A Dynamic Model of Bank Runs

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Introduction

- Banks subject to the refinance risk in a dynamic model
  - Collective actions in the financial crisis
    - ★ Runs on short-term debts observed in 2008–2009 (Lucas 2009, Gorton and Metrick 2009, Adrian and Shin 2009)
    - ★ Models of Financial Frictions – Borrowing constraints (Christiano et al 2009, Gertler-Karadi 2009, Gertler-Kiyotaki 2010; Carlstrom-Fuerst 1997, Bernanke-Gertler-Gilchrist 1999)
    - ★ Static models of bank runs (Diamond and Dybvig 1983, Diamond and Rajan 2001, Allen and Gale 1998, Uhlig 2009)
    - ★ Dynamic debt runs (He and Xiong 2010, Acharya, Gale, and Yorulmazer 2009)
  - Complement to Optimal Default literature
    - ★ Sovereign debt crisis, bankruptcies in business cycles
    - ★ Optimal defaults – Borrowers choose to default (Hopenhayn and Werning 2008, Arellano 2008, Chatterjee et al. 2007)
    - ★ Collective runs – Lenders choose to refuse refinancing.

Summary

- We introduce the banks subject to refinance risk in the infinite-horizon business cycle model.
- We show the existence and uniqueness of the equilibrium with an endogenous debt capacity, which is the threshold of the bank run.
- Debt capacity is higher as variance of productivity is smaller.
- Trade-off in debt restructuring after the bank run.
  - If the depositors' bargaining power is high, the debt capacity is high, whereas once the bank run occurs it is likely to recur.
  - If the depositors' bargaining power is low, the debt capacity is low, whereas once the bank run occurs it is not likely to recur.

Setup (1/2)

- Closed economy with discrete time: $t = 0, 1, 2, \cdots, \infty$.
- Unit mass of the consumers and the banks
- Fixed amount of land endowed initially to the consumers.
  - Banks have human capital to produce the goods from land. One bank can operate only one unit of land.
  - Consumers cannot use land. (Land is useless for consumers.)
  - Land produces A_t units of the goods, where A_t is the aggregate productivity, which is a random variable with $A_t \in [0, +\infty]$ and $\int_0^\infty f(A)dA = 1$, where $f(A)$ is the PDF of A_t.
  - Banks can walk away (without producing A_t), leaving z units of the goods and the land.
- In the initial period, consumers give land to banks in exchange for bank deposits (or short-term debt).

Setup (2/2)

- Consumer's problem:

$$E_0\left[\sum_{t=0}^{\infty}\beta^t\{u(c_t)-h_t\}\right], \quad (1)$$

subject to

$$c_t+b_t+d_t \leq h_t+(1+r_{t-1})b_{t-1}+\tilde{R}_{t-1}d_{t-1}. \quad (2)$$

- Participation Constraint of depositors for the banks:

$$E_t[\tilde{R}_t] \geq 1+r_t=\beta^{-1}. \quad (3)$$

Bank Runs

- Banks try to issue new debt D_t and consume $C_t = A_t - R_{t-1}D_{t-1} + D_t$, conditional on no bank runs, to

$$\max E_0 \left[\sum_{t=0}^{\infty} U(C_t) \right]$$

- Bank Run: All consumers refuse to refinance D_t.
- If the bank fails to refinance D_t, it must walk away from the depositors, leaving z units of the goods and the land.
- Once the banks walk away, they get zero utility thereafter; and the same number of new-born banks enter the economy.
- The group of depositors sells the land to a new-born bank in exchange for X units of bank deposits. (X is endogenous.)
- During the bank run, a depositor gets the following returns:
  - R_t with probability $\frac{z}{R_{t-1}D_{t-1}}$.
  - $\frac{X}{R_{t-1}D_{t-1}-z}$ with probability $1 - \frac{z}{R_{t-1}D_{t-1}}$.

Debt Capacity

- The consumers refinance the banks' debt, D_t, if

$$E_t[\tilde{R}_t] \geq \beta^{-1}. \qquad \text{(Participation Constraint)} \qquad (4)$$

- There is an endogenous threshold, $\overline{D}$, such that $E_t[\tilde{R}_t] < \beta^{-1}$ if and only if

$$D_t > \overline{D}.$$

- Since $D_{t+1} > \overline{D} \Leftrightarrow A_{t+1} < R_t D_t - \overline{D}$, (4) can be rewritten as

$$R_t \int_{R_t D_t - \overline{D}}^{\infty} df(A_{t+1}) + \left(\frac{z}{R_t D_t} R_t + \frac{R_t D_t - z}{R_t D_t} \frac{X}{R_t D_t - z} \right) \int_0^{R_t D_t - \overline{D}} df(A_{t+1})$$
$$= R_t \int_{R_t D_t - \overline{D}}^{\infty} f(A_{t+1}) dA_{t+1} + \left(\frac{z}{D_t} + \frac{X}{R_t D_t} \right) \int_0^{R_t D_t - \overline{D}} f(A_{t+1}) dA_{t+1} \geq \beta^{-1}. \qquad (5)$$

- The determination of $\overline{D}$ is specified later.

How to determine X

- After the bank run, the depositors seize the land and sell it to a new-born bank in exchange for X units of bank deposits.
- Maximum liability that a bank with one land can sustain is $\overline{D}$.
- X should be between 0 and $\overline{D}$.
- Bargaining between the group of the depositors and the new-born bank determines X. As a reduced form, we assume

$$X = \theta\overline{D}, \tag{6}$$

where θ $(0 < \theta < 1)$ is the bargaining power of the depositors.

Optimization for Banks

- Bellman equation for the value at the beginning of period t:

$$V(x_{t-1}) = \int_{x_{t-1}-\overline{D}}^{\infty} \max_{R_t(A_t),\ D_t(A_t)} \{U(A_t - x_{t-1} + D_t) + \beta V(R_t D_t)\} f(A_t) dA_t, \tag{7}$$

subject to

$$E[\tilde{R}_t] \geq 1 + r_t, \tag{8}$$

$$\max\{0, x_{t-1} - A_t\} \leq D_t, \tag{9}$$

$$D_t \leq \overline{D}, \tag{10}$$

where $U(C_t)$ is the bank's utility, $C_t = A_t - x_{t-1} + D_t$ is the consumption, and $x_{t-1} = R_{t-1} D_{t-1}$.

Existence and Uniqueness (1/2)

- **Proposition 1:** The Bellman equation has a unique solution.
- (Proof)
  - Define the operator T by

$$Tv(x) = \int_{x-\overline{D}}^{\infty} \max_{R,\, D}\{U(A - x + D) + \beta v(RD)\} f(A)dA, \tag{11}$$

subject to

$$\max\{0, x - A\} \le D \le \overline{D}, \tag{12}$$

$$R \int_{RD-\overline{D}}^{\infty} f(A')dA' + \left(\frac{z}{D} + \frac{X}{RD}\right) \int_{0}^{RD-\overline{D}} f(A')dA' \ge \beta^{-1}. \tag{13}$$

  - Define a set of functions $F \subset C[0, \overline{RD}]$ by

$$F = \left\{ v(x) \,\middle|\, 0 \le v(x) \le \frac{\beta E[U(A)]}{1-\beta} \text{ for all } x \in [0, \overline{RD}] \right\}. \tag{14}$$

Existence and Uniqueness (2/2)

- Proof (continued)
  - The solution to the Bellman equation is the fixed point of T.
  - F is closed and bounded.
  - $T : F \to F$ is continuous and monotone.
  - $T(F)$ is an equicontinuous family (Lemma 5).
  - Theorem 17.7 in Stokey and Lucas (1989) implies the existence of the fixed points of T in F.
  - The uniqueness established by that the minimal and maximal fixed points are identical (Lemma 6).

Determination of R_t

- **Lemma 1.** R_t is pinned down by

$$E_t[\tilde{R}_t] = \beta^{-1}.$$

  - The FOC and the envelope condition imply

$$\beta D_t V'(R_t D_t) + \mu_t(A_t)[\cdots] = 0, \tag{15}$$

$$V'(x) \leq \int_{x-\overline{D}}^{\infty} U'(A_t - x + D_t) d f(A) < 0. \tag{16}$$

  - $\mu(A_t)$ cannot be zero. Otherwise $V'(RD) = 0$ from (15) *and* $V'(RD) < 0$ from (16), a contradiction.
  - Therefore, $\mu(A_t) > 0$ and $E_t[\tilde{R}_t] \geq \beta^{-1}$ is binding.

Determination of $\overline{D}$ (1/2)

- $\overline{D}$ is uniquely determined by

$$\overline{D} = \max \left\{ D \,\middle|\, \max_R R \int_{(R-1)D}^{\infty} f(A)dA + \left(\frac{z}{D} + \frac{X}{RD} \right) \int_0^{(R-1)D} f(A)dA \geq \frac{1}{\beta} \right\}, \tag{17}$$

  and $X = \theta\overline{D}$.

- **Lemma 2:** Suppose that $\overline{D}$ is given by (17). This $\overline{D}$ is the unique value that satisfies the following claim: Given that the depositors in period $t+1$ refuse to refinance D_{t+1} if and only if $D_{t+1} > \overline{D}$, it is optimal for the depositors in period t to refuse to refinance D_t if and only if $D_t > \overline{D}$.

Determination of $\overline{D}$ (2/2)

- Proof of Lemma 2
- Define $\overline{D}_t(\overline{D}_{t+1})$, the best response of the depositors in period t to those in period $t+1$:

$$\overline{D}_t(\overline{D}_{t+1}) = \max\left\{D \,\middle|\, \max_R G(R, D; \overline{D}_{t+1}) \geq \frac{1}{\beta}\right\}, \tag{18}$$

where

$$G(R, D; \overline{D}_{t+1}) = R\int_{RD-\overline{D}_{t+1}}^{\infty} df(A) + \left(\frac{z}{D} + \frac{X}{RD}\right)\int_0^{RD-\overline{D}_{t+1}} df(A). \tag{19}$$

- $\overline{D}_t$ is maximum amount in t that satisfies PC for depositors t, given that $\overline{D}_{t+1}$ is maximum amount in $t+1$.
- It is shown that

$$\overline{D}_t(\overline{D}_{t+1}) > \overline{D}_{t+1} \text{ for } \overline{D}_{t+1} < \overline{D}, \tag{20}$$

$$\overline{D}_t(\overline{D}_{t+1}) < \overline{D}_{t+1} \text{ for } \overline{D}_{t+1} > \overline{D}. \tag{21}$$

- $\overline{D}$ is the unique threshold consistent with Rational Expectations.

Debt Capacity and Depositors' Bargaining Power

Parameter values: $\beta = 0.96$, $E[A] = 1$, $z = 0.1$.

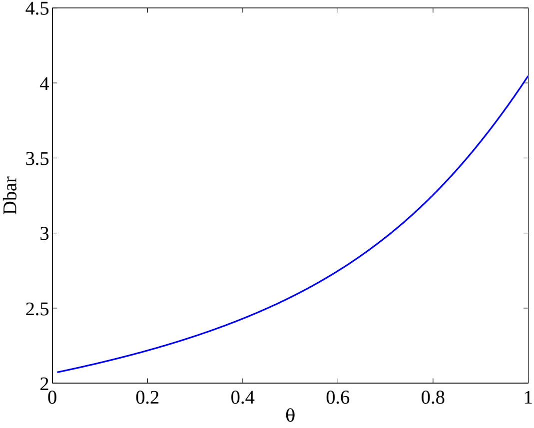

Figure 1. Debt capacity, $\overline{D}$, and bargaining power of depositors, θ.

Debt Capacity and Variance of Productivity

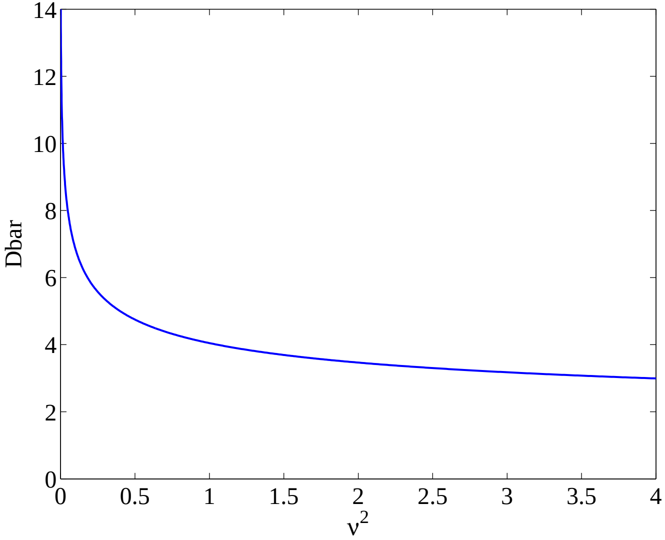

Figure 2. Debt capacity, $\overline{D}$, and the variance, ν^2 ($\theta = 1$).

R_t and Probability of Bank Runs

- R_t is uniquely pinned down by the following, given D_t,

$$R_t \int_{R_t D_t - \overline{D}}^{\infty} f(A_{t+1}) dA_{t+1} + \left(\frac{z}{D_t} + \frac{\theta \overline{D}}{R_t D_t} \right) \int_0^{R_t D_t - \overline{D}} f(A_{t+1}) dA_{t+1} = \beta^{-1}. \tag{22}$$

- Probability, as of t, of occurrence of bank run in $t+1$:

$$Pr(D_t) \equiv \int_0^{R_t D_t - \overline{D}} f(A) dA. \tag{23}$$

- **Lemma 3:**
  - If $D_t \leq \beta \overline{D}$, then $R_t = \beta^{-1}$ and $Pr(D_t) = 0$.
  - If $\beta \overline{D} < D_t \leq \overline{D}$, then $R_t > \beta^{-1}$ and $Pr(D_t) > 0$.

Illustration of Path of Bank Debt

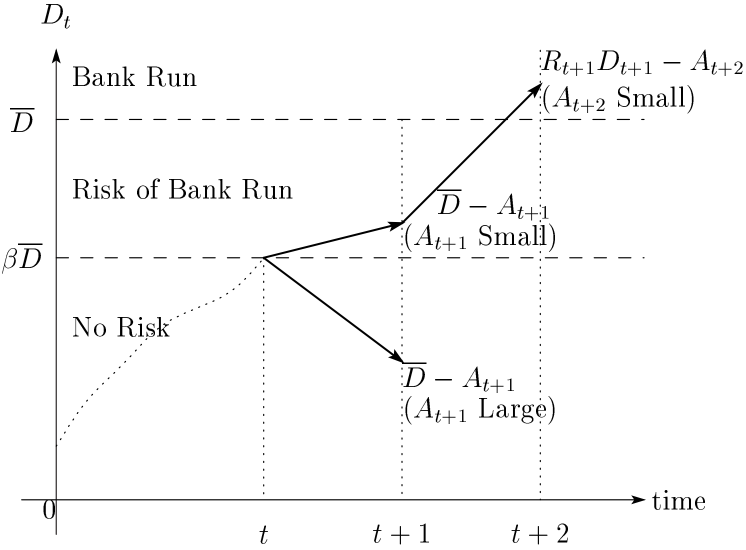

Figure 3. Illustration of Path of Bank Debt.

Probability of Bank Runs for a Small θ

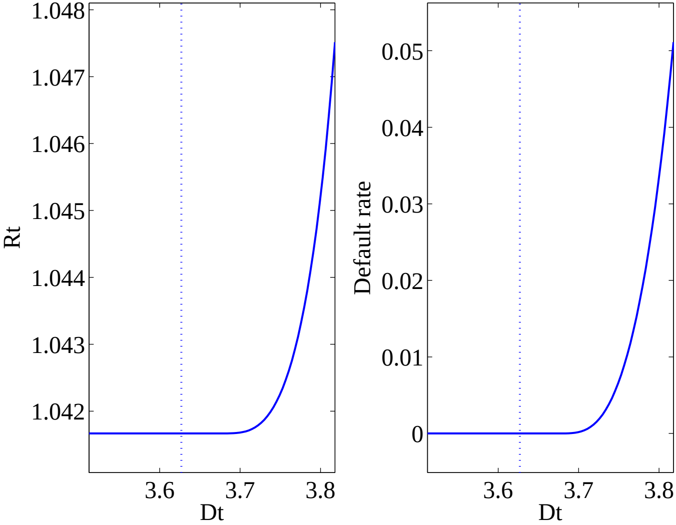

Figure 4. R_t and $Pr(D_t)$ in the case where $\theta = 0.95 < \beta$

Probability of Bank Runs for a Large θ

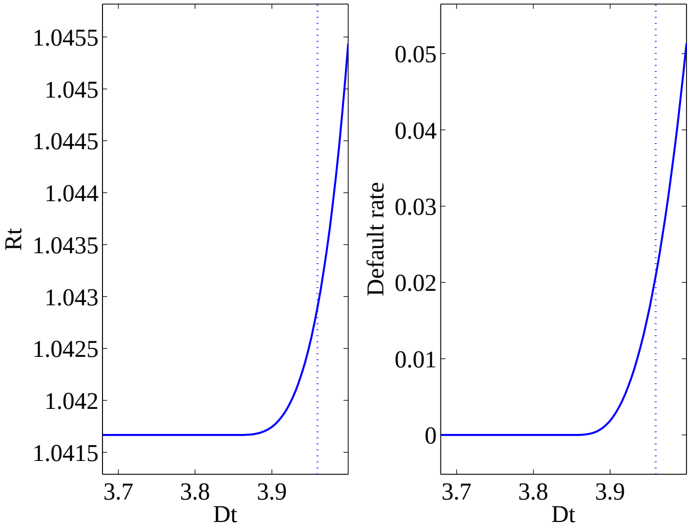

Figure 5. R_t and $Pr(D_t)$ in the case where $\beta < \theta = 0.99$

Trade-off between $\overline{D}$ and $Pr(D_t)$

- If θ is large,
  - Debt capacity, $\overline{D}$, is large,
  - After the bank run, the bank debt becomes $D_t = X = \theta\overline{D} > \beta\overline{D}$. Since $Pr(X) > 0$, the risk of another bank run occurring in the next period is positive.
- If θ is small,
  - Debt capacity, $\overline{D}$, is small,
  - After the bank run, the bank debt becomes $D_t = X = \theta\overline{D} < \beta\overline{D}$. Since $Pr(X) = 0$, the risk of another bank run occurring in the next period is zero.

Dynamics of D_t

- Dynamics of D_t depend on the form of function $U(C)$.
- If $U(C)$ is sufficiently concave, D_t may exhibit the mean-reversion.

- **Lemma 4:** If $U(C) = C$, then
  $D_t = R_{t-1}D_{t-1} - A_t$, when $R_{t-1}D_{t-1} - A_t \le \overline{D}$;
  and $D_t = \theta\overline{D}$, when $R_{t-1}D_{t-1} - A_t > \overline{D}$.
  - If the banks maximize the discounted sum of profits, the banks minimize the remaining debt in each period.
  - The banks repay the debt as much as possible, and consume zero if $D_t > 0$.
  - The banks consume $A_t - R_{t-1}D_{t-1}$ only if D_t becomes zero.

Generalized Model – Setup

- Introducing labor: $y_t = A_t l_t^{1-\alpha}$. (cf. $y_t = A_t$ in the basic model.)
- Consumers, Banks, and Firms.
- Firms produce goods using labor from consumers and capital from banks.
- Firms' production is feasible if the bank runs do not occur.

$$\max_{l_t, k_t} A_t k_t^{\alpha} l_t^{1-\alpha} - \tilde{r}_t k_t - \tilde{w}_t l_t. \tag{24}$$

The FOCs imply

$$\tilde{r}_t = \alpha A_t l_t^{1-\alpha}, \tag{25}$$

$$\tilde{w}_t = (1-\alpha) A_t l_t^{-\alpha}, \tag{26}$$

Generalized Model – Consumer's problem

- Consumers choose the labor supply l_t *before* A_t is realized.

$$\max_{c_t, l_t, h_t} E_0 \left[\sum_{t=0}^{\infty} \beta^t \{ u(c_t) - c(l_t) - h_t \} \right], \tag{27}$$

subject to

$$c_t + b_t + d_t = (1 + r_{t-1}) b_{t-1} + \tilde{R}_t d_t + \tilde{w}_t l_t + h_t, \tag{28}$$

where l_t is chosen before $\tilde{w}_t$ is realized. The FOC for l_t is

$$c'(l_t) = E_t[\tilde{w}_t]. \tag{29}$$

Generalized Model – Bank's problem

- The state variables are $x_{t-1} = R_{t-1}D_{t-1}$ and l_t.
- Banks take l_{t+1} as given, and choose R_t and D_t:

$$V(x_{t-1}, l_t) = \int_{A(x_{t-1},l_t)}^{\infty} \max_{R_t,D_t} \left\{ U(\alpha A_t l_t^{1-\alpha} - x_{t-1} + D_t) + \beta V(R_t D_t, l_{t+1}) \right\} d f(A_t) \tag{30}$$

where $A(x_{t-1}, l_t) \equiv \frac{x_{t-1} - \overline{D}}{\alpha l_t^{1-\alpha}}$, subject to

$$\max\{0, x_{t-1} - \alpha A_t l_t^{1-\alpha}\} \leq D_t, \tag{31}$$

$$D_t \leq \overline{D}, \tag{32}$$

$$R_t \int_{A(R_t D_t, l_{t+1})}^{\infty} f(A_{t+1}) dA_{t+1} + \left(\frac{z}{D_t} + \frac{X}{R_t D_t} \right) \int_0^{A(R_t D_t, l_{t+1})} f(A_{t+1}) dA_{t+1} \geq \beta^{-1}. \tag{33}$$

Generalized Model – Equilibrium Conditions

- l_{t+1} is determined by

$$c'(l_{t+1}) = \int_{A(R_t D_t, l_{t+1})}^{\infty} (1-\alpha) A_{t+1} l_{t+1}^{-\alpha} f(A_{t+1}) dA_{t+1}. \quad (34)$$

- $\overline{D}$ is determined by

$$\overline{D} = \max \left\{ D \,\middle|\, \max_R R \int_{\frac{(R-1)D}{\alpha l^{1-\alpha}}}^{\infty} f(A) dA + \left(\frac{z}{D} + \frac{\theta}{R} \right) \int_0^{\frac{(R-1)D}{\alpha l^{1-\alpha}}} f(A) dA \geq \frac{1}{\beta} \right\} \quad (35)$$

where l is determined by

$$c'(l) = \int_{\frac{(\overline{R}-1)\overline{D}}{\alpha l^{1-\alpha}}}^{\infty} (1-\alpha) A l^{-\alpha} f(A) dA. \quad (36)$$

Generalized Model – Labor supply and Bank Runs

- Lemma 3 applies to the generalized model.
  - If $D_t \leq \beta\overline{D}$, then $R_t = \beta^{-1}$ and $Pr(D_t) = 0$. $l_{t+1} = l^*$, where

$$c'(l^*) = (1-\alpha)E[A](l^*)^{-\alpha}. \tag{37}$$

  - If $\beta\overline{D} < D_t \leq \overline{D}$, then $R_t > \beta^{-1}$ and $Pr(D_t) > 0$. $l_{t+1} < l^*$
- Labor supply decreases as D_t exceeds $\beta\overline{D}$.

- Parameter values: $\alpha = 0.3$, $\nu^2 = 0.01$, $c(l) = l^2/2$.

Debt Capacity and Depositors' Bargaining Power

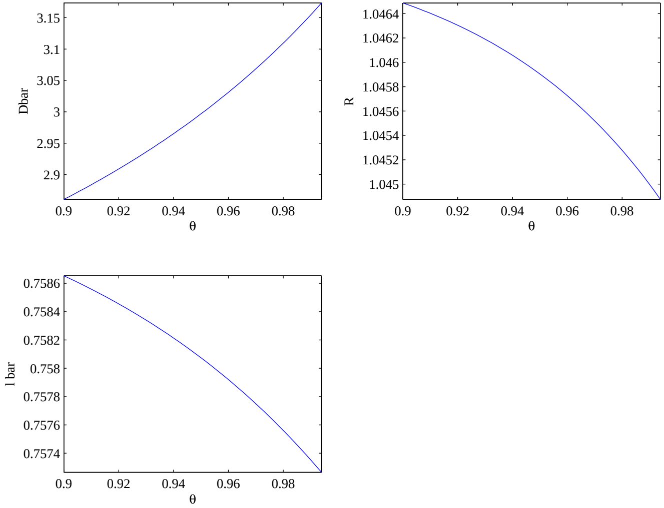

Figure 6. Debt capacity, $\overline{D}$, and bargaining power, θ ($\nu^2 = 0.01$).

Debt Capacity and Variance

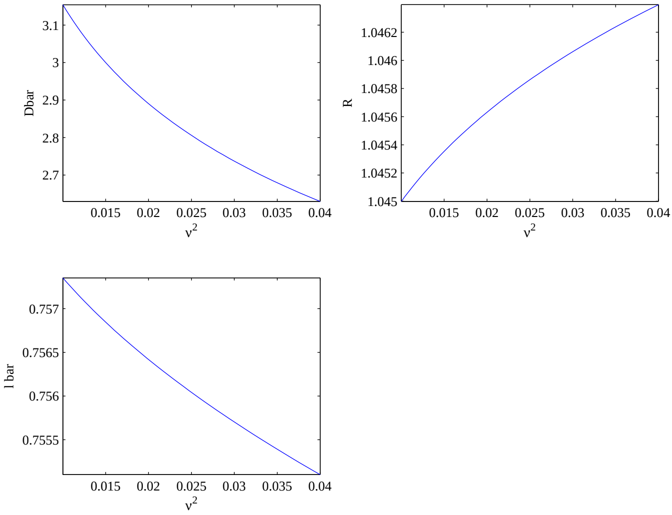

Figure 7. Debt capacity, $\overline{D}$, and variance of productivity, ν^2 ($\theta = 0.99$).

Low bargaining power and risk of bank runs

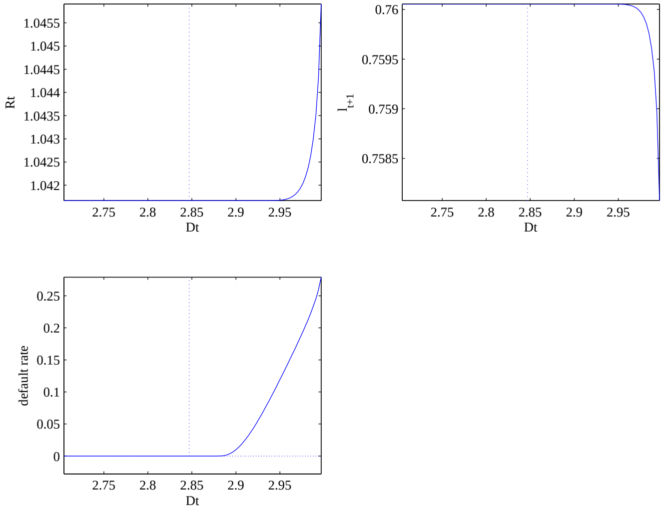

Figure 8. R_t, $Pr(D_t)$, l_{t+1} in the case where $\theta = 0.95 < \beta$.

High bargaining power and risk of bank runs

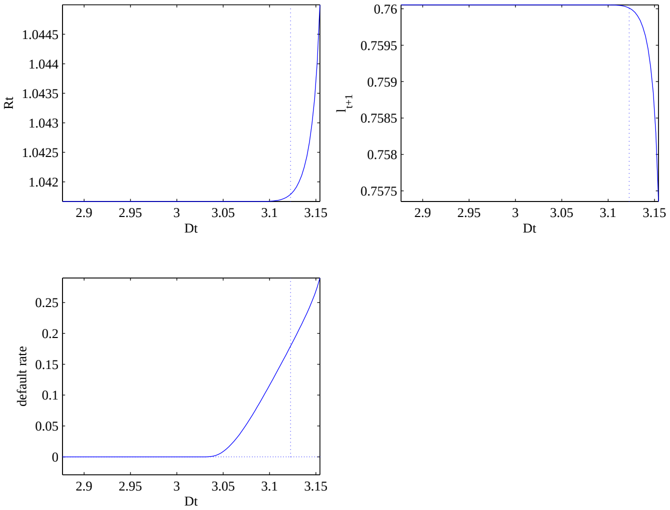

Figure 9. R_t, $Pr(D_t)$, l_{t+1} in the case where $\beta < \theta = 0.99$.

Conclusion

- Banks are subject to refinance risk.
- Existence and uniqueness of the equilibrium in which the depositors run on the banks if the bank deposits exceed the debt capacity.
- There is a trade-off between the debt capacity and the risk of recurrence of the bank runs.
  - If θ is high, the debt capacity is high, whereas the risk of recurrence of the bank runs is positive.
  - If θ is low, the debt capacity is low, whereas the risk of recurrence of the bank runs is zero.

Topic for Future Research

- Bubble and bank runs
  If X includes the bubble component and declines suddenly, then $\overline{D}$ declines suddenly and the bank run is triggered.
- Distortion in productivity
  Risk aversion may lead to TFP declines when $\beta\overline{D} < D_t \leq \overline{D}$, through agents' choice of the production technology.
- Monetary Policy
  Introduce Nominal Rigidities. The nominal interest rate should be lower when $\beta\overline{D} < D_t \leq \overline{D}$, than when $D_t \leq \beta\overline{D}$.