Recurrent Bubbles and Economic Growth*

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Abstract

We propose a model that generates permanent effects on economic growth following a recession (super hysteresis). Recurrent bubbles are introduced to an otherwise standard infinite-horizon business-cycle model with liquidity scarcity and endogenous productivity. In our setup, bubbles promote growth because they provide liquidity to constrained investors. Bubbles are sustained only when the financial system is under-developed. If the financial development is in an intermediate stage, recurrent bubbles can be harmful in the sense that they decrease the unconditional mean and increase the unconditional volatility of the growth rate relative to the fundamental equilibrium in the same economy. Through the lens of an estimated version of our model fitted to U.S. data, we argue that 1) there is evidence of recurrent bubbles; 2) the Great Moderation results from the collapse of the monetary bubble in the late 1970s; and 3) the burst of the housing bubble is partially responsible for the post-Great Recession dismal recovery of the U.S. economy.

1 Introduction

Recent crises have left a lasting effect on the level of output around the world. The Great Recession is an example of this scaring impact on economy activity. But recessions may also have an enduring impact on the growth rate of the economy, a phenomenon referred as super hysteresis (Ball (2014)). Indeed, Blanchard, Cerutti, and Summers (2014), using a sample of 23 advanced countries over 50 years, report that "about two thirds of recessions are followed by lower output relative to its

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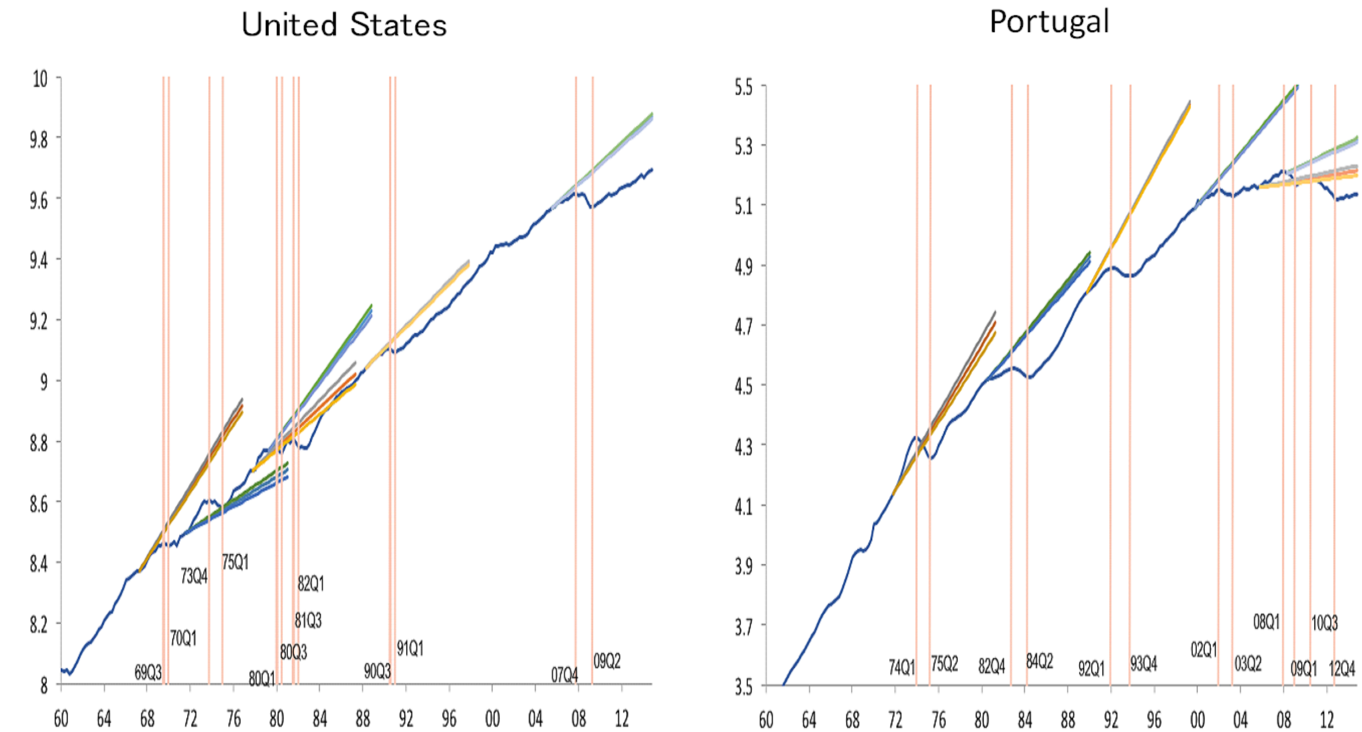

Figure 1: Super hysteresis in action. Real GDP taken from Blanchard, Cerutti, and Summers (2014). Straight lines correspond to pre-recession trends.

pre-recession trend." More important, "in about one half of those cases, the recession is followed not just by lower output, but by lower output growth relative to its pre-recession growth rate." In the U.S., the 5-year average growth rate of GDP before the 2001 and 2008 crises were 4.3% and 2.9%, respectively. In the five years following the crisis, the average growth rates went down to 2.9% and 1.9% (right panel in Figure 1). Portugal offers a more sobering example of super hysteresis in action in recent years (left panel in Figure 1). Interestingly, Blanchard, Cerutti, and Summers argue that it is "difficult to think mechanisms that lead to super-hysteresis." In this paper, we take on the task of understanding how and when hysteresis and super hysteresis arise in the economy.

We generate super hysteresis using a tractable model of recurrent bubbles, liquidity scarcity, and endogenous productivity. Investors are liquidity constrained in the sense of Kiyotaki and Moore (2012), resulting in depressed investment and low growth. In this environment, bubbles may mitigate the problem by providing extra liquidity, which in our endogenous growth model enhances economic growth. But if bubbles are helpful, their burst is harmful because it is followed by a sharp economic contraction and then prolonged low growth. The stagnation only ends when a new bubble emerges.

Bubbles are intrinsically useless assets in our model; they contribute neither to production nor households' utility. However, bubbles are particularly liquid, nothing prevents their trades in spot markets when they exist, and liquidity service may convince people to hold them even though their returns are clearly dominated by other less liquid assets. Yet the existence of bubbles requires special circumstances. They exist only if aggregate liquidity is in short supply, and only if everyone believes that bubbles are traded at a positive price. For tractability, we assume that there is a period of time in which bubbles cannot arise and be traded for exogenous reasons. Under this assumption, we analyze an interesting regime-switching equilibrium in which bubbles exist and are traded in one regime, and no such assets exist in the other.

In a calibrated version of our model, we find that the impact of bubbles on economic growth critically depends on the fundamentals of the economy. Particularly important is the degree of financial development, which is represented in the model by the tightness of the liquidity constraints. If the economy is financially underdeveloped, investors cannot get enough funds from selling equity on their capital. Because bubbles are liquid, they mitigate the problem of a weak financial system and hence they enhance both growth and employment. These benefits, however, come at the expense of volatility emanating from two sources. First, the economy switches between periods of bubbles with high growth and bubbleless periods with low growth. Second, the bubbly economy is more responsive to supply and demand shocks, implying that volatility is higher in the bubbly regime than in the fundamental.[1]

In contrast, if the financial market is relatively developed from the beginning, bubbles lower the growth rate of the economy. This is because bubbles strengthen the household's incentive to raise the capacity utilization rate, which results from bubbles and capital being substitutes as sources of liquidity. As a result, investors depend less on capital to obtain funds. Excessive capital utilization leads to fast depreciation, lowering net investment, and hence the growth of the economy, even though gross investment increases. Interestingly, this channel operates not only when bubbles actually exist but also in the bubbleless period because the price of capital is affected by the possibility of bubbles arising in the future through the Euler equation.

We exploit these previous insights to map our model to the post World War II data in the U.S. Our estimation reveals the existence of a persistent bubble prior to 1980. As we move through the 1980s and forward, bubbles became less persistent with one coinciding with the housing boom and a second one at the end of our sample. Through the eyes of our model, lower volatility post-84 results in part from the absence of persistent bubbles. One possible interpretation of a pre-Moderation bubble is the central bank's attempt to exploit the Phillips curve by providing easy money. The Fed's realization that this was not possible led to the bubble burst, leading to lower volatility but also lower growth. When we estimate our model, we find that demand shocks are more volatile than technology shocks. In spite of the moderating effect of the bubble burst, there is also a significant decline in the volatility of shocks post-1984.

Our model also provides an intuitive explanation of the slowdown in growth over the past

[1] This is a natural consequence of the liquidity services provided by bubbles. In our model, bubbles increase liquidity so investors find it easier to invest more. During recessions, liquidity is tighter, leading to deeper downturns.

decades. As will become clear, bubbles enhances growth in our framework. To the extend that the 1970s, 1990s, and mid 2000s were periods associated with monetary, IT, and housing bubbles, the collapse of these bubbles lead inevitably to slower growth. Furthermore, growth will remain depress until a new bubble arises in the economy. Our model is rich enough that it can account for the post-Great Recession downward shift in the trend of economic activity in the U.S. Indeed, a temporary financial shock results in lower investment, which through an endogenous productivity channel leads to permanently lower trend in output even though the growth rate of the economy returns to its pre-crisis level.

Dealing with bubbles in DSGE models is intrinsically complicated. This is so because one must track the history of booms and bursts to characterize the current state of the economy. In our model, the states are capital, exogenous shocks, and an indicator of the regime: fundamental or bubble. Since the economy switches between the two regimes, capital is regime dependent. But because of endogenous productivity, capital is a sufficient statistic for the history of bubbles. So once we de-trend the model using capital, there is no longer regime dependence and the equilibrium conditions depend on only the exogenous states of the economy. This model can be easily solved by standard methods and is amenable to estimation.

The rest of the paper proceeds as follows. Next, we highlight the contributions of our model to the existing literature. We describe the baseline model in section 3. In section 4 and 5, we discuss issues such as existence of bubbles, their effect on growth and show dynamic responses implied by our model. The empirical results with a discussion of the Great Moderation and the Great Recession are in section 6.

2 Related Work in the Literature

Our paper is in line with the literature on rational bubbles in infinite horizon economies with imperfect financial markets. The seminal papers are Bewley (1980), Townsend (1980), Scheinkman and Weiss (1986), and Woodford (1990).[2] These papers study deterministic fiat money (or government bonds) in an endowment economy when borrowing and lending are not allowed. Although these studies prove the existence of deterministic bubbles in infinite horizon economies, they do not necessarily show the necessary conditions explicitly for the reason asset bubbles can occur. Kocherlakota (1992) derive the necessary conditions for deterministic bubbles in an endowment economy when borrowing is allowed. Kocherlakota (2009) extends Kocherlakota (1992) to include a production economy without growth, and examines the effects of land bubbles on production.

Based on these seminal papers, we develop an endogenous growth model with financial frictions, and examine recurrent asset bubbles and their impact on long run economic growth. In this regard,

[2] Samuelson (1958) and **?** are the seminal papers showing rational bubbles in an overlapping generations model. See Farhi and Tirole (2012), Miao (2014), and Allen, Barlevy, and Gale (2017) for the recent devepment on rational bubbles in overlapping generations models.

our paper is related to Hirano and Yanagawa (2017). There are, however, substantial differences. First, we consider recurrent bubbles, i.e., bubbles are expected to arise and to collapse recurrently in the future, while Hirano and Yanagawa study the stochastic bubbles developed by Weil (1987). That is, a bubble is expected to collapse, but its reappearance is not expected at all. Second, the role of bubbles is different between Hirano and Yanagawa's paper and ours. Hirano and Yanagawa emphasize the role of bubbles as speculative vehicles. Agents buy and sell bubble assets mainly because they provide a high rate of return. In contrast, our paper emphasizes the role of bubbles as liquid assets, i.e., bubbles can be sold quickly compared with illiquid capital. Our formulation of bubbles is based on Kiyotaki and Moore (2012) where deterministic fiat money is described as a liquid asset. We show under what conditions recurrent bubbles with high liquidity can arise in equilibrium, and examine their impact on business cycles and the long-run economic growth rate.

Regarding recurrent bubbles, our paper is related to Gali (2014) and Miao and Wang (2017). In their papers, only a fraction of the existing bubbles collapses every period, and new bubbles are created right away so that aggregate supply of bubble assets is kept constant over time. This means that the economy is always in the bubbly regime. There is no entire collapse of bubbles. In our model, the emergence and entire collapse of bubbles is recurrent. As a consequence, the economy repeatedly switches between the bubbly regime and the bubbleless regime.

Moreover, Gali (2014) and Miao and Wang (2017) focus on a local analysis of the bubbly steady state. In our model, however, the entire breaking of bubbles implies that the economy no longer stays around the neighborhood of the bubbly steady-state. That is, the collapse of bubbles causes a sudden regime shift to the bubbleless economy, generating highly non-linear effects on macroeconomic activity. Importantly, these non-linear effects are anticipated by agents ex-ante and have major consequences on the model's dynamics. In this regard, our paper shares a similarity with the non-linear effects emphasized by Brunnermeier and Sannikov (2014), He and Krishnamurthy (2013), and Gertler and Kiyotaki (2015). In these papers, relatively large shocks to an economy cause the economy to jump far away from steady state, producing highly non-linear effects. They emphasize that this non-linearity is important to account for financial crisis phenomena.

The recurrent bubbles in Martin and Ventura (2012) are more similar to ours, in the sense that there is an entire collapse of bubbles. However, our papers differ in important dimensions. First, their model is based on an overlapping generations model, and agents live for only two periods. Hence anticipations about reappearance and recollapsing of bubbles in the future do not affect decisions of the current young agents at all. Their recurrent bubbles are essentially the same as the stochastic bubbles developed by Weil (1987), in which agents consider only the probability of the bubble bursts. Unlike theirs, in our model, infinitely-lived agents fully anticipate both the probability of reappearance and recollapsing in the future. Thus expectations about recurrent bubbles affect consumption, investment, and economic growth in the current period, which in turn affects bubble prices in the future. In this sense, there is a two-way feedback effect between macroeconomic activity and recurrent bubbles across time. This is a unique property in our model with infinitely-lived agents.

Furthermore, Martin and Ventura (2012) use a linear utility function, and agents consume only in old periods. Because of this assumption, agents do not care about the volatility arising from the collapse of bubbles. In our paper, however, agents are risk averse, and they fully anticipate the probability of recurrent bubbles. Hence, agents care about volatility arising from recurrent bubbles, which is crucial in our welfare analysis. Finally, Martin and Ventura's model does not have mechanisms that are standard in the business cycle literature such as the intertemporal Euler equation, endogenous labor supply, and endogenous capacity utilization. In contrast, our model is a standard real business cycle model, in which both propagation through dynamic optimization and amplification through intra-temporal optimization of time allocation and capacity utilization are present. These features are important because we estimate our model using U.S. data.

Our paper is also related to the effects on long run economic growth of various types of financial crises. For example, Cerra and Saxena (2008) show that most financial crises are associated with a decline in growth that leaves output permanently below its pre-crisis trend. Our paper shows that the collapse of bubbles causes permanently lower output level than its pre-bubble burst trend, but also generates permanently lower long run economic growth, i.e., super-hysteresis. Furthermore we relate to studies on the role of financial development and growth as in Aghion, Howitt, and Mayer-Foulkes (2005). Unlike their paper, ours focuses on 1) the provision of liquidity as a way to overcome underdeveloped financial systems; and 2) the impact of bubbles in economic growth.

Our study of hysteresis is connected to previous work such as Gali (2016). This paper studies hysteresis in labor markets and the design of monetary policy. We view our papers as complementary since we highlight the role that bubbles may have in creating not only hysteresis but also super hysteresis in economic activity. Finally, we relate to the literature on the solution and estimation of Markov switching models as in Farmer, Waggoner, and Zha (2009), Bianchi (2013), and Hamilton (2016).

3 Model

Our description of the model consists of regimes, firms, households, and endogenous productivity.

3.1 Regimes

Let z_t denote a realization of the regime $z_t \in \{b, f\}$ where b and f denote a bubbly and a fundamental regime, respectively. Their defining features are the existence or the lack of bubbly assets, which are intrinsically useless assets contributing to neither production nor households' utilities directly. There are no bubbly assets in the economy in the fundamental regime. When the regime switches to a bubbly one, M units of bubbly assets are created and given to households in a lump-sum way. There is no creation of bubbly assets in other contingencies. Bubbly assets last without depreciation as long as the economy stays in the bubbly regime. They disappear suddenly and completely once the regime switches back to the fundamental one. We assume that z_t follows a Markov process satisfying

$$\Pr\left(z_t = f | z_{t-1} = f\right) = 1 - \sigma_f \tag{1}$$

and

$$\Pr\left(z_t = b | z_{t-1} = b\right) = 1 - \sigma_b. \tag{2}$$

3.2 Firms

Output is produced using capital and labor services denoted by KS_t^D and L_t^D, respectively. The production function is

$$Y_t = A_t \left(KS_t^D\right)^{\alpha} \left(L_t^D\right)^{1-\alpha}$$

where A_t is the technology level which agents in the economy take as given. Competitive firms choose KS_t^D and L_t^D to maximize profits defined as

$$Y_t - r_t KS_t^D - w_t L_t^D$$

where r_t is rental price of capital and w_t is wage rate. First order conditions are

$$r_t = \alpha \frac{Y_t}{KS_t^D}$$

and

$$w_t = (1-\alpha) \frac{Y_t}{L_t^D}.$$

3.3 Households

The economy is populated by a continuum of households, with measure one. Each household has a unit measure of members who are identical at the beginning of a period. During the period, members are separated from each other, and each member receives a shock that determines the role of the member in the period. A member will be an investor with probability $\pi \in [0, 1]$ and will be a saver with probability $1 - \pi$. These shocks are i.i.d. among the members and across time.

A period is divided into four stages: household's decisions, production, investment, and consumption. In the household's decision stage, all members of a household are together and pool their assets: n_t units of equities and $\tilde{m}_t$ units of bubbly assets. An equity is the ownership of a unit of capital. Aggregate shocks to exogenous state variables are realized. The capacity utilization rate u_t is decided. Because all the members of the household are identical in this stage, the household head evenly divides the assets among the members. The household head also gives contingency plans to each member as follows. If one becomes an investor, he or she spends i_t units of final goods to invest, and brings back home x_t^i units of final goods, n_{t+1}^i units of equity claims, and $\tilde{m}_{t+1}^i$ units of bubbly assets before the consumption stage. Likewise, if the member becomes a saver, he or she supplies l_t units of labor, and brings back home x_t^s units of final goods, n_{t+1}^s units of equity claims, and $\tilde{m}_{t+1}^s$ units of bubbly assets before the consumption stage. After receiving these instructions, members go to the market and remain separated from each other until the consumption stage.

At the beginning of the production stage, each member receives the shock determining his or her role in the period. Competitive firms produce final goods. Compensations to productive factors are paid to their owners. A fraction $\delta(u_t)$ of capital depreciates, where

$$\delta(u_t) = \delta_0 + \frac{\delta_1}{1+\zeta}\left(u_t^{1+\zeta} - 1\right).$$

Note that the elasticity of $\delta(u_t)$ is constant at ζ;

$$\frac{u_t \delta''(u_t)}{\delta'(u_t)} = \zeta$$

for all u_t.

Investors seek finance and undertake investment projects in the investment stage. The technology is linear; they transform any amount i_t units of final goods into i_t units of new capital. Asset markets close at the end of this stage.

Members of the household meet again in the consumption stage. An investor consumes c_t^i units of final goods and a saver consumes c_t^s units of final goods.

The instructions must meet a set of constraints. First, they have to satisfy the intra-temporal budget constraints, i.e.,

$$x_t^i + i_t + q_t\left(n_{t+1}^i - i_t - (1-\delta(u_t))\,n_t\right) + \mathbf{1}_{\{z_t=b\}}\tilde{p}_t\left(\tilde{m}_{t+1}^i - \tilde{m}_t\right) = u_t r_t n_t \tag{3}$$

for an investor and

$$x_t^s + q_t\left(n_{t+1}^s - (1-\delta(u_t))\,n_t\right) + \mathbf{1}_{\{z_t=b\}}\tilde{p}_t\left(\tilde{m}_{t+1}^s - \tilde{m}_t\right) = u_t r_t n_t + w_t l_t \tag{4}$$

for a saver, where q_t and $\tilde{p}_t$ denote prices of equities and bubbly assets, respectively. The indicator function in front of $\tilde{p}_t$ captures the idea that there is neither spot nor future market for bubbly assets in the fundamental regime. Without markets, none can purchase bubbly assets, which is formally stated as follows;

$$\mathbf{1}_{\{z_t=f\}}\tilde{m}^i_{t+1} = \mathbf{1}_{\{z_t=f\}}\tilde{m}^s_{t+1} = 0. \tag{5}$$

A feasibility constraint in the consumption stage is given by

$$\pi x^i_t + (1-\pi)\, x^s_t = \pi c^i_t + (1-\pi)\, c^s_t. \tag{6}$$

An investor can issue new equity on at most a fraction ϕ of investment. In addition, she can sell at most a fraction ϕ of existing capital in the market.[3] Effectively, these constraints introduce a lower bound to the capital holdings of an entrepreneur at the end of the period:

$$n^i_{t+1} \geq (1-\phi)\left(i_t + (1-\delta(u_t))\, n_t\right). \tag{7}$$

Following Shi (2015), we call equation (7) a liquidity constraint. A similar constraint applies to savers, but we omit it because it does not bind in equilibrium (they are net buyers of equities). We also omit non-negativity constraints for u_t, c^i_t, i_t, n^i_{t+1}, x^s_t, c^s_t, l_t, n^s_{t+1}, and $\tilde{m}^s_{t+1}$ for the same reason. Exceptions are both a short-sale constraint for investors

$$\tilde{m}^i_{t+1} \geq 0 \tag{8}$$

and a borrowing constraint for investors

$$x^i_t \geq 0. \tag{9}$$

The household problem is written as follows. They choose a sequence of u_t, x^i_t, c^i_t, i_t, n^i_{t+1}, $\tilde{m}^i_{t+1}$, x^s_t, c^s_t, l_t, n^s_{t+1}, and $\tilde{m}^s_{t+1}$ to maximize

$$E_0\left[\sum_{t=0}^{\infty}\beta^t e^{b_t}\left(\pi\frac{\left[c^i_t\right]^{1-\rho}-1}{1-\rho} + (1-\pi)\,\frac{\left[c^s_t\,(1-l_t)^{\eta}\right]^{1-\rho}-1}{1-\rho}\right)\right] \tag{10}$$

subject to (3), (4), (5), (6), (7), (8), (9), and the laws of motions of assets given by

$$n_{t+1} = \pi n^i_{t+1} + (1-\pi)\, n^s_{t+1} \tag{11}$$

and

$$\tilde{m}_{t+1} = \pi\tilde{m}^i_{t+1} + (1-\pi)\,\tilde{m}^s_{t+1} + \mathbf{1}_{\{z_t=f,z_{t+1}=b\}}M \tag{12}$$

for all $t \geq 0$. Initial portfolio is $\{n_0, \tilde{m}_0\} = \left\{K_0, \mathbf{1}_{\{z_t=b\}}M\right\}$ where K_t is capital stock in the economy in period t. b_t is a preference shock.

[3] These two constraints are different in nature as Kiyotaki and Moore (2012) carefully distinguish. Our assumption that a single parameter ϕ governs both is just for simplicity.

3.4 Learning-by-doing

We assume that the technology level A_t is endogenous;

$$A_t = \bar{A}\left(K_t\right)^{1-\alpha} e^{a_t}.$$

a_t is a productivity shock and $\bar{A}$ is a scale parameter. Following Arrow (1962), Sheshinski (1967), and Romer (1986), we interpret the dependency of A_t on K_t as learning-by-doing; namely, knowledge is a by-product of investment and in addition, it is a public good that anyone can access at zero cost. With it, the long-run tendency for capital to experience diminishing returns is eliminated. We want to stress that the details behind the endogenous productivity mechanism are largely irrelevant for our purposes. Similar results would attain if we relied on expanding-variety or creative-destruction framework.

3.5 Market Clearing

Competitive equilibrium is defined in a standard way; all economic agents optimize given prices, and markets clear;

$$n_{t+1} = K_{t+1}, \tag{13}$$

$$L_t^D = (1-\pi)\, l_t,$$

$$KS_t^D = u_t K_t,$$

and

$$\pi c_t^i + (1-\pi)\, c_t^s + \pi i_t = Y_t$$

for all t, and

$$\pi \tilde{m}_{t+1}^i + (1-\pi)\, \tilde{m}_{t+1}^s = M$$

in a bubble regime. Because $\pi \tilde{m}_{t+1}^i + (1-\pi)\, \tilde{m}_{t+1}^s = 0$ holds in a fundamental regime, we have

$$\pi \tilde{m}_{t+1}^i + (1-\pi)\, \tilde{m}_{t+1}^s = \mathbf{1}_{\{z_t=b\}} M \tag{14}$$

for all t. The law of motion for capital is

$$K_{t+1} = (1-\delta(u_t))\, K_t + \pi i_t,$$

which automatically holds by Walras' law.

4 Permanent Fundamental

We first consider a special case in which the economy is always in the fundamental regime. Specifically, we assume that $z_0 = f$ and $\sigma_f = 0$, implying that $z_t = f$ for all $t \geq 0$. Guerron-Quintana and Jinnai (2015) use a variant of this fundamental model to study the implications of the 2008/2009 financial crisis on the level of output in the U.S. economy, showing that a temporary financial shock can trigger a secular stagnation in an estimated model.

4.1 Equilibrium with no binding liquidity constraint

We guess and subsequently verify numerically that the price of capital is always equal to one if the liquidity constraint is sufficiently loose (ϕ is large). Let us consider the household's problem in this environment.

If the price of capital is equal to one, the household is indifferent between investing in capital in-house and purchasing capital in the market. Hence the liquidity constraint (7) does not bind. The borrowing constraint (9) does not bind either; if it does, the household can make it loose without affecting other constraints or the amount of equity holding at the end of period t by increasing x_t^i by $\Delta > 0$, decreasing x_t^s by $(\pi / (1 - \pi)) \Delta$, decreasing n_{t+1}^i by Δ, and increasing n_{t+1}^s by $(\pi / (1 - \pi)) \Delta$. These observations allow us to summarize the constraints to a single equation;

$$\pi c_t^i + (1 - \pi) c_t^s + n_{t+1} = [u_t r_t + (1 - \delta(u_t))] n_t + w_t (1 - \pi) l_t. \tag{15}$$

This is a standard budget constraint. The first order conditions are standard too;

$$\left(c_t^i\right)^{-\rho} = \left(c_t^s\right)^{-\rho} (1 - l_t)^{\eta(1-\rho)}, \tag{16}$$

$$\eta \frac{c_t^s}{1 - l_t} = w_t, \tag{17}$$

$$r_t - \delta'(u_t) = 0,$$

and

$$1 = E_t \left[\beta e^{b_{t+1} - b_t} \left(\frac{c_t^i}{c_{t+1}^i} \right)^{\rho} (u_{t+1} r_{t+1} + 1 - \delta(u_{t+1})) \right].$$

The first equation states that the marginal utility from consumption has to be equalized across members. The second equation states that the marginal rate of substitution of leisure for consumption has to be equal to the real wage. The third equation states that the marginal benefit of raising the capacity utilization rate has to be equal to its opportunity cost, i.e, the amount of depreciated capital at the margin. The fourth equation is the Euler equation.

4.2 Equilibrium with binding liquidity constraint

We continue guessing that the price of capital exceeds one if the liquidity constraint is sufficiently tight. We focus our attention to the case in which the price of capital always satisfies $1 < q_t < 1/\phi$ following the literature (e.g., Shi (2015)). These inequalities guarantee that investing to capital is profitable (return is bigger than marginal cost) but investment cannot be made without down-payments. Let us consider the household's problem in this environment.

The inequality constraints (7) and (9) always bind in this equilibrium for the following reasons. If (7) is not binding, households can increase utility without violating any constraints or affecting their portfolio at the end of the period by increasing i_t by $\Delta > 0$, increasing n_{t+1}^i by $(q_t - 1)\Delta/q_t$, increasing both x_t^s and c_t^s by $(\pi/(1-\pi))(q_t-1)\Delta$, and decreasing n_{t+1}^s by $(\pi/(1-\pi))((q_t-1)/q_t)\Delta$, which is a contradiction to the household's optimization. If (9) is not binding, households can relax (7) without violating any constraints or affecting their portfolio at the end of the period by decreasing x_t^i by Δ, increasing x_t^s by $(\pi/(1-\pi))\Delta$, increasing n_{t+1}^i by $(1/q_t)\Delta$, and decreasing n_{t+1}^s by $(\pi/(1-\pi))(1/q_t)\Delta$. This is a contradiction to the household's optimization because they can increase utility if (7) is not binding.

With (7) and (9) binding, the optimal investment level is given by

$$i_t = \frac{[u_t r_t + \phi q_t (1 - \delta(u_t))] n_t}{1 - \phi q_t}. \tag{18}$$

Substituting (7) and (18) into (11), we find

$$n_{t+1} = \pi \frac{1}{q_t} (1 + \lambda_t) [u_t r_t + \phi q_t (1 - \delta(u_t))] n_t + \pi (1 - \phi)(1 - \delta(u_t)) n_t + (1 - \pi) n_{t+1}^s \tag{19}$$

where λ_t is given by

$$\lambda_t = \frac{q_t - 1}{1 - \phi q_t}. \tag{20}$$

Substituting (6) and (19) into (4), we can summarize the constraints by a single equation;

$$\pi c_t^i + (1 - \pi) c_t^s + q_t n_{t+1} = [u_t r_t + (1 - \delta(u_t)) q_t] n_t + \pi \lambda_t (u_t r_t + \phi q_t (1 - \delta(u_t))) n_t + (1 - \pi) w_t l_t. \tag{21}$$

An important variable in this equation is λ_t, which Shi (2015) calls the liquidity service. It measures how much value an investor can create using a unit of liquidity. The reason is the following. An investor can create $1/(1 - \phi q_t)$ units of capital from a unit of liquidity, which is the reciprocal of the downpayment. A fraction ϕ of the new capital is equity financed, and the rest is added to the investor's portfolio. The latter is worth $(1 - \phi) q_t / (1 - \phi q_t)$ at the market price. Finally, substracting the costs of the investment from it, we find

$$\frac{(1 - \phi) q_t}{1 - \phi q_t} - 1 = \frac{q_t - 1}{1 - \phi q_t}.$$

Note that the right-hand side is the liquidity service. In (21), we see that the second term in the right-hand side is the product of the liquidity service, the fraction of the household's members becoming investors, π, and the amount of liquidity each investor can obtain, $(u_t r_t + \phi q_t (1 - \delta(u_t))) n_t$. It is the value added of leveraged investment.

Now we discuss the first order conditions. The first order conditions for consumption and labor are the same as before, i.e., (16) and (17). The optimality condition with respect to the capacity utilization rate is

$$r_t - \delta'(u_t) q_t + \pi\lambda_t (r_t - \phi q_t \delta'(u_t)) = 0. \tag{22}$$

The Euler equation is

$$q_t = E_t \left[\beta e^{b_{t+1}-b_t} \left(\frac{c_t^i}{c_{t+1}^i} \right)^\rho (u_{t+1} r_{t+1} + (1 - \delta(u_{t+1})) q_{t+1} + \pi\lambda_{t+1} (u_{t+1} r_{t+1} + \phi q_{t+1} (1 - \delta(u_{t+1})))) \right]. \tag{23}$$

Note that both q_t and λ_t are involved in (22) and (23). q_t appears in the second term in (22) because the opportunity costs of raising the capacity utilization rate is the market value of depreciated capital at the margin. λ_t appears in the third term in (22) because raising the capacity utilization rate provides additional liquidity to investors. λ_t appears in the right-hand side of (23) because capital plays dual roles in the current environment. Namely, capital is not only a production factor but also a means of providing liquidity to its owners. (23) states that the capital is valued based on both of these services.

4.3 Calibration

Table 1 summarizes the parameter values. We set the discount factor at $\beta = 0.99$, set the inverse of the intertemporal elasticity of substitution at $\rho = 1$, set the capital share at $\alpha = 0.33$, set the elasticity of $\delta'(u_t)$ at $\zeta = 0.33$ following Comin and Gertler (2006), and set the probability of having an investment opportunity at $\pi = 0.06$ following Shi (2015).

The rest of the parameters is calibrated. We assume that if there were no binding liquidity constraint, the growth rate would be 2 percent per annum, the hours worked would be 27 percent of the available time, and the depreciation rate would be 5 percent per quarter along the balanced growth path. We then solve for the three parameters δ_0, $\delta_1 u^{1+\zeta}$, and η jointly. The scale parameter in the production function $\bar{A}u^\alpha$ is found by the equilibrium condition. We set $u = 1$, which is just a normalization.

The targeted depreciation rate (5 percent per quarter) is large, but remember that this is the depreciation in an extreme situation in which ϕ is so large that the liquidity constraint never binds. The previous studies in the literature assume a smaller ϕ (Kiyotaki and Moore (2012) and Shi (2015)). If we follow Kiyotaki and Moore (2012) and set it at $\phi = 0.19$ in our model just calibrated above, the implied depreciation rate is 2.4 percent per quarter. We are however agnostic about

Parameter	Value	Calibration Target
β	0.99	Exogenously Chosen
ρ	1	Exogenously Chosen
ζ	0.33	Exogenously Chosen
α	0.33	Capital Share=0.33
π	0.06	Shi (2015)
δ_0	0.001	Frinctionless Growth $g^4 = 1.02$
$\delta_1 u^{1+\zeta}$	0.065	Frinctionless Depreciation $\delta(u) = 0.05$
η	2.67	Frinctionless Hours $l = 0.27$
$\bar{A}u^{\alpha}$	0.49	Equilibrium Condition
u	1	Normalization

Table 1: Parameters and Calibration Targets

the value of ϕ at this point. We show the comparative statics with respect to this parameter.

We use the model with no binding liquidity constraint for the calibration because the equilibrium is unique in this situation. We will discuss the issue of multiple equilibria later, which arises only if the liquidity constraint is sufficiently tight.

4.4 Comparative Statics

Figure 2 shows how the degree of the liquidity constraint, ϕ, influences the growth rate along the balanced growth path denoted by g. We assume that both productivity and preference shocks are constant at $a_t = b_t = 0$ for all t in this exercise. The green flat line on the right part of the figure shows that the growth rate is constant once ϕ reaches a certain threshold. Beyond this point, neither liquidity nor borrowing constraints bind because investors obtain the desired level of liquidity.

On the left part of the envelope (blue line), we see a nonlinear relation between liquidity and growth. That is, when the liquidity in the economy is scarce (ϕ is small), providing additional liquidity (a marginal increase in ϕ) enhances growth, but when the liquidity is relatively abundant, it is harmful. We interpret ϕ as the degree of financial development in the economy, because this parameter governs how much money investors can borrow using capital as collateral. The result suggests that neither too immature nor too advanced financial market is beneficial for growth, but the most growth-enhancing level of financial development lies somewhere in the middle.

The nonlinearity is caused by competing effects of the parameter ϕ. On one hand, a marginal increase in ϕ promotes gross investment as shown in the top-right panel of Figure 3. But on the other hand, it accelerates capital depreciation, suppressing net investment. To understand the second effect, we plot the relation between ϕ and the price of capital q in the bottom-right panel of Figure 3. In the unconstrained region (green line), the price of capital is one; the capital is nothing but a production factor there. In the constained region (blue line), however, the price of capital exceeds one, becuase the capital is now not only a production factor but also the source of

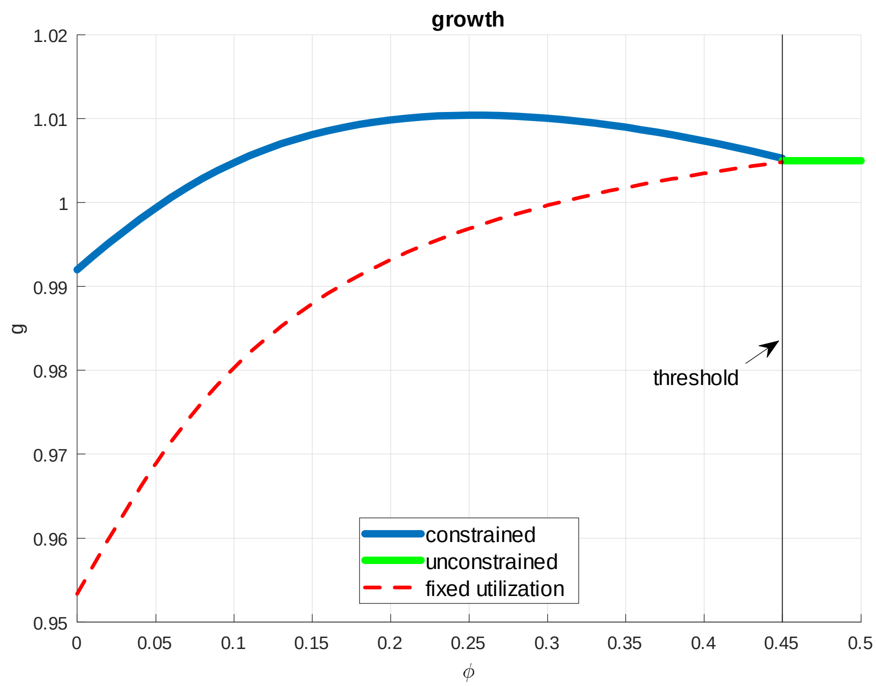

Figure 2: Liquidity and Growth in Permanent Fundamental

liquidity. Facing the high price, the households become reluctant to raise capacity utilization rate, because the market value of depreciated capital is the costs of raising capacity utilization rate. Production is small with low capacity utilization rate as shown in the top-left panel of Figure 3. But the capital accumulation may be accelerated due to low capital depreciation rate. In other words, our model suggests that the economy may use capital too loosely if capital is overly cheap. This is possible because people do not internalize the externality from capital. A mild liquidity constraint might mitigate this problem by raising the price of capital.

We conduct the same experiments in an otherwise identical economy with fixed capacity utilization rate. The red dashed line in Figures 2 shows that an increase in ϕ monotonically increases growth if capacity utilization rates are fixed. This model, however, is not only arguably unrealistic, but it suffers from the well known comovement problem (Barro and King (1984)). In our study, the model with fixed utilization does not have a recession when the bubble collapses suddenly and completely. We will discuss this point later.

5 Recurrent Bubble

We now flesh out the general case with regime switches. As in the previous section, we guess that the price of capital is always equal to one if the liquidity constraint is sufficiently loose. In

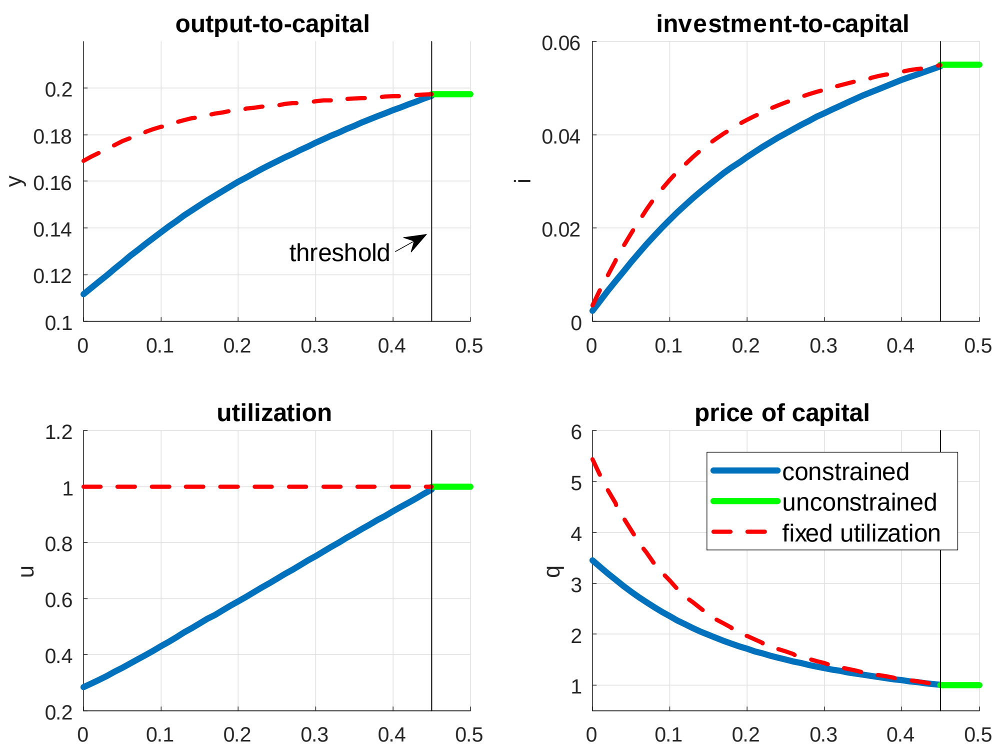

Figure 3: Effects of Liquidity in Permanent Fundamental

this case, $\tilde{p}_t = 0$ always hold in the bubbly regime.[4] Therefore, the price of capital being strictly greater than one at least in some periods is necessary for bubbly assets being traded at a strictly positive value.

5.1 Bubbly Equilibrium

We now characterize the equilibrium in which bubbly assets are traded at a strictly positive price. We guess and numerically verify that if the liquidity constraint is sufficiently tight, there is an equilibrium having the following properties: (i) $1 < q_t < 1/\phi$ always hold and (ii) $\tilde{p}_t > 0$ for all t with $z_t = b$. We consider the household's problem in this environment. The inequality constraints (7), (8), and (9) always bind in such an equilibrium for the following reasons. If (7) is not binding, households can increase utility without violating any constraints or affecting their portfolio at the end of the period by increasing i_t by $\Delta > 0$, increasing n^i_{t+1} by $(q_t - 1)\Delta/q_t$, increasing both x^s_t and c^s_t by $(\pi/(1-\pi))(q_t - 1)\Delta$, and decreasing n^s_{t+1} by $(\pi/(1-\pi))((q_t-1)/q_t)\Delta$, which is a contradiction to the household's optimization. (8) holds with equality in a fundamental regime due to (5). If (8) is not binding in a bubble regime, households can relax (7) without violating any constraints or affecting their portfolio at the end of the period by decreasing $\tilde{m}^i_{t+1}$ by Δ, increasing $\tilde{m}^s_{t+1}$ by $(\pi/(1-\pi))\Delta$, increasing n^i_{t+1} by $\tilde{p}_t\Delta/q_t$, and decreasing n^s_{t+1} by $(\pi/(1-\pi))(\tilde{p}_t/q_t)\Delta$. This is a contradiction to the household's optimization because they can increase utility if (7) is not binding. If (9) is not binding, households can relax (7) without violating any constraints or affecting their portfolio at the end of the period by decreasing x^i_t by Δ, increasing x^s_t by $(\pi/(1-\pi))\Delta$, increasing n^i_{t+1} by $(1/q_t)\Delta$, and decreasing n^s_{t+1} by $(\pi/(1-\pi))(1/q_t)\Delta$. This is a contradiction to the household's optimization because they can increase utility if (7) is not binding.

Because (7), (8), and (9) hold with equality, optimal investment level is given by

$$i_t = \frac{[u_t r_t + \phi q_t (1 - \delta(u_t))] n_t + \mathbf{1}_{\{z_t = b\}} \tilde{p}_t \tilde{m}_t}{1 - \phi q_t}. \tag{24}$$

Substituting (7) and (24) into (11), we find

$$n_{t+1} = \pi \frac{1}{q_t} (1 + \lambda_t) \left[(u_t r_t + \phi q_t (1 - \delta(u_t))) n_t + \mathbf{1}_{\{z_t = b\}} \tilde{p}_t \tilde{m}_t \right] + \pi (1 - \phi)(1 - \delta(u_t)) n_t + (1 - \pi) n^s_{t+1}. \tag{25}$$

[4] If $q_t = 1$, the household finds it indifferent between investing in capital in-house or purchasing capital in the market. As a result, the liquidity constraint (7) does not bind. Furthermore, the borrowing constraint (9) does not bind either. If it did, the household could make it loose without affecting other constraints or the amount of capital at the end of period t by increasing x^i_t by $\Delta > 0$, decreasing x^s_t by $(\pi/(1-\pi))\Delta$, decreasing n^i_{t+1} by Δ, and increasing n^s_{t+1} by $(\pi/(1-\pi))\Delta$. With both (7) and (9) unbound, the equilibrium price of bubble assets must be $\tilde{p}_t = 0$ because otherwise the demand for bubble assets is zero.

Substituting (6) and (25) into (4), we find the budget constraint at the household level;

$$\begin{aligned} & \pi c_t^i + (1-\pi) c_t^s + q_t n_{t+1} + \mathbf{1}_{\{z_t=b\}} \tilde{p}_t (1-\pi) \tilde{m}_{t+1}^s \\ = & \left[u_t r_t + (1-\delta(u_t)) q_t\right] n_t + \pi \lambda_t (u_t r_t + \phi q_t (1-\delta(u_t))) n_t \\ & + \mathbf{1}_{\{z_t=b\}} \tilde{p}_t (1+\pi\lambda_t) \tilde{m}_t + (1-\pi) w_t l_t. \end{aligned} \tag{26}$$

The household maximizes the expected utility (10) subject to (26), the accumulation rule for bubbly assets,

$$\tilde{m}_{t+1} = (1-\pi) \tilde{m}_{t+1}^s + \mathbf{1}_{\{z_t=f, z_{t+1}=b\}} M,$$

and the absence of bubbly-asset market in the fundamental regime,

$$\mathbf{1}_{\{z_t=f\}} \tilde{m}_{t+1}^s = 0.$$

The first order conditions with respect to the consumption allocation within household (16), the labor supply (17), the capacity utilization rate (22), and the Euler equation for capital (23) are the same as before. The Euler equation for the bubbly asset is

$$\mathbf{1}_{\{z_t=b\}} \tilde{p}_t = \mathbf{1}_{\{z_t=b\}} E_t \left[\beta e^{b_{t+1}-b_t} \left(\frac{c_t^i}{c_{t+1}^i} \right)^{\rho} (1+\pi\lambda_{t+1}) \tilde{p}_{t+1} \mathbf{1}_{\{z_{t+1}=b\}} \right].$$

This is a key equation in our model. Two observations are worth noting. First, the price of bubbly assets $\tilde{p}_t$ can be strictly positive only if there is a chance that they will be traded at a strictly positive value in the future. In other words, it is future resalability of bubbly assets that justifies a positive price of bubbly assets in the present. Second, the parameter ϕ is absent in the equation. Bubbly assets are more liquid than capital, and with this relative advantage, savers find the two assets indifferent at the margin even though bubbly assets do not provide dividends.

Let us discuss crowding-in and crowding-out effects of bubbly assets being traded at a positive price. In equilibrium $\mathbf{1}_{\{z_t=b\}} \tilde{m}_t = \mathbf{1}_{\{z_t=b\}} M$ holds,[5] and therefore, (24) is rewritten as follows:

$$i_t = \frac{\left[u_t r_t + \phi q_t (1-\delta(u_t))\right] K_t + \tilde{p}_t \mathbf{1}_{\{z_t=b\}} M}{1-\phi q_t}. \tag{27}$$

In the right-hand side, the last term in the numerator is positive if and only if the current regime is bubbly and the bubbly assets are traded at a positive value. This is the crowding-in effect; that

[5] This is because the following relation holds in the equilibrium,

$$\begin{aligned} \mathbf{1}_{\{z_t=b\}} \tilde{m}_t &= \mathbf{1}_{\{z_t=b\}} \left[\pi \tilde{m}_t^i + (1-\pi) \tilde{m}_t^s + \mathbf{1}_{\{z_{t-1}=f, z_t=b\}} M\right] \\ &= \mathbf{1}_{\{z_t=b\}} \left[\mathbf{1}_{\{z_{t-1}=b\}} M + \mathbf{1}_{\{z_{t-1}=f, z_t=b\}} M\right] \\ &= \mathbf{1}_{\{z_t=b\}} M. \end{aligned}$$

is, bubbles provide liquidity to investors, hence increasing gross investment i_t. We will discuss that equation (27) plays a key role in determining whether bubbles are sustainable or not.

Next, by substituting the budget constraint (26) forward, we find the following equation in equilibrium;

$$\begin{aligned}
& \pi c_0^i + (1-\pi) c_0^s + E_0 \left[\sum_{t=1}^{\infty} \frac{1}{R_{n,1} \cdots R_{n,t}} \left(\pi c_t^i + (1-\pi) c_t^s \right) \right] \\
= \; & \left(u_0 r_0 + (1 - \delta(u_0)) q_0 + \pi \lambda_0 \left(u_0 r_0 + \phi q_0 (1 - \delta(u_0)) \right) \right) n_0 \\
& + (1-\pi) \left(w_0 l_0 + E_0 \left[\sum_{t=1}^{\infty} \frac{1}{R_{n,1} \cdots R_{n,t}} w_t l_t \right] \right) \\
& + \pi \left(\lambda_0 \tilde{p}_0 \mathbf{1}_{\{z_0 = b\}} M + E_0 \left[\sum_{t=1}^{\infty} \frac{1}{R_{1,t} \cdots R_{n,t}} \lambda_t \tilde{p}_t \mathbf{1}_{\{z_t = b\}} M \right] \right),
\end{aligned} \tag{28}$$

where $R_{n,t}$ is the return from capital from the household's perspective, defined as

$$R_{n,t} \equiv \frac{u_t r_t + (1 - \delta(u_t)) q_t + \pi \lambda_t \left(u_t r_t + \phi q_t (1 - \delta(u_t)) \right)}{q_{t-1}}.$$

(28) is interpreted as an intertemporal budget constraint. The left-hand side is the present value of the consumption stream. The first term in the right-hand side is the value of the current equity holdings. The second term is the present value of the labor income stream. Finally, and most importantly, the third term is the present value of the "income" stream from bubbly assets, reflecting the value created from the liquidity provided by the current and the future bubbles. Because the third term is generally positive, it relaxes the budget constraint, increasing consumption, decreasing labor supply, and hence leaving less resource for investment than the case in which this term is absent, i.e., either the case in which the regime is always fundamental or the case in which the bubbly assets are always traded at zero price. This is the crowding-out effect of bubbles in our model. Notice that the parenthesis in the third term has two terms, associated with the present and the future bubbles, respectively. Importantly, future bubbles have crowding-out effect even if the current regime is fundametal. Both recurrent bubbles and infinitely lived household are crucial to obtain this result.

5.2 Comparative Statics

5.2.1 Growth Impact

Figure 4 plots the degree of the liquidity constraint, ϕ, on the horizontal axis and the capital growth, $g_t \equiv K_{t+1}/K_t$, on the vertical axis. We assume that the probabilities of regime switches are $\sigma_f = \sigma_b = 0.015$. The solid blue line shows the relation in an equilibrium in which the price of bubbly assets is always zero whenever they exist. This is possible because bubbly assets are intrinsically useless by definition. We call it the fundamental equilibrium, because it is essentially the same as the economy always being in the fundamental regime. Not surprisingly, the blue line in Figure 4 is identical to the one in Figure 2. But this may not be the unique rational equilibrium.

The red circles and crosses show growth rates in the other extreme. Namely, the price of bubbly assets is always positive whenever they exist. We call it the recurrent-bubble equilibrium because bubbles occur repeatedly as the regime switches back and forth between the two. The growth rates are regime dependent; the red circles denote the bubbly regime, whereas the red crosses denote the fundamental regime. The bubbly equilibrium exists only if the liquidity constraint is sufficiently tight. The vertical line in the figure shows the threshold value for the existence.

Red circles are located above red crosses at any level of ϕ below the threshold, meaning that the emergence of bubbles in this equilibrium accelerates economic growth from the bubbleless period. The key for this result is the inter-temporal substitution, or more precisely, the inter-regime substitution. As shown in Figure 5, the households work harder and invest more in the bubbly regime than in the fundamental regime. The bubbly regime is favorable time for investment, and the households recognizing it optimally allocate both time and resources not only across time but also regimes.

Having seen the two polar cases, we now consider an intermediate one. Let us assume that the economy starts from the bubbly regime in which bubbly assets are traded at a positive price. The bubble however bursts with a positive probability, triggered by the regime switch. After the burst, there is no re-emergence of bubbles. This is one of the multiple equilibria in our model economy; i.e., it is perfectly consistent with the definition of the competitive equilibrium that the price of bubbly assets is positive only in the initial bubbly regime and zero thereafter even though the bubbly regime is revisited. Alternatively, we can think of it as a bubbly equilibrium in an otherwise same model economy with different stochastic process with the fundamental regime being the absorbing state ($\sigma_f = 0$). They are isomorphic. This kind of bubbles, i.e., bursting stochastically after which the economy is free from bubbles forever, is studied by a pioneering work of Weil (1987) and known as the stochastic bubble in the literature.

The green circles and crosses in Figure 4 show the observed growth rates in the stochastic-bubble equilibrium. The speed of the economic growth in the initial bubbly regime is faster than the one after the burst of the bubble except for ϕ being very close to the threshold for the existence of the bubbly equilibrium. The inter-regime substitution plays an important role again. Note that green crosses are perfectly aligned with the solid blue line. This is because, with no chance of re-emergence of bubbles, the economy after the burst of the stochastic bubble is essentially identical to the fundamental equilibrium.

In constrast, the fundamental regime in the recurrent-bubble equilibrium is clearly different from the fundamental equilibrium; the speed of the economic growth is consistently slower in the former (red cross) than the latter (blue line). Two channels are important for this result. First, there is the wealth effect of future bubbles. That is, as shown in Figure 5, people expecting the re-emergence of bubbles consume more, work less (spend more time on leisure), and invest less than people with no such expectation. In short, the expectation for future bubbles effectively makes people lazy in the fundamental regime. Second, there is the general equilibrium effect. That is, the price of capital is low if people expect re-emergence of bubbles (the bottom-right panel of Figure 5) because bubbles provide liquidity to the economy, diluting the collateral value of capital. Note that the price is affected even if it is currently in the fundamental regime and even if re-emergence of bubbles is in the distant future. Low price of capital leads to high capacity utilization rate (the bottom-left panel of Figure 5), which slows down the capital accumulation and so does the economic growth.

Low growth in the fundamental regime has an important implication in the long run. Please see the locations of the red circle and the blue line at $\phi = 0.15$ in Figure 4. The red circle is above the blue line, meaning that the speed of the economic growth in the bubbly regime is higher with bubbles (the recurrent-bubble equilibrium) than without (the fundamental equilibrium). But interestingly, the fast growth in the bubbly regime alone does not necessarily mean the fast growth in the long run. The red triangles in Figure 4 show the unconditional mean, i.e., the expected speed of the economic growth in the recurrent-bubble equilibrium calculated with the stationary distribution.[6] The red triangle is below the solid blue line at $\phi = 0.15$, meaning that the speed of the economic growth in the long run is slower in the recurrent-bubble equilibrium than in the fundamental equilibrium. The cause is obviously the slow economic growth in the fundamental regime.

Things are simpler if bubbles are not recurrent but stochastic. Because the initial bubble is the only bubble in the stochastic-bubble equilibrium, the bubbly period is nothing but a temporary deviation from the fundamental equilibrium. Anything good in the initial bubbly regime is therefore good in the long run too. The speed of the economic growth is no exception.

The red triangles are above the blue line if ϕ is sufficiently small. This means that countries having low ϕ grow fast in the long run if they are in the recurrent-bubble equilibrium than in the fundamental equilibrium. But the growth in the recurrent-bubble equilibrium is bumpy, disrupted by occasional bursts of bubbles. Were the same economy in the fundamental equilibrium, the speed of the economic growth would be slow but stable on average. This result is reminiscent of Ranciere, Tornell, and Westermann (2008), who document that countries having experienced occasional financial crises have grown faster on average. Our model is consistent with their findings if we interpret the burst of bubbles as financial crisis at least for countries with small ϕ. For more advanced economies in which investors relatively easily obtain funds, however, our model provides a different prediction. Please see that the red triangles are below the blue line in the middle part of the figure. This means that in the advanced economies, bubbles are harmful to the economic growth in the long run.

[6] Formally it is given by $\left(\sigma_f g_{br} + \sigma_b g_{fr}\right) / \left(\sigma_f + \sigma_b\right)$, where g_{br} and g_{fr} denote the speed of the economic growth in the bubbly and fundamental regimes respectively.

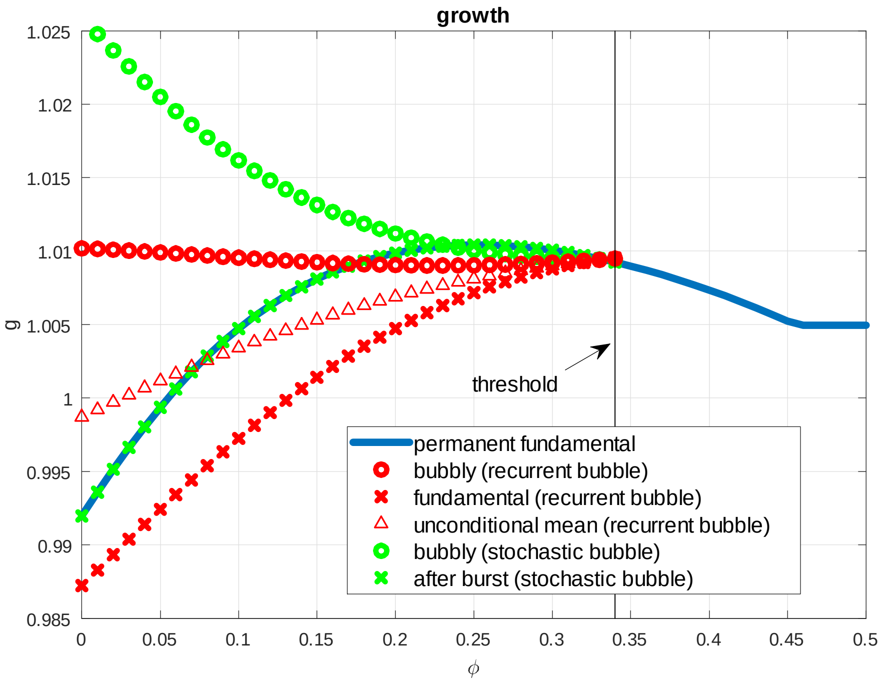

Figure 4: Liquidity and Growth in Recurrent-Bubble Model

The previous discussion suggests an interesting possibility; bubbles can be either growth-enhancing or the opposite, and it crucially depends on the maturity of the financial system. But the reference we compared the bubbly equilibrium with was the fundamental equilibrium, which may be unrealistic because it is completely bubble-proof. We therefore consider a less exteme alternative, namely, infrequent bubbles. Specifically, we reduce the probability of regime switch to the bubbly regime to $\sigma_f = 0.3\%$ per quarter, which is one fifth of the benchmark calibration. As in the previous cases, we can think of it as one of the multiple equilibria in the benchmark model too.

The light blue triangles in Figure 6 show the expected speed of the economic growth in the equilibrium with infrequent bubbles. They are roughly in the middle of the red triangles and the solid blue line. This means that our finding was robust to the infrequent bubbles. Namely, for the economies with weak financial system, bubbles are growth-enhancing, and the more frequent bubbles, the better. For the economies with developed financial system, the opposite is the case. It is also interesting that both light blue circles and crosses are shifted up from their red counterparts. This is because the dynamic substitution operates at different magnitudes. That is, people rush to invest in the bubbly period if they think bubbles are rare. By the same token, people are more disciplined in the bubbleless period if they think bubbles are rare.[7]

[7] We also experiment durable bubbles, setting the parameters at $(\sigma_f, \sigma_b) = (1.5\%, 0.3\%)$. The message is basically the same; for the economies with weak financial system, durable bubbles promote the growth in the long-run, while the opposite is the case for the economies with developed financial system.

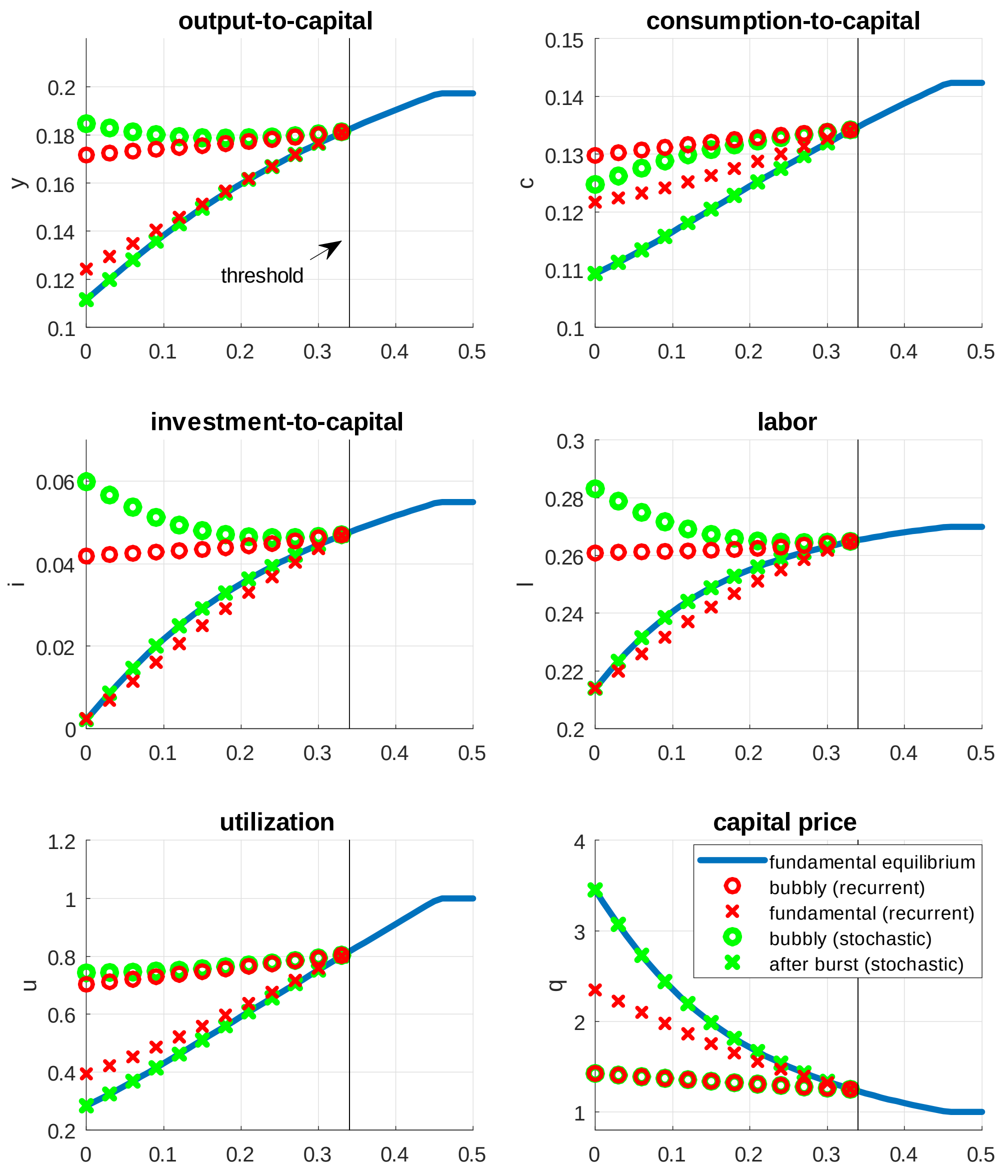

Figure 5: Effects of Liquidity in Recurrent-Bubble Model

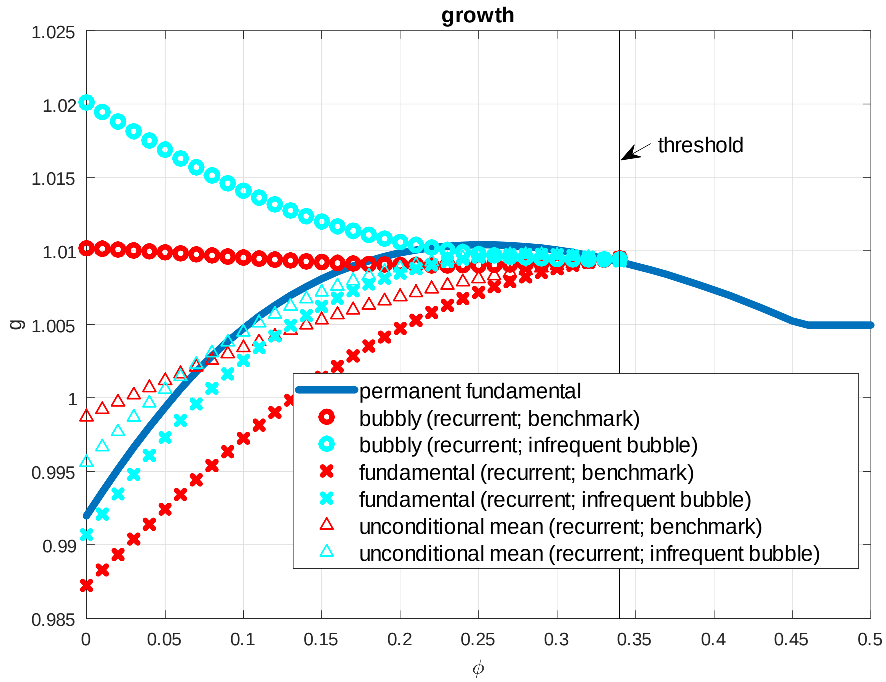

Figure 6: Bubble Frequency and Growth

5.2.2 Welfare Impact

The previous section discusses the growth impact of the recurrent bubbles. But the welfare impact may be different. This section examines it. We first define the welfare measure.

Let us rewrite the utility function (10) in the recursive form,

$$V_t = (1-\beta)\{\log[c_t] + (1-\pi)\eta\log[1-l_t]\} + E_t[\beta V_{t+1}].$$

Remember that we set the intertemporal elasticity of substition at one in the baseline calibration. c_t is the common consumption level across members of the household ($c_t \equiv c_t^i = c_t^s$), which is an implication of the log utility. We keep assuming that $a_t = b_t = 0$ for all t in this section. Both the continuation utility value V_t and the consumption c_t have growth trends. Detrended, the equation becomes

$$\hat{V}_t = (1-\beta)\{\log[\hat{c}_t] + (1-\pi)\eta\log[1-l_t]\} + \beta\log[g_t] + E_t\left[\beta\hat{V}_{t+1}\right],$$

where $\hat{V}_t$ and $\hat{c}_t$ are defined as $\hat{V}_t \equiv V_t - \log K_t$ and $\hat{c}_t \equiv c_t/K_t$ respectively, and g_t is the capital growth $g_t \equiv K_{t+1}/K_t$. $\hat{V}_t$ is our welfare measure.

In Figure 7, we plot the welfare in the fundamental equilibrium as a function of ϕ, which is given by

$$\hat{V}_{fq}(\phi) = \log[\hat{c}_{fq}(\phi)] + (1-\pi)\eta\log[1-l_{fq}(\phi)] + \frac{\beta}{1-\beta}\log[g_{fq}(\phi)]. \tag{29}$$

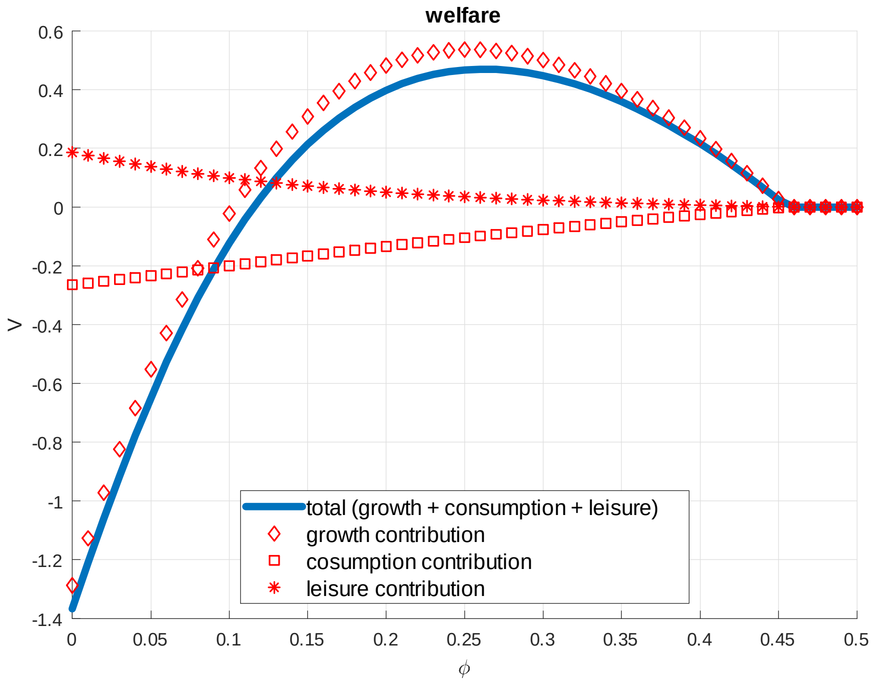

Figure 7: Liquidity and Welfare in Fundamental Equilibrium

The subscript fq denotes the fundamental equilibrium. We add a constant to the welfare measure before plotting it so that it takes zero in the case with no binding liquidity constraints (sufficiently large ϕ).[8] Needless to say, adding a constant does not change the welfare ranking.

The solid blue line is the benchmark. The overall shape of the welfare plot looks very similar to the growth plot in Figure 4, suggesting the importance of the economic growth to the welfare. We confirm it by factor decomposition. Namely, we vary the derended level of consumption, hours worked, and the speed of the economic growth one by one while keeping the other two variables constant at their values in the environment in which the liquidity constraints do not bind. We plot the welfare in each of the three exercises in red squares (consumption contribution), stars (leisure contribution), and diamonds (growth contribution), respectively. If they are added up, we obtain the solid blue line. The consumption contribution to the welfare monotonically increases with ϕ, but the leisure contribution decreases. These tendencies are obvious from Figure 5, in which we see that people both consume and work more as ϕ gets larger. With consumption and leisure offsetting each other, the speed of the economic growth turns out to be crucial to the welfare.

In Figure 8, we plot the welfare in the stochastic-bubble equilibrium with green circles and crosses, denoting the welfare in the initial bubbly regime and after the burst, respectively. Two things are worth noting. First, the welfare after the burst is identical to the welfare in the

[8] Namely, we plot $\hat{V}_{fq}(\phi) - \hat{V}^*_{fq}$ where $\hat{V}^*_{fq}$ is the continuation utility value in the economy in which ϕ is sufficiently large so that the liquidity constraints do not bind in the equilibrium any longer.

fundamental equilibrium (blue line). Second, the welfare in the bubbly regime is higher than the welfare in the fundamental regime except for a narrow parameter region near the threshold for the existence of the bubbly equilibrium. These implications are similar to their growth counterparts shown in Figure 4. In fact, they are almost identical in the following sense; if the speed of the economic growth is higher in the bubbly regime than in the fundamental regime, almost surely so is the welfare.

In the recurrent-bubble equilibrium, however, the relation between the growth and the welfare is more nuanced. Red circles and crosses in Figure 8 plot the welfare in the equilibrium. Please see the locations of the red circle and the blue line at $\phi = 0.15$ for instance; the blue line is above the red circle in Figure 8 while the exact opposite is the case in Figure 4. This means that the speed of the economic growth in the bubbly regime is higher in the recurrent-bubble equilibrium than in the fundamental equilibrium, but despite the high growth, the welfare in the bubbly regime is lower in the former equilibrium than in the latter. Expecation about the regime switch plays a key role. Namely, people know that bubbles will eventually collapse, and this expectation makes them unhappy. Formally, the welfare in the bubbly regime is given by

$$\begin{aligned}\hat{V}_{br}(\phi) &= \frac{1-\beta(1-\sigma_f)}{1-\beta(1-\sigma_b-\sigma_f)}\left(\log[\hat{c}_{br}(\phi)] + (1-\pi)\eta\log[1-l_{br}(\phi)] + \frac{\beta}{1-\beta}\log[g_{br}(\phi)]\right) \\ &\quad + \frac{\beta\sigma_b}{1-\beta(1-\sigma_b-\sigma_f)}\left(\log[\hat{c}_{fr}(\phi)] + (1-\pi)\eta\log[1-l_{fr}(\phi)] + \frac{\beta}{1-\beta}\log[g_{fr}(\phi)]\right).\end{aligned}$$

where the subscripts br and fr denote the bubbly and the fundamental regimes, respectively. If σ_b is strictly positive, the second term in the right-hand side takes a non-zero value, meaning that the welfare in the bubbly regime depends on the situation in the fundamental regime as well. Knowing that the speed of the economic growth in the fundamental regime is slow, people in the bubbly regime are not as happy as the speed of the economic growth in the bubbly regime suggests.

Similarly, low growth in the fundamental regime does not necessarily mean low welfare in the same regime, according to the recurrent-bubble equilibrium. Please see the locations of the red cross and the blue line at $\phi = 0$. The blue line is below the red cross in Figure 8 while the exact opposite is the case in Figure 4. Expecting re-emergence of bubbles, people are still happy amid the poor growth performance in the fundamental regime. The same expectation however makes people lazy at the same time, the point we discussed in the previous section. In other words, seemingly puzzling coexistence of both the high welfare and the miserable growth performance in the fundamental regime is not a coincidence, but they are two sides of the same coin.

The red triangles in Figure 8 show the unconditional mean, i.e., the expected welfare level in

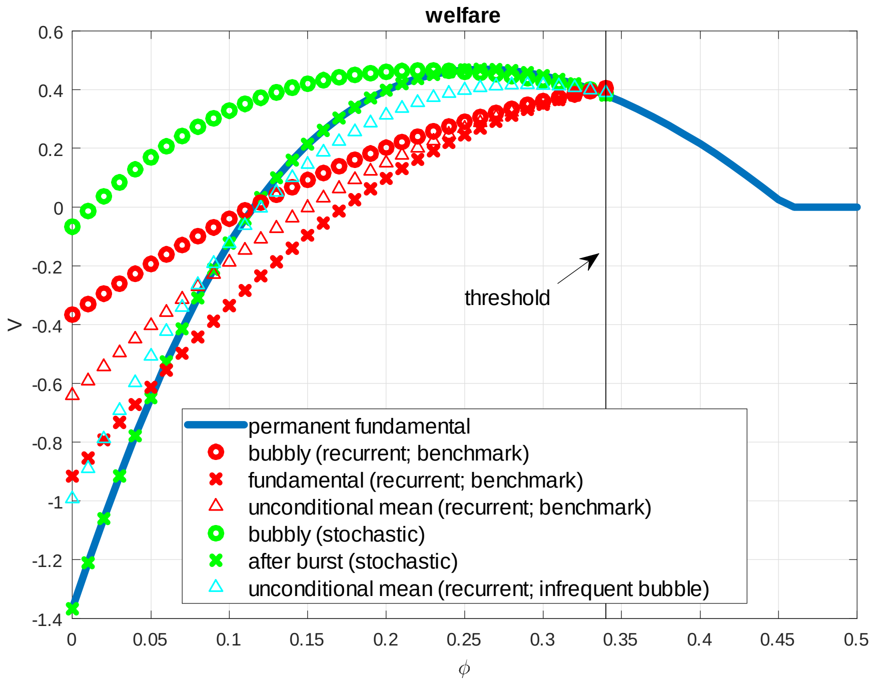

Figure 8: Liquidity and Welfare in Recurrent-Bubble Model

the recurrent-bubble equilibrium calculated with the stationary distribution.[9] They are similar to their growth counterparts in Figure 4. This result implies that for the economies with weak financial system, recurrent bubbles are not only growth-enhancing but also welfare-improving in the long run. But for the advanced economies in which investors can relatively easily obtain funds, the exact opposite is the case. Finally, the light blue triangles in Figure 8 show the expected welfare level in the case with infrequent bubbles, with σ_f set at $\sigma_f = 0.3\%$. Again, being in the middle of the red triangles and the solid blue line, they are similar to their growth counterparts in Figure 6. Therefore, for the economies with weak financial system, frequent bubbles are better because they raise the speed of the economic growth as well as the welfare. For the advanced economies with developed financial system, the exact opposite is the case.

[9]It is given by

$$\begin{aligned}\frac{\sigma_f \hat{V}_{br}(\phi) + \sigma_b \hat{V}_{fr}(\phi)}{\sigma_b + \sigma_f} &= \frac{\sigma_f}{\sigma_b + \sigma_f}\left(\log\left[\hat{c}_{br}(\phi)\right] + (1-\pi)\,\eta \log\left[1 - l_{br}(\phi)\right] + \frac{\beta}{1-\beta}\log\left[g_{br}(\phi)\right]\right) \\ &\quad + \frac{\sigma_b}{\sigma_b + \sigma_f}\left(\log\left[\hat{c}_{fr}(\phi)\right] + (1-\pi)\,\eta \log\left[1 - l_{fr}(\phi)\right] + \frac{\beta}{1-\beta}\log\left[g_{fr}(\phi)\right]\right).\end{aligned}$$

6 Taking the model to the data

We use our model to revisit the post-world war II U.S.'s economic performance. Specifically, we use U.S. data on the growth rate of output and the consumption-to-investment ratio for the period 1947.Q2 - 2016.Q4 to estimate the paths of supply and demand shocks in our model. We choose these observables because in our model these variables are sensitive to both the regime switch and shock processes. Specifically, we estimate the persistence and volatility of productivity and preference shocks. For this section, except for the liquidity parameter, ϕ, all other parameter values are those in table 1. Recall that the liquidity parameter was a free parameter in the previous sections since our objective was to analyze its impact on different versions of our model. For our quantitative section, we choose $\phi = 0.19$, which is in line with Del Negro, Eggertsson, Ferrero, and Kiyotaki (2016).

Our model follows within the class of MS-DSGE models discussed in Farmer, Waggoner, and Zha (2009). We find a fundamental minimum-state-variable equilibrium. The absence of endogenous state variables greatly simplifies the solution method as otherwise we would have to rely on the methods in Farmer, Waggoner, and Zha (2011).

6.1 A Regime Switching World

As an initial step, we estimate the model using maximum likelihood and Kim's filter,[10] assuming that the economy is in the recursive-bubble equilibrium. Our identification of the regimes relies on the implications we showed in both Table 2 and Figure 4; i.e., the bubbly regime is characterized by both higher volatility and higher economic growth. Although these two elements were present in the pre-1980s sample, there is a clear tension in the last decades. For instance, the housing boom epoch displayed higher growth but lower volatility. Hence, the importance of taking the model to the data to discipline the switches in the model.

The left upper panel in Figure 9 presents the filtered and smoothed probabilities of the economy being in a bubble regime. They suggest that the economy had been in a bubble regime prior to the 1980s, had moved to the fundamental regime and stayed until the late 1990s, and have returned to the bubble regime again. These patterns are reminiscent of the long-run trend in output growth reported by Comin and Gertler (2006). That is, as a secular trend, the U.S. output growth was generally robust until the 1970s, was generally weak until the mid-1990s, and reversed course again until the Great Recession. Identifying the cause of these medium-term cycles is a challenging task. The pioneering work of Comin and Gertler (2006) attribute them to exogenous changes to wage markups, which in their model are amplified by prominent mechanisms in the growth literature such as product innovations and costly implementations of new ideas. Our model offers a novel explanation; the medium-term cycles might be caused by non-fundamental factors, i.e., regime

[10] We thank Dongho Song for helping setting up the Markov Switching estimation routine.

switches between bubble and fundamental. It does not have to assume changes to the underlying technologies or the model parameters.

The timings of the regime switches in our model look different from estimates from alternative regime switching models. For example, Sims and Zha (2006) fit U.S. data to a regime switching VAR with drifting coefficients and variances. They report the existence of four distinct regimes: the Greenspan state prevailing during the 1990s and early 2000s; the second most common regime emerges in the early 1960s and parts of the 1970s; the last two regimes corresponds to sporadic events such as 9/11. Our regimes are unlike those estimated to account for the Great Moderation with a high volatility regime prior to 1984s and a calmer one post 1984 (Stock and Watson (2002)). Finally, our bubble regime bears little resemblance to recession regimes (See Hamilton (2016) for an extensive review of regime switching in macroeconomics).

Going into the details, the estimated probability path suggests that prior to the 1980s the likehood of being in a bubble regime was consistently high. But as we move through the 1980s and forward, the fundamental regime became more prevalent. Indeed, the bubble regime is less likely during the 1990s with a short-lived increase during the mid 1990s. Importantly, the estimated model captures the rise of the housing bubble during the early 2000s and its subsequent collapse in 2008-2009. Additionally, our estimation points to the rise of a post Great Recession bubble, which some economic commentators have attributed to the extremely loose monetary policy of the last years. However, our model struggles to pinpoint the rise and collapse of the IT bubble. This is most likely a result of the short duration of this bubble and that the housing bubble arose fairly close. That is, the estimation gives more weight to the housing bubble over the IT bubble. The curious reader may have noticed the bubble's temporary and abrupt collapse in the early 1960s. This seems to capture Kennedy's Slide of 1962 (the stock market flash crash from December 1961 to June 1962).

The right upper panel in Figure 9 shows the filtered path of the bubble's price (in red the HP-filtered trend). It plots the expected value calculated by the probability of the economy being in the bubble regime in a period times the price of bubble assets realized if the economy is in the bubbly regime in the same period. Measured this way, a unit of bubble asset was priced highly during most of the pre-1980s sample. But with the arrival of the Great Moderation epoch (the mid-1980s), the bubble's price became more volatile. In addition, remember that trading bubble assets occurs only in the bubble regime. Because the estimated probabilities suggest that there were regime switches to the fundamental in the mid-1980s and to the bubble in the late-1990s, the actual trade volume of bubble assets is likely to be even more volatile in the latter half of the sample. Interestingly, both the estimated probabilities and the bubble price correctly capture the housing boom-bust episode. As a trend over the entire sample, we observe that the bubble price has been declining since the 1960s. This is precisely at the core of our model. Namely, because periods of high valuation are associated with periods of faster growth in our model, the growth slowdown of the recent decades could be attributed in part to smaller size the bubbles.

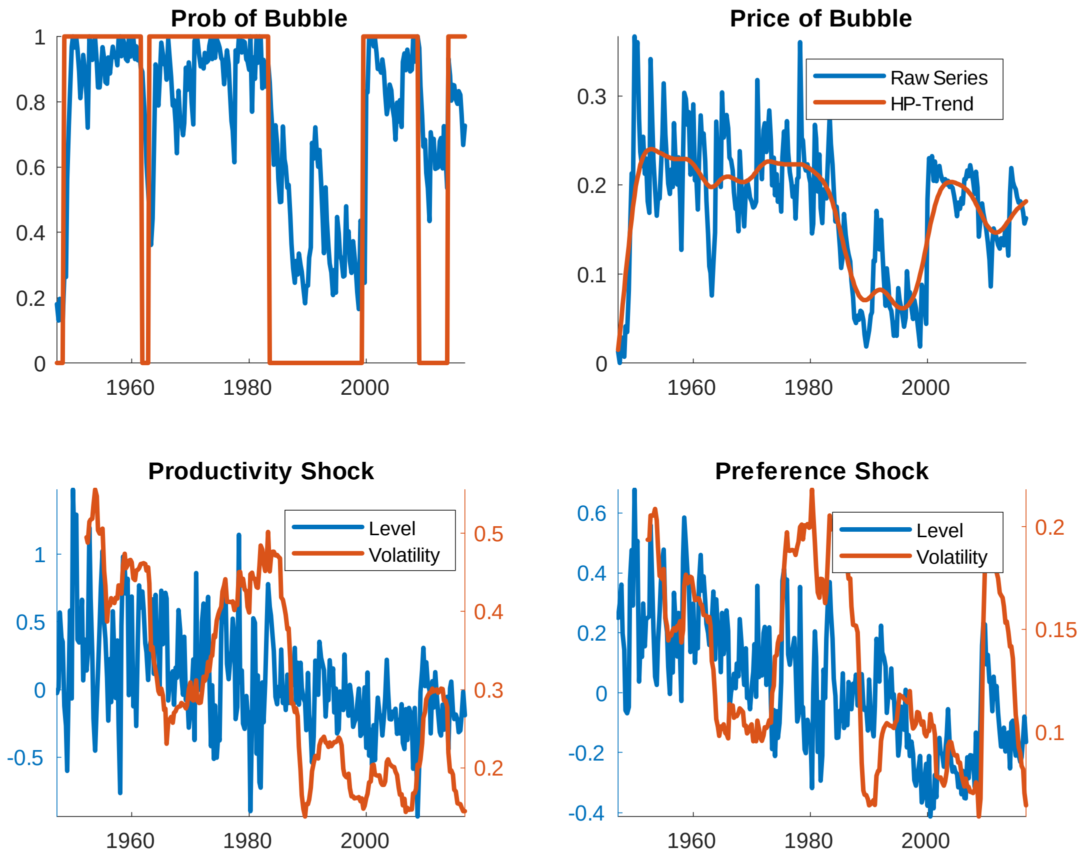

Figure 9: Variables from Recurrent Bubble Model

A natural question at this point is what these bubbles are capturing in reality. Although there is very little arguing about the housing and IT bubbles, it is less clear where the bubbles arose prior to 1980. For the 1970s, the obvious candidate is loose monetary policy (the Burns-Miller's dove regime estimated in Fernandez-Villaverde, Guerron-Quintana, and Rubio-Ramirez (2015)). Interestingly, Contessi and Kerdnunvong (2015) report stock and housing markets exuberance (based on cyclically adjusted price earning and cyclically adjusted price rent ratios) during the period 1965:Q3-1968Q4. Shiller (2015) also pointed out an instance of a high price-earnings ratio occurring in January 1966, calling it the "Kennedy-Johnson Peak."

The bottom panels in Figure 9 display the paths of productivity and preference shocks (the red lines correspond to 5-year rolling window volatilities). In spite of the moderating effect of the fundamental regime, there is still a role for less volatile shocks to account for the Great Moderation. By the same token, the high volatility episode during the 2008-2009 recession calls for larger disturbances, particularly so in the demand side of the economy.

6.2 A Permanent-Bubble World

We read the same observations through a different lens. Specifically, we assume that the economy is best described by the permanent bubble model. In this variant, the volatility of both supply and demand shocks declined by a factor of 2 during the Great Moderation. The bubble is about 6 times more volatile than output. The post-1984 moderation results from less volatile structural shocks. This view of the Great Moderation is consistent with Stock and Watson (2002).

Figure 10 displays the bubble's real valuation over the entire sample. In general, the bubble is more valuable during expansions like the 1970s, 1990s, and the first part of 2000s. Crucial for our purposes, the bubble's value declines in the early 1980s just as the Great Moderation started. It recovered during the technology bubble of the 1990s, which in our model implies higher growth. Except for this episode, the bubble's path is broadly consistent with the one estimated in our benchmark formulation (Figure 9).

The housing bubble in the early 2000s is captured by our model both during the pre-crisis years and the bust. At the end of the sample, we see some recovery but it is far from previous other recoveries. As we argued above, the less valuable the bubble is, the lower the liquidity services it provides, which results in weaker growth. By the end of our sample, we observe that the bubble's price is recovering. This finding is consistent with some economic observers' view that the quantitative easing measures implemented by the Federal Reserve Board are fueling a new bubble.[11]

[11] See for example the PBS column http://www.pbs.org/newshour/making-sense/column-the-monetary-bubble-to-end-all-bubbles-is-coming/

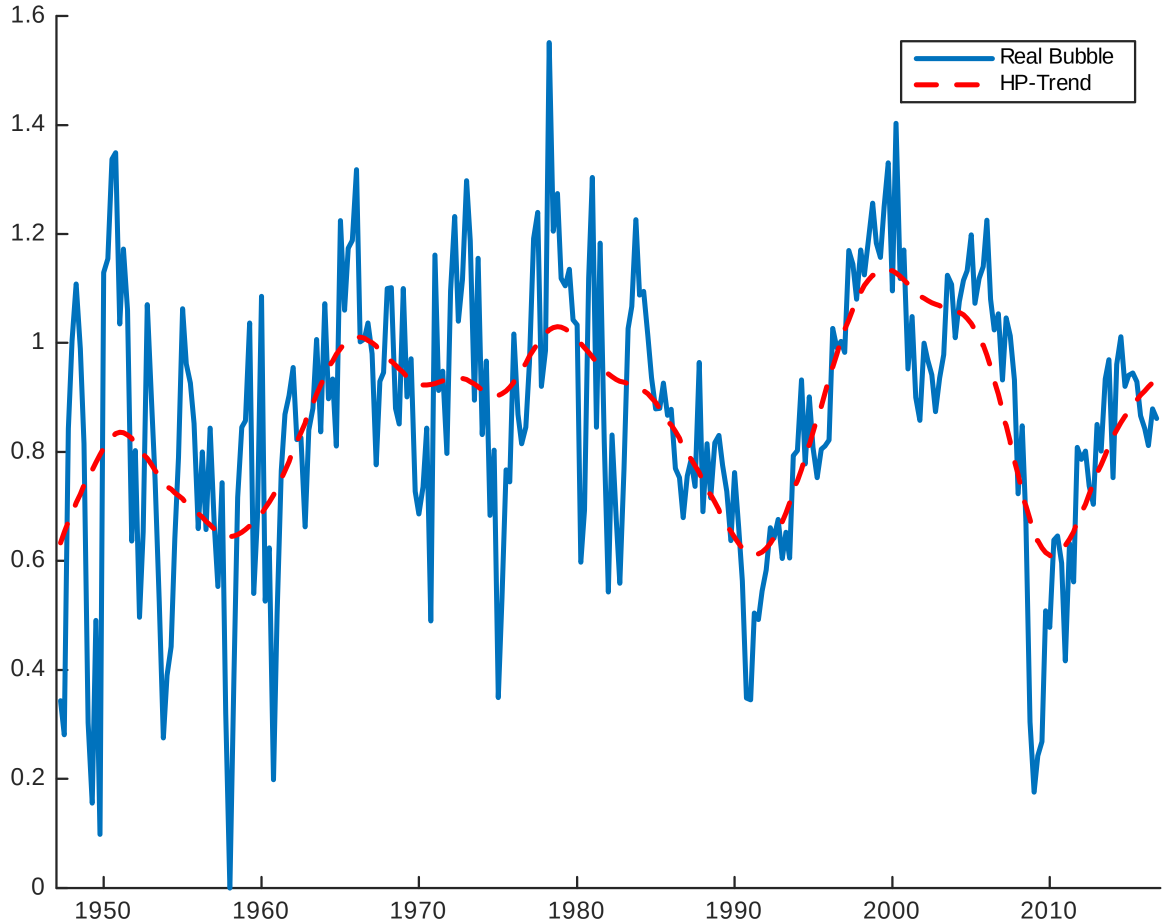

Figure 10: Implicit Real Value Bubble: $m_{b,t}$

6.3 A Bubbleless World

Now, let's imagine there was never a bubble in the economy. Supply and demand shocks became less volatile post-1984, with the volatility of the productivity declining by a factor of 1.7 while the volatility of the second disturbances shrinking by half. In all, this version of our model reflects the good-luck hypothesis behind the Great Moderation as in the permanent bubble model.

6.4 Three Models Side-By-Side

The smoothed shocks for the three versions of our model are in Figure 11. The dynamic paths for preference shocks are very similar across the different variants although the shocks from the bubbleless model are slightly more volatile. Productivity shocks in the permanent bubble model display significantly more variability than demand shocks. Moreover, supply shocks in this model are more volatile than the same shocks but in the other two variants. Finally, the post-1984 moderation is apparent in the two shocks.

7 Conclusions

We advance a model of recurrent bubbles, liquidity, and endogenous productivity. Unlike previous work in the literature (Martin and Ventura (2012)), we introduce recurrent bubbles in an infinite horizon business cycle model. We find that recurrent bubbles in this environment have non-trivial impact on the model's dynamics because prominent mechanisms emphasized in the business cycle literature, such as the intertemporal substitution of consumption and leisure, the endogenous time allocation, and the endogenous capacity utilization rate, are greatly influenced by bubbles. We find that bubbles enhances long-run growth when the degree of financial development is limited. However, if the financial sector is developed enough from the beginning, bubbles may be detrimental to growth due to its general equilibrium effect through the price of capital and endogenous capacity utilization rate. Our model of recurrent bubbles and endogenous productivity attributes the slowdown post-1984 to the collapse of an unproductive bubble.

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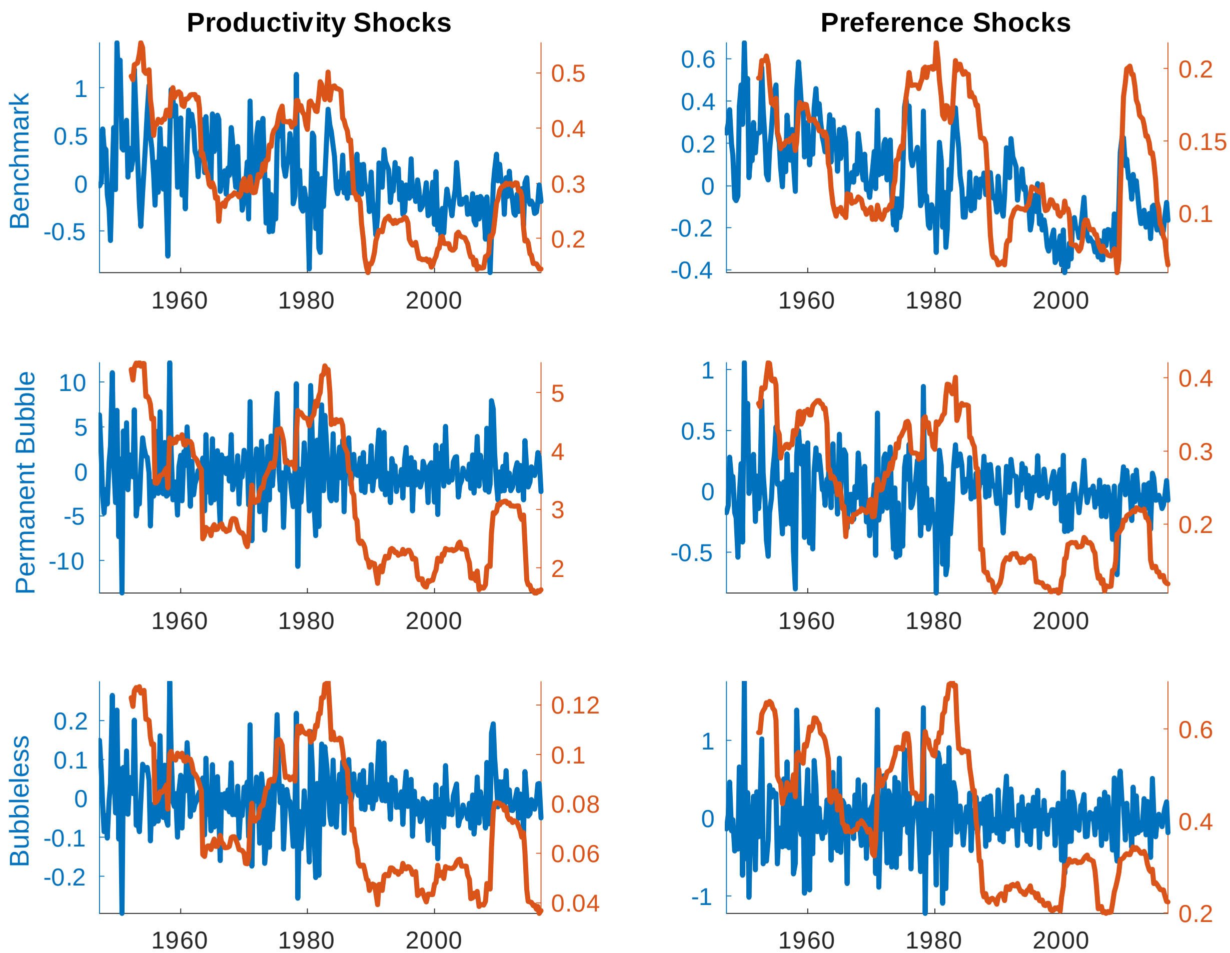

Figure 11: Smoothed Shocks

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8 Appendix

8.1 Permanent Fundamental

8.1.1 Without Binding Inequality Constraints

The household's problem is

$$\max E_0 \left[\sum_{t=0}^{\infty} \beta^t e^{b_t} \left(\pi \frac{\left[c_t^i\right]^{1-\rho}}{1-\rho} + (1-\pi) \frac{\left[c_t^s (1-l_t)^{\eta}\right]^{1-\rho}}{1-\rho} \right) \right]$$

subject to

$$\pi c_t^i + (1-\pi) c_t^s + n_{t+1} - (1-\delta(u_t)) n_t = u_t r_t n_t + w_t (1-\pi) l_t.$$

The equilibrium conditions are summarized as follows;

$$Y_t = \bar{A} e^{a_t} u_t^{\alpha} K_t ((1-\pi) l_t)^{1-\alpha},$$

$$\left(c_t^i\right)^{-\rho} = \left(c_t^s\right)^{-\rho} (1-l_t)^{\eta(1-\rho)},$$

$$\eta \frac{c_t^s}{1-l_t} = w_t,$$

$$\delta'(u_t) = r_t,$$

$$1 = E_t \left[\beta e^{b_{t+1}-b_t} \left(\frac{c_t^i}{c_{t+1}^i} \right)^{\rho} (u_{t+1} r_{t+1} + 1 - \delta(u_{t+1})) \right],$$

$$r_t = \alpha \frac{Y_t}{u_t K_t},$$

$$w_t = (1-\alpha) \frac{Y_t}{(1-\pi) l_t},$$

and

$$\pi c_t^i + (1-\pi) c_t^s + K_{t+1} - (1-\delta(u_t)) K_t = Y_t$$

for all t.

Detrend variables by K_t;

$$\hat{Y}_t = \bar{A} e^{a_t} u_t^{\alpha} ((1-\pi) l_t)^{1-\alpha},$$

$$\left(\hat{c}_t^i\right)^{-\rho} = \left(\hat{c}_t^s\right)^{-\rho} (1-l_t)^{\eta(1-\rho)},$$

$$\eta \frac{\hat{c}_t^s}{1-l_t} = \hat{w}_t,$$

$$\delta'(u_t) = r_t,$$

$$1 = E_t \left[\beta e^{b_{t+1}-b_t} \left(\frac{\hat{c}_t^i}{\hat{c}_{t+1}^i} \frac{1}{g_t} \right)^{\rho} (u_{t+1} r_{t+1} + 1 - \delta(u_{t+1})) \right],$$

$$r_t = \alpha \frac{\hat{Y}_t}{u_t},$$

$$\hat{w}_t = (1-\alpha) \frac{\hat{Y}_t}{(1-\pi) l_t},$$

and

$$\pi \hat{c}_t^i + (1-\pi) \hat{c}_t^s + g_t - (1 - \delta(u_t)) = \hat{Y}_t.$$

8.1.2 With Binding Inequality Constraints

The household's problem is

$$\max E_0 \left[\sum_{t=0}^{\infty} \beta^t e^{b_t} \left(\pi \frac{\left[c_t^i\right]^{1-\rho}}{1-\rho} + (1-\pi) \frac{\left[c_t^s (1-l_t)^{\eta}\right]^{1-\rho}}{1-\rho} \right) \right]$$

subject to

$$\pi c_t^i + (1-\pi) c_t^s + q_t n_{t+1} = \left[u_t r_t + (1-\delta(u_t)) q_t + \pi \lambda_t (u_t r_t + \phi q_t (1-\delta(u_t)))\right] n_t + (1-\pi) w_t l_t$$

A competitive equilibrium is defined as a sequence of prices, w_t, r_t, and q_t, and quantities, Y_t, i_t, K_{t+1}, c_t^i, c_t^s, l_t, and u_t, that satisfy the following conditions:

$$Y_t = \bar{A} e^{a_t} u_t^{\alpha} K_t ((1-\pi) l_t)^{1-\alpha},$$

$$\left(c_t^i\right)^{-\rho} = \left(c_t^s\right)^{-\rho} (1-l_t)^{\eta(1-\rho)},$$

$$\eta \frac{c_t^s}{1-l_t} = w_t,$$

$$r_t - \delta'(u_t) q_t + \pi \lambda_t (r_t - \phi q_t \delta'(u_t)) = 0,$$

$$q_t = E_t \left[\beta e^{b_{t+1} - b_t} \left(\frac{c_t^i}{c_{t+1}^i} \right)^{\rho} (u_{t+1} r_{t+1} + (1-\delta(u_{t+1})) q_{t+1} + \pi \lambda_{t+1} (u_{t+1} r_{t+1} + \phi q_{t+1} (1-\delta(u_{t+1})))) \right],$$

$$r_t = \alpha \frac{Y_t}{u_t K_t},$$

$$w_t = (1-\alpha) \frac{Y_t}{(1-\pi) l_t},$$

$$Y_t = \pi c_t^i + (1-\pi) c_t^s + \pi \frac{\left[u_t r_t + \phi q_t (1-\delta(u_t))\right] K_t}{1-\phi q_t},$$

and

$$K_{t+1} = (1-\delta(u_t)) K_t + \pi \frac{\left[u_t r_t + \phi q_t (1-\delta(u_t))\right] K_t}{1-\phi q_t}$$

for all t.

Since the model displays endogenous productivity, it is necessary to detrend it before we solve it numerically. Dividing quantities by K_t, we obtain the following equations.

$$\hat{Y}_t = \bar{A}e^{a_t}u_t^{\alpha}\left(\left(1-\pi\right)l_t\right)^{1-\alpha},$$

$$\left(\hat{c}_t^i\right)^{-\rho} = \left(\hat{c}_t^s\right)^{-\rho}\left(1-l_t\right)^{\eta(1-\rho)},$$

$$\eta\frac{\hat{c}_t^s}{1-l_t} = \hat{w}_t,$$

$$r_t - \delta'\left(u_t\right)q_t + \pi\lambda_t\left(r_t - \phi q_t\delta'\left(u_t\right)\right) = 0,$$

$$q_t = E_t\left[\beta e^{b_{t+1}-b_t}\left(\frac{\hat{c}_t^i}{\hat{c}_{t+1}^i}\frac{1}{g_t}\right)^{\rho}\left(u_{t+1}r_{t+1} + \left(1-\delta\left(u_{t+1}\right)\right)q_{t+1} + \pi\lambda_{t+1}\left(u_{t+1}r_{t+1} + \phi q_{t+1}\left(1-\delta\left(u_{t+1}\right)\right)\right)\right)\right],$$

$$r_t = \alpha\frac{\hat{Y}_t}{u_t},$$

$$\hat{w}_t = \left(1-\alpha\right)\frac{\hat{Y}_t}{\left(1-\pi\right)l_t},$$

$$\hat{Y}_t = \pi\hat{c}_t^i + \left(1-\pi\right)\hat{c}_t^s + \pi\frac{u_t r_t + \phi q_t\left(1-\delta\left(u_t\right)\right)}{1-\phi q_t},$$

and

$$g_t = 1-\delta\left(u_t\right) + \pi\frac{u_t r_t + \phi q_t\left(1-\delta\left(u_t\right)\right)}{1-\phi q_t}$$

for all t, where hat variables denote the original variable divided by K_t, i.e., $\hat{Y}_t \equiv Y_t/K_t$ and so on, and $g_t \equiv K_{t+1}/K_t$.

8.2 Recurrent Bubble Model

Competitive equilibrium is summarized by the following equations;

$$Y_t = \bar{A}e^{a_t}u_t^{\alpha}K_t\left(\left(1-\pi\right)l_t\right)^{1-\alpha},$$

$$\left(c_t^i\right)^{-\rho} = \left(c_t^s\right)^{-\rho}\left(1-l_t\right)^{\eta(1-\rho)},$$

$$\eta\frac{c_t^s}{1-l_t} = w_t,$$

$$r_t - \delta'\left(u_t\right)q_t + \pi\lambda_t\left(r_t - \phi q_t\delta'\left(u_t\right)\right) = 0,$$

$$q_t = E_t\left[\beta e^{b_{t+1}-b_t}\left(\frac{c_t^i}{c_{t+1}^i}\right)^{\rho}\left(u_{t+1}r_{t+1} + \left(1-\delta\left(u_{t+1}\right)\right)q_{t+1} + \pi\lambda_{t+1}\left(u_{t+1}r_{t+1} + \phi q_{t+1}\left(1-\delta\left(u_{t+1}\right)\right)\right)\right)\right],$$

$$\mathbf{1}_{\{z_t=b\}}\tilde{p}_t = \mathbf{1}_{\{z_t=b\}}E_t\left[\beta e^{b_{t+1}-b_t}\left(\frac{c_t^i}{c_{t+1}^i}\right)^{\rho}\left(1+\pi\lambda_{t+1}\right)\tilde{p}_{t+1}\mathbf{1}_{\{z_{t+1}=b\}}\right],$$

$$r_t = \alpha \frac{Y_t}{u_t K_t},$$

$$w_t = (1-\alpha) \frac{Y_t}{(1-\pi) l_t},$$

$$Y_t = \pi c_t^i + (1-\pi) c_t^s + \pi \frac{\left[u_t r_t + \phi q_t (1-\delta(u_t))\right] K_t + \tilde{p}_t \mathbf{1}_{\{z_t=b\}} M}{1-\phi q_t},$$

$$K_{t+1} = (1-\delta(u_t)) K_t + \pi \frac{\left[u_t r_t + \phi q_t (1-\delta(u_t))\right] K_t + \tilde{p}_t \mathbf{1}_{\{z_t=b\}} M}{1-\phi q_t},$$

and

$$\lambda_t = \frac{q_t - 1}{1-\phi q_t}.$$

Dividing variables by K_t, we find

$$\hat{Y}_t = \bar{A} e^{a_t} u_t^{\alpha} \left((1-\pi) l_t\right)^{1-\alpha},$$

$$\left(\hat{c}_t^i\right)^{-\rho} = \left(\hat{c}_t^s\right)^{-\rho} (1-l_t)^{\eta(1-\rho)},$$

$$\eta \frac{\hat{c}_t^s}{1-l_t} = \hat{w}_t,$$

$$r_t - \delta'(u_t) q_t + \pi \lambda_t \left(r_t - \phi q_t \delta'(u_t)\right) = 0,$$

$$q_t = E_t \left[\beta e^{b_{t+1}-b_t} \left(\frac{\hat{c}_t^i}{\hat{c}_{t+1}^i} \frac{1}{g_t}\right)^{\rho} \left(u_{t+1} r_{t+1} + (1-\delta(u_{t+1})) q_{t+1} + \pi \lambda_{t+1} \left(u_{t+1} r_{t+1} + \phi q_{t+1} (1-\delta(u_{t+1}))\right)\right)\right],$$

$$m_t = \mathbf{1}_{\{z_t=b\}} E_t \left[\beta e^{b_{t+1}-b_t} \left(\frac{\hat{c}_t^i}{\hat{c}_{t+1}^i} \frac{1}{g_t}\right)^{\rho} (1+\pi \lambda_{t+1}) m_{t+1} g_t\right],$$

$$r_t = \alpha \frac{\hat{Y}_t}{u_t},$$

$$\hat{w}_t = (1-\alpha) \frac{\hat{Y}_t}{(1-\pi) l_t},$$

$$\hat{Y}_t = \pi \hat{c}_t^i + (1-\pi) \hat{c}_t^s + \pi \frac{u_t r_t + \phi q_t (1-\delta(u_t)) + m_t}{1-\phi q_t},$$

$$g_t = 1 - \delta(u_t) + \pi \frac{u_t r_t + \phi q_t (1-\delta(u_t)) + m_t}{1-\phi q_t},$$

and

$$\lambda_t = \frac{q_t - 1}{1-\phi q_t}$$

where $m_t \equiv \tilde{p}_t \mathbf{1}_{\{z_t=b\}} M / K_t$.

It is convenient to make the dependence on the regime explicit;

$$\hat{Y}_{f,t} = \bar{A} e^{a_t} (u_{f,t})^{\alpha} \left((1-\pi) l_{f,t}\right)^{1-\alpha}, \tag{30}$$

$$\hat{Y}_{b,t} = \bar{A} e^{a_t} \left(u_{b,t}\right)^{\alpha} \left(\left(1-\pi\right) l_{b,t}\right)^{1-\alpha}, \tag{31}$$

$$\left(\hat{c}^i_{f,t}\right)^{-\rho} = \left(\hat{c}^s_{f,t}\right)^{-\rho} \left(1-l_{f,t}\right)^{\eta(1-\rho)}, \tag{32}$$

$$\left(\hat{c}^i_{b,t}\right)^{-\rho} = \left(\hat{c}^s_{b,t}\right)^{-\rho} \left(1-l_{b,t}\right)^{\eta(1-\rho)}, \tag{33}$$

$$\eta \frac{\hat{c}^s_{f,t}}{1-l_{f,t}} = \hat{w}_{f,t}, \tag{34}$$

$$\eta \frac{\hat{c}^s_{b,t}}{1-l_{b,t}} = \hat{w}_{b,t}, \tag{35}$$

$$r_{f,t} - \delta'\left(u_{f,t}\right) q_{f,t} + \pi \lambda_{f,t} \left(r_{f,t} - \phi q_{f,t} \delta'\left(u_{f,t}\right)\right) = 0, \tag{36}$$

$$r_{b,t} - \delta'\left(u_{b,t}\right) q_{b,t} + \pi \lambda_{b,t} \left(r_{b,t} - \phi q_{b,t} \delta'\left(u_{b,t}\right)\right) = 0, \tag{37}$$

$$\begin{aligned} q_{f,t} &= E_t \left[\begin{array}{c} \left(1-\sigma_f\right) \beta e^{b_{t+1}-b_t} \left(\frac{\hat{c}^i_{f,t}}{\hat{c}^i_{f,t+1}} \frac{1}{g_{f,t}} \right)^{\rho} \\ \left(u_{f,t+1} r_{f,t+1} + \left(1-\delta\left(u_{f,t+1}\right)\right) q_{f,t+1} + \pi \lambda_{f,t+1} \left(u_{f,t+1} r_{f,t+1} + \phi q_{f,t+1} \left(1-\delta\left(u_{f,t+1}\right)\right)\right)\right) \end{array} \right] \\ &\quad + E_t \left[\begin{array}{c} \sigma_f \beta e^{b_{t+1}-b_t} \left(\frac{\hat{c}^i_{f,t}}{\hat{c}^i_{b,t+1}} \frac{1}{g_{f,t}} \right)^{\rho} \\ \left(u_{b,t+1} r_{b,t+1} + \left(1-\delta\left(u_{b,t+1}\right)\right) q_{b,t+1} + \pi \lambda_{b,t+1} \left(u_{b,t+1} r_{b,t+1} + \phi q_{b,t+1} \left(1-\delta\left(u_{b,t+1}\right)\right)\right)\right) \end{array} \right], \end{aligned} \tag{38}$$

$$\begin{aligned} q_{b,t} &= E_t \left[\begin{array}{c} \left(1-\sigma_b\right) \beta e^{b_{t+1}-b_t} \left(\frac{\hat{c}^i_{b,t}}{\hat{c}^i_{b,t+1}} \frac{1}{g_{b,t}} \right)^{\rho} \\ \left(u_{b,t+1} r_{b,t+1} + \left(1-\delta\left(u_{b,t+1}\right)\right) q_{b,t+1} + \pi \lambda_{b,t+1} \left(u_{b,t+1} r_{b,t+1} + \phi q_{b,t+1} \left(1-\delta\left(u_{b,t+1}\right)\right)\right)\right) \end{array} \right] \\ &\quad + E_t \left[\begin{array}{c} \sigma_b \beta e^{b_{t+1}-b_t} \left(\frac{\hat{c}^i_{b,t}}{\hat{c}^i_{f,t+1}} \frac{1}{g_{b,t}} \right)^{\rho} \\ \left(u_{f,t+1} r_{f,t+1} + \left(1-\delta\left(u_{f,t+1}\right)\right) q_{f,t+1} + \pi \lambda_{f,t+1} \left(u_{f,t+1} r_{f,t+1} + \phi q_{f,t+1} \left(1-\delta\left(u_{f,t+1}\right)\right)\right)\right) \end{array} \right], \end{aligned} \tag{39}$$

$$m_{f,t} = 0, \tag{40}$$

$$\begin{aligned} m_{b,t} &= E_t \left[\left(1-\sigma_b\right) \beta e^{b_{t+1}-b_t} \left(\frac{\hat{c}^i_{b,t}}{\hat{c}^i_{b,t+1}} \frac{1}{g_{b,t}} \right)^{\rho} \left(1+\pi \lambda_{b,t+1}\right) m_{b,t+1} g_{b,t} \right] \\ &\quad + E_t \left[\sigma_b \beta e^{b_{t+1}-b_t} \left(\frac{\hat{c}^i_{b,t}}{\hat{c}^i_{f,t+1}} \frac{1}{g_{b,t}} \right)^{\rho} \left(1+\pi \lambda_{f,t+1}\right) m_{f,t+1} g_{b,t} \right], \end{aligned} \tag{41}$$

$$r_{f,t} = \alpha \frac{\hat{Y}_{f,t}}{u_{f,t}}, \tag{42}$$

$$r_{b,t} = \alpha \frac{\hat{Y}_{b,t}}{u_{b,t}}, \tag{43}$$

$$\hat{w}_{f,t} = \left(1-\alpha\right) \frac{\hat{Y}_{f,t}}{\left(1-\pi\right) l_{f,t}}, \tag{44}$$

$$\hat{w}_{b,t} = (1-\alpha)\frac{\hat{Y}_{b,t}}{(1-\pi)\, l_{b,t}}, \tag{45}$$

$$\hat{Y}_{f,t} = \pi \hat{c}^i_{f,t} + (1-\pi)\,\hat{c}^s_{f,t} + \pi \frac{u_{f,t} r_{f,t} + \phi q_{f,t}\left(1-\delta\left(u_{f,t}\right)\right) + m_{f,t}}{1-\phi q_{f,t}}, \tag{46}$$

$$\hat{Y}_{b,t} = \pi \hat{c}^i_{b,t} + (1-\pi)\,\hat{c}^s_{b,t} + \pi \frac{u_{b,t} r_{b,t} + \phi q_{b,t}\left(1-\delta\left(u_{b,t}\right)\right) + m_{b,t}}{1-\phi q_{b,t}}, \tag{47}$$

$$g_{f,t} = 1-\delta\left(u_{f,t}\right) + \pi \frac{u_{f,t} r_{f,t} + \phi q_{f,t}\left(1-\delta\left(u_{f,t}\right)\right) + m_{f,t}}{1-\phi q_{f,t}}, \tag{48}$$

$$g_{b,t} = 1-\delta\left(u_{b,t}\right) + \pi \frac{u_{b,t} r_{b,t} + \phi q_{b,t}\left(1-\delta\left(u_{b,t}\right)\right) + m_{b,t}}{1-\phi q_{b,t}}, \tag{49}$$

$$\lambda_{f,t} = \frac{q_{f,t}-1}{1-\phi q_{f,t}}, \tag{50}$$

and

$$\lambda_{b,t} = \frac{q_{b,t}-1}{1-\phi q_{b,t}} \tag{51}$$

where subscripts f and b denote realizations of the variables in a fundamental and bubble regime, respectively; for instance, $\hat{Y}_{f,t}$ is the realization of $\hat{Y}_t$ in a fundamental regime.

The impulse response functions are calculated by linearizing the equations (30) to (51) around $\hat{Y}_{f,t} = \hat{Y}_f$, $\hat{c}^i_{f,t} = \hat{c}^i_f$, $\hat{c}^s_{f,t} = \hat{c}^s_f$, $l_{f,t} = l_f$, $g_{f,t} = g_f$, $q_{f,t} = q_f$, $\lambda_{f,t} = \lambda_f$, $u_{f,t} = u_f$, $r_{f,t} = r_f$, $\hat{w}_{f,t} = \hat{w}_f$, $\hat{Y}_{b,t} = \hat{Y}_b$, $\hat{c}^i_{b,t} = \hat{c}^i_b$, $\hat{c}^s_{b,t} = \hat{c}^s_b$, $l_{b,t} = l_b$, $g_{b,t} = g_b$, $q_{b,t} = q_b$, $\lambda_{b,t} = \lambda_b$, $u_{b,t} = u_b$, $r_{b,t} = r_b$, $\hat{w}_{b,t} = \hat{w}_b$ and $m_{b,t} = m_b$.

8.3 Existence Condition

From the discussion above, it should be apparent that depending on the degree of financial tightness bubbles may or may not be valuable. In this section, we highlight other elements that may affect bubbles' valuation.

8.3.1 Permanent Bubble

The steady state investment condition (27) is useful to understand when bubbles arise (are valued positively). To this end, let's re-write it as follows:

$$m = \hat{\imath}(1-\phi q) - ur - \phi q(1-\delta\left(u\right)). \tag{52}$$

Here, m and $\hat{\imath}$ are the size of the bubbles and investment relative to the capital stock, i.e., $m_t = \tilde{p}_t M/K_t$ and $\hat{\imath}_t = i_t/K_t$, in the steady state respectively. The first term in the right-hand side of equation (52) is the down payment each investor pays for investment. The second term is the rental rate of capital, and the third term is the proceeds from selling capital up to the limit.

Therefore, this equation says that bubbles have positive valuation (the left-hand side is positive) if and only if the amount of liquidity an investor can withdraw from capital is less than the amount of liquidity investors need to undertake investment project.

To convey more intuition, let's assume that utilization is 1 and there is full depreciation. Under these assumptions, equation (52) is rewritten as

$$m = g(1 - \phi q) - r \tag{53}$$

because $\hat{\imath} = g$ where g is the growth rate of the economy in the steady state. Bubbles are valued when the rental rate of capital is sufficiently low. This implication is in line with the previous work on bubbles; if we further assume that ϕ is equal to $\phi = 0$, the first term in the right-hand side collapses to g, and $g > r$ is the familiar dynamic inefficiency condition for the existence of bubbles in OLG models.

If ϕ is strictly positive, investors can borrow money from savers using capital as collateral. By making the first term in the right-hand side smaller, a larger value of ϕ makes it more difficult to support bubbles. This implication is also in line with previous work; i.e., Tirole (1982) shows that bubbles cannot arise in infinite horizon economies in which agents can borrow and lend freely. In other words, a tight enough friction in the financial market is necessary for the economy to have bubble equilibrium.

8.3.2 Recurrent Bubble

Let us briefly discuss the existence condition when bubbles come and go. Assuming full depreciation and fixing the utilization at one, we arrive to the following expression,

$$m_b = \hat{\imath}_b(1 - \phi q_b) - r_b.$$

Other things being equal, bubbles are sustained (m_b is positive) when the liquidity constraint is tight, the rental price of capital is high, and/or the investment (and hence the growth rate) in the bubble regime is high. These implications are similar to the permanent bubble model.

But people take the possibility of the bubble burst into account when they are in the bubble regime, evaluating assets accordingly. The opposite is true in the fundamental regime. Therefore, both prices and behaviors are affected not only by the actual occurrence of the regime switch but also by the sheer possibility of the regime switch. For instance, under full depreciation the steady state price of equity in the bubbly regime is

$$q_b = (1 - \sigma_b)\,\beta g_b^{-\rho}\,(r_b + \pi\lambda_b r_b) + \sigma_b\beta\left(\frac{\hat{c}_b^i}{\hat{c}_f^i}\frac{1}{g_b}\right)^{\rho}(r_f + \pi\lambda_f r_f)\,.$$

Clearly, the dynamic link between the two regimes makes the existence condition complicated, but it sheds a new light on the study of bubbles.

8.4 Impulse Responses

Let us bring back supply and demand shocks into the analysis. We continue to assume that $\sigma_f = \sigma_b = 0.01$ and $\phi = 0.15$. There are multiple equilibria under these parameter values, i.e., the recurrent-bubble equilibrium in which the price of bubble assets is always positive in the bubble regime, and the fundamental equilibrium in which the price of bubble assets is always zero in the bubble regime. Computing the impulse response functions in the fundamental equilibrium is just standard. For the recurrent-bubble equilibrium, we compute impulse response functions by linearizing the system of equations summarizing the equilibrium around the regime-dependent steady states (please see the appendix for detail).

The top panel in Table 2 shows the impact of the productivity shock. We assume that the exogenous component of the productivity a_t increases by 1 percent in period t, slowly coming back to the steady state level thereafter with the autocorrelation coefficient being 0.95 quarterly. We report the contemporaneous responses in period t alone because they are enough to summarize the impulse responses. This is because there is no endogenous state variables in our model once endogenous variables are detrended by K_t, implying that both the regime $z_t \in \{f, b\}$ and the levels of the exogenous shocks $\{a_t, b_t\}$ are sufficient to pin down detrended-endogenous variables. Note, however, that the persistence of the shock does affect responses in period t because the model has forward looking variables and households are infinitely lived.

The first two columns show the IRFs in the recurrent-bubble equilibrium. A positive productivity shock generally raises macroeconomic variables in both regimes, but the magnitudes are different. Specifically, output, consumption, investment, hours worked, and capacity utilization all increase more in the bubble regime than in the fundamental regime. Asset prices play an important role. Namely, the size of the bubble increases when a positive productivity shock hits the economy in the bubble regime. This is because the demand for liquidity is strong when productivity is high. With more liquidity provided by bubbles, the price of capital does not rise as much as in the fundamental regime. With the price of capital cheaper, households are less reluctant to raise the capacity utilization rate, making the fluctuation larger in the bubble regime. The productivity shock, however, increases the growth rate of the capital $g_t = K_{t+1}/K_t$ because net investment increases.

The right column in Table 2 shows the impulse responses in the fundamental equilibrium. Broadly speaking, the responses are similar to those in the fundamental regime in the recurrent-bubble equilibrium. Looking closer, however, we see the wealth effects working in the recurrent-bubble equilibrium. Namely, households in the recurrent-bubble equilibrium enjoy more consumption and leisure because they understand that new bubble will arise in the future making them rich.

Supply Shock ($\Delta a_t = 1\%$, $\mathrm{Corr}(a_t, a_{t-1}) = 0.95$)			
	Recurrent-Bubble Equilibrium		Fundamental Equilibrium
	Bubble Regime	Fundamental Regime	Both Regimes
output	1.24%	1.09%	1.09%
consumption	1.08%	1.04%	1.03%
investment	1.69%	1.28%	1.32%
hours	0.12%	0.04%	0.05%
utilization	0.41%	0.16%	0.16%
capital price	0.74%	0.96%	0.98%
bubble size	2.29%	0%	0%
capital growth	0.033%	0.019%	0.022%

Demand Shock ($\Delta b_t = 1\%$, $\mathrm{Corr}(b_t, b_{t-1}) = 0.8$)			
	Recurrent-Bubble Equilibrium		Fundamental Equilibrium
	Bubble Regime	Fundamental Regime	Both Regimes
output	0.03%	0.11%	0.09%
consumption	0.31%	0.30%	0.31%
investment	-0.78%	-0.71%	-0.73%
hours	-0.22%	-0.15%	-0.17%
utilization	0.39%	0.49%	0.47%
capital price	-0.53%	-0.60%	-0.60%
bubble size	-0.87%	0%	0%
capital growth	-0.034%	-0.024%	-0.025%

Table 2: Effects of Supply and Demand Shocks

The bottom panel in Table 2 shows responses to the preference shock. b_t increases by 1 percent in period t, slowly coming back to the steady state level with the autocorrelation coefficient being 0.8 quarterly. Tilting the relative weights on the utility flow, this shock effectively makes the households impatient, consume more, work less, and invest less. But the magnitudes of the responses are again larger in the bubble regime than in the fundamental regime. Asset prices are important. That is, the size of the bubble shrinks in the bubble regime after the shock because the households become impatient. With the amount of liquidity provided by the bubble decreases, the price of capital does not drop as much as it does in the fundamental regime. Because the price of capital is relatively high, the households are reluctant to raise the capacity utilization rate, making drops in investment and hours worked larger in the bubble regime than in the fundamental regime.