Online Appendix to “Debt overhang and lack of lender’s commitment”

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A Simple example with debt dilution

In this appendix, we extend the Simple Example in Section 1 to analyze the effects and implications of debt dilution (Hatchondo et al. 2014, 2016) on our model of the lack of lenders’ commitment. The implications from this example are summarized as the following four points:

- Debt restructuring is ex-post efficient in restoring the lenders’ commitment.
- Prohibition of debt restructuring is ex-ante welfare improving: it increases the initial borrowing ex ante by reducing the risk of debt dilution, whereas it makes the economy unable to escape from the debt overhang, ex post.
- In this example, the first-best policy is to set an upper limit on the total borrowing to prevent the dilution of the initial debt from occurring.
- The above results can be overturned if there is a high possibility of debt overhang due to exogenous negative shocks, for example, low productivity shocks, and a low possibility of debt dilution. In such a case, debt restructuring can be welfare enhancing.

A.1 Setup

The economy continues for three periods: period 1, period 2, and period 3. There is a borrower and infinitely many lenders. The lenders are risk neutral and willing to lend as long as the expected profits are nonnegative. The lenders’ subjective rate of time preference is 1, which means that the market rate of interest on a risk-free asset is 0.

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While the lenders are patient, the borrower is impatient. The ex-ante utility for the borrower as of period 1 is given by

$$U = C_1 + \beta_2 C_2 + \beta_3 C_3,$$

where C_i is the consumption in period i and β_i is the time preference, where we assume for simplicity that $\beta_2 = 0.5$ and $\beta_3 = 0$. Thus, the borrower values the consumption in period 2 as half of that in period 1, and the consumption in period 3 as nothing. The borrower's possible actions are as follows:

- In period 1, the borrower sell bonds with face value D_1 at the price q_1, which promises to pay D_1 dollars in period 3, and he immediately consumes the proceeds from issuance of the bonds. The face value D_1 is the choice variable for the borrower, and the price q_1 is determined as an equilibrium outcome.

- In period 2, the borrower can sell additional bonds with face value D_2 at the price q_2, which promises to pay D_2 dollars in period 3, and he consumes the proceeds. D_2 is the choice variable, while q_2 is determined by the market equilibrium.

- At the beginning of period 3, there is an opportunity of debt restructuring. The borrower can reduce the outstanding debt $D_1 + D_2$ to $\hat{D}$, where $\hat{D} \leq D_1 + D_2$. We assume that the value of $\hat{D}$ is determined such that the total value of the lenders' payoff is maximized. Debt reduction is on the pro rata basis, that is, D_i is reduced to γD_i for $i = 1, 2$, with $\gamma = \hat{D}/(D_1 + D_2)$.

- In the middle of period 3, after debt restructuring, the borrower is given the opportunity of producing output. If the borrower works hard, he can earn USD 1000 at the end of period 3, and if he does not, he earns USD 100. The borrower does not work hard unless his expected payoff at the end of period 3 is no less than 200. This setting is the same as Simple Example in Section 1 in the main text of the paper.

- At the end of period 3, the borrower has the revenue of either USD 1000 or USD 100. The lenders obtain the repayment of $\min\{\hat{D},\ R\}$, where R is the borrower's revenue. Each lender receives the repayment on the pro-rata basis.

A.2 Equilibrium

We consider the equilibrium backward.

Lack of lenders' commitment in period 3: The analysis in Simple Example of Section 1 is applicable. If $\hat{D}$, the outstanding debt in period 3, is larger than 800, the borrower does not work hard and earns only 100, and if $\hat{D} \leq 800$, the borrower works hard and earns 1000, from which he repays $\hat{D}$ to the lenders. This result demonstrates that the debt restructuring to make $\hat{D} \leq 800$ is ex-post efficient to resolve the inefficiencies due to the lack of lenders' commitment.

Debt dilution in period 2: Suppose that the debt restructuring at the beginning of period 3 is feasible. It is foreseen that the restructured debt in period 3 will be $\hat{D} = \min\{D_1 + D_2,\ 800\}$. Given the expectation $\hat{D} = \min\{D_1 + D_2,\ 800\}$, for any value of D_1, the optimal choice for the borrower in period 2 is to borrow $D_2 = N \times D_1$ with $q_2 D_2 = 800$, where N is set very large such that $N \to +\infty$, and to consume the proceed of 800 in period 2. It is feasible because the debt restructuring on the pro-rata basis implies that all repayment $\hat{D} = 800$ is accrued to the lenders of period 2, as $\hat{D} \times \frac{D_1}{D_1+D_2} = \frac{1}{N+1} \to 0$ and $\hat{D} \times \frac{D_2}{D_1+D_2} = \frac{N}{N+1}\hat{D} \to \hat{D} = 800$. Thus, the lenders of period 2 obtain everything and the lenders of period 1 get nothing. This is the debt dilution by the borrowing in period 2. Anticipating the debt dilution in period 2 and the debt restructuring in period 3, the lenders in period 1 lend zero to the borrower, because they know that they will have zero repayment in period 3. In this case, the borrower consumes 0 in periods 1 and 3, and 800 in period 2. The present value of the borrower's utility as of period 1 is $U = 0.5 \times 800 = 400$. As we see in the following paragraph, the ex-ante utility (U = 400) is lower than that in the case where the debt restructuring is prohibited (U = 750).

If debt restructuring is prohibited: Suppose that the debt restructuring is prohibited or infeasible for some technical reasons. In this case $\hat{D} = D_1 + D_2$, which may be larger than 800. Note that if $\hat{D} > 800$, the borrower does not work hard and earns only 100, and if $\hat{D} \leq 800$, the borrower works hard and earns 1000. Let us consider backward. First, we consider the choice of D_2 in period 2, given the value of D_1. Suppose $D_1 \leq 700$. The best response for the borrower is to choose $D_2 = 800 - D_1$ (> 100) and consume the proceed D_2 in period 2. The lenders obtain full repayments D_1 and D_2 in period 3. Suppose $D_1 > 700$. In this case, the best response for the borrower is to choose $D_2 = N \times D_1$, where $N \to +\infty$, such that $q_2 D_2 = 100$, and consume 100, the proceed, in period 2. The lenders accept to lend 100 in period 2 because the period-2 lenders receive the repayment of 100 in period 3. The repayment to the period-2 lenders is 100, because the borrower's revenue is 100 due to debt overhang, that is, $\hat{D} > 800$, and the period-2 lenders receive $100 \times \frac{D_2}{D_1+D_2} = 100$, as $D_2 \to +\infty$, whereas the period-1 lenders receive no repayment. Anticipating that the repayment will be zero if the period-1 debt exceeds 700, the period-1 lenders lend no more than 700 in the first place. The above arguments imply that the best choice for the borrower is $D_1 = 700$ and $D_2 = 100$. In this case, the borrower consumes 700 in period 1, 100 in period 2, and 0 in period 3. The borrower's ex-ante utility is $U = 700 + 100 \times 0.5 = 750$.

If debt limit is set to prevent debt dilution: Suppose that, at the beginning of period 1, an upper limit for total borrowing is set at the maximum repayable amount, that is, $D_1 + D_2 \leq 800$. The best choice for the borrower is obviously to set $D_1 = 800$ and $D_2 = 0$. The borrower consumes 800 in period 1 and zero in periods 2 and 3. In this case, the debt dilution is prevented from occurring, and the borrower consumes 800 in period 1, implying that $U = 800$.

A.3 Discussion

The key takeaways of this example are as follows.

First, the analysis on period 3 implies that debt restructuring is ex-post efficient in restoring the lenders' commitment.

Second, given that there is a possibility of debt dilution, the prohibition of debt restructuring is ex-ante welfare improving. On one hand, it increases the initial borrowing because the lenders of initial debt become willing to lend more, as there is a smaller possibility of their debt being diluted.[1] On the other hand, the prohibition of debt restructuring makes the economy unable to escape from the second type of debt overhang, ex post, once the economy is trapped in it.

Third, in this example, the first-best policy is to set an upper limit on the total borrowing to prevent the dilution of the initial debt from occurring. If the upper limit is set at the maximum repayable amount, the inefficiency of the lack of lenders' commitment is also avoided and the economy attains the first best outcome.

Fourth, in this example, elimination of debt dilution is most significant in improving welfare, because we assumed that the debt dilution is the major source of inefficiency. In general, however, the distortion of debt dilution may not be dominant in the debt problems. It seems obvious that the conclusion of this example may be overturned, if the inefficiency of debt overhang is more significant. Suppose that the debt can easily accumulate overly due to negative external shocks, for example, low productivity shocks, whereas the possibility of debt dilution is low. In such a case, the policy interventions that promote debt restructuring can be welfare enhancing, as it is in the main text of this paper.

[1] A more detailed explanation is the following: the prohibition of debt restructuring increases the risk of output decline due to debt overhang, which reduces the expected payoff for the period-2 lenders. This expectation makes the period-2 lenders less willing to lend and reduces the period-2 debt, implying a smaller possibility of debt dilution of the period-1 debt.

B Existence of equilibrium

This appendix proves the existence of the equilibrium in the discrete version of the model.

B.1 Discretization of the model

Discretization: Denote the set of integers by $\mathbb{Z}$, and define

$$\Delta = \{0, \delta, 2\delta, \cdots, N_{\max}\delta\},$$
$$\Delta_{+1} = \{0, \delta, 2\delta, \cdots, n_\delta[(1+r)N_{\max}\delta]\}.$$

Here, δ is the minimum unit of debt, $N_{\max} \in \mathbb{Z}$ is a sufficiently large integer, and $n_\delta(x) = n\delta$ for $x > 0$, where n is the integer satisfying $(n-1)\delta < x \leq n\delta$. We assume that the amount of debt, D, must be an element of Δ:

$$D \in \Delta.$$

For each $s \in \{s_L, s_H\}$, the set of possible values of k, $\Delta_k(s)$, is defined as

$$\Delta_k(s) = \left\{ k \;\middle|\; \exists n \in \mathbb{Z}, \text{ s.t. } F(s,k) - Rk - G(s,k) = n \times \frac{\delta}{1+r} \right\}.$$

Then, $k^*(s)$ and $k^{npl}(s)$ are defined as

$$k^*(s) = \arg \max_{k \in \Delta_k(s)} F(s,k) - Rk,$$
$$k^{npl}(s_H) = \arg \max_{k \in \Delta_k(s_H)} F(s_H,k) - Rk - G(s_H,k),$$
$$k^{npl}(s_L) = \arg \max_{k \in \Delta_k(s_L)} F(s_L,k) - Rk - G(s_L,k),$$

Here, we assume that the parameter values are selected such that

$$G^{npl}(s_H) > \beta[\pi_{HH}G^{npl}(s_H) + \pi_{HL}G^{npl}(s_L)], \tag{1}$$
$$G^{npl}(s_L) > \beta[\pi_{LL}G^{npl}(s_L) + \pi_{LH}G^{npl}(s_H)], \tag{2}$$

where $\pi_{HH} = \Pr(s_{t+1} = s_H | s_t = s_H)$, $\pi_{HL} = 1 - \pi_{HH}$, $\pi_{LL} = \Pr(s_{t+1} = s_L | s_t = s_L)$, and $\pi_{LH} = 1 - \pi_{LL}$. We also let $G^{npl}(s) \equiv G(s, k^{npl}(s))$.

Our arguments in this paper can be easily modified for the case where the inequalities (1) and/or (2) do not hold.[2] For each $s \in \{s_L, s_H\}$, the repayment in the NPL equilibrium, $b^{npl}(s)$, is defined by

$$b^{npl}(s) = F(s, k^{npl}(s)) - Rk^{npl}(s) - G^{npl}(s) + \beta\mathbb{E}[G^{npl}(s_{+1})|s].$$

[2] For this purpose, it suffices to redefine
$k^{npl}(s_H) = \max\{k \in \Delta_k(s_H) |\; G(s, k_H) \leq \beta[\pi_{HH}G(s_H, k) + \pi_{HL}G(s_H, k^{npl}(s_L))]\}$,
and/or
$k^{npl}(s_L) = \max\{k \in \Delta_k(s_L) |\; G(s, k_L) \leq \beta[\pi_{LL}G(s_L, k) + \pi_{LH}G(s_H, k^{npl}(s_H))]\}$. In the case where $k^{npl}(s)$ is redefined, $b^{npl}(s)$ is also redefined as $b^{npl}(s) = F(s, k^{npl}(s)) - Rk^{npl}(s)$.

The set of possible values of repayments, $\Delta_b(s, D)$, depends on D:

$$\Delta_b(s, D) = \left\{ b \in \mathbb{R} \;\middle|\; \exists \tilde{D}_{+1} \in \Delta_{+1} \text{ s.t. } b = D - \frac{1}{1+r}\tilde{D}_{+1}, \text{ and } b \geq 0 \right\} \cup \{b^{npl}(s)\}.$$

At each state (s, D), b and k must satisfy

$$b \in \Delta_b(s, D), \quad \text{and} \quad k \in \Delta_k(s).$$

Bank's problem: Let $V^e(s, D)$ denote the bank's expectation regarding the value of the firm as a function of the current state (s, D). Then, the bank's profit maximization is formulated as the Bellman equation:

$$d(s, D) = \max_{b \in \Gamma(s,D)} \quad b + \beta \mathbb{E} d(s_{+1}, D_{+1}), \tag{3}$$

where

$$\begin{aligned} \Gamma(s, D) = &\{b \in \Delta_b(s, D) \mid \exists k \in \Delta_k(s) \text{ s.t.} \\ &D_{+1} = \min\{N_{\max}\delta,\; n_\delta[(1+r)(D-b)]\}, \\ &F(s,k) - Rk - b + \beta \mathbb{E} V^e(s_{+1}, D_{+1}) \geq G(s,k), \\ &F(s,k) - Rk - b \geq 0\}. \end{aligned}$$

Here, $n_\delta[(1+r)(D-b)] = n \times \delta$, where n is the integer that satisfies $(n-1)\delta < (1+r)(D-b) \leq n\delta$.

Let $\Sigma(s, D)$ denote the set of (b, D_{+1}) that solves the maximization problem in (3). The bank then decides k and $V(s, D)$ by solving the following problem:

$$V(s, D) = \max_{k \in \Delta_k(s),\; (b, D_{+1}) \in \Sigma(s,D)} F(s,k) - Rk - b + \beta \mathbb{E} V^e(s_{+1}, D_{+1}), \tag{4}$$

subject to

$$\begin{aligned} &F(s,k) - Rk - b + \beta \mathbb{E} V^e(s_{+1}, D_{+1}) \geq G(s,k), \\ &F(s,k) - Rk - b \geq 0. \end{aligned}$$

Let $\Lambda(s, D)$ denote the set of (k, b, D_{+1}) that solves the maximization problem in (4).

Given $\Lambda(s, D)$, the equilibrium values of (k, b, D_{+1}) at (s, D) are selected as follows. First, $b(s, D)$ and $D_{+1}(s, D)$ are decided as

$$b(s, D) = \max_{(k, b, D_{+1}) \in \Lambda(s,D)} b, \tag{5}$$

$$D_{+1}(s, D) = \min\{N_{\max}\delta,\; n_\delta[(1+r)\{D - b(s,D)\}]\}. \tag{6}$$

Then, $k(s, D)$ is determined by

$$k(s, D) = \max_{(k, b(s,D), D_{+1}(s,D)) \in \Lambda(s,D)} k.$$

Then, the value of the firm must satisfy

$$V(s,D) = F(s,k(s,D)) - Rk(s,D) - b(s,D) + \beta \mathbb{E} V^e(s_{+1}, D_{+1}(s,D)). \tag{7}$$

Assuming rational expectations, the bank's belief $V^e(s,D)$ should be consistent with $V(s,D)$ given in (7):

$$V(s,D) = V^e(s,D). \tag{8}$$

Definition of the threshold, $D_{\max}(s)$: Given the existence of an equilibrium, we define $D_{\max}(s)$ as follows:

$$D_{\max}(s_H) \equiv \max\{D \in \Delta \,|\, D_{+1}(s_H, D) < D\}, \tag{9}$$

$$D_{\max}(s_L) \equiv \max\{D \in \Delta \,|\, D_{+1}(s_L, D) < D_{\max}(s_H)\}. \tag{10}$$

Thus, if D exceeds $D_{\max}(s_H)$ at s_H, the amount of debt in the next period is greater than or equal to D. Similarly, if D exceeds $D_{\max}(s_L)$ at state s_L, the next period's debt is greater than or equal to $D_{\max}(s_H)$. The following lemma demonstrates that if $D > D_{\max}(s_L)$, then $D_{+1}(s_L, D) \geq D$. As a result, once D exceeds $D_{\max}(s)$ at each s, D will never decrease.

Lemma 1. *If $D > D_{\max}(s_L)$, then $D_{+1}(s_L, D) \geq D$.*

Proof. Let $D > D_{\max}(s_L)$, and suppose, for the sake of contradiction, that $D_{+1}(s_L, D) < D$. Then,

$$D_{+1}(s_L, D_{+1}(s_L, D)) < D_{+1}(s_L, D).$$

However, since $D > D_{\max}(s_L)$, $D_{+1}(s_L, D) \geq D_{\max}(s_H)$. By the definition of $D_{\max}(s_H)$, we have

$$D_{+1}(s_H, D_{+1}(s_L, D)) \geq D_{+1}(s_L, D).$$

We also have

$$D_{+1}(s_H, D_{+1}(s_L, D)) \leq D_{+1}(s_L, D_{+1}(s_L, D)).$$

Combining these inequalities, we obtain

$$D_{+1}(s_L, D) \leq D_{+1}(s_H, D_{+1}(s_L, D)) \leq D_{+1}(s_L, D_{+1}(s_L, D)) < D_{+1}(s_L, D),$$

which is a contradiction. □

We can confirm that $D_{\max}(s) < \infty$ as follows. For $D > \bar{D}$, it is obvious that, for any $b \leq \max_k\{F(s,k) - Rk\}$, the debt never decreases over time, that is, $D_{+1} = (1+r)(D-b) > D$. Thus, there exists $D_{\max}(s_H)$ such that $D_{\max}(s_H) \leq \bar{D} < \infty$. As $D_{\max}(s_H) < \infty$, it follows from (9)-(10) that $D_{\max}(s_L) \leq D_{\max}(s_H)$.

B.2 Equilibrium of the discrete model

In this section, we assume that the interest rate in the debt contract is equal to the market rate for the risk-free bond:

$$\beta = \frac{1}{1+r}. \tag{11}$$

Note that even under assumption (11), the bank can still make the expected payoff non-negative, by adjusting the initial amount of the principal of the loan.[3] In Sections B.2.1, B.2.2, and B.2.3, we characterize the equilibrium, taking the existence of an equilibrium as given. In Section B.2.4, we prove the existence. All proofs are provided in Section B.4.

B.2.1 The repayment in the case of small D

Two working assumptions: In Sections B.2.1 and B.2.2, we make the following two assumptions. They are verified later in Lemma 12 in Section B.2.4.

Assumption 1. For $D < D_{\max}(s)$, $V^e(s, D+\delta) \leq V^e(s, D) - \delta$.

Assumption 2. For all s and $D \geq \delta$, $b(s, D)$ satisfies

$$b(s, D) \geq \delta. \tag{12}$$

We first characterize the equilibrium repayment function $b(s, D)$ for $D \leq D_{\max}(s)$.

Lemma 2. *For all $D \geq 0$, $d(s, D+\delta) \leq d(s, D) + \delta$.*

Lemma 3. *For $D \leq D_{\max}(s)$, $b(s, D) = \bar{b}(s, D)$, where $\bar{b}(s, D)$ is the maximum feasible value, that is, $\bar{b}(s, D) = \max\{b \mid b \in \Gamma(s, D)\}$. It also holds that $k(s, D) > k^{npl}(s)$ for $D \leq D_{\max}(s)$.*

Lemma 3 directly implies the following corollary.

Corollary 4. *If (s, D) is a state such that $k(s, D) = k^*(s)$, then*

$$b(s, D) = \min\ \{D,\ b^*(s, D)\},$$

where

$$b^*(s, D) = \max_{n \in \mathbb{Z}} D - \beta n \delta,$$
$$s.t. \quad D - \beta n \delta \leq F(s, k^*(s)) - Rk^*(s).$$

[3] The initial principal of the debt D_0 may not be fully repaid in equilibrium, so that the expected PDV of repayments, $d(s_0, D_0) \equiv \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t b_t$, may be smaller than D_0. Let I_0 denote the initial amount of lending. The zero-profit condition for the bank is satisfied if the contractual amount of initial debt, D_0, is set as

$$I_0 = d(s_0, D_0).$$

Now, we define

$$
\begin{aligned}
&f(s,k) \equiv F(s,k) - Rk - G(s,k),\\
&\delta_f \equiv \max_{k\in\Delta_k(s),\ k^{npl}(s)\le k\le k^*(s)} F'(s,k) - R,\\
&\delta_k \equiv \max\{k'-k \mid k\in\Delta_k(s),\ k'\in\Delta_k(s),\\
&\qquad\qquad k^{npl}(s)\le k<k'<k^*(s),\ |f(s,k)-f(s,k')|=\beta\delta\},\\
&\delta_g \equiv \max\{G(s,k')-G(s,k) \mid k\in\Delta_k(s),\ k'\in\Delta_k(s),\\
&\qquad\qquad k^{npl}(s)\le k<k'<k^*(s),\ |f(s,k)-f(s,k')|=\beta\delta\}.
\end{aligned}
$$

Note that $\delta_f = O(1)$, $\delta_k = O(\delta)$, and $\delta_g = O(\delta)$. Then, the following lemma holds.

Lemma 5. *For (s,D) such that $k^{npl}(s) < k(s,D) < k^*(s)$, it holds that $0 \le F(s,k(s,D)) - Rk(s,D) - b(s,D) < \xi + \beta\delta$, where $\xi = \delta_f\delta_k$.*

As $\xi = O(\delta)$, Corollary 4 and Lemma 5 implies that $b(s,D) \approx \min\{D,\ F(s,k(s,D)) - Rk(s,D)\}$ for small δ. This means that the optimal contract involves backloaded payment to the firm; that is, the firm repays debt as fast as possible by setting its dividend at almost zero, that is, $b \approx \min\{D,\ F(s,k) - Rk\}$, when D is smaller than or equal to $D_{\max}(s)$.

B.2.2 Equilibrium at large D

Here, we demonstrate that when D is large so that $D > D_{\max}(s)$, the equilibrium exhibits the feature that we call the NPL equilibrium. For that, the minimum unit δ is sufficiently small such that the following assumption is satisfied.

Assumption 3. The value of δ and the function $G(s,k)$ satisfy

$$
\min_s\ G^{npl}(s) > \frac{\xi + \beta(\delta + \delta_g)}{1-\beta},
$$

where $\xi = \delta_f\delta_k$.

Lemma 6. *For $k(s,D) < k^*(s)$, the binding no-default constraint implies that*

$$
V(s,D) - \delta_g < G(s,k(s,D)) \le V(s,D).
$$

Proof. The first inequality holds because otherwise the bank can obtain a positive gain by changing $k(s,D)$ to k', where $k' > k(s,D)$ and $|f(k(s,D)) - f(k')| = \beta\delta$. □

Lemma 7. *For all $D > D_{\max}(s)$, it holds that $k(s,D) = k^{npl}(s)$.*

Proposition 8. *For all (s,D) with $D > D_{\max}(s)$, $d(s,D) = d^{npl}(s)$, $k(s,D) = k^{npl}(s)$, $b(s,D) = b^{npl}(s)$, and $V(s,D) = G^{npl}(s)$.*

This proposition[4] is similar to Proposition 3 in Section 3.3 of the main text, but stronger because $D_{\max}(s) \leq \bar{D}$. Once D exceeds $D_{\max}(s)$ at any s, the contractual amount of debt will keep on growing and the constraint $b \leq D$ will never bind. Thus, D becomes irrelevant for the choice of k and b, and the equilibrium variables depend solely on the exogenous state s, given as the NPL equilibrium. The intuition is that when D is larger than $D_{\max}(s)$, it becomes impossible to pay back D in full, and thus the contractual amount of debt becomes payoff irrelevant. It follows that the lender can no longer commit to any future repayment plans. The loss of the bank's credibility leads to an inefficient outcome referred to as the NPL equilibrium.

B.2.3 Characterization of the equilibrium

Here, we summarize the analytical results obtained for the discrete model with $1+r = \beta^{-1}$. First, there exist endogenously determined thresholds, $D_{\max}(s)$, which are defined by (9) and (10).

Define $D_{\min}(s_L)$ by

$$D_{\min}(s_L) = \max\left\{D \in \Delta \,|\, \forall D' \leq D, D_{+1}(s_L, D') < D'\right\}.$$

Since $D_{+1}(s_H, D) \leq D_{+1}(s_L, D)$ for all D, once D becomes sufficiently small that $D \leq D_{\min}(s_L)$, D declines over time thereafter, regardless of the realization of the exogenous state s.

Thus, if the initial debt D_0 satisfies $D_0 \leq D_{\min}(s_L)$, there is no chance that the economy will fall into the NPL equilibrium. In this case, the equilibrium dynamics are qualitatively the same as those of the AH model. The borrower repays as much debt as possible in every period by setting dividend (almost) zero, that is, $F(s,k) - Rk - b \approx 0$ (Lemma 5), where the qualification "almost" is required because of the discretization. Functions $k(s,D)$ and $V(s,D)$ are both non-increasing in D.[5] As the current debt D

[4] In Proposition 8, we have assumed that the parameter values are restricted such that $k^{npl}(s)$ is defined by $k^{npl}(s) \equiv \arg\max_{k \in \Delta_k(s)} F(s,k) - Rk - G(s,k)$. It is generalized as follows, in the case where $k^{npl}(s_L)$ is defined by $k^{npl}(s_L) = \max\{k \in \Delta_k(s_L)|\ G(s,k_L) \leq \beta[\pi_{LL}G(s_L,k) + \pi_{LH}G(s_H, k^{npl}(s_H))]\}$: We define $V^{npl}(s)$ by

$$V^{npl}(s_H) = G^{npl}(s_H),$$
$$V^{npl}(s_L) = \beta\mathbb{E}[V^{npl}(s_{+1})|s = s_L].$$

Then, we redefine $b^{npl}(s)$ by $b^{npl}(s) = F(s, k^{npl}(s)) - Rk^{npl}(s) - G^{npl}(s) + \beta\mathbb{E}[V^{npl}(s_{+1})|s]$. Then, the modified version of Proposition 8 states: For all (s,D) with $D > D_{\max}(s)$, $d(s,D) = d^{npl}(s)$, $k(s,D) = k^{npl}(s)$, $b(s,D) = b^{npl}(s)$, and $V(s,D) = V^{npl}(s)$. The proof of the modified version is similar to that of Proposition 8.

[5] First, Lemma 12 in Section B.2.4 implies that $V(s,D)$ is non-increasing in D. Second, $k(s,D)$ is non-increasing in D, because $k(s,D) = \max\{k \in \Delta_k(s) \,|\, V(s,D) \geq G(s,k)\}$ and $V(s,D)$ is non-increasing.

satisfies $D \leq D_{\min}(s_L)$, the next period debt D_{+1} is smaller than D. Thus, along the equilibrium path, $D_{t+1} = \beta^{-1}[D_t - b(s_t, D_t)]$ converges to 0 within finite periods. When $D = 0$, the bank takes 0 because $b \leq D$ binds at $D = 0$, and the problem (for the bank) is to maximize the firm's profits by selecting $k = k^*(s) = \arg\max_k F(s,k) - Rk$. Thus, the economy converges to a first-best allocation, $\{D, k\} = \{0,\ k^*(s)\}$, within finite periods. In this case, the state variable, D, remains payoff-relevant along the whole equilibrium path.

If the initial debt satisfies $D_0 \geq D_{\max}(s_H)$, debt D_t always increases regardless of the exogenous state s, that is, $D_{t+1} \geq D_t$ with probability one for all t. Then, D_t is no longer a payoff-relevant state variable, and the bank is unable to make a commitment to future repayment plans. As a result, the economy falls into the NPL equilibrium: $\{k(s,D),\ b(s,D),\ d(s,D),\ V(s,D)\} = \{k^{npl}(s),\ b^{npl}(s),\ d^{npl}(s),\ V^{npl}(s)\}$. In the NPL equilibrium, the firm's output is "minimized" in the sense that $k^{npl}(s) = \min_{D \in \Delta} k(s,D)$.

For initial debt D_0 in the intermediate region, $D_{\min}(s_L) < D_0 \leq D_{\max}(s_H)$, the economy may end up with either the first best or NPL equilibrium. Both can occur with a positive probability. While D is in this region, the dividend to the firm is $F(s,k) - Rk - b \approx 0$ (Lemma 5). D remains to be payoff-relevant.

B.2.4 Existence of equilibrium

In this subsection, we demonstrate the existence of an equilibrium, which is characterized as a fixed point of an operator, T, on the functions of (s, D). As the space for (s, D) is discrete and finite, the existence of an equilibrium is proved by finding a fixed point of the operator T in a finite-dimensional vector space.

Define the operator T by

$$(d^{(n+1)}(s,D), V^{(n+1)}(s,D), \bar{D}^{(n+1)}(s)) = T(d^{(n)}(s,D), V^{(n)}(s,D), \bar{D}^{(n)}(s)),$$

where $(d^{(n+1)}(s,D), V^{(n+1)}(s,D), \bar{D}^{(n+1)}(s))$ is generated from $(d^{(n)}(s,D), V^{(n)}(s,D), \bar{D}^{(n)}(s))$, as follows. Define $\Gamma^{(n+1)}(s,D)$ by

$$\begin{aligned}\Gamma^{(n+1)}(s,D) \equiv \{b \in \Delta_b(s,D) \mid \exists k \in \Delta_k(s) \ \text{ s.t.}\\ & D_{+1} = \min\{N_{\max}\delta,\ n_\delta[(1+r)(D-b)]\},\\ & F(s,k) - Rk - b + \beta \mathbb{E} V^{(n)}(s_{+1}, D_{+1}) \geq G(s,k),\\ & F(s,k) - Rk - b \geq 0\}.\end{aligned}$$

Given state (s,D) and expectations $(V^{(n)}(s,D),\ d^{(n)}(s,D))$, the bank solves

$$d^{(n+1)}(s,D) = \max_{b \in \Gamma^{(n+1)}(s,D)} \ b + \beta \mathbb{E} d^{(n)}(s_{+1}, D_{+1}). \tag{13}$$

Denote by $\Sigma^{(n+1)}(s,D)$ the set of (b, D_{+1}) that solves the maximization in (13). The bank decides k and $V^{(n+1)}(s,D)$ by solving the following problem.

$$V^{(n+1)}(s,D) = \max_{k \in \Delta_k(s),\ (b, D_{+1}) \in \Sigma^{(n+1)}(s,D)} F(s,k) - Rk - b + \beta \mathbb{E} V^{(n)}(s_{+1}, D_{+1}), \tag{14}$$

subject to

$$F(s,k) - Rk - b + \beta \mathbb{E} V^{(n)}(s_{+1}, D_{+1}) \geq G(s,k),$$
$$F(s,k) - Rk - b \geq 0.$$

Let $\Lambda^{(n+1)}(s, D)$ denote the set of (k, b, D_{+1}) that solves the maximization in (14).

The equilibrium values of (k, b, D_{+1}) are selected as follows. First, $b^{(n+1)}(s, D)$ and $D^{(n+1)}_{+1}(s, D)$ are determined as

$$b^{(n+1)}(s,D) = \max_{(k,b,D_{+1}) \in \Lambda^{(n+1)}(s,D)} b, \tag{15}$$

$$D^{(n+1)}_{+1}(s,D) = \min\{N_{\max}\delta,\ n_\delta((1+r)[D - b^{(n+1)}(s,D)])\}. \tag{16}$$

Then, $k^{(n+1)}(s, D)$ is decided as

$$k^{(n+1)}(s,D) = \max_{(k,\ b^{(n+1)}(s,D),\ D^{(n+1)}_{+1}(s,D)) \in \Lambda^{(n+1)}(s,D)} k,$$

and $\bar{D}^{(n+1)}(s)$ is provided by

$$\bar{D}^{(n+1)}(s_H) = \max\left\{ D \in \Delta \,|\, D^{(n+1)}_{+1}(s_H, D) < \bar{D}^{(n)}(s_H) \right\},$$
$$\bar{D}^{(n+1)}(s_L) = \max\left\{ D \in \Delta \,|\, D^{(n+1)}_{+1}(s_L, D) < \bar{D}^{(n)}(s_H) \right\}.$$

Define $V^*_H \equiv \frac{1}{1-\beta}[F(s_H, k^*(s_H)) - Rk^*(s_H)]$.

We set the initial values $(\bar{D}^{(0)}(s), d^{(0)}(s,D), V^{(0)}(s,D))$ as follows.

$$\bar{D}^{(0)}(s) = \bar{D}^{(0)} \equiv V^*_H - G^{npl}(s_H),$$

$$d^{(0)}(s,D) = \begin{cases} D & \text{for } D \leq \bar{D}^{(0)}, \\ d^{npl}(s) & \text{for } D > \bar{D}^{(0)}, \end{cases}$$

$$V^{(0)}(s,D) = \begin{cases} V^*_H - D & \text{for } D \leq \bar{D}^{(0)}, \\ G^{npl}(s) & \text{for } D > \bar{D}^{(0)}. \end{cases}$$

Now, the existence of a fixed point of operator T is established by demonstrating the convergence of the sequence $\{d^{(n)}, V^{(n)}, \bar{D}^{(n)}\}_{n=0}^{\infty}$.

Theorem 9. *There exists a fixed point $(d(s,D), V(s,D), D_{\max}(s))$ of the operator T, that is, $(d, V, D_{\max}) = T(d, V, D_{\max})$.*

This fixed point is an equilibrium of the economy. The proof of this theorem is as follows. The following lemmas demonstrate that $(d^{(n)}(s,D),\ V^{(n)}(s,D),\ \bar{D}^{(n)}(s))$ satisfies

$$\begin{aligned}(d^{npl}(s), G^{npl}(s), d^{npl}(s)) &\leq (d^{(n+1)}(s,D), V^{(n+1)}(s,D), \bar{D}^{(n+1)}(s)) \\ &\leq (d^{(n)}(s,D), V^{(n)}(s,D), \bar{D}^{(n)}(s))\end{aligned}$$

for $D > d^{npl}(s)$, and that

$$(0,\ G^{npl}(s), d^{npl}(s)) \le (d^{(n+1)}(s,D), V^{(n+1)}(s,D), \bar{D}^{(n+1)}(s)) \\ \le (d^{(n)}(s,D), V^{(n)}(s,D), \bar{D}^{(n)}(s))$$

for $D \le d^{npl}(s)$. Thus, the sequence $\left\{d^{(n)}(s,D), V^{(n)}(s,D), \bar{D}^{(n)}(s)\right\}_{n=0}^{\infty}$ at any fixed (s,D) converges pointwise, because it is a weakly decreasing sequence of real numbers, which is bounded from below: $\exists(d(s,D), V(s,D), D_{\max}(s))$ such that

$$(d^{(n)}(s,D), V^{(n)}(s,D), \bar{D}^{(n)}(s)) \to (d(s,D), V(s,D), D_{\max}(s))$$

as $n \to \infty$. This $(d(s,D), V(s,D), D_{\max}(s))$ is a fixed point of the operator T by construction.

The proof is by induction. The first step of the induction is provided by the following lemma.

Lemma 10. *Denote* $(d^{(1)}(s,D), V^{(1)}(s,D), \bar{D}^{(1)}(s)) = T(d^{(0)}(s,D), V^{(0)}(s,D), \bar{D}^{(0)}(s))$. *Let* $(b^{(1)}(s,D), k^{(1)}(s,D))$ *be the value of* (b,k) *that solves* (13) *and* (14) *with* $n = 0$. *Then,* $(d^{(1)}(s,D),\ V^{(1)}(s,D),\ \bar{D}^{(1)}(s),\ b^{(1)}(s,D),\ k^{(1)}(s,D))$ *satisfies*

(i) $d^{(1)}(s, D+\delta) \le d^{(1)}(s,D) + \delta$,

(ii) $d^{npl}(s) \le d^{(1)}(s,D) \le d^{(0)}(s,D)$ *for* $D > d^{npl}(s)$, *and* $0 \le d^{(1)}(s,D) \le d^{(0)}(s,D)$ *for* $D \le d^{npl}(s)$,

(iii) $\forall D > \bar{D}^{(1)}(s)$, $d^{(1)}(s,D) = d^{npl}(s)$, $V^{(1)}(s,D) = V^{npl}(s)$, $b^{(1)}(s,D) = b^{npl}(s)$, $k^{(1)}(s,D) = k^{npl}(s)$,

(iv) $V^{(1)}(s, D+\delta) \le -\delta + V^{(1)}(s,D)$ *for* $D < \bar{D}^{(1)}(s)$,

(v) $\forall(s,D)$, $G^{npl}(s) \le V^{(1)}(s,D) \le V^{(0)}(s,D)$,

(vi) $d^{npl}(s) < \bar{D}^{(1)}(s) < \bar{D}^{(0)}$.

The second step of the induction is provided by the following lemma.

Lemma 11. *Denote* $(d^{(n+1)}(s,D), V^{(n+1)}(s,D), \bar{D}^{(n+1)}(s)) = T(d^{(n)}(s,D), V^{(n)}(s,D), \bar{D}^{(n)}(s))$. *Let* $(b^{(n+1)}(s,D),\ k^{(n+1)}(s,D))$ *be the value of* (b,k) *that solves* (13) *and* (14). *Suppose that* $(d^{(n)}(s,D),\ V^{(n)}(s,D),\ \bar{D}^{(n)}(s),\ b^{(n)}(s,D),\ k^{(n)}(s,D))$ *satisfies*

(i') $d^{(n)}(s, D+\delta) \le d^{(n)}(s,D) + \delta$,

(ii') $d^{npl}(s) \le d^{(n)}(s,D) \le d^{(n-1)}(s,D)$ *for* $D > d^{npl}(s)$, *and* $0 \le d^{(n)}(s,D) \le d^{(n-1)}(s,D)$ *for* $D \le d^{npl}(s)$

(iii') $\forall D > \bar{D}^{(n)}(s)$, $d^{(n)}(s,D) = d^{npl}(s)$ *and* $V^{(n)}(s,D) = V^{npl}(s)$,

(iv') $V^{(n)}(s, D+\delta) \leq -\delta + V^{(n)}(s, D)$ *for* $D < \bar{D}^{(n)}(s)$*,*

(v') $\forall(s, D), \quad G^{npl}(s) \leq V^{(n)}(s, D) \leq V^{(n-1)}(s, D)$*,*

(vi') $0 < \bar{D}^{(n)}(s) \leq \bar{D}^{(n-1)}(s)$*.*

Then, $(d^{(n+1)}(s, D)$*,* $V^{(n+1)}(s, D)$*,* $\bar{D}^{(n+1)}(s)$*,* $b^{(n+1)}(s, D)$*,* $k^{(n+1)}(s, D))$ *satisfies*

(i) $d^{(n+1)}(s, D+\delta) \leq d^{(n+1)}(s, D) + \delta$*,*

(ii) $d^{npl}(s) \leq d^{(n+1)}(s, D) \leq d^{(n)}(s, D)$ *for* $D > d^{npl}(s)$*, and* $0 \leq d^{(n+1)}(s, D) \leq d^{(n)}(s, D)$ *for* $D \leq d^{npl}(s)$*,*

(iii) $\forall D > \bar{D}^{(n+1)}(s)$*,* $d^{(n+1)}(s, D) = d^{npl}(s)$ *and* $V^{(n+1)}(s, D) = V^{npl}(s)$*,*

(iv) $V^{(n+1)}(s, D+\delta) \leq -\delta + V^{(n+1)}(s, D)$ *for* $D < \bar{D}^{(n+1)}(s)$*,*

(v) $\forall(s, D), \quad G^{npl}(s) \leq V^{(n+1)}(s, D) \leq V^{(n)}(s, D)$*,*

(vi) $0 < \bar{D}^{(n+1)}(s) \leq \bar{D}^{(n)}(s)$*.*

In Sections B.2.1 and B.2.2, we have assumed Assumptions 1 and 2 to establish some equilibrium properties. The next lemma demonstrates that those assumptions are indeed satisfied by the equilibrium constructed as the fixed point of T.

Lemma 12. *For* $D \leq D_{\max}(s)$*,* $V(s, D+\delta) \leq V(s, D) - \delta$*. For all* $D \geq \delta$*,* $b(s, D)$ *satisfies* $b(s, D) \geq \delta$*.*

B.3 Discrete model with stochastic debt restructuring

In the baseline model, debt restructuring is prohibited. We modify the model in this section such that debt restructuring is feasible with some friction. For simplicity, we adopt a reduced-form approach: In each period t, the bank may be able to reduce the contractual amount of debt D_t. However, this option of debt restructuring arrives with an exogenously given probability $p \in (0, 1)$ in each period. With this option in hand, the bank can reduce D_t to any value $D \in [0, D_t]$. The probability p is a fixed parameter and represents the friction in debt restructuring.

When the bank with contractual amount of debt D_t restructures debt, it reduces D_t to $\hat{D}(s, D_t)$ defined by

$$\hat{D}(s, D_t) = \arg \max_{0 \leq D \leq D_t} d(s, D).$$

Here, $d(s, D)$ is the PDV of repayments, given as the solution to (20) below. Clearly, $\hat{D}(s, D) = D$ for a small value of D, because the bank has no incentive to reduce the debt if it is sufficiently small.

Definitions: Given the possibility of debt restructuring, we modify the formulation of the discrete model, because the NPL equilibrium, $\{k^{npl}(s),\ b^{npl}(s),\ d^{npl}(s),\ G^{npl}(s)\}$ now depends on when and by how much debt is reduced. The grid points for D, D_{+1}, and k are the same as in the previous sections, but we modify the grid points for b, $\Delta_b(s, D)$.

Take as given the beliefs $\{V^e(s, D), k^e_{npl}(s), \hat{D}^e(s, D)\}$, where $V^e(s, D)$ describes the expected value of the firm, $k^e_{npl}(s)$ the expected value of working capital in the NPL equilibrium, and $\hat{D}^e(s, D)$ the expected amount of debt after debt restructuring. We use the same parameter values as in the baseline model. For the probability p of a certain size, the candidate for $k^{npl}(s)$ makes the enforcement constraint nonbinding, that is, $\tilde{k}^{npl}(s) \equiv \arg\max_{k\in\Delta_k(s)} F(s, k) - Rk - G(s, k)$ does not satisfy

$$G(s, k) > \beta\mathbb{E}[(1 - p)V^{npl}(s_{+1}) + pV^e(s_{+1}, \hat{D}^e_{+1})|s], \tag{17}$$

where we define $V^{npl}(s_{+1})$ by

$$V^{npl}(s) = F(s, k^{npl}(s)) - Rk^{npl}(s) - b^{npl}(s) + \beta\mathbb{E}[(1 - p)V^{npl}(s_{+1}) + pV^e(s_{+1}, \hat{D}^e_{+1})|s],$$

and $D^e_{+1} = \hat{D}^e(s_{+1}, D_{+1})$.[6] Therefore, not as in the baseline case, we define $k^{npl}(s)$ for the case where $\tilde{k}^{npl}(s)$ does not satisfy (17) as

$$k^{npl}(s) = \max\{k \in \Delta_k(s)|\ G(s, k) \leq \beta\mathbb{E}[(1 - p)V^{npl}(s_{+1}) + pV^e(s_{+1}, \hat{D}^e_{+1})|s]. \tag{18}$$

Note that $k^{npl}(s)$ depends on the given beliefs $\{V^e(s, D), k^e_{npl}(s), \hat{D}^e(s, D)\}$. Of course, $k^{npl}(s) = k^e_{npl}(s)$ must hold in equilibrium. We define $b^{npl}(s)$ by

$$b^{npl}(s) = F(s, k^{npl}(s)) - Rk^{npl}(s) + \beta\mathbb{E}[(1 - p)V^{npl}(s_{+1}) + pV^e(s_{+1}, \hat{D}^e_{+1})|s] - G^{npl}(s),$$

in the case where $k^{npl}(s) = \tilde{k}^{npl}(s)$, and by

$$b^{npl}(s) = F(s, k^{npl}(s)) - Rk^{npl}(s), \tag{19}$$

in the case where $k^{npl}(s)$ is defined by (18).

Now, we define the grid points for b as

$$\Delta_b(s, D) = \left\{b \in \mathbb{R} \mid \exists D_{+1} \in \Delta_{+1} \ \text{ s.t. } \ b = D - \frac{1}{1 + r}D_{+1}, \ \text{ and } b \geq 0\right\} \cup \{b^{npl}(s)\}.$$

As stated above, the NPL equilibrium, $\{k^{npl}(s),\ b^{npl}(s),\ d^{npl}(s),\ V^{npl}(s)\}$, is defined given the beliefs $\{V^e(s, D),\ k^e_{npl}(s),\ \hat{D}^e(s, D)\}$.

[6] Note that in the NPL equilibrium where $D > D_{\max}(s)$, $\hat{D}(s, D)$ is independent of D, that is, $\hat{D}(s, D) = \hat{D}(s)$, which is defined by $\hat{D}(s) \equiv \arg\max_{D\in\Delta} d(s, D)$. Thus, for $D > D_{\max}(s)$, $\hat{D}^e(s, D)$ should also be independent of D.

The bank's problem: Given beliefs $\{V^e(s,D),\ k^e_{npl}(s),\ \hat{D}^e(s,D)\}$, the bank solves

$$d(s,D) = \max_{b\in\Gamma(s,D)} \quad b + \beta\mathbb{E}[(1-p)d(s_{+1},D_{+1}) + pd(s_{+1},\hat{D}^e_{+1})], \tag{20}$$

where

$$\begin{aligned}
\Gamma(s,D) = \{b \in \Delta_b(s,D) \mid \exists k \in \Delta_k(s) \ \text{ s.t.}&\\
& D_{+1} = \min\{N_{\max}\delta,\ (1+r)(D-b)\},\\
& F(s,k) - Rk - b + \beta\mathbb{E}[(1-p)V^e(s_{+1},D_{+1}) + pV^e(s_{+1},\hat{D}^e_{+1})] \geq G(s,k),\\
& F(s,k) - Rk - b \geq 0\}.
\end{aligned}$$

Let $\Sigma(s,D)$ denote the set of (b, D_{+1}) that solves the maximization problem in (20). The bank decides on k and $V(s,D)$ by solving the following problem:

$$\begin{aligned}
V(s,D) = \max_{k\in\Delta_k(s),\ (b,D_{+1})\in\Sigma(s,D)} & F(s,k) - Rk - b\\
& + \beta\mathbb{E}[(1-p)V^e(s_{+1},D_{+1}) + pV^e(s_{+1},\hat{D}^e_{+1})],
\end{aligned} \tag{21}$$

subject to

$$\begin{aligned}
& F(s,k) - Rk - b + \beta\mathbb{E}[(1-p)V^e(s_{+1},D_{+1}) + pV^e(s_{+1},\hat{D}^e_{+1})] \geq G(s,k),\\
& F(s,k) - Rk - b \geq 0.
\end{aligned}$$

Let $\Lambda(s,D)$ denote the set of (k, b, D_{+1}) that solves the maximization problem in (21).

The equilibrium values of (k, b, D_{+1}) are determined as follows. First, $b(s,D)$ and $D_{+1}(s,D)$ are given by

$$b(s,D) = \max_{(k,b,D_{+1})\in\Lambda(s,D)} b, \tag{22}$$

$$D_{+1}(s,D) = \min\{N_{\max}\delta,\ (1+r)\{D - b(s,D)\}\}. \tag{23}$$

Then, $k(s,D)$ is determined by

$$k(s,D) = \max_{(k,b(s,D),D_{+1}(s,D))\in\Lambda(s,D)} k,$$

$\hat{D}(s,D)$ by

$$\hat{D}(s,D) = \arg\max_{D'\leq D} d(s,D'),$$

and $d^{npl}(s)$ is

$$d^{npl}(s) = b^{npl}(s) + \beta\mathbb{E}[(1-p)d^{npl}(s_{+1}) + pd(s_{+1},\hat{D}^e_{+1})].$$

For consistency, we require that

$$V(s,D) = V^e(s,D), \quad k^{npl}(s) = k^e_{npl}(s), \quad \text{and} \quad \hat{D}(s,D) = \hat{D}^e(s,D). \tag{24}$$

B.4 Proofs

B.4.1 Proof of Lemma 2

There exists $D_{+1} \in \Delta$ such that

$$d(s, D+\delta) = b' + \beta \mathbb{E} d(s_{+1}, D_{+1}),$$
$$b' = D + \delta - \beta D_{+1}.$$

Note that Assumption 2 implies that $b' \geq \delta$. Consider $b = D - \beta D_{+1}$. Then, $b \geq 0$, and therefore, $b \in \Delta_b(s, D)$, while b may not be an element of $\Delta_b(s, D+\delta)$. It is easily confirmed that $b \in \Gamma(s, D)$. Thus,

$$\begin{aligned} d(s, D+\delta) &= b + \delta + \beta \mathbb{E} d(s_{+1}, D_{+1}) \\ &= \delta + [b + \beta \mathbb{E} d(s_{+1}, D_{+1})] \\ &\leq \delta + \max_{\tilde{b} \in \Gamma(s,D)} [\tilde{b} + \beta \mathbb{E} d(s_{+1}, \beta^{-1}(D - \tilde{b})] \\ &= \delta + d(s, D). \end{aligned}$$

B.4.2 Proof of Lemma 3

Suppose that $b(s, D)$ is not the maximum feasible value. Then, $b(s, D) + \beta\delta \in \Gamma(s, D)$. We compare $d(s, D)$ and $X(b(s, D) + \beta\delta, s, D)$, where $X(b, s, D) \equiv b + \beta \mathbb{E} d(s_{+1}, \beta^{-1}[D - b])$. Lemma 2 implies that

$$\begin{aligned} X(b(s,D) + \beta\delta, s, D) &= b(s, D) + \beta\delta + \beta \mathbb{E} d(s_{+1}, \beta^{-1}(D - b(s, D)) - \delta) \\ &= b(s, D) + \beta \mathbb{E}\{\delta + d(s_{+1}, \beta^{-1}(D - b(s, D)) - \delta)\} \\ &\geq b(s, D) + \beta \mathbb{E} d(s_{+1}, \beta^{-1}(D - b(s, D))) \\ &= d(s, D) = \max_b X(b, s, D). \end{aligned}$$

If $X(b(s,D) + \beta\delta, s, D) > d(s, D)$, it contradicts (3), which defines $b(s, D)$. If $X(b(s, D) + \beta\delta, s, D) = d(s, D)$, Assumption 1 implies that $F(s, k(s, D)) - Rk(s, D) - b(s, D) - \beta\delta + \beta \mathbb{E} V^e(s_{+1}, D_{+1}(s, D) - \delta) \geq F(s, k(s, D)) - Rk(s, D) - b(s, D) + \beta \mathbb{E} V^e(s_{+1}, D_{+1}(s, D)) = V(s, D)$. Then, $b(s, D) + \beta\delta$ should be the equilibrium value of b. This is a contradiction. Therefore, $b(s, D)$ is the maximum feasible value in $\Gamma(s, D)$, that is, $b(s, D) = \bar{b}(s, D)$.

Next, we prove $k(s, D) > k^{npl}(s)$ for $D \leq D_{\max}(s)$. For $D \leq D_{\max}(s)$, we have $V(s, D) \geq G^{npl}(s) + \delta$, as $V(s, D) \geq V(s, D+\delta) + \delta$ from Assumption 1 and $V(s, D+\delta) \geq G^{npl}(s)$ due to Lemma **??** in Appendix **??**. Now, we prove $k(s, D) > k^{npl}(s)$ by contradiction. Suppose that $k(s, D) = k^{npl}(s)$. Then, since $(b(s, D),\ k(s, D))$ satisfy the above inequality and the limited liability constraint, we have

$$V(s, D) = F(s, k^{npl}(s)) - Rk^{npl}(s) - b(s, D) + \beta \mathbb{E} V(s_{+1}, D_{+1}(s, D)) \geq G^{npl}(s) + \delta,$$
$$F(s, k^{npl}(s)) - Rk^{npl}(s) - b(s, D) \geq 0.$$

Pick $k^{npl+}(s)$ $(> k^{npl}(s))$, which is defined by $f(s, k^{npl}(s)) - f(s, k^{npl+}(s)) = \beta\delta$, where $f(s, k) \equiv F(s, k) - Rk - G(s, k)$. Then, $k^{npl+}(s)$ satisfies

$$F(s, k^{npl+}(s)) - Rk^{npl+}(s) - b(s, D) + \beta\mathbb{E}V(s_{+1}, D_{+1}(s, D)) \geq G(s, k^{npl+}(s)) + (1-\beta)\delta,$$
$$F(s, k^{npl+}(s)) - Rk^{npl+}(s) - b(s, D) \geq 0.$$

Therefore, $k(s, D)$ should be $k^{npl+}(s)$, not $k^{npl}(s)$, because $k^{npl+}(s)$ is feasible without changing $b(s, D)$ and $D_{+1}(s, D)$. This is a contradiction. Thus, we have demonstrated that for $D \leq D_{\max}(s)$, $k(s, D) > k^{npl}(s)$.

B.4.3 Proof of Lemma 5

Suppose that $F(s, k(s, D)) - Rk(s, D) - b(s, D) \geq \xi + \beta\delta$ for $k(s, D) \in (k^{npl}(s),\ k^*(s))$. In this case, the bank can choose $\hat{k} < k(s, D)$, where $\hat{k} \in \Delta_k(s)$, so that $F(s, \hat{k}) - R\hat{k} - b(s, D) \geq \beta\delta$. We know that $F(s, k(s, D)) - Rk(s, D) - G(s, k(s, D)) - b(s, D) + \beta\mathbb{E}V^e(s, D_{+1}(s, D)) \geq 0$, where $D_{+1}(s, D) = \beta^{-1}[D - b(s, D)]$. As $F(s, k) - Rk - G(s, k)$ is strictly decreasing in k for $k > k^{npl}(s)$, it must be the case that

$$F(s, \hat{k}) - R\hat{k} - G(s, \hat{k}) \geq F(s, k(s, D)) - Rk(s, D) - G(s, k(s, D)) + \beta\delta.$$

Thus, $\hat{b} = b(s, D) + \beta\delta$ satisfies

$$F(s, \hat{k}) - R\hat{k} - \hat{b} \geq 0,$$
$$F(s, \hat{k}) - R\hat{k} - \hat{b} - G(s, \hat{k}) + \beta\mathbb{E}V^e(s_{+1}, \beta^{-1}(D - \hat{b})) \geq 0.$$

Then, $\hat{b} = b(s, D) + \beta\delta$ is feasible and Lemma 3 implies that $\hat{b}$ should be the solution to (3). This is a contradiction.

B.4.4 Proof of Lemma 7

For any s and $D > D_{\max}(s)$, we consider a stochastic sequence $\{s_t, k_t, b_t, D_t\}$, where $k_t = k(s_t, D_t)$, $b_t = b(s_t, D_t)$, $D_t = n_\delta[(1+r)(D_{t-1} - b_{t-1})]$, $s_0 = s$, and $D_0 = D$, given that s_t is an exogenous stochastic variable.

First, we consider the case where $s = s_H$. Suppose there exists D, which satisfies $D > D_{\max}$, such that $k(s, D) \neq k^{npl}(s)$. Then, Lemma **??** implies $k(s, D) > k^{npl}(s)$. Then, Lemma 5 implies that $0 \leq F(s, k) - Rk - b < \xi + \beta\delta$, which implies, together with $V \geq G(s, k)$, that

$$G(s, k(s, D)) \leq V(s, D) \leq \xi + \beta\delta + \beta\mathbb{E}V(s_{+1}, D_{+1})$$

As it is obvious that $V(s_L, D) \leq V(s_H, D)$, it must be the case that $\mathbb{E}V(s_{+1}, D_{+1}) \leq V(s_H, D_{+1})$. Then,

$$G(s, k(s, D)) \leq V(s, D) \leq \xi + \beta\delta + \beta V(s_H, D_{+1}), \tag{25}$$

where $D_{+1} > D$ as $D > D_{\max}(s)$. Lemma 6 implies that $V(s_H, D_{+1}) < \delta_g + G(s_H, k(s_H, D_{+1}))$. Thus,

$$G(s_H, k(s_H, D)) < \xi + \beta(\delta + \delta_g) + \beta G(s_H, k(s_H, D_{+1})). \tag{26}$$

Assumption 3 and the inequality (26) imply that $G(s_H, k(s_H, D)) < (1-\beta)G^{npl}(s) + \beta G(s_H, k(s_H, D_{+1})) \leq G(s_H, k(s_H, D_{+1}))$, because $G^{npl}(s) \leq G(s_H, k(s_H, D_{+1}))$. Thus, $k(s_H, D) < k(s_H, D_{+1})$. Let us set $(s_0, D_0) = (s, D)$ and consider the sequence $\{s_t, D_t, k(s_t, D_t)\}$. Given (26), we can prove the following inequality:

$$k^{npl}(s_H) < k(s_H, D_t) < k(s_H, D_{t+1}), \tag{27}$$

$$G(s_H, k(s_H, D_0)) < \frac{\{\xi + \beta(\delta + \delta_g)\}(1-\beta^t)}{1-\beta} + \beta^t G(s_H, k(s_H, D_t)) \tag{28}$$

The proof is by induction. The above argument has proven (27) and (28) for $t = 0$. Suppose that (27) holds for $t-1$. (26) applies for D_t and implies that

$$G(s_H, k(s_H, D_t)) < \xi + \beta(\delta + \delta_g) + \beta G(s_H, k(s_H, D_{t+1})), \tag{29}$$

which, together with Assumption 3, implies that $G(s_H, k(s_H, D_{t+1})) > G(s_H, k(s_H, D_t))$, or $k(s_H, D_{t+1}) > k(s_H, D_t)$. Thus, (27) has been proven for t. Suppose that (28) holds for t. This inequality, together with (29), implies that

$$\begin{aligned} G(s_H, k(s_H, D_0)) &< \frac{\{\xi + \beta(\delta + \delta_g)\}(1-\beta^t)}{1-\beta} + \beta^t G(s_H, k(s_H, D_t)) \\ &< \frac{\{\xi + \beta(\delta + \delta_g)\}[1-\beta^t + \beta^t(1-\beta)]}{1-\beta} + \beta^{t+1} G(s_H, k(s_H, D_{t+1})) \\ &= \frac{\{\xi + \beta(\delta + \delta_g)\}(1-\beta^{t+1})}{1-\beta} + \beta^{t+1} G(s_H, k(s_H, D_{t+1})). \end{aligned}$$

Thus, (28) has been proven for $t+1$. We have demonstrated that (27) and (28) hold for all t.

Assumption 3 and (28) imply that in the limit of $t \to \infty$, we have $V(s_t, D_t) \to \infty$. This is a contradiction because $V(s, D)$ is bounded from above: $V(s, D) < V_{\max}$. Thus, it cannot be the case that $k(s_H, D) \neq k^{npl}(s_H)$.

Next, we consider the case where $s = s_L$. Suppose that $k(s_L, D) \neq k^{npl}(s_L)$. Then, Lemma **??** implies that $k(s_L, D) > k^{npl}(s_L)$. In this case, Lemmas 5 and 6 imply that for $D_0 = D$ and the sequence $\{s_t, D_t, k(s_t, D_t)\}$,

$$\begin{aligned} G(s_L, k(s_L, D_t)) &< \xi + \beta(\delta + \delta_g) + \beta \mathbb{E}_t G(s_{t+1}, k(s_{t+1}, D_{t+1})) \\ &= \xi + \beta(\delta + \delta_g) + \beta[p_L G(s_L, k(s_L, D_{t+1})) + (1-p_L)G^{npl}(s_H)], \end{aligned}$$

where $p_L = \Pr(s_{t+1} = s_L | s_t = s_L)$ and $G(s_H, k(s_H, D_{t+1})) = G^{npl}(s_H)$ for $D_{t+1} > D_{\max}$, as shown above. Let $k(s_L, D) = k_0$ and define $\{k_t\}_{t=0}^{\infty}$ by the following law of motion,

$$G(s_L, k_t) = \xi + \beta(\delta + \delta_g) + \beta[p_L G(s_L, k_{t+1}) + (1-p_L)G^{npl}(s_H)].$$

Lemma 2 in Section 3.3 of the main text implies that $k(s_L, D_t) \geq k^{npl}(s_L)$ for all $t \geq 1$. In the case where $k(s_L, D) = k_0 > k^{npl}(s_L)$, the sequence $\{k_t\}_{t=0}^{\infty}$ is such that $\lim_{t\to\infty} k_t = \infty$. Thus, $V(s_L, D_t) > G(s_L, k(s_L, D_t)) - \delta_g$ goes to infinity, and eventually violates the condition $V(s_L, D_t) < V_{\max}$. This is a contradiction. Thus, $k(s_L, D)$ must be $k^{npl}(s_L)$.

Therefore, if $D > D_{\max}$, then $k(s, D) = k^{npl}(s)$ for all $s \in \{s_L, s_H\}$.

B.4.5 Proof of Proposition 8

The proof consists of two parts. First, we prove the existence of one equilibrium, in which $V^e(s, D) = G(s, k^{npl}(s)) \equiv G^{npl}(s)$. Second, we demonstrate that this equilibrium is the unique equilibrium that maximizes $d(s, D)$ subject to the no-default condition.

Existence: we guess and later verify that $V^e(s, D) = G^{npl}(s)$. Given this expectation, the bank solves

$$d(s, D) = \max_{b\in\Delta_b(s,D), k\in\Delta_k(s)} b + \beta\mathbb{E}d(s_{+1}, D_{+1}),$$

$$\text{s, t.} \begin{cases} F(s, k) - Rk - b + \beta\mathbb{E}G(s_{+1}, k^{npl}(s_{+1})) \geq G(s, k), \\ F(s, k) - Rk - b \geq 0. \end{cases}$$

Given that $V^e(s, D) = G^{npl}(s)$, it is easily shown that $\Gamma(s, D) = \{b \mid b \in \Delta_b(s, D),\ 0 \leq b \leq b^{npl}(s)\}$.

Claim: The solution to the bank's problem is $b(s, D) = b^{npl}(s)$ and $k(s, D) = k^{npl}(s)$.
(Proof of Claim)
Because $b(s, D) \leq b^{npl}(s)$, there exists a nonnegative integer m and a nonnegative real number ε, where $0 \leq \varepsilon < \beta\delta$, such that $b(s, D) = b^{npl}(s) - \varepsilon - m\beta\delta$. Then, $D_{+1}(s, D) = \min\{N_{\max}\delta, \beta^{-1}[D - b(s, D)]\} = D^{npl}_{+1} + m'\delta$, where $0 \leq m' \leq m$ and we define $D^{npl}_{+1} = \min\{N_{\max}\delta, n_\delta(\beta^{-1}[D - b^{npl}(s)])\}$. Thus,

$$\begin{aligned} d(s, D) &= b(s, D) + \beta\mathbb{E}d(s_{+1}, D^{npl}_{+1} + m'\delta) \\ &= b^{npl}(s) - \varepsilon - m\beta\delta + \beta\mathbb{E}d(s_{+1}, D^{npl}_{+1} + m'\delta) \\ &= b^{npl}(s) - \varepsilon - (m - m')\beta\delta + \beta\mathbb{E}[-m'\delta + d(s_{+1}, D^{npl}_{+1} + m'\delta)] \\ &\leq b^{npl}(s) - \varepsilon - (m - m')\beta\delta + \beta\mathbb{E}d(s_{+1}, D^{npl}_{+1}) \\ &\leq b^{npl}(s) + \beta\mathbb{E}d(s_{+1}, D^{npl}_{+1}). \end{aligned}$$

The first inequality is from Lemma 2. Therefore, $b(s, D) = b^{npl}(s)$ and $k(s, D) = k^{npl}(s)$.
(End of Proof of Claim)
Thus, the solution to the bank's problem is $k = k^{npl}(s)$ and $b = b^{npl}(s)$. It is also easily confirmed that $V(s, D) = F(s, k^{npl}(s)) - Rk^{npl}(s) - b^{npl}(s) + \beta\mathbb{E}G(s_{+1}, k^{npl}(s_{+1})) = G(s, k^{npl}(s))$, which verifies the expectation.

Uniqueness: In what follows, we demonstrate that $d^{npl}(s)$ is the maximum amount of the present discounted value (PDV) of repayments that satisfies the enforcement constraint, and the above equilibrium is the unique equilibrium that attains $d^{npl}(s)$. We consider the following planner's problem, assuming that $k(s,D) = k^{npl}(s)$. We set this assumption because Lemma 7 shows that $k(s,D) = k^{npl}(s)$ for $D > D_{\max}(s_H)$ in any equilibrium that exists. Given $k(s,D) = k^{npl}(s)$, the planner's problem is

$$\begin{aligned} d(s,D) = &\max_{b,V(s,D)} b + \beta \mathbb{E} d(s_{+1}, \beta^{-1}(D-b)), \\ &\text{s. t. } \; V(s,D) = F(s,k^{npl}(s)) - Rk^{npl}(s) - b + \beta \mathbb{E} V(s_{+1}, \beta^{-1}(D-b)) \geq G^{npl}(s), \\ &\qquad F(s,k^{npl}(s)) - Rk^{npl}(s) - b \geq 0. \end{aligned}$$

Define $W^{npl}(s) = F(s,k^{npl}(s)) - Rk^{npl}(s) + \beta \mathbb{E} W^{npl}(s_{+1})$. Then, $d(s,D) = W^{npl}(s) - V(s,D)$. Thus, the planner's problem can be rewritten as

$$\begin{aligned} &\max_{b,V(s,D)} d(s,D) = W^{npl}(s) - V(s,D), \\ &\text{s. t. } \; d(s,D) \leq W^{npl}(s) - G^{npl}(s), \\ &\qquad F(s,k^{npl}(s)) - Rk^{npl}(s) - b \geq 0. \end{aligned}$$

We temporarily omit the limited liability constraint, $F(s,k^{npl}(s)) - Rk^{npl}(s) - b \geq 0$, and later justify that it is satisfied. Without this constraint, it is obvious that the maximum PDV of repayments is $W^{npl}(s) - G^{npl}(s) = d^{npl}(s)$, and it is attained by setting $b = d(s,D) - \beta \mathbb{E} d(s_{+1}, D_{+1}) = W^{npl}(s) - G^{npl}(s) - \beta \mathbb{E}[W^{npl}(s_{+1}) - G^{npl}(s_{+1})] = F^{npl}(s) - Rk^{npl}(s) - G^{npl}(s) + \beta \mathbb{E} G^{npl}(s_{+1}) = b^{npl}(s)$. Therefore, the value of the firm becomes $V(s,D) = G^{npl}(s)$. By definition of $k^{npl}(s)$, it is obvious that the limited liability constraint is satisfied in this equilibrium. Thus, the unique equilibrium that maximizes the PDV of repayments is the NPL equilibrium.

B.4.6 Proof of Lemma 10

We prove Lemma 10 by explicitly deriving $\{d^{(1)}(s,D), V^{(1)}(s,D), b^{(1)}(s,D), k^{(1)}(s,D)\}$. For $D < D^{**}(s) \equiv F(s,k^*(s)) - Rk^*(s)$,

$$\begin{aligned} &d^{(1)}(s,D) = D, \\ &V^{(1)}(s,D) = F(s,k) - Rk + \beta V_H^* - D, \end{aligned}$$

as $d^{(1)}(s,D) = \max_b b + \beta[\beta^{-1}(D-b)]$ and $b = D$ is feasible because $F(s,k) - Rk + \beta V_H^* - D \geq G(s,k)$ is satisfied at $k = k^*(s)$. Thus, for $0 \leq D \leq D^{**}(s)$, $(d^{(1)}(s,D), V^{(1)}(s,D))$ are given as above, with $k = k^*(s)$ and $b = D$.

For $D \in (D^{**}(s),\ D^*(s)]$, where $D^*(s)$ is the solution to $D^{**}(s) + \beta[\beta^{-1}(D - D^{**}(s))] =$

$D = F(s, k^*(s)) - Rk^*(s) + \beta V_H^* - G(s, k^*(s))$,

$$d^{(1)}(s, D) = D,$$
$$V^{(1)}(s, D) = F(s, k) - Rk + \beta V_H^* - D,$$

where $k = k^*(s)$ and $b = D^{**}(s)$.

For $D \in (D^*(s),\ \hat{D}^{(1)}(s)]$, where $\hat{D}^{(1)}(s) = F(s, k^{npl}(s)) - Rk^{npl}(s) - G(s, k^{npl}(s)) + \beta V_H^*$, the solution $(d^{(1)}(s, D), V^{(1)}(s, D))$ is given as follows.

$$d^{(1)}(s, D) = D,$$
$$V^{(1)}(s, D) = F(s, k(s, D)) - Rk(s, D) + \beta V_H^* - D,$$

where

$$k(s, D) = \arg \max_{k \in \Delta_k(s)} F(s, k) - Rk - D + \beta V_H^*,$$
$$\text{s.t. } F(s, k) - Rk - D + \beta V_H^* \geq G(s, k). \tag{30}$$

Then, it is obvious that $k(s, D)$ is decreasing in D. $D_{+1}(s, D)$ is given by

$$D_{+1}(s, D) = \min_{D_{+1} \in \Delta} D_{+1},$$
$$\text{s. t. } D - \beta D_{+1} \leq F(s, k(s, D)) - Rk(s, D).$$

Note that if $D = \hat{D}^{(1)}(s)$, then $D_{+1} = V_H^* - \beta^{-1} G^{npl}(s) < \bar{D}^{(0)}$. Note that if $D > \hat{D}^{(1)}(s)$, the enforcement constraint (30) is never satisfied for any value of k, if $V^{(1)}(s, D) = F(s, k(s, D)) - Rk(s, D) + \beta V_H^* - D$.

For $D > \hat{D}^{(1)}(s)$, it must be the case that $D_{+1} \geq \bar{D}^{(0)}$, since otherwise $V^{(1)}(s, D)$ becomes $F(s, k(s, D)) - Rk(s, D) + \beta V_H^* - D$ and the enforcement constraint (30) is never satisfied because $\hat{D}^{(1)}(s)$ is the maximum value that is feasible under (30). $D_{+1} \geq \bar{D}^{(0)}$ is feasible for D $(> \hat{D}^{(1)}(s))$, because $\beta^{-1} \hat{D}^{(1)}(s) > \bar{D}^{(0)}$ is easily shown. Given that $D_{+1} > \bar{D}^{(0)}$, we have $d^{(0)}(s, D_{+1}) = d^{npl}(s)$ and $V^{(0)}(s, D_{+1}) = G^{npl}(s)$. Thus, the values of $(d^{(1)}(s, D),\ V^{(1)}(s, D),\ b(s, D), k(s, D))$ are given as the solution to the following problem.

$$d^{(1)}(s, D) = \max_{b \in \Delta_b(s, D),\ k \in \Delta_k(s)} b + \beta \mathbb{E} d^{npl}(s),$$
$$\text{s.t.} \begin{cases} F(s, k) - Rk - b + \beta \mathbb{E} G^{npl}(s) \geq G(s, k), \\ F(s, k) - Rk \geq b. \end{cases}$$

Then,

$$V^{(1)}(s, D) = F(s, k(s, D)) - Rk(s, D) - b(s, D) + \beta \mathbb{E} G^{npl}(s).$$

The solution is

$$b(s,D) = b^{npl}(s), \quad k(s,D) = k^{npl}(s), \quad d^{(1)}(s,D) = d^{npl}(s), \quad V^{(1)}(s,D) = G^{npl}(s),$$

for $D > \hat{D}^{(1)}(s)$. It is also easily confirmed that

$$\hat{D}^{(1)}(s) = \bar{D}^{(1)}(s),$$

where $\bar{D}^{(1)}(s)$ is defined by

$$\begin{aligned}\bar{D}^{(1)}(s_H) = &\max D,\\ &\text{s.t. } D_{+1}(s_H, D) < \bar{D}^{(0)},\\ \bar{D}^{(1)}(s_L) = &\max D,\\ &\text{s.t. } D_{+1}(s_L, D) < \bar{D}^{(0)}.\end{aligned}$$

Now, we can show the following claim.

Claim 1. $\bar{D}^{(1)}(s_L) \leq \bar{D}^{(1)}(s_H) < \bar{D}^{(0)}$.

(Proof of Claim 1)
We have $\bar{D}^{(1)}(s_L) \leq \bar{D}^{(1)}(s_H)$, and

$$\begin{aligned}\bar{D}^{(1)}(s_H) &= F(s_H, k^{npl}(s_H)) - Rk^{npl}(s_H) - G(s_H, k^{npl}(s_H)) + \beta V_H^*\\ &< F(s_H, k^*(s_H)) - Rk^*(s_H) + \beta V_H^* - G(s_H, k^{npl}(s_H))\\ &= V_H^* - G(s_H, k^{npl}(s_H)) = \bar{D}^{(0)}.\end{aligned}$$

(End of proof of Claim 1)
Note that $d^{npl}(s) < \bar{D}^{(1)}(s)$ because $V_H^* > G^{npl}(s_H) + d^{npl}(s_H)$ implies that $d^{npl}(s) = b^{npl}(s) + \beta\mathbb{E}d^{npl}(s_{+1}) = F(s, k^{npl}(s)) - Rk^{npl}(s) - G^{npl}(s) + \beta\mathbb{E}[G^{npl}(s_{+1}) + d^{npl}(s_{+1})] < F(s, k^{npl}(s)) - Rk^{npl}(s) - G^{npl}(s) + \beta V_H^* = \bar{D}^{(1)}(s)$.

These explicit solutions directly imply (i)–(vi) of Lemma 10.

B.4.7 Proof of Lemma 11

Proof of (ii). The assumption (ii') implies that $\mathbb{E}d^{(n)}(s_{+1}, D_{+1}) \leq \mathbb{E}d^{(n-1)}(s_{+1}, D_{+1})$, and the assumption (v') implies that $\Gamma^{(n+1)}(s,D) \subset \Gamma^{(n)}(s,D)$. These facts imply that

$$d^{(n+1)}(s,D) = \max_{b\in\Gamma^{(n+1)}(s,D)} b + \beta\mathbb{E}d^{(n)}(s_{+1}, D_{+1}) \leq \max_{b\in\Gamma^{(n)}(s,D)} b + \beta\mathbb{E}d^{(n-1)}(s_{+1}, D_{+1}) = d^{(n)}(s,D).$$

Since $b^{npl}(s) \in \Gamma^{(n+1)}(s,D)$ and $d^{(n)}(s,D) \geq d^{npl}(s)$ for $D > d^{npl}(s)$,

$$d^{(n+1)}(s,D) = \max_{b\in\Gamma^{(n+1)}(s,D)} b + \beta\mathbb{E}d^{(n)}(s_{+1}, D_{+1}) \geq b^{npl}(s) + \beta\mathbb{E}d^{npl}(s_{+1}) = d^{npl}(s),$$

for $D > d^{npl}(s)$. It is obvious that $d^{(n+1)}(s, D) \geq 0$ for $D \leq d^{npl}(s)$.

Proof of (iii). Assumption (iii') implies that for $D \geq \bar{D}^{(n+1)}(s)$, the values of $(d^{(n+1)}(s, D)$, $V^{(n+1)}(s, D)$, $b^{(n+1)}(s, D)$, $k^{(n+1)}(s, D))$ are given as the solution to the following problem.

$$d^{(n+1)}(s, D) = \max_{b \in \Delta_b(s,D),\ k \in \Delta_k(s)} b + \beta \mathbb{E} d^{npl}(s),$$
$$\text{s.t.} \begin{cases} F(s, k) - Rk - b + \beta \mathbb{E} G^{npl}(s) \geq G(s, k), \\ F(s, k) - Rk \geq b. \end{cases}$$

Then,

$$V^{(n+1)}(s, D) = F(s, k(s, D)) - Rk(s, D) - b(s, D) + \beta \mathbb{E} G^{npl}(s).$$

It is easily shown that the solution is given by

$$b(s, D) = b^{npl}(s), \quad k(s, D) = k^{npl}(s), \quad d^{(n+1)}(s, D) = d^{npl}(s), \quad V^{(n+1)}(s, D) = G^{npl}(s).$$

Proof of (i). For $D \geq \bar{D}^{(n+1)}(s)$, it is the case that $d^{(n+1)}(s, D + \delta) = d^{npl}(s) \leq d^{(n+1)}(s, D) + \delta$ by the part (iii) above. Next, we consider the case where $D < \bar{D}^{(n+1)}(s)$. We can prove the following claim.

Claim 2. For $D < \bar{D}^{(n+1)}(s)$, $b^{(n+1)}(s, D + \delta)$ is the maximum feasible value, that is,

$$b^{(n+1)}(s, D + \delta) = \max_{b \in \Gamma^{(n+1)}(s, D+\delta)} b.$$

(Proof of Claim 2). Suppose that $b^{(n+1)}(s, D + \delta)$ is not the maximum feasible value. Then, $b^{(n+1)}(s, D+\delta)+\beta\delta \in \Gamma^{(n+1)}(s, D+\delta)$. We compare $d^{(n+1)}(s, D+\delta)$ and $X^{(n+1)}(b^{(n+1)}(s, D+\delta) + \beta\delta, s, D + \delta)$, where $X^{(n+1)}(b, s, D) \equiv b + \beta \mathbb{E} d^{(n)}(s_{+1}, \beta^{-1}(D - b))$. Assumption (i') implies that

$$\begin{aligned}
&X^{(n+1)}(b^{(n+1)}(s, D + \delta) + \beta\delta, s, D + \delta) \\
&= b^{(n+1)}(s, D + \delta) + \beta\delta + \beta \mathbb{E} d^{(n)}(s_{+1}, \beta^{-1}(D + \delta - b^{(n+1)}(s, D + \delta)) - \delta) \\
&= b^{(n+1)}(s, D + \delta) + \beta \mathbb{E}\{\delta + d^{(n)}(s_{+1}, \beta^{-1}(D + \delta - b^{(n+1)}(s, D + \delta)) - \delta)\} \\
&\geq b^{(n+1)}(s, D + \delta) + \beta \mathbb{E} d^{(n)}(s_{+1}, \beta^{-1}(D + \delta - b^{(n+1)}(s, D + \delta))) \\
&= d^{(n+1)}(s, D + \delta) = \max_b X^{(n+1)}(b, s, D + \delta).
\end{aligned}$$

Assumption (iv') implies that

$V^{(n+1)}(s, D + \delta) =$

$F(s, k(s, D + \delta)) - Rk(s, D + \delta) - b(s, D + \delta) + \beta \mathbb{E} V^{(n)}(s_{+1}, D_{+1}^{(n+1)}(s, D + \delta)) \leq$

$F(s, k(s, D + \delta)) - Rk(s, D + \delta) - b(s, D + \delta) + \beta \mathbb{E}(-\delta + V^{(n)}(s_{+1}, D_{+1}^{(n+1)}(s, D + \delta) - \delta)) =$

$F(s, k(s, D + \delta)) - Rk(s, D + \delta) - b(s, D + \delta) - \beta\delta + \beta \mathbb{E} V^{(n)}(s_{+1}, D_{+1}^{(n+1)}(s, D + \delta) - \delta).$

Assumption (iv') applies here as $D+\delta \leq \bar{D}^{(n+1)}(s)$, which implies $D_{+1}^{(n+1)}(s, D+\delta) \leq \bar{D}^{(n)}(s)$. These two inequalities imply that the equilibrium value of b should be $b(s, D+\delta)+\beta\delta$. This contradicts the definition of $b^{(n+1)}(s, D+\delta)$. Therefore, $b^{(n+1)}(s, D+\delta)$ is the maximum feasible value. *(End of proof of Claim 2)*
This claim implies that it suffices to consider the region $b \geq \delta$, when we evaluate $d^{(n+1)}(s, D+\delta)$. If $b+\delta \in \Gamma^{(n+1)}(s, D+\delta)$ then $b \in \Gamma^{(n+1)}(s, D)$ for $D > F(s, k^*(s)) - Rk^*(s)$.[7] Defining $\hat{b}$ by $\hat{b} = b(s, D+\delta) - \delta$, it is easily demonstrated that $\hat{b} \in \Gamma^{(n+1)}(s, D)$. Thus,

$$\begin{aligned} d^{(n+1)}(s, D+\delta) &= b(s, D+\delta) + \beta \mathbb{E} d^{(n)}(s_{+1}, \beta^{-1}(D+\delta - b(s, D+\delta))) \\ &= \delta + \hat{b} + \beta \mathbb{E} d^{(n)}(s_{+1}, \beta^{-1}(D - \hat{b})), \\ &\leq \delta + \max_{b \in \Gamma^{(n+1)}(s,D)} b + \beta \mathbb{E} d^{(n)}(s_{+1}, \beta^{-1}(D-b)) = \delta + d^{(n+1)}(s, D). \end{aligned}$$

Proof of (iv). We consider the case where $D+\delta \leq \bar{D}^{(n+1)}(s)$. Define $\tilde{\Delta}_b(s, D) = \{b \in \mathbb{R} | b = D - \beta D_{+1},\ \text{where } D_{+1} \in \Delta_{+1},\ \text{ and } b \geq 0\} \cup \{b^{npl}(s) - \delta\}$. Define $\tilde{\Gamma}^{(n+1)}(s, D) = \{b \in \tilde{\Delta}_b(s, D) \mid \exists k \in \Delta_k(s),\ \text{s.t. } F(s,k) - Rk - b - \delta + \beta \mathbb{E} V^{(n)}(s_{+1}, \beta^{-1}(D-b)) \geq G(s,k),\ \text{ and } F(s,k) - Rk - b - \delta \geq 0\}$. Let $\tilde{b}(s, D)$ be the maximum value of $\tilde{\Gamma}^{(n+1)}(s, D)$. It is obvious that $\tilde{b}(s, D) \leq b(s, D)$, as $b(s, D)$ is the maximum value of $\Gamma^{(n+1)}(s, D)$. $V^{(n+1)}(s, D+\delta)$ can be written as

$$V^{(n+1)}(s, D+\delta) = -\delta + \tilde{V}^{(n+1)}(s, D), \tag{31}$$

where

$$\begin{aligned} \tilde{V}^{(n+1)}(s, D) = \max_{k \in \Delta_k(s)} &F(s,k) - Rk - \tilde{b}(s, D) + \beta \mathbb{E} V^{(n)}(s_{+1}, \beta^{-1}(D - \tilde{b}(s, D))), \qquad (32) \\ \text{s.t. } &F(s,k) - Rk - \tilde{b}(s, D) - \delta + \beta \mathbb{E} V^{(n)}(s_{+1}, \beta^{-1}(D - \tilde{b}(s, D))) \geq G(s,k), \\ &F(s,k) - Rk - \tilde{b}(s, D) - \delta \geq 0. \end{aligned}$$

Let $\tilde{k}(s, D)$ be the solution to (32). The following claim holds:

Claim 3. $\tilde{b}(s, D)$ and $\tilde{k}(s, D)$ satisfy $\tilde{b}(s, D) \leq b(s, D)$ and $\tilde{k}(s, D) \leq k(s, D)$.

(Proof of Claim 3). We know $\tilde{b}(s, D) \leq b(s, D)$ from the above argument. Now, $k(s, D)$ is the maximum k that satisfies

$$\begin{aligned} &F(s,k) - Rk - b(s, D) + \beta \mathbb{E} V^{(n)}(s_{+1}, \beta^{-1}(D - b(s, D))) \geq G(s,k), \\ &F(s,k) - Rk - b(s, D) \geq 0, \end{aligned}$$

[7] For $D \leq F(s, k^*(s)) - Rk^*(s)$, $(b, D_{+1}) = (D, 0)$ is feasible. Let $d^{(n+1)}(s, D) = b + \beta \mathbb{E} d^{(n)}(s_{+1}, \beta^{-1}(D - b))$. Assumption (i') implies that, for any $b \geq 0$, $\beta \mathbb{E} d^{(n)}(s_{+1}, \beta^{-1}(D-b)) \leq \beta[\beta^{-1}(D-b)] + \beta \mathbb{E} d^{(n)}(s_{+1}, 0)$. Thus, it must be the case that $d^{(n+1)}(s, D) = D + \beta \mathbb{E} d^{(n)}(s_{+1}, 0)$. Therefore, $d^{(n+1)}(s, D+\delta) = \delta + d^{(n+1)}(s, D)$, for $D \leq F(s, k^*(s)) - Rk^*(s)$.

while $\tilde{k}(s,D)$ is the maximum k that satisfies

$$F(s,k) - Rk - \tilde{b}(s,D) - \delta + \beta\mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - \tilde{b}(s,D))) \geq G(s,k),$$
$$F(s,k) - Rk - \tilde{b}(s,D) - \delta \geq 0.$$

We will demonstrate that $\tilde{k}(s,D) \leq k(s,D)$ by contradiction. Suppose that $\tilde{k}(s,D) > k(s,D)$. Then, $F(s,\tilde{k}(s,D)) - R\tilde{k}(s,D) - b(s,D) \geq 0$ is satisfied. The condition for $\tilde{b}(s,D)$ implies

$$F(s,\tilde{k}(s,D)) - R\tilde{k}(s,D) - \tilde{b}(s,D) - \delta + \beta\mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - \tilde{b}(s,D))) \geq G(s,\tilde{k}(s,D)). \tag{33}$$

By definition of $\tilde{\Gamma}^{(n+1)}(s,D)$, the fact that $\tilde{b}(s,D) \leq b(s,D)$ implies that there exists an integer $m\,(\geq 0)$ such that $\tilde{b}(s,D) + m\beta\delta = b(s,D)$. Then,

$$\begin{aligned}
&-\tilde{b}(s,D) + \beta\mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - \tilde{b}(s,D))) = -b(s,D) + m\beta\delta + \beta\mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - b(s,D) + m\beta\delta)) \\
&\leq -b(s,D) + \beta\mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - b(s,D))),
\end{aligned}$$

where the inequality is due to assumption (iv'). This inequality together with (33) implies that

$$F(s,\tilde{k}(s,D)) - R\tilde{k}(s,D) - b(s,D) - \delta + \beta\mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - b(s,D))) \geq G(s,\tilde{k}(s,D)).$$

This condition and the nonnegativity condition $(F(s,\tilde{k}(s,D)) - R\tilde{k}(s,D) - b(s,D) \geq 0)$ imply that $\tilde{k}(s,D) \in \Gamma^{(n+1)}(s,D)$, which implies that $\tilde{k}(s,D) \leq k(s,D)$, a contradiction. Thus, it must be the case that $\tilde{k}(s,D) \leq k(s,D)$. *(End of proof of Claim 3).*

Let $(k,b) = (k(s,D), b(s,D))$ and $(\tilde{k},\tilde{b}) = (\tilde{k}(s,D), \tilde{b}(s,D))$. Then, Claim 3 implies that there exist a nonnegative integer m and a nonnegative real number ε such that

$$F(s,\tilde{k}) - R\tilde{k} = F(s,k) - Rk - \varepsilon,$$
$$\tilde{b} = b - m\beta\delta.$$

Thus,

$$\begin{aligned}
\tilde{V}^{(n+1)}(s,D) &= F(s,\tilde{k}) - R\tilde{k} - \tilde{b} + \beta\mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - \tilde{b})), \\
&= F(s,k) - Rk - \varepsilon - b + m\beta\delta + \beta\mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - b) + m\delta), \\
&= -\varepsilon + F(s,k) - Rk - b + \beta\mathbb{E}[m\delta + V^{(n)}(s_{+1}, \beta^{-1}(D - b) + m\delta) \\
&\leq -\varepsilon + F(s,k) - Rk - b + \beta\mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - b)) \\
&= -\varepsilon + V^{(n+1)}(s,D) \leq V^{(n+1)}(s,D),
\end{aligned}$$

where the first inequality is from Assumption (iv'). Note that Assumption (iv') applies, since $\beta^{-1}(D - \tilde{b}) < D^{(n)}(s)$ because $D + \delta < D^{(n+1)}(s)$. (31) implies that $V^{(n+1)}(s, D+\delta) =$

$-\delta + \tilde{V}^{(n+1)}(s,D) \leq -\delta + V^{(n+1)}(s,D).$

Proof of (v). For $D > \bar{D}^{(n+1)}(s)$, it is the case that $V^{(n+1)}(s,D) = G^{npl}(s)$ as proven at part (iii). Next, we consider the case where $D \leq \bar{D}^{(n+1)}(s)$. For a fixed (s,D), Assumption (v') implies that $\Gamma^{(n+1)}(s,D) \subset \Gamma^{(n)}(s,D)$ and $\Lambda^{(n+1)}(s,D) \subset \Lambda^{(n)}(s,D)$. The following claim holds.

Claim 4. The variables for $(n+1)-$th problem satisfy $b^{(n+1)}(s,D) \leq b^{(n)}(s,D)$ and $k^{(n+1)}(s,D) \leq k^{(n)}(s,D)$.

(Proof of Claim 4). Since $\Gamma^{(n+1)}(s,D) \subset \Gamma^{(n)}(s,D)$, Claim 2 implies that $b^{(n+1)}(s,D) \leq b^{(n)}(s,D)$. Next, we prove $k^{(n+1)}(s,D) \leq k^{(n)}(s,D)$. Denote by $(C^{(n)})$ and $(C^{(n+1)})$ the following conditions:

$$(C^{(n)}) \qquad \begin{cases} F(s,k) - Rk - b + \beta \mathbb{E} V^{(n-1)}(s_{+1}, \beta^{-1}(D-b)) \geq G(s,k), \\ F(s,k) - Rk - b \geq 0, \end{cases}$$

$$(C^{(n+1)}) \qquad \begin{cases} F(s,k) - Rk - b + \beta \mathbb{E} V^{(n)}(s_{+1}, \beta^{-1}(D-b)) \geq G(s,k), \\ F(s,k) - Rk - b \geq 0, \end{cases}$$

- Case 1: Suppose that $b^{(n+1)} = b^{(n)}$

  In this case, $k^{(n+1)} \leq k^{(n)}$ should hold because $(C^{(n+1)})$ is (weakly) tighter than $(C^{(n)})$ for $b = b^{(n+1)} = b^{(n)}$.

- Case 2: Suppose that $b^{(n+1)} < b^{(n)}$.

  In this case, we first prove that the following condition holds:

  $$0 \leq F(s, k^{(n+1)}(s,D)) - Rk^{(n+1)}(s,D) - b^{(n+1)}(s,D) < \delta(s, k^{(n+1)}(s,D)) + \beta\delta, \tag{34}$$

  where $\delta(s, k^{(n+1)}(s,D))$ is defined by $\delta(s, k^{(n+1)}(s,D)) \equiv F(s, k^{(n+1)}(s,D)) - Rk^{(n+1)}(s,D) - F(s, k_-^{(n+1)}(s,D)) + Rk_-^{(n+1)}(s,D)$, where $k_-^{(n+1)}(s,D)$ is defined by $f(s, k_-^{(n+1)}(s,D)) - f(s, k^{(n+1)}(s,D)) = \beta\delta$. Thus, $k_-^{(n+1)}(s,D)$ is the value of k, which is smaller than and adjacent to $k^{(n+1)}(s,D)$. The condition (34) is proven by contradiction.[8] Then, as $b^{(n)}(s,D) \geq b^{(n+1)}(s,D) + \beta\delta$, the condition (34) implies that

  $$F(s, k_-^{(n+1)}(s,D)) - Rk_-^{(n+1)}(s,D) - b^{(n)}(s,D) < 0,$$

  which implies that $k^{(n)}(s,D) > k_-^{(n+1)}(s,D)$, which means $k^{(n)}(s,D) \geq k^{(n+1)}(s,D)$.

[8] Suppose that $F(s, k^{(n+1)}(s,D)) - Rk^{(n+1)}(s,D) - b^{(n+1)}(s,D) \geq \delta(s, k^{(n+1)}(s,D)) + \beta\delta$. Then, $k = k_-^{(n+1)}(s,D)$ and $b = b^{(n+1)}(s,D) + \beta\delta$ satisfies $(C^{(n+1)})$, as follows. First, the limited liability $(F(s,k) - Rk - b \geq 0)$ is obviously satisfied. Second, since $F(s, k^{(n+1)}(s,D)) - Rk^{(n+1)}(s,D) - G(s, k^{(n+1)}(s,D)) = F(s, k_-^{(n+1)}(s,D)) - Rk_-^{(n+1)}(s,D) - G(s, k_-^{(n+1)}(s,D)) - \beta\delta$ and $V^{(n)}(s_{+1}, \beta(D - b^{(n+1)}(s,D))) \leq V^{(n)}(s_{+1}, \beta(D - b^{(n+1)}(s,D) - \beta\delta))$, the enforcement constraint is satisfied for $k = k_-^{(n+1)}(s,D)$ and $b = b^{(n+1)}(s,D) + \beta\delta$. Thus, they are in $\Gamma^{(n+1)}(s,D)$. Then, the solution to $(n+1)$-th problem should be $b^{(n+1)}(s,D) + \beta\delta$, instead of $b^{(n+1)}(s,D)$. This is a contradiction.

(End of proof of Claim 4).

Let $(k, b) = (k^{(n)}(s, D), b^{(n)}(s, D))$ and $(\tilde{k}, \tilde{b}) = (k^{(n+1)}(s, D), b^{(n+1)}(s, D))$. The above claim implies that there exists a nonnegative integer m and a nonnegative real number ε such that $F(s, \tilde{k}) - R\tilde{k} = F(s, k) - Rk - \varepsilon$ and $\tilde{b} = b - m\beta\delta$. Thus,

$$\begin{aligned} V^{(n+1)}(s, D) &= F(s, \tilde{k}) - R\tilde{k} - \tilde{b} + \beta\mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - \tilde{b})), \\ &\leq F(s, k) - Rk - \varepsilon - b + m\beta\delta + \beta\mathbb{E}V^{(n-1)}(s_{+1}, \beta^{-1}(D - b) + m\delta), \\ &= -\varepsilon + F(s, k) - Rk - b + \beta\mathbb{E}[m\delta + V^{(n-1)}(s_{+1}, \beta^{-1}(D - b) + m\delta)] \\ &\leq -\varepsilon + F(s, k) - Rk - b + \beta\mathbb{E}V^{(n-1)}(s_{+1}, \beta^{-1}(D - b)) \\ &= -\varepsilon + V^{(n)}(s, D) \leq V^{(n)}(s, D), \end{aligned}$$

where the first inequality is from Assumption (v') and the second inequality is from Assumption (iv'). Note that Assumption (iv') applies since $D \leq \bar{D}^{(n+1)}(s)$, which implies that $\beta^{-1}(D - b) \leq \bar{D}^{(n)}(s) \leq \bar{D}^{(n-1)}(s)$. The fact that $k^{(n+1)}(s, D) \geq k^{npl}(s)$ and the enforcement constraint $[V^{(n+1)}(s, D) \geq G(s, k^{(n+1)}(s, D))]$ directly imply that

$$V^{(n+1)}(s, D) \geq G^{npl}(s).$$

Proof of (vi). First, we prove $\bar{D}^{(n+1)}(s) \leq \bar{D}^{(n)}(s)$ by contradiction. Suppose that $\exists s,\ \bar{D}^{(n+1)}(s) > \bar{D}^{(n)}(s)$. Then, we can pick D such that $\bar{D}^{(n)}(s) < D \leq \bar{D}^{(n+1)}(s)$, which satisfies

$$\begin{aligned} D_{+1}^{(n+1)}(s, D) &= \beta^{-1}[D - b^{(n+1)}(s, D)] < \bar{D}^{(n)}(s_H) \leq \bar{D}^{(n-1)}(s_H), \\ D_{+1}^{(n)}(s, D) &= \beta^{-1}[D - b^{(n)}(s, D)] \geq \bar{D}^{(n-1)}(s_H). \end{aligned}$$

These inequalities imply $b^{(n+1)}(s, D) > b^{(n)}(s, D)$, while $b^{(n+1)}(s, D)$ is feasible in (n)-th problem:

$$b^{(n+1)}(s, D) \in \Gamma^{(n+1)}(s, D) \subset \Gamma^{(n)}(s, D).$$

Therefore, $b^{(n)}(s, D)$ and $b^{(n)}(s, D) + \beta\delta$ are both feasible in (n)-th problem. Assumption (i') implies

$$\begin{aligned} d^{(n)}(s, D) &= b^{(n)}(s, D) + \beta\mathbb{E}d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s, D)) \\ &\leq b^{(n)}(s, D) + \beta\mathbb{E}[\delta + d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s, D) - \delta)] \\ &= b^{(n)}(s, D) + \beta\delta + \beta\mathbb{E}d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s, D) - \delta). \end{aligned}$$

If $d^{(n)}(s, D) < b^{(n)}(s, D) + \beta\delta + \beta\mathbb{E}d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s, D) - \delta)$, then $b^{(n)} + \beta\delta$ should be the solution to the (n)-th problem. This is a contradiction because $b^{(n)}(s, D)$ is the solution. If $d^{(n)}(s, D) = b^{(n)}(s, D) + \beta\delta + \beta\mathbb{E}d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s, D) - \delta)$, then the fact that $d^{(n)}(s, D) = d^{npl}(s)$ and $b^{(n)}(s, D) = b^{npl}(s)$ for $D > \bar{D}^{(n)}(s)$, together with $d^{npl}(s) = b^{npl}(s) + \beta\mathbb{E}d^{npl}(s_{+1})$, implies that

$$\mathbb{E}d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s, D) - \delta) < \mathbb{E}d^{npl}(s_{+1}),$$

which, in turn, implies that $\exists s_{+1},\ d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s,D)-\delta) < d^{npl}(s_{+1})$. On the other hand, $D > \bar{D}^{(n)}(s) > d^{npl}(s_H)$ implies that $D \geq d^{npl}(s_H) + 2\delta$, which, in turn, implies that $D_{+1}^{(n)}(s,D) - \delta \geq D - \delta > d^{npl}(s_H)$. Then, Assumption (ii') implies that $d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s,D)-\delta) \geq d^{npl}(s_{+1})$. Thus, we have demonstrated that $\exists s_{+1}$, such that $d^{npl}(s_{+1}) \leq d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s,D)-\delta) < d^{npl}(s_{+1})$, which is a contradiction. Therefore, it cannot be the case that $\exists s,\ \bar{D}^{(n+1)}(s) > \bar{D}^{(n)}(s)$.

B.4.8 Proof of Lemma 12

Claim 2 implies that $b(s,D) = \lim_{n\to\infty} b^{(n)}(s,D)$ satisfies $b(s,D) \geq \delta$ for $D < D_{\max}(s)$. For $D \geq D_{\max}(s)$, Lemmas 10 and 11 imply $b(s,D) = b^{npl}(s) \geq \delta$. Therefore, $b(s,D) \geq \delta$ for all (s,D).

Lemmas 10 and 11 imply that $V(s,D) = \lim_{n\to\infty} V^{(n)}(s,D)$ and $D_{\max}(s) = \lim_{n\to\infty} \bar{D}^{(n)}(s)$ satisfy that $V(s, D+\delta) \leq V(s,D) - \delta$ for $D < D_{\max}(s)$.

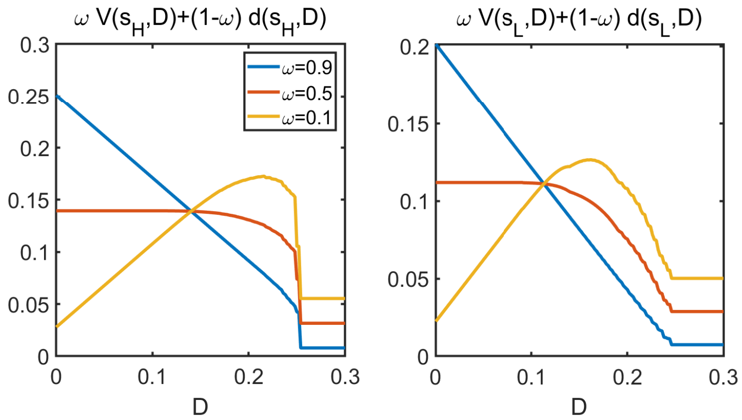

Figure 5: The case with $1 + r = \beta^{-1}$.

C Welfare analysis

In this appendix, we compare the social welfare, $W(s, D) \equiv \omega V(s, D) + (1 - \omega)d(s, D)$, for the three values of Pareto weight: $\omega = 0.1, 0.5$, and 0.9. Figure 5 shows the social welfare for the case with $1 + r = \beta^{-1}$, the value and policy functions of which are given in Figure 1, whereas Figure 6 shows welfare for the case with $1 + r > \beta^{-1}$, given in Figure 2. Figure 7 presents welfare for the case with frictional debt restructuring ($p = 0.2$), given in Figure 3. For these figures, the left-hand-side panel shows welfare when productivity is high ($s = s_H$), while the right-hand-side panel shows welfare when productivity is low ($s = s_L$). Figure 8 compares the baseline case ($p = 0$) with $1 + r > \beta^{-1}$, given in Figure 6, and the case with frictional debt restructuring ($p = 0.2$), given in Figure 7. This comparison allows us to isolate the effect of frictional debt restructuring.

The results are summarized as follows. First, the social optimal level of debt increases (or does not decrease) as the weight to the borrower, ω, decreases. Second, debt restructuring improves social welfare when debt is large. Debt restructuring increases the optimal level of debt when the weight to the borrower is low.

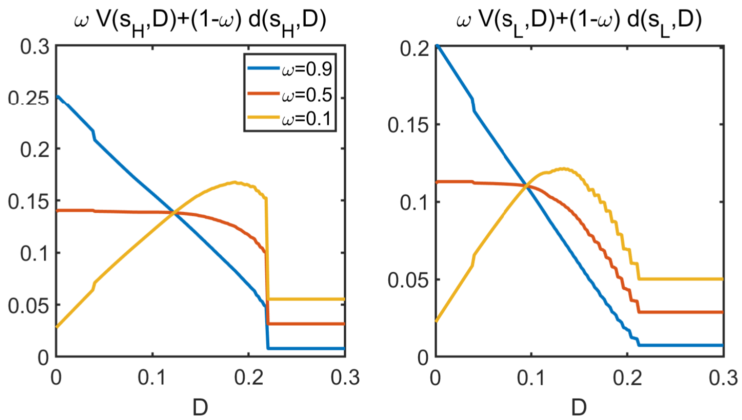

Figure 6: The case with $1 + r > \beta^{-1}$.

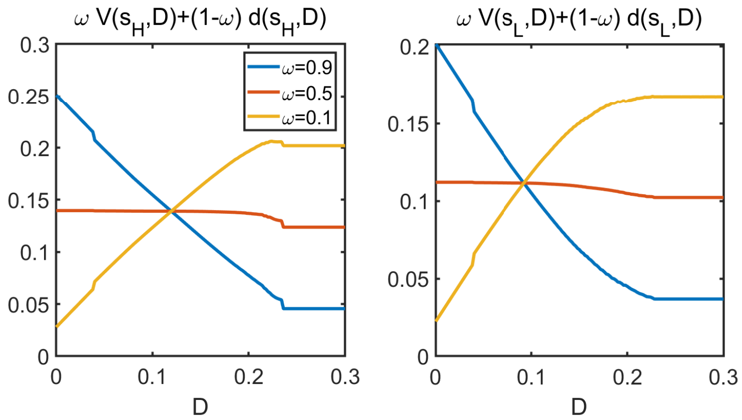

Figure 7: The case with frictional debt restructuring ($p = 0.2$, $1 + r > \beta^{-1}$).

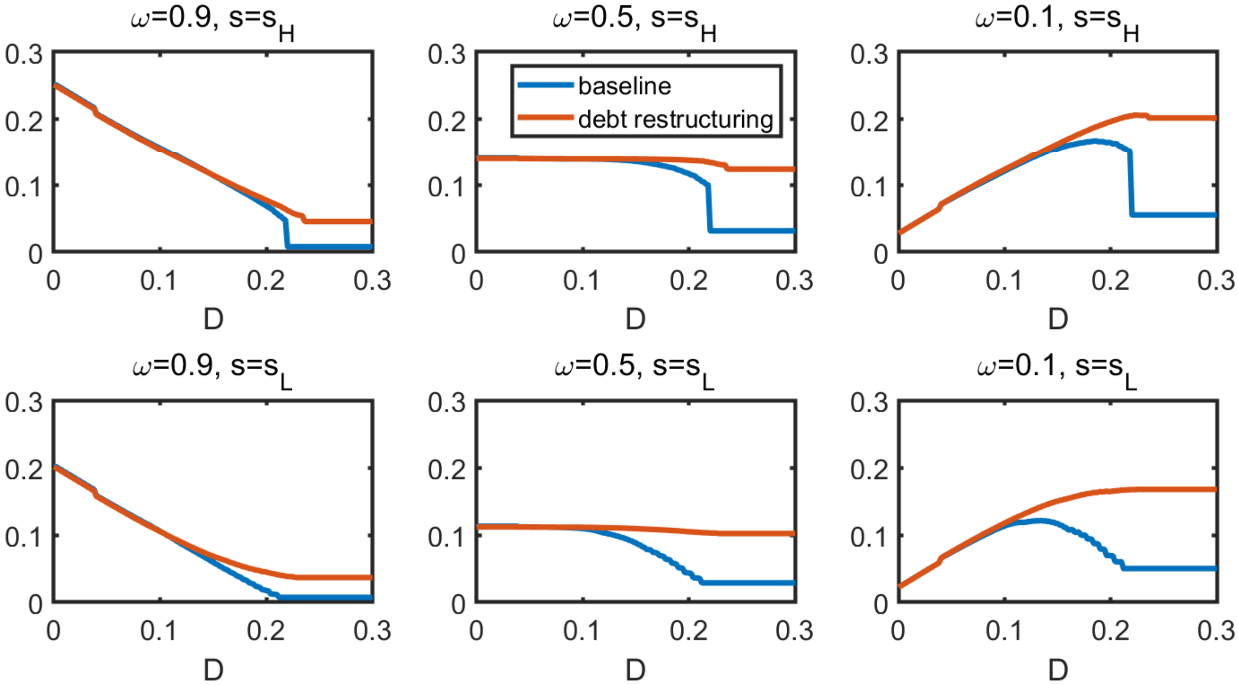

Figure 8: Comparison between the baseline case with $1 + r > \beta^{-1}$ and the case with frictional debt restructuring ($p = 0.2$ and $1 + r > \beta^{-1}$).