Equilibrium Default

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Default: pervasive and costly

- firms:
  - Average 5% loans in Spain, 3% in Norway (Fernandez de Lis, Martinez Pages and Saurina, 2000; SEBRA panel Bernhandsen, Norges Bank, 2001)
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- sovereign countries:
  - 106 countries defaulted, a total of 250 times, since the end of the Napoleonic Wars. Over 4% probability of default (Tomz and Wright, 2007)
  - Disruptive events, lengthy negotiations: average 8.8 years, output losses (Pitchford and Wright, 2007)

Models of default

1. Optimal contract literature (investment with limited commitment):
   - Outside option/default used to renegotiate terms
   - No default/inefficient termination
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2. Non-contingent debt/default
   - Risk adjusted non-contingent one period loans
   - Default when not repaying is preferred
   - References:
     - Sovereign debt: Eaton and Gersovitz, Arellano
     - Consumer loans: Chaterjee, Corbae and Rios Rull
     - Firm financing: Arellano, Bai and Zhang

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- Results...
  - Hazard rate decreases over time
  - Capital/growth rate increases over time—investment weighted against default risk
  - Implemented by risk adjusted loans
  - Extensions

Related Application

Human Capital investment

- General and Specific costs shared by firm and worker
  e.g. cost of german apprentice $10,600
- risk of separations affects investment
- Becker, Hashimoto, Acemoglu and Pishke

Outline

- Model
- Special case
- General properties
- Implementation with risk adjusted loans
- Extensions

General model

- Project: $R(k, \dot{k})$ flow of profits or output
  - increasing and strictly concave in k
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  - Homogeneous case: $R(k, \dot{k}) = Ak - c(\frac{\dot{k}}{k})k$

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- Common discount rate r
- outside opportunity: s
  - arrives with Poisson arrival rate λ and distribution $F(s, k)$
  - Default leaves zero residual value to principal
  - Separation not efficient

Interpretation

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Default = Inefficient Separation

Special case

- $R(k, \dot{k}) = f(k) - \delta k - \dot{k}$
- Outside option independent of k: distribution $F(s)$ on support $[0, \bar{s}]$ (contrast: Albuquerque-Hopenhayn)

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- First best:

$$f'(k^*) = r + \delta$$
$$rW^* = f(k^*) - (r+\delta)k^*$$
$$W^* \geq \bar{s}$$

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- Assume...
  - $W^* - I > 0$ (project jointly efficient)
  - $W^* - \bar{s} < I$ (no commitment binding)

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- Maximum value of debt = W^*- minimum feasible $V_{\min}$

$$rV_{\min} = \lambda \int_{V_{\min}} (s' - V_{\min}) F(ds')$$

  - Minimum feasible value without forced termination
  - Can be delivered exactly without need of separation
- Contract is financially feasible iff $B_{\max} = W^* - V_{\min} \geq I_0$
- If feasible, no default and no growth

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- evolution of value

$$\begin{aligned} rV(t) &= \dot{V}(t) + c(t) + \lambda \int_{V(t)} (s - V(t))F(ds) \\ &= \dot{V}(t) + c(t) + \int_{V(t)} h(z)dz \end{aligned}$$

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- incentive compatibility
  ➡ $V(t)$ independent of s ➡ continuation contracts cannot separate
- if V sufficiently high ➡ no separation

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- Decreasing in relevant region → $B(V(t))$ decreasing over time

Proposition. Suppose $F(s)$ independent on capital k
→ declining default hazard $h(t)$
without adjustment costs → capital $k(t)$ rises over time, debt $B(t)$ falls

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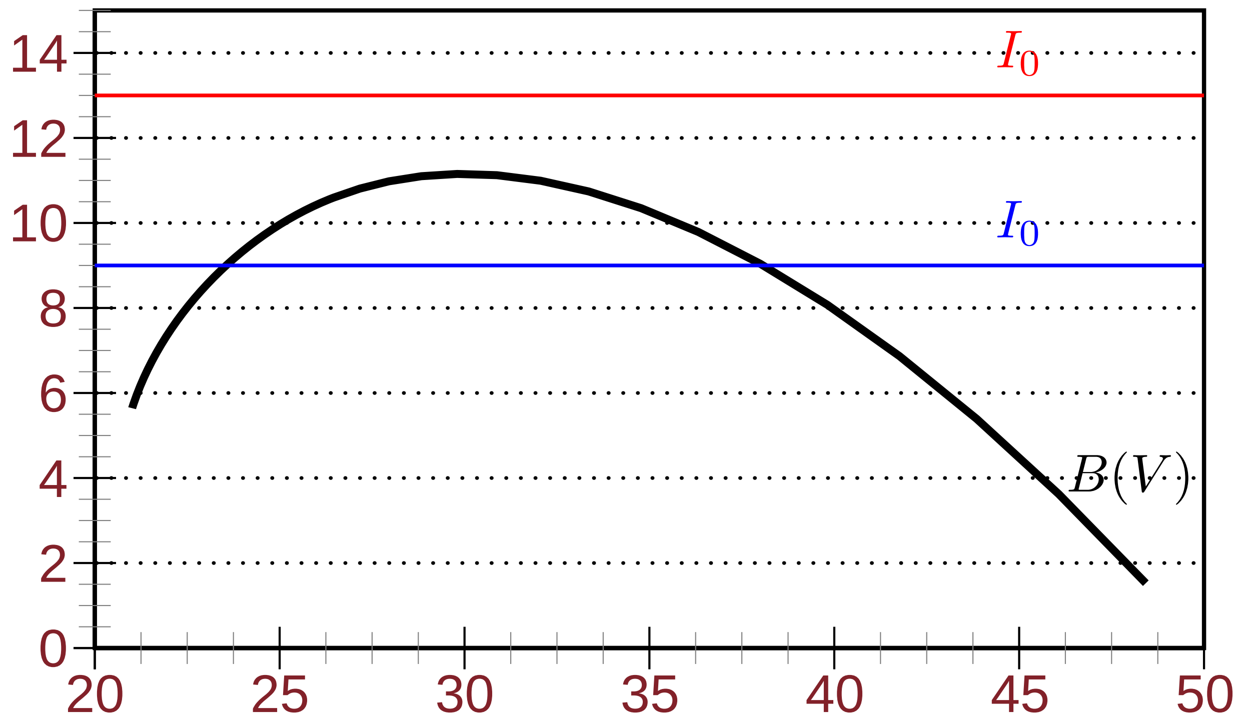

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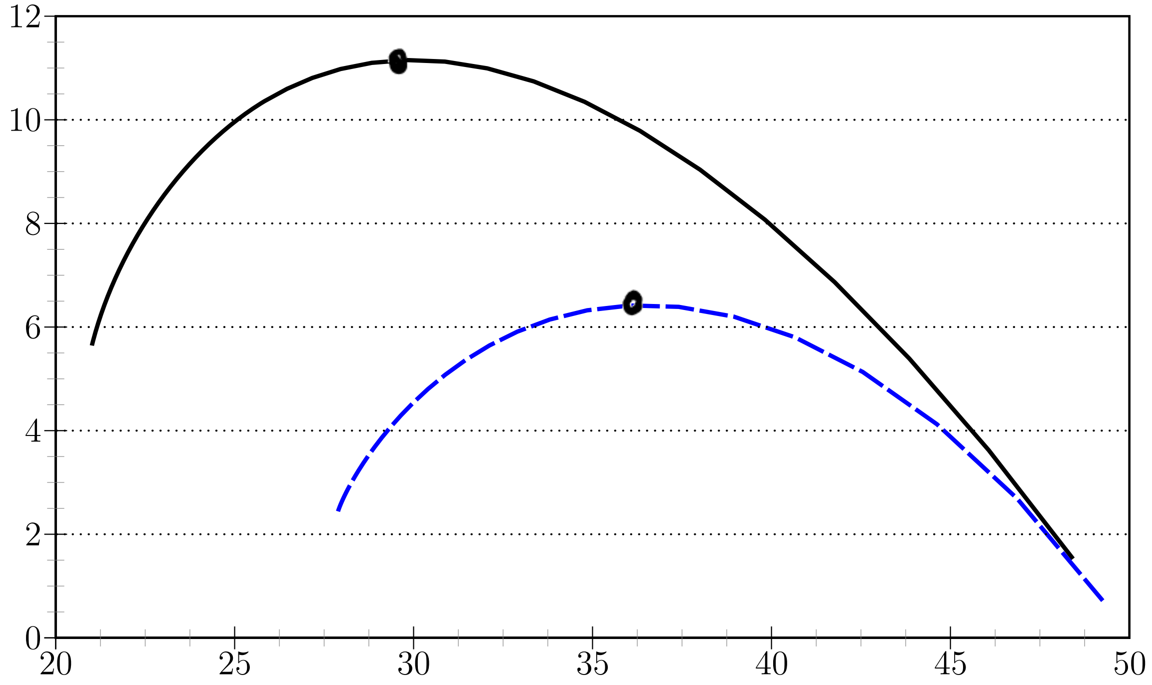
- if $B_{\max} < I_0$ then
  - need own funds: $I_0 - B_{\max}$
  - or a secured loan component

Proposition. Suppose no adjustment costs and $\tilde{F}(s) \leq F(s)$: optimum for $\tilde{F}$ → higher default rates and lower investment.



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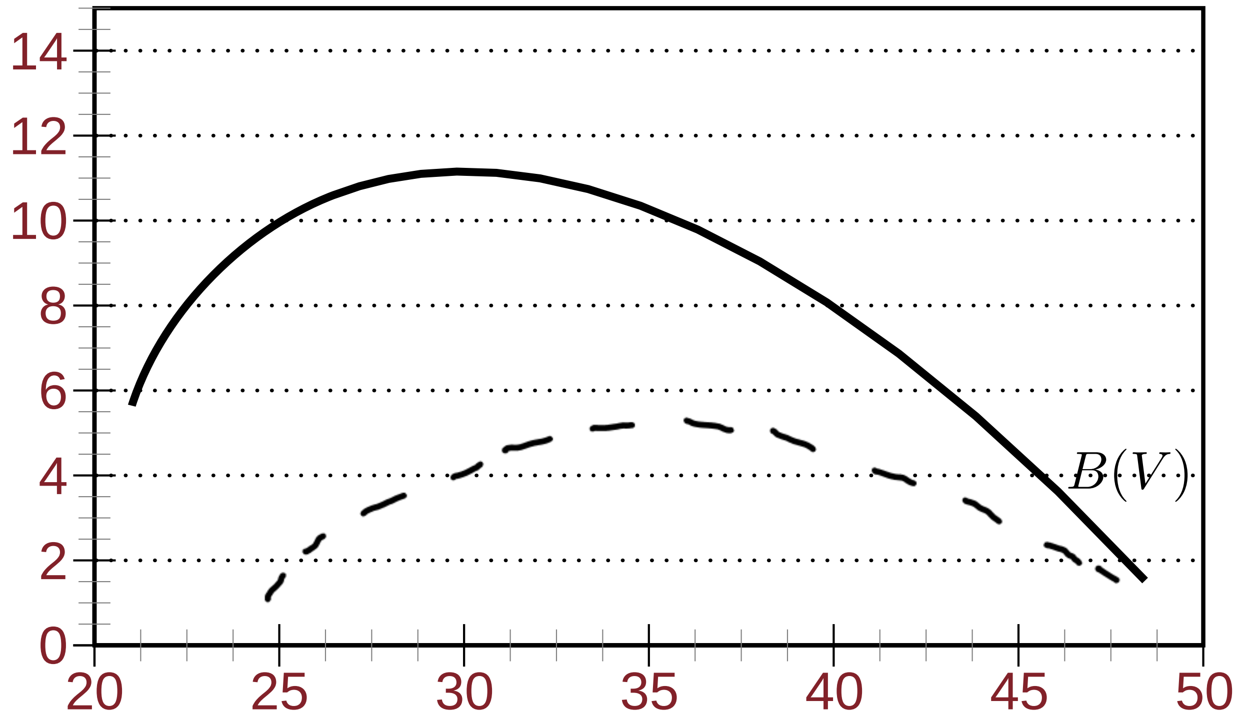
- two scenarios: $\tilde{F}(s) \leq F(s)$



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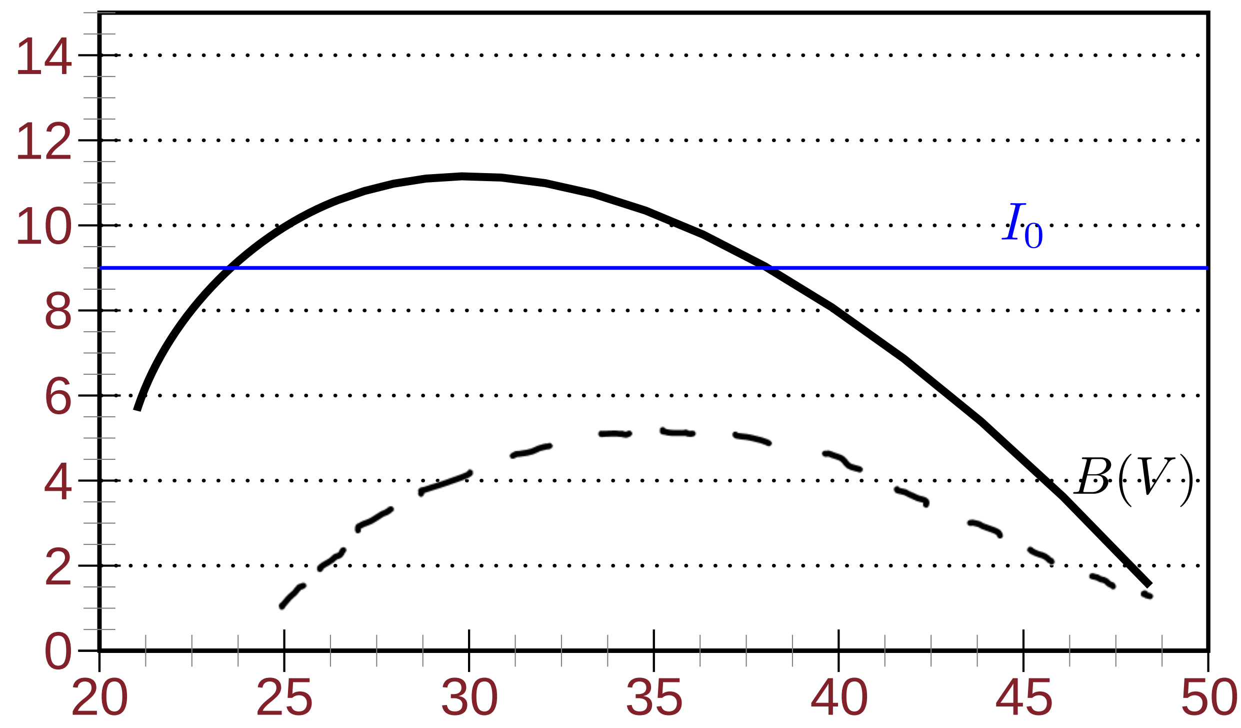

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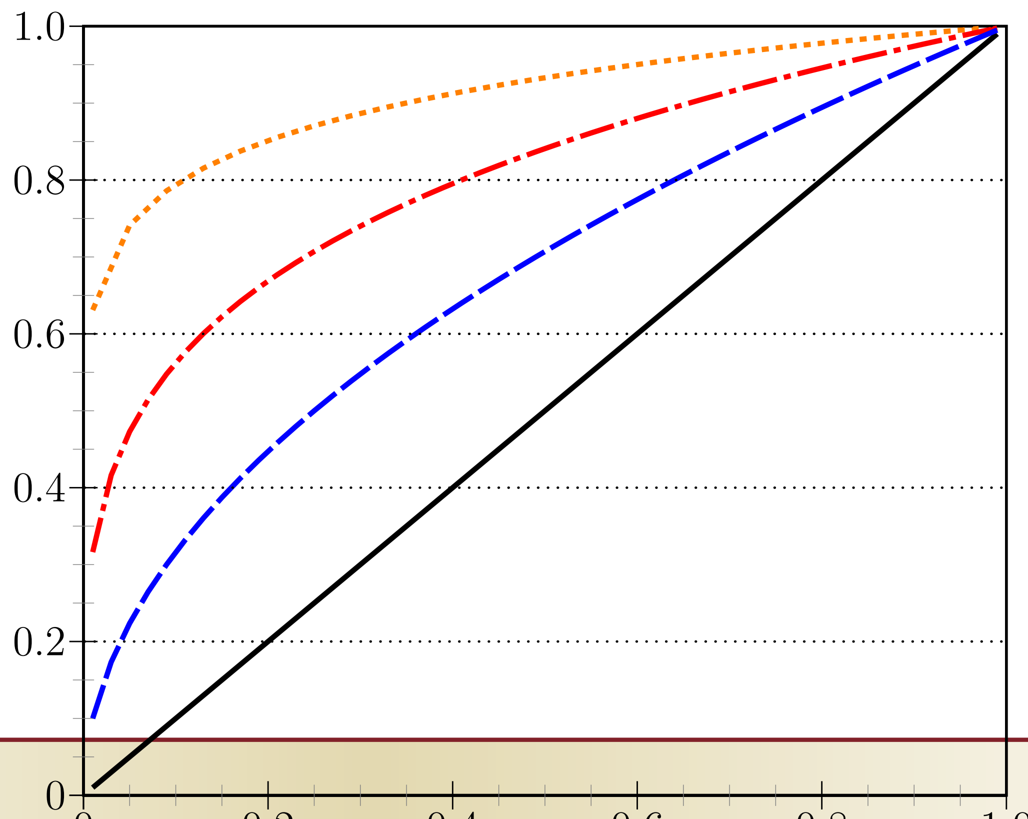
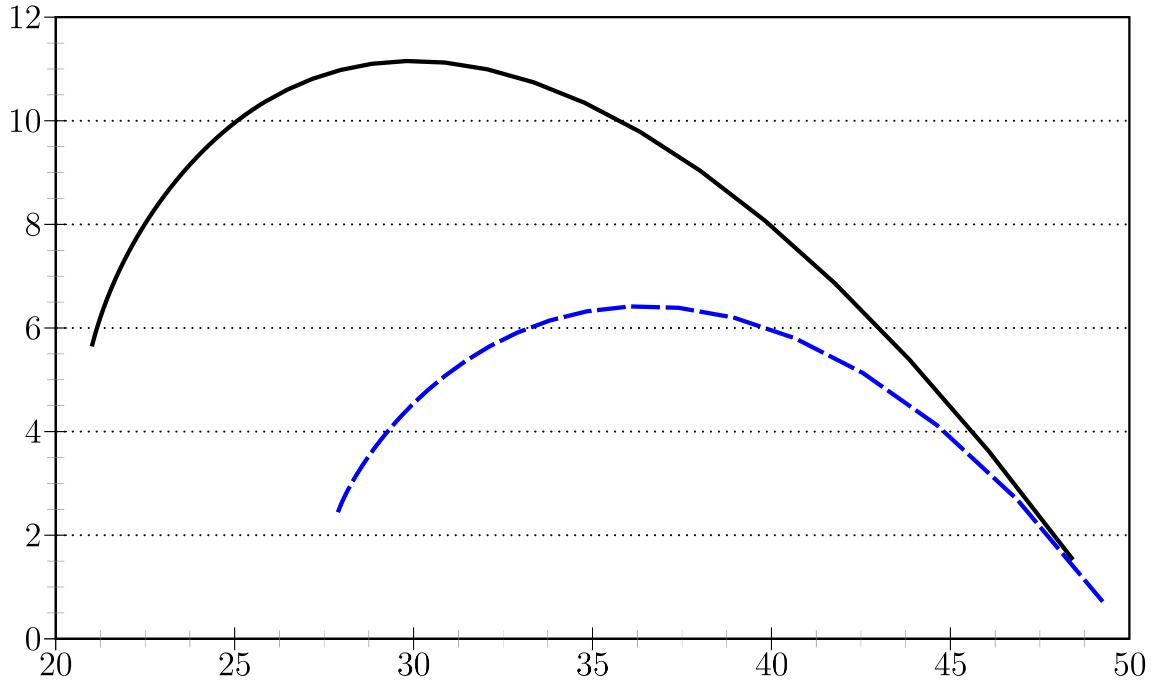
- $R(k, \dot{k}) = k^{1/2} - \delta k - \dot{k}$
- $r = \delta = 5\%$
- Steady state $k^* = 25$, $W^* = 50$.
- Outside value: sW^*, with $F(s) = s^\gamma$, $\gamma \in \{0.1, 0.25, 0.5, 1\}$



Value functions B(V)



- Section where B is increasing
- Maximum debt (peak) higher for lower γ

Evolution of debt ($\gamma = 0.25$)

- start at maximum debt...
- ...follow optimal contract

t	$W = V + B$	h	Survival	$r + h$	k
0	42.9	7.6	1.00	12.6%	8.1
1	44.2	6.7	0.93	11.7%	9.0
3	46.9	4.7	0.83	9.7%	11.6
5	49.0	2.4	0.77	7.4%	16.2
7	50.0	0	0.75	5.0%	25.0

- Assets and default

Leverage and outside options

Starting at maximum debt...

γ	V_0	B_0	W_0	k_0	Leverage excl. k	Leverage incl. k	T_{eff}	Survival
0.10	30.0	11.2	41.1	11.1	27%	42.6%	11.5	74%
0.25	36.5	6.4	42.9	8.1	15%	28.5%	7.0	75%
0.50	40.3	4.0	44.3	6.1	9%	20.1%	4.7	77%
1.00	43.2	2.4	45.6	4.5	5%	13.8%	3.2	80%

- Leverage increases with size
  consistent with Arellano et al. for Ecuador
- suppose $I_0 \geq B_0$ → $I_0 - B_0$ as secured debt
- higher γ implies more secured debt, shorter horizon and smaller total debt

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- Recursive formulation

$$rB(V,k) = \max_{\dot{k},w} R(k,\dot{k}) - h(V,k)B(V,k) + B_V(V,k)m(V,k) + B_k(V,k)\dot{k}$$

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- first-order condition for $\dot{k}$...

$$R_2(k, \dot{k}) + B_2(V, k) = 0$$

Marginal returns to capital

- Modified q equation:

$$[r + h(V,k)]\, q = R_1\left(k, \dot{k}\right) + \dot{q} - h_2(V,k)\, B(V,k) + B_1 m_2(V,k)$$

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- Special case: no adj cost: $q = 1$ and $R\left(k, \dot{k}\right) = f(k) - \delta k - \dot{k}$:

$$f'(k) = r + \delta + h(V,k) + h_2(V,k)\, B(V,k) - B_1 m_2(V,k)\,.$$

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$$f'(k)=r+\delta+h(V,k)+h_2(V,k)\,B(V,k)-B_1 m_2(V,k)\,.$$

- In addition if $F(ds)$ independent of k :

$$f'(k)=r+\delta+h(V)$$

- Assets and default risk: h, k and V

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and $B \leq \bar{B}(k)$

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$$\dot{B} = c + r(B,k) - R(k,\dot{k})$$

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- Suppose agent faces a flow payment schedule $r(B,k)$

$$rV(B,k) = \max_{\dot{k},c} c + V_2(B,k)\dot{k} + V_1(B,k)\left(r(B,k) - R(k,\dot{k}) + c\right) + \int_{V(B,k)} h(s,k)\,ds.$$

- Borrower owes $r(B,k)$, sets $c = 0$ and pays $R(k,\dot{k})$...
- if $r(B,k) - R(k,\dot{k}) < 0$ debt falls

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Requirements

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- idea → V is inverse of B
- $V(B, k)$: Invertible in relevant range
  assume is quasi concave in B for all k

1. Generate Same allocations

- recall FOC from principal's problem...

$$0 = B_2(V,k) + R_2(k,\dot{k})$$

- now, FOC from agent...

$$\begin{aligned} 0 &= V_2(B,k) - V_1(B,k)R_2(k,\dot{k}) \\ &= -\frac{V_2(B,k)}{V_1(B,k)} + R_2(k,\dot{k}) \\ &= B_2(V,k) + R_2(k,\dot{k}) \end{aligned}$$

- last step: $V(B,k)$ inverses of $B(V,k)$:

$$V(B(V,k),k) = V \quad \rightarrow \quad V_2(B,k) + V_1(B,k)B_2(V,k) = 0$$

2. Same path of debt...

$$\dot{B} = r(B,k) - R(k,\dot{k})$$

Value of debt:

$$rB = R(k,\dot{k}) - h(V,k)B + \dot{B}$$

substituting...

$$r(B,k) = [r + h(V(B,k),k)]\,B$$

Incomplete contracts vs. short term debt

- Can risk adjusted one term debt be rationalized by incomplete contingent contracts?
- Example shows that long term non-contingent debt may improve.
- Two periods, no discounting ($r = 0$)
- Technology
  - Indivisible project $k = 1$
  - Full depreciation
  - output $y = 3$ with probability p and zero otherwise
- default leads to liquidation

One period debt

- First period: bank lends one dollar
  - if $y = 3$ pay back and self-finance
  - if zero, borrow two dollars and rollover
- Second period $(y_1 = 0)$
  - face value of debt D
  - default if $y_2 = 0$
  - risk adjusted debt: $pD = 2$
  - Since $D \leq 3$ works $\iff p \geq 2/3$

Long term debt

- Advance 2 dollars
- Pay debt D at the end of second period
- Only default with probability $(1-p)^2$ when $y_1 = y_2 = 0$
- risk adjusted debt: $(1-(1-p)^2)D = 2$
- works $\iff (1-(1-p)^2) \geq 2/3$
- Weaker condition than $p \geq 2/3$
- Implicit cross-subsidizing from $y_1 = 3$ to $y_1 = 0$

Risk Aversion

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- Bellman...

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$$+ B_V(V,k)(\dot{V}) + B_k(V,k)\dot{k}$$

$$-u'(c) = 1/B_V(V,k)$$

Discounting

- subjective discounting → $\rho > 0$
- ... up to now: $r = \rho$
- instead: $r < \rho$ (general equilibrium?)
- no converge to first best
- with z shocks → long run distribution of (V, k)

Conclusions

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- special cases ➞ reasonable properties
- Implementation with one period risk adjusted loans
- Extensions:
  - risk aversion and discounting
  - Hybrid models: public and private info