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Abstract

When individuals' labor and capital income are subject to uninsurable idiosyncratic risks, should capital and labor be taxed, and if so how? In a two period general equilibrium model with production, we derive a decomposition formula of the welfare effects of these taxes into insurance and distribution effects. This allows us to determine how the sign of the optimal taxes on capital and labor depend on the nature of the shocks, the degree of heterogeneity among consumers' income as well as on the way in which the tax revenue is used to provide lump sum transfers to consumers. When shocks affect primarily labor income and heterogeneity is small, the optimal tax on capital is positive. However in other cases a negative tax on capital is welfare improving. (JEL codes: D52, H21. Keywords: optimal linear taxes, incomplete markets, constrained efficiency)

1 Introduction

The main objective of this paper is to investigate the effects and the optimal taxation of investment and labor income in a two period production economy with uninsurable background risk. More precisely, we examine whether the introduction of linear, distortionary taxes or subsidies on labor income and/or on the returns from savings are welfare improving and what is then the optimal sign of such taxes. This amounts to studying the *Ramsey problem* in a general equilibrium set-up. We depart however from most of the literature on the subject[1] for the fact that we consider an environment with no public expenditure, where there is no need to raise tax revenue. Nonetheless, optimal taxes are typically nonzero; even distortionary taxes can improve the allocation of risk in the face of incomplete markets. Then the question is which production factor should be taxed: we want to identify the economic properties which determine the signs of the optimal taxes on production factors.

A possible answer may come from the following consequence of the agents' precautionary motive for saving: under uninsurable risk, this motive implies that savings and hence capital accumulation

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[1] See Chari and Kehoe (1999) for a survey.

will be higher compared to the situation where markets are complete. This point was made in an influential paper by Aiyagari (1995, p.1160) and in fact various papers thereafter suggested that with incomplete markets, the *precautionary saving motive* leads to *over-accumulation of capital* and hence that a *positive tax on capital is welfare improving.*[2]

However, the comparison between the level of capital accumulation with and without complete markets has no clear welfare implication. If there were a policy tool which could implement the complete market allocation, there would be little doubt for the policy maker to adopt such a policy as far as attaining efficiency is concerned. But since a tax and subsidy scheme of the kind mentioned above is not sufficient for completing the markets, this comparison tells little about the welfare effects of taxation, not to mention whether or not capital should be taxed.

To properly assess whether or not positive taxes on capital are welfare improving when markets are incomplete, one should rather compare the competitive equilibria with and without taxes, keeping the other parts of the market structure, in particular the set of available financial assets, fixed. In this paper we carry out this second best exercise. We find that subsidizing capital may indeed be welfare improving in a situation where the equilibrium stock of capital is higher than when markets are complete. This finding does not rely on the presence of upward sloping demand curves, so that subsidizing capital further increases its level but nevertheless raises consumers' welfare. More generally, we identify conditions, regarding the nature of the shocks and the degree of heterogeneity among consumers, under which it is optimal to tax or subsidize capital and labor, for different specifications of the tax policy. In the rest of this section we outline the structure of this paper and summarize our findings.

Section 2 describes the economy. Production occurs only in the second period, using capital and labor, traded in competitive markets. There is a continuum of consumers, who may differ in terms of their initial income as well as of their preferences. Each consumer has to choose how much to work in the second period and his savings in the first period, invested to obtain capital, the only saving instrument. Idiosyncratic shocks affect the productivity of the work of each consumer and possibly the return on his savings.

A first component of the welfare effects of the introduction of linear taxes on capital and labor is the effect of the change of equilibrium factor prices induced by the change in savings and labor supply. To isolate and analyze this effect in Section 3 we consider an ideal situation where consumers' savings or labor supply can be directly modified. This problem was previously investigated by Davila *et al* (2012) in a similar setup, but they assume an exogenous labor supply and labor productivity shocks only.

The main and novel finding we report here is the derivation of a *decomposition formula*: the welfare effects per marginal changes of equilibrium factor prices are decomposed into an *insurance* and a *distribution* effect for each factor. Roughly speaking, if the price of a factor changes by $\triangle p$ at the margin, the welfare changes by $\triangle p$ times the sum of the insurance effect and the distribution effect associated with the factor. The insurance effect is related to the fact that with incomplete markets price changes affect the risk allocation agents can attain by trading in markets, a pecuniary externality first noticed by Hart (1975) and Stiglitz (1982). We show that the insurance effect tends to be negative and vanishes if there is no productivity shock; intuitively speaking, if the price of a production factor increases, the risk in a consumer's future income from the factor is magnified, reducing welfare level. The distribution effect is due to the reallocation of income, among agents differently endowed with capital and labor, induced by a change in relative factor prices. Thus this

[2] See for instance, Ljungqvist and Sargent (2004, Chapter 15, pp. 535-536), Mankiw, Weinzierl and Yagan (2009, pp. 167-8).

effect depends on the heterogeneity of consumers.

The decomposition formula is a powerful tool for investigating the signs of the welfare effects of policy changes. When the shocks affect primarily labor, e.g. there is no capital productivity shock (a case we shall refer to as the benchmark case), the insurance effect tells us that the utility of each consumer increases when the total investment made by consumers is decreased, or alternatively their total labor supply is increased. Indeed, since in this case the insurance effect of capital is zero and only the negative insurance effect of labor is present, a change in factors' usage which reduces the marginal productivity of labor, and hence the wage level, contributes positively to welfare through the insurance effect. On the other hand, the distribution effect has a different sign across consumers: in the case of a decrease in the investment level, the relative price of capital increases, and so this effect contributes negatively to welfare for the relatively poor consumers who rely primarily on labor income, and positively for the relatively rich ones who rely primarily on capital income. The contribution to social welfare, evaluated in terms of a utilitarian social welfare function, turns out to be negative and its magnitude increases with the heterogeneity of income. Hence the decomposition result tells us that in the benchmark case, when the insurance effect prevails over the distribution effect, a (marginal) reduction of the investment is welfare improving, which means that there is over investment at the equilibrium. As noted above, the prevalence of the insurance effect occurs when the heterogeneity among consumers is small.

In Section 4, the optimal taxation problem is examined. The net revenue from the introduction of linear taxes on the wage income and investment income is redistributed to consumers via lump sum transfers and we consider different specifications of the way in which such transfer takes place. These transfers generate an additional component of the welfare effect of the tax policy, besides the change in equilibrium prices induced by the policy. Since it is difficult to provide qualitative results about the actual level of the optimal tax rates, we focus first on the effects of introducing taxes at the margin, to identify whether a positive or a negative tax on labor and/or on capital is welfare improving at a competitive equilibrium: thus the decomposition result sharply characterizes the welfare effects of the price changes induced by the taxes. In Section 5 we follow up on each of these cases with a numerical example, where we compute the optimal tax rates and show that the sign of the optimal tax rates are typically in accord with our findings from the local analysis.

We begin by studying in Section 4.1 the case where the tax paid by each consumer is transferred to him in each individual state. Here the effects of taxes are analogous to the case where investment and labor supply can be directly controlled: capital should be taxed (and labor should be subsidized) exactly when there is over investment. So in the benchmark case, capital should be taxed when the heterogeneity among consumers is small, but it should rather be subsidized if the heterogeneity is large.

In Sections 4.2 and 4.3 we turn our attention to the case where the tax revenue is redistributed to each consumer via deterministic transfers, so that the tax scheme also provides some insurance and, possibly, some income redistribution among consumers. In this situation the decomposition formula, suitably extended to include the welfare effects of the transfers, tells us that the provision of insurance strengthens the case for a positive tax, especially for the factor whose income is more affected by the shocks. In contrast, the provision of redistribution tends to strengthen the case for taxing capital and weakens that for taxing labor, since the main source of income is typically capital for wealthy consumers and labor for poor consumers and transfers from rich to poor are beneficial for social welfare. Thus the sign of the optimal tax also depends on the relative importance of these two effects. We also consider in Section 4.4 the case where lump sum transfers are not available, so that the revenue of the tax on one factor is redistributed to consumers via a subsidy on the other factor. Curiously enough, we find that capital should be taxed whenever there is *under* investment in equilibrium, exactly the opposite of Section 4.1.

The final Section 6 presents some concluding remarks and a discussion of the extension of the analysis to general non-linear taxes and to an infinite horizon set up.

2 The Economy

We consider a two period competitive market economy as follows. The economic agents consist of one representative firm and I types of consumers, with a continuum of consumers of size 1 for each type.

The firm has a constant returns to scale technology described by a production function $F(K, L)$, where the output is the amount produced of the single consumption good, K is the amount of capital input, and L is the amount of labor input, all measured per capita. The inputs are in efficiency units, as is made clearer in what follows. We assume that F is smooth, homogeneous, strictly increasing, concave and satisfies the standard boundary conditions, $\lim_{K\to 0} \partial F/\partial K = \infty$ for all $L > 0$ and $\lim_{L\to 0} \partial F/\partial L = \infty$ for all $K > 0$.

The firm maximizes profits taking prices as given: writing r for the cost of capital per efficiency unit and w for the wage per efficiency unit, K and L will be chosen so that $F_K(K, L) = r$ and $F_L(K, L) = w$. The firm operates in the second period, when both the production activity and the purchases of inputs take place, although other interpretations are possible.

Consumers of the same type are ex ante identical, i.e., have the same preferences and endowment profile and consequently make the same choices in the first period. Each consumer of type i is endowed with $e_i > 0$ units of consumption good in the first period, which may be consumed or invested. If invested, it will yield some amount of the capital good next period (which may also be interpreted as human capital), to be sold to the firm at price r. Denote by k_i the amount invested by type i, thus $e_i - k_i$ is the consumption in the first period. In the second period, any type i consumer is endowed with $\bar{H}_i$ units of labor hour ($\bar{H}_i > 0$) which can be supplied in the market.

Each consumer is subject to idiosyncratic risk. For each i, denote by (Θ_i, P_i) the probability space which describes the shock affecting type i consumers. We assume that the shock is independently and identically distributed across the consumers of type i, and independently distributed across different types.

The idiosyncratic shock affects both the return of the consumers' investment and the efficiency of the consumers' labor. In state $\theta_i \in \Theta_i$, an investment of k_i units in the first period by a type i consumer yields

$$K_i^{\theta_i} := \rho_i^K(\theta_i)\, k_i \tag{1}$$

in efficiency units of capital supplied in the second period. The level of the labor supply is chosen after $\theta_i \in \Theta_i$ is realized: writing $h_i^{\theta_i}$ for the labor hours supplied after the consumer observed θ_i, the labor supply in efficiency units $L_i^{\theta_i}$ is defined by

$$L_i^{\theta_i} := \rho_i^L(\theta_i)\, h_i^{\theta_i}. \tag{2}$$

Both ρ_i^K and ρ_i^L are random variables on (Θ_i, P_i), taking strictly positive values with probability 1. To preserve the uninsurable nature of the consumers' idiosyncratic risks, we shall also assume that the two random variables ρ_i^L and ρ_i^K are comonotonic, i.e., $\left(\rho_i^L(\theta_i) - \rho_i^L(\theta_i')\right)\left(\rho_i^K(\theta_i) - \rho_i^K(\theta_i')\right) \geq 0$ for any pair of states θ_i and θ_i'. That is, if the labor endowment of a type i household is relatively large, the productivity of capital tends to be high as well. We can then say that consumer i is (relatively) rich at state θ_i if the corresponding $\rho_i^L(\theta_i)$ is (relatively) large.

We normalize units so that $\mathbf{E}\left[\rho_i^L(\theta_i)\right] = 1$ for every i and let then $\gamma_i := \mathbf{E}\left[\rho_i^K(\theta_i)\right]$. We further assume that the i.i.d. assumption of the shocks guarantees that the total supply of capital of type i consumers is given by $K_i = \mathbf{E}\left[\rho_i^K(\theta_i)\,k_i\right]$ while the total supply of labor is given by $L_i = \mathbf{E}\left[L_i^{\theta_i}\right]$ in efficiency units[3]. Hence the aggregate per-capita supply of capital K is equal to $\frac{1}{I}\sum_i K_i = \sum_i \gamma_i k_i$, and of labor is $L = \frac{1}{I}\sum_i \mathbf{E}\left[L_i^{\theta_i}\right]$.

The structure of the uncertainty thus allows both for idiosyncratic labor income risk, as in Aiyagari (1994), and idiosyncratic capital income risk, as in Angeletos (2007). Allowing for both capital and labor risks generates some symmetry and allows us to identify the role played by each type of risk in the comparative statics and welfare analysis. The special case where there is no shock to capital income, i.e. ρ_i^K is constant and only labor efficiency is subject to idiosyncratic shocks, constitutes an important benchmark and we will refer to it as the *benchmark case*. It will be the main focus of the parts of the analysis where the sign of the optimal tax rates are determined.

A type i consumer's risk preferences are represented by a time additively separable utility function: the first period utility is given by a function v_i of the first period consumption of the good, and the second period utility is given by a function u_i of the consumption of the good and leisure. So when a type i individual chooses to invest k_i and supply $h_i^{\theta_i}$ at θ_i, his consumption of the good is $e_i - k_i$ in the first period and $wL_i^{\theta_i} + rK_i^{\theta_i}$ in the second period, and his leisure consumption is $\bar{H}_i - h_i^{\theta_i}$ in the second period at state θ_i. Thus his choice problem is given as follows:

$$\max_{k_i,\left(h_i^{\theta_i}\right)_{\theta_i\in\Theta_i}} \quad v_i\left(e_i - k_i\right) + \mathbf{E}\left[u_i\left(wL_i^{\theta_i} + rK_i^{\theta_i}, \bar{H}_i - h_i^{\theta_i}\right)\right], \tag{3}$$

where $K_i^{\theta_i}$ and $L_i^{\theta_i}$ are understood as functions of k_i and $h_i^{\theta_i}$ as in (1) and (2). We assume that both v_i and u_i are smooth and concave, strictly increasing in the consumption good, non-decreasing in leisure and again that $u_i(.)$ satisfies standard boundary conditions. We also assume that the random variables are well behaved so that the first order approach is valid: i.e., we assume that the following first order condition completely characterizes the solution to the consumer's choice problem[4]:

$$-v_i'\left(e_i - k_i\right) + \mathbf{E}\left[u_{ic}\left(wL_i^{\theta_i} + rK_i^{\theta_i}, \bar{H}_i - h_i^{\theta_i}\right)\cdot r\rho_i^K\left(\theta_i\right)\right] = 0. \tag{4}$$

$$u_{ic}\left(wL_i^{\theta_i} + rK_i^{\theta_i}, \bar{H}_i - h_i^{\theta_i}\right)\rho_i^L\left(\theta_i\right)w - u_{il}\left(wL_i^{\theta_i} + rK_i^{\theta_i}, \bar{H}_i - h_i^{\theta_i}\right) = 0, \text{ at every } \theta_i, \tag{5}$$

where u_{ic} and u_{il} stand for the partial derivatives with respect to consumption and leisure, respectively. Similar convention will be used throughout the paper, e.g., u_{icc} stands for the second derivative with respect to consumption, and u_{icl} stands for the cross derivative. Furthermore, for the special case where u_{il} is always equal to zero, labor is inelastically supplied and condition (5) is replaced with

$$\rho_i^L\left(\theta_i\right)\bar{H}_i - L_i^{\theta_i} = 0 \text{ at every } \theta_i. \tag{6}$$

Note that, since all individuals of the same type solve the same strictly convex problem, their optimal decisions are also the same.

It can be readily verified that the consumption good markets clear when all the factor markets clear. So in this economy a competitive equilibrium occurs when the firm's profit maximization condition is satisfied at a level of the aggregate supply of inputs. Formally,

[3] Since the shocks are independent, the meaning of the expectation will be clear and so we shall omit the reference to the underlying measure P_i.

[4] This assumption is satisfied, for instance, if each state space is finite. Note that we allow consumers to borrow at the same rate of return of savings.

Definition 1 *A collection* $\left(\hat{w}, \hat{r}, \left(\hat{k}_i, \left(\hat{h}_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}\right)$ *constitutes a competitive equilibrium if, for each* i, $\left(\hat{k}_i, \left(\hat{h}_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)$ *is a solution to (3), and the profit maximization conditions,* $F_K\left(\hat{K}, \hat{L}\right) = \hat{r}$ *and* $F_L\left(\hat{K}, \hat{L}\right) = \hat{w}$, *hold for* $\hat{K} = \frac{1}{I}\sum_{i=1}^{I}\gamma_i \hat{k}_i$ *and* $\hat{L} = \frac{1}{I}\sum_{i=1}^{I}\mathbf{E}\left[\hat{L}_i^{\theta_i}\right]$.

By construction, equilibrium labor hours $\left(\hat{h}_i^{\theta_i}\right)_{\theta_i \in \Theta_i}$ for every type i consumer are a function of the realization of θ_i, so are capital in efficiency units, second period consumption, and labor supply in efficiency units. Let, in particular, $\hat{c}_i^{\theta_i} := \hat{w}\hat{L}_i^{\theta_i} + \hat{r}\hat{K}_i^{\theta_i}$ and $l_i^{\theta_i} := \bar{H}_i - h_i^{\theta_i}$. The comparative statics properties of the equilibrium and the sign of the optimal taxes depend on how these variables vary with respect to θ_i, $i = 1, .., I$. In this regard, we shall focus on the following case:

Definition 2 *A competitive equilibrium* $\left(\hat{w}, \hat{r}, \left(\hat{k}_i, \left(\hat{h}_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}\right)$ *is said to exhibit a* ***standard response to shocks*** *(in short, is* ***standard to shocks****) if, for every* i:
i) $u_{ic}\left(\hat{c}_i^{\theta_i}, \hat{l}_i^{\theta_i}\right)$ *is decreasing in* $\rho_i^L(\theta_i)$*; i.e.,* $u_{ic}\left(\hat{c}_i^{\theta_i}, \hat{l}_i^{\theta_i}\right) < u_{ic}\left(\hat{c}_i^{\theta_i'}, \hat{l}_i^{\theta_i'}\right)$ *holds whenever* $\rho_i^L(\theta_i') < \rho_i^L(\theta_i)$.
ii) $\hat{K}_i^{\theta_i}$ *is non-decreasing and* $\hat{L}_i^{\theta_i}$ *is increasing in* $\rho_i^L(\theta_i)$.

We argue that a competitive equilibrium is standard to shocks in 'normal cases'. Regarding condition i), intuitively speaking, a high realization of $\rho_i^L(\theta_i)$ implies that the consumer is rich ex post, and so consumption should be relatively high and hence the marginal utility from consumption should be relatively low. Also, the amount of labor in efficiency units should be relatively high. Formally (the proof, applying usual consumer theory, is provided for completeness in the Appendix):

Lemma 1 *Assume that* u_i *is strictly concave* $(u_{icc}u_{ill} - (u_{icl})^2 > 0$ *everywhere) and consumption is a normal good* $(u_{icc}u_{il} - u_{icl}u_{ic} < 0$ *everywhere). Then in any competitive equilibrium* $\left(\hat{w}, \hat{r}, \left(\hat{k}_i, \left(\hat{h}_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}\right)$, *condition i) of Definition 2 holds.*

For condition ii), let us focus on the benchmark case where $\rho_i^K(\theta_i)$ is constant, so $\hat{K}_i^{\theta_i}$ is non-decreasing trivially. A higher level of $\rho_i^L(\theta_i)$ means a higher effective wage which induces more labor, but it also generates a higher income from capital which induces more leisure. So $\hat{L}_i^{\theta_i}$ should be increasing when the income effect from the higher revenue from the capital investment is not excessively large. A formal sufficient condition is stated in the following (also proved in the Appendix):

Lemma 2 *Assume that* $\rho_i^K(\theta_i) = \gamma_i$ *for all* $\theta_i \in \Theta_i$, *and that* u_i *is strictly concave and leisure is a normal good* $(u_{ill}u_{ic} - u_{icl}u_{il} < 0$ *everywhere). Then in any competitive equilibrium* $\left(\hat{w}, \hat{r}, \left(\hat{k}_i, \left(\hat{h}_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}\right)$, *condition ii) of Definition 2 holds.*

An alternative sufficient condition is the case of inelastic labor hour supply. Indeed in such case condition ii) is obviously satisfied, and i) holds since u_{icc} is negative and independent of $l_i^{\theta_i}$.[5]

[5] Davila et al. (2012) consider the case where labor supply is inelastic and $\rho_i^K(\theta_i)$ is constant, thus an equilibrium in their set up is automatically standard to shocks.

3 Welfare Analysis of Price Changes

We present here the key tool for our welfare analysis of taxation, the **decomposition result**. We will see that, around a competitive equilibrium, the first order welfare effects of the introduction of a tax scheme can be decomposed in a simple and economically meaningful way.

Taxation in our setup is ultimately an instrument which affects individual choices of capital and labor. So suppose the social planner could directly control the amounts of investment, k_i, as well as of labor hours, $h_i^{\theta_i}$, for all consumers in every state θ_i. That is, the policy instruments available to the planner can be identified with a tuple $\left(k_i, \left(h_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}$. This may be interpreted as an extreme form of personalized non-linear taxation without direct income transfers. All the other, non-policy variables are determined in competitive markets: the planner cannot control the firm's decisions, and so the prices r and w are endogenously determined in such a way that the firm's demand for inputs equal the aggregate supply of inputs set by the planner. Thus, the associated market clearing prices are given by $r(K,L) = F_K(K,L)$ and $w(K,L) = F_L(K,L)$, for $K = \frac{1}{I}\sum_{i=1}^{I} \gamma_i k_i$ and $L = \frac{1}{I}\sum_{i=1}^{I} \mathbf{E}\left[\rho_i^L(\theta_i) h_i^{\theta_i}\right]$,

A feasible policy must warrant non-negative consumption to every consumer. So a policy $\left(k_i, \left(h_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}$ is said to be **feasible** if for every i, we have non-negative first period consumption $e_i - k_i \geq 0$, non-negative second period leisure and consumption, $\bar{H}_i - h_i^{\theta_i} \geq 0$ and $w(K,L) L_i^{\theta_i} + r(K,L) K_i^{\theta_i} \geq 0$ at every θ_i. By construction, the utility level of a type i consumer induced by a feasible policy $\left(k_i, \left(h_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}$ with the associated aggregate quantities, K and L, is given by:

$$U_i\left(\left(k_i, \left(h_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}\right) := v_i(e_i - k_i) + \mathbf{E}\left[u_i\left(w(K,L) L_i^{\theta_i} + r(K,L) K_i^{\theta_i}, \bar{H}_i - h_i^{\theta_i}\right)\right]. \quad (7)$$

From now on, fix a competitive equilibrium $\left(\hat{w}, \hat{r}, \left(\hat{k}_i, \left(\hat{h}_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}\right)$. Clearly, $\left(\hat{k}_i, \left(\hat{h}_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}$ is a feasible policy. We shall study how the agents' expected utility level given in (7) behaves with marginal changes in policy, differentiating the equilibrium values with respect to the policy and evaluating them at the equilibrium values. Notice that a change in policy has two effects on U_i: the first is of course the direct effect of the change in the values of k_i and $\left(h_i^{\theta_i}\right)_{\theta_i \in \Theta_i}$; the second is an indirect effect due to the change in the values of the equilibrium prices r and w. At a competitive equilibrium, however, the direct effect has no first order effect on welfare by the envelope property; in view of (3), the values $\hat{k}_i$ and $\left(\hat{h}_i^{\theta_i}\right)_{\theta_i \in \Theta_i}$ already maximize the utility of consumer i at the prices $(\hat{w}, \hat{r})$. Therefore, the only first order welfare effect of the policy change is the indirect effect, that is, only the *pecuniary externality* of the change in prices. This in particular means that the only welfare effect of the policy change is the change in aggregate variables, K and L, via its effect on equilibrium prices. In conclusion, any policy changes which induce the same values of K and L are equivalent in this context.

For this reason, *as long as we are concerned with the first order effects of a policy change evaluated at an equilibrium allocation,* we can take U_i in (7) as a function of K and L only, taking $\left(k_i, \left(h_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}$, and so also $L_i^{\theta_i}, K_i^{\theta_i}$, as fixed constants, equal to the equilibrium values. Since the first period utility v_i does not depend on (K,L), the marginal changes in U_i coincides with the partial derivative of the second period expected utility with respect to (K,L). For a change in K

this is given by:

$$\begin{aligned}
\left.\frac{\partial U_i}{\partial K}\right|_{(\hat{K},\hat{L})} &= \left.\frac{\partial}{\partial K}\mathbf{E}\left[u_i\left(w\left(K,L\right)\hat{L}_i^{\theta_i}+r\left(K,L\right)\hat{K}_i^{\theta_i},\bar{H}_i-\hat{h}_i^{\theta_i}\right)\right]\right|_{(\hat{K},\hat{L})} \\
&= \mathbf{E}\left[u_{ic}\cdot\left(\frac{\partial w}{\partial K}\cdot\hat{L}_i^{\theta_i}+\frac{\partial r}{\partial K}\cdot\hat{K}_i^{\theta_i}\right)\right], \qquad (8) \\
&= \mathbf{E}\left\{u_{ic}\cdot\left[\left(\frac{\partial w}{\partial K}\cdot\hat{L}_i^{\theta_i}+\frac{\partial r}{\partial K}\cdot\hat{K}_i^{\theta_i}\right)-\left(\frac{\partial w}{\partial K}\cdot\hat{L}+\frac{\partial r}{\partial K}\cdot\hat{K}\right)\right]\right\} \\
&= \left\{\mathbf{E}\left[u_{ic}\cdot\left(\hat{K}_i^{\theta_i}-\hat{K}_i\right)\right]+\mathbf{E}\left[u_{ic}\right]\left(\hat{K}_i-\hat{K}\right)\right\}\frac{\partial r}{\partial K} \qquad (9) \\
&\quad +\left\{\mathbf{E}\left[u_{ic}\cdot\left(\hat{L}_i^{\theta_i}-\hat{L}_i\right)\right]+\mathbf{E}\left[u_{ic}\right]\left(\hat{L}_i-\hat{L}\right)\right\}\frac{\partial w}{\partial K},
\end{aligned}$$

where u_{ic} is evaluated at the equilibrium levels of leisure $\bar{H}_i-\hat{h}_i^{\theta_i}$ and consumption $\hat{w}\hat{L}_i^{\theta_i}+\hat{r}\hat{K}_i^{\theta_i}$ in the second period, and $\frac{\partial w}{\partial K}$ and $\frac{\partial r}{\partial K}$ are both evaluated at $\left(\hat{K},\hat{L}\right)$. A similar convention will be used throughout the paper. In the third step above, we used the relation

$$\frac{\partial r}{\partial K}\cdot K+\frac{\partial w}{\partial K}\cdot L=0, \qquad (10)$$

which follows since $F_K\left(K,L\right)=r\left(K,L\right)$ and $F_K\left(K,L\right)K+F_L\left(K,L\right)L=F\left(K,L\right)$ by the homogeneity of F.

Applying the same logic to a change in L, utilizing

$$\frac{\partial r}{\partial L}\cdot K+\frac{\partial w}{\partial L}\cdot L=0, \qquad (11)$$

we have:

$$\begin{aligned}
\left.\frac{\partial U_i}{\partial L}\right|_{(\hat{K},\hat{L})} &= \mathbf{E}\left[u_{ic}\cdot\left(\frac{\partial w}{\partial L}\cdot\hat{L}_i^{\theta_i}+\frac{\partial r}{\partial L}\cdot\hat{K}_i^{\theta_i}\right)\right] \qquad (12) \\
&= \left\{\mathbf{E}\left[u_{ic}\cdot\left(\hat{K}_i^{\theta_i}-\hat{K}_i\right)\right]+\mathbf{E}\left[u_{ic}\right]\left(\hat{K}_i-\hat{K}\right)\right\}\frac{\partial r}{\partial L} \qquad (13) \\
&\quad +\left\{\mathbf{E}\left[u_{ic}\cdot\left(\hat{L}_i^{\theta_i}-\hat{L}_i\right)\right]+\mathbf{E}\left[u_{ic}\right]\left(\hat{L}_i-\hat{L}\right)\right\}\frac{\partial w}{\partial L}
\end{aligned}$$

3.1 The decomposition result

The terms $\mathbf{E}\left[u_{ic}\cdot\left(\hat{K}_i^{\theta_i}-\hat{K}_i\right)\right]$ and $\mathbf{E}\left[u_{ic}\cdot\left(\hat{L}_i^{\theta_i}-\hat{L}_i\right)\right]$ in (9) and (13) describe the relationship between an agent's marginal utility and the idiosyncratic shocks affecting him. To get an economic interpretation, imagine an agent with VNM utility v and consider the marginal benefit of insuring a risky source of income Y by substituting an infinitesimal part of it with its average $\mathbf{E}Y$:

$$\left.\frac{d\mathbf{E}\left[v\left(\left(1-t\right)Y+t\mathbf{E}Y\right)\right]}{dt}\right|_{t=0}=-\mathbf{E}\left[v'\left(Y\right)\left(Y-\mathbf{E}Y\right)\right].$$

Capital and labor income are risky ways to provide for agents' second period consumption, hence a reduction of r (or w) has a similar effect, reducing the risk faced by the agent. So in what follows, we shall refer to these terms as *insurance effect* and use the following short-hand notation:

$$I_i^K:=\mathbf{E}\left[u_{ic}\cdot\left(\hat{K}_i^{\theta_i}-\hat{K}_i\right)\right], \qquad (14)$$

$$I_i^L:=\mathbf{E}\left[u_{ic}\cdot\left(\hat{L}_i^{\theta_i}-\hat{L}_i\right)\right], \qquad (15)$$

where I stands for "insurance". Note that their magnitude depends on the size of income risks borne by each consumer at the equilibrium: when individual shocks are fully insured, e.g. $\hat{K}_i^{\theta_i}=\hat{K}_i$ for all θ_i, the corresponding term, I_i^K, is zero.

Lemma 3 *At an equilibrium which is standard to shocks the insurance effects are negative:* $I_i^K \leq 0$ *and* $I_i^L < 0$.

Proof. By the definition of the standard response to shocks, u_{ic} and $\hat{K}_i^{\theta_i}, \hat{L}_i^{\theta_i}$ move in the opposite direction when θ_i varies, hence $COV\left[u_{ic} \cdot \left(\hat{K}_i^{\theta_i} - \hat{K}_i\right)\right] \leq 0$ and $COV\left[u_{ic} \cdot \left(\hat{L}_i^{\theta_i} - \hat{L}_i\right)\right] < 0$. Recall that for two random variables X and Y, we have $\mathbf{E}(XY) = \mathbf{E}(X)\mathbf{E}(Y) + COV(X,Y)$. Here $\mathbf{E}\left(K_i^{\theta_i} - K_i\right) = 0 = \mathbf{E}\left(\hat{L}_i^{\theta_i} - \hat{L}_i\right)$ by construction, so the negative correlation implies that $\mathbf{E}\left[u_{ic} \cdot \left(\hat{K}_i^{\theta_i} - \hat{K}_i\right)\right] \leq 0$ and $\mathbf{E}\left[u_{ic} \cdot \left(\hat{L}_i^{\theta_i} - \hat{L}_i\right)\right] < 0$ hold. ■

The remaining terms in (9), $\mathbf{E}\left[u_{ic}\right]\left(\hat{K}_i - \hat{K}\right)$ and $\mathbf{E}\left[u_{ic}\right]\left(\hat{L}_i - \hat{L}\right)$, are the deviations of type i per capita supply of capital and labor from the aggregate per capita supply, weighted by the expected marginal utility of these consumers. These terms measure the effect on type i's utility of the redistribution of income induced by the change in factor prices, determined by the relative size of i's trades with respect to those of the whole economy, i.e., the relative position of type i in the market. Thus, for future reference, we shall denote them *distribution effect* and write:

$$D_i^K := \mathbf{E}\left[u_{ic}\right]\left(\hat{K}_i - \hat{K}\right), \tag{16}$$

$$D_i^L := \mathbf{E}\left[u_{ic}\right]\left(\hat{L}_i - \hat{L}\right), \tag{17}$$

where D stands for "distribution". Evidently, when the economy consists of ex ante homogeneous types, or $I = 1$, these terms will be zero and their magnitude depends on the degree of heterogeneity among consumers in the economy at the equilibrium.

Consider then the welfare effects of a change in K: this modifies both factor prices, hence the *total* insurance (distribution) effect due to a change in K is given by the sum of the insurance (distribution) effects of the two price changes, each multiplied by the corresponding change in prices. From (9) and (13), by substituting $\frac{\partial r}{\partial K}$, $\frac{\partial w}{\partial K}$ with F_{KK}, F_{KL} and $\frac{\partial r}{\partial L}$, $\frac{\partial w}{\partial L}$ with F_{LK}, F_{LL}, using (14) - (17) and (10), (11) we obtain so the following **decomposition result**:

Proposition 4 *The first order effect on the welfare of type i consumers of a change in policy at a competitive equilibrium can be decomposed into a total insurance effect and a total distribution effect as follows:*

$$\left.\frac{\partial U_i}{\partial K}\right|_{(\hat{K},\hat{L})} = \left(I_i^K + D_i^K\right) F_{KK} + \left(I_i^L + D_i^L\right) F_{KL} = \hat{K} F_{KK} \left\{ \left(\frac{I_i^K}{\hat{K}} - \frac{I_i^L}{\hat{L}} \right) + \left(\frac{D_i^K}{\hat{K}} - \frac{D_i^L}{\hat{L}} \right) \right\}, \tag{18}$$

$$\left.\frac{\partial U_i}{\partial L}\right|_{(\hat{K},\hat{L})} = \left(I_i^K + D_i^K\right) F_{LK} + \left(I_i^L + D_i^L\right) F_{LL} = \hat{L} F_{LL} \left\{ \left(\frac{I_i^L}{\hat{L}} - \frac{I_i^K}{\hat{K}} \right) + \left(\frac{D_i^L}{\hat{L}} - \frac{D_i^K}{\hat{K}} \right) \right\}. \tag{19}$$

where all terms are evaluated at the equilibrium values.

This decomposition result provides a number of economic insights and we can use it to obtain sharp results on the sign of the welfare changes. Note, first of all, that those consumers who favor a decrease in the stock of capital are exactly those who favor an increase in the amount of labor:

Corollary 5 *For any type i,* $\frac{\partial U_i}{\partial K} \leq 0$ *if and only if* $\frac{\partial U_i}{\partial L} \geq 0$.

Proof. Compare (18) and (19): both F_{KK} and F_{LL} are negative by assumption, and the terms multiplying them are identical but with opposite sign. ■

Consider next the insurance effect associated to either a decrease in r or in w : when the equilibrium is standard to shocks, this effect yields a *unanimous* increase in individual welfare, since $I_i^K \leq 0$ and $I_i^L < 0$ for every i by Lemma 3. However a change in K has the opposite effect on factor prices w and r. Its sign depends on the relative importance of the two price changes and the two distribution effects. Roughly speaking, the riskier factor will matter more, because its insurance term is larger in magnitude. In particular, in the benchmark case of no capital accumulation shock, $I_i^K = 0$, and thus a marginal increase in aggregate investment will *reduce* the induced utility of *every* household since $\frac{\partial w}{\partial K} > 0$. So as far as the insurance effect is concerned, the households unanimously prefer a *reduction* of capital.

To find the overall effect on welfare we need then to consider also the distribution effects. We can say that when the (absolute) magnitude of the distribution effects is bigger than that of the insurance effects (always true in the benchmark case where $I_i^K = 0$ for every i), a reduction of the stock of capital benefits those types who invest more and work less than the economy average.

Corollary 6 *When* $\left|I_i^K\right| < \left|D_i^K\right|$, $\frac{\partial U_i}{\partial K} < 0$ *holds if* $\hat{K}_i > \hat{K}$ *and* $\hat{L}_i < \hat{L}$. *When* $\left|I_i^L\right| < \left|D_i^L\right|$, $\frac{\partial U_i}{\partial K} > 0$ *if* $\hat{K}_i < \hat{K}$ *and* $\hat{L}_i > \hat{L}$.

Proof. Note first that $\hat{K}_i - \hat{K} > 0$ and $\hat{L}_i - \hat{L} < 0$ imply that $D_i^K > 0$ and $D_i^L < 0$. Hence by Proposition 4 $\frac{\partial U_i}{\partial K} < 0$ follows when $\left|I_i^K\right| < \left|D_i^K\right|$, since $F_{KK} < 0$ and $F_{KL} > 0$. The second claim is established by a symmetric argument: when $\hat{K}_i - \hat{K} < 0$ and $\hat{L}_i - \hat{L} > 0$ we have $D_i^K < 0$ and $D_i^L > 0$ and hence $\frac{\partial U_i}{\partial K} > 0$ follows from $\left|I_i^L\right| < \left|D_i^L\right|$. ■

3.2 Effects on Social Welfare

We saw in the previous section that while the insurance effect has typically the same sign across agents, this is not true for the distribution effect, since its sign depends on the relative size of an agent's trade with respect to the average trades in the market. Hence we cannot expect a change in K or L to generate a Pareto improvement when agents are sufficiently heterogeneous.[6] We shall use then in what follows the utilitarian criterion with the social welfare function defined as the average utility $W\left(\left(k_i, \left(h_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}\right) := \frac{1}{I}\sum_{i=1}^{I} U_i\left(\left(k_i, \left(h_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}\right)$ to analyze the welfare effects of a change in K or L. This welfare function may be viewed as evaluation "under the veil of ignorance" before the type of an agent is determined, where an arbitrary consumer may be assigned to any type i with equal probability.

It is a benefit of the utilitarian function that the decomposition result readily extends by simply summing up the derivatives of the individual utility functions found in Proposition 4 as well as (18) and (19) as follows:

$$\left.\frac{\partial W}{\partial K}\right|_{(\hat{K},\hat{L})} = \hat{K} F_{KK} \sum_i \frac{1}{I}\left[\left(\frac{I_i^K}{\hat{K}} - \frac{I_i^L}{\hat{L}}\right) + \left(\frac{D_i^K}{\hat{K}} - \frac{D_i^L}{\hat{L}}\right)\right], \tag{20}$$

$$\left.\frac{\partial W}{\partial L}\right|_{(\hat{K},\hat{L})} = \hat{L} F_{LL} \sum_i \frac{1}{I}\left[\left(\frac{I_i^L}{\hat{L}} - \frac{I_i^K}{\hat{K}}\right) + \left(\frac{D_i^L}{\hat{L}} - \frac{D_i^K}{\hat{K}}\right)\right]. \tag{21}$$

With this, we are now ready to provide a rigorous and precise definition of over/under investment.

[6] The general constrained inefficiency result of Citanna - Kajii - Villanacci (1998) basically shows that with incomplete markets a competitive equilibrium can be Pareto improved by the first order effects if the planner has at least as many policy tools as the number of households plus one. In our framework, the number of policy tools which can have first order welfare effects is effectively two, K and L, whatever the number of households.

Definition 3 *A competitive equilibrium exhibits* ***over investment*** *if* $\frac{\partial W}{\partial K} < 0$*, and under investment if* $\frac{\partial W}{\partial K} > 0$*. Similarly, there is* ***under supply*** *of labor if* $\frac{\partial W}{\partial L} > 0$ *and over supply of labor if* $\frac{\partial W}{\partial L} < 0$.

Even though the distribution effect terms have different signs for different individuals, the average values $\frac{1}{I}\sum_i D_i^K$ and $\frac{1}{I}\sum_i D_i^L$ can be signed. To see this, think of assigning a type i to an agent at random; $\hat{K}_i$, $\hat{L}_i$ and $\mathbf{E}[u_{ic}]$ can thus be regarded as random variables over states $i = 1, .., I$ which are equally likely. Since $\sum_i \left(\hat{K}_i - \hat{K}\right) = 0$ and $\sum_i \left(\hat{L}_i - \hat{L}\right) = 0$, we have $\frac{1}{I}\sum_i D_i^K = Cov\left[\hat{K}_i, \mathbf{E}(u_{ic})\right]$ and $\frac{1}{I}\sum_i D_i^L = Cov\left[\hat{L}_i, \mathbf{E}(u_{ic})\right]$ by construction. We should therefore expect that at a competitive equilibrium the relatively "rich" types of households, whose consumption level tends to be higher than the economy average, also tend to invest more than the average and work less than the average. This property relies on some normality of consumers' demands and so we shall use again the term 'standard' to refer to it:

Definition 4 *A competitive equilibrium is said be* ***standard in distribution*** *if* $\mathbf{E}[u_{ic}]$ *is negatively correlated with* $\hat{K}_i$ *and positively correlated with* $\hat{L}_i$.

When the equilibrium is standard both to shocks and in distribution, we simply call it a **standard equilibrium**. An immediate implication of the above property is as follows:

Lemma 7 *If* $I > 1$*, in an equilibrium standard in distribution the average distribution effect for capital is negative and that for labor is positive: i.e.,* $\frac{1}{I}\sum_i D_i^K < 0$ *and* $\frac{1}{I}\sum_i D_i^L > 0$.

Hence at a standard equilibrium, the total average distribution effect of an increase in K, given by the first term of (20), has a positive sign. On the other hand, in the benchmark case the total average insurance effect has a negative sign in (20) since $I_i^L F_{KL} < 0$ for all i by Lemma 3.

Thus there is a clear trade-off: whether there is under or over investment depends on the relative size of the total average distribution effect over the total average insurance effect. Intuitively, the distribution effect gets magnified as the heterogeneity of income across types of households increases, whereas the insurance effect has no direct link to this heterogeneity. It means that we should expect to see over investment when the income disparity is small, and under investment when it is large enough: subsidizing capital might be welfare improving indeed. Formally, we have the following[7]:

Proposition 8 *Assume that* $\rho_i^K(\theta_i) = \gamma_i$ *for all* $\theta_i \in \Theta_i$*. A standard equilibrium exhibits over investment if the average distribution effect is smaller than the average insurance effect in the sense that* $\sum_i \left|\frac{I_i^L}{L}\right| > \left|\sum_i \left(\frac{D_i^K}{K} - \frac{D_i^L}{L}\right)\right|$*. It exhibits under investment if the reverse inequality holds. In particular, if* $I = 1$*, a standard equilibrium exhibits over investment.*

Proof. The fact that the productivity of the investment is not subject to idiosyncratic shocks implies that $I_i^K = 0$ for every i. Using (20), we obtain that $\frac{\partial W}{\partial K} < 0$ if and only if $\sum_i \left(-\frac{I_i^L}{L} + \left(\frac{D_i^K}{K} - \frac{D_i^L}{L}\right)\right) > 0$. Since $I_L^i < 0$ by Lemma 3 and $\sum_i D_i^K < 0$ and $\sum_i D_i^L > 0$ by Lemma 7, the latter inequality is equivalent to $\sum_i \left|\frac{I_i^L}{L}\right| > \left|\sum_i \left(\frac{D_i^K}{K} - \frac{D_i^L}{L}\right)\right|$. If $I = 1$, the distribution terms vanish by construction, so the condition holds trivially. ■

Remark 1 *The notion of over investment as well as the underlying logic of Proposition 8 are rather different from those in Aiyagari (1995). To see this, notice that our results hold irrespective*

[7]The special case of $I = 1$ constitues a generalization of a constrained inefficiency result in Davila et al. (2012), which assumes inelastically supplied labor.

of whether a precautionary motive is present or not and the level of the equilibrium interest rate - whether it is lower or higher than when agents are able to trade in a complete set of contingent markets at the initial date - plays no role. What is crucial, on the other hand, is the nature of the shocks, whether they hit primarily capital or labor income.

Remark 2 *Notice that our local characterization results go through even when there are no shocks at all, so that markets are complete. In this special case, competitive equilibria are Pareto efficient and so a change in capital or labor will result in an efficiency loss. Ex ante welfare can be improved nonetheless, since redistribution from relatively rich to relatively poor types provides consumers with some insurance against 'bad' realizations of their type.*

4 Optimal Taxation

We consider now the optimal taxation problem where the policy instruments are anonymous, linear taxes on labor and capital, with the net revenue of such taxes redistributed to consumers via lump sum taxes or transfers. In this case, consumers choose optimally the level of their investment and labor supply, while the firm still chooses the level of its inputs so as to maximize profits and prices r and w are set at a level such that markets clear. An optimal tax scheme maximizes the social welfare function under these conditions.

Our goal in this section is to study the welfare effects of introducing, at the margin, a tax and subsidy scheme. This will allow us to conclude that at least locally a positive/negative tax on capital/labor is welfare improving upon the equilibrium. So throughout this section, we shall fix a competitive equilibrium $\left(\hat{w}, \hat{r}, \left(\hat{k}_i, \left(\hat{h}_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}\right)$. We will denote by τ_K the tax rate on the revenue from the investment in capital, by τ_L the tax rate on labor income, and by $T_i(\theta_i)$ the lump sum transfer received by consumer i in state θ_i. Various scenarios can be considered, in particular concerning the specific definition of the transfer policy $T_i(\theta_i)$, and we shall study a few cases which will clarify the essence of the optimal taxation problem.

When consumer i invests k_i and chooses $h_i^{\theta_i}$ in each state θ_i, an amount $w\tau_L L_i^{\theta_i} + r\tau_K K_i^{\theta_i}$ of his revenue next period in state θ_i must be paid in taxes. The choice problem of a type i consumer is therefore as follows:

$$\max_{k_i, \left(h_i^{\theta_i}\right)_{\theta_i \in \Theta_i}} \quad v_i(e_i - k_i) + \mathbf{E}\left[u_i\left(w(1-\tau_L) L_i^{\theta_i} + r(1-\tau_K) K_i^{\theta_i} + T_i(\theta_i), \bar{H}_i - h_i^{\theta_i}\right)\right], \tag{22}$$

where $K_i^{\theta_i}$ and $L_i^{\theta_i}$ are still as defined in (1) and (2). The maximization problem (22) remains a concave problem, and so the following first order conditions characterize its solutions:

$$-v_i'(e_i - k_i) + \mathbf{E}\left[u_{ic} \cdot \rho_i^K(\theta_i)\, r(1-\tau_K)\right] = 0. \tag{23}$$

$$u_{ic} \cdot w(1-\tau_L)\, \rho_i^L(\theta_i) - u_{il} = 0, \text{ at every } \theta_i, \tag{24}$$

where the derivatives of u_i are evaluated at $\left(w(1-\tau_L) L_i^{\theta_i} + r(1-\tau_K) K_i^{\theta_i} + T_i(\theta_i), \bar{H}_i - h_i^{\theta_i}\right)_{\theta_i \in \Theta_i}$.

We will derive decomposition results analogous to Proposition 4 for the welfare effects of the various tax schemes considered. Notice that the terms I_i^K, D_i^K , and I_i^L, D_i^L in Proposition 4 describe the marginal effects on agents' utility of unit changes in prices at the equilibrium under consideration. The marginal change in factor prices then depends on the change in the levels of K and L induced by the policy. The lump sum transfer scheme may then affect the income distribution and generate additional welfare effects.

4.1 Taxes with no redistribution of income

In order to isolate the pure substitution effect of the tax, we shall consider first a tax-subsidy scheme which does not induce any redistribution of income across different types of agents nor even across realizations of the idiosyncratic state θ_i. That is, the lump sum transfer is both agent and state specific so that the taxes paid in state θ_i by an agent is returned as a lump sum transfer in that same state. The following budget balance condition then holds for every realization of θ_i:

$$w\tau_L L_i^{\theta_i} + r\tau_K K_i^{\theta_i} = T_i(\theta_i), \tag{25}$$

for every i. The implementation of this tax scheme is informationally very demanding since it requires knowledge of individual trades and of the realization θ_i of the idiosyncratic shocks, which may be private information of the agent. However, it provides a useful theoretical starting point for the subsequent analyses.

Definition 5 *A* tax equilibrium with no redistribution of income *is a collection* $(w, r, (\tau_K, \tau_L), \left(T_i(\cdot), k_i, \left(h_i^{\theta_i}\right)_{\theta_i\in\Theta_i}\right)_{i=1}^{I})$ *such that: i) for each i,* $\left(k_i, \left(h_i^{\theta_i}\right)_{\theta_i\in\Theta_i}\right)$ *satisfies (23), (24), ii) profit maximization holds, i.e.,* $F_K(K,L) = r$, $F_L(K,L) = w$, *iii) markets clear,* $K = \frac{1}{I}\sum_{i=1}^{I}\gamma_i k_i$ *and* $L = \frac{1}{I}\sum_{i=1}^{I} E\left[L_i^{\theta_i}\right]$, *and iv) the budget balance (25) holds, for each i at every* θ_i.

Social welfare can then be written, analogously to the previous section, as a function of the policy instruments, given now by the tax rates (τ_K, τ_L):

$$W(\tau_K, \tau_L) := \sum_{i=1}^{I}\frac{1}{I}\left\{v_i(e_i - k_i) + \mathbf{E}\left[u_i\left(wL_i^{\theta_i} + rK_i^{\theta_i}, \bar{H}_i - h_i^{\theta_i}\right)\right]\right\}, \tag{26}$$

where the variables are determined as in Definition 5.

By construction, a competitive equilibrium is a tax equilibrium, where $\tau_K = \tau_L = 0$, and $T_i(\cdot) \equiv 0$ for every i. Our goal is to find whether social welfare is increasing in τ_K and/or τ_L at $\tau_K = \tau_L = 0$. To this end we assume the variables at an equilibrium with taxes are smooth functions of (τ_K, τ_L) around $(\tau_K, \tau_L) = 0$.[8] We shall say that capital should be taxed if $\frac{\partial}{\partial\tau_K}W(\tau_K, \tau_L) > 0$ and it should be subsidized if $\frac{\partial}{\partial\tau_K}W(\tau_K, \tau_L) < 0$. A similar analysis can be done for labor.

Since individual income in every state is not affected by the tax scheme and the envelope property again implies that the marginal changes of individual choices have no welfare effect, the first order effect on welfare of the scheme is only due to the change in equilibrium prices. Therefore, as far as the local welfare analysis is concerned, the effects of the tax policy are the same as those obtained in the previous section, except that the changes in equilibrium prices are induced by a change in τ_K and τ_L, not by a direct change of K and L:

Proposition 9 *The (first order) welfare effects of taxes with no redistribution of income at a competitive equilibrium can be decomposed as follows:*

$$\left.\frac{\partial W}{\partial\tau_K}\right|_{\tau=0} = \hat{K}\frac{\partial r}{\partial\tau_K}\sum_i\frac{1}{I}\left\{\left(\frac{I_i^K}{\hat{K}} - \frac{I_i^L}{\hat{L}}\right) + \left(\frac{D_i^K}{\hat{K}} - \frac{D_i^L}{\hat{L}}\right)\right\}, \tag{27}$$

$$\left.\frac{\partial W}{\partial\tau_L}\right|_{\tau=0} = \hat{L}\frac{\partial w}{\partial\tau_L}\sum_i\frac{1}{I}\left\{\left(\frac{I_i^L}{\hat{L}} - \frac{I_i^K}{\hat{K}}\right) + \left(\frac{D_i^L}{\hat{L}} - \frac{D_i^K}{\hat{K}}\right)\right\}, \tag{28}$$

where the terms I_i^K, I_i^L, D_i^K, *and* D_i^L *are the same as in (14), (15), (16), and (17).*

[8] This will be generically the case at least if the underlying state space is finite. The number of equilibrium variables exceeds in fact the number of equations defining an equilibrium by two.

It remains so to identify the signs of the changes in equilibrium prices in response to the tax change. In general, prices could move in any direction, depending on the signs of the derivatives of the excess demand functions for capital and labor. We shall consider the case where prices change in the *natural direction* as in the standard case of a market equilibrium with downward sloping demand and upward sloping supply curves: at the margin, an increase in the tax on the revenue from the sale of an input increases the gross unit revenue (i.e., cum tax) of the input but reduces the net unit revenue (i.e., net of tax)[9]; that is:

$$\left.\frac{\partial r(\tau_K,\tau_L)}{\partial \tau_K}\right|_{\tau=0} > 0, \quad \left.\frac{\partial}{\partial \tau_K}\left[(1-\tau_K)r(\tau_K,\tau_L)\right]\right|_{\tau=0} < 0 \tag{29}$$

$$\left.\frac{\partial w(\tau_K,\tau_L)}{\partial \tau_L}\right|_{\tau=0} > 0, \quad \left.\frac{\partial}{\partial \tau_L}\left[(1-\tau_L)w(\tau_K,\tau_L)\right]\right|_{\tau=0} < 0 \tag{30}$$

We shall refer to (29) and (30) as the *natural signs* for the changes in equilibrium factor prices.

We obtain so a corollary similar to Corollary 5, which says whenever it is good to tax capital, it should be good to subsidize labor as well.

Corollary 10 *Assume the natural signs as above. Then taxing capital and taxing labor has opposite effects on welfare:* $\frac{\partial W}{\partial \tau_K} \geq 0$ *if and only if* $\frac{\partial W}{\partial \tau_L} \leq 0$.

We shall therefore focus our attention on identifying the conditions under which capital should be taxed. The next result is an analogue of Proposition 8:

Proposition 11 *Assume the natural signs (29) and (30). Then capital should be taxed with personal transfers at a standard competitive equilibrium if and only if there is over investment. Suppose, in addition, that* $\rho_i^K(\theta_i) = \gamma_i$ *for all* $\theta_i \in \Theta_i$ *and the competitive equilibrium is standard. Then capital should be taxed if* $\sum_i \left|\frac{I_i^L}{L}\right| > \left|\sum_i \left(\frac{D_i^K}{K} - \frac{D_i^L}{L}\right)\right|$ *and subsidized if the reverse inequality holds; when* $I = 1$*, capital should always be taxed.*

Proof. Under (29), $\partial r/\partial \tau_K$ has always the opposite sign of $\partial r/\partial K$. Hence the same is true for the expression for $\frac{\partial W}{\partial \tau_K}$ in (27) and that for $\frac{\partial W}{\partial K}$ in (20), which establishes the first part. Given this, the second part is an immediate corollary of Proposition 8. ■

4.2 Type dependent lump-sum transfers as insurance

We consider next the case of type dependent lump sum transfers. The amount T_i of transfer per capita is type dependent and equal to the per capita tax paid by type i consumers, a deterministic amount by the i.i.d. assumption:

$$w\tau_L L_i + r\tau_K K_i = T_i, \tag{31}$$

for every i. Notice that this form of transfer provides some insurance to the agents, since the transfer received exceeds the tax paid whenever the return on capital and labor is below its expected value, and is smaller in the opposite case. Although consumers' types need to be observable for this scheme, the informational requirement is less demanding than in the previous case since the transfer is independent of the individual shock.

An *equilibrium with type dependent lump-sum transfers* can be defined, analogously to Definition 5, by suitably replacing the budget balance condition (25) with (31). The corresponding welfare

[9]This property holds for instance with Cobb Douglas production functions and CRRA preferences, as we see in the example in Section 5.

function analogous to (26) has the following form:

$$W^I(\tau_K,\tau_L) := \sum_i \frac{1}{I}\left\{v_i(e_i-k_i)+\mathbf{E}\left[u_i\left(wL_i^{\theta_i}+rK_i^{\theta_i}-\left\{w\tau_L\left(L_i^{\theta_i}-L_i\right)+r\tau_K\left(K_i^{\theta_i}-K_i\right)\right\},\bar{H}_i-h_i^{\theta_i}\right)\right]\right\}, \tag{32}$$

where we used (31) to substitute for T_i in the consumers' objective function (22) and the superscript I highlights the fact this is a different function and the new insurance aspect of the policy.

Differentiating $W^I(\tau_K,\tau_L)$ with respect to τ_K and τ_L and evaluating it at $\tau_K=\tau_L=0$, we obtain the following decomposition formula for the policy under consideration:

$$\begin{aligned}\left.\frac{\partial W^I}{\partial\tau_K}\right|_{\tau=0} &= \sum_i \frac{1}{I}\left\{\left(I_i^K+D_i^K\right)\frac{\partial r^I}{\partial\tau_K}+\left(I_i^L+D_i^L\right)\frac{\partial w^I}{\partial\tau_K}-\mathbf{E}\left[u_{ic}\cdot\left(\hat{K}_i^{\theta_i}-\hat{K}_i\right)\right]\hat{r}\right\}\\ &= \sum_i \frac{1}{I}\left\{I_i^K\left(\frac{\partial r^I}{\partial\tau_K}-\hat{r}\right)+D_i^K\frac{\partial r^I}{\partial\tau_K}+\left(I_i^L+D_i^L\right)\frac{\partial w^I}{\partial\tau_K}\right\}\end{aligned} \tag{33}$$

$$\left.\frac{\partial W^I}{\partial\tau_L}\right|_{\tau=0} = \sum_i \frac{1}{I}\left\{\left(I_i^K+D_i^K\right)\frac{\partial r^I}{\partial\tau_L}+I_i^L\left(\frac{\partial w^I}{\partial\tau_L}-\hat{w}\right)+D_i^L\frac{\partial w^I}{\partial\tau_L}\right\}. \tag{34}$$

where $r^I(\tau_K,\tau_L)$ and $w^I(\tau_K,\tau_L)$ denote the equilibrium price functions for this case. Comparing these expressions with (27) and (28), we see that there is an additional term in each of them. This is due to the fact that the expression of agents' second period consumption in (32) differs from the one in (26) for one, extra term, $-(w\tau_L\left(L_i^{\theta_i}-L_i\right)+r\tau_K\left(K_i^{\theta_i}-K_i\right))$. The derivative of this term with respect to prices, when evaluated at $\tau_L=\tau_K=0$, is zero - hence this term does not contribute to the welfare effect of the price change. But its derivatives with respect to taxes, $-w\left(L_i^{\theta_i}-L_i\right)$ and $-r\left(K_i^{\theta_i}-K_i\right)$, do not vanish. Hence changes in taxes have now a direct effect on agents' utility, equal to $-\mathbf{E}\left[u_{ic}\cdot\left(\hat{L}_i^{\theta_i}-\hat{L}_i\right)\right]\hat{w}$ and $-\mathbf{E}\left[u_{ic}\cdot\left(\hat{K}_i^{\theta_i}-\hat{K}_i\right)\right]\hat{r}$, which are the effects of the lump sum rebates for the two taxes and are equal to the insurance effects, I_i^L and I_i^K, multiplied by the opposite of the respective factor prices. Indeed, the decomposition result clearly reveals the additional insurance role played by the type dependent transfer.

Since I_i^K and I_i^L are both negative by Lemma 3, we conclude that the additional term in both (33) and (34) is positive. Hence the claim in Corollary 10 is not valid in the present situation and it is possible that both the optimal tax on capital and that on labor are positive. In contrast to Proposition 11 we find that, under the same conditions, in the benchmark case the optimal tax on labor is always positive when tax rebates have an insurance role. On the other hand, the properties of the sign of the optimal tax on capital are unchanged.

Proposition 12 *Assume the natural signs*[10] *(29), (30), and that* $\rho_i^K(\theta_i)=\gamma_i$ *for all* $\theta_i\in\Theta_i$. *At a standard competitive equilibrium, labor should always be taxed with type dependent transfer while capital should be taxed whenever there is over investment, that is if* $\sum_i\left|\frac{I_i^L}{L}\right|>\left|\sum_i\left(\frac{D_i^K}{K}-\frac{D_i^L}{L}\right)\right|$, *and subsidized otherwise. If* $I=1$, *capital should always be taxed.*

Proof. No productivity shock for capital implies $I_i^K=0$ for every i. At a standard equilibrium, by Lemma 7 the aggregate distribution effects are respectively negative and positive, $\sum_i D_i^K<0$ and $\sum_i D_i^L>0$, while by Lemma 3 the insurance effect I_i^L is negative for all i. The natural sign assumption means then $\hat{w}>\frac{\partial w^I}{\partial\tau_L}>0$ and so also $\frac{\partial r^I}{\partial\tau_L}<0$ by the profit maximization conditions $r=F_K$, $w=F_L$. Hence all the three terms in (34) are positive, which establishes the first claim.

Comparing (33) with (27), we see they differ only for the term multiplying I_i^K. Since by assumption $I_i^K=0$ for all i, the derivative with respect to τ_K has the same form at an equilibrium without and with insurance. So the result follows from Proposition 11. ∎

[10] Strictly speaking, the natural sign assumption in the present framework should be stated by replacing $\frac{\partial r}{\partial\tau_K}$ and $\frac{\partial w}{\partial\tau_L}$ in (29) and (30) with $\frac{\partial r^I}{\partial\tau_K}$ and $\frac{\partial w^I}{\partial\tau_L}$ to reflect the fact that the equilibrum price maps are different. With a slight abuse of language we avoid to make this explicit, here and in what follows.

Remark 3 *One might wonder why labor should be taxed even when there is under supply of labor (when $\sum_i \left|\frac{I_i^L}{L}\right| > \left|\sum_i \left(\frac{D_i^K}{K} - \frac{D_i^L}{L}\right)\right|$ there is over investment and hence also, by Corollary 5, under supply of labor). But under (31) the lump sum tax transfer may provide insurance against private idiosyncratic risks. In the benchmark case, where $\rho_i^K(\theta_i) = \gamma_i$ for all θ_i, this insurance role kicks in only if labor is taxed since the idiosyncratic shocks only affect labor productivity. The result in Proposition 12 shows that, under the natural sign assumption, the benefits from such direct insurance exceeds the welfare loss from further discouraging already under supplied labor.*

4.3 Anonymous lump-sum transfers as insurance and redistribution

Next, we shall consider the case where the lump sum transfer is not only deterministic but also the same for all types, equal to the average tax payment across types:

$$w\tau_L L + r\tau_K K = T. \tag{35}$$

This form of transfer not only provides insurance, but also redistributes wealth across different types. The transfer requires no knowledge of the realizations of the individual shocks nor of individual types, and hence it is completely anonymous.

An *equilibrium with anonymous transfers* is similarly defined and the welfare function takes the form:

$$W^{IR}(\tau_L, \tau_K) := \sum_i \frac{1}{I}\left\{v_i(e_i - k_i) + \mathbf{E}\left[u_i\left(c_i^{\theta_i}, \bar{H}_i - h_i^{\theta_i}\right)\right]\right\}, \tag{36}$$

where for each i, the expression of second period consumption, obtained by using (35) to substitute for T:

$$c_i^{\theta_i} = wL_i^{\theta_i} + rK_i^{\theta_i} - \left\{w\tau_L\left[\left(L_i^{\theta_i} - L_i\right) + (L_i - L)\right] + r\tau_K\left[\left(K_i^{\theta_i} - K_i\right) + (K_i - K)\right]\right\}. \tag{37}$$

Here the additional superscript R marks the new, redistribution element of the tax scheme.

We have so the following decomposition result for this case:

$$\begin{aligned} \left.\frac{\partial W^{IR}}{\partial \tau_K}\right|_{\tau=0} &= \sum_i \tfrac{1}{I}\left\{\left(I_i^K + D_i^K\right)\tfrac{\partial r^{IR}}{\partial \tau_K} + \left(I_i^L + D_i^L\right)\tfrac{\partial w^{IR}}{\partial \tau_K} - \mathbf{E}\left[u_{ic}\left(K_i^{\theta_i} - K_i\right)\right]\hat{r} - \mathbf{E}\left[u_{ic}\right](K_i - K)\,\hat{r}\right\} \\ &= \sum_i \tfrac{1}{I}\left\{\left(I_i^K + D_i^K\right)\left(\tfrac{\partial r^{IR}}{\partial \tau_K} - \hat{r}\right) + \left(I_i^L + D_i^L\right)\tfrac{\partial w^{IR}}{\partial \tau_K}\right\} \\ &= \hat{K}\tfrac{\partial r^{IR}}{\partial \tau_K}\sum_i \tfrac{1}{I}\left\{\left(\tfrac{I_i^K}{\hat{K}} - \tfrac{I_i^L}{\hat{L}}\right) + \left(\tfrac{D_i^K}{\hat{K}} - \tfrac{D_i^L}{\hat{L}}\right)\right\} - \hat{r}\sum_i\left(I_i^K + D_i^K\right) \end{aligned} \tag{38}$$

$$\begin{aligned} \left.\frac{\partial W^{IR}}{\partial \tau_L}\right|_{\tau=0} &= \sum_i \tfrac{1}{I}\left\{\left(I_i^K + D_i^K\right)\tfrac{\partial r^{IR}}{\partial \tau_L} + \left(I_i^L + D_i^L\right)\left(\tfrac{\partial w^{IR}}{\partial \tau_L} - \hat{w}\right)\right\} \\ &= \hat{L}\tfrac{\partial w^{IR}}{\partial \tau_L}\sum_i \tfrac{1}{I}\left\{\left(\tfrac{I_i^L}{\hat{L}} - \tfrac{I_i^K}{\hat{K}}\right) + \left(\tfrac{D_i^L}{\hat{L}} - \tfrac{D_i^K}{\hat{K}}\right)\right\} - \hat{w}\sum_i\left(I_i^L + D_i^L\right). \end{aligned} \tag{39}$$

They differ from the corresponding terms in the previous section, (33) and (34), by $-\sum_i\left\{\mathbf{E}\left[u_{ic}\right](K_i - K)\right\}\hat{r}$ and $-\sum_i\left\{\mathbf{E}\left[u_{ic}\right](L_i - L)\right\}\hat{w}$. This is due to the fact that the expression for $c_i^{\theta_i}$ in (37) also has two additional terms, $-w\tau_L(L_i - L)$ and $-r\tau_K(K_i - K)$, which describe the redistributive component of the transfer. Differentiating them with respect to taxes[11] we get $-\hat{r}\mathbf{E}\left[u_{ic}\right](K_i - K)$ and $-\hat{w}\mathbf{E}\left[u_{ic}\right](L_i - L)$, which are equal to $-\hat{r}D_i^K$ and $-\hat{w}D_i^L$: the above decomposition result pins down the additional distributive role of the transfer. By Lemma 7 the sum of these terms, constituting the new terms in (33) and (34), has respectively a positive and a negative sign. That is, the new redistributive effect of the tax scheme strengthens the case for taxing capital and weakens that for taxing labor.

[11] The derivative with respect to prices, for the same reasons as in the previous section equals zero.

Proposition 13 *Assume the natural signs (29), (30) and that $\rho_i^K(\theta_i) = \gamma_i$ for all $\theta_i \in \Theta_i$. At a standard competitive equilibrium, both capital and labor should be taxed with anonymous transfers if $\left|\sum_i I_i^L\right| > \left|\sum_i D_i^L\right|$, i.e., the total distribution effect of labor is smaller than the total insurance effect of labor. Moreover, capital should be taxed if there is over investment.*

Proof. Under the stated assumptions, for the same argument as in the proof of Proposition 12, we have again $I_i^K = 0$ and $I_i^L < 0$ for every i, and $\sum_i D_i^K < 0$ and $\sum_i D_i^L > 0$. Since the natural signs assumption implies $(\frac{\partial r^{IR}}{\partial \tau_K} - \hat{r}) < 0$ and $\frac{\partial w^{IR}}{\partial \tau_K} < 0$, from (38) we see that $\left.\frac{\partial W^{IR}}{\partial \tau_K}\right|_{\tau=0} > 0$ if $\sum_i \left(I_i^L + D_i^L\right) < 0$, i.e., $\left|\sum_i I_i^L\right| > \left|\sum_i D_i^L\right|$. Moreover, since the natural signs assumption also implies $(\frac{\partial w^{IR}}{\partial \tau_L} - \hat{w}) < 0$ and $\frac{\partial r^{IR}}{\partial \tau_L} < 0$, from (39) we see that $\left.\frac{\partial W^{IR}}{\partial \tau_L}\right|_{\tau=0} > 0$ follows from $\sum_i \left(I_i^L + D_i^L\right) < 0$. ■

Compare the condition, obtained from (38), for the positivity of the optimal tax on capital in the present framework, with the one for over investment we get from (20). Since both $F_{KK} > 0$ and $\frac{\partial r^{IR}}{\partial \tau_K} > 0$, we see the first condition is now weaker, because of the additional redistributive effect of this tax scheme. Consequently, whereas the optimal tax on capital is always positive when there is over investment (e.g., when $I = 1$), the optimal tax on capital may also be positive when there is under investment (in contrast with Propositions 11 and 12).

Let us understand why one might want to tax under invested capital to further discourage capital accumulation. . When the lump sum transfer is equal for all types, the tax has an additional redistributive element, since wealthier consumers tend to have a higher income from capital. That is, the tax on capital in this scheme effectively creates an income transfer from wealthier to poorer consumers, which is beneficial for the social welfare. Therefore, if the distribution effect $\left|\sum_i D_i^K\right|$ is large enough, an additional loss from reducing capital further, can be compensated by an additional welfare gain from equalizing income, justifying taxing under invested capital accumulation and subsidizing over supplied labor in this case. This result indeed shows the power of decomposition result, clarifying the underlying economics.

4.4 No transfer: taxing capital or labor?

It is useful to briefly conclude this analysis by considering the extreme case where no lump sum transfer can be made, so that the budget balance condition is:

$$w\tau_L L + r\tau_K K = 0, \tag{40}$$

and the two tax rates can no longer be independently set, since $\tau_L = -\tau_K \frac{rK}{wL}$. The corresponding welfare function is obtained simply by replacing τ_L with this expression in W^{IR}, $W^{IR}\left(\tau_K, -\tau_K \frac{rK}{wL}\right)$, whose derivative with respect to τ_K is $\frac{\partial W^{IR}}{\partial \tau_K} - \frac{\hat{r}\hat{K}}{\hat{w}\hat{L}} \cdot \frac{\partial W^{IR}}{\partial \tau_L}$. From (38) and (39) we obtain so the following decomposition formula:

$$\left.\frac{dW^{IR}}{d\tau_K}\right|_{\tau=0} = \hat{K}\hat{r}\left(\frac{\partial r^{IR}}{\partial \tau_K}/\hat{r} + \frac{\partial w^{IR}}{\partial \tau_L}/\hat{w} - 1\right)\sum_i \frac{1}{I}\left\{\left(\frac{I_i^K}{\hat{K}} - \frac{I_i^L}{\hat{L}}\right) + \left(\frac{D_i^K}{\hat{K}} - \frac{D_i^L}{\hat{L}}\right)\right\}. \tag{41}$$

In view of (20) and noting that

$$\left.\frac{d\left[r^{IR}(1-\tau_K)\right]}{d\tau_K}\right|_{\tau=0} = \frac{\partial r^{IR}}{\partial \tau_K} - \frac{\partial r^{IR}}{\partial \tau_L}\left(\frac{\hat{r}\hat{K}}{\hat{w}\hat{L}}\right) - \hat{r} = \frac{\partial r^{IR}}{\partial \tau_K} + \frac{\partial w^{IR}}{\partial \tau_L}\left(\frac{\hat{r}}{\hat{w}}\right) - \hat{r},$$

we have so:

Proposition 14 *Assume that* $d\left[r^{IR}(1-\tau_K)\right]/d\tau_K\big|_{\tau=0} < 0$. *If no lump sum transfer is allowed, capital should be* ***taxed*** *if and only if there is* ***under*** *investment.*

This result is somewhat surprising as the sign of the optimal tax on capital, under a condition slightly stronger than the natural signs (29) and (30), is *exactly* the opposite of what we found in Proposition 11 and, partly, also in Proposition 12. So when we are in the benchmark case and $I = 1$, capital should be *subsidized* even though there is over investment. In the present environment, with no lump sum rebates the tax has still a redistribution effect, given by the total net amount of the tax paid by each individual, which is welfare relevant. The decomposition result reveals that what matters in this case is the effect of the tax on the net, after tax return of capital rather than its gross return, in contrast to Proposition 11.

5 Numerical Example

In this section we consider a simple numerical example for which we derive the level of the optimal capital and labor tax rates for the different types of tax transfer schemes investigated in the previous section. In this framework we will also illustrate the findings of the local analysis carried out in the previous section, and compare them to the globally optimal tax rates. In principle, there is little reason to believe that the local information around the competitive equilibrium is sufficient to identify the properties of the optimal tax rates in the general set up we considered. But it will be seen that, under the functional forms and the specification of parameters which are commonly used in the literature, the results of the local analysis turn out to be useful to infer the properties of the global maximum.

There are two types of consumers and so $I = 2$. Consumers have the same preferences and second period endowments, as well as the same distribution of the idiosyncratic shock, they only differ for their initial endowments. We set $e_1 > e_2$ without loss of generality. So type 1 consumers are richer than type 2 consumers and we shall refer to the first ones as *rich* and the second ones as *poor*. There are two equally likely individual states, $\Theta = \{\theta_H, \theta_L\}$, and the common expected utility is

$$U\left(c^0, \left(c^\theta, h^\theta\right)_{\theta\in\Theta}\right) = \frac{1}{1-\sigma}\left(c^0\right)^{1-\sigma} + \mathbf{E}\left[\frac{B}{1-\sigma}\left(c^\theta\right)^{1-\sigma} - \frac{\chi}{1+\varphi}\left(h^\theta\right)^{1+\varphi}\right]$$

where c^0 is consumption in the first period, and c^θ and h^θ are consumption and labor supply in the second period in state θ. There is no shock to the productivity of investments in capital, as in the benchmark case, and the shock to the productivity of labor is the identity map:

$$\rho_i^K(\theta) = 1, \qquad \text{and} \qquad \rho_i^L(\theta) = \theta \text{ for all } \theta \in \Theta$$

The production technology is Cobb-Douglas:

$$F(K, L) = AK^\alpha L^{1-\alpha}.$$

The values of the parameters are set as follows: $\sigma = 3$, $\varphi = 1$, $B = 8$, $\chi = 2.5$, $A = 1$, $\alpha = 0.36$. We also fix the average value of the labor productivity shocks, $\bar{\theta}$, at unity, and the average initial endowment, $\bar{e} = (e_1 + e_2)/2$, at 3.7162,[12] while allowing for different values for the magnitude of

[12] The values of σ, φ, α are in line with those commonly used in the macroeconomics literature. The value of $\bar{e}$ has been chosen so that, when the consumers are identical and there are no shocks, $e_1 = e_2$ and $\theta_H = \theta_L$, the equilibrium level of consumption is the same in the two periods, as in the steady state of a dynamic economy.

the shocks, identified by θ_H, and the degree of heterogeneity, identified by $e_1/\bar{e}$. Note that under all parameter configurations considered, our example yields a standard equilibrium, and the sign conditions assumed in Propositions 11, 12, 13, and 14 are satisfied.

Figure 1 shows how market incompleteness affects capital accumulation and labor supply. There we fix the endowment distribution at $e_1/\bar{e} = 1.42$ and $e_2/\bar{e} = 0.58$, which implies a standard deviation of about 60 percent. Then we let θ_H vary from 1 to 1.4 with $\theta_L = 2 - \theta_H$. Thus the standard deviation of the idiosyncratic shock θ changes from zero to about 57 percent. The solid lines in the two panels of Figure 1 portray, respectively, the aggregate capital stock and the aggregate labor supply at the competitive equilibrium of the economy for the different values of the standard deviation of θ. The dotted lines in the same panels depict also the aggregate capital and labor supply but at a competitive equilibrium where agents can trade in a complete market for contingent claims. We see that aggregate capital is greater with incomplete asset markets than with complete markets (due to the precautionary saving motive exhibited by the utility function considered). On the other hand, aggregate labor supply is lower, due to the income effect caused by the higher aggregate stock of capital. Moreover, the difference between the values with incomplete and complete markets is larger when the standard deviation of the idiosyncratic shock is also larger, that is, the uninsurable shock is more significant.

In the subsequent figures we fix the values of the idiosyncratic shocks at $\theta_H = 1.2$ and $\theta_L = 0.8$, and let the standard deviation of the distribution of the initial endowments, $e_i/\bar{e}$, vary from zero to 45 percent. Figure 2 plots the values of $\partial W/\partial \tau_K$ and $\partial W/\partial \tau_L$ evaluated at $\tau_K = \tau_L = 0$, that is, the marginal effects on welfare of introducing taxation on capital and labor income at the competitive equilibrium, as in the local analysis carried out in the previous section. Figure 3 plots the optimal tax rates, that is, the tax rates that maximize social welfare $W(\tau_K, \tau_L)$. In both figures, we examine alternative specifications of the lump-sum transfers as considered in the previous section.

In each of the two figures the north-west panel corresponds to the tax scheme with no redistribution of income discussed in Section 4.1 (see equation (25)). As shown in Proposition 11, taxing capital or subsidizing labor is welfare enhancing (marginally at the competitive equilibrium), when the average distribution effect is relatively small. In the example here, the average distribution effect increases as the inequality in the initial endowments rises. This can be seen in the north-west panel of Figure 2: $\partial W/\partial \tau_K > 0$ and $\partial W/\partial \tau_L < 0$ when the inequality in initial distribution is sufficiently small, and vice versa when it is large. This local result is in accord with the values obtained for the optimal tax rates, displayed in the north-west panel of Figure 3. When the standard deviation of income distribution is close to zero the optimal tax rate on capital is around 2 percent. The optimal tax rate then decreases monotonically as inequality increases, becomes negative at the same point found in Figure 2 and equals approximately -3 percent when the standard deviation of income distribution is 45 percent. The reverse properties hold for the optimal tax on labor (whose level, in absolute value, is a bit lower).

Secondly, we consider the case with type dependent transfers examined in Section 4.2 (see equation (31)). The north-east panel of Figure 2 plots again the marginal effects of capital and labor taxes, illustrating the claim in Proposition 12. The sign of the marginal effect on welfare of capital taxation at the competitive equilibrium is the same as in the previous tax scheme.[13] This is in accord with this proposition, since in our example there is no idiosyncratic shock to the return on capital. Also, the marginal effect of labor taxation is now positive regardless of the degree of inequality in the initial endowment distribution. The north-east panel of Figure 3 shows how the optimal tax rates vary with the inequality in the initial endowment. With no inequality the optimal tax rates on

[13]The values are different since the derivatives of equilibrium price functions are different.

capital and labor are both positive, equal respectively to around 6 and 10 percent. As the degree of inequality increases, the optimal tax rate on capital decreases. Note however that it stays positive even when the marginal effect of capital taxation at an equilibrium with no taxes becomes negative. This is the region where the suggestions provided by the local analysis turns out to be misleading for the globally optimal tax rates, but the size of the region appears to be small. The optimal tax rate on labor is positive and increases with the degree of inequality, in accord with what suggested by the local analysis, reaching a level of around 12 percent when the standard deviation of the endowment distribution is 45 percent.

Third, we consider the case of anonymous transfers, discussed in Section 4.3 (see equation (35)). The south-west panel of Figure 2 plots the marginal effects of taxation for this case, which illustrate the claim in Proposition 13. The marginal welfare effect of capital taxation turns out to be always positive and, in contrast with the previous cases, to increase with the degree of initial inequality as we see in Figure 2. So in this case, in the economy considered in this example capital should be taxed even when there is under investment. We see in Figure 2 that the effect of labor taxation is also positive but falls as the degree of inequality increases, reflecting the fact that the distribution effect works against taxing labor. The south-west panel of Figure 3 shows that the optimal capital and labor tax rates are again in line with what the local analysis suggests: the optimal tax rate on capital income is positive and increases with the degree of inequality, reaching a level of around 35 percent when the standard deviation of income is the highest considered. The optimal tax rate on labor income is also positive though considerably lower and declines slightly with the degree of inequality. As argued in Section 4.3, the nature of the tax transfer in this case strengthens the case for taxing capital.

The last tax scheme we consider is the one without lump-sum transfers, analyzed in Section 4.4 (equation (40)). The south-east panel of Figure 2 plots the marginal effects of taxation for this case. By construction, the tax rates on capital and labor income move in opposite directions to balance the government's budget. By comparing the south-east with the north-east panels we see that under this tax scheme the sign of the marginal effect of capital taxation is exactly the opposite to the one found for the tax schemes without insurance nor redistribution analyzed in Section 4.1, in accord with what shown in Proposition 14. Thus the marginal effect of capital taxation on welfare is negative with no inequality in the initial endowment, but increases as inequality rises and becomes positive for a sufficiently large degree of inequality. In the south-east panel of Figure 3 we confirm that the optimal tax rates on capital and labor behave as our local analysis suggests: the optimal tax on capital ranges from -10 to 10 percent as the standard deviation of income distribution varies from 0 to 45 percent.

6 Concluding remarks

Our results demonstrate that the optimal tax rate on capital may be negative. Moreover this finding applies for equilibria exhibiting standard properties and hence does not rely on any pathological properties, as the existence of upward sloping demand curves. In these cases, by subsidizing capital and taxing labor the level of capital accumulation increases beyond the already "excessively high" level of the equilibrium with no taxes. This arises because the determination of the optimal taxation level is a second best problem, constrained by the fact that the attainable allocations are those obtained as competitive equilibria for a suitably designed tax system, in the presence of incomplete markets.

In the environment considered competitive equilibria are generally Pareto inefficient and the inefficiency can be decomposed in two parts: allocational inefficiency and production inefficiency. The latter is caused by over/under use of capital/labor in our model. If the capital/labor ratio is above the efficient, complete market level, a positive tax on capital should move this ratio towards its efficient level, thus generate an economic surplus. Hence, it may seem hard to believe it could ever worsen the welfare.

This line of argument, while correct in a partial equilibrium analysis, is insufficient in a general equilibrium environment with incomplete markets for at least two reasons. First, it ignores the fact that the introduction of taxes may modify equilibrium prices and, with incomplete markets, the level of equilibrium prices affects the consumers' ability to hedge the risk they face, production efficiency may be improved at the expense of the allocational efficiency. It is indeed the insurance effect in our analysis what captures this intuition.

Secondly, the surplus can only be distributed via taxes and lump sum transfers as well as price changes. The distribution effects capture these features. As our analysis has shown, these effects will be different and have opposite signs for agents with sufficiently different income levels and preferences. Furthermore, since the utilitarian social welfare favors equality in income, social welfare may be improved by increasing income equality while sacrificing production efficiency, and this is true even when markets are complete.

The answer to the question of what is the optimal taxation level with incomplete markets is a delicate one, and one cannot deduce it from the partial equilibrium intuition. It is our contribution to identify the insurance and distribution effects of the tax and the determinants of the sign and magnitude of these effects. This in turn provides useful information about the sign and magnitude of the optimal tax on capital and labor, in a general equilibrium setting.

We conclude with a discussion of a couple of extensions, the first one to the case where non-linear taxes are also allowed. Our decomposition results show that the marginal welfare change is given by the insurance and distribution effects multiplied by the changes in factor prices. Linear taxes simplify the problem since equilibrium prices are simply a function of two tax rates in our model. With non-linear taxes they would be a more complex function of the tax schedule. However, the decomposition result will still hold for the same set of insurance and distribution effect terms, suitably multiplied by the derivatives of equilibrium price functions, as we saw in Section 3 for a general policy which could be interpreted as personalized, non linear taxation. Although it might be hard to characterize optimal taxes, our first order analysis readily extends in principle.

In our two period environment we cannot address important issues, as the intertemporal allocation of the tax burden, examined in the literature, starting with Judd (1985) and Chamley (1986) in a complete market setting and with Aiyagari (1995) when markets are incomplete. So an extension to an infinite horizon set up is an important next step. Still, we contend that the findings of our present analysis allow to gain some insights on the properties of optimal taxes in dynamic, infinite horizon economies. In particular, the effects of taxes we find are quite similar to the effects of these taxes on a steady state equilibrium of a dynamic economy. Building on this work in fact, in a companion paper Gottardi, Kajii and Nakajima (2011) study the solutions of the dynamic Ramsey problem of finding the optimal path of the level of public debt and linear taxes on capital and labor in an infinite horizon model with incomplete markets.

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Appendix

Proofs for Lemmas 1 and 2. Consider the second-period utility maximization problem for a type i consumer at θ_i:

$$\max_{c,l} u_i(c,l)$$

subject to

$$c + w\rho_i^L(\theta_i)l = r\rho_i^K(\theta_i)k_i + w\rho_i^L(\theta_i)\bar{H}_i,$$

where k_i has already been chosen in the first period. This problem can be restated as a standard consumer problem in general equilibrium:

$$\max_{c,l} u(c,l)$$
$$\text{subject to } c + pl = m + p\bar{H},$$

where $p = w\rho_i^L(\theta_i)$ and $m = r\rho_i^K(\theta_i)k_i$ are taken as given. Writing $c(p,m)$ and $l(p,m)$ for the derived demand functions for the consumption good and leisure respectively, and denoting by $\lambda(p,m)$ the Lagrange multiplier, the following first-order conditions characterize the demand functions:

$$\begin{aligned} u_c(c(p,m), l(p,m)) - \lambda(p,m) &= 0, \\ u_l(c(p,m), l(p,m)) - \lambda(p,m)\,p &= 0, \\ -(c(p,m) + pl(p,m)) &= -\left(m + p\bar{H}\right). \end{aligned}$$

Recall that ρ_i^L and ρ_i^K are comonotonic, and so are both p and m. Therefore, to establish Lemma 1, it suffices to show that $\lambda(p,m)$ is decreasing in m and p. Since $\lambda(p,m)$ is the derivative of the indirect utility function with respect to income and the indirect utility function is concave in income for a concave utility, it readily follows that λ is decreasing in m.

We shall now show that $\lambda(p,m)$ is decreasing in p as well. To simplify the notation, we shall omit reference to (p,m) below. To find the derivatives of λ (as well as those for c and l), we follow the standard technique of differentiating the system of the first order conditions:

$$\begin{bmatrix} u_{cc} & u_{cl} & -1 \\ u_{cl} & u_{ll} & -p \\ -1 & -p & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial p}c \\ \frac{\partial}{\partial p}l \\ \frac{\partial}{\partial p}\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \\ l - \bar{H} \end{bmatrix}.$$

The strict concavity assumption implies that the determinant of the square matrix above is positive:

$$\triangle := -u_{cc}p^2 + 2u_{cl}p - u_{ll} > 0,$$

and we have:

$$\begin{bmatrix} u_{cc} & u_{cl} & -1 \\ u_{cl} & u_{ll} & -p \\ -1 & -p & 0 \end{bmatrix}^{-1} = \frac{1}{\triangle} \begin{bmatrix} -p^2 & p & u_{ll} - pu_{cl} \\ p & -1 & u_{cc}p - u_{cl} \\ u_{ll} - pu_{cl} & u_{cc}p - u_{cl} & u_{cc}u_{ll} - (u_{cl})^2 \end{bmatrix}.$$

Thus we have:

$$\frac{\partial}{\partial p}\lambda = \frac{1}{\triangle}\left((u_{cc}p - u_{cl})\lambda + \left(u_{cc}u_{ll} - (u_{cl})^2\right)\left(l - \bar{H}\right)\right),$$

which is negative. Indeed, using the first order condition, $(u_{cc}p - u_{cl})\lambda = u_{cc}u_l - u_{cl}u_c < 0$ where the inequality holds by the normality of consumption good, and $\left(u_{cc}u_{ll} - (u_{cl})^2\right) > 0$ by concavity and $\left(l - \bar{H}\right) < 0$. Therefore, Lemma 1 has been established.

Notice that the labor supply in efficiency units corresponds to $\left(\bar{H} - l\right)p/w$ in the consumer problem above, so in order to establish Lemma 2 it suffices to show that $\left(\bar{H} - l\right)p$ is increasing in p. From the system of equations above,

$$\frac{\partial}{\partial p}l = \frac{1}{\triangle}\left(-\lambda + (u_{cc}p - u_{cl})\left(l - \bar{H}\right)\right),$$

and so

$$
\begin{aligned}
\frac{d}{dp}\left(\left(\bar{H}-l\right)p\right) &= \left(\bar{H}-l\right) - p\frac{\partial}{\partial p}l, \\
&= \left(\bar{H}-l\right) - \frac{p}{\triangle}\left(-\lambda + \left(u_{cc}p - u_{cl}\right)\left(l-\bar{H}\right)\right), \\
&= \left(\bar{H}-l\right)\left(1+\frac{p}{\triangle}\left(u_{cc}p-u_{cl}\right)\right) + \frac{p\lambda}{\triangle}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
1+\frac{1}{\triangle}\left(u_{cc}p^2 - pu_{cl}\right) &= \frac{1}{\triangle}\left(\left(-u_{cc}p^2 + 2u_{cl}p - u_{ll}\right) + \left(u_{cc}p^2 - pu_{cl}\right)\right), \\
&= \frac{1}{\lambda\triangle}\left(u_{cl}u_l - u_c u_{ll}\right), \\
&> 0,
\end{aligned}
$$

where the last inequality follows from the normality of leisure. This proves Lemma 2.