha\beta}\left(\frac{\alpha G_H^{(1-\rho)t} + (1-\alpha)G_X^{(1-\rho)t}}{\alpha G_H^{(1-\rho)(t-1)} + (1-\alpha)G_X^{(1-\rho)(t-1)}}\right)^{\frac{\rho}{1-\rho}} \frac{G_X^{(1-\rho)t}}{G_H^{(1-\rho)(t-1)}} \\ &= \frac{1-\alpha}{\alpha\beta}\left(\frac{\alpha(G_H/G_X)^{(1-\rho)t} + 1-\alpha}{\alpha(G_H/G_X)^{(1-\rho)(t-1)} + 1-\alpha}\right)^{\frac{\rho}{1-\rho}} G_H^{1-\rho}G_X^{\rho}(G_H/G_X)^{(\rho-1)t}, \end{aligned}$$

which tends to ∞ as $t \to \infty$ because $G_H > G_X$ and $\rho > 1$. An interest rate diverging to infinity is counterfactual and pathological.

The third justification is that when the marginal product of labor is bounded away from zero, the elasticity of substitution necessarily exceeds 1 at high input levels, as the following lemma shows.

Lemma 3.2. *Let F be a neoclassical production function with $\lim_{H\to\infty} F_H(H,1) = m > 0$. Then*

$$\liminf_{H\to\infty} \sigma_F(H,1) \geq 1.$$

Example 3. To illustrate Lemma 3.2, for parameters $A, B > 0$, $\alpha \in (0,1)$, and $\rho > 0$, consider the neoclassical production function

$$F(H,X) = A\left(\alpha H^{1-\rho} + (1-\alpha)X^{1-\rho}\right)^{\frac{1}{1-\rho}} + BH. \tag{3.6}$$

This functional form is common in applied works. For instance, H could be capital and $B = 1-\delta$ could be the fraction remaining after depreciation. Alternatively, (3.6) can be thought of as a generalization of the main model in Section 2 by identifying A as $A_2H_2^{\alpha}$ and B as A_1H_1. To simplify notation, let

$$Y = \left(\alpha H^{1-\rho} + (1-\alpha)X^{1-\rho}\right)^{\frac{1}{1-\rho}}.$$

Then

$$\begin{aligned}
F &= AY + BH, \\
F_H &= A\alpha Y^\rho H^{-\rho} + B, \\
F_X &= A(1-\alpha)Y^\rho X^{-\rho}, \\
F_{HX} &= \rho A\alpha(1-\alpha)Y^{2\rho-1}H^{-\rho}X^{-\rho}.
\end{aligned}$$

Applying Lemma 3.1, the elasticity of substitution becomes

$$\begin{aligned}
\sigma = \frac{F_H F_X}{F F_{HX}} &= \frac{(A\alpha Y^\rho H^{-\rho} + B)(A(1-\alpha)Y^\rho X^{-\rho})}{(AY+BH)\rho A\alpha(1-\alpha)Y^{2\rho-1}H^{-\rho}X^{-\rho}} \\
&= \frac{1}{\rho}\frac{1+\frac{B}{A\alpha}(H/Y)^\rho}{1+\frac{B}{A}(H/Y)}.
\end{aligned}$$

Clearly, as $H \to \infty$ we have

$$\frac{Y}{H} = \left(\alpha + (1-\alpha)(X/H)^{1-\rho}\right)^{\frac{1}{1-\rho}} \to \begin{cases} \alpha^{\frac{1}{1-\rho}} & \text{if } \rho < 1, \\ 0 & \text{if } \rho \geq 1, \end{cases}$$

where the case $\rho = 1$ follows because $Y = H^\alpha X^{1-\alpha}$. Therefore

$$\sigma(H,1) \to \begin{cases} 1/\rho & \text{if } \rho < 1, \\ 1/\alpha & \text{if } \rho = 1, \\ \infty & \text{if } \rho > 1 \end{cases}$$

as $H \to \infty$. In all cases, we have $\liminf_{H\to\infty} \sigma(H,1) > 1$, satisfying Assumption 2.

3.3 Unbalanced growth and land overvaluation

We now extend the land overvaluation result in Proposition 2 to a general setting. The following theorem is the main result of this paper.

Theorem 1 (Land Overvaluation)**.** *Suppose Assumptions 1, 2 hold and*

$$\mathrm{E}_0 \sum_{t=0}^{\infty} (A_{Ht}/A_{Xt})^{1/\sigma-1} < \infty \tag{3.7}$$

almost surely. Then land is overvalued in equilibrium.

The proof of Theorem 1 is deferred to the appendix. The condition (3.7) can be understood as follows. Suppose for simplicity that $A_{Ht} = G_H^t$ and $A_{Xt} = G_X^t$, so productivity growth is exponential. Then the t-th term in the sum (3.7) is $(G_H/G_X)^{(1/\sigma-1)t}$, which is summable if $\sigma > 1$ and $G_H > G_X$. Thus condition (3.7) roughly says that labor productivity growth is higher than land productivity growth in the long run. The intuition for Theorem 1 is similar to the one noted in the introduction, so we do not repeat it. It is important to note that since the equilibrium is unique by Proposition 3, under the conditions in Theorem 1, there are no equilibria in which the land price equals its fundamental value.

Theorem 1 has three important implications. First, it clarifies the role of unbalanced growth and elasticity of substitution for generating land overvaluation, which was previously overlooked. Regarding the assumption of elasticity of substitution between land and non-land exceeding 1, we justify it on empirical and theoretical grounds as discussed in Section 3.2.

Second, we can derive a new insight on the long-run behavior of land prices in a modern economy. The conventional view is that on the long-run trend, the land price should reflect its fundamental value, even if it may deviate from the fundamental value temporarily. In sharp contrast with this widely-held view, Theorem 1 implies that during the process of economic development characterized by unbalanced productivity growth, land overvaluation will naturally and necessarily arise.[6]

Before discussing the third implication in Section 3.4, we show that all examples in Section 2 are special cases of Theorem 1.

Corollary 2. *Proposition 2 is true.*

Proof. Define the aggregate production function by

$$F_t(H, X) = A_{1t}H_1H + A_{2t}(H_2H)^\alpha X^{1-\alpha},$$

[6]Of course, if $G_H = G_X$ holds in the long run, which is often assumed to ensure a balanced growth path in the growth literature, there will be no overvaluation in land prices but this is obviously a knife-edge case.

where $H_1, H_2 > 0$ are constants. Define

$$F(H, X) = H_2 H + H_2^\alpha H^\alpha X^{1-\alpha},$$
$$(A_{Ht}, A_{Xt}) = (A_{1t}, (A_{2t}/A_{1t}^\alpha)^{\frac{1}{1-\alpha}}).$$

Then clearly $F_t(H, X) = F(A_{Ht}H, A_{Xt}X)$ and Assumption 1 holds. Assumption 2 holds by Example 3. If $A_{jt} = G_j^t$ with $G_1 > G_2$, then

$$A_{Ht}/A_{Xt} = (A_{1t}/A_{2t})^{\frac{1}{1-\alpha}} = (G_1/G_2)^{\frac{t}{1-\alpha}},$$

so (3.7) holds. □

Corollary 3. *Land is overvalued in Example 1 if* $G_1 > G_2$*.*

Proof. As is well known, profit maximization at the individual sector or firm level is equivalent to that at the aggregate level. Consider the aggregation of the two production functions in (2.2). Suppressing the t subscript and setting $(X_1, X_2) = (0, X)$, the Lagrangian for the maximization problem

$$F(H, X) := \max \left\{ \sum_{j=1}^{2} F_j(H_j, X_j) : \sum_{j=1}^{2} H_j = H, \sum_{j=1}^{2} X_j = X \right\}$$

is

$$\mathcal{L}(H_1, H_2, \lambda) = A_1 H_1 + A_2 H_2^\alpha X^{1-\alpha} + \lambda(H - H_1 - H_2),$$

where λ is the Lagrange multiplier. Applying the Karush-Kuhn-Tucker theorem, we obtain $\lambda = A_1$, $H_2 = (\alpha A_2/A_1)^{\frac{1}{1-\alpha}} X$, and the aggregate production function

$$F_t(H, X) = A_{1t} H + (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} (A_{2t}/A_{1t}^\alpha)^{\frac{1}{1-\alpha}} X,$$

which is linear. Therefore if we define

$$F(H, X) = H + (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} X$$

and (A_{Ht}, A_{Xt}) as in the proof of Corollary 2, the same argument applies. □

Corollary 4. *Land is overvalued in Example 2 if* $G_H > G_X$*.*

Proof. Trivial. □

3.4 Recurrent stochastic fluctuations

We discuss the third implication of Theorem 1 by specializing it.

The production function takes the CES form (2.12). Let $A_t := A_{Ht}/A_{Xt}$ be the relative productivity of labor. The state of the economy at time t is denoted by n_t, which evolves over time according to a Markov chain with transition probability matrix $\Pi = (\pi_{nn'})$, where $\pi_{nn'} = \Pr(n_t = n' \mid n_{t-1} = n)$. The relative productivity A_t evolves over time as a Markov multiplicative process

$$A_t = G_t A_{t-1}, \tag{3.8}$$

where G_t conditional on $(n_{t-1}, n_t) = (n, n')$ is an IID copy of some random variable $G_{nn'} > 0$.[7] Let $S_n(A)$ be the value of (3.7) when $(A_0, n_0) = (A, n)$. Due to the multiplicative nature of shocks and homogeneity, we may write $S_n(A) = s_n A^{\rho-1}$ for some constant $s_n > 0$, where $\rho = 1/\sigma$. A dynamic programming argument shows

$$s_n = 1 + \sum_{n'=1}^{N} \pi_{nn'} \operatorname{E}[G_{nn'}^{\rho-1}] s_{n'}. \tag{3.9}$$

Defining the $N \times 1$ vector $s = (s_1, \dots, s_N)'$, the vector of ones $1 = (1, \dots, 1)'$, and the $N \times N$ nonnegative matrix $K = (\pi_{nn'} \operatorname{E}[G_{nn'}^{\rho-1}])$, we may rewrite (3.9) as

$$s = 1 + Ks \iff s = (I - K)^{-1} 1. \tag{3.10}$$

A positive and finite solution to (3.10) exists if and only if the spectral radius of K (the maximum modulus of all eigenvalues) is less than 1.[8] Therefore we obtain the following proposition.

Proposition 4. *Suppose the production function is CES with elasticity of substitution $\sigma > 1$ and the relative labor productivity $A_t := A_{Ht}/A_{Xt}$ follows the Markov multiplicative process (3.8). Let $K = (\pi_{nn'} \operatorname{E}[G_{nn'}^{1/\sigma-1}])$. Then land is overvalued if the spectral radius of K is less than 1.*

[7] See Beare and Toda (2022, §2) for more details.

[8] This argument is similar to Borovička and Stachurski (2020).

As a numerical example, we set $\beta = 0.5$, $\alpha = 0.8$, $\sigma = 1.25$, $N = 2$, $\pi_{nn'} = 1/3$ if $n \neq n'$, and $(G_{1n'}, G_{2n'}) = (1.1, 0.95)$ for all n', which implies that the spectral radius of K is less than 1 and land is overvalued. Figure 1 shows one simulation for 200 periods. The land price exhibits boom-bust cycles. The price-rent ratio steadily increases, consistent with Theorem 1.

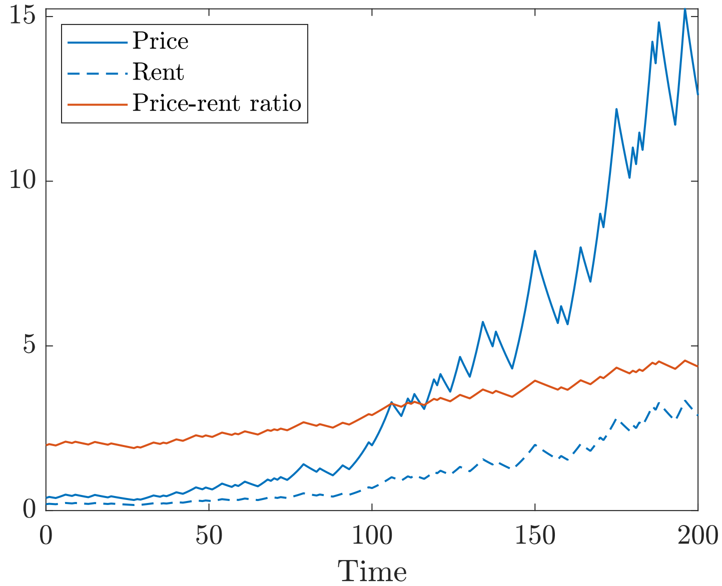

Figure 1: Simulation of the numerical example of Proposition 4.

Proposition 4 and this numerical example provide the third implication of Theorem 1. When productivities increase and remain to be high, land prices will continue to rise relative to the trend, which may look like an emergence of a large land price bubble. Conversely, if productivities decrease and remain to be so for an extended period of time, land prices will fall, which may appear to be a bursting of a land bubble. Thus, land prices exhibit recurrent booms and busts driven by fluctuations in productivities. Nonetheless, as long as the relative productivity growth of land is low, land will always be overvalued, with the size of land overvaluation fluctuating over time and a steady upward trend in the price-rent ratio. Our model provides a theoretical foundation for recurrent stochastic bubbles.

4 Concluding remarks

This paper has studied the long-run behavior of land prices in a modern economy. We have established the Land Overvaluation Theorem showing the surprising link between unbalanced growth, elasticity of substitution, and land overvaluation in an economy with aggregate risk. This Theorem provides new insights on both short-run and long-run behaviors of land prices. Unlike the conventional view that land overvaluation (sometimes called land bubbles) may occur only as short-run phenomena, our paper shows that it will naturally and necessarily arise along the process of economic development with unbalanced growth. Moreover, driven by stochastic fluctuations in productivities, land prices experience large swings, with expansions and contractions in the size of land overvaluation that may appear to be the emergence and collapse of large land bubbles.

To derive these results, unbalanced growth together with elasticity of substitution exceeding 1 plays an important role. Although unbalanced growth may not seem to be common in the standard growth models, it is a general feature in the growth process in reality because different sectors or production factors have different growth rates. At the same time, in reality, as economies develop, the importance of land as a factor of production usually diminishes, yet its role as a store of value continues to be high. What our Theorem shows is that under such circumstances, land overvaluation will arise as the equilibrium outcome. In this sense, we believe that the results of our paper capture an important aspect of the modern economy.

Finally, we would like to add two remarks. In the present paper, for simplicity we only considered an overlapping generations model with exogenous growth. However, whether growth is exogenous or endogenous is not important for asset overvaluation. In a parallel work, Hirano, Jinnai, and Toda (2022) show that growth and asset overvaluation endogenously emerge as the leverage of entrepreneurs is relaxed. They also employ a Bewley-type model with infinitely-lived agents, implying that the overlapping generations structure is inessential.

In our model, land is the primary store of value and overvaluation necessarily occurs in land. If there are multiple assets that serve as a store of value (such as gold and cryptocurrency), the extent of overvaluation in individual assets could be indeterminate. Nonetheless, the aggregate amount of overvaluation and the equilibrium outcome are determinate, as in the present paper. However, as noted in the introduction, we think land is a focal point as a store of value due to its characteristics. Moreover, in the macro-finance model of Hirano, Jinnai, and Toda (2022) with credit frictions, there are means of savings other than land, which are lending to other economic agents or investing in capital. Even in that setting, although there is no aggregate risk, land overvaluation necessarily emerges as the unique equilibrium outcome under sufficiently lax leverage.

A Proofs

Proof of Proposition 3. The first-order condition for profit maximization implies (3.2a) and (3.2b). Define the return on land by

$$R_{t+1} = \frac{P_{t+1} + r_{t+1}}{P_t}.$$

Then the budget constraints (2.6) can be combined into one as

$$z_{t+1} = R_{t+1}(w_t - y_t).$$

Suppressing the time subscripts and substituting into the objective function, the young seek to maximize

$$(1-\beta)\log y + \mathrm{E}[\log z] = (1-\beta)\log y + \beta \log(w-y) + \beta\,\mathrm{E}[\log R].$$

Clearly this function is strictly concave in y and achieves a unique maximum characterized by the first-order condition

$$\frac{1-\beta}{y} - \frac{\beta}{w-y} = 0 \iff y = (1-\beta)w,$$

which is (3.2d). Since in equilibrium we have $x_t = 1$, the land price satisfies $P_t = P_t x_t = w_t - y_t = \beta w_t$, which is (3.2c). □

Proof of Lemma 3.1. Since F is homogeneous of degree 1, F_H is homogeneous of degree 0. Therefore differentiating both sides of

$$F(\lambda H, \lambda X) = \lambda F(H, X),$$
$$F_H(\lambda H, \lambda X) = F_H(H, X)$$

with respect to λ and setting $\lambda = 1$, we obtain

$$HF_H + XF_X = F, \tag{A.1a}$$
$$HF_{HH} + XF_{HX} = 0. \tag{A.1b}$$

Let $h = \log(H/X)$. Using the definition (3.4) and (A.1), we obtain

$$\begin{aligned}\frac{1}{\sigma_F} &= \frac{\partial}{\partial h}\log\frac{F_X(X\mathrm{e}^h, X)}{F_H(X, \mathrm{e}^h, X)} = \frac{X\mathrm{e}^h F_{HX}}{F_X} - \frac{X\mathrm{e}^h F_{HH}}{F_H} \\ &= \frac{HF_{HX}}{F_X} - \frac{HF_{HH}}{F_H} = \frac{HF_{HX}}{F_X} + \frac{XF_{HX}}{F_H} \\ &= \frac{F_{HX}}{F_H F_X}(HF_H + XF_X) = \frac{FF_{HX}}{F_H F_X}. \qquad \square\end{aligned}$$

Proof of Lemma 3.2. Let $X = 1$ and $h = \log H$. Using (3.4) and applying l'Hôpital's rule, we obtain

$$\limsup_{H\to\infty} \rho(H, 1) = \limsup_{H\to\infty} \frac{\log(F_X/F_H)}{\log H} = 1 + \limsup_{H\to\infty} \frac{\log\frac{F_X}{HF_H}}{\log H}.$$

Therefore to prove the claim, it suffices to show $F_X \leq HF_H$ for large enough H. Since $X = 1$ and F is homogeneous of degree 1, we have $F = HF_H + F_X$, so

$$\frac{1}{H}(HF_H - F_X) = \frac{1}{H}(2HF_H - F) = 2F_H - \frac{F}{H} \to 2m - m = m > 0,$$

implying $F_X < HF_H$ for large enough H. $\square$

We prove Theorem 1 by establishing a series of lemmas.

Lemma A.1. *Let $A > 0$ and suppose that $\sigma_F(H, 1) \geq \sigma$ for $H \geq A$. Let $\rho = 1/\sigma$. If $A_H/A_X \geq A$, then*

$$\frac{F_X}{F_H}(A_H, A_X) \leq \frac{F_X}{F_H}(A, 1)A^{-\rho}(A_H/A_X)^{\rho}. \tag{A.2}$$

Proof. By Assumption 1, F is homogeneous of degree 1. Therefore F_H, F_X are homogeneous of degree 0, and so is $\rho(H, X)$ in (3.4).

Let $B := A_H/A_X \geq A$. Setting $H = \mathrm{e}^h$ and $X = 1$ in (3.4), we obtain

$$\rho(\mathrm{e}^h, 1) = \frac{\mathrm{d}}{\mathrm{d}h} \log \frac{F_X}{F_H}(\mathrm{e}^h, 1).$$

Integrating both sides from $h = \log A$ to $h = \log B$ and applying the intermediate value theorem for integrals, there exists $h_1 \in (\log A, \log B)$ such that

$$\begin{aligned} \rho(\mathrm{e}^{h_1}, 1) \log(B/A) &= \int_{\log A}^{\log B} \rho(\mathrm{e}^h, 1) \, \mathrm{d}h \\ &= \log \frac{F_X}{F_H}(B, 1) - \log \frac{F_X}{F_H}(A, 1). \end{aligned} \tag{A.3}$$

Taking the exponential of both sides of (A.3), letting $M := (F_X/F_H)(A, 1)$, and using the homogeneity of F_H, F_X, we obtain

$$\frac{F_X}{F_H}(A_H, A_X) = \frac{F_X}{F_H}(B, 1) = M(B/A)^{\rho(\mathrm{e}^{h_1}, 1)}.$$

Since $B \geq A$ and $\rho(\mathrm{e}^{h_1}, 1) \leq \rho := 1/\sigma$, it follows that

$$\frac{F_X}{F_H}(A_H, A_X) \leq M(B/A)^\rho = MA^{-\rho}(A_H/A_X)^\rho,$$

which is (A.2). □

Lemma A.2. *In equilibrium, the fundamental value of land is bounded above as*

$$V_t \leq w_t \, \mathrm{E}_t \left[\sum_{s=1}^{\infty} \frac{r_{t+s}}{w_{t+s}} \right]. \tag{A.4}$$

Proof. The stochastic discount factor between time t and $t+1$ equals the marginal rate of substitution

$$\begin{aligned} m_{t \to t+1} := \frac{\beta / z_{t+1}}{(1-\beta)/y_t} &= \frac{\beta y_t}{(1-\beta) z_{t+1}} \\ &= \frac{\beta w_t}{\beta w_{t+1} + r_{t+1}} \leq \frac{w_t}{w_{t+1}}, \end{aligned}$$

where the last line uses (3.2) and $r_{t+1} \geq 0$. Then we can bound the stochastic discount factor between time t and $t+s$ from above as

$$m_{t\to t+s} := \prod_{j=0}^{s-1} m_{t+j\to t+j+1} \le \frac{w_t}{w_{t+s}}.$$

Therefore we can bound the fundamental value of land from above as

$$V_t := \mathrm{E}_t\left[\sum_{s=1}^{\infty} m_{t\to t+s} r_{t+s}\right] \le \mathrm{E}_t\left[\sum_{s=1}^{\infty} \frac{w_t}{w_{t+s}} r_{t+s}\right] = w_t \, \mathrm{E}_t\left[\sum_{s=1}^{\infty} \frac{r_{t+s}}{w_{t+s}}\right]. \qquad \square$$

Lemma A.3. *We have* $\lim_{t\to\infty} V_t/P_t = 0$ *almost surely.*

Proof. By (3.2c) and Lemma A.2, we have

$$0 \le \frac{V_t}{P_t} \le \frac{1}{\beta} \mathrm{E}_t\left[\sum_{s=1}^{\infty} \frac{r_{t+s}}{w_{t+s}}\right].$$

Therefore to show the claim, it suffices to show that $\mathrm{E}_t[\sum_{s=1}^{\infty} r_{t+s}/w_{t+s}] \to 0$ almost surely as $t\to\infty$.

By Assumption 2, we can take a constant $A>0$ such that $\sigma(H,1) \ge \sigma > 1$ for all $H \ge A$. Let $A_t := A_{Ht}/A_{Xt}$ and $\rho = 1/\sigma \in (0,1)$. Since the expectation of the infinite sum (3.7) is finite, the sum converges with probability 1 and hence we must have $A_t^{\rho-1} \to 0$ and $A_t \to \infty$ because $\rho < 1$. In particular, there exists $T > 0$ such that $A_t \ge A$ for $t \ge T$. For such t, by Lemma A.1 we have

$$\frac{r_t}{w_t} = \frac{F_X(A_{Ht}, A_{Xt})A_{Xt}}{F_H(A_{Ht}, A_{Xt})A_{Ht}} \le \frac{F_X}{F_H}(A,1)A^{-\rho}A_t^{\rho-1}.$$

Therefore

$$\mathrm{E}_t\left[\sum_{s=1}^{\infty} \frac{r_{t+s}}{w_{t+s}}\right] \le \frac{F_X}{F_H}(A,1)A^{-\rho}\,\mathrm{E}_t \sum_{s=1}^{\infty} A_{t+s}^{\rho-1}.$$

Letting $t\to\infty$ and using condition (3.7), we obtain $\mathrm{E}_t[\sum_{s=1}^{\infty} r_{t+s}/w_{t+s}] \to 0$ almost surely as $t \to \infty$. $\square$

Proof of Theorem 1. The absence of arbitrage and the definition of the fundamental value imply

$$P_t = \mathrm{E}_t[m_{t\to t+1}(P_{t+1} + r_{t+1})],$$
$$V_t = \mathrm{E}_t[m_{t\to t+1}(V_{t+1} + r_{t+1})].$$

Taking the difference, we obtain

$$P_t - V_t = \mathrm{E}_t[m_{t\to t+1}(P_{t+1} - V_{t+1})].$$

Iterating this equation and applying the law of iterated expectations, we obtain

$$P_t - V_t = \mathrm{E}_t[m_{t\to t+s}(P_{t+s} - V_{t+s})].$$

Lemma A.3 implies $V_{t+s}/P_{t+s} \to 0$ almost surely as $s \to \infty$ and hence $P_{t+s} > V_{t+s}$ for large enough s with probability 1. Therefore $P_t > V_t$ for all t, and land is overvalued. □

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