Uncertainty aversion & heterogeneous beliefs in linear models

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Basic idea

- Models with heterogeneous ambiguity averse agents
  - agents use different worst case beliefs to evaluate future plans
  - equilibria look like in models with disagreement
- Multiple priors utility with ambiguity about means
  - first order effects of uncertainty
  - linear approximation works well

$\Rightarrow$ Use linear models to study effects of uncertainty

Leading example

- Recent history
  - low real interest rate + stock market boom
- Candidate story: higher uncertainty about middle class incomes
  - type A agents perceive uncertain labor income
    = behave as if mean labor income will be low
  - type B agents confident about labor income
  - agents trade trees with safe payoff, safe bonds collateralized by trees
- Equilibrium
  - type B agents rich: own trees & supply safe bonds to type A agents
  - type A agents poor: perceive tree price uncertainty, hold no trees
  - asset pricing: low riskless rate & *high* price of trees
- Effects of policy
  - government debt can substitute for safe private debt
  - nominal credit: inflation uncertainty reduces gains from trade

Computation

- System of stochastic difference equations
  - as typically used to characterize equilibria in macro models
  - but: different expectation operators for each agent
- Solution strategy
  - linearize around steady state
  - jointly determine steady state & dynamics
- Models with heterogeneous beliefs
  - agents may agree to disagree in the long run
- Models with differences in ambiguity aversion
  - similar in many ways to models with differences in risk aversion
  - average effects and response to uncertainty shocks easy to characterize
  - can be accurately solved by linearization

Outline

- Ambiguity aversion & precautionary savings
- Two agent model with heterogenous beliefs
  - computational approach
  - numerical example

Aversion to uncertainty

- Two dates
  - date 2 states $\omega \in \Omega$
  - consumption plans $(c, \tilde{c}(\omega))$
- Preferences over consumption plans

$$u(c) + \beta u(CE(\tilde{c}))$$

  - for certain plans $(\tilde{c}(\omega) = \bar{c})$, certainty equivalent $CE(\tilde{c}) = \bar{c}$
  - u captures desire to smooth consumption, IES $= -cu''(c) / u'(c)$
- Certainty equivalent with risk aversion

$$v(CE(\tilde{c})) = E^{Q}[v(\tilde{c})]$$

  - subjective belief Q
  - v captures risk aversion; CRRA $= -cv''(c) / v'(c)$
- Certainty equivalent with ambiguity aversion (multiple priors utility)

$$v(CE(\tilde{c})) = \min_{P \in \mathcal{P}} E^{P}[v(\tilde{c})]$$

  - set of subjective beliefs $\mathcal{P}$ captures ambiguity

Precautionary savings

- Choose optimal savings s s.t.

$$c + s = w \qquad\qquad \tilde{c} = Rs + \tilde{y}$$

- Risky labor income $\tilde{y} = \bar{y} + \sigma\tilde{z}$, $E^Q[\tilde{z}] = 0, var^Q(\tilde{z}) = 1$
  - assume $u = v$ & interior solution

$$u'(c) = \beta R E^Q\left[u'(\tilde{c})\right]$$

  - response of savings to risk at $\sigma = 0$

$$\frac{ds}{d\sigma} = \Delta^{-1} R E^Q\left[u''(\tilde{c})\,\tilde{z}\right] = \Delta^{-1} R u'''(\bar{c}) + \ldots$$

  - $\Delta > 0$ from 2nd order condition; response positive if $u''' > 0$
- Multiple priors with ambiguous labor income $\tilde{y} = \bar{y} + \tilde{z}$, $E\tilde{z} \in [-a, a]$
  - same FOC, but $Q =$ worst case belief with $E\tilde{z} = -a$
  - response of savings to ambiguity at $a = 0$

$$\frac{ds}{da} = \Delta^{-1} R E^Q\left[-u''(\tilde{c})\right] = -\Delta^{-1} R u''(\bar{c}) + \ldots$$

  - response positive if $u'' < 0$
  - precautionary savings will show up in linearized FOC!

Recursive multiple priors utility

- Ω = state space
  - one element $\omega \in \Omega$ realized every period
  - histories $\omega^t \in \Omega^t$
- Preferences over consumption plans $c = (c_t(\omega^t))$
- Utility process solves

$$U_t(c;\omega^t) = u(c_t(\omega^t)) + \beta \min_{p \in \mathcal{P}_t(\omega^t)} E^p\left[U_{t+1}(c;\omega^t,\omega_{t+1})\right]$$

- Primitives
  - felicity u, discount factor β
  - one-step-ahead belief sets $\mathcal{P}_t(\omega^t)$; may depend on history
- Consider equilibrium with RMP agents
  - at optimal plan, minimizers $p \in \mathcal{P}_t(\omega^t)$ support choice

  $\Rightarrow$ always obs equivalent EU model; possibly heterogeneous beliefs
  - if worst case belief easy to find, just study het beliefs model

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Model of ambiguity sharing

- Two types of infinitely lived agents; preferences

$$E_0^j\left[\sum_{t=0}^{\infty}\beta^t u\left(c_t\right)\right]$$

- Beliefs about income (endowment of goods)
  - type B agents always get $\bar{y}^B$; all agents know this
  - at date t, type A agents believe they get $y_{t+1}=\bar{y}^A\exp\left(-a_t\right)$
  - in fact, type A agents get $y_{t+1}=\bar{y}^A$; type B agents know this
  - $a_t=$ ambiguity perceived by type A agents (stochastic)
- Assets
  - One period noncontingent debt, price q_t
  - Trees: dividend d, price p_t; no short sales
  - Leverage $\ell_t=-q_t b_t/p_t\theta_t$; cost to borrower $k\left(\ell_t\right)q_t b_t$
    $k\left(\ell\right)=0$ for $\ell\leq 0$; $k\left(\ell\right),k'\left(\ell\right),k''\left(\ell\right)>0$ for $\ell>0$
- Date t budget constraint

$$c_t+p_t\theta_t+q_t b_t\left(1+k\left(\ell_t\right)\right)=y_t+\left(p_t+d_t\right)\theta_{t-1}+b_{t-1}=:w_t$$

Recursive equilibrium

- Market clearing
  - goods market: $c_t^A + c_t^B = \bar{y}^A + \bar{y}^B - q_t b_t k(\ell_t)$
  - debt market: $b^A + b^B = 0$
- State variables
  - type A income y^A
  - ambiguity a
  - distribution of asset holdings; here just type B debt $b = -b^B = b^A$
- Allocations & prices
  - find $c^i(y^A, a, b)$, $b'(y^A, a, b)$, $q(y^A, a, b)$, $p(y^A, a, b)$
  - agents disagree only about income, know equilibrium map

  $\Rightarrow$ find functions from Euler equations + budget constraints

System of stochastic difference equations

- Equilibria s.t. only type B agents hold tree
  - type B agents: Euler equations for both assets
  - type A agents: Euler equation for bonds, tree too expensive
- Bond pricing

$$q_t = \beta E_t^A \left[\left(\frac{c_{t+1}^A}{c_t^A} \right)^{-\gamma} \right] = \beta E_t^B \left[\left(\frac{c_{t+1}^B}{c_t^B} \right)^{-\gamma} \right] + q_t \left(k\left(\ell_t\right) + \ell_t k'\left(\ell_t\right) \right)$$

  - lender pessimism, borrower leverage cost $\Rightarrow$ lower bond price
- Tree pricing

$$p_t = \beta E_t^B \left[\left(\frac{c_{t+1}^B}{c_t^B} \right)^{-\gamma} \left(p_{t+1} + d_{t+1} \right) \right] + \ell_t^2 k'\left(\ell_t\right) p_t$$

  - collateral benefit increases tree price
- With budget constrs and $\ell = qb/p$: 6 equations in c^A, c^B, q, p, b, ℓ

Computational challenge

- Steady state = solution to system when vol of shocks is zero
  - usually just remove E_t and solve for deterministic steady state
- Expectations in Euler equations without shocks
  - Type B expects all state variables to remain constant
  - Type A expects next period income $\bar{y}^A \exp(-\bar{a})$ and consumption

$$c^A\left(\bar{y}^A \exp(-\bar{a}), \bar{a}, \bar{b}\right)$$

  - behaves as if perpetually surprised by high income
  - problem: policy function c^A matters for steady state
- With different expectation operators for each agent
  - agents expect different dynamics of endogenous state variables
  - cannot solve for steady state independently of dynamics

Solution strategy

- Guess coefficients of loglinear approximation
  - e.g.type A consumption

$$\hat{c}_A = \log c^A - \log \bar{c}^A = \varepsilon^{cA}_{yA} \hat{y}^A + \varepsilon^{cA}_{a} \hat{a} + \varepsilon^{cA}_{b} \hat{b}$$

- Compute steady state
  - use guessed coefficients to model expectations, e.g.

$$c^A \left(\bar{y}^A \exp\left(-\bar{a}\right), \bar{a}, \bar{b} \right) \approx \bar{c}^A \exp\left(-\varepsilon^{cA}_{yA} \bar{a} \right)$$

- Linearize around candidate steady state
- Iterate until fixed point in coefficients!

- Remarks
  - steady state typically $\neq$ any agent's long run expectation
  - can check accuracy via Euler equation errors

Gains from trade: sharing ambiguity vs risk

- Comparison model with risk
  - type B agents as before
  - at date t, type A agents get $y_t = \bar{y}^A (1 + \tilde{z}_t)$, $\tilde{z}$ iid with mean zero
- Risk sharing in equilibrium
  - type B sells safe claim to type A
  - type A gives up goods today since CE of future income low
  - A's precautionary savings lowers interest rate
- Ambiguity sharing in equilibrium
  - type A gives up goods today since he acts as if future income low

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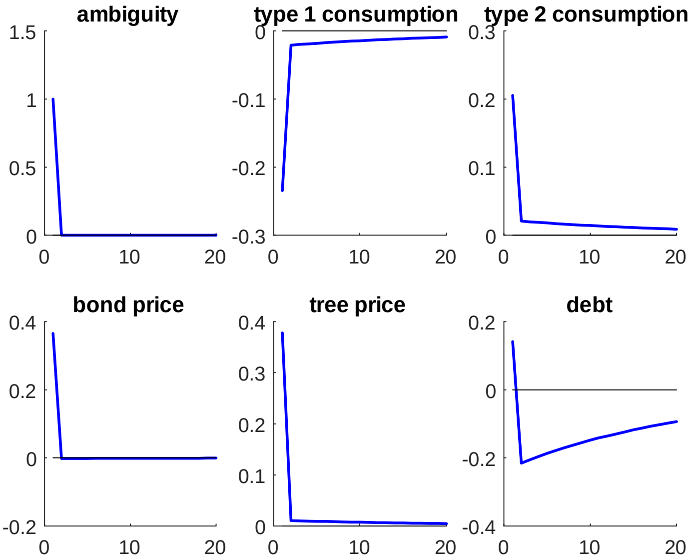
Numerical example

- Baseline parameters
  - output = 1, income & dividends: $\bar{y}^A = \bar{y}^B = .45$, $d = .1$
  - preferences: $u(c) = \log c$, $\beta = .96$, ambiguity $\bar{a} = 10\%$
  - $k(\ell) = (.075)\,\ell^2 \longrightarrow$ steady state leverage cost $k(\ell)\,qb = .016$
- Steady state

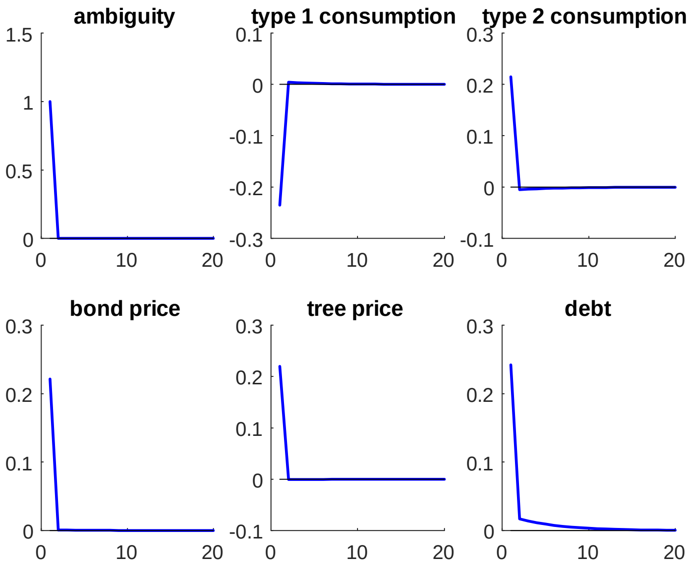
	$\bar{c}^A$	$\bar{c}^B$	$\bar{q}$	$\bar{p}$	debt $\bar{q}\bar{b}$	$\bar{\ell} = \bar{q}\bar{b}/\bar{p}$
baseline	.454	.530	.997	3.2	1.3	.40
$\bar{a} = 0$			.960	2.4	0	0
$\gamma = 2$	.428	.539	1.012	4.0	1.9	.48

- Ambiguity generates gain from trade, type A "late consumer"
  - lower interest rate, higher tree price
- Lower IES $\longrightarrow$ ambiguity matters more
  - negative interest rate, $\bar{c}^A < \bar{y}^A$

Impulse response to increase in ambiguity



Impulse response to ambiguity (lo discount factor)



Government debt

- Government
  - issues safe debt b^g & levies lump sum taxes; budget constraint

$$b^g_{t-1} = q_t b^g_t + \tau^A_t + \tau^B_t$$

  - bond market clearing $b^A_t + b^B_t = b^g_t$
  - parameters: $\bar{b}^g = .6$, equal taxes for both types
- Steady state comparison

	$\bar{c}^A$	$\bar{c}^B$	$\bar{q}$	$\bar{p}$	debt $\bar{q}\bar{b}$ (private)	$\bar{\ell}$
baseline	.454	.530	.997	3.2	1.3 (1.3)	.40
government	.463	.527	.994	3.1	1.8 (1.2)	.32

- Fiscal policy here alternative safe asset scheme
  - less private debt, higher interest rate, lower collateral values

Nominal credit & uncertain inflation

- Nominal credit
  - payoff of bond $= (1+\pi)^{-1}$; q now nominal bond price
  - actual inflation zero
  - endogenous worst cases: borrower believes $\pi = -.01$, lender believes $\pi = .01$
    (worst case beliefs with ambiguous inflation)
- Steady state comparison

	$\bar{c}^A$	$\bar{c}^B$	$\bar{q}$	$\bar{p}$	debt $\bar{q}\bar{b}$	$\bar{\ell} = \bar{q}\bar{b}/\bar{p}$
baseline	.454	.530	.997	3.2	1.3	.40
amb. inflation	.456	.537	.993	2.7	0.9	.32

- Inflation uncertainty lowers gains from trade
  - uncertainty premium lowers price of nominal bonds
  - less debt, lower value of collateral

Conclusion

- Stochastic difference equations with disagreement
  - transition dynamics important for behavior in steady state
  - solve jointly for steady state & coefficients of loglinear approximation
  - paper provides general formulation
- Multiple priors utility with ambiguity about means
  - find worst case beliefs for all agents
  - solve implied model with disagreement
- Example: differences in ambiguity about income
  - precautionary saving by scared poor type A agents
  - trees valuable as collateral for confident rich type B agents
  - low interest rate and stock market boom
  - ambiguity shocks may persistently lower type A consumption
  - government debt may substitute for safe private debt
  - inflation uncertainty lowers gains from trade