Self-enforcing Debt Limits and Costly Default in General Equilibrium

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Abstract

We establish a novel determination of self-enforcing debt limits at the present value of default cost in a general competitive equilibrium. Agents can trade state-contingent debt but cannot commit to repay. If an agent defaults, she loses a fraction of her current and future endowments. Moreover, she is excluded from borrowing but is still allowed to save, as in Bulow and Rogoff (1989). Competition implies that debt limits are not-too-tight, as in Alvarez and Jermann (2000). Under a mild condition that the endowment loss from default is bounded away from zero, we show that the equilibrium interest rates must be sufficiently high that the present value of aggregate endowments is finite. We show that equilibrium debt limits are exactly equal to the present value of endowment loss due to default. The determination of competitive debt limits based on endowment loss is isomorphic to the determination of public debt sustainable by tax revenues. We also show that competitive equilibria with

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self-enforcingdebt and costly default are equivalent to Arrow-Debreu equilibria with limitedpledgeability, as defined by Gottardi and Kubler (2015).

1 Introduction

This paper addresses a classic question in macroeconomics: how much debt can be sustained if borrowers cannot commit to repay? We develop a general equilibrium model of risk sharing under multilateral lack of commitment. Agents would like to borrow and lend to hedge against heterogeneous shocks to their endowments. If a borrower defaults, the assumption is that she is excluded from borrowing but is still allowed to save, as in Bulow and Rogoff (1989) and Hellwig and Lorenzoni (2009). Competition implies that the endogenous debt limits must be *not too tight*, i.e., they must be set at the largest possible levels such that repayment is individually rational, as in Alvarez and Jermann (2000). A key departure is that, in addition to credit exclusion, *defaulting agents lose a fraction of their current and future endowments*. This assumption is relevant in the context of sovereign default, as it has been documented that countries tend to suffer economic contractions during default episodes.[1]

Within this framework, we establish strong characteristics of the joint behavior of the equilibrium interest rates (or intertemporal prices) and endogenous debt limits. Under a mild condition on primitives, asserting that the total endowment loss across agents is non-negligible (but the loss can be arbitrarily small and can flexibly vary across agents and time), we provide a non-trivial insight about the forces that pin down the implied interest rates and debt limits in a competitive equilibrium.

First, the interest rates must be sufficiently high such that the present value of endowments across all agents in the economy is finite. As a consequent, there cannot exist bubbly equilibria where debt is indefinitely rolled over. Second, it is *endogenous* in equilibrium that the debt limits of any agent must not exceed the present value of her endowments (the natural debt limit). Based on these two results, we establish the

[1] See, e.g., Trebesch et al. (2012), Tomz and Wright (2013), and Aguiar et al. (2014) for surveys of recent empirical research in the sovereign debt literature.

third, which states that the debt limits in any competitive equilibrium must exactly be equal to the present value of endowment loss. This is not a mere coincidence, as the determination of not-too-tight debt limits is akin to the pricing of assets in a competitive equilibrium. The intuition is as follows. On the one hand, high implied interest rates rule out the possibility that borrowers can indefinitely roll over their debt as a Ponzi scheme. Therefore, equilibrium debt limits cannot exceed the present value of output loss, as in Bulow and Rogoff (1989). On the other hand, in the presence of direct costs, debtors would always choose to honour any debt not exceeding the present value of pledgeable resources. Together, those forces imply that equilibrium debt limits should exactly reflect the present value of output loss.

The argument above resembles the no-arbitrage argument that determines the prices of long-lived assets, such as a Lucas tree, to the present value of the streams of dividends, in any non-bubbly equilibrium under full commitment. In our environment with limited commitment, the not-too-tight debt limits reflect the opportunity cost of honouring the contractual arrangements of contingent debt, which is akin to the value of a claim to debtors' resources that would be lost due to default.

The results also provide an insight to an isomorphism between private debt and public debt. We map the current model to an environment where agents cannot issue debt; however, they can instead purchase public debt that is issued by a fiscal authority. We show that as long as the authority can impose a non-negligible tax rate on private agent endowments, the equilibrium interest rates are sufficiently high such that bubbly equilibria, where the authority can indefinitely roll over debt, cannot arise. Moreover, in any competitive equilibrium, the amount of public debt bought by private agents must be exactly equal to the present value of tax revenues, i.e., public debt must be fully backed by taxation. This result stands in a stark contrast to that in Hellwig and Lorenzoni (2009), where the authors show that in the absence of any tax base, any positive amount of public debt in equilibrium must be indefinitely rolled over as a bubble. Our result thus points to the fragility of bubbly equilibria, in a similar sense that the presence of a Lucas tree with non-vanishing dividends can rule out the possibility of rational bubbles (see Santos and Woodford, 1997 and also

Tirole, 1985).

Our general equilibrium model also nests the partial equilibrium model of Bulow and Rogoff (1989) (henceforth BR) as a special case, and our results imply as a corollary their celebrated debt impossibility theorem. They show that, if (i) the interest rates are such that present value of endowments of an agent (or country) is finite, and (ii) the agent's borrowing cannot exceed the natural debt limit, then it is impossible to sustain positive borrowing when the only punishment for default is exclusion from future credit. A key difference is that while BR impose assumptions (i) and (ii) on endogenous variables (equilibrium prices and quantities), these two properties arise naturally in our environment.

From a technical standpoint, our proofs exploit a new characterization of not-too-tight debt limits that can be of independent interest. The interesting feature of our characterization result, which has no analogue in the absence of output costs, is its implication about the level of interest rates. As stressed above, we do not impose a priori any restriction on interest rates. We instead show that no matter how small is the output loss, interest rates must be higher than agents' growth rates if debt is self-enforcing. This, in turn, allows us to show that any process of self-enforcing debt limits is the sum of the present value of output losses and a "bubble" component. Taking into account market clearing conditions, at equilibrium, the self-enforcing debt limits coincide with the present value of output losses (i.e., equilibrium debt limits are bubble free).

Related literature. Our paper is closely related to the general equilibrium models of risk sharing with limited commitment and endogenous borrowing constraints. Kehoe and Levine (1993), Kocherlakota (1996), Alvarez and Jermann (2000, 2001), and Kehoe and Perri (2004) develop models where agents are excluded from financial markets after default, while Hellwig and Lorenzoni (2009) and Martins-da-Rocha and Vailakis (2016, 2017a) develop models where defaulters are only excluded from future credit, as in BR.[2] To the best of our knowledge, our paper is the first to introduce

[2]See also Kletzer and Wright (2000) for a game theoretic analysis of the threat of credit exclusion.

recourse or endowment loss from default into this general equilibrium environment. From a substantive point of view, while most existing papers impose assumptions on endogenous variables, namely the equilibrium interest rates are high and/or the debt limits are bounded by wealth, the introduction of the default loss allows us to generate these properties endogenously.

Our paper is also related to the rational bubbles literature (Diamond, 1965, Tirole, 1985, Miao and Wang, 2011, Farhi and Tirole, 2012, Martin and Ventura, 2012, Hirano and Yanagawa, 2016) and papers on the shortage of assets (Caballero et al., 2008, J. Caballero and Farhi, 2017). A common theme of these papers is that a shortage of savings instruments or a shortage of collateral depresses the interest rates, raising the present value of aggregate output or endowments, leading to the possibility of rational bubbles. Our paper shows that the introduction of arbitrarily small but non-negligible endowment loss from default guarantees finite wealth and rules out the possibility of bubbles. Thus, our paper complements Santos and Woodford (1997) in highlighting the fragility of the conditions for the existence of bubbles.

Finally, our paper is also related to Woodford (1990) and Holmström and Tirole (1998, 2011) in highlighting the relationship between private liquidity and public liquidity in environments with financial frictions that lead to scarce collateral.

2 Environment

2.1 Uncertainty

Time and uncertainty are both discrete. We use an event tree Σ to describe time, uncertainty and the revelation of information over an infinite horizon. There is a unique initial date-0 event $s^0 \in \Sigma$ and for each date $t \in \{0, 1, 2, \ldots\}$ there is a finite set $S^t \subseteq \Sigma$ of date-t events s^t. Each s^t has a unique predecessor $\sigma(s^t)$ in S^{t-1} and a finite number of successors s^{t+1} in S^{t+1} for which $\sigma(s^{t+1}) = s^t$. We use the notation $s^{t+1} \succ s^t$ to specify that s^{t+1} is a successor of s^t. Event $s^{t+\tau}$ is said to follow event s^t, also denoted $s^{t+\tau} \succ s^t$, if $\sigma^{(\tau)}(s^{t+\tau}) = s^t$. The set $S^{t+\tau}(s^t) := \{s^{t+\tau} \in S^{t+\tau} \ : \ s^{t+\tau} \succ s^t\}$

denotes the collection of all date-$(t+\tau)$ events following s^t. Abusing notation, we let $S^t(s^t) := \{s^t\}$. The subtree of all events starting from s^t is then

$$\Sigma(s^t) := \bigcup_{\tau \geq 0} S^{t+\tau}(s^t).$$

We use the notation $s^\tau \succeq s^t$ when $s^\tau \succ s^t$ or $s^\tau = s^t$. In particular, we have $\Sigma(s^t) = \{s^\tau \in \Sigma : \ s^\tau \succeq s^t\}$.

2.2 Endowments and Preferences

There is a single perishable consumption good, and the economy consists of a finite set I of household types, each type consists of a unit measure of identical agents. Agents are infinitely lived. They cannot commit to future actions.

The preferences of agents over (non-negative) consumption processes $c = (c(s^t))_{s^t \in \Sigma}$ are represented by the lifetime expected and discounted utility functional

$$U(c) := \sum_{t \geq 0} \beta^t \sum_{s^t \in S^t} \pi(s^t) u(c(s^t))$$

where $\beta \in (0,1)$ is the discount factor, $\pi(s^t)$ is the unconditional probability of s^t and $u : \mathbb{R}_+ \to [-\infty, \infty)$ is a Bernoulli function assumed to be strictly increasing, concave, continuous on $\mathbb{R}_+$, differentiable on $(0, \infty)$, bounded and satisfying Inada's condition at the origin.[3] Given an event s^t, we denote by $U(c|s^t)$ the lifetime continuation utility conditional to s^t defined by

$$U(c|s^t) := u(c(s^t)) + \sum_{\tau \geq 1} \beta^\tau \sum_{s^{t+\tau} \succ s^t} \pi(s^{t+\tau}|s^t) u(c(s^{t+\tau}))$$

[3]The function u is said to satisfy the Inada's condition at the origin if $\lim_{\varepsilon \to 0}[u(\varepsilon) - u(0)]/\varepsilon = \infty$. We assume that agents' preferences are homogeneous. This is only for the sake of simplicity. All arguments can be adapted to handle the heterogeneous case where the preferences differ among agents. The boundedness of u is only used to establish a necessary condition of the equilibrium debt limits. It can be relaxed if we impose some mild restrictions on the endowments and default punishment. We refer to Remark 1 for the detailed discussion.

where $\pi(s^{t+\tau}|s^t) := \pi(s^{t+\tau})/\pi(s^t)$ is the conditional probability of $s^{t+\tau}$ given s^t.

Agents receive endowments in each period that are subject to random shocks. We denote by $y^i = (y^i(s^t))_{s^t \in \Sigma}$ the process of positive endowments $y^i(s^t) > 0$ of the representative agent of type i.[4] A collection $(c^i)_{i \in I}$ of consumption processes is called a consumption allocation. It is said to be resource feasible if $\sum_{i \in I} c^i = \sum_{i \in I} y^i$. We also fix an allocation $(a^i(s^0))_{i \in I}$ of initial financial claims $a^i(s^0) \in \mathbb{R}$ that satisfies the usual market clearing condition: $\sum_{i \in I} a^i(s^0) = 0$. The initial financial claim $a^i(s^0)$ can be interpreted the consequence of (un-modeled) past transactions.

2.3 Markets

At any event s^t, agents can issue and trade a complete set of one-period contingent bonds which promise to pay one unit of the consumption good contingent on the realization of any successor event $s^{t+1} \succ s^t$. Let $q(s^{t+1}) > 0$ denote the price at event s^t of the s^{t+1}-contingent bond. Agent i's holding of this bond is $a^i(s^{t+1})$. The amount of state-contingent debt agent i can issue is observable and subject to state-contingent (non-negative and finite) debt limits $D^i = (D^i(s^t))_{s^t \in \Sigma}$. Given the initial financial claim $a^i(s^0)$, we denote by $B^i(D^i, a^i(s^0))$ the budget set of an agent who never defaults. It consists of all pairs (c^i, a^i) of consumption and bond holdings satisfying the following constraints: for any event s^t,

$$c^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) a^i(s^{t+1}) \leq y^i(s^t) + a^i(s^t) \tag{2.1}$$

and

$$a^i(s^{t+1}) \geq -D^i(s^{t+1}), \quad \text{for all } s^{t+1} \succ s^t. \tag{2.2}$$

Fix an event s^τ and some initial claim $a \in \mathbb{R}$. We denote by $V^i(D^i, a|s^\tau)$ the value function defined by

$$V^i(D^i, a|s^\tau) := \sup\{U(c^i|s^\tau) \ : \ (c^i, a^i) \in B^i(D^i, a|s^\tau)\}$$

[4] For brevity, instead of saying "the representative agent of type i", we simply say "agent i" in the remaining of the paper.

where $B^i(D^i, a|s^\tau)$ is the set of all plans (c^i, a^i) satisfying $a^i(s^\tau) = a$ together with restrictions (2.1) and (2.2) at every successor node $s^t \succeq s^\tau$.

Without any loss of generality, we restrict attention to debt limits $(D^i(s^t))_{s^t \succeq s^0}$ that are *consistent*, meaning that at every event s^t, the maximal debt can be repaid out of the current resources and the largest possible debt contingent on future events, i.e.,

$$D^i(s^t) \leq y^i(s^t) + \sum_{s^{t+1} \succeq s^t} q(s^{t+1}) D(s^{t+1}), \quad \text{for all } s^t \succeq s^0. \tag{2.3}$$

The above condition is necessary for the budget set $B^i(D^i, -D^i(s^t)|s^t)$ to be non-empty.[5]

2.4 Default Costs

An agent might not honor her debt obligations and decide to default if it is optimal for her. The decision to default depends on the consequences of default. As in Bulow and Rogoff (1989), we assume that defaulters start with neither assets nor liabilities, are excluded from future credit, but retain the ability to purchase bonds. In addition, they suffer a (possibly zero) fractional loss in income upon default.[6] Specifically, debt repudiation leads to a contraction $\tau^i(s^t) \in [0, 1]$ of agent i's endowment, where τ may vary across agents and events. Formally, agent i's default option at event s^t is the largest continuation utility when starting with zero financial claims, cannot borrow and her income reduces by a fraction $\tau^i(s^\tau)$ at every $s^\tau \succeq s^t$:

$$V_d^i(0, 0|s^t) := \sup\{U(c^i|s^t) \ : \ (c^i, a^i) \in B_d^i(0, 0|s^t)\}, \tag{2.4}$$

[5] It is straightforward to check that any process of self-enforcing debt limits (defined in Section 2.5) and any process of natural debt limits (defined in Section 2.6) satisfies the consistency property (2.3).

[6] Disruption of international trade or of the domestic financial system can lead to output drops if either trade or banking credit is essential for production. Among others, Mendoza and Yue (2012), Gennaioli et al. (2014), and Phan (2017) model explicitly how sovereign default may lead to an efficiency loss in production. We follow the tradition in the sovereign debt literature (see for instance Cohen and Sachs (1986), Cole and Kehoe (2000), Aguiar and Gopinath (2006), Arellano (2008), Abraham and Carceles-Poveda (2010), and Bai and Zhang (2010, 2012)) and model the negative implications on output by the loss of an exogenous fraction of income.

where $B^i_d(0,0|s^t)$ denotes the budget set corresponding to $B^i(0,0|s^t)$ when the endowment $y^i(s^\tau)$ is replaced by $(1-\tau^i(s^\tau))y^i(s^\tau)$ at any event $s^\tau \succeq s^t$.

Remark 1. We have assumed that the Bernoulli function u is bounded. This assumption can be dropped if the primitives of the economy satisfy the following properties:

(a) the default option $V^i_d(0,0|s^t)$ is finite for all agent i and event s^t;

(b) If a resource feasible consumption allocation $(c^i)_{i\in I}$ satisfies the participation constraints $U(c^i|s^t) \geq V^i_d(0,0|s^t)$ for all agent i and event s^t, then the continuation utility $U(c^i|s^t)$ is finite for all agent i and event s^t.[7]

The above conditions are obviously true when the Bernoulli function is bounded. However, they are also satisfied if for all agent i and every event s^t we have

$$-\infty < U((1-\tau^i)y^i|s^t) \quad \text{and} \quad U(\bar{y}|s^t) < \infty,$$

where $\bar{y} := \sum_{i\in I} y^i$ is the aggregate endowment. The above inequalities are valid if the endowments are uniformly bounded from above and away from zero and the contraction coefficients $(\tau^i)_{i\in I}$ are uniformly bounded away from 1.

2.5 Self-enforcing Debt Limits

We now incorporate the fact that agents have the option to default. Since borrowers issue contingent bonds, lenders have no incentives to provide credit contingent to some event if they anticipate that a debtor will default.[8] The maximum amount of debt $D^i(s^t)$ at any event $s^t \succeq s^0$ should reflect this property. If agent i's initial financial claim at event s^t corresponds to the maximum debt $-D^i(s^t)$, then she prefers to repay her debt if, and only if, $V^i(D^i, -D^i(s^t)|s^t) \geq V^i_d(0,0|s^t)$. When a process of bounds satisfies the above inequality at every node $s^t \succeq s^0$, it is called

[7]Observe that any consumption allocation derived from a competitive equilibrium (defined in Section 4) necessarily satisfies the participation constraints.

[8]Since the default punishment is independent of the default level, there is no partial default. Agents either repay or default totally.

self-enforcing.[9] Competition among lenders naturally leads to consider the largest self-enforcing bound $D^i(s^t)$ defined by the equation

$$V^i(D^i, -D^i(s^t)|s^t) = V_d^i(0, 0|s^t). \tag{2.5}$$

We follow Alvarez and Jermann (2000) and refer to such debt limits as *not-too-tight*.

Remark 2. If we fix an agent i and assume that $\tau^i(s^t) = 1$ for all every event s^t, then the agent has no incentive to ever default. Equivalently, in this case, the agent can effectively commit to her financial promises. This is because the value of the default option is $V_d^i(0, 0|s^t) = U(0|s^t)$, and consequently any process of consistent debt limits is self-enforcing. Therefore, the specification of output costs encompasses a *mixed* environment where some agents can perfectly commit to financial contracts while others have limited commitment.[10] In case where $\tau^i(s^t) = 1$ for all event s^t and agent i, the model collapses to a standard risk-sharing model with full commitment.

2.6 Natural Debt Limits

Given state-contingent bond prices $q = (q(s^t))_{s^t \succ s^0}$, we denote by $p(s^t)$ the associated date-0 price of consumption at s^t defined recursively by $p(s^0) = 1$ and $p(s^{t+1}) = q(s^{t+1})p(s^t)$ for all $s^{t+1} \succ s^t$. We use $\mathrm{PV}(x|s^t)$ to denote the present value at event s^t of a process x restricted to the subtree $\Sigma(s^t)$ and defined by

$$\mathrm{PV}(x|s^t) := \frac{1}{p(s^t)} \sum_{s^{t+\tau} \in \Sigma(s^t)} p(s^{t+\tau})x(s^{t+\tau}).$$

When the present value $\mathrm{PV}(y^i|s^0)$ is finite, we say that agent i has *finite wealth*[11]

[9] Indeed, since the function $a \mapsto V^i(D^i, a|s^t)$ is increasing, for any bond holding $a^i(s^t)$ satisfying the restriction $a^i(s^t) \geq -D^i(s^t)$, agent i prefers honouring her obligation than defaulting on $a^i(s^t)$.

[10] In particular, our model nests both the model in Bulow and Rogoff (1989) (unilateral lack of commitment) and the one in Hellwig and Lorenzoni (2009) (multilateral lack of commitment).

[11] Some authors use the terminology "interest rates are higher (respectively lower or equal) than agent i's growth rates" when the agent's wealth is finite (respectively infinite). The choice of this terminology is driven by the following particular case. Assume that interest rates and bounds

and define

$$W^i(s^t) := \mathrm{PV}(y^i|s^t).$$

The process $(W^i(s^t))_{s^t\in\Sigma}$ is called the process of *natural debt limits*. In a Arrow–Debreu economy starting at event s^t, the limit $W^i(s^t)$ is the largest amount agent i can consume at event s^t if she can commit to deliver all of her future resources and consume zero forever. We say that a process of debt limits $(D^i(s^t))_{s^t\succeq s^0}$ is *naturally bounded* when it is tighter than the natural debt limits, i.e., $D^i(s^t) \le W^i(s^t)$ for all event s^t.

3 Partial Equilibrium

This section presents a characterization of self-enforcing debt limits that can be obtained as a corollary of the original results in Bulow and Rogoff (1989).[12]

They consider a partial equilibrium set up.[13] The following result asserts that equilibrium debt can only be sustained by the present value of endowment loss. In the absence of endowment loss, debt cannot be sustained by the threat of credit exclusion alone.

Lemma 1 (Bulow and Rogoff)**.** *Consider an arbitrary agent i constrained by a process $(D^i(s^t))_{s^t\succeq s^0}$ of self-enforcing debt limits. Assume that*

(i) prices are such that the agent's wealth is finite, i.e., $W^i(s^0) < \infty$, and

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on growth rates are time and state invariant, i.e., $q(s^{t+1}) = \pi(s^{t+1}|s^t)(1+r)^{-1}$ and $y^i(s^{t+1}) = (1+g^i(s^{t+1}))y^i(s^t)$ for all $s^{t+1} \succ s^t$, where the stochastic growth rate is such that $m^i g^i \le g^i(s^{t+1}) \le M^i g^i$ for some parameter g^i with $0 < m^i \le M^i < \infty$. Then, the agent i's wealth is finite if, and only if, $r > g^i$, and infinite if, and only if, $r \le g^i$.

[12] We refer to Martins-da-Rocha and Vailakis (2016) for the detailed connection. Here we only remark that the model in Bulow and Rogoff (1989) is slightly different than ours since they consider a sovereign who trades at the initial node a complete set of state-contingent contracts that specify the net transfers to foreign investors in all future periods and events. Contracts are restricted to be compatible with repayment incentives and allow investors to break even in present value terms (this environment is in the spirit of Kehoe and Levine (1993) but with a different default option).

[13] A small open economy issues debt taking as given an exogenous, time-invariant world interest rate.

(ii) D^i is naturally bounded, i.e., $D^i(s^t) \leq W^i(s^t)$ at every event s^t.

Then debt limits cannot exceed the present value of endowment loss:

$$D^i(s^t) \leq \mathrm{PV}(\tau^i y^i | s^t), \quad \textit{for all } s^t \succeq s^0.$$

In particular, if $\tau^i = 0$, then $D^i = 0$, i.e., if there is no endowment loss, then debt is not sustainable.

In a general equilibrium set up, conditions (i) and (ii) impose ad-hoc restrictions on endogenous variables (prices and debt limits). A natural question that arises is whether those restrictions hold true at a competitive equilibrium. A contribution of this paper is to identify assumptions on primitives such that, at a competitive equilibrium, the aggregate wealth of the economy is finite and the debt limits are naturally bounded.

4 General Equilibrium

Instead of imposing conditions on endogenous variables (as prices and debt limits), we now address the issue of debt sustainability in the context of a general equilibrium setup.

Definition 1. For given initial asset positions $a^i(s^0)$ such that $\sum_{i \in I} a^i(s^0) = 0$, a *competitive equilibrium with self-enforcing debt* $(q, (c^i, a^i, D^i)_{i \in I})$ consists of state-contingent bond prices q, a resource feasible consumption allocation $(c^i)_{i \in I}$, a market clearing allocation of bond holdings $(a^i)_{i \in I}$ and an allocation of consistent and finite debt limits $(D^i)_{i \in I}$ such that:

(a) for each agent $i \in I$, taking prices and the debt limits as given, the plan (c^i, a^i) is optimal among budget feasible plans in $B^i(D^i, a^i(s^0)|s^0)$;

(b) for each agent $i \in I$, the sequence of debt limits D^i are not-too-tight, i.e., equation (2.5) is satisfied at all events.

Our first result establishes a lower bound on equilibrium debt limits:

Lemma 2. *Not-too-tight debt limits are at least as large as the present value of endowment loss:* $D^i(s^t) \geq \mathrm{PV}(\tau^i y^i | s^t)$ *for each event* s^t *and each agent* i*.*

This result provides an intuitive lower bound on equilibrium debt limits, which must be not-too-tight, and serves as an important result for the analysis in the remaining of the paper. A natural approach to prove this result is to show that $D^i(s^t) \geq \tau^i(s^t)y^i(s^t) + \widetilde{D}^i(s^t)$ where $\widetilde{D}^i(s^t) := \sum_{s^{t+1} \succ s^t} q(s^{t+1})D^i(s^{t+1})$ is the present value of next period's debt limits, and then use a standard iteration argument. Because in equilibrium, debt limits are not-too-tight, this is equivalent to proving that agent i does not have an incentive to default when her net asset position is $\tau^i(s^t)y^i(s^t) + \widetilde{D}^i(s^t)$, i.e.,

$$V^i(D^i, -\tau^i(s^t)y^i(s^t) - \widetilde{D}^i(s^t)|s^t) \geq V_d^i(0, 0|s^t). \tag{4.1}$$

By definition, the default value function V_d^i satisfies:

$$V_d^i(0, 0|s^t) \geq u((1 - \tau^i(s^t))y^i(s^t)) + \beta \sum_{s^{t+1} \succ s^t} \pi(s^{t+1}|s^t)V_d^i(0, 0|s^{t+1}). \tag{4.2}$$

If we had an equality in (4.2), then inequality (4.1) would be straightforward. Indeed, consuming $(1 - \tau^i(s^t))y^i(s^t)$ and borrowing up to each debt limit $D^i(s^{t+1})$ at event s^t leads to the right hand side continuation utility in (4.2), and satisfies the solvency constraint at event s^t in the budget set defining the left hand side of (4.1). However, in our environment where an agent can save upon default, (4.2) need not be satisfied with an equality.[14] Dealing with this problem is non-trivial and constitutes the technical contribution in the proof presented in Appendix A.1.

[14]In the simpler environment where saving is not possible after default (as it is the case in Alvarez and Jermann (2000)) we always have an equality in (4.2).

4.1 Wealth must be finite

From now on we impose the following minimal assumption on the structure of endowment loss:

Assumption 1. The aggregate endowment loss from default is *non-negligible* (with respect to aggregate endowments), in the sense that there exists $\varepsilon > 0$ such that:

$$\sum_{i \in I} \tau^i(s^t) y^i(s^t) \geq \varepsilon \sum_{i \in I} y^i(s^t), \quad \text{for all } s^t \succeq s^0. \tag{4.3}$$

The above property is satisfied, for example, if the fraction of endowment loss is uniformly positive for all agents, i.e., for all agent $i \in I$, there exists $\varepsilon > 0$ such that $\tau^i(s^t) \geq \varepsilon$ at any event $s^t \in \Sigma$. Moreover, the property is still satisfied even if only a subset of agents face endowment loss. For example, assume a subset of agents $I^c \subseteq I$ (the "committed types") face the maximum endowment loss: $\tau^i = 1$ for all $i \in I^c$; the remaining agents (the "uncommitted types") do not face any endowment loss: $\tau^i = 0$ for all $i \in I \setminus I^c$. Condition (4.3) still holds as long as the total endowment of the committed types is non-negligible, i.e., there exists $\varepsilon > 0$ such that $\sum_{i \in I^c} y^i(s^t) \geq \varepsilon \sum_{i \in I} y^i(s^t)$.

The combination of Lemma 2 and Assumption 1 yields an important result about equilibrium interest rates. The result states that no matter how small the output costs are, as long as they are non-negligible, the present value of the stream of endowments – discounted at the equilibrium interest rates – must be finite. More informally, this result implies that the interest rates in any competitive equilibrium with non-negligible endowment loss must be at least as high as the growth rate of the economy.

Theorem 1. *Assume non-negligible endowment loss. In any competitive equilibrium with self-enforcing debt, the wealth of any agent is finite, i.e., $W^i(s^0) < \infty$ for all $i \in I$. As a consequence, the aggregate wealth of the economy is finite, i.e., $\sum_{i \in I} W^i(s^0) < \infty$.*

Proof. Since the aggregate endowment loss is *non-negligible* (Assumption 1), we have

$$\sum_{i\in I} \mathrm{PV}(\tau^i y^i|s^0) \geq \varepsilon \sum_{i\in I} \mathrm{PV}(y^i|s^0) = \varepsilon \sum_{i\in I} W^i(s^0).$$

On the other hand, Lemma 2 implies that

$$\sum_{i\in I} D^i(s^0) \geq \sum_{i\in I} \mathrm{PV}(\tau^i y^i|s^0).$$

Combining the two inequalities above yields the following inequality:

$$\sum_{i\in I} D^i(s^0) \geq \varepsilon \sum_{i\in I} W^i(s^0).$$

Since the debt limits D^i are finite by definition, the inequality implies that the aggregate wealth of the economy $\sum_{i\in I} W^i(s^0)$ is finite. As a corollary, each individual's wealth $W^i(s^0)$ is finite. □

This result stands in stark contrast with the result of Hellwig and Lorenzoni (2009), which states that in the absence of endowment loss, any equilibrium with positive debt requires low interest rates, so that each agent's wealth is infinite. In other words, the result state that if we introduce even just very small but non-negligible endowment loss, we can effectively rule out bubbly equilibria. This fragility of bubbly equilibria is related to Santos and Woodford (1997) and Tirole (1985), who find that the existence conditions for bubbles are relatively fragile.

4.2 Equilibrium debt limits must be naturally bounded

Finite aggregate wealth implies the necessity of a market transversality condition (see equation (4.4) below). This condition in turn allows us to show that the present value of debt limits satisfies an asymptotic property (see 4.7 below). We then deduce from the consistency property (2.3) that debt limits are bounded from above by the natural debt limits, plus a non-negative process that satisfies exact roll-over.[15] Taking into account market clearing conditions, we can show that the exact roll-over component must be zero, i.e., debt cannot contain a bubbly component in equilibrium. We then deduce that the equilibrium self-enforcing debt limits are tighter than the natural debt limits. This is established formally in the following theorem.

Theorem 2. *Assume non-negligible endowment loss. In any competitive equilibrium with self-enforcing debt, debt limits are naturally bounded, i.e.,* $D^i(s^t) \leq W^i(s^t)$ *for all event* s^t *and agent* $i \in I$*.*

Proof. Let $(q, (c^i, a^i, D^i)_{i \in I})$ be a competitive equilibrium. Fix a date-event s^t and consider another period $\tau > t$. The proof proceeds in three steps.

First, since consumption markets clear, we obtain from Theorem 1 that the present value of optimal consumption is finite for all agents. In addition, due to the Inada's condition, the optimal consumption is strictly positive. Lemma A.1 in Martins-da-Rocha and Vailakis (2017a) then implies that the following market transversality condition holds true:[16]

$$\lim_{\tau \to \infty} \sum_{s^\tau \in S^\tau(s^t)} p(s^\tau)[a^i(s^\tau) + D^i(s^\tau)] = 0. \tag{4.4}$$

Second, consolidating agent i's (binding) flow budget constraints along the subtree $\Sigma(s^t)$ up to period $\tau - 1$ produces

$$\begin{aligned} &\sum_{s^r \in \Sigma^{\tau-1}(s^t)} p(s^r) c^i(s^r) + \sum_{s^\tau \in S^\tau(s^t)} p(s^\tau)[a^i(s^\tau) + D^i(s^\tau)] \\ &\qquad = a^i(s^t) + \sum_{s^r \in \Sigma^{\tau-1}(s^t)} p(s^r) y^i(s^r) + \sum_{s^\tau \in S^\tau(s^t)} p(s^\tau) D^i(s^\tau). \end{aligned} \tag{4.5}$$

[15]A process satisfying exact roll-over can be interpreted as a bubble. We refer to Hellwig and Lorenzoni (2009) for details.

[16]The market transversality condition differs from the individual transversality condition. Indeed, due to the lack of commitment, agent i's debt limits may bind, in which case we do not necessarily have that $p(s^t) = \beta^t \pi(s^t) u'(c^i(s^t))/u'(c^i(s^0))$.

By letting τ converge to infinity, transversality condition (4.4) and consolidated budget equation (4.5) imply:

$$\mathrm{PV}(c^i|s^t) = a^i(s^t) + W^i(s^t) + M^i(s^t), \tag{4.6}$$

where:

$$M^i(s^t) := \frac{1}{p(s^t)} \lim_{\tau \to \infty} \sum_{s^\tau \in S^\tau(s^t)} p(s^\tau) D^i(s^\tau) \quad \text{exists in } \mathbb{R}_+. \tag{4.7}$$

Observe that the process M^i satisfies the following Martingale or exact-roll-over property

$$\forall s^t \in \Sigma, \quad M^i(s^t) = \sum_{s^{t+1} \succ s^t} q(s^{t+1}) M^i(s^{t+1}). \tag{4.8}$$

Summing over i in equation (4.6) and exploiting market clearing implies that

$$\forall s^t \in \Sigma, \quad \sum_{i \in I} M^i(s^t) = 0.$$

Since each M^i is non-negative we get that $M^i = 0$ for all $i \in I$.

Finally, since the debt limits are consistent (i.e., they obey condition (2.3)), for all arbitrary date-event s^t and every period $\tau > t$, we have

$$p(s^t)D^i(s^t) \leq \sum_{r=t}^{\tau} \sum_{s^t \in S^r(s^t)} p(s^r) y^i(s^r) + \sum_{s^\tau \in S^\tau(s^t)} p(s^\tau) D^i(s^\tau).$$

Passing to the limit when τ tends to infinite, we get that

$$D^i(s^t) \leq W^i(s^t) + M^i(s^t). \tag{4.9}$$

Because $M^i = 0$, inequality (4.9) then implies that $D^i \leq W^i$ for all agents.

□

Remark 3. If Assumption 1 is dropped, then Theorem 1 and Theorem 2 may fail. For instance, if there is no output drop upon default, Hellwig and Lorenzoni (2009) provide an example where, at a competitive equilibrium, each agent's wealth is infinite. Even if some agents can fully commit to honor their debt obligations, pledging all of their income as collateral, the equilibrium debt limits of those agents may not be naturally bounded. We refer to Appendix B.2 for an example.

4.3 Exact characterization of equilibrium debt limits

Theorems 1 and 2 together with Lemma 1 allow us to have an exact characterization of the debt limits in equilibrium, without having to impose assumptions on endogenous variables (the interest rates or debt limits). Specifically, because Theorem 1 states that each agent's wealth is finite and Theorem 2 states that each agent's debt limit is naturally bounded, both conditions (i) and (ii) in Lemma 1 are automatically satisfied. This allows us to establish the following result:

Theorem 3. *Assume non-negligible endowment loss. In any competitive equilibrium with self-enforcing debt, the debt limit must be exactly equal to the present value of endowment loss:*

$$D^i(s^t) = \mathrm{PV}(\tau^i y^i | s^t), \quad \textit{for all } s^t \succeq s^0. \tag{4.10}$$

Proof. Theorems 1 and 2 imply that conditions (i) and (ii) of Lemma 1 are satisfied. As a consequence, Lemma 1 implies that we have an upper bound on the debt limits:

$$D^i(s^t) \leq \mathrm{PV}(\tau^i y^i | s^t), \quad \text{for all } s^t \text{ and every } i.$$

On the other hand, from Lemma 2, we have a lower bound on the debt limits:

$$D^i(s^t) \geq \mathrm{PV}(\tau^i y^i | s^t), \quad \text{for all } s^t \text{ and every } i.$$

As the bounds coincide, it must be that equilibrium debt limits are exactly equal to the present value of endowment loss, i.e., equation (4.10) holds. □

This theorem formalizes a non-trivial insight. There are three forces that pin down the debt limits in any competitive equilibrium. First, the presence of non-negligible endowment loss implies that interest rates are sufficiently high that aggregate wealth (and thus the present value of endowment loss) is finite. Second, the threat of default and high interest rates imply that self-enforcing debt limits cannot exceed the present value of endowment loss. Third, competition among lenders implies that debt limits must be not too tight, which, in equilibrium, must be at least as large as the present value of endowment loss. Together, they imply that equilibrium debt limits must exactly coincide with the present value of endowment loss.

This theorem formalizes a non-trivial insight. There are four forces that pin down the debt limits in any competitive equilibrium. First, competition among lenders imply that debt limits must be not too tight, which, in equilibrium, must be at least as large as the present value of endowment loss (Lemma 2). Second, the assumption of non-negligible endowment loss implies that interest rates are sufficiently high that aggregate wealth is finite (Theorem 1). Third, high interest rates and market clearing imply that debt limits are naturally bounded (Theorem 2). Fourth, the possibility to save after default imply that self-enforcing debt limits cannot exceed the present value of endowment loss (Lemma 1). Together, they imply that equilibrium debt limits must exactly coincide with the present value of endowment loss.

The theorem also generalizes (and strengthen) Bulow and Rogoff result to the general equilibrium framework (without having to impose ad-hoc assumptions on endogenous quantities and prices). It implies that as long as the endowment loss is non-negligible, BR claim in Lemma 1 that $D^i \leq \mathrm{PV}(\tau^i y^i)$ holds with equality. A special case is the celebrated no-debt result:

Corollary 1. *Assume non-negligible endowment loss. If an agent i faces no cost of default ($\tau^i = 0$) then she cannot borrow in equilibrium ($D^i = 0$).*

5 Simple example

To clearly illustrate the main results of the paper, we turn to a special case of our general environment.

Consider a deterministic economy with two agents $I := \{i_1, i_2\}$. The agents' income alternate between a high and a low value $y_{\mathrm{H}} > y_{\mathrm{L}} > 0$. Agent i_1 starts with high income. We assume for simplicity that the output loss fraction $\tau \in [0,1]$ is neither type nor time specific. We restrict attention to symmetric stationary competitive equilibria as defined below.

Definition 2. A competitive equilibrium with self-enforcing debt $(q, (c^i, a^i, D^i)_{i\in I})$ is said to be *symmetric stationary* when there exists $(c_{\mathrm{H}}, d_{\mathrm{H}}) \in \mathbb{R}_+ \times \mathbb{R}_+$ such that

$$\forall i \in I, \quad (c^i_t, a^i_t) = (c_{\mathrm{H}}, -d_{\mathrm{H}}) \quad \text{when} \quad y^i_t = y_{\mathrm{H}}.$$

If a competitive equilibrium with self-enforcing debt is symmetric stationary, then we necessarily have that

$$(c^i_t, a^i_t) = (\underbrace{\bar{y} - c_{\mathrm{H}}}_{=:c_{\mathrm{L}}}, d_{\mathrm{H}}) \quad \text{when} \quad y^i_t = y_{\mathrm{L}}$$

where we recall that $\bar{y} := y_{\mathrm{H}} + y_{\mathrm{L}}$. Moreover, interest rates are constant and satisfy

$$q_t = \beta \frac{u'(c_{\mathrm{L}})}{u'(c_{\mathrm{H}})} =: q.$$

Lemma 3. *If a competitive equilibrium with self-enforcing debt is symmetric stationary, then there exists $(D_{\mathrm{H}}, D_{\mathrm{L}}) \in \mathbb{R}^2_+$ such that*

$$D^i_t = \begin{cases} D_{\mathrm{H}} & \text{if } y^i_t = y_{\mathrm{H}}, \\ D_{\mathrm{L}} & \text{if } y^i_t = y_{\mathrm{L}}, \end{cases}$$

for each i and all $t \geq 0$.

Proof. If $\tau > 0$, we can apply our results to conclude that each agent's wealth must be finite. This implies that the interest rate must be positive (i.e., $q < 1$) and debt limits satisfy

$$D_t^i = \tau\,\mathrm{PV}_t(y^i) = \begin{cases} \tau \dfrac{y_{\mathrm{H}} + q y_{\mathrm{L}}}{1-q^2} =: D_{\mathrm{H}} & \text{if } y_t^i = y_{\mathrm{H}}, \\ \\ \tau \dfrac{y_{\mathrm{L}} + q y_{\mathrm{H}}}{1-q^2} =: D_{\mathrm{L}} & \text{if } y_t^i = y_{\mathrm{L}}. \end{cases}$$

If $\tau = 0$, we can apply Hellwig and Lorenzoni (2009) to conclude that not-too-tight debt limits must allow for exact roll-over, i.e.,

$$\forall t \geq 0, \quad D_t^i = q D_{t+1}^i.$$

If $q < 1$, then each agent's wealth must be finite and we can show that must have $D_t^i = 0$ for each i and all $t \geq 0$. If $q = 1$, then we get that $D_t^i = D_{t+1}^i$ for each $t \geq 0$. □

Proposition 1. *Under the following condition*

$$\tau \geq \tau^\star := (1-\beta)\frac{(y_{\mathrm{H}} - y_{\mathrm{L}})/2}{y_{\mathrm{H}} + \beta y_{\mathrm{L}}} \tag{5.1}$$

a competitive equilibrium with self-enforcing debt is symmetric stationary if, and only if, it is efficient. In particular, we have

$$q = \beta \quad \textit{and} \quad c_{\mathrm{H}} = c_{\mathrm{L}} = \frac{y_{\mathrm{H}} + y_{\mathrm{L}}}{2}.$$

Proof. We first prove that first best can be the outcome of a competitive equilibrium with self-enforcing debt. Assume that $q = \beta$ and $c_{\mathrm{H}} = c_{\mathrm{L}}$. The only difficult step is to show that debt constraints are satisfied (or, equivalently, there are no default incentives).[17] If an agent's income is high, she holds a long position on the bond and the debt constraint is obviously satisfied. If an agent's income is low, her bond holding is $-d_{\mathrm{H}}$. Since we assumed that $c_{\mathrm{H}} = c_{\mathrm{L}}$, we then get that

$$y_{\mathrm{H}} - (1+\beta)d_{\mathrm{H}} = y_{\mathrm{L}} + (1+\beta)d_{\mathrm{H}}$$

[17]Indeed, Euler equations and the transversality condition are trivially satisfied.

or, equivalently,

$$d_{\mathrm{H}} = \frac{(y_{\mathrm{H}} - y_{\mathrm{L}})/2}{1+\beta}.$$

Equation (5.1) then implies that $d_{\mathrm{H}} \leq D_{\mathrm{H}}$.

We now show that first best is the only possible outcome of a symmetric stationary competitive equilibrium with self-enforcing debt. First observe that first order conditions imply

$$q = \beta \frac{u'(c_{\mathrm{L}})}{u'(c_{\mathrm{H}})} \quad \text{and} \quad q \geq \beta \frac{u'(c_{\mathrm{H}})}{u'(c_{\mathrm{L}})}.$$

The above conditions imply that $c_{\mathrm{L}} \leq c_{\mathrm{H}}$. To prove that $c_{\mathrm{L}} = c_{\mathrm{H}}$, we assume, by way of contradiction, that $c_{\mathrm{L}} < c_{\mathrm{H}}$. Recall that budget feasibility implies

$$d_{\mathrm{H}} = \frac{y_{\mathrm{H}} - c_{\mathrm{H}}}{1+q} = \frac{c_{\mathrm{L}} - y_{\mathrm{L}}}{1+q}.$$

Since $c_{\mathrm{L}} < c_{\mathrm{H}}$, we deduce that

$$d_{\mathrm{H}} < \frac{(y_{\mathrm{H}} - y_{\mathrm{L}})/2}{1+q}. \tag{5.2}$$

Moreover, condition (5.1) implies that

$$D_{\mathrm{H}} = \tau \frac{y_{\mathrm{H}} + q y_{\mathrm{L}}}{1-q^2} \geq \frac{y_{\mathrm{H}} + q y_{\mathrm{L}}}{y_{\mathrm{H}} + \beta y_{\mathrm{L}}} \times \frac{1-\beta}{1-q} \times \frac{(y_{\mathrm{H}} - y_{\mathrm{L}})/2}{1+q}.$$

Since $c_{\mathrm{L}} < c_{\mathrm{H}}$, we deduce that $q > \beta$ which implies that

$$D_{\mathrm{H}} > \frac{(y_{\mathrm{H}} - y_{\mathrm{L}})/2}{1+q}.$$

Combining the above inequality with (5.2), we get that $d_{\mathrm{H}} < D_{\mathrm{H}}$. We have thus proved that none of the debt constraints is binding and we deduce from the Euler equations the following contradiction: $c_{\mathrm{H}} = c_{\mathrm{L}}$. $\square$

When $\tau = 0$, no-trade is always a symmetric stationary competitive equilibrium with self-enforcing debt. We identify two cases: either it is the only symmetric stationary competitive equilibrium, or there exists an additional symmetric stationary competitive equilibrium with zero interest rates where debt limits form a bubble.

Proposition 2. *Assume that $\tau = 0$.*

(a) If $\beta u'(y_{\mathrm{L}})/u'(y_{\mathrm{H}}) > 1$, then there is a unique symmetric stationary competitive equilibrium with self-enforcing debt that involves trade. It is characterized by

$$q = 1 \quad \text{and} \quad D_t^i = d_{\mathrm{H}}, \quad \text{for all } i \text{ and } t$$

where $d_{\mathrm{H}} \in (0, (y_{\mathrm{H}} - y_{\mathrm{L}})/2)$ is the only solution to the following equation

$$u'(y_{\mathrm{H}} - 2d_{\mathrm{H}}) = \beta u'(y_{\mathrm{L}} + 2d_{\mathrm{H}}). \tag{5.3}$$

(b) If $\beta u'(y_{\mathrm{L}})/u'(y_{\mathrm{H}}) \leq 1$, then no-trade is the only symmetric stationary competitive equilibrium with self-enforcing debt.

Proof. Consider a symmetric stationary competitive equilibrium with self-enforcing debt that involves trade. Hellwig and Lorenzoni (2009) proved that debt limits must satisfy exact roll-over. In particular, we must have

$$D_{\mathrm{H}} = q^2 D_{\mathrm{H}}.$$

Since there is trade, we must have $D_{\mathrm{H}} > 0$. This implies that $q = 1$ and we deduce that

$$1 = \beta \frac{u'(c_{\mathrm{L}})}{u'(c_{\mathrm{H}})} = \beta \frac{u'(y_{\mathrm{L}} + 2d_{\mathrm{H}})}{u'(y_{\mathrm{H}} - 2d_{\mathrm{H}})}.$$

If $u'(y_{\mathrm{H}}) \geq \beta u'(y_{\mathrm{L}})$, then the above equation does not have a solution and we get a contradiction. If $u'(y_{\mathrm{H}}) < \beta u'(y_{\mathrm{L}})$, then the above equation has a unique solution in the interval $(0, (y_{\mathrm{H}} - y_{\mathrm{L}})/2)$. Moreover, we must have $d_{\mathrm{H}} = D_{\mathrm{H}}$ otherwise the consumption allocation would be efficient and we would have $q = \beta$.

Reciprocally, assume that there exists $d_{\mathrm{H}} \in (0, (y_{\mathrm{H}} - y_{\mathrm{L}})/2)$ solving Equation (5.3).

We claim that the bubbly equilibrium defined by

$$D_t^i := d_{\mathrm{H}}, \quad \text{for all } i \text{ and } t$$

is a competitive equilibrium with self-enforcing debt. By construction debt limits allow for exact roll-over and therefore are not-too-tight. Budget constraints are satisfied with equality and Euler equations are also satisfied. The transversality condition is satisfied since debt limits bind infinitely many times for each agent. □

Proposition 3. *Assume that*

$$0 < \tau < \tau^\star. \tag{5.4}$$

Then any symmetric stationary competitive equilibrium with self-enforcing debt displays positive interest rates ($q < 1$) and is characterized by the following equations

$$u'(y_{\mathrm{H}} - (1+q)d_{\mathrm{H}})q = \beta u'(y_{\mathrm{L}} + (1+q)d_{\mathrm{H}}) \quad \text{and} \quad d_{\mathrm{H}} = \tau \frac{y_{\mathrm{H}} + q y_{\mathrm{L}}}{1 - q^2}. \tag{5.5}$$

Proof. Consider a symmetric stationary competitive equilibrium with self-enforcing debt. Since $\tau > 0$, the wealth of each agent must be finite. This implies that $q < 1$ and

$$D_{\mathrm{H}} = \tau \frac{y_{\mathrm{H}} + q y_{\mathrm{L}}}{1 - q^2}.$$

The first order condition associated to the optimal decision of the high income agent is

$$q = \beta \frac{u'(c_{\mathrm{L}})}{u'(c_{\mathrm{H}})} = \beta \frac{u'(y_{\mathrm{L}} + (1+q)d_{\mathrm{H}})}{u'(y_{\mathrm{H}} - (1+q)d_{\mathrm{H}})}.$$

To get (5.5), we only have to show that the debt constraint contingent to high income binds. Assume by way of contradiction that $d_{\mathrm{H}} < D_{\mathrm{H}}$. This implies that none of the debt constraints is binding and we get an efficient allocation. We then deduce that $q = \beta$ and

$$\tau \frac{y_{\mathrm{H}} + \beta y_{\mathrm{L}}}{1 - \beta^2} = D_{\mathrm{H}} > d_{\mathrm{H}} = \frac{(y_{\mathrm{H}} - y_{\mathrm{L}})/2}{1 + \beta} = d_{\mathrm{H}}.$$

The above inequality leads to the following contradiction $\tau > \tau^\star$.

Reciprocally, let $q < 1$ and d_{H} satisfying the conditions of (5.5). Consider the family $(q, (c^i, a^i, D^i)_{i \in I})$ defined by

$$(c^i_t, a^i_t, D^i_t) := \begin{cases} (c_{\text{H}}, -d_{\text{H}}, D_{\text{H}}) & \text{when } y^i_t = y_{\text{H}}, \\ (\underbrace{\bar{y} - c_{\text{H}}}_{=: c_{\text{L}}}, d_{\text{H}}, D_{\text{L}}) & \text{when } y^i_t = y_{\text{L}}, \end{cases}$$

where debt limits are defined by

$$D_{\text{H}} := \tau \frac{y_{\text{H}} + q y_{\text{L}}}{1 - q^2} \quad \text{and} \quad D_{\text{L}} := \tau \frac{y_{\text{L}} + q y_{\text{H}}}{1 - q^2}.$$

Debt limits are not-too-tight by construction. Consumption and bond markets clear by construction. Debt constraints are satisfied since $d^i_t = -D^i_t$ when $y^i_t = y_{\text{H}}$ (this corresponds to the second property in (5.5)) and $d^i_t = d_{\text{H}} > 0 \geq -D^i_t$ when $y^i_t = y_{\text{L}}$. The flow budget constraints are satisfied by construction. The transversality condition is satisfied since debt limits bind infinitely many times. We only have to check that the Euler equations are satisfied. The first condition of (5.5) implies that the first order condition corresponding the high income agent is satisfied. The one corresponding the low income agent is satisfied if we prove that

$$y_{\text{L}} + (1 + q) d_{\text{H}} = c_{\text{L}} \leq c_{\text{H}} = y_{\text{H}} - (1 + q) d_{\text{H}}.$$

Actualy, we claim that we must have $c_{\text{L}} < c_{\text{H}}$. Assume, by way of contradiction, that $c_{\text{L}} \geq c_{\text{H}}$, or equivalently,

$$y_{\text{L}} + (1 + q) d_{\text{H}} \geq y_{\text{H}} - (1 + q) d_{\text{H}}.$$

The above inequality implies that

$$\tau \frac{y_{\text{H}} + q y_{\text{L}}}{1 - q} \geq \frac{y_{\text{H}} - y_{\text{L}}}{2}. \tag{5.6}$$

We also deduce from (5.5) that

$$q = \beta \frac{u'(c_{\mathrm{L}})}{u'(c_{\mathrm{H}})} \leq \beta.$$

Combining the above inequality with (5.6), we get that

$$\tau \frac{y_{\mathrm{H}} + \beta y_{\mathrm{L}}}{1 - \beta} \geq \frac{y_{\mathrm{H}} - y_{\mathrm{L}}}{2}$$

which implies the contradiction: $\tau \geq \tau^\star$. $\square$

5.1 Existence of equilibria

For any given $\tau \in (0, \tau^\star)$, the issue of existence of a symmetric Markovian equilibrium with self-enforcing debt reduces to the following problem: can we find $q_\tau \in (0, 1)$ solving the equation

$$u'\left(y_{\mathrm{H}} - \tau \frac{y_{\mathrm{H}} + q_\tau y_{\mathrm{L}}}{1 - q_\tau}\right) q_\tau = \beta u'\left(y_{\mathrm{L}} + \tau \frac{y_{\mathrm{H}} + q_\tau y_{\mathrm{L}}}{1 - q_\tau}\right)? \tag{5.7}$$

To analyze the existence, uniqueness and asymptotic properties of the solution of the above equation, we introduce the following notations.

Let $\xi : [0, 1) \to \mathbb{R}$ be defined by

$$\xi(q) := \frac{y_{\mathrm{H}} + q y_{\mathrm{L}}}{1 - q}.$$

Observe that $\xi(q)/(1 + q)$ is the present value of income conditional to high income. The function ξ is twice continuously differentiable and strictly increasing. Moreover, we have $\xi([0, 1)) = [y_{\mathrm{H}}, \infty)$. In particular, there exists $\bar{q}_\tau \in (0, 1)$ such that

$$\xi(\bar{q}_\tau) = \frac{y_{\mathrm{H}}}{\tau}.$$

Let $\eta : [0, y_{\mathrm{H}}) \to \mathbb{R}$ be defined by

$$\eta(x) := \beta \frac{u'(y_{\mathrm{L}} + x)}{u'(y_{\mathrm{H}} - x)}.$$

The number $\eta(x)$ is the marginal rate of substitution of consumption contingent to high for consumption to contingent to low income when the agent saves the amount $x/(1+q) \geq 0$. The function η is twice continuously differentiable. Since u is strictly increasing and strictly concave, the function η is strictly decreasing and $\eta([0, y_{\mathrm{H}})) = [\beta u'(y_{\mathrm{L}})/u'(y_{\mathrm{H}}), 0)$.

We denote by D the set of all pairs $(\tau, q) \in (0, \tau^\star) \times [0, 1)$ such that $q < \bar{q}_\tau$, i.e.,

$$D := \{(\tau, q) \in (0, \tau^\star) \times [0, 1) \ : \ \tau\xi(q) < y_{\mathrm{H}}\}.$$

We can let $\psi : D \to \mathbb{R}$ be defined by

$$\psi(\tau, q) := \eta(\tau\xi(q)).$$

This is the marginal rate of substitution of consumption contingent to high for consumption to contingent to low income when the agent's saving correspond to the debt limit contingent to high income. The function ψ is twice continuously differentiable.

Fix $\tau \in (0, \tau^\star)$ and observe that $\psi(\tau, \cdot)$ is strictly decreasing with

$$\psi(\tau, 0) = \beta \frac{u'(y_{\mathrm{L}} + \tau y_{\mathrm{H}})}{u'(y_{\mathrm{H}} - \tau y_{\mathrm{H}})} \quad \text{and} \quad \lim_{q \to \bar{q}_\tau} \psi(\tau, q) = 0.$$

Applying the Intermediate Value Theorem, we deduce that there exists a unique $q_\tau \in (0, \bar{q}_\tau)$ such that

$$\psi(\tau, q_\tau) = q_\tau.$$

Observe moreover that we must have

$$\forall \tau \in (0, \tau^\star), \quad q_\tau < \beta \frac{u'(y_{\mathrm{L}} + \tau y_{\mathrm{H}})}{u'(y_{\mathrm{H}} - \tau y_{\mathrm{H}})}. \tag{5.8}$$

We let $d_{\mathrm{H},\tau} := \tau\xi(q_\tau)/(1+q_\tau)$ be the equilibrium debt level contingent to high income.

For any $q \in [0,1)$, we let $\bar{\tau}_q := y_{\mathrm{H}}/\xi(q)$. The function $\psi(\cdot, q) : (0, \bar{\tau}_q) \to \mathbb{R}$ is strictly decreasing. We then deduce that the function $\tilde{q} : (0, \tau^\star) \to \mathbb{R}$ defined by

$$\tilde{q}(\tau) := q_\tau$$

is strictly increasing. Since $\tilde{q}(\tau) \in (0, \bar{q}_\tau) \subseteq (0,1)$, we deduce that there exists $q_0 \in (0,1]$ such that

$$\lim_{\tau \to 0} q_\tau = \sup_{\tau \in (0,\tau^\star)} q_\tau =: q_0.$$

5.2 Limiting case as output loss vanishes

An interesting question arises: how do equilibrium prices and quantities converge as the default cost τ converges to zero? Specifically, do we converge to the equilibrium with no trade or the bubbly equilibrium where debt is being issued as a Ponzi scheme?

Proposition 4. *The asymptotic behavior of the symmetric Markovian equilibrium when τ vanishes is described by:*

(a) If $\beta u'(y_{\mathrm{L}})/u'(y_{\mathrm{H}}) \leq 1$, then $q_0 = \beta u'(y_{\mathrm{L}})/u'(y_{\mathrm{H}})$ and the symmetric Markovian equilibrium associated to q_τ converges to the no-trade equilibrium (which is the unique symmetric Markovian equilibrium when $\tau = 0$).

(b) If $\beta u'(y_{\mathrm{L}})/u'(y_{\mathrm{H}}) > 1$, then $q_0 = 1$ and the symmetric Markovian equilibrium associated to q_τ converges to the unique bubbly symmetric Markovian equilibrium when $\tau = 0$. In particular, we have

$$\lim_{\tau \to 0} \tau\xi(q_\tau) = 2d_{\mathrm{H}} > 0$$

where d_{H} is the solution of the equation

$$u'(y_{\mathrm{H}} - 2d_{\mathrm{H}}) = \beta u'(y_{\mathrm{L}} + 2d_{\mathrm{H}}).$$

Proof. We split the proof in two parts.

(a) Assume that $\beta u'(y_{\mathrm{L}})/u'(y_{\mathrm{H}}) < 1$. Passing to the limit in (5.8), we get that

$$q_0 = \lim_{\tau \to 0} q_\tau \le \lim_{\tau \to 0} \beta \frac{u'(y_{\mathrm{L}} + \tau y_{\mathrm{H}})}{u'(y_{\mathrm{H}} - \tau y_{\mathrm{H}})} = \beta \frac{u'(y_{\mathrm{L}})}{u'(y_{\mathrm{H}})} < 1.$$

This implies that

$$\lim_{\tau \to 0} \xi(q_\tau) = \xi(q_0) < \infty$$

and we deduce that

$$\lim_{\tau \to 0} d_{\mathrm{H},\tau} = \lim_{\tau \to 0} \frac{\tau \xi(q_\tau)}{1 + q_\tau} = 0.$$

We then get that the symmetric Markovian equilibrium associated to q_τ converges to the no-trade equilibrium. Moreover, since $q_\tau = \eta(\tau \xi(q_\tau))$, passing to the limit when τ vanishes, we get that

$$q_0 = \eta(0) = \beta u'(y_{\mathrm{L}})/u'(y_{\mathrm{H}}).$$

(b) Assume that $\beta u'(y_{\mathrm{L}})/u'(y_{\mathrm{H}}) \ge 1$. Since η is strictly decreasing with $\eta([0, y_{\mathrm{H}})) = (0, \beta u'(y_{\mathrm{L}})/u'(y_{\mathrm{L}})]$, we can consider the inverse function

$$\eta^{-1} : \left(0, \beta \frac{u'(y_{\mathrm{L}})}{u'(y_{\mathrm{H}})}\right] \longrightarrow [0, y_{\mathrm{H}})$$

which is also continuous and strictly decreasing. Observe that for any $\tau \in (0, \tau^\star)$, we have

$$q_\tau = \eta(\tau \xi(q_\tau)).$$

Since $\tau \mapsto q_\tau$ is increasing and η^{-1} is strictly decreasing, we deduce that $\tau \mapsto \tau \xi(\tau)$ is decreasing and therefore converges to some value $\ell \in [0, y_{\mathrm{H}})$, i.e.,

$$\lim_{\tau \to 0} \tau \xi(\tau) = \lim_{\tau \to 0} \eta^{-1}(q_\tau) = \inf_{\tau \in (0, \tau^\star)} \tau \xi(\tau) = \ell.$$

If we pose $d_{\mathrm{H}} := \ell/(1+q_0)$, we have that

$$\lim_{\tau\to 0} d_{\mathrm{H},\tau} = \lim_{\tau\to 0} \frac{\tau\xi(q_\tau)}{1+q_\tau} = \frac{\ell}{1+q_0} = d_{\mathrm{H}}.$$

Moreover, since $q_\tau = \eta(\tau\xi(\tau))$, passing to the limit when τ vanishes, we get that

$$q_0 = \eta(\ell) = \eta((1+q_0)d_{\mathrm{H}}) = \beta\frac{u'(y_{\mathrm{L}} + (1+q_0)d_{\mathrm{H}})}{u'(y_{\mathrm{H}} - (1+q_0)d_{\mathrm{H}})}.$$

We claim that $q_0 = 1$. Assume, by way of contradiction, that $q_0 < 1$. We then get that $\xi(q_0) < \infty$ which implies that

$$(1+q_0)d_{\mathrm{H}} = \lim_{\tau\to\infty} \tau\xi(q_\tau) = \xi(q_0)\lim_{\tau\to 0}\tau = 0.$$

We then get the contradiction

$$1 \le \beta\frac{u'(y_{\mathrm{L}})}{u'(y_{\mathrm{H}})} = q_0 < 1.$$

We have thus proved that $q_0 = 1$ and d_{H} satisfies

$$1 = \beta\frac{u'(y_{\mathrm{L}} + 2d_{\mathrm{H}})}{u'(y_{\mathrm{H}} - 2d_{\mathrm{H}})}.$$

If $\beta u'(y_{\mathrm{L}})/u'(y_{\mathrm{H}}) = 1$, then we must have $d_{\mathrm{H}} = 0$. If $\beta u'(y_{\mathrm{L}})/u'(y_{\mathrm{H}}) > 1$, then we must have $d_{\mathrm{H}} > 0$.

□

6 Isomorphisms

The result that the competitive debt limits must equate the present value of default costs implies several interesting equivalence results. We will establish payoff-equivalence mappings between the current environment with self-enforcing debt and other environments with alternative institutional arrangements.

6.1 Backed Public Debt

First, we establish a mapping between the current environment and an environment with government debt backed by income tax revenues. This mapping is akin to the mapping between an environment with private liquidity (namely debt issued by private agents) and public liquidity (debt issued by a government).

Consider the same model, but assume individual agents can no longer issue debt, i.e., $D^i(s^t) = 0$ for all event s^t. Instead, only a government can issue debt. The government debt is backed by tax rate $\tau(s^t)$ on labor income. We assume τ is not type-specific in this section for simplicity.[18] Under this specification, the non-negligible Assumption 1 simplifies to $\tau(s^t) \geq \varepsilon$ for all s^t, and we impose it throughout the section (though it is not essential for our result).

Let $\widehat{B}^i(\theta^i(s^0))$ denote the budget set of an agent in this economy with initial endowment $\theta^i(s^0) \geq 0$. It consists of all pairs (c^i, θ^i) of consumption $c^i = (c^i(s^t))_{s^t \succeq s^0}$ and public debt holdings $\theta^i = (\theta^i(s^t))_{s^t \succ s^0}$ satisfying the (after-tax) budget constraints: for all event s^t,

$$c^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})\theta^i(s^{t+1}) \leq (1 - \tau(s^t))y^i(s^t) + \theta^i(s^t), \tag{6.1}$$

and

$$\theta^i(s^{t+1}) \geq 0 \quad \text{for all successor } s^{t+1} \succ s^t. \tag{6.2}$$

Observe that $\widehat{B}^i(a^i(s^0))$ coincides with $B^i_d(0, a^i(s^0)|s^0)$.

Definition 3. For a given allocation $(\theta^i(s^0))_{i\in I}$ of initial asset positions, a *competitive equilibrium with backed public debt* $(q, d, (c^i, \theta^i)_{i\in I})$ consists of state-contingent bond prices q, a resource feasible consumption allocation $(c^i)_{i\in I}$, an allocation of *non-negative* government bond holdings $(\theta^i)_{i\in I}$ and government net liability positions d

[18] This assumption is not essential. The proofs below can be easily modified for tax rates τ^i that vary across types.

such that:

(i) for each agent $i \in I$, taking prices as given, the plan (c^i, θ^i) is optimal among budget feasible plans in $\widehat{B}^i(\theta^i(s^0)|s^0)$;

(ii) the government debt market clears

$$\sum_{i \in I} \theta^i(s^t) = d(s^t), \quad \text{for all } s^t; \tag{6.3}$$

(iii) the government's budget constraint with tax τ is satisfied:[19]

$$d(s^t) = \tau(s^t)\bar{y}(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})d(s^{t+1}), \quad \text{for all } s^t. \tag{6.4}$$

When tax is non-negligible, we can show that the interest rates in a backed public debt equilibrium are necessarily sufficiently high to prevent bubbles:

Lemma 4. *Assume that tax is non-negligible, i.e., there exists $\varepsilon > 0$ such that $\tau(s^t) \geq \varepsilon$ for all event s^t. In any competitive equilibrium with backed public debt, the agents' wealth is finite:*

$$\sum_{i \in I} W^i(s^0) < \infty.$$

Proof. From the government's budget constraint, we have that for any $T \geq 1$,

$$d(s^0) = \sum_{s^t \in \Sigma^{T-1}} p(s^t)\tau(s^t)\bar{y}(s^t) + \sum_{s^T \in S^T} p(s^T)d(s^T).$$

Since d is non-negative, we deduce that the partial sums

$$\sum_{s^t \in \Sigma^{T-1}} p(s^t)\tau(s^t)\bar{y}(s^t)$$

[19]Recall that $\bar{y}$ denotes the aggregate endowment $\sum_{i \in I} y^i$.

are bounded from above by $d(s^0)$. This implies that the series converge and we get that

$$\mathrm{PV}(\tau\bar{y}|s^0) \leq d(s^0). \tag{6.5}$$

Since tax is non-negligible, we deduce that

$$\varepsilon\,\mathrm{PV}(\bar{y}|s^0) \leq \mathrm{PV}(\tau\bar{y}|s^0) \leq d(s^0)$$

and we get the desired result. $\square$

The following result is the public debt counterpart of Theorem 3:

Proposition 5. *Assume that tax is non-negligible. In any competitive equilibrium with public debt, the government debt must be exactly equal to the present value of tax revenues:*

$$d(s^t) = \mathrm{PV}(\tau\bar{y}|s^t), \quad \textit{for all event } s^t.$$

Proof. The proof is similar to the proof of Theorem 2. Let $(q, d, (c^i, \theta^i)_{i \in I})$ be a competitive equilibrium with backed public debt. Fix an event s^t and consider $T > t$. Consolidating the government's budget constraint along the sub-tree $\Sigma(s^t)$ up to period $T-1$, we have

$$d(s^t) = \sum_{s^r \in S^{T-1}} p(s^r)\tau(s^r)\bar{y}(s^r) + \sum_{s^T \in S^T(s^t)} p(s^T)d(s^T). \tag{6.6}$$

Since we proved that $\mathrm{PV}(\tau\bar{y}|s^t)$ is finite, we deduce from the above equation that

$$M(s^t) := \lim_{T \to \infty} \sum_{s^T \in S^T(s^t)} p(s^T)d(s^T) \quad \text{exists in } \mathbb{R}.$$

Passing equation (6.6) to the limit, we also get that

$$d(s^t) = \mathrm{PV}(\tau\bar{y}|s^t) + M(s^t), \quad \text{for all } s^t \succeq s^0. \tag{6.7}$$

Observe that the process M is non-negative and satisfies exact roll-over in the sense that

$$M(s^t) = \sum_{s^{t+1} \succ s^t} q(s^{t+1}) M(s^{t+1}), \quad \text{for all } s^t. \tag{6.8}$$

We would like to show that $M = 0$. Given the exact roll-over property, it is sufficient to show that $M(s^0) = 0$.

Fix $t \geq 1$. Consolidating agent i's (binding) flow budget constraints up to period $t - 1$ produces

$$\sum_{s^r \in \Sigma^{t-1}} p(s^r) c^i(s^r) + \sum_{s^t \in S^t} p(s^t) \theta^i(s^t) = \theta^i(s^0) + \sum_{s^r \in \Sigma^{t-1}} p(s^r)(1 - \tau(s^r)) y^i(s^r). \tag{6.9}$$

Since consumption markets clear, we obtain from Lemma 4 that the present value of optimal consumption is finite for all agents. In addition, due to the Inada's condition, the optimal consumption is strictly positive. Lemma A.1 in Martins-da-Rocha and Vailakis (2017a) then implies that the following market transversality condition holds true:

$$\lim_{t \to \infty} \sum_{s^t \in S^t} p(s^t) \theta^i(s^t) = 0. \tag{6.10}$$

Passing to the limit in equation (6.9) (as t tends to infinite) gives

$$\mathrm{PV}(c^i | s^0) = \theta^i(s^0) + \mathrm{PV}((1 - \tau) y^i | s^0). \tag{6.11}$$

Summing equation (6.11) over i and exploiting market clearing imply that

$$\sum_{i \in I} M^i(s^0) = 0$$

and we get the desired result. $\square$

Finally, we establish the following proposition, which is similar to Theorem 2 in Hellwig and Lorenzoni (2009) and establishes an isomorphism between a competitive equilibrium with public debt and a competitive equilibrium with self-enforcing debt.

Proposition 6. *Let $\tau = (\tau(s^t))_{s^t \succeq s^0}$ be a non-negligible tax process. The family $(q, d, (c^i, \theta^i)_{i \in I})$ constitutes a competitive equilibrium with public debt backed by tax τ if, and only if, $(q, (c^i, a^i, D^i)_{i \in I})$ constitutes a competitive equilibrium with self-enforcing debt and endowment loss τ, where*

$$D^i = \mathrm{PV}(\tau y^i) \quad \textit{and} \quad a^i = \theta^i - D^i. \tag{6.12}$$

Proof. We will make use of the following straightforward property:

$$(c^i, a^i) \in B^i(D^i, a^i(s^0)|s^0) \Longleftrightarrow (c^i, \theta^i) \in \widehat{B}^i(\theta^i(s^0)). \tag{6.13}$$

Step 1. Let $(q, d, (c^i, \theta^i)_{i \in I})$ be a competitive equilibrium with public debt backed by tax τ. Since tax is non-negligible, we can apply Lemma 4 to deduce that agents' wealth is finite. We can then pose $D^i := \mathrm{PV}(\tau y^i)$ and $a^i := \theta^i - D^i$ for each agent i. By construction debt limits are not-too-tight when τ represents the endowment loss fraction (this follows from a translation invariance property of the budget sets). Moreover, it follows from (6.13) that (c^i, a^i) is optimal in $B^i(D^i, a^i(s^0)|s^0)$. The consumption markets obviously clear. We also have from Proposition 5 that

$$d(s^t) = \mathrm{PV}(\tau \bar{y}|s^t), \quad \text{for all } s^t.$$

This implies that the bond markets also clear. We have thus proved that the family $(q, (c^i, a^i, D^i)_{i \in I})$ constitutes a competitive equilibrium with self-enforcing debt.

Step 2. Let $(q, (c^i, a^i, D^i)_{i \in I})$ be a competitive equilibrium with self-enforcing debt where τ is the endowment loss fraction. We pose

$$\theta^i := a^i + D^i, \quad \text{for each agent } i \in I.$$

Observe that θ^i is a non-negative process. We can apply Theorem 1 to deduce that the agents' wealth is finite. We can then pose

$$d(s^t) := \mathrm{PV}(\tau \bar{y}|s^t), \quad \text{for all } s^t.$$

By construction, the government's budget constraint is satisfied. Moreover, it follows from Theorem 3 that $D^i = \mathrm{PV}(\tau y^i)$ for each agent i. We can then deduce that the markets for government bonds clear. Consumption markets obviously clear. The last property we have to check is the optimality of (c^i, θ^i) in the budget set $\widehat{B}^i(\theta^i(s^0))$. This follows from (6.13). $\square$

6.2 Collateral constraints with complete markets

This section shows that our competitive model with costly default is equivalent to an Arrow–Debreu equilibrium with limited pledgeability as defined by Gottardi and Kubler (2015).

Consider an economy with the same primitives: endowments and preferences. Assume now that only the fraction $\tau^i(s^t)$ of agent i's endowment $y^i(s^t)$ can be sold in advance to finance consumption or savings. The part $e^i(s^t) := (1 - \tau^i(s^t))y^i(s^t)$ constitutes the non-pledgeable endowments. We follow Gottardi and Kubler (2015) and introduce the following abstract equilibrium notion.

Definition 4. For given initial transfers $a^i(s^0)$ such that $\sum_{i\in I} a^i(s^0) = 0$, an *Arrow–Debreu equilibrium with limited pledgeability* $(p, (c^i)_{i\in I})$ consists of time-0 prices $p = (p(s^t))_{s^t \succeq s^0}$ and a resource feasible consumption allocation $(c^i)_{i\in I}$ such that:[20]

(a) each agent's wealth is finite

$$\sum_{s^t \succeq s^0} p(s^t) y^i(s^t) = \mathrm{PV}(y^i | s^0) < \infty, \quad \text{for all agent } i \tag{6.14}$$

(b) the budget restriction is satisfied at the initial event:

$$\mathrm{PV}(c^i | s^0) \leq a^i(s^0) + \mathrm{PV}(y^i | s^0) \tag{6.15}$$

[20]Since $p(s^t)$ is the date-0 price of consumption at s^t, we normalize p such that $p(s^0) = 1$.

(c) the limited pledgeability constraints are satisfied at all $s^t \succeq s^0$:

$$\mathrm{PV}(c^i|s^t) \geq \mathrm{PV}(e^i|s^t). \tag{6.16}$$

The definition is the same as that of an Arrow–Debreu competitive equilibrium, where agents are able to trade at the initial date $t = 0$ in a complete set of contingent commodity markets, except for the additional constraints (6.16). These constraints express precisely the condition that $e^i(s^t) = (1 - \tau^i(s^t))y^i(s^t)$ is unalienable. This component of the endowment can only be used to finance consumption in the event s^t in which it is received or in any successor event.

Gottardi and Kubler (2015) proved that the above equilibrium notion captures the allocations attained as competitive equilibria in models where agents trade sequentially in financial markets and short positions must be backed by collateral as in Geanakoplos and Zame (2002) and Kubler and Schmedders (2003). We show below that it also captures the allocation attained in an environment with self-enforcing debt and costly default.

Theorem 4. *The family $(p, (c^i)_{i \in I})$ is an Arrow–Debreu equilibrium with limited pleadgeability, if and only if, $(q, (c^i, a^i, D^i)_{i \in I})$ constitutes a competitive equilibrium with self-enforcing debt such that each agent's wealth is finite and where bond holdings and debt limits are defined by*

$$a^i(s^t) := \mathrm{PV}(c^i - y^i|s^t) \quad \textit{and} \quad D^i(s^t) := \mathrm{PV}(y^i - e^i|s^t) = \mathrm{PV}(\tau^i y^i|s^t),$$

for all event $s^t \succ s^0$

Proof. We split the proof in two parts.

Step 1. Let $(p, (c^i)_{i \in I})$ be an Arrow–Debreu equilibrium with limited pleadgeability. By definition, the wealth $W^i(s^0) = \mathrm{PV}(y^i|s^0)$ is finite. Since consumption markets clear, we deduce that the present value $\mathrm{PV}(c^i|s^0)$ of each agent i's consumption is finite. We can then define for any event s^t

$$\theta^i(s^t) := \mathrm{PV}(c^i - e^i | s^t).$$

The limited pleadgeability constraint (6.16) implies that $\theta^i(s^t) \geq 0$ for all event s^t. Moreover, by construction, we have

$$c^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \theta^i(s^{t+1}) = (1 - \tau^i(s^t)) y^i(s^t) + \theta^i(s^t)$$

where the bond prices are defined by $q(s^{t+1}) := p(s^{t+1})/p(s^t)$ for any $s^{t+1} \succ s^t$. We have thus proved that $(c^i, \theta^i) \in \widehat{B}^i(\theta^i(s^0))$. Observe moreover that the Arrow–Debreu budget constraint (6.15) must be satisfied with equality. This implies that

$$\theta^i(s^0) = \mathrm{PV}(c^i - e^i | s^t) = a^i(s^0) + \mathrm{PV}(y^i - e^i | s^0) = a^i(s^0) + \mathrm{PV}(\lambda^i y^i | s^0). \qquad (6.17)$$

Denote by d the process defined by

$$d(s^t) := \sum_{i \in I} \mathrm{PV}((1 - \lambda^i) y^i | s^t)$$

which can be interpreted as the government debt when each agent i's labor tax is λ^i. Observe that by construction, the government's budget constraint is satisfied. Since consumption markets clear, we have that

$$\sum_{i \in I} \theta^i(s^t) = \sum_{i \in I} \mathrm{PV}(y^i - e^i | s^t) = d(s^t)$$

and the government debt markets clear. We claim that (c^i, θ^i) is optimal in $\widehat{B}^i(\theta^i(s^0))$. Indeed, let $(\tilde{c}^i, \tilde{\theta}^i)$ be an arbitrary choice in $\widehat{B}^i(\theta^i(s^0))$. $\square$

Appendices

A Omitted Proofs

A.1 Proof of Lemma 2

Let D be a process of not-too-tight bounds. We first show that there exists a non-negative process $\underline{D}$ satisfying

$$\underline{D}(s^t) = \tau(s^t)y(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \min\{D(s^{t+1}), \underline{D}(s^{t+1})\}, \quad \text{for all } s^t \in \Sigma. \tag{A.1}$$

Indeed, let Φ be the mapping $B \in \mathbb{R}^\Sigma \longmapsto \Phi B \in \mathbb{R}^\Sigma$ defined by

$$(\Phi B)(s^t) := \tau(s^t)y(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \min\{D(s^{t+1}), B(s^{t+1})\}, \quad \text{for all } s^t \in \Sigma.$$

Denote by $[0, \bar{D}]$ the set of all processes $B \in \mathbb{R}^\Sigma$ satisfying $0 \leq B \leq \bar{D}$ where

$$\bar{D}(s^t) := \tau(s^t)y(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D(s^{t+1}), \quad \text{for all } s^t \in \Sigma.$$

The mapping Φ is continuous (for the product topology) and we have $\Phi[0, \bar{D}] \subseteq [0, \bar{D}]$. Since $[0, \bar{D}]$ is convex and compact (for the product topology), it follows that Φ admits a fixed point $\underline{D}$ in $[0, \bar{D}]$.

Claim 1. The process $\underline{D}$ is tighter than the process D, i.e., $\underline{D} \leq D$.

Proof of Claim 1. Fix a node s^t. It is sufficient to show that $V(D, -\underline{D}(s^t)|s^t) \geq V_d(0, 0|s^t)$.[21] Denote by $(\tilde{c}, \tilde{a})$ the optimal consumption and bond holdings associated to the default option at s^t, i.e., $(\tilde{c}, \tilde{a}) \in d(0, 0|s^t)$.[22] We let $\widehat{D}$ be the process defined

[21]Recall that $V_d(0, 0|s^t) = V(D, -D(s^t)|s^t)$ and $V(D, \cdot|s^t)$ is strictly increasing.

[22]Equivalently, $(\tilde{c}, \tilde{a})$ satisfies $U(\tilde{c}|s^t) := V_d(0, 0|s^t)$ and belongs to $B_d(0, 0|s^t)$.

by $\widehat{D}(s^t) := \min\{D(s^t), \underline{D}(s^t)\}$ for all s^t. Observe that

$$\begin{aligned} y(s^t) - \underline{D}(s^t) &= (1-\tau(s^t))y(s^t) - \sum_{s^{t+1}\succ s^t} q(s^{t+1})\widehat{D}(s^{t+1}) \\ &= \tilde{c}(s^t) + \sum_{s^{t+1}\succ s^t} q(s^{t+1})[\tilde{a}(s^{t+1}) - \widehat{D}(s^{t+1})] \\ &= \tilde{c}(s^t) + \sum_{s^{t+1}\succ s^t} q(s^{t+1})a(s^{t+1}) \end{aligned}$$

where $a(s^{t+1}) := \tilde{a}(s^{t+1}) - \widehat{D}(s^{t+1})$. Since $\widehat{D} \leq D$ we have $a(s^{t+1}) \geq -D(s^{t+1})$. At any successor event $s^{t+1} \succ s^t$, we have

$$\begin{aligned} y(s^{t+1}) + a(s^{t+1}) &= y(s^{t+1}) + \tilde{a}(s^{t+1}) - \widehat{D}(s^{t+1}) \\ &\geq y(s^{t+1}) + \tilde{a}(s^{t+1}) - \underline{D}(s^{t+1}) \\ &\geq (1-\tau(s^{t+1}))y(s^{t+1}) + \tilde{a}(s^{t+1}) - \sum_{s^{t+2}\succ s^{t+1}} q(s^{t+2})\widehat{D}(s^{t+2}) \\ &\geq \tilde{c}(s^{t+2}) + \sum_{s^{t+2}\succ s^{t+1}} q(s^{t+2})[\tilde{a}(s^{t+2}) - \widehat{D}(s^{t+2})] \\ &\geq \tilde{c}(s^{t+2}) + \sum_{s^{t+2}\succ s^{t+1}} q(s^{t+2})a(s^{t+2}) \end{aligned}$$

where $a(s^{t+2}) := \tilde{a}(s^{t+2}) - \widehat{D}(s^{t+2})$.[23] Observe that $a(s^{t+2}) \geq -D(s^{t+2})$ (since $\widehat{D} \leq D$).

Defining $a(s^\tau) := \tilde{a}(s^\tau) - \widehat{D}(s^\tau)$ for any successor $s^\tau \succ s^t$ and iterating the above argument, we can show that $(\tilde{c}, a)$ belongs to the budget set $B(D, -\underline{D}(s^t)|s^t)$. It follows that

$$V(D, -\underline{D}(s^t)|s^t) \geq U(\tilde{c}|s^t) = V_d(0, 0|s^t)$$

implying the desired result: $\underline{D}(s^t) \leq D(s^t)$. $\square$

[23]To get the second weak inequality we make use of equation (A.1).

It follows from Claim 1 that $\underline{D}$ satisfies

$$\underline{D}(s^t) = \tau(s^t)y(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})\underline{D}(s^{t+1}), \quad \text{for all } s^t \in \Sigma. \tag{A.2}$$

Applying equation (A.2) recursively we get

$$\begin{aligned} p(s^t)\underline{D}(s^t) = p(s^t)\tau(s^t)y(s^t) + &\sum_{s^{t+1} \in S^{t+1}(s^t)} p(s^{t+1})\tau(s^{t+1})y(s^{t+1}) + \dots \\ &\dots + \sum_{s^T \in S^T(s^t)} p(s^T)\tau(s^T)y(s^T) + \sum_{s^{T+1} \in S^{T+1}(s^t)} p(s^{T+1})\underline{D}(s^{T+1}) \end{aligned}$$

for any $T > t$. Since $\underline{D}$ is non-negative, it follows that

$$p(s^t)\underline{D}(s^t) \geq \sum_{r=0}^{T-t} \sum_{s^{t+r} \in S^{t+r}(s^t)} p(s^{t+r})\tau(s^{t+r})y(s^{t+r}).$$

Passing to the limit when T goes to infinite we get that $\text{PV}(\tau y|s^t)$ is finite for any event s^t (in particular for s^0). Recalling that $D \geq \underline{D}$, we also get that $D(s^t) \geq \text{PV}(\tau y|s^t)$. $\square$

B Examples

Following Hellwig and Lorenzoni (2009) we consider a model with two agents who wish to smooth their endowment risks but cannot commit to their promises. The endowment y^i of each agent $i \in I := \{a, b\}$ is stochastic. Uncertainty is captured by a Markov process with state space $Z = \{z^a, z^b\}$. The state z^i corresponds to the situation where agent i's endowment is $\bar{y} \in (1/2, 1)$ and agent j's endowment, with $j \neq i$, is $\underline{y} := 1 - \bar{y} < \bar{y}$. The transition probabilities are symmetric where $\alpha \in (0, 1)$ is the probability of switching states, i.e., $\text{Prob}(z^i|z^j) = \alpha$ for $i \neq j$. The Bernoulli function $u : [0, \infty) \to \mathbb{R}$ is strictly concave, strictly increasing, continuously differentiable and bounded; and the discount factor β belongs to $(0, 1)$. The parameters $(u, \beta, \alpha, \bar{y})$ are chosen such that there exists $\bar{c} > 0$ satisfying

$$\bar{c} < \bar{y} \quad \text{and} \quad 1 - \beta(1-\alpha) = \beta\alpha \frac{u'(1-\bar{c})}{u'(\bar{c})}. \tag{B.1}$$

The corresponding event tree Σ can be defined as follows. The initial event is $s^0 := z^a$ and a date-t event is a history of state realizations $s^t = (s^0, s_1, \ldots, s_t)$ with $s_r \in Z$ for each $1 \leq r \leq t$. The transition probabilities are defined by $\pi(s^{t+1}|s^t) = \alpha$ if $s_{t+1} \neq s_t$ and $\pi(s^{t+1}|s^t) = 1 - \alpha$ if $s_{t+1} = s_t$. Agent i's endowment process y^i is defined by $y^i(s^{t+1}) := \bar{y}$ if $s_{t+1} = z^i$ and $y^i(s^{t+1}) := \underline{y}$ if $s_{t+1} \neq z^i$. Since the initial state is z^a, agent a begins with the high endowment $\bar{y}$ while agent b's initial endowment is $\underline{y}$.

In the absence of output costs, Hellwig and Lorenzoni (2009) show that the above economy has a competitive equilibrium with positive levels of debt. The state-contingent bond prices q are as follows

$$q(s^{t+1}) := \begin{cases} q^c := \beta\alpha \dfrac{u'(1-\bar{c})}{u'(\bar{c})} & \text{if} \quad s_{t+1} \neq s_t \\ q^{nc} := \beta(1-\alpha) & \text{if} \quad s_{t+1} = s_t. \end{cases}$$

Agent i's consumption process is defined by $c^i(s^{t+1}) := \bar{c}$ if $s_{t+1} = z^i$ and $c^i(s^{t+1}) := 1 - \bar{c}$ if $s_{t+1} \neq z^i$. His bond holdings are $a^i(s^{t+1}) := -\omega$ if $s_{t+1} = z^i$ and $a^i(s^{t+1}) := \omega$ if $s_{t+1} \neq z^i$, where ω is defined by the equation $(1 - q^{nc} + q^c)\omega = \bar{y} - \bar{c}$. The initial bond holding allocation $(a^i(s^0))_{i \in I}$ is defined by $a^a(s^0) := -\omega$ and $a^b(s^0) := \omega$.

The not-too-tight debt limits $(D_0^i)_{i \in I}$ are given by $D_0^i(s^t) := \omega$. Observe that the state-contingent bond prices are such that $q^c + q^{nc} = 1$, that is, the non-contingent interest rate is zero. In particular, the wealth of each agent is infinite. Finally, observe that D_0^i allows for exact roll-over in the sense that $D_0^i(s^t) = \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D_0^i(s^{t+1})$. This is sufficient to get that the process of debt limits D_0^i is not-too-tight.

B.1 Non-negligible Output Drop

We propose to modify the default punishment in the above example assuming that, in addition to credit exclusion, a fraction λ of income is lost upon default. When the output contraction parameter is small enough, we show that the economy has a symmetric Markov equilibrium where the consumption allocations $\bar{c}_\lambda$ and $\underline{c}_\lambda := 1 - \bar{c}_\lambda$ are defined as follows.[24]

Lemma 5. *There exists $\bar{\lambda} > 0$ small enough such that for every $\lambda \in (0, \bar{\lambda})$, there exists $\bar{c}_\lambda \in (1/2, \bar{c})$ satisfying*

$$[1 - q^{nc} + q^c(\bar{c}_\lambda)]\lambda w(\bar{c}_\lambda) = \bar{y} - \bar{c}_\lambda \tag{B.2}$$

where for any $x \in (1/2, \bar{c})$

$$q^c(x) := \beta\alpha \frac{u'(1-x)}{u'(x)} \quad \text{and} \quad w(x) := \frac{(1-q^{nc})\bar{y} + q^c(x)\underline{y}}{(1-q^{nc})^2 - (q^c(x))^2}.$$

Moreover, we have

$$\lim_{\lambda \to 0} \bar{c}_\lambda = \bar{c}, \tag{B.3}$$

where $\bar{c}$ is equilibrium consumption contingent to high income in the economy with credit exclusion and no output costs, i.e., $\bar{c}$ solves Equation (B.1).

Fix a $\lambda \in (0, \bar{\lambda})$ and denote by $\mathcal{E}_\lambda$ the economy where agents' initial asset holdings are defined by $a^a_\lambda(s^0) := -\lambda w(\bar{c}_\lambda)$ and $a^b_\lambda(s^0) := \lambda w(\bar{c}_\lambda)$. All the other primitives remain the same as in the economy $\mathcal{E}$.[25] We now describe a symmetric Markov equilibrium of the economy $\mathcal{E}_\lambda$.

Let q_λ be the price process defined by

[24] See Martins-da-Rocha and Vailakis (2017b) for the detailed proof of this claim.

[25] This choice of initial holdings simplifies both the existence of a symmetric Markov equilibrium and the asymptotic analysis. In particular, we show below that $a^i_\lambda(s^0) \longrightarrow a^i(s^0)$ when $\lambda \to 0$.

$$q_\lambda(s^{t+1}) := \begin{cases} q^c(\bar{c}_\lambda) & \text{if } s_{t+1} \neq s_t \\ q^{nc} & \text{if } s_{t+1} = s_t. \end{cases} \tag{B.4}$$

We claim that each agent has a finite wealth. Indeed, since endowments are uniformly bounded from above, it is sufficient to show that $q^c(\bar{c}_\lambda)+q^{nc} < 1$. This is true because the map $x \mapsto q^c(x)$ is increasing, $\bar{c}_\lambda < \bar{c}$ and $q^c(\bar{c}) + q^{nc} = 1$ (recall that $q^c(\bar{c}) = q^c$). Observe that computing the present value of future endowments y^i at any event s^t using the prices q_λ gives

$$\mathrm{PV}(q_\lambda, y^i|s^t) = \begin{cases} w(\bar{c}_\lambda) & \text{if } s_t = z^i \\ w(\bar{c}_\lambda) - \dfrac{\bar{y} - \underline{y}}{1 - q^{nc} + q^c(\bar{c}_\lambda)} & \text{if } s_t \neq z^i. \end{cases}$$

Consider the following consumption profiles and asset holdings.

$$c^i_\lambda(s^{t+1}) := \begin{cases} \bar{c}_\lambda & \text{if } s_{t+1} = z^i \\ 1 - \bar{c} & \text{if } s_{t+1} \neq z^i \end{cases} \quad \text{and} \quad a^i_\lambda(s^{t+1}) := \begin{cases} -\lambda w(\bar{c}_\lambda) & \text{if } s_{t+1} = z^i \\ \lambda w(\bar{c}_\lambda) & \text{if } s_{t+1} \neq z^i \end{cases} \tag{B.5}$$

The following result is proven Martins-da-Rocha and Vailakis (2017b).

Proposition 7. *For every $\lambda > 0$ small enough, the collection $(q_\lambda, (c^i_\lambda, a^i_\lambda, D^i_\lambda)_{i\in I})$ of state contingent prices, consumption profiles, asset holdings and not-too-tight debt limits defined by*

$$D^i_\lambda(s^t) := \lambda\, \mathrm{PV}(q_\lambda; y^i|s^t) = \mathrm{PV}(q_\lambda; \lambda y^i|s^t), \quad \textit{for all } s^t \in \Sigma \tag{B.6}$$

is a competitive equilibrium of the economy $\mathcal{E}_\lambda$.

B.2 Dropping Assumption 1

Our aim is to show that there exists a competitive equilibrium for which the conclusion of Theorem 2 is not valid, if assumption 1 does not hold.

We consider below an economy with stochastic endowments that has a stationary Markovian equilibrium with zero risk-less rates.

Example 1. We modify the example in Hellwig and Lorenzoni (2009) with two agents (R_1 and R_2) by adding a third agent (agent P). Agents R_1 and R_2 do not suffer endowment loss upon default ($\tau^{\mathrm{R}_1} = \tau^{\mathrm{R}_2} = 0$) while agent P can fully commit to her financial contracts ($\tau^{\mathrm{P}} = 1$). The primitives $(\beta, u(\cdot), \pi)$ are chosen such that there exists a pair $(\underline{c}, \overline{c})$ satisfying

$$0 < \underline{c} < \overline{c}, \quad \underline{c} + \overline{c} = 1 \quad \text{and} \quad 1 - \beta(1-\pi) = \beta\pi\frac{u'(\underline{c})}{u'(\overline{c})}.$$

We let (q^c, q^{nc}) be defined by

$$q^c := \beta\pi\frac{u'(\underline{c})}{u'(\overline{c})} \quad \text{and} \quad q^{nc} := \beta(1-\pi). \tag{B.7}$$

Observe that $q^c + q^{nc} = 1$. We fix some arbitrary number $\delta > 0$ such that $q^c\delta < \underline{c}$ and let $(\underline{y}, \overline{y})$ be the pair defined by

$$\underline{y} := \underline{c} - q^c\delta \quad \text{and} \quad \overline{y} := \overline{c} + q^c\delta.$$

Observe that

$$0 < \underline{y} < \underline{c} < \overline{c} < \overline{y} \quad \text{and} \quad \underline{y} + \overline{y} = 1.$$

In each period, one of the agents R_1 and R_2 receives the high endowment $\overline{y}$ with the other receiving the low endowment $\underline{y}$. Those agents switch endowment with probability π from one period to the next. Formally, uncertainty is captured by the Markov process s_t with state space $\{z_1, z_2\}$ and symmetric transition probabilities

$$\pi := \mathrm{Prob}(s_{t+1} = z_1 | s_t = z_2) = \mathrm{Prob}(s_{t+1} = z_2 | s_t = z_1).$$

The event s^t corresponds to the sequence $(s_0, s_1, \dots, s_t)$ and the endowments $y^{\text{R}_k}(s^t)$, for $k \in \{1, 2\}$, only depend on the current realization of s_t, with

$$y^{\text{R}_k}(s^t) := \begin{cases} \overline{y}, & \text{if } s_t = z_k \\ \underline{y}, & \text{otherwise.} \end{cases}$$

Agent's P endowment is defined by $y^{\text{P}}(s^0) := \overline{y}$ and for each event $s^t \succ s^0$,

$$y^{\text{P}}(s^t) := \begin{cases} y^{\text{P}}(s^{t-1}), & \text{if } s_t = s_{t-1} \\ \gamma y^{\text{P}}(s^{t-1}), & \text{otherwise,} \end{cases}$$

where $\gamma \in (0, 1)$ is chosen such that

$$\frac{u'(\gamma y^{\text{P}}(s^t))}{u'(y^{\text{P}}(s^t))} = \frac{u'(\underline{c})}{u'(\overline{c})}, \quad \text{for all } s^t. \tag{B.8}$$

Remark 4. Fix $\alpha \in (0, 1)$. If we let u be such that $u(c) := c^{1-\alpha}/(1-\alpha)$ in the interval $[0, \overline{y}]$ and extend this function on $[\overline{y}, \infty)$ such that the assumptions on u are satisfied, then the equality (B.8) is true for $\gamma = \underline{c}/\overline{c}$.

To focus on a stationary equilibrium, we assume that the economy begins in state $s^0 = s_0 = z_1$ (agent R_1 has the highest endowment) and the initial asset positions are

$$a^{\text{P}}(s^0) := -\delta, \quad a^{\text{R}_1}(s^0) := 0 \quad \text{and} \quad a^{\text{R}_2}(s^0) := \delta.$$

Observe that this economy does not satisfy Assumption 1. Although agent P can commit to honour her financial contacts, her endowment is negligible with respect to the aggregate endowment of the economy. Indeed, for any infinite path $(s_t)_{t \geq 0}$ displaying infinitely many switches, we have

$$\lim_{t \to \infty} y^{\text{P}}(s^t) = 0$$

while the aggregate endowment of agents R_1 and R_2 is always equal to 1.

Proposition 8. *The economy of Example 1 admits a competitive equilibrium with self-enforcing debt in which agent* P*'s debt limits are not naturally bounded. Specifically, we have that* $D^{\text{P}}(s^t) = W^{\text{P}}(s^t) + \delta$ *at any event* s^t.

Proof. We first describe the equilibrium prices, debt limits and allocations.

Let the price process $(q(s^t))_{s^t \succ s^0}$ be as follows:

$$q(s^t) := \begin{cases} q^c, & \text{if } s_t \neq s_{t-1} \\ q^{nc}, & \text{otherwise.} \end{cases}$$

Since by assumption $q^c + q^{nc} = 1$, the risk-less interest rate is zero. Observe that

$$\sum_{s^{t+1} \succ s^t} q(s^{t+1}) y^{\text{P}}(s^{t+1}) \leq (q^c \gamma + q^{nc}) y^{\text{P}}(s^t).$$

Since $\gamma \in (0,1)$ and $q^c + q^{nc} = 1$, this implies that agent P's wealth $W^{\text{P}}(s^t)$ is finite at any contingency s^t. Indeed, if we let $\chi := (q^c\gamma + q^{nc})$, we can show that $W^{\text{P}}(s^t) \leq y^{\text{P}}(s^t) \sum_{t \geq 0} \chi^t = y^{\text{P}}(s^t)/(1-\chi)$.

Consider the following debt limits:

$$D^{\text{R}_1}(s^t) = D^{\text{R}_2}(s^t) := 0 \quad \text{and} \quad D^{\text{P}}(s^t) := W^{\text{P}}(s^t) + \delta.$$

The debt limits of agents R_1 and R_2 are not-too-tight by definition. The fact that D^{P} is not-too-tight follows from a standard translation invariance of the flow budget constraints.

Let $(c^{\text{P}}, a^{\text{P}})$ be defined as follows: $c^{\text{P}}(s^t) := y^{\text{P}}(s^t)$ and $a^{\text{P}}(s^t) := -\delta$ for all event s^t. At the initial period, the agent P repays the inherited debt δ by issuing the non-contingent debt δ. At the subsequent periods, instead of repaying, she rolls over this debt at infinite.

We also let $(c^{\mathrm{R}_k}, a^{\mathrm{R}_k})$ be defined as follows:

$$c^{\mathrm{R}_k}(s^t) := \begin{cases} \overline{c}, & \text{if } s_t = z_k \\ \underline{c}, & \text{otherwise} \end{cases} \quad \text{and} \quad a^{\mathrm{R}_k}(s^{t+1}) := \begin{cases} \delta, & \text{if } s_{t+1} \neq z_k \\ 0, & \text{otherwise.} \end{cases}$$

Agents R_1 and R_2 save to transfer resources against the low income shock. They do not issue debt since they are credit-constrained.

We next show that equilibrium allocations are indeed optimal.

Observe that $(c^{\mathrm{P}}, a^{\mathrm{P}})$ is optimal since it is budget feasible (with equality) and satisfies the Euler equations together with the individual transversality condition:

$$\lim_{t\to\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u'(c^{\mathrm{P}}(s^t))[a^{\mathrm{P}}(s^t) + D^{\mathrm{P}}(s^t)] = u'(c^{\mathrm{P}}(s^0)) \lim_{t\to\infty} \sum_{s^t \in S^t} p(s^t) W(s^t) = 0.$$

The plan $(c^{\mathrm{R}_k}, a^{\mathrm{R}_k})$ is also optimal since it is budget feasible (with equality), it satisfies the Euler equations and the individual transversality condition.[26]

Finally, all markets clear by construction. □

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[26]The transversality condition is satisfied because the equilibrium is Markovian stationary. Formally, we have

$$\sum_{s^t \in S^t} \beta^t \pi(s^t) u'(c^{\mathrm{R}_k}(s^t)) a^{\mathrm{R}_k}(s^t) \leq \beta^t u'(\underline{c})\delta \xrightarrow[t\to\infty]{} 0.$$

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