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Lack of debt restructuring and lender's credibility

– A theory of nonperforming loans –*

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Abstract

Firms and households occasionally accumulate debt beyond the level they can repay, particularly at times of financial distress. In such cases, debt restructuring can take a considerable amount of time. In this paper, we propose a model for a long-term debt contract with a time-consuming debt restructuring process, and demonstrate that large debt can cause persistent inefficiency. The key is that if the debt is accumulated beyond a threshold level, the lender can no longer commit to any future repayment plans. The loss of lender's credibility then discourages the borrower's *demand* for new loans and leads to an inefficient outcome. This contrasts with the existing theory based on credit crunch or debt overhang, where it is the *supply* of new loans that is dampened. Our model generates a debt Laffer curve, i.e., the lender's payoff may decrease with the contractual amount of debt. The efficiency of equilibrium can be improved if debt restructuring is facilitated by policy measures.

Key words: Optimal contract, backloading, two-sided lack of commitment, secular stagnation.

JEL Classification: E30, G01, G30.

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1 Introduction

Let us consider a borrower and a lender who are in a long-term financial relationship, and suppose that, due to a sequence of bad shocks, the borrower's debt has exceeded the maximum amount that s/he can repay. For simplicity, let us also suppose that the borrower's liquidation value is so small that the lender never chooses to liquidate the borrower. In that case, since the borrower is no longer able to repay the debt, the lender would need to reduce the amount of debt ("debt forgiveness," or "debt restructuring" more broadly).¹ In practice, however, debt restructuring can be very time-consuming and the extent of its significance reflected in the data can be measured by the quantity of so-called *nonperforming loans*.²

In this paper, we demonstrate how a lengthy debt restructuring process can create nonperforming loans and lead to a persistent decline in the borrower's economic activity. Our theory is distinct from the existing theory based on credit crunch or debt overhang. In our theory, it is the depressed demand for new loans by the borrower that causes the inefficiency, whereas in the extant theory, it is the dampened supply of new loans. Given that a financial crisis typically prompts an increase in nonperforming loans, our theory sheds new light on why financial crises tend to cause long-lasting recession.

Formally, our model builds on that of Albuquerque and Hopenhayn (2004) (hereafter, AH), who study the constrained efficient allocation of funds between a borrower and lender in a model in which the borrower is unable to commit to repaying the debt. As is standard in this literature, they formulate the problem recursively using the borrower's value as a state variable, where the borrower's value comprises the present discounted value (PDV) of dividend payments. One key implication of their model is that, if we abstract from the possibility of liquidation, the supply of working capital to the borrower reaches the first-best level in finite time. This constrained efficient allocation is implemented by backloading payoffs to the borrower: All borrower's profits are paid to the lender and the borrower receives no dividends until the first-best allocation is attained.³

In the AH model, the debt is assumed to be state-contingent, and equals the PDV of repayments from the borrower to the lender ("the lender's value"). In our context, this is interpreted as an environment in which the borrower's debt is immediately and costlessly

¹Debt forgiveness is one of many ways to restructure debt. However, in this paper, we focus on the role of debt forgiveness, and sometimes use these two terms interchangeably.

²We review related evidence and literature on this in Section 2.

³In a more general model of long-term relationship between two parties with one-sided lack of commitment, Ray (2002) demonstrates that the optimal contract involves backloading the payoff of the party who lacks the ability to commit. Furthermore, as demonstrated, for instance, by Clementi and Hopenhayn (2006), backloading payoffs to the borrower also plays a crucial role in the dynamic optimal contract problem with asymmetric information.

adjusted in response to shocks to the profitability of the borrower.

To analyze the effects of lengthy debt restructuring processes, however, we assume that the debt is not state-contingent. Then, given the possibility that the debt exceeds the repayable amount, it is no longer equal to the PDV of repayments. In this paper, by “debt” we mean the “contractual amount of debt,” rather than the PDV of repayments.

We then modify the AH model in two respects. First, we assume that neither the borrower’s nor the lender’s value is verifiable so that they cannot be used as a state variable in the financial contracting problem. Thus, the borrower’s value is not predetermined but is instead selected in each period. The contractual amount of debt, on the other hand, is predetermined, and it is the only variable that can be used as a state variable. Second, we assume that debt restructuring takes time. For this, we start with an extreme assumption that debt restructuring never occurs, no matter how large the borrower’s debt becomes. This assumption is later relaxed so that debt restructuring is feasible but occurs only probabilistically over time.

To illustrate our main results, let us consider the benchmark case in which debt restructuring is not at all feasible. Then, there are three regions for the value of debt: “small,” “intermediate,” and “large.” When the borrower’s debt is “small,” our model behaves similarly to the AH model. The borrower does not receive dividends and all profits are used to repay the debt until the amount of debt reaches some target level. Once the target level is attained, the equilibrium allocation of funds becomes first best. However, it becomes drastically different when the debt becomes “large.” In this region, backloading the payoffs to the borrower becomes infeasible, and, as a result, the equilibrium allocation of funds becomes permanently inefficient (or until debt restructuring is made when it occurs probabilistically). In the intermediary region, the debt can either become “small” or “large” in the future. The borrower’s payoffs are backloaded in this region, but, due to the latter possibility, the equilibrium provision of funds is less efficient than when the debt is “small.”

When the debt becomes “large,” the amount of debt exceeds the PDV of repayments, and increases monotonically over time regardless of the realizations of the shocks to the borrower’s profitability. We interpret the debt in this region as “nonperforming loans.” To emphasize this, we use the term “nonperforming loans (NPL) equilibrium” to describe the equilibrium behavior of the model in this region of the debt.

Why does the equilibrium allocation of funds become persistently inefficient when the debt becomes large? To illustrate this, consider the following simple example.⁴ A borrower earns one million dollars every period. S/he owes a certain amount of debt to a lender. The borrower chooses to default if the PDV of repayments exceeds one million dollars. The interest rate is normalized to zero. If the amount of debt is less than or equal to one

⁴This example is elaborated further in Section 3.

million dollars, there are many feasible repayment plans without default by the borrower. For instance, one million dollars today and none afterwards, or a half million today and the other half million in the next period, etc. Now, suppose, for whatever reason, that the amount of debt has become two million, and consider the lender's options without reducing the contractual amount of debt. Can the lender simply tell the borrower that s/he only needs to repay the lender one million today and nothing afterwards? Such a proposal is not credible because in the next period the contractual value of the debt will be one million and the lender has the right to (and thus will) demand that the borrower repay the remaining one million dollars. Anticipating the lender's behavior in advance, the borrower will reject the lender's offer and choose to cease operation immediately.⁵

What this example demonstrates is that the lender may not be able to commit to dynamic repayment plans when the debt exceeds a certain threshold. Thus, in such a case, the contractual problem between lender and borrower is characterized as *two-sided lack of commitment*. Without the credible commitment to dynamic repayment plans, it is not possible to backload payoffs to the borrower. This is why the borrower's economic activity is depressed persistently in the NPL equilibrium.

In our model, the lender's ability to commit changes endogenously with the amount of debt. When the debt is "small," the lender can commit to dynamic repayment plans so that the equilibrium behavior is similar to that in the AH model. When it becomes "large," however, the lender cannot commit to such plans. In the "intermediate" region, the commitment is possible only insofar as the amount of debt stays away from the "large" region.

Another implication of our model that is worth stressing is the debt Laffer curve. In the AH model or other models of long-term debt, the lender's value is a weakly increasing function of the contractual amount of debt. In our model, however, the lender's value has an inverted U-shaped relationship with debt. It is interpreted as a debt Laffer curve, and illustrates that the payoff for the lender can be increased by reducing the contractual amount of debt.

The above results are obtained under the assumption that debt restructuring is not possible. To examine their robustness, we extend the model so that debt restructuring is possible but occurs only probabilistically. This is intended to capture frictions in the bargaining between the lender and borrower in a reduced-form way. In this extension, we demonstrate that, as long as the probability of debt restructuring is sufficiently small, the basic features of the equilibrium remain the same except that the NPL equilibrium lasts until the debt restructuring occurs rather than permanently.

The rest of this paper is organized as follows. In Section 2, we briefly review evidence

⁵In our model, the operation does not stop but its level becomes inefficiently low in the "NPL equilibrium."

and literature related to this paper. In Section 3, the example discussed above is elaborated further to illustrate the intuition regarding how over-accumulation of debt impairs the credibility of repayment plans offered by the lender and leads to an inefficient outcome. In Section 4, we describe the baseline model, in which debt restructuring never occurs. In Section 5, we discretize the model to prove the existence of the equilibrium. In Section 6, equilibrium dynamics of the discrete model are described. The results of numerical simulations are also provided in this section. In Section 7, the baseline model is modified in such a way that debt restructuring occurs stochastically. Section 8 concludes.

2 Related evidence and literature

2.1 Nonperforming loans

As discussed in the introduction, the main result of this paper is the inefficiency in the relationship between a borrower and a lender caused by the delay in (or lack of) debt restructuring. How relevant is this empirically? One way to see the significance of the delay in debt restructuring in the data is to examine the amount of nonperforming loans.

Loans are classified as nonperforming if payments of interest and/or principals are overdue by 90 days or more (IMF 2019). Such loans are not yet defaulted, as they are not written off from the banks' balance sheet. Since the borrowers are not bankrupt, they continue operations, which typically requires the bank to provide additional loans (working capital).

In Japan, for instance, the amount of nonperforming loans had surged since the beginning of the 1990s, when the Japanese economy experienced a historic collapse of stock and real-estate prices. A large number of nonperforming loans have remained in the Japanese economy for approximately 15 years. Sekine, Kobayashi, and Saita (2003) demonstrate that additional lending of working capital to the borrowers of nonperforming loans was indeed widespread during this period, which they call “forbearance lending.” This practice is also called “evergreening” (Peek and Rosengren, 2005) and “zombie lending” (Caballero, Hoshi, and Kashyap, 2008). Peek and Rosengren (2005) and Caballero, Hoshi, and Kashyap (2008) argue that nonperforming loans have caused a huge inefficiency in the Japanese economy.

Some European countries experienced a similar problem after the Global Financial Crisis of 2008–2009. Figure 1 plots the fraction of nonperforming loans in five European countries, Greece, Ireland, Italy, Portugal, and Spain, during the period 2005–2017. Nonperforming loans increased in those countries following the financial crisis, and in 2017, they still exceeded the pre-crisis levels. These observations indicate that lack of timely restructuring of nonperforming loans is common in post-crisis recessions, and a significant number of nonperforming loans can remain for years in crisis-hit economies.

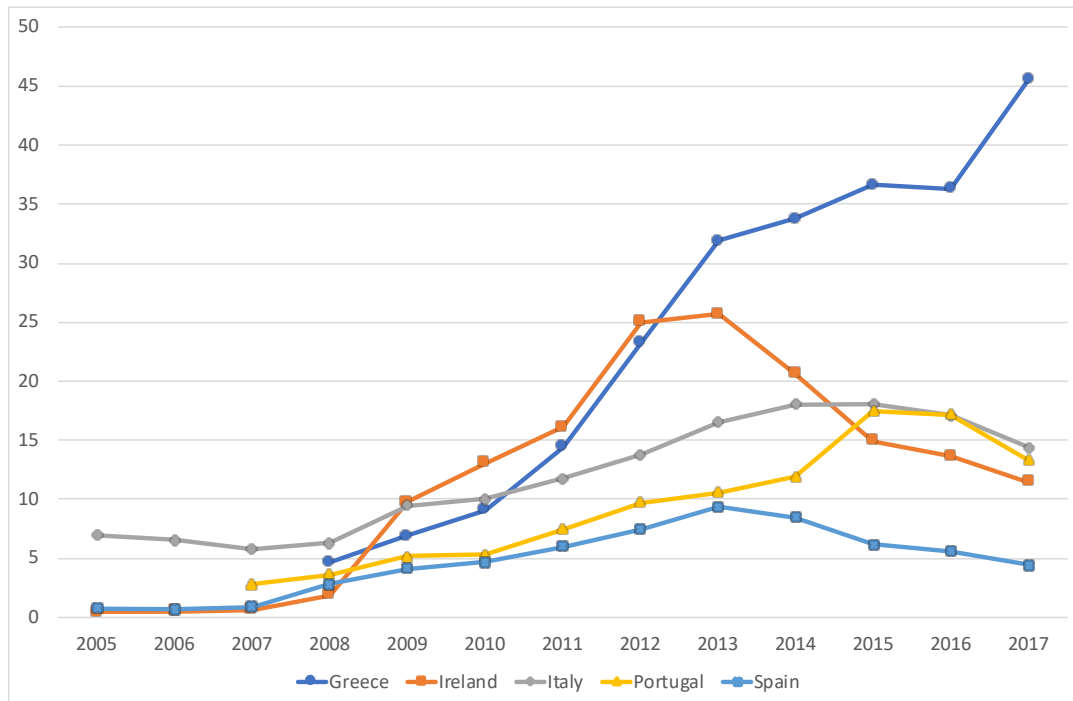


Figure 1: Fraction of non-performing loans in total gross loans. Source: World Bank.

The problem of nonperforming loans appears not to be severe in the United States, as the ratio of nonperforming loans lowered promptly to the normal level following the global financial crisis (5.0% in 2009 to 1.85% in 2014).⁶ However, these figures refer to loans in the corporate sector. In the U.S., the nonperforming loan problem can be serious for student loans, because federal and private student loans cannot be expunged through bankruptcy (Lochner and Monge-Naranjo, 2016). Indeed, the delinquency rate of student loans has risen quite significantly since the crisis. Our theory might be used to examine the (potential) inefficiency associated with student loans for the U.S. economy.

2.2 Why is debt restructuring delayed?

The data on nonperforming loans suggests that debt restructuring is a time-consuming process, in particular during times of financial distress.⁷ There must be various reasons as to why debt restructuring is delayed and nonperforming loans linger on. Here, we discuss three of these reasons that we consider relevant for the recent rise in nonperforming loans

⁶Source: World Bank.

⁷Related evidence is provided by Fukuda and Nakamura (2011), who consider the Japanese firms that are classified as “zombies” by Caballero, Hoshi, and Kashyap (2008). In our context, zombie firms are interpreted as borrowers of nonperforming loans. Fukuda and Nakamura (2011) demonstrate that firms have a higher tendency to remain zombie if they have been zombie for longer periods.

in several countries; namely, political factors, the Bank for International Settlements (BIS) regulations, and bargaining frictions.

First, political factors might have played a crucial role in causing delays to debt restructuring in the 1990s in Japan and in the 2010s in some European countries. If mass bankruptcies of borrowers occurred in various industries, the economic and political costs would be huge. Thus, government officials and politicians tend to be reluctant to reveal the problem. They may support, either directly or indirectly, banks' postponement with respect to writing off nonperforming loans. Peek and Rosengren (2005) and Caballero, Hoshi, and Kashyap (2008) provide detailed descriptions of the Japanese case.

Second, the BIS regulation tends to incentivize banks to hide and evergreen nonperforming loans by providing additional funds to debt-ridden borrowers (see, e.g., Caballero, Hoshi, and Kashyap, 2008). The BIS regulation implies that if a bank's capital is impaired and falls below a threshold, then the bank will be excluded from international operations. Writing off nonperforming loans and realizing the losses might reduce a bank's capital below the threshold, thus making banks reluctant to undertake debt restructuring.

Third, bargaining frictions can cause a delay in the restructuring of nonperforming loans. Debt restructuring involves bargaining between the lender and borrower, or among the lenders if there are multiple lenders. Although efficiency requires an immediate settlement, various sources of inefficiency are known to generate delays (see, e.g., Abreu and Gul 2000, Fuchs and Skrzypacz 2010). A variety of anecdotal evidence suggests the importance of bargaining frictions. One example would be the negotiation over the Argentine bond in the 2000s. Argentina defaulted on the sovereign bond in 2001, but it took 16 years for it to complete the negotiation with its major bondholders and to return to the international capital market. One reason for this protracted delay is the *pari passu* litigation raised by one of the bondholders (Argentina v. NML Capital), which overturned an otherwise-agreed settlement of debt restructuring and caused the years-long delay.

2.3 Literature

As discussed in the introduction, our model is based on AH. We impose a restriction on the possibility of debt forgiveness in their model, and establish that such a restriction brings a novel type of inefficiency into the relationship between borrower and lender. Like AH, we use recursive contracts to formulate our equilibrium. Golosov, Tsyvinski, and Werquin (2016) survey the literature on recursive contracts. Although we employ dynamic programming to solve for our equilibrium, the Lagrange multiplier method may also be used, as described by Marcet and Marimon (2019).⁸

⁸Commonly used frameworks for borrowing constraints in macroeconomics are provided by Kiyotaki and Moore (1997), Carlstrom and Fuerst (1997), and Bernanke, Gertler, and Gilchrist (1999). The financial contracts in these papers are essentially static, whereas in our model they are dynamic.

As the key source of inefficiency in this paper is the over-accumulation of debt, it is closely related to the literature on debt overhang. Since the seminal contribution by Myers (1977) in corporate finance, the notion of debt overhang has been applied to different fields, for instance, macroeconomics by Lamont (1995) and Philippon (2009), and sovereign debt by Krugman (1988) and Kovrijnykh and Szentes (2007). The theory of debt overhang also predicts under-investment by a heavily indebted borrower. Its argument is that it is difficult for such a borrower to find a new lender because (at least a part of) the proceeds from the new investment would be used for the repayment to the incumbent lenders. This is a hold-up problem between the incumbent and new lenders. Our theory, however, is different. First, we consider a long-term relationship between a single lender and a single borrower. Second, debt overhang provides a theory as to why the *supply* of funds is limited for heavily indebted borrowers. In contrast, our theory is intended to reveal why the *demand* for funds by those borrowers is often depressed. The key is that, absent from debt forgiveness, over-accumulation of debt makes the lender unable to commit to future repayment plans. Third, our theory generates a debt Laffer curve; that is, the PDV of repayments to the lender can decrease with the contractual value of the debt it holds. This is also a unique feature of our theory.⁹

Our model can be interpreted as a formalization of the notion of “zombie lending” (Caballero, Hoshi, and Kashyap, 2008). This notion refers to a practice whereby banks extend loans to nonviable firms (“zombie firms”) to keep them operating and to help them repay previous loans. As discussed in the previous section, we share their views regarding why zombie lending persists. However, our views differ in terms of who become zombie firms. The presumption by Caballero, Hoshi, and Kashyap (2008) is that such firms are intrinsically inefficient. Thus, zombie firms should be liquidated because lending to them prevents the entry of more efficient firms into the market. In contrast, our emphasis is on the fact that even efficient firms can be zombie firms. In such a case, debt forgiveness would be more suitable than liquidation to restore efficiency. In this respect, Fukuda and Nakamura (2011) provide useful evidence. They extend the data used by Caballero, Hoshi, and Kashyap (2008) for later periods, and find that a majority of the zombie firms indeed subsequently recovered. They argue that swift corporate restructuring under substantial financial support is essential in the recovery of zombie firms. Their evidence seems to be consistent with our theory.

We often observe persistent recessions for years in the aftermath of financial crises (see, e.g., Reinhart and Rogoff 2008). In particular, the Global Financial Crisis of 2008–2009 and the subsequent recession raised the growing concern about *secular stagnation*; that is, economic growth in developed countries may have decelerated permanently (Summers 2013, Eggertsson and Mehrotra 2014). Since our theory predicts persistent inefficiency

⁹In contrast, Krugman (1988) assumes a debt Laffer curve exogenously.

caused by the accumulation of nonperforming loans, we expect it to be helpful in understanding the persistence of the recession following a financial crisis. In particular, it suggests that policies that facilitate debt restructuring can be effective for the recovery from a crisis. Such policies have not been paid due attention in the recent research.¹⁰ Therefore, one purpose of our study is to call more attention to these issues.

To simplify the exposition, we assume in the baseline model that debt restructuring (or debt forgiveness) is infeasible. However, we extend the model in Section 7 so that debt restructuring is possible but only with a certain probability. The intention is to capture bargaining frictions in a reduced form. An example of inefficient delays in bargaining is provided by Abreu and Gull (2000), who demonstrate that the belief that the opponent of negotiation may be irrational causes a delay in the settlement even between rational players. Another example is Fuchs and Skrzypacz (2010), who demonstrate that asymmetric information with a stochastic arrival of new players creates a delay in the bargaining. In the context of sovereign debt restructuring, Benjamin and Wright (2009) demonstrate that option value of waiting creates a delay in sovereign debt restructuring, which may or may not be efficient. Pitchford and Wright (2012) examine a model in which sovereign debt restructuring is delayed by the holdout by creditors.

3 A simple example

Why does the lender lose credibility when the debt is too large? Why does this distrust of the lender lead to an inefficient outcome? A simple example in this section provides an intuitive account for these questions, which may be helpful to understand the full model in Section 4.

Consider a simple economy where the net rate of interest is zero, $r = 0$. Time is discrete and goes from zero to infinity, $t = 0, 1, 2, \dots, \infty$. There is a firm (borrower) and a bank (lender), and the firm owes initial debt, D , to the bank in period 0. Let b_t denote the repayment to the lender in period t , and $d_0 \equiv \sum_{t=0}^{\infty} b_t$ the present discounted value (PDV) of repayments (as of period 0).

The firm can default on the debt D at any time, and it stops operations and walks away if it defaults. When the firm continues operations, it earns 1 million dollars in each period. With an (unspecified) outside opportunity, the firm will choose to default and walk away if the PDV of repayments to the bank exceeds 1 million dollars. The liquidation value of the firm for the bank is zero.

In this environment, the maximum repayable debt, d_{\max} , equals 1 million dollars, i.e., $d_{\max} = 1$ million. This is because, first, the repayment of any amount that is not greater than 1 million is feasible because the firm earns 1 million every period, and, second, the

¹⁰The few exceptions include Geanakoplos (2014).

repayment of any amount greater than 1 million is not feasible because the firm would default and walk away if the bank insisted on a repayment greater than 1 million.

Case of small debt: Suppose that $D \leq d_{\max}$. In this case, there is no problem with repayments. The bank can receive the full repayment of D , i.e., $D = d_0$, and the firm continues operations so that the economy achieves an efficient outcome. This result is attained by, for example, the following repayment plan: $b_0 = D$ and $b_t = 0$ for $t \geq 1$. This repayment plan is feasible because b_t is (weakly) smaller than the borrower's earnings (1 million dollars) for all $t \geq 0$, and the PDV of repayments is also (weakly) smaller than the maximum repayable amount (1 million dollars). The plan $\{b_t\}_{t=0}^{\infty}$ is credible because the bank has no legal right to demand any amount in excess of D . The state variable D is obviously payoff-relevant, as the payoff for the bank is D .

Case of large debt: What would happen if the contractual amount of debt, D , exceeded the maximum repayable amount d_{\max} , and D could not be reduced? To be specific, suppose that $D = 2$ million. Obviously, the bank is unable to collect 2 million from the firm, but what about 1 million ($= d_{\max}$)? For example, suppose that the bank offers a repayment plan $\{b'_t\}_{t=0}^{\infty}$, where $b'_0 = 1$ million and $b'_t = 0$ for all $t \geq 1$. This plan is feasible, as it satisfies that all repayments b'_t are less than the borrower's earnings and the PDV of repayments is no greater than d_{\max} .

However, this repayment plan $\{b'_t\}_{t=0}^{\infty}$ is not credible. To see this, suppose that the firm accepts $\{b'_t\}$. The firm then pays 1 million to the bank in period 0, and they enter period 1 with the remaining debt $D_1 = 2 - 1 = 1$ million. As the contractual amount of debt is $D_1 = 1$ million, the bank has the legitimate right to demand that the firm repay 1 million in period 1. Then, the argument in the above paragraph (Case of small debt) applies in period 1, and the bank will offer a new repayment plan $\{b''_t\}_{t=1}^{\infty}$, where $b''_1 = 1$ million and $b''_t = 0$ for $t \geq 2$, and the firm has to accept it in period 1. Thus, the promise $\{b'_t\}_{t=0}^{\infty}$ in period 0 is not credible, because it will necessarily be broken in period 1.

Anticipating what will happen in period 1, the firm evaluates in period 0 that the PDV of repayments will be $b'_0 + b''_1 = 1 + 1 = 2$, which exceeds d_{\max} . Therefore, the firm chooses to default and walk away in period 0, resulting in the closure of the firm's business. As the firm stops operations in period 0, the repayment is zero for all periods and the PDV of repayments is also zero in equilibrium.

In sum, if $D > d_{\max}$, the bank can no longer commit to any future repayment plans. The loss of the bank's credibility causes the borrower to stop operations, an inefficient outcome. It is worth noting that the contractual amount of debt D is no longer payoff-relevant when $D > d_{\max}$ in the sense that neither the lender's nor the borrower's value depends on D . As discussed more formally below, this is the obverse of the coin of the

loss of the bank's credibility.

4 Baseline Model

We modify the AH model of long-term debt contract by restricting the possibility of debt restructuring. Here, we focus on debt forgiveness as a means of debt restructuring. In the baseline model, we consider the case in which debt forgiveness is not at all feasible. This is extended in Section 7 to allow for stochastic debt restructuring.

As in AH, the borrowing constraint arises because the borrower may default at any time. The amount of debt accumulates over time as negative productivity shocks hit the borrower. If the debt exceeds a threshold value, it is no longer repayable. Then, as we discuss, the lender loses credibility regarding the future repayment plans it offers, which leads to an equilibrium outcome that is constrained inefficient. The loss of the lender's credibility is permanent in the baseline model, whereas the credibility can be restored stochastically in the extended model in Section 7.

4.1 Setup

We consider an economy where time is discrete and goes from zero to infinity, i.e., $t = 0, 1, 2, \dots, \infty$. There is a bank (lender) and a firm (borrower), who have the common discount factor β , where $0 < \beta < 1$. At the beginning of period 0, the borrower owes D_0 to the lender as the initial debt. We do not make a specific assumption as to where the initial debt comes from. The interest rate for the debt D_0 is fixed at r in the debt contract. We assume that the value of r is given exogenously, satisfies $\beta \geq \frac{1}{1+r}$ as there exists the default risk, and is constant over time. In the general equilibrium setting, the value of r would be determined by the bank's zero-profit condition. See footnote 12.

The debt at the beginning of period t , D_t , evolves as:

$$D_{t+1} = (1+r)(D_t - b_t), \quad \text{for } t \geq 0, \quad (1)$$

where b_t is the repayment in period t . In each period t , the firm needs to borrow capital service (working capital), k_t , to generate revenue, $F(s_t, k_t)$, where $s_t \in \mathbb{R}_+$ is the productivity of the firm in period t .

The production function $F(s, k)$ is a continuously differentiable function that satisfies $F(s, 0) = 0$, and $F_k(s, k) > 0$, $F_s(s, k) > 0$, $F_{sk}(s, k) > 0$, and $F_{kk}(s, k) < 0$ for $k > 0$, where $F_k \equiv \frac{\partial F}{\partial k}$, $F_s \equiv \frac{\partial F}{\partial s}$, $F_{sk} \equiv \frac{\partial^2 F}{\partial k \partial s}$, and $F_{kk} \equiv \frac{\partial^2 F}{\partial k^2}$. The productivity s_t is either s_H or s_L , where $0 \leq s_L < s_H$, and changes over time following a stationary Markov process with $\Pr(s_{t+1} = s_j | s_t = s_i) = \pi_{ij}$, where $\pi_{ij} > 0$ for $i, j \in \{L, H\}$. The firm finances the input k_t by borrowing the amount Rk_t of an intra-period loan from the bank.¹¹ The firm

¹¹In this paper, we assume for simplicity that the firm borrows the intra-period loan, Rk_t , from the

borrowes Rk_t at the beginning of period t and repays Rk_t at the end of the same period t , where the price of capital input R is constant.

The borrower: The dividend to the firm owner is $F(s_t, k_t) - Rk_t - b_t$. The firm owner in our economy is protected by the limited liability so that the dividend is nonnegative:

$$F(s_t, k_t) - Rk_t - b_t \geq 0, \quad \forall t \geq 0. \quad (2)$$

Let V_t denote the expected value of the PDV of dividends:

$$V_t \equiv \mathbb{E}_t \left\{ \sum_{j=0}^{\infty} \beta^j [F(s_{t+j}, k_{t+j}) - Rk_{t+j} - b_{t+j}] \right\} = F(s_t, k_t) - Rk_t - b_t + \beta \mathbb{E}_t V_{t+1}, \quad (3)$$

where \mathbb{E}_t is the expectation operator as of time t .

In any period t , the firm can choose to default after receiving working capital k_t . If the firm defaults on the debt $D_t + Rk_t$, it has the outside opportunity to use k_t and earn $G(s_t, k_t)$. The value of outside opportunity $G(s, k)$ is a continuously differentiable function that satisfies $G(s, 0) = 0$, and $G_k(s, k) > 0$, $G_s(s, k) \geq 0$ and $G_{kk}(s, k) \leq 0$ for $k > 0$. It is furthermore assumed that $F(s, k)$ and $G(s, k)$ satisfy

$$F_{kk}(s, k) - G_{kk}(s, k) < 0, \quad \text{and} \quad F_{ks}(s, k) - G_{ks}(s, k) > 0,$$

for all s and $k (> 0)$. To prevent the firm from defaulting, the equity value of the firm, V_t , must satisfy:

$$V_t \geq G(s_t, k_t), \quad \forall t \geq 0, \quad (4)$$

which yields the borrowing limit on k_t .

The bank: The bank takes as given the market rates of interest for the inter-period debt, r , and the intra-period loan, R . In each period t , it chooses an offer $\{b_{t+j}, k_{t+j}\}_{j=0}^{\infty}$ to maximize the expected value of the PDV of repayments, d_t , which is defined as

$$d_t = \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j b_{t+j}. \quad (5)$$

The offer is made to the firm in a take-it-or-leave-it manner. If the firm declines the offer, it will be liquidated. We assume for simplicity that the liquidation value of the firm is zero. In equilibrium, therefore, the bank never chooses an offer that will be rejected by the firm.

same bank from which it borrowed the initial loan, D_0 . It can be easily confirmed that our result does not change even if the firm borrows Rk_t from other banks.

As explained above, the firm has an option to default after it accepts an offer and receives working capital k_t . We assume that the bank obtains nothing when the firm chooses to default. (In this case, $G(s_t, k_t)$ can be interpreted as the liquidation value of the firm, and all of it is taken away by the firm.)

As the contractual value of the debt, D_t , is verifiable, the bank has no legal right to require a repayment that exceeds the outstanding debt. Thus, the following constraint must be satisfied:

$$b_t \leq D_t, \tag{6}$$

for all $t \geq 0$.

Debt restructuring: In this paper, by the term *debt restructuring* we mean debt forgiveness; that is, a reduction in the contractual amount of debt, D_t . In the baseline model, we consider the case in which debt restructuring is not possible. Thus, D_t cannot deviate from the law of motion (1), and thus it is in general different from the PDV of repayments, d_t . This is the most crucial difference between our model and the AH model. In the AH model, there is no distinction between D_t and d_t , which reflects their assumption that debt restructuring is made immediately at every instant of time.

4.2 The bank's problem and the NPL equilibrium

Throughout this paper, we focus on the Markov Perfect Equilibrium, in which all actions of agents and the value functions in each period t are functions of the state variables $(s_t, D_t) \in \{s_L, s_H\} \times \mathbb{R}_+$. We formulate it in the recursive way. In doing so, we omit the time subscript, and use the subscript, $+1$, for the variables in the next period, and the subscript, -1 , for the variables in the previous period.

The bank's problem: Given a belief in the borrower's value, $V^e(s, D)$, the bank solves the following problem,

$$d(s, D) = \max_{b, k} b + \beta \mathbb{E}d(s_{+1}, D_{+1}) \tag{7}$$

$$\text{s.t.} \quad \begin{cases} F(s, k) - Rk - b + \beta \mathbb{E}V^e(s_{+1}, D_{+1}) \geq G(s, k), \\ F(s, k) - Rk - b \geq 0, \\ D_{+1} = (1 + r)(D - b), \\ b \leq D. \end{cases}$$

The solution to this problem is written as

$$b = b(s, D),$$

$$k = k(s, D).$$

We define

$$\begin{aligned} D_{+1}(s, D) &\equiv (1+r)(D - b(s, D)), \\ V(s, D) &\equiv F(s, k(s, D)) - Rk(s, D) - b(s, D) + \beta \mathbb{E}V^e(s_{+1}, D_{+1}(s, D)). \end{aligned}$$

Assuming rational expectations, the following condition must be satisfied in equilibrium:

$$V(s, D) = V^e(s, D), \quad (8)$$

and

$$V^e(s, D) \leq \frac{1}{1-\beta} \{F(s_H, k^*(s_H)) - Rk^*(s_H)\},$$

where $k^*(s)$ is the first-best level of working capital at $s = \{s_H, s_L\}$:

$$k^*(s) \equiv \arg \max_k F(s, k) - Rk. \quad (9)$$

Assumption 1. If there exist multiple solutions to the maximization problem in (7) for some (s, D) , the bank selects the solution that maximizes $k(s, D)$. Thus, if both (b_1, k_1) and (b_2, k_2) solve the problem and $k_2 < k_1$, then $k(s, D) = k_1$ and $b(s, D) = b_1$.

An *equilibrium* is defined as a solution to (7), $\{k(s, D), b(s, D), d(s, D), V(s, D)\}$, that satisfies (8).¹² As discussed in Section 4.3 below, there is a fundamental difficulty in proving the existence of an equilibrium and characterizing it for general values of D . To overcome the difficulty, we consider a discretized version of the model from the next section.

Meanwhile, in this section, we take the existence of an equilibrium as given, and restrict attention to the behavior of the equilibrium for large values of D . Define \bar{D} by

$$\bar{D} \equiv \frac{1+r}{r} \{F(s_H, k^*(s_H)) - Rk^*(s_H)\},$$

where $k^*(s_H)$ is the first-best level of working capital at s_H , as defined in (9). Clearly, there is no way for the firm to repay more than \bar{D} . Here, we consider what would happen when

$$D > \bar{D}. \quad (10)$$

¹² This is a partial equilibrium model, in which r is given exogenously. In the general equilibrium, the value of r would be determined by the zero-profit condition for the bank, given the initial amount of lending D_0 :

$$D_0 = d(s_0, D_0),$$

where D_0 is the amount that the bank lends to the firm in period 0, and $d(s_0, D_0)$ is the payoff for the bank, defined as the solution to (7).

We define $k^{npl}(s)$, and $G^{npl}(s) \equiv G(s, k^{npl}(s))$ as follows. First, define $\tilde{k}^{npl}(s)$ as

$$\tilde{k}^{npl}(s) \equiv \arg \max_k F(s, k) - Rk - G(s, k).$$

Then, if the following inequality holds for each $s \in \{s_L, s_H\}$

$$G(s, \tilde{k}^{npl}(s)) \geq \beta \mathbb{E}[G(s_{+1}, \tilde{k}^{npl}(s_{+1}))|s], \quad (11)$$

set $k^{npl}(s) = \tilde{k}^{npl}(s)$. Note that (11) is necessarily satisfied for $s = s_H$ under our assumption. To see this, since $F_{kk} - G_{kk} < 0$ and $F_{ks} - G_{ks} > 0$, $\tilde{k}^{npl}(s_H) > \tilde{k}^{npl}(s_L)$.¹³ Then, since $G(s, k)$ is increasing in both s and k , $G(s_H, \tilde{k}^{npl}(s_H)) > G(s_L, \tilde{k}^{npl}(s_L))$. It follows that (11) is satisfied for $s = s_H$.

If (11) is not satisfied for $s = s_L$, then redefine $k^{npl}(s)$ by

$$k^{npl}(s_H) = \tilde{k}^{npl}(s_H), \quad (12)$$

$$G(s_L, k^{npl}(s_L)) = \beta \mathbb{E}[G(s_{+1}, k^{npl}(s_{+1}))|s_L]. \quad (13)$$

Given $k^{npl}(s_H) = \tilde{k}^{npl}(s_H)$, there is a unique solution $k^{npl}(s_L)$ that solves equation (13).

In Sections 4–6, we focus on the case where condition (11) is satisfied and $k^{npl}(s) = \tilde{k}^{npl}(s)$ for all $s \in \{s_H, s_L\}$. The next lemma demonstrates that k is no less than k^{npl} in equilibrium.

Lemma 1. *In equilibrium, $k(s, D) \geq k^{npl}(s)$ for all $s \in \{s_L, s_H\}$ and $D \in \mathbb{R}_+$.*

Proof. Suppose $k(s, D) < k^{npl}(s)$ for some $(s, D) \in \{s_L, s_H\} \times \mathbb{R}_+$. Then, the bank can increase both k and b without violating any constraints. This contradicts Assumption 1. Thus, $k(s, D) \geq k^{npl}(s)$. \square

The next proposition is one of the main results in this paper. It states that if the contractual amount of debt D exceeds a threshold value \bar{D} , then (i) the equilibrium values $\{k(s, D), b(s, D), d(s, D), V(s, D)\}$ do not depend on D ; (ii) their values correspond to $k^{npl}(s)$ defined above; and (iii) the contractual amount of debt D will never decrease. A similar but stronger result is obtained for the discrete version of the model in Proposition 10 below.

Proposition 2. *For $D > \bar{D}$, the equilibrium values of the variables do not depend on D , and satisfy $\{k(s, D), b(s, D), d(s, D), V(s, D)\} = \{k^{npl}(s), b^{npl}(s), d^{npl}(s), G^{npl}(s)\}$, where*

$$b^{npl}(s) \equiv F(s, k^{npl}(s)) - Rk^{npl}(s) - G^{npl}(s) + \beta \mathbb{E}G^{npl}(s_{+1}),$$

$$d^{npl}(s) \equiv b^{npl}(s) + \beta \mathbb{E}d^{npl}(s_{+1}).$$

¹³By differentiating the first-order condition for the definition of $\tilde{k}^{npl}(s)$, we obtain

$$\frac{d\tilde{k}^{npl}}{ds} = -\frac{F_{ks} - G_{ks}}{F_{kk} - G_{kk}} > 0.$$

Proof is given in Appendix A. As Lemma 1 shows, k^{npl} is the lowest level of working capital provision that can occur in equilibrium. Thus, once D becomes greater than \bar{D} , the equilibrium level of production falls to the lowest level *permanently*. This creates a sharp contrast with the property of the constrained efficient equilibrium analyzed by AH, where the first-best provision of working capital is attained in a finite period of time with probability one in the absence of liquidation.

Intuitively, the proposition follows from the fact that the contractual amount of debt D is no longer payoff relevant if it becomes so large that there is no way for the firm to pay it back in full. Thus, the offer $\{k(s, D), b(s, D), d(s, D), V(s, D)\}$ made by the bank cannot depend on D in the region where $D > \bar{D}$. In other words, the bank loses its ability to commit to future repayment plans when the debt becomes “too large.” The loss of the bank’s credibility forces its offer to be “static,” depending solely on the current exogenous state s . As discussed by AH, constrained efficiency requires the offer to be dynamic. In particular, the payoff to the firm must be backloaded until the amount of debt becomes sufficiently small. In the absence of debt restructuring, too much debt makes the dynamic provision of incentives infeasible, leading to an inefficiently low level of production by the firm.

NPL equilibrium: In what follows, we refer to the set of values of endogenous variables defined above, $\{k^{npl}(s), b^{npl}(s), d^{npl}(s), G^{npl}(s)\}$, as the *nonperforming loans (NPL) equilibrium*. This term might be somewhat confusing, because $\{k^{npl}(s), b^{npl}(s), d^{npl}(s), G^{npl}(s)\}$ does not constitute an equilibrium by itself. Instead, it comprises the set of values that those variables take in the region where the contractual amount of debt exceeds a certain threshold. In this sense, it is a “part” of equilibrium. When we say that the economy falls into the NPL equilibrium, we do not mean that the economy switches to a new equilibrium called the NPL equilibrium. What we mean is, rather, that as a result of accumulation of debt, the equilibrium values of endogenous variables become particular levels provided by $\{k^{npl}(s), b^{npl}(s), d^{npl}(s), G^{npl}(s)\}$. Despite the potential confusion this causes, we find it a convenient term for our expositional purpose.

4.3 Note on equilibrium for a small D

We have established that the economy falls into the NPL equilibrium for sufficiently large D . For a smaller value of D , however, it is difficult to provide any further characterization of equilibrium. There are two reasons for this, which we discuss in this subsection.

Discontinuity of the value functions: One of the difficulties is that a solution to the (7) involves many (possibly an infinite number of) jumps in $\{b(s, D), k(s, D), V(s, D), d(s, D)\}$. First of all, there is a discontinuous jump in those variables when D equals the

threshold value above which the NPL equilibrium occurs. It then defines another, smaller threshold for D at which the next period's debt is just equal to the threshold for the NPL equilibrium. It then defines another threshold, and so on. Such discontinuity makes the application of the standard result for dynamic programming difficult. To overcome this difficulty, we consider a discrete version of the model in the following sections.

Competing forces of back loading and front loading: Another difficulty arises if we assume that $\beta > \frac{1}{1+r}$. This is a natural assumption because the debt is risky. However, it provides an incentive for the bank to front load the payment to the firm, at least when the debt is sufficiently small. To see this, suppose that D is sufficiently small in period 0 so that the firm can repay D at once. If D is repaid in period 0, then the value for the bank is D . On the other hand, if the firm repays nothing in period 0 and $(1+r)D$ in period 1, then the present value for the bank in period 0 is $\beta(1+r)D$, which is larger than D , because $\beta(1+r) > 1$. Thus, for a small value of D , the bank may choose repayment b such that D_{+1} is greater than D . Therefore, if $\beta > \frac{1}{1+r}$, there are competing forces that induce backloading and frontloading the payoff of the firm. This complicates the dynamics of the contract. Because of this difficulty, we assume that $\beta = \frac{1}{1+r}$ to obtain analytical results in Section 6. The case of $\beta > \frac{1}{1+r}$ is analyzed numerically in Section 6.5.

5 Discretization of the model

As discussed above, in the following, we consider a discrete version of the model.

Discretization: Denote the set of integers by \mathbb{Z} , and define

$$\begin{aligned}\Delta &= \{0, \delta, 2\delta, \dots, N_{\max}\delta\}, \\ \Delta_{+1} &= \{0, \delta, 2\delta, \dots, n_{\delta}[(1+r)N_{\max}\delta]\}.\end{aligned}$$

Here, δ is the minimum unit of debt, $N_{\max} \in \mathbb{Z}$ is a sufficiently large integer, and $n_{\delta}(x) = n\delta$ for $x > 0$, where n is the integer satisfying $(n-1)\delta < x \leq n\delta$. We assume that the amount of debt, D , must be an element of Δ :

$$D \in \Delta.$$

For each $s \in \{s_L, s_H\}$, the set of possible values of k , $\Delta_k(s)$, is defined as

$$\Delta_k(s) = \left\{ k \mid \exists n \in \mathbb{Z}, \text{ s.t. } F(s, k) - Rk - G(s, k) = n \times \frac{\delta}{1+r} \right\}.$$

Then, $k^*(s)$ and $k^{npl}(s)$ are defined as

$$k^*(s) = \arg \max_{k \in \Delta_k(s)} F(s, k) - Rk,$$

$$k^{npl}(s_H) = \arg \max_{k \in \Delta_k(s_H)} F(s_H, k) - Rk - G(s_H, k),$$

$$k^{npl}(s_L) = \arg \max_{k \in \Delta_k(s_L)} F(s_L, k) - Rk - G(s_L, k),$$

Here, we are assuming that the parameter values are selected such that

$$G^{npl}(s_H) > \beta[\pi_{HH}G^{npl}(s_H) + \pi_{HL}G^{npl}(s_L)], \quad (14)$$

$$G^{npl}(s_L) > \beta[\pi_{LL}G^{npl}(s_L) + \pi_{LH}G^{npl}(s_H)], \quad (15)$$

where $\pi_{HH} = \Pr(s_{t+1} = s_H | s_t = s_H)$, $\pi_{HL} = 1 - \pi_{HH}$, $\pi_{LL} = \Pr(s_{t+1} = s_L | s_t = s_L)$, and $\pi_{LH} = 1 - \pi_{LL}$. We also let $G^{npl}(s) \equiv G(s, k^{npl}(s))$.

Our arguments in this paper can be easily modified for the case where the inequalities (14) and/or (15) do not hold.¹⁴ For each $s \in \{s_L, s_H\}$, the repayment in the NPL equilibrium, $b^{npl}(s)$, is defined by

$$b^{npl}(s) = F(s, k^{npl}(s)) - Rk^{npl}(s) - G^{npl}(s) + \beta \mathbb{E}[G^{npl}(s_{+1}) | s].$$

The set of possible values of repayments, $\Delta_b(s, D)$, depends on D :

$$\Delta_b(s, D) = \left\{ b \in \mathbb{R} \mid \exists \tilde{D}_{+1} \in \Delta_{+1} \text{ s.t. } b = D - \frac{1}{1+r} \tilde{D}_{+1}, \text{ and } b \geq 0 \right\} \cup \{b^{npl}(s)\}.$$

At each state (s, D) , b and k must satisfy

$$b \in \Delta_b(s, D), \quad \text{and} \quad k \in \Delta_k(s).$$

Bank's problem: Let $V^e(s, D)$ denote the bank's expectation regarding the value of the firm as a function of the current state (s, D) . Then, the bank's profit maximization is formulated as the Bellman equation:

$$d(s, D) = \max_{b \in \Gamma(s, D)} b + \beta \mathbb{E}d(s_{+1}, D_{+1}), \quad (16)$$

where

$$\Gamma(s, D) = \{b \in \Delta_b(s, D) \mid \exists k \in \Delta_k(s) \text{ s.t.}$$

$$D_{+1} = \min\{N_{\max}\delta, n_\delta[(1+r)(D-b)]\},$$

$$F(s, k) - Rk - b + \beta \mathbb{E}V^e(s_{+1}, D_{+1}) \geq G(s, k),$$

$$F(s, k) - Rk - b \geq 0\}.$$

¹⁴For this purpose, it suffices to redefine

$$k^{npl}(s_H) = \max\{k \in \Delta_k(s_H) \mid G(s, k_H) \leq \beta[\pi_{HH}G(s_H, k) + \pi_{HL}G(s_H, k^{npl}(s_L))]\},$$

and/or

$$k^{npl}(s_L) = \max\{k \in \Delta_k(s_L) \mid G(s, k_L) \leq \beta[\pi_{LL}G(s_L, k) + \pi_{LH}G(s_H, k^{npl}(s_H))]\}. \text{ In the case where } k^{npl}(s)$$

is redefined, $b^{npl}(s)$ is also redefined as $b^{npl}(s) = F(s, k^{npl}(s)) - Rk^{npl}(s)$.

Here, $n_\delta[(1+r)(D-b)] = n \times \delta$, where n is the integer that satisfies $(n-1)\delta < (1+r)(D-b) \leq n\delta$.

Let $\Sigma(s, D)$ denote the set of (b, D_{+1}) that solves the maximization problem in (16). The bank then decides k and $V(s, D)$ by solving the following problem:

$$V(s, D) = \max_{k \in \Delta_k(s), (b, D_{+1}) \in \Sigma(s, D)} F(s, k) - Rk - b + \beta \mathbb{E}V^e(s_{+1}, D_{+1}), \quad (17)$$

subject to

$$\begin{aligned} F(s, k) - Rk - b + \beta \mathbb{E}V^e(s_{+1}, D_{+1}) &\geq G(s, k), \\ F(s, k) - Rk - b &\geq 0. \end{aligned}$$

Let $\Lambda(s, D)$ denote the set of (k, b, D_{+1}) that solves the maximization problem in (17).

Given $\Lambda(s, D)$, the equilibrium values of (k, b, D_{+1}) at (s, D) are selected as follows. First, $b(s, D)$ and $D_{+1}(s, D)$ are decided as

$$b(s, D) = \max_{(k, b, D_{+1}) \in \Lambda(s, D)} b, \quad (18)$$

$$D_{+1}(s, D) = \min\{N_{\max}\delta, n_\delta[(1+r)\{D - b(s, D)\}]\}. \quad (19)$$

Then, $k(s, D)$ is determined by

$$k(s, D) = \max_{(k, b(s, D), D_{+1}(s, D)) \in \Lambda(s, D)} k.$$

Then, the value of the firm must satisfy

$$V(s, D) = F(s, k(s, D)) - Rk(s, D) - b(s, D) + \beta \mathbb{E}V^e(s_{+1}, D_{+1}(s, D)). \quad (20)$$

Assuming rational expectations, the bank's belief $V^e(s, D)$ should be consistent with $V(s, D)$ given in (20):

$$V(s, D) = V^e(s, D). \quad (21)$$

Definition of the threshold, $D_{\max}(s)$: Given the existence of an equilibrium, we define $D_{\max}(s)$ as follows:

$$D_{\max}(s_H) \equiv \max\{D \in \Delta \mid D_{+1}(s_H, D) < D\}, \quad (22)$$

$$D_{\max}(s_L) \equiv \max\{D \in \Delta \mid D_{+1}(s_L, D) < D_{\max}(s_H)\}. \quad (23)$$

Thus, if D exceeds $D_{\max}(s_H)$ at s_H , the amount of debt in the next period is greater than or equal to D . Similarly, if D exceeds $D_{\max}(s_L)$ at state s_L , the next period's debt is greater than or equal to $D_{\max}(s_H)$. The following lemma demonstrates that if $D > D_{\max}(s_L)$, then $D_{+1}(s_L, D) \geq D$. As a result, once D exceeds $D_{\max}(s)$ at each s , D will never decrease.

Lemma 3. *If $D > D_{\max}(s_L)$, then $D_{+1}(s_L, D) \geq D$.*

Proof. Let $D > D_{\max}(s_L)$, and suppose, for the sake of contradiction, that $D_{+1}(s_L, D) < D$. Then,

$$D_{+1}(s_L, D_{+1}(s_L, D)) < D_{+1}(s_L, D).$$

However, since $D > D_{\max}(s_L)$, $D_{+1}(s_L, D) \geq D_{\max}(s_H)$. By the definition of $D_{\max}(s_H)$, we have

$$D_{+1}(s_H, D_{+1}(s_L, D)) \geq D_{+1}(s_L, D).$$

We also have

$$D_{+1}(s_H, D_{+1}(s_L, D)) \leq D_{+1}(s_L, D_{+1}(s_L, D)).$$

Combining these inequalities, we obtain

$$D_{+1}(s_L, D) \leq D_{+1}(s_H, D_{+1}(s_L, D)) \leq D_{+1}(s_L, D_{+1}(s_L, D)) < D_{+1}(s_L, D),$$

which is a contradiction. \square

We can confirm that $D_{\max}(s) < \infty$ as follows. For $D > \bar{D}$, it is obvious that, for any $b \leq \max_k \{F(s, k) - Rk\}$, the debt never decreases over time, i.e., $D_{+1} = (1+r)(D-b) > D$. Thus, there exists $D_{\max}(s_H)$ such that $D_{\max}(s_H) \leq \bar{D} < \infty$. As $D_{\max}(s_H) < \infty$, it follows from (22)-(23) that $D_{\max}(s_L) \leq D_{\max}(s_H)$.

6 Equilibrium of the discrete model

In this section, we assume that the interest rate in the debt contract is equal to the market rate for the risk-free bond:

$$\beta = \frac{1}{1+r}. \quad (24)$$

As discussed in Section 4.3, it simplifies the analysis on the equilibrium dynamics in our model. Note, however, that even under assumption (24), the bank can still make the expected payoff nonnegative, by adjusting the initial amount of the principal of the loan.¹⁵ In Sections 6.1, 6.2, and 6.3, we characterize the equilibrium, taking the existence

¹⁵ The initial principal of the debt D_0 may not be fully repaid in equilibrium, so that the expected PDV of repayments, $d(s_0, D_0) \equiv \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t b_t$, may be smaller than D_0 . Let I_0 denote the initial amount of lending. The zero profit condition for the bank is satisfied if the contractual amount of initial debt, D_0 , is set as

$$I_0 = d(s_0, D_0).$$

of an equilibrium as given. In Section 6.4, we prove the existence. In Section 6.5 we show numerical results. There, we also consider the case where $\beta > \frac{1}{1+r}$ and confirm the robustness of the results.

6.1 The repayment in the case of small D

Two working assumptions: In Sections 6.1 and 6.2, we proceed by making the following two assumptions. They are verified later in Lemma 14 in Section 6.4. All proofs are provided in the Appendix.

Assumption 2. For $D < D_{\max}(s)$, $V^e(s, D + \delta) \leq V^e(s, D) - \delta$.

Assumption 3. For all s and $D \geq \delta$, $b(s, D)$ satisfies

$$b(s, D) \geq \delta. \quad (25)$$

We first characterize the equilibrium repayment function $b(s, D)$ for $D \leq D_{\max}(s)$.

Lemma 4. For all $D \geq 0$, $d(s, D + \delta) \leq d(s, D) + \delta$.

Lemma 5. For $D \leq D_{\max}(s)$, $b(s, D) = \bar{b}(s, D)$, where $\bar{b}(s, D)$ is the maximum feasible value, i.e., $\bar{b}(s, D) = \max\{b \mid b \in \Gamma(s, D)\}$. It also holds that $k(s, D) > k^{npl}(s)$ for $D \leq D_{\max}(s)$.

Lemma 5 directly implies the following corollary.

Corollary 6. If (s, D) is a state such that $k(s, D) = k^*(s)$, then

$$b(s, D) = \min \{D, b^*(s, D)\},$$

where

$$\begin{aligned} b^*(s, D) &= \max_{n \in \mathbb{Z}} D - \beta n \delta, \\ \text{s.t. } D - \beta n \delta &\leq F(s, k^*(s)) - Rk^*(s). \end{aligned}$$

Now, we define

$$\begin{aligned} f(s, k) &\equiv F(s, k) - Rk - G(s, k), \\ \delta_f &\equiv \max_{k \in \Delta_k(s), k^{npl}(s) \leq k \leq k^*(s)} F'(s, k) - R, \\ \delta_k &\equiv \max\{k' - k \mid k \in \Delta_k(s), k' \in \Delta_k(s), \\ &\quad k^{npl}(s) \leq k < k' < k^*(s), |f(s, k) - f(s, k')| = \beta\delta\}, \\ \delta_g &\equiv \max\{G(s, k') - G(s, k) \mid k \in \Delta_k(s), k' \in \Delta_k(s), \\ &\quad k^{npl}(s) \leq k < k' < k^*(s), |f(s, k) - f(s, k')| = \beta\delta\}. \end{aligned}$$

Note that $\delta_f = O(1)$, $\delta_k = O(\delta)$, and $\delta_g = O(\delta)$. Then, the following lemma holds.

Lemma 7. For (s, D) such that $k^{npl}(s) < k(s, D) < k^*(s)$, it holds that $0 \leq F(s, k(s, D)) - Rk(s, D) - b(s, D) < \xi + \beta\delta$, where $\xi = \delta_f \delta_k$.

As $\xi = O(\delta)$, Corollary 6 and Lemma 7 implies that $b(s, D) \approx \min\{D, F(s, k(s, D)) - Rk(s, D)\}$ for small δ . This means that the optimal contract involves backloaded payment to the firm; that is, the firm repays debt as fast as possible by setting its dividend at almost zero, i.e., $b \approx \min\{D, F(s, k) - Rk\}$, when D is smaller than or equal to $D_{\max}(s)$.

6.2 Equilibrium at large D

Here, we demonstrate that when D is large so that $D > D_{\max}(s)$, the equilibrium exhibits the feature that we call the NPL equilibrium. For that, the minimum unit δ is sufficiently small such that the following assumption is satisfied.

Assumption 4. The value of δ and the function $G(s, k)$ satisfy

$$\min_s G^{npl}(s) > \frac{\xi + \beta(\delta + \delta_g)}{1 - \beta},$$

where $\xi = \delta_f \delta_k$.

Lemma 8. For $k(s, D) < k^*(s)$, the binding no-default constraint implies that

$$V(s, D) - \delta_g < G(s, k(s, D)) \leq V(s, D).$$

Proof. The first inequality holds because otherwise the bank can obtain a positive gain by changing $k(s, D)$ to k' , where $k' > k(s, D)$ and $|f(k(s, D)) - f(k')| = \beta\delta$. \square

Lemma 9. For all $D > D_{\max}(s)$, it holds that $k(s, D) = k^{npl}(s)$.

Proposition 10. For all (s, D) with $D > D_{\max}(s)$, $d(s, D) = d^{npl}(s)$, $k(s, D) = k^{npl}(s)$, $b(s, D) = b^{npl}(s)$, and $V(s, D) = G^{npl}(s)$.

This proposition¹⁶ is similar to Proposition 2 in Section 4.2, but stronger because $D_{\max}(s) \leq \bar{D}$. Once D exceeds $D_{\max}(s)$ at any s , the contractual amount of debt will

¹⁶ In Proposition 10, we have assumed that the parameter values are restricted such that $k^{npl}(s)$ is defined by $k^{npl}(s) \equiv \arg \max_{k \in \Delta_k(s)} F(s, k) - Rk - G(s, k)$. It is generalized as follows, in the case where $k^{npl}(s_L)$ is defined by $k^{npl}(s_L) = \max\{k \in \Delta_k(s_L) \mid G(s, k_L) \leq \beta[\pi_{LL}G(s_L, k) + \pi_{LH}G(s_H, k^{npl}(s_H))]\}$: We define $V^{npl}(s)$ by

$$\begin{aligned} V^{npl}(s_H) &= G^{npl}(s_H), \\ V^{npl}(s_L) &= \beta\mathbb{E}[V^{npl}(s_{+1}) \mid s = s_L]. \end{aligned}$$

Then, we redefine $b^{npl}(s)$ by $b^{npl}(s) = F(s, k^{npl}(s)) - Rk^{npl}(s) - G^{npl}(s) + \beta\mathbb{E}[V^{npl}(s_{+1}) \mid s]$. Then, the modified version of Proposition 10 states: For all (s, D) with $D > D_{\max}(s)$, $d(s, D) = d^{npl}(s)$, $k(s, D) = k^{npl}(s)$, $b(s, D) = b^{npl}(s)$, and $V(s, D) = V^{npl}(s)$. The proof of the modified version is similar to that of Proposition 10.

keep on growing and the constraint $b \leq D$ will never bind. Thus, D becomes irrelevant for the choice of k and b , and the equilibrium variables depend solely on the exogenous state s , given as the NPL equilibrium. The intuition is that when D is larger than $D_{\max}(s)$, it becomes impossible to pay back D in full, and thus the contractual amount of debt becomes payoff irrelevant. It follows that the lender can no longer commit to any future repayment plans. The loss of the banks credibility leads to an inefficient outcome referred to as the NPL equilibrium.

6.3 Characterization of the equilibrium

Here, we summarize the analytical results obtained for the discrete model with $1+r = \beta^{-1}$. First, there exist endogenously determined thresholds, $D_{\max}(s)$, which are defined by (22) and (23).

Define $D_{\min}(s_L)$ by

$$D_{\min}(s_L) = \max \{D \in \Delta \mid \forall D' \leq D, D_{+1}(s_L, D') < D'\}.$$

Since $D_{+1}(s_H, D) \leq D_{+1}(s_L, D)$ for all D , once D becomes sufficiently small that $D \leq D_{\min}(s_L)$, D declines over time thereafter, regardless of the realization of the exogenous state s .

Thus, if the initial debt D_0 satisfies $D_0 \leq D_{\min}(s_L)$, there is no chance that the economy will fall into the NPL equilibrium. In this case, the equilibrium dynamics are qualitatively the same as those of the AH model. The borrower repays as much debt as possible in every period by setting dividend (almost) zero, i.e., $F(s, k) - Rk - b \approx 0$ (Lemma 7), where the qualification ‘‘almost’’ is required because of the discretization. Functions $k(s, D)$ and $V(s, D)$ are both non-increasing in D .¹⁷ As the current debt D satisfies $D \leq D_{\min}(s_L)$, the next period debt D_{+1} is smaller than D . Thus, along the equilibrium path, $D_{t+1} = \beta^{-1}[D_t - b(s_t, D_t)]$ converges to 0 within finite periods. When $D = 0$, the bank takes 0 because $b \leq D$ binds at $D = 0$, and the problem (for the bank) is to maximize the firm’s profits by selecting $k = k^*(s) = \arg \max_k F(s, k) - Rk$. Thus, the economy converges to a first-best allocation, $\{D, k\} = \{0, k^*(s)\}$, within finite periods. In this case, the state variable, D , remains payoff-relevant along the whole equilibrium path.

If the initial debt satisfies $D_0 \geq D_{\max}(s_H)$, debt D_t always increases regardless of the exogenous state s , i.e., $D_{t+1} \geq D_t$ with probability one for all t . Then, D_t is no longer a payoff-relevant state variable, and the bank is unable to make a commitment to future repayment plans. As a result, the economy falls into the NPL equilibrium: $\{k(s, D), b(s, D), d(s, D), V(s, D)\} = \{k^{npl}(s), b^{npl}(s), d^{npl}(s), V^{npl}(s)\}$. In the NPL equilibrium, the firm’s output is ‘‘minimized’’ in the sense that $k^{npl}(s) = \min_{D \in \Delta} k(s, D)$.

¹⁷ First, Lemma 14 in Section 6.4 implies that $V(s, D)$ is non-increasing in D . Second, $k(s, D)$ is non-increasing in D , because $k(s, D) = \max\{k \in \Delta_k(s) \mid V(s, D) \geq G(s, k)\}$ and $V(s, D)$ is non-increasing.

For initial debt D_0 in the intermediate region, $D_{\min}(s_L) < D_0 \leq D_{\max}(s_H)$, the economy may end up with either the first best or NPL equilibrium. Both can occur with a positive probability. While D is in this region, the dividend to the firm is $F(s, k) - Rk - b \approx 0$ (Lemma 7). D remains to be payoff-relevant.

6.4 Existence of equilibrium

In this subsection, we demonstrate the existence of an equilibrium, which is characterized as a fixed point of an operator, T , on the functions of (s, D) . As the space for (s, D) is discrete and finite, the existence of an equilibrium is proved by finding a fixed point of the operator T in a finite-dimensional vector space.

Define the operator T by

$$(d^{(n+1)}(s, D), V^{(n+1)}(s, D), \bar{D}^{(n+1)}(s)) = T(d^{(n)}(s, D), V^{(n)}(s, D), \bar{D}^{(n)}(s)),$$

where $(d^{(n+1)}(s, D), V^{(n+1)}(s, D), \bar{D}^{(n+1)}(s))$ is generated from $(d^{(n)}(s, D), V^{(n)}(s, D), \bar{D}^{(n)}(s))$, as follows. Define $\Gamma^{(n+1)}(s, D)$ by

$$\begin{aligned} \Gamma^{(n+1)}(s, D) \equiv & \{b \in \Delta_b(s, D) \mid \exists k \in \Delta_k(s) \text{ s.t.} \\ & D_{+1} = \min\{N_{\max}\delta, n_\delta[(1+r)(D-b)]\}, \\ & F(s, k) - Rk - b + \beta\mathbb{E}V^{(n)}(s_{+1}, D_{+1}) \geq G(s, k), \\ & F(s, k) - Rk - b \geq 0\}. \end{aligned}$$

Given state (s, D) and expectations $(V^{(n)}(s, D), d^{(n)}(s, D))$, the bank solves

$$d^{(n+1)}(s, D) = \max_{b \in \Gamma^{(n+1)}(s, D)} b + \beta\mathbb{E}d^{(n)}(s_{+1}, D_{+1}). \quad (26)$$

Denote by $\Sigma^{(n+1)}(s, D)$ the set of (b, D_{+1}) that solves the maximization in (26). The bank decides k and $V^{(n+1)}(s, D)$ by solving the following problem.

$$V^{(n+1)}(s, D) = \max_{k \in \Delta_k(s), (b, D_{+1}) \in \Sigma^{(n+1)}(s, D)} F(s, k) - Rk - b + \beta\mathbb{E}V^{(n)}(s_{+1}, D_{+1}), \quad (27)$$

subject to

$$\begin{aligned} F(s, k) - Rk - b + \beta\mathbb{E}V^{(n)}(s_{+1}, D_{+1}) & \geq G(s, k), \\ F(s, k) - Rk - b & \geq 0. \end{aligned}$$

Let $\Lambda^{(n+1)}(s, D)$ denote the set of (k, b, D_{+1}) that solves the maximization in (27).

The equilibrium values of (k, b, D_{+1}) are selected as follows. First, $b^{(n+1)}(s, D)$ and $D_{+1}^{(n+1)}(s, D)$ are determined as

$$b^{(n+1)}(s, D) = \max_{(k, b, D_{+1}) \in \Lambda^{(n+1)}(s, D)} b, \quad (28)$$

$$D_{+1}^{(n+1)}(s, D) = \min\{N_{\max}\delta, n_\delta((1+r)[D - b^{(n+1)}(s, D)])\}. \quad (29)$$

Then, $k^{(n+1)}(s, D)$ is decided as

$$k^{(n+1)}(s, D) = \max_{(k, b^{(n+1)}(s, D), D_{+1}^{(n+1)}(s, D)) \in \Lambda^{(n+1)}(s, D)} k,$$

and $\bar{D}^{(n+1)}(s)$ is provided by

$$\begin{aligned} \bar{D}^{(n+1)}(s_H) &= \max \left\{ D \in \Delta \mid D_{+1}^{(n+1)}(s_H, D) < \bar{D}^{(n)}(s_H) \right\}, \\ \bar{D}^{(n+1)}(s_L) &= \max \left\{ D \in \Delta \mid D_{+1}^{(n+1)}(s_L, D) < \bar{D}^{(n)}(s_H) \right\}. \end{aligned}$$

Define $V_H^* \equiv \frac{1}{1-\beta}[F(s_H, k^*(s_H)) - Rk^*(s_H)]$.

We set the initial values $(\bar{D}^{(0)}(s), d^{(0)}(s, D), V^{(0)}(s, D))$ as follows.

$$\begin{aligned} \bar{D}^{(0)}(s) &= \bar{D}^{(0)} \equiv V_H^* - G^{npl}(s_H), \\ d^{(0)}(s, D) &= \begin{cases} D & \text{for } D \leq \bar{D}^{(0)}, \\ d^{npl}(s) & \text{for } D > \bar{D}^{(0)}, \end{cases} \\ V^{(0)}(s, D) &= \begin{cases} V_H^* - D & \text{for } D \leq \bar{D}^{(0)}, \\ G^{npl}(s) & \text{for } D > \bar{D}^{(0)}. \end{cases} \end{aligned}$$

Now, the existence of a fixed point of operator T is established by demonstrating the convergence of the sequence $\{d^{(n)}, V^{(n)}, \bar{D}^{(n)}\}_{n=0}^{\infty}$.

Theorem 11. *There exists a fixed point $(d(s, D), V(s, D), D_{\max}(s))$ of the operator T , that is, $(d, V, D_{\max}) = T(d, V, D_{\max})$.*

This fixed point is an equilibrium of the economy. The proof of this theorem is as follows. The following lemmas demonstrate that $(d^{(n)}(s, D), V^{(n)}(s, D), \bar{D}^{(n)}(s))$ satisfies

$$\begin{aligned} (d^{npl}(s), G^{npl}(s), d^{npl}(s)) &\leq (d^{(n+1)}(s, D), V^{(n+1)}(s, D), \bar{D}^{(n+1)}(s)) \\ &\leq (d^{(n)}(s, D), V^{(n)}(s, D), \bar{D}^{(n)}(s)) \end{aligned}$$

for $D > d^{npl}(s)$, and that

$$\begin{aligned} (0, G^{npl}(s), d^{npl}(s)) &\leq (d^{(n+1)}(s, D), V^{(n+1)}(s, D), \bar{D}^{(n+1)}(s)) \\ &\leq (d^{(n)}(s, D), V^{(n)}(s, D), \bar{D}^{(n)}(s)) \end{aligned}$$

for $D \leq d^{npl}(s)$. Thus, the sequence $\{d^{(n)}(s, D), V^{(n)}(s, D), \bar{D}^{(n)}(s)\}_{n=0}^{\infty}$ at any fixed (s, D) converges pointwise, because it is a weakly decreasing sequence of real numbers, which is bounded from below: $\exists(d(s, D), V(s, D), D_{\max}(s))$ such that

$$(d^{(n)}(s, D), V^{(n)}(s, D), \bar{D}^{(n)}(s)) \rightarrow (d(s, D), V(s, D), D_{\max}(s))$$

as $n \rightarrow \infty$. This $(d(s, D), V(s, D), D_{\max}(s))$ is a fixed point of the operator T by construction.

The proof is by induction. The first step of the induction is provided by the following lemma.

Lemma 12. Denote $(d^{(1)}(s, D), V^{(1)}(s, D), \bar{D}^{(1)}(s)) = T(d^{(0)}(s, D), V^{(0)}(s, D), \bar{D}^{(0)}(s))$. Let $(b^{(1)}(s, D), k^{(1)}(s, D))$ be the value of (b, k) that solves (26) and (27) with $n = 0$. Then, $(d^{(1)}(s, D), V^{(1)}(s, D), \bar{D}^{(1)}(s), b^{(1)}(s, D), k^{(1)}(s, D))$ satisfies

- (i) $d^{(1)}(s, D + \delta) \leq d^{(1)}(s, D) + \delta$,
- (ii) $d^{npl}(s) \leq d^{(1)}(s, D) \leq d^{(0)}(s, D)$ for $D > d^{npl}(s)$, and $0 \leq d^{(1)}(s, D) \leq d^{(0)}(s, D)$ for $D \leq d^{npl}(s)$,
- (iii) $\forall D > \bar{D}^{(1)}(s)$, $d^{(1)}(s, D) = d^{npl}(s)$, $V^{(1)}(s, D) = V^{npl}(s)$, $b^{(1)}(s, D) = b^{npl}(s)$, $k^{(1)}(s, D) = k^{npl}(s)$,
- (iv) $V^{(1)}(s, D + \delta) \leq -\delta + V^{(1)}(s, D)$ for $D < \bar{D}^{(1)}(s)$,
- (v) $\forall (s, D)$, $G^{npl}(s) \leq V^{(1)}(s, D) \leq V^{(0)}(s, D)$,
- (vi) $d^{npl}(s) < \bar{D}^{(1)}(s) < \bar{D}^{(0)}$.

The second step of the induction is provided by the following lemma.

Lemma 13. Denote $(d^{(n+1)}(s, D), V^{(n+1)}(s, D), \bar{D}^{(n+1)}(s)) = T(d^{(n)}(s, D), V^{(n)}(s, D), \bar{D}^{(n)}(s))$. Let $(b^{(n+1)}(s, D), k^{(n+1)}(s, D))$ be the value of (b, k) that solves (26) and (27). Suppose that $(d^{(n)}(s, D), V^{(n)}(s, D), \bar{D}^{(n)}(s), b^{(n)}(s, D), k^{(n)}(s, D))$ satisfies

- (i') $d^{(n)}(s, D + \delta) \leq d^{(n)}(s, D) + \delta$,
- (ii') $d^{npl}(s) \leq d^{(n)}(s, D) \leq d^{(n-1)}(s, D)$ for $D > d^{npl}(s)$, and $0 \leq d^{(n)}(s, D) \leq d^{(n-1)}(s, D)$ for $D \leq d^{npl}(s)$
- (iii') $\forall D > \bar{D}^{(n)}(s)$, $d^{(n)}(s, D) = d^{npl}(s)$ and $V^{(n)}(s, D) = V^{npl}(s)$,
- (iv') $V^{(n)}(s, D + \delta) \leq -\delta + V^{(n)}(s, D)$ for $D < \bar{D}^{(n)}(s)$,
- (v') $\forall (s, D)$, $G^{npl}(s) \leq V^{(n)}(s, D) \leq V^{(n-1)}(s, D)$,
- (vi') $0 < \bar{D}^{(n)}(s) \leq \bar{D}^{(n-1)}(s)$.

Then, $(d^{(n+1)}(s, D), V^{(n+1)}(s, D), \bar{D}^{(n+1)}(s), b^{(n+1)}(s, D), k^{(n+1)}(s, D))$ satisfies

- (i) $d^{(n+1)}(s, D + \delta) \leq d^{(n+1)}(s, D) + \delta$,
- (ii) $d^{npl}(s) \leq d^{(n+1)}(s, D) \leq d^{(n)}(s, D)$ for $D > d^{npl}(s)$, and $0 \leq d^{(n+1)}(s, D) \leq d^{(n)}(s, D)$ for $D \leq d^{npl}(s)$,
- (iii) $\forall D > \bar{D}^{(n+1)}(s)$, $d^{(n+1)}(s, D) = d^{npl}(s)$ and $V^{(n+1)}(s, D) = V^{npl}(s)$,
- (iv) $V^{(n+1)}(s, D + \delta) \leq -\delta + V^{(n+1)}(s, D)$ for $D < \bar{D}^{(n+1)}(s)$,

$$(v) \forall (s, D), \quad G^{npl}(s) \leq V^{(n+1)}(s, D) \leq V^{(n)}(s, D),$$

$$(vi) 0 < \bar{D}^{(n+1)}(s) \leq \bar{D}^{(n)}(s).$$

In Sections 6.1 and 6.2, we have assumed Assumptions 2 and 3 to establish some equilibrium properties. The next lemma demonstrates that those assumptions are indeed satisfied by the equilibrium constructed as the fixed point of T .

Lemma 14. *For $D \leq D_{\max}(s)$, $V(s, D + \delta) \leq V(s, D) - \delta$. For all $D \geq \delta$, $b(s, D)$ satisfies $b(s, D) \geq \delta$.*

6.5 Numerical experiment

In this subsection, we report numerical solutions to our model. We obtain a fixed point of operator T , defined in Section 6.4, by iterating $(d^{(n+1)}(s, D), V^{(n+1)}(s, D), \bar{D}^{(n+1)}(s)) = T(d^{(n)}(s, D), V^{(n)}(s, D), \bar{D}^{(n)}(s))$. The numerical examples here illustrate the properties of our model discussed in previous subsections. Furthermore, they demonstrate that our model generates a *debt Laffer curve*; that is, the bank's value $d(s, D)$ has an inverted-U shaped relationship with the contractual amount of debt D .

We assume the following functional forms: $F(s, k) = sAk^\alpha$, and $G(s, k) = Bk$. The parameter values are set as shown in Table 1.¹⁸ Our purpose here is to confirm the properties of the model, and thus the parameter values are set somewhat arbitrarily, without much empirical grounding.¹⁹

6.5.1 Baseline case with $1 + r = \beta^{-1}$

Figure 2 plots the bank's value function, $d(s, D)$, the firm's value function, $V(s, D)$, the schedule for the working capital provision, $k(s, D)$, and the repayment schedule, $b(s, D)$. The reader may be puzzled about the discrete jumps in these functions. These jumps at smaller values of D are due to discretization, e.g., $b = (m - \beta n)\delta$, where $D = m\delta$ and $D_{+1} = n\delta$, with $n, m \in \mathbb{Z}$.²⁰ In addition, some jumps at larger values of D are caused by the discontinuity at the boundary of the NPL equilibrium, as discussed in Section 4.3.

The bank's value function $d(s, D)$ in Figure 2 displays a debt Laffer curve for each s . For a sufficiently small value of D , $d(s, D) = D$, that is, D is repaid in full with probability

¹⁸We set $\delta = 0.002$ for discretization of D , and use 5 grid points for k .

¹⁹In particular, the value for α might appear too high. Note, however, that α does not correspond to the capital share here. To be specific, suppose that k finances the capital input K and the production function exhibits decreasing-return to scale, i.e., $Y = K^\beta L^\gamma$, where $\beta + \gamma < 1$ and L is the labor input. The revenue of the firm is then given by

$$F(K) = \max_L K^\beta L^\gamma - wL = CK^{\frac{\beta}{1-\gamma}},$$

where $C > 0$ is a constant and $\frac{\beta}{1-\gamma} < 1$. Then, α in our model is given by $\alpha \equiv \frac{\beta}{1-\gamma}$, which is greater than

Table 1: Parameter values

	Description	Value
A	Normalization	0.1
B	Outside option	0.1
α	Production function	0.8
R	Rental rate of capital	0.1
β	Discount factor	0.96
s_H, s_L	Productivity	1.15, 0.85
π_{HH}, π_{LL}	Transition probability	0.9, 0.9

one, and the economy never falls into the NPL equilibrium. When D is in this region, working capital is provided at the first-best level, $k(s, D) = k^*(s)$, and the firm repays as much as possible to the bank: $b(s, D) \approx \min(D, F(s, k^*(s)) - Rk^*(s))$. As D becomes larger, $k(s, D)$ and $b(s, D)$ start to decrease with D , and $d(s, D)$ exhibits an inverted-U shape in D . When D exceeds the threshold, the economy falls into the NPL equilibrium. In this example, $d(s_H, D) = d^{npl}(s_H)$ for $D > 0.252$ and $d(s_L, D) = d^{npl}(s_L)$ for $D > 0.244$. We should note that in this example, the difference between $k^{npl}(s)$ and $k^*(s)$ is very large. It may be too large to be justified by evidence. One reason for this is that the NPL equilibrium continues permanently under our assumption that debt restructuring never occurs. In Section 7, we will see that the difference between $k^*(s)$ and $k(s, D)$ for large D becomes much more modest with stochastic debt restructuring.

6.5.2 Case with $1 + r > \beta^{-1}$

For ease of theoretical analysis, we have so far assumed that $1 + r = \beta^{-1}$. Here, we numerically examine the case where $1 + r > \beta^{-1}$. Specifically, we set $r = 0.05$, and $\beta = 0.96$. All other parameters are given the same values as before. The bank's value function, $d(s, D)$, the firm's value function, $V(s, D)$, the schedule for working capital provision, $k(s, D)$, and the repayment schedule, $b(s, D)$, in this case are plotted in Figure 3.

the capital share.

²⁰ To illustrate, suppose that b is selected as a function of $D = m\delta$ to solve $b = \max_n(m - \beta n)\delta$ subject to $b < C$. Suppose, in addition, that $\delta = 1$, $\beta = 0.9$, and C is an integer. Then, $D = m$, $D_{+1} = n$, and the solution to the above problem becomes

$$b(D) = C - 0.9 + 0.1x,$$

where $x = D - C \pmod 9$, that is, x is an integer with $0 \leq x \leq 8$ and there exists an integer i such that $D = C + 9i + x$. Thus, $b(D)$ has discrete jumps at $D = C + 9i$, where $i = 1, 2, 3, \dots$.

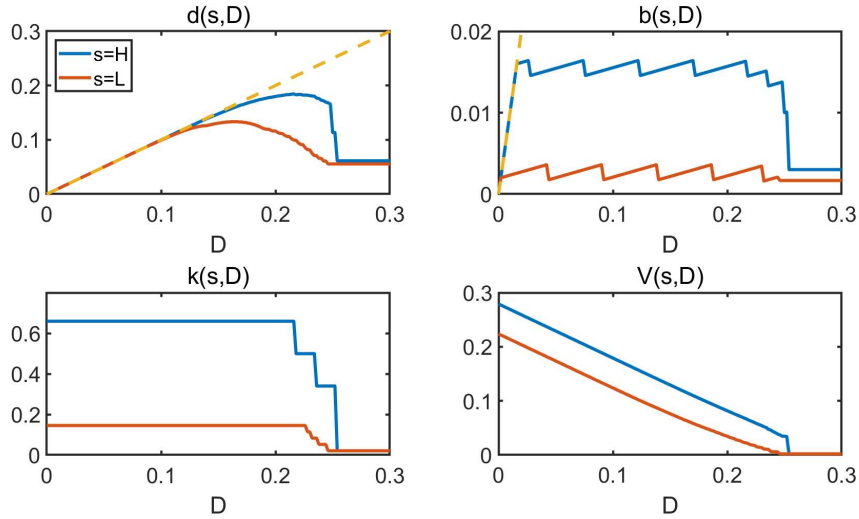


Figure 2: The case with $1 + r = \beta^{-1}$.

Figure 3 also demonstrates the debt Laffer curve relationship between the bank's value and the contractual amount of debt. Major differences from Figure 2 are that $b(s, D) \approx 0$ and $d(s, D) > D$ for small values of D . Setting $b(s, D) \approx 0$ is optimal, because with $1 + r > \beta^{-1}$, the bank can increase $d(s, D)$ by delaying the repayment when D is small, as discussed in Section 4.3. As a result, $d(s, D) > D$ for small values of D . Except for these two differences, the qualitative features of the model with $1 + r > \beta^{-1}$ are the same as the model with $1 + r = \beta^{-1}$. Here, the NPL equilibrium occurs when $D > 0.218$ for $s = s_H$, and when $D > 0.210$ for $s = s_L$.

7 Discrete model with stochastic debt restructuring

In the baseline model, debt restructuring is prohibited. We modify the model in this section such that debt restructuring is feasible with some friction. For simplicity, we adopt a reduced-form approach: In each period t , the bank may be able to reduce the contractual amount of debt D_t . However, this option of debt restructuring arrives with an exogenously given probability $p \in (0, 1)$ in each period. With this option in hand, the bank can reduce D_t to any value $D \in [0, D_t]$. The probability p is a fixed parameter and represents the friction in debt restructuring.

When the bank with contractual amount of debt D_t restructures debt, it reduces D_t to $\hat{D}(s, D_t)$ defined by

$$\hat{D}(s, D_t) = \arg \max_{0 \leq D \leq D_t} d(s, D).$$

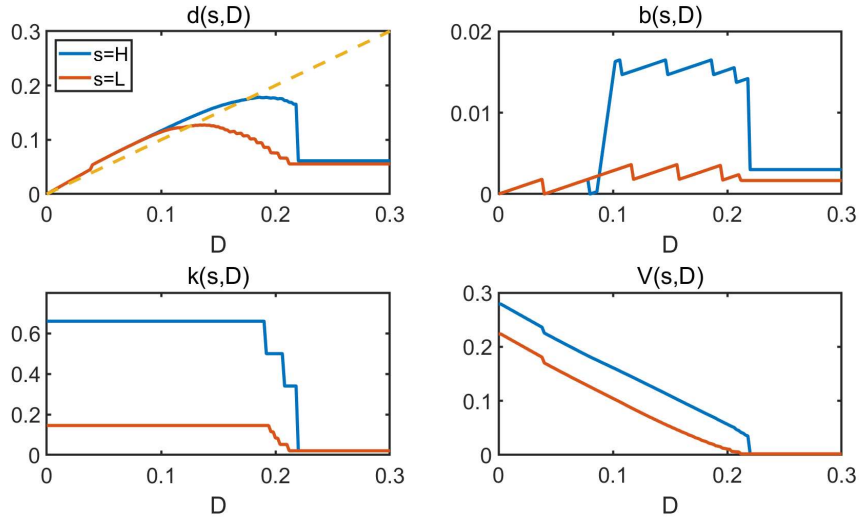


Figure 3: The case with $1 + r > \beta^{-1}$.

Here, $d(s, D)$ is the PDV of repayments, given as the solution to (33) below. Clearly, $\hat{D}(s, D) = D$ for a small value of D , because the bank has no incentive to reduce the debt if it is sufficiently small.

Definitions: Given the possibility of debt restructuring, we modify the formulation of the discrete model, because the NPL equilibrium, $\{k^{npl}(s), b^{npl}(s), d^{npl}(s), G^{npl}(s)\}$ now depends on when and by how much debt is reduced. The grid points for D , D_{+1} , and k are the same as in the previous sections, but we modify the grid points for b , $\Delta_b(s, D)$.

Take as given the beliefs $\{V^e(s, D), k_{npl}^e(s), \hat{D}^e(s, D)\}$, where $V^e(s, D)$ describes the expected value of the firm, $k_{npl}^e(s)$ the expected value of working capital in the NPL equilibrium, and $\hat{D}^e(s, D)$ the expected amount of debt after debt restructuring. We use the same parameter values as in the baseline model. For the probability p of a certain size, the candidate for $k^{npl}(s)$ makes the enforcement constraint nonbinding, that is, $\tilde{k}^{npl}(s) \equiv \arg \max_{k \in \Delta_k(s)} F(s, k) - Rk - G(s, k)$ does not satisfy

$$G(s, k) > \beta \mathbb{E}[(1 - p)V^{npl}(s_{+1}) + pV^e(s_{+1}, \hat{D}_{+1}^e)|s], \quad (30)$$

where we define $V^{npl}(s_{+1})$ by

$$V^{npl}(s) = F(s, k^{npl}(s)) - Rk^{npl}(s) - b^{npl}(s) + \beta \mathbb{E}[(1 - p)V^{npl}(s_{+1}) + pV^e(s_{+1}, \hat{D}_{+1}^e)|s],$$

and $D_{+1}^e = \hat{D}^e(s_{+1}, D_{+1})$.²¹ Therefore, not as in the baseline case, we define $k^{npl}(s)$ for

²¹Note that in the NPL equilibrium where $D > D_{\max}(s)$, $\hat{D}(s, D)$ is independent of D , i.e., $\hat{D}(s, D) =$

the case where $\tilde{k}^{npl}(s)$ does not satisfy (30) as

$$k^{npl}(s) = \max\{k \in \Delta_k(s) \mid G(s, k) \leq \beta\mathbb{E}[(1-p)V^{npl}(s_{+1}) + pV^e(s_{+1}, \hat{D}_{+1}^e) \mid s]\}. \quad (31)$$

Note that $k^{npl}(s)$ depends on the given beliefs $\{V^e(s, D), k_{npl}^e(s), \hat{D}^e(s, D)\}$. Of course, $k^{npl}(s) = k_{npl}^e(s)$ must hold in equilibrium. We define $b^{npl}(s)$ by

$$b^{npl}(s) = F(s, k^{npl}(s)) - Rk^{npl}(s) + \beta\mathbb{E}[(1-p)V^{npl}(s_{+1}) + pV^e(s_{+1}, \hat{D}_{+1}^e) \mid s] - G^{npl}(s),$$

in the case where $k^{npl}(s) = \tilde{k}^{npl}(s)$, and by

$$b^{npl}(s) = F(s, k^{npl}(s)) - Rk^{npl}(s), \quad (32)$$

in the case where $k^{npl}(s)$ is defined by (31).

Now, we define the grid points for b as

$$\Delta_b(s, D) = \left\{ b \in \mathbb{R} \mid \exists D_{+1} \in \Delta_{+1} \text{ s.t. } b = D - \frac{1}{1+r}D_{+1}, \text{ and } b \geq 0 \right\} \cup \{b^{npl}(s)\}.$$

As stated above, the NPL equilibrium, $\{k^{npl}(s), b^{npl}(s), d^{npl}(s), V^{npl}(s)\}$, is defined given the beliefs $\{V^e(s, D), k_{npl}^e(s), \hat{D}^e(s, D)\}$.

The bank's problem: Given beliefs $\{V^e(s, D), k_{npl}^e(s), \hat{D}^e(s, D)\}$, the bank solves

$$d(s, D) = \max_{b \in \Gamma(s, D)} b + \beta\mathbb{E}[(1-p)d(s_{+1}, D_{+1}) + pd(s_{+1}, \hat{D}_{+1}^e)], \quad (33)$$

where

$$\begin{aligned} \Gamma(s, D) &= \{b \in \Delta_b(s, D) \mid \exists k \in \Delta_k(s) \text{ s.t.} \\ &D_{+1} = \min\{N_{\max}\delta, (1+r)(D-b)\}, \\ &F(s, k) - Rk - b + \beta\mathbb{E}[(1-p)V^e(s_{+1}, D_{+1}) + pV^e(s_{+1}, \hat{D}_{+1}^e)] \geq G(s, k), \\ &F(s, k) - Rk - b \geq 0\}. \end{aligned}$$

Let $\Sigma(s, D)$ denote the set of (b, D_{+1}) that solves the maximization problem in (33). The bank decides on k and $V(s, D)$ by solving the following problem:

$$\begin{aligned} V(s, D) &= \max_{k \in \Delta_k(s), (b, D_{+1}) \in \Sigma(s, D)} F(s, k) - Rk - b \\ &\quad + \beta\mathbb{E}[(1-p)V^e(s_{+1}, D_{+1}) + pV^e(s_{+1}, \hat{D}_{+1}^e)], \end{aligned} \quad (34)$$

subject to

$$\begin{aligned} F(s, k) - Rk - b + \beta\mathbb{E}[(1-p)V^e(s_{+1}, D_{+1}) + pV^e(s_{+1}, \hat{D}_{+1}^e)] &\geq G(s, k), \\ F(s, k) - Rk - b &\geq 0. \end{aligned}$$

$\hat{D}(s)$, which is defined by $\hat{D}(s) \equiv \arg \max_{D \in \Delta} d(s, D)$. Thus, for $D > D_{\max}(s)$, $\hat{D}^e(s, D)$ should also be independent of D .

Let $\Lambda(s, D)$ denote the set of (k, b, D_{+1}) that solves the maximization problem in (34).

The equilibrium values of (k, b, D_{+1}) are determined as follows. First, $b(s, D)$ and $D_{+1}(s, D)$ are given by

$$b(s, D) = \max_{(k, b, D_{+1}) \in \Lambda(s, D)} b, \quad (35)$$

$$D_{+1}(s, D) = \min\{N_{\max}\delta, (1+r)\{D - b(s, D)\}\}. \quad (36)$$

Then, $k(s, D)$ is determined by

$$k(s, D) = \max_{(k, b(s, D), D_{+1}(s, D)) \in \Lambda(s, D)} k,$$

$\hat{D}(s, D)$ by

$$\hat{D}(s, D) = \arg \max_{D' \leq D} d(s, D'),$$

and $d^{npl}(s)$ is

$$d^{npl}(s) = b^{npl}(s) + \beta \mathbb{E}[(1-p)d^{npl}(s_{+1}) + pd(s_{+1}, \hat{D}_{+1}^e)].$$

For consistency, we require that

$$V(s, D) = V^e(s, D), \quad k^{npl}(s) = k_{npl}^e(s), \quad \text{and} \quad \hat{D}(s, D) = \hat{D}^e(s, D). \quad (37)$$

7.1 Numerical experiment

Here, we report the results of numerical experiments for the extended model. Except for the probability of debt restructuring, p , all parameter values and the functional forms are set in the same way as in the baseline model with $1+r > \beta^{-1}$.

Figure 4 plots the main equilibrium functions for the case with $p = 0.2$. In this case, $k^{npl}(s)$ is defined by (31) and $b^{npl}(s)$ is defined by (32) for both s_H and s_L . The bank's value function $d(s, D)$ increases with D when D is small, and stays constant when D is large. Thus, the debt-Laffer curve is not inverted U shaped, but inverted L shaped. The economy enters the NPL equilibrium when $D > 0.234$ for $s = s_H$ and when $D > 0.226$ for $s = s_L$. Thus, the thresholds for the NPL equilibrium become larger than in the baseline case, where the NPL equilibrium arises when $D > 0.218$ for $s = s_H$ and $D > 0.210$ for $s = s_L$. In addition, $d(s, D)$ becomes larger for each s and D compared to the baseline case. These are intuitive results, because the possibility of debt restructuring increases the firm's value, relaxes the borrowing constraint, and thus raises the amount of debt that the firm can repay. In addition, the difference between $k^*(s)$ and $k^{npl}(s)$ becomes more modest than in the baseline case: $k^{npl}(s_H)/k^*(s_H) = 0.313$ and $k^{npl}(s_L)/k^*(s_L) = 1$.

Figure 5 shows how the equilibrium is affected by the possibility of debt restructuring, where we examine three values of p : 0, 0.002, 0.2. The left panels show the equilibrium

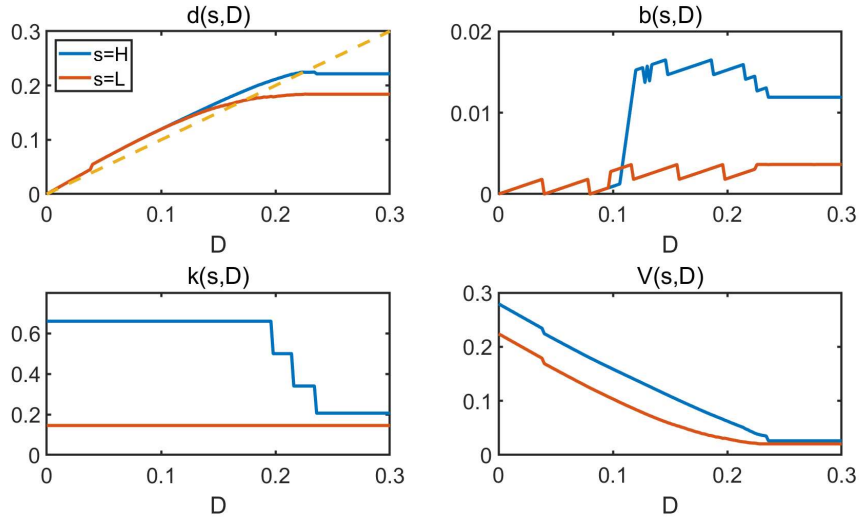


Figure 4: The case with frictional debt restructuring ($p = 0.2$).

functions corresponding to state s_H , and the right panels show those corresponding to state s_L . Although $k^{npl}(s)$ is defined by (31) and $b^{npl}(s)$ is defined by (32) for both s_H and s_L in the case with $p = 0.002$, the variables in this case are almost the same as those in the baseline case with $p = 0$. They show that an increase in the possibility of debt restructuring leads to upward shifts in the bank's value function, the working capital provision, and the firm's value function.

8 Concluding remarks

In this paper, a model of long-term debt contract has been analyzed and it has been demonstrated that nonperforming loans can cause persistent inefficiency. To the extent that debt restructuring is delayed due to political and/or bargaining frictions, a borrower's debt may grow to an unrepayable level. Without reducing the debt to some repayable level, the lender loses its credibility with respect to any future repayment plans it may offer to the borrower. This impairment of the lender's credibility discourages the borrower from expending effort, leading to a decrease in the loan demand and persistent inefficiency. Although the optimal contract features a backloaded payoff to the borrower until the amount of debt becomes sufficiently small, it is no longer possible to provide incentives dynamically in our model when the debt becomes "too large." Our model generates a debt Laffer curve; that is, the payoff to the lender, which is the present discounted value of repayments, can decrease with the contractual amount of debt if it exceeds some threshold value. In the baseline case, we assume for simplicity that debt restructuring is not possible. However, we extend the model to allow for stochastic debt restructuring, and

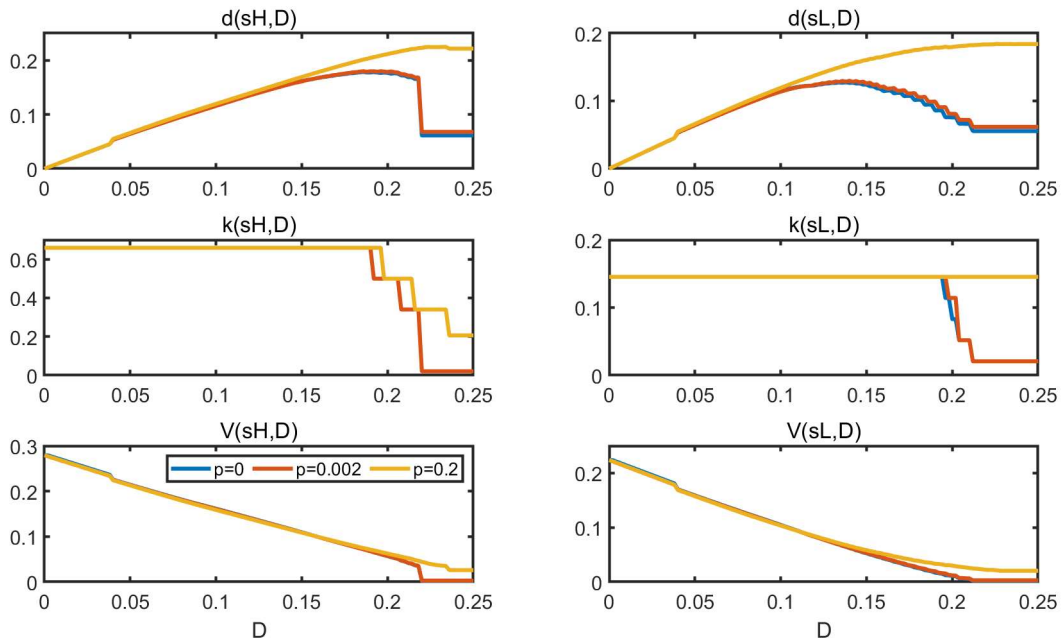


Figure 5: Frictional debt restructuring ($p = 0.0, 0.002, 0.2$).

confirm that the main results remain invariant qualitatively. The efficiency of equilibrium can be improved if debt restructuring is facilitated by policy measures.

This paper has a number of limitations. As the focus of our analysis is solely theoretical, the model used is simplistic and stylized. Thus, it needs further elaboration to be applicable to real episodes of financial crises and business fluctuations; for instance, the (possibility of) secular stagnation in the aftermath of the Global Financial Crisis. Bargaining frictions of debt restructuring could be modeled more explicitly, as opposed to the reduced-form approach adopted in this paper. All these extensions are left for future research.

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A Proof of Proposition 2

In this appendix, we characterize the NPL equilibrium in the continuous-value model. First, we prove the following lemma.

Lemma 15. *If $k(s, D) < k^*(s) \equiv \arg \max_k F(s, k) - Rk$, then*

$$\begin{aligned}\beta \mathbb{E}G(s_{+1}, k(s_{+1}, D_{+1})) &\leq G(s, k(s, D)), \\ V(s, D) &= G(s, k(s, D)).\end{aligned}$$

Proof. For any s , any b that is feasible for $k < k^{npl}(s)$ is also feasible for $k = k^{npl}(s)$. Thus, $k(s, D) \geq k^{npl}(s)$. Suppose $\beta \mathbb{E}G(s_{+1}, k(s_{+1}, D_{+1})) > G(s, k(s, D))$. Then, there exists $\varepsilon > 0$ such that $k = k(s, D) + \varepsilon (\leq k^*(s))$ is feasible under the borrowing constraint ($F(s, k) - Rk - b + \beta \mathbb{E}V^e(s_{+1}, D_{+1}) \geq G(s, k)$), because $V^e(s, D) = V(s, D) \geq G(s, D)$ in equilibrium. Then, Assumption 1 implies that the equilibrium value of k should be $k(s, D) + \varepsilon$, not $k(s, D)$. This is a contradiction. Therefore, $\beta \mathbb{E}G(s_{+1}, k(s_{+1}, D_{+1})) \leq G(s, k(s, D))$. Suppose that $V(s, D) > G(s, k(s, D))$. Then, there exists $\varepsilon > 0$ such that $k = k(s, D) + \varepsilon (\leq k^*(s))$ is feasible. As above, Assumption 1 implies that $k = k(s, D) + \varepsilon$ should be the equilibrium value. This is a contradiction. Thus, $V(s, D) = G(s, k(s, D))$ in equilibrium. \square

Since $D > \bar{D}$ implies that $D_{+1} = (1+r)(D-b) > D$ for any feasible b , the lender's commitment constraint ($b \leq D$) never binds for D . Thus, the bank's problem can be rewritten as

$$\begin{aligned}d(s, D) &= \max_{b, k} b + \beta \mathbb{E}d(s_{+1}, D_{+1}) & (38) \\ \text{s.t. } V &= F(s, k) - Rk - b + \beta \mathbb{E}V^e(s_{+1}, D_{+1}), \\ V &\geq G(s, k), \\ F(s, k) - Rk - b &\geq 0,\end{aligned}$$

with the equilibrium conditions: $V(s, D) = V^e(s, D)$ and $V^e(s, D) \leq V_{\max}$, where

$$V_{\max} \equiv \frac{1}{1-\beta} \{F(s_H, k^*(s_H)) - Rk^*(s_H)\}.$$

Lemma 16. *Consider the case where $D \geq \bar{D}$. Suppose that $F(s, k(s, D)) - Rk(s, D) - b(s, D) > 0$ for some (s, D) . Then, $k(s, D) = k^{npl}(s)$ for the same (s, D) .*

Proof. The proof of this claim is by contradiction. Suppose that $k(s, D) \neq k^{npl}(s)$ for a particular value of (s, D) , for which $F(s, k(s, D)) - Rk(s, D) - b(s, D) > 0$. Then, Lemma 1 implies $k(s, D) > k^{npl}(s)$. Define $\varepsilon(s, D) \equiv F(s, k(s, D)) - Rk(s, D) - b(s, D)$. Define $k^\varepsilon(s, D) = \max \{k(s, D; \varepsilon), k^{npl}(s)\}$, where $k(s, D; \varepsilon)$ is the solution to $F(s, k) - Rk -$

$b(s, D) = \frac{1}{2}\varepsilon(s, D)$. Obviously, $k^\varepsilon(s, D) < k(s, D)$. Now, define $b^\varepsilon(s, D) = \min\{b_1(s, D), b_2(s, D)\}$, where

$$b_1(s, D) = F(s, k^\varepsilon(s, D)) - Rk^\varepsilon(s, D),$$

and

$$b_2(s, D) = \max\{b \mid F(s, k^\varepsilon(s, D)) - Rk^\varepsilon(s, D) - b + \beta\mathbb{E}V^e(s_{+1}, (1+r)(D-b)) \geq G(s, k^\varepsilon(s, D))\}.$$

Note that $b_2(s, D) = +\infty$ may be possible for some (s, D) . Obviously, $b_1(s, D) > b(s, D)$, because $b_1(s, D) = b(s, D) + \frac{1}{2}\varepsilon$ when $k^\varepsilon(s, D) = k(s, D; \varepsilon)$, and $b_1(s, D) > b(s, D) + \frac{1}{2}\varepsilon$ when $k^\varepsilon(s, D) = k^{npl}(s) > k(s, D; \varepsilon)$. Furthermore, it is easily confirmed that $b_2(s, D) > b(s, D)$, because $b_2(s, D)$ is the maximum value of b that satisfies

$$b \leq F(s, k^\varepsilon(s, D)) - Rk^\varepsilon(s, D) - G(s, k^\varepsilon(s, D)) + \beta\mathbb{E}V^e(s_{+1}, (1+r)(D-b)),$$

and $b(s, D)$ is the maximum value of b that satisfies

$$b \leq F(s, k(s, D)) - Rk(s, D) - G(s, k(s, D)) + \beta\mathbb{E}V^e(s_{+1}, (1+r)(D-b)),$$

where $F(s, k) - Rk - G(s, k)$ is decreasing in k for $k > k^{npl}(s)$. Here, we used $k(s, D) > k^{npl}(s)$ to show $b_2(s, D) > b(s, D)$. Since $b_1(s, D) > b(s, D)$ and $b_2(s, D) > b(s, D)$, it is obvious that $b^\varepsilon(s, D) > b(s, D)$ for the particular (s, D) . Since $\{b^\varepsilon(s, D), k^\varepsilon(s, D)\}$ satisfies all constraints of (38), it is feasible. As formally stated in the following Claim 1, $\{b^\varepsilon(s, D), k^\varepsilon(s, D)\}$ should be the solution to (38) instead of $\{b(s, D), k(s, D)\}$, and this result contradicts the fact that $\{b(s, D), k(s, D)\}$ is the solution to (38). Therefore, $k(s, D) = k^{npl}(s)$ should hold, if $F(s, k(s, D)) - Rk(s, D) - b(s, D) > 0$.

The reason why $\{b^\varepsilon(s, D), k^\varepsilon(s, D)\}$ should be the solution to (38) is formally described in the following Claim 1. First, we define the sequential problem corresponding to the recursive problem (38), as follows. For $(s_0, D_0) = (s, D)$,

$$\begin{aligned} d^*(s, D) &= \max_{\{b_t, k_t\}_{t=0}^\infty} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t b_t \right], \\ \text{s.t. } & F(s_t, k_t) - Rk_t - b_t + \beta\mathbb{E}_t V^e(s_{t+1}, D_{t+1}) \geq G(s_t, k_t), \\ & F(s_t, k_t) - Rk_t - b_t \geq 0, \\ & D_{t+1} = (1+r)(D_t - b_t). \end{aligned} \tag{39}$$

We know that the solution to (38) is $\{b(s, D), k(s, D)\}$ and $d(s, D)$ is written as

$$d(s, D) = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t b(s_t, D(s^t)) \right], \tag{40}$$

where $s^t = \{s_0, s_1, s_2, \dots, s_t\}$, $s_0 = s$, $s^0 = \{s\}$, $D(s^0) = D$, and

$$D(s^t) = (1+r)\{D(s^{t-1}) - b(s_{t-1}, D(s^{t-1}))\}$$

for $t \geq 1$. We define $d^\varepsilon(s, D)$ by

$$d^\varepsilon(s, D) = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t b^\varepsilon(s_t, D^\varepsilon(s^t)) \right], \quad (41)$$

where $D^\varepsilon(s^0) = D$, $D^\varepsilon(s^t) = (1+r)\{D^\varepsilon(s^{t-1}) - b^\varepsilon(s_{t-1}, D^\varepsilon(s^{t-1}))\}$ for $t \geq 1$, and, for $t = 0$, $b^\varepsilon(s_0, D(s^0)) = b^\varepsilon(s, D) > b(s, D)$ for the particular (s, D) and

$$b^\varepsilon(s_t, D_t^\varepsilon(s^t)) = b(s_t, D(s^t)), \quad (42)$$

for $t \geq 1$. Note that the right-hand side of (42) is $b(s_t, D(s^t))$, not $b(s_t, D^\varepsilon(s^t))$. The claim is as follows.

Claim 1. For the particular (s, D) where $F(s, k(s, D)) - Rk(s, D) - b(s, D) > 0$, it must be the case that $d^*(s, D) \geq d^\varepsilon(s, D) > d(s, D)$.

(Proof of Claim 1) We have shown that $b^\varepsilon(s_0, D(s^0))$ is feasible for the particular $(s_0, D(s^0)) = (s, D)$. For $t \geq 1$, it is obvious that $D^\varepsilon(s^t) < D(s^t)$, because $b^\varepsilon(s_0, D_0) > b(s_0, D_0)$ for $t = 0$. As we assumed that $V^e(s_t, D_t)$ is a (weakly) decreasing function of D_t , it is the case that

$$V^e(s_t, D^\varepsilon(s^t)) \geq V^e(s_t, D(s^t)).$$

Then, at the state $(s_t, D^\varepsilon(s^t))$, the pair $\{b(s_t, D(s^t)), k(s_t, D(s^t))\}$ is feasible as it satisfies both constraints. (Note that this argument holds for all t , because the constraint, $b_t \leq D_t$, never binds for $D_t > \bar{D}$.) Therefore, $d^\varepsilon(s, D)$ is feasible and, by definition of $d^*(s, D)$, (39), it must be the case that $d^*(s, D) \geq d^\varepsilon(s, D)$. It is obvious that $d^\varepsilon(s, D) > d(s, D)$, from (40) and (41) by definition of $b^\varepsilon(s_t, D^\varepsilon(s^t))$ and $b^\varepsilon(s, D) > b(s, D)$ for the particular (s, D) . *(End of the proof of Claim 1)*

This claim contradicts the theorem of dynamic programming that the solutions to the recursive problem (38) and the sequential problem (39) are identical, i.e., $d(s, D) = d^*(s, D)$. Thus, $k(s, D) = k^{npl}(s)$ should hold, if $F(s, k(s, D)) - Rk(s, D) - b(s, D) > 0$. □

For any s and $D > \bar{D}$, we consider a stochastic sequence $\{s_t, k_t, b_t, D_t\}$, where $k_t = k(s_t, D_t)$, $b_t = b(s_t, D_t)$, $D_t = (1+r)(D_{t-1} - b_{t-1})$, $s_0 = s$, and $D_0 = D$, given that s_t is an exogenous stochastic variable. We will prove $k(s_0, D_0) = k^{npl}(s_0)$ in what follows.

For $s_t = s_H$, we have the following lemma.

Lemma 17. Consider the case where $D \geq \bar{D}$. For all $t \geq 0$, if $s_t = s_H$, then

$$G(s_H, k(s_H, D_t)) > \beta \mathbb{E}_t[G(s_{t+1}, k(s_{t+1}, D_{t+1}))].$$

This inequality implies from Lemma 16 that $k(s_H, D_t) = k^{npl}(s_H)$ for all t .

Proof. The latter half of the lemma is proven as follows. Lemma 15 states $V(s, D) = G(s, k)$, which implies that $F(s, k) - Rk - b \geq G(s, k) - \beta \mathbb{E}G(s_{+1}, k_{+1})$. The inequality $G(s, k) - \beta \mathbb{E}G(s_{+1}, k_{+1}) > 0$ implies that $F(s, k) - Rk - b > 0$, which implies that $k = k^{npl}(s)$ by Lemma 16.

The proof of the inequality $G(s, k) - \beta \mathbb{E}G(s_{+1}, k_{+1}) > 0$ is by contradiction. Suppose that this inequality does not hold. Then, for $s_0 = s_H$, Lemma 15 implies

$$G(s_H, k(s_H, D_0)) = \beta \mathbb{E}_0[G(s_1, k(s_1, D_1))],$$

where $D_1 \geq D_0$ as $D_0 \geq \bar{D}$. Then, in the case where $s_1 = s_H$,

$$G(s_H, k(s_H, D_0)) < \beta G(s_H, k(s_H, D_1)),$$

because obviously $k(s_H, D) > k(s_L, D)$ for any D , $G(s, k) > G(s, k')$ for $k > k'$, and $G(s_H, k) \geq G(s_L, k)$. As $G(s_H, k(s_H, D_0)) \geq G^{npl}(s_H)$, it is the case that $G(s_H, k(s_H, D_1)) > G^{npl}(s_H)$. This inequality implies that $k(s_H, D_1) > k^{npl}(s_H)$. Then,

$$G(s_H, k(s_H, D_1)) = \beta \mathbb{E}_1[G(s_2, k(s_2, D_2))],$$

because it should be the case that $k(s_H, D_1) = k^{npl}(s_H)$ and $G(s_H, k(s_H, D_1)) = G^{npl}(s_H)$ if $G(s_H, k(s_H, D_1)) > \beta \mathbb{E}_1[G(s_2, k(s_2, D_2))]$. Iterating this argument, it is easily shown that for any integer t ,

$$G(s_H, k(s_H, D_0)) < \beta^t G(s_H, k(s_H, D_t)).$$

This inequality holds for an arbitrarily large t , as the above iteration can continue indefinitely, because $D_t \geq D_{t-1} \geq \bar{D}$ for any t , and the constraint, $b_t \leq D_t$, never binds. Then, for a sufficiently large t ,

$$G(s_H, k(s_H, D_t)) > \beta^{-t} G(s_H, k(s_H, D_0)) > V_{\max},$$

which contradicts that $k(s_H, D_t)$ is the equilibrium value. Therefore, this lemma should hold. \square

For $s_t = s_L$, we have the following lemma.

Lemma 18. *Consider the case where $D \geq \bar{D}$. For all $t \geq 0$, if $s_t = s_L$, then $k(s_L, D_t) = k^{npl}(s_L)$.*

Proof. If $G(s_L, k(s_L, D_t)) > \beta \mathbb{E}_t[G(s_{t+1}, k(s_{t+1}, D_{t+1}))]$, the limited liability constraint is nonbinding and $k(s_L, D_t)$ must be $k^{npl}(s_L) = \arg \max F(s_L, k) - Rk - G(s_L, k)$.

Suppose that $G(s_L, k(s_L, D_t)) > \beta \mathbb{E}_t[G(s_{t+1}, k(s_{t+1}, D_{t+1}))]$ is not satisfied. Then, Lemma 15 implies that $G(s_L, k(s_L, D_t)) = \beta \mathbb{E}_t[G(s_{t+1}, k(s_{t+1}, D_{t+1}))]$. In this case, since

Lemma 17 implies that $k(s_H, D_t) = k^{npl}(s_H)$ for $t \geq 0$, the following equation must hold for all $t \geq 0$:

$$G(s_L, k_t^L) = \beta[\pi_{LL}G(s_L, k_{t+1}^L) + \pi_{LH}G^{npl}(s_H)],$$

where $\pi_{LL} = \Pr(s_{t+1} = s_L | s_t = s_L)$, $\pi_{LH} = 1 - \pi_{LL}$, and $k_t^L = k(s_L, D_t)$. Here, we define \bar{k}_L as the fixed point of this equation, i.e., $G(s_L, \bar{k}_L) = \beta[\pi_{LL}G(s_L, \bar{k}_L) + \pi_{LH}G^{npl}(s_H)]$. Let us consider what would happen if $k_0^L \neq \bar{k}_L$. If $k_0^L < \bar{k}_L$, then the sequence $\{k_t^L\}_{t=0}^\infty$ that satisfies the above equation and $k_t^L \geq 0$ for all t cannot exist, because k_t^L becomes a negative number for a finite t . If $k_0^L > \bar{k}_L$, then $\lim_{t \rightarrow \infty} k_t^L = +\infty$, which cannot satisfy the condition for an equilibrium: $G(s_L, k_t^L) \leq V_{\max}$ for all t . Therefore, it must be that $k_t^L = \bar{k}_L$ for all t . As we assumed (11) is satisfied by $k^{npl}(s_L)$, i.e., $G(s_L, k^{npl}(s_L)) \geq \beta\mathbb{E}[G(s_{+1}, k^{npl}(s_{+1})|s_L)]$, it is the case that $k^{npl}(s_L) \geq \bar{k}_L$. If $k_t^L < k^{npl}(s_L)$ for any t , Lemma 1 is violated, a contradiction. Therefore, it must be the case that either $G(s_L, k(s_L, D_t)) > \beta\mathbb{E}_t[G(s_{t+1}, k(s_{t+1}, D_{t+1}))]$ or $\bar{k}_L = k^{npl}(s_L)$. Thus, $k_t^L = k^{npl}(s_L)$ for any $D_0 > \bar{D}$. \square

Lemmas 17 and 18 imply that $k(s, D) = k^{npl}(s)$ for any s and D , that satisfies $D > \bar{D}$. Since the no-default constraint is binding, $V(s, D) = G(s, k^{npl}(s)) = G^{npl}(s)$ for all s and $D > \bar{D}$. The equilibrium condition implies that $V^e(s, D) = G^{npl}(s)$. Thus, $b(s, D) = b^{npl}(s)$ and $d(s, D) = d^{npl}(s)$.

B Proof of Lemma 4

There exists $D_{+1} \in \Delta$ such that

$$\begin{aligned} d(s, D + \delta) &= b' + \beta\mathbb{E}d(s_{+1}, D_{+1}), \\ b' &= D + \delta - \beta D_{+1}. \end{aligned}$$

Note that Assumption 3 implies that $b' \geq \delta$. Consider $b = D - \beta D_{+1}$. Then, $b \geq 0$, and therefore, $b \in \Delta_b(s, D)$, while b may not be an element of $\Delta_b(s, D + \delta)$. It is easily confirmed that $b \in \Gamma(s, D)$. Thus,

$$\begin{aligned} d(s, D + \delta) &= b + \delta + \beta\mathbb{E}d(s_{+1}, D_{+1}) \\ &= \delta + [b + \beta\mathbb{E}d(s_{+1}, D_{+1})] \\ &\leq \delta + \max_{\tilde{b} \in \Gamma(s, D)} [\tilde{b} + \beta\mathbb{E}d(s_{+1}, \beta^{-1}(D - \tilde{b}))] \\ &= \delta + d(s, D). \end{aligned}$$

C Proof of Lemma 5

Suppose that $b(s, D)$ is not the maximum feasible value. Then, $b(s, D) + \beta\delta \in \Gamma(s, D)$. We compare $d(s, D)$ and $X(b(s, D) + \beta\delta, s, D)$, where $X(b, s, D) \equiv b + \beta\mathbb{E}d(s_{+1}, \beta^{-1}[D - b])$.

Lemma 4 implies that

$$\begin{aligned}
X(b(s, D) + \beta\delta, s, D) &= b(s, D) + \beta\delta + \beta\mathbb{E}d(s_{+1}, \beta^{-1}(D - b(s, D)) - \delta) \\
&= b(s, D) + \beta\mathbb{E}\{\delta + d(s_{+1}, \beta^{-1}(D - b(s, D)) - \delta)\} \\
&\geq b(s, D) + \beta\mathbb{E}d(s_{+1}, \beta^{-1}(D - b(s, D))) \\
&= d(s, D) = \max_b X(b, s, D).
\end{aligned}$$

If $X(b(s, D) + \beta\delta, s, D) > d(s, D)$, it contradicts (16), which defines $b(s, D)$. If $X(b(s, D) + \beta\delta, s, D) = d(s, D)$, Assumption 2 implies that $F(s, k(s, D)) - Rk(s, D) - b(s, D) - \beta\delta + \beta\mathbb{E}V^e(s_{+1}, D_{+1}(s, D) - \delta) \geq F(s, k(s, D)) - Rk(s, D) - b(s, D) + \beta\mathbb{E}V^e(s_{+1}, D_{+1}(s, D)) = V(s, D)$. Then, $b(s, D) + \beta\delta$ should be the equilibrium value of b . This is a contradiction. Therefore, $b(s, D)$ is the maximum feasible value in $\Gamma(s, D)$, i.e., $b(s, D) = \bar{b}(s, D)$.

Next, we prove $k(s, D) > k^{npl}(s)$ for $D \leq D_{\max}(s)$. For $D \leq D_{\max}(s)$, we have $V(s, D) \geq G^{npl}(s) + \delta$, as $V(s, D) \geq V(s, D + \delta) + \delta$ from Assumption 2 and $V(s, D + \delta) \geq G^{npl}(s)$ due to Lemma 15 in Appendix A. Now, we prove $k(s, D) > k^{npl}(s)$ by contradiction. Suppose that $k(s, D) = k^{npl}(s)$. Then, since $(b(s, D), k(s, D))$ satisfy the above inequality and the limited liability constraint, we have

$$\begin{aligned}
V(s, D) &= F(s, k^{npl}(s)) - Rk^{npl}(s) - b(s, D) + \beta\mathbb{E}V(s_{+1}, D_{+1}(s, D)) \geq G^{npl}(s) + \delta, \\
F(s, k^{npl}(s)) - Rk^{npl}(s) - b(s, D) &\geq 0.
\end{aligned}$$

Pick $k^{npl+}(s) (> k^{npl}(s))$, which is defined by $f(s, k^{npl}(s)) - f(s, k^{npl+}(s)) = \beta\delta$, where $f(s, k) \equiv F(s, k) - Rk - G(s, k)$. Then, $k^{npl+}(s)$ satisfies

$$\begin{aligned}
F(s, k^{npl+}(s)) - Rk^{npl+}(s) - b(s, D) + \beta\mathbb{E}V(s_{+1}, D_{+1}(s, D)) &\geq G(s, k^{npl+}(s)) + (1 - \beta)\delta, \\
F(s, k^{npl+}(s)) - Rk^{npl+}(s) - b(s, D) &\geq 0.
\end{aligned}$$

Therefore, $k(s, D)$ should be $k^{npl+}(s)$, not $k^{npl}(s)$, because $k^{npl+}(s)$ is feasible without changing $b(s, D)$ and $D_{+1}(s, D)$. This is a contradiction. Thus, we have demonstrated that for $D \leq D_{\max}(s)$, $k(s, D) > k^{npl}(s)$.

D Proof of Lemma 7

Suppose that $F(s, k(s, D)) - Rk(s, D) - b(s, D) \geq \xi + \beta\delta$ for $k(s, D) \in (k^{npl}(s), k^*(s))$. In this case, the bank can choose $\hat{k} < k(s, D)$, where $\hat{k} \in \Delta_k(s)$, so that $F(s, \hat{k}) - R\hat{k} - b(s, D) \geq \beta\delta$. We know that $F(s, k(s, D)) - Rk(s, D) - G(s, k(s, D)) - b(s, D) + \beta\mathbb{E}V^e(s, D_{+1}(s, D)) \geq 0$, where $D_{+1}(s, D) = \beta^{-1}[D - b(s, D)]$. As $F(s, k) - Rk - G(s, k)$ is strictly decreasing in k for $k > k^{npl}(s)$, it must be the case that

$$F(s, \hat{k}) - R\hat{k} - G(s, \hat{k}) \geq F(s, k(s, D)) - Rk(s, D) - G(s, k(s, D)) + \beta\delta.$$

Thus, $\hat{b} = b(s, D) + \beta\delta$ satisfies

$$\begin{aligned} F(s, \hat{k}) - R\hat{k} - \hat{b} &\geq 0, \\ F(s, \hat{k}) - R\hat{k} - \hat{b} - G(s, \hat{k}) + \beta\mathbb{E}V^e(s_{+1}, \beta^{-1}(D - \hat{b})) &\geq 0. \end{aligned}$$

Then, $\hat{b} = b(s, D) + \beta\delta$ is feasible and Lemma 5 implies that \hat{b} should be the solution to (16). This is a contradiction.

E Proof of Lemma 9

For any s and $D > D_{\max}(s)$, we consider a stochastic sequence $\{s_t, k_t, b_t, D_t\}$, where $k_t = k(s_t, D_t)$, $b_t = b(s_t, D_t)$, $D_t = n_\delta[(1+r)(D_{t-1} - b_{t-1})]$, $s_0 = s$, and $D_0 = D$, given that s_t is an exogenous stochastic variable.

First, we consider the case where $s = s_H$. Suppose there exists D , which satisfies $D > D_{\max}$, such that $k(s, D) \neq k^{npl}(s)$. Then, Lemma 1 implies $k(s, D) > k^{npl}(s)$. Then, Lemma 7 implies that $0 \leq F(s, k) - Rk - b < \xi + \beta\delta$, which implies, together with $V \geq G(s, k)$, that

$$G(s, k(s, D)) \leq V(s, D) \leq \xi + \beta\delta + \beta\mathbb{E}V(s_{+1}, D_{+1})$$

As it is obvious that $V(s_L, D) \leq V(s_H, D)$, it must be the case that $\mathbb{E}V(s_{+1}, D_{+1}) \leq V(s_H, D_{+1})$. Then,

$$G(s, k(s, D)) \leq V(s, D) \leq \xi + \beta\delta + \beta V(s_H, D_{+1}), \quad (43)$$

where $D_{+1} > D$ as $D > D_{\max}(s)$. Lemma 8 implies that $V(s_H, D_{+1}) < \delta_g + G(s_H, k(s_H, D_{+1}))$. Thus,

$$G(s_H, k(s_H, D)) < \xi + \beta(\delta + \delta_g) + \beta G(s_H, k(s_H, D_{+1})). \quad (44)$$

Assumption 4 and the inequality (44) imply that $G(s_H, k(s_H, D)) < (1 - \beta)G^{npl}(s) + \beta G(s_H, k(s_H, D_{+1})) \leq G(s_H, k(s_H, D_{+1}))$, because $G^{npl}(s) \leq G(s_H, k(s_H, D_{+1}))$. Thus, $k(s_H, D) < k(s_H, D_{+1})$. Let us set $(s_0, D_0) = (s, D)$ and consider the sequence $\{s_t, D_t, k(s_t, D_t)\}$. Given (44), we can prove the following inequality:

$$k^{npl}(s_H) < k(s_H, D_t) < k(s_H, D_{t+1}), \quad (45)$$

$$G(s_H, k(s_H, D_0)) < \frac{\{\xi + \beta(\delta + \delta_g)\}(1 - \beta^t)}{1 - \beta} + \beta^t G(s_H, k(s_H, D_t)) \quad (46)$$

The proof is by induction. The above argument has proven (45) and (46) for $t = 0$. Suppose that (45) holds for $t - 1$. (44) applies for D_t and implies that

$$G(s_H, k(s_H, D_t)) < \xi + \beta(\delta + \delta_g) + \beta G(s_H, k(s_H, D_{t+1})), \quad (47)$$

which, together with Assumption 4, implies that $G(s_H, k(s_H, D_{t+1})) > G(s_H, k(s_H, D_t))$, or $k(s_H, D_{t+1}) > k(s_H, D_t)$. Thus, (45) has been proven for t . Suppose that (46) holds for t . This inequality, together with (47), implies that

$$\begin{aligned} G(s_H, k(s_H, D_0)) &< \frac{\{\xi + \beta(\delta + \delta_g)\}(1 - \beta^t)}{1 - \beta} + \beta^t G(s_H, k(s_H, D_t)) \\ &< \frac{\{\xi + \beta(\delta + \delta_g)\}[1 - \beta^t + \beta^t(1 - \beta)]}{1 - \beta} + \beta^{t+1} G(s_H, k(s_H, D_{t+1})) \\ &= \frac{\{\xi + \beta(\delta + \delta_g)\}(1 - \beta^{t+1})}{1 - \beta} + \beta^{t+1} G(s_H, k(s_H, D_{t+1})). \end{aligned}$$

Thus, (46) has been proven for $t + 1$. We have demonstrated that (45) and (46) hold for all t .

Assumption 4 and (46) imply that, in the limit of $t \rightarrow \infty$, we have $V(s_t, D_t) \rightarrow \infty$. This is a contradiction because $V(s, D)$ is bounded from above: $V(s, D) < V_{\max}$. Thus, it cannot be the case that $k(s_H, D) \neq k^{npl}(s_H)$.

Next, we consider the case where $s = s_L$. Suppose that $k(s_L, D) \neq k^{npl}(s_L)$. Then, Lemma 1 implies that $k(s_L, D) > k^{npl}(s_L)$. In this case, Lemmas 7 and 8 imply that for $D_0 = D$ and the sequence $\{s_t, D_t, k(s_t, D_t)\}$,

$$\begin{aligned} G(s_L, k(s_L, D_t)) &< \xi + \beta(\delta + \delta_g) + \beta \mathbb{E}_t G(s_{t+1}, k(s_{t+1}, D_{t+1})) \\ &= \xi + \beta(\delta + \delta_g) + \beta[p_L G(s_L, k(s_L, D_{t+1})) + (1 - p_L)G^{npl}(s_H)], \end{aligned}$$

where $p_L = \Pr(s_{t+1} = s_L | s_t = s_L)$ and $G(s_H, k(s_H, D_{t+1})) = G^{npl}(s_H)$ for $D_{t+1} > D_{\max}$, as shown above. Let $k(s_L, D) = k_0$ and define $\{k_t\}_{t=0}^{\infty}$ by the following law of motion,

$$G(s_L, k_t) = \xi + \beta(\delta + \delta_g) + \beta[p_L G(s_L, k_{t+1}) + (1 - p_L)G^{npl}(s_H)].$$

Lemma 1 implies that $k(s_L, D_t) \geq k^{npl}(s_L)$ for all $t \geq 1$. In the case where $k(s_L, D) = k_0 > k^{npl}(s_L)$, the sequence $\{k_t\}_{t=0}^{\infty}$ is such that $\lim_{t \rightarrow \infty} k_t = \infty$. Thus, $V(s_L, D_t) > G(s_L, k(s_L, D_t)) - \delta_g$ goes to infinity, and eventually violates the condition $V(s_L, D_t) < V_{\max}$. This is a contradiction. Thus, $k(s_L, D)$ must be $k^{npl}(s_L)$.

Therefore, if $D > D_{\max}$, then $k(s, D) = k^{npl}(s)$ for all $s \in \{s_L, s_H\}$.

F Proof of Proposition 10

The proof consists of two parts. First, we prove the existence of one equilibrium, in which $V^e(s, D) = G(s, k^{npl}(s)) \equiv G^{npl}(s)$. Second, we demonstrate that this equilibrium is the unique equilibrium that maximizes $d(s, D)$ subject to the no-default condition.

Existence: we guess and later verify that $V^e(s, D) = G^{npl}(s)$. Given this expectation,

the bank solves

$$d(s, D) = \max_{b \in \Delta_b(s, D), k \in \Delta_k(s)} b + \beta \mathbb{E}d(s+1, D_{+1}),$$

$$\text{s. t. } \begin{cases} F(s, k) - Rk - b + \beta \mathbb{E}G(s+1, k^{npl}(s+1)) \geq G(s, k), \\ F(s, k) - Rk - b \geq 0. \end{cases}$$

Given that $V^e(s, D) = G^{npl}(s)$, it is easily shown that $\Gamma(s, D) = \{b \mid b \in \Delta_b(s, D), 0 \leq b \leq b^{npl}(s)\}$.

Claim: The solution to the bank's problem is $b(s, D) = b^{npl}(s)$ and $k(s, D) = k^{npl}(s)$.

(Proof of Claim)

Because $b(s, D) \leq b^{npl}(s)$, there exists a nonnegative integer m and a nonnegative real number ε , where $0 \leq \varepsilon < \beta\delta$, such that $b(s, D) = b^{npl}(s) - \varepsilon - m\beta\delta$. Then, $D_{+1}(s, D) = \min\{N_{\max}\delta, \beta^{-1}[D - b(s, D)]\} = D_{+1}^{npl} + m'\delta$, where $0 \leq m' \leq m$ and we define $D_{+1}^{npl} = \min\{N_{\max}\delta, n_\delta(\beta^{-1}[D - b^{npl}(s)])\}$. Thus,

$$\begin{aligned} d(s, D) &= b(s, D) + \beta \mathbb{E}d(s+1, D_{+1}^{npl} + m'\delta) \\ &= b^{npl}(s) - \varepsilon - m\beta\delta + \beta \mathbb{E}d(s+1, D_{+1}^{npl} + m'\delta) \\ &= b^{npl}(s) - \varepsilon - (m - m')\beta\delta + \beta \mathbb{E}[-m'\delta + d(s+1, D_{+1}^{npl} + m'\delta)] \\ &\leq b^{npl}(s) - \varepsilon - (m - m')\beta\delta + \beta \mathbb{E}d(s+1, D_{+1}^{npl}) \\ &\leq b^{npl}(s) + \beta \mathbb{E}d(s+1, D_{+1}^{npl}). \end{aligned}$$

The first inequality is from Lemma 4. Therefore, $b(s, D) = b^{npl}(s)$ and $k(s, D) = k^{npl}(s)$.

(End of Proof of Claim)

Thus, the solution to the bank's problem is $k = k^{npl}(s)$ and $b = b^{npl}(s)$. It is also easily confirmed that $V(s, D) = F(s, k^{npl}(s)) - Rk^{npl}(s) - b^{npl}(s) + \beta \mathbb{E}G(s+1, k^{npl}(s+1)) = G(s, k^{npl}(s))$, which verifies the expectation.

Uniqueness: In what follows, we demonstrate that $d^{npl}(s)$ is the maximum amount of the present discounted value (PDV) of repayments that satisfies the enforcement constraint, and the above equilibrium is the unique equilibrium that attains $d^{npl}(s)$. We consider the following planner's problem, assuming that $k(s, D) = k^{npl}(s)$. We set this assumption because Lemma 9 shows that $k(s, D) = k^{npl}(s)$ for $D > D_{\max}(s_H)$ in any equilibrium that exists. Given $k(s, D) = k^{npl}(s)$, the planner's problem is

$$d(s, D) = \max_{b, V(s, D)} b + \beta \mathbb{E}d(s+1, \beta^{-1}(D - b)),$$

$$\text{s. t. } \begin{cases} V(s, D) = F(s, k^{npl}(s)) - Rk^{npl}(s) - b + \beta \mathbb{E}V(s+1, \beta^{-1}(D - b)) \geq G^{npl}(s), \\ F(s, k^{npl}(s)) - Rk^{npl}(s) - b \geq 0. \end{cases}$$

Define $W^{npl}(s) = F(s, k^{npl}(s)) - Rk^{npl}(s) + \beta \mathbb{E}W^{npl}(s_{+1})$. Then, $d(s, D) = W^{npl}(s) - V(s, D)$. Thus, the planner's problem can be rewritten as

$$\begin{aligned} & \max_{b, V(s, D)} d(s, D) = W^{npl}(s) - V(s, D), \\ \text{s. t. } & d(s, D) \leq W^{npl}(s) - G^{npl}(s), \\ & F(s, k^{npl}(s)) - Rk^{npl}(s) - b \geq 0. \end{aligned}$$

We temporarily omit the limited liability constraint, $F(s, k^{npl}(s)) - Rk^{npl}(s) - b \geq 0$, and later justify that it is satisfied. Without this constraint, it is obvious that the maximum PDV of repayments is $W^{npl}(s) - G^{npl}(s) = d^{npl}(s)$, and it is attained by setting $b = d(s, D) - \beta \mathbb{E}d(s_{+1}, D_{+1}) = W^{npl}(s) - G^{npl}(s) - \beta \mathbb{E}[W^{npl}(s_{+1}) - G^{npl}(s_{+1})] = F^{npl}(s) - Rk^{npl}(s) - G^{npl}(s) + \beta \mathbb{E}G^{npl}(s_{+1}) = b^{npl}(s)$. Therefore, the value of the firm becomes $V(s, D) = G^{npl}(s)$. By definition of $k^{npl}(s)$, it is obvious that the limited liability constraint is satisfied in this equilibrium. Thus, the unique equilibrium that maximizes the PDV of repayments is the NPL equilibrium.

G On the proof of Theorem 11

G.1 Proof of Lemma 12

We prove Lemma 12 by explicitly deriving $\{d^{(1)}(s, D), V^{(1)}(s, D), b^{(1)}(s, D), k^{(1)}(s, D)\}$. For $D < D^{**}(s) \equiv F(s, k^*(s)) - Rk^*(s)$,

$$\begin{aligned} d^{(1)}(s, D) &= D, \\ V^{(1)}(s, D) &= F(s, k) - Rk + \beta V_H^* - D, \end{aligned}$$

as $d^{(1)}(s, D) = \max_b b + \beta[\beta^{-1}(D - b)]$ and $b = D$ is feasible because $F(s, k) - Rk + \beta V_H^* - D \geq G(s, k)$ is satisfied at $k = k^*(s)$. Thus, for $0 \leq D \leq D^{**}(s)$, $(d^{(1)}(s, D), V^{(1)}(s, D))$ are given as above, with $k = k^*(s)$ and $b = D$.

For $D \in (D^{**}(s), D^*(s)]$, where $D^*(s)$ is the solution to $D^{**}(s) + \beta[\beta^{-1}(D - D^{**}(s))] = D = F(s, k^*(s)) - Rk^*(s) + \beta V_H^* - G(s, k^*(s))$,

$$\begin{aligned} d^{(1)}(s, D) &= D, \\ V^{(1)}(s, D) &= F(s, k) - Rk + \beta V_H^* - D, \end{aligned}$$

where $k = k^*(s)$ and $b = D^{**}(s)$.

For $D \in (D^*(s), \hat{D}^{(1)}(s)]$, where $\hat{D}^{(1)}(s) = F(s, k^{npl}(s)) - Rk^{npl}(s) - G(s, k^{npl}(s)) + \beta V_H^*$, the solution $(d^{(1)}(s, D), V^{(1)}(s, D))$ is given as follows.

$$\begin{aligned} d^{(1)}(s, D) &= D, \\ V^{(1)}(s, D) &= F(s, k(s, D)) - Rk(s, D) + \beta V_H^* - D, \end{aligned}$$

where

$$\begin{aligned} k(s, D) &= \arg \max_{k \in \Delta_k(s)} F(s, k) - Rk - D + \beta V_H^*, \\ \text{s.t. } & F(s, k) - Rk - D + \beta V_H^* \geq G(s, k). \end{aligned} \quad (48)$$

Then, it is obvious that $k(s, D)$ is decreasing in D . $D_{+1}(s, D)$ is given by

$$\begin{aligned} D_{+1}(s, D) &= \min_{D_{+1} \in \Delta} D_{+1}, \\ \text{s.t. } & D - \beta D_{+1} \leq F(s, k(s, D)) - Rk(s, D). \end{aligned}$$

Note that if $D = \hat{D}^{(1)}(s)$, then $D_{+1} = V_H^* - \beta^{-1}G^{npl}(s) < \bar{D}^{(0)}$. Note that if $D > \hat{D}^{(1)}(s)$, the enforcement constraint (48) is never satisfied for any value of k , if $V^{(1)}(s, D) = F(s, k(s, D)) - Rk(s, D) + \beta V_H^* - D$.

For $D > \hat{D}^{(1)}(s)$, it must be the case that $D_{+1} \geq \bar{D}^{(0)}$, since otherwise $V^{(1)}(s, D)$ becomes $F(s, k(s, D)) - Rk(s, D) + \beta V_H^* - D$ and the enforcement constraint (48) is never satisfied because $\hat{D}^{(1)}(s)$ is the maximum value that is feasible under (48). $D_{+1} \geq \bar{D}^{(0)}$ is feasible for $D (> \hat{D}^{(1)}(s))$, because $\beta^{-1}\hat{D}^{(1)}(s) > \bar{D}^{(0)}$ is easily shown. Given that $D_{+1} > \bar{D}^{(0)}$, we have $d^{(0)}(s, D_{+1}) = d^{npl}(s)$ and $V^{(0)}(s, D_{+1}) = G^{npl}(s)$. Thus, the values of $(d^{(1)}(s, D), V^{(1)}(s, D), b(s, D), k(s, D))$ are given as the solution to the following problem.

$$\begin{aligned} d^{(1)}(s, D) &= \max_{b \in \Delta_b(s, D), k \in \Delta_k(s)} b + \beta \mathbb{E}d^{npl}(s), \\ \text{s.t. } & \begin{cases} F(s, k) - Rk - b + \beta \mathbb{E}G^{npl}(s) \geq G(s, k), \\ F(s, k) - Rk \geq b. \end{cases} \end{aligned}$$

Then,

$$V^{(1)}(s, D) = F(s, k(s, D)) - Rk(s, D) - b(s, D) + \beta \mathbb{E}G^{npl}(s).$$

The solution is

$$b(s, D) = b^{npl}(s), \quad k(s, D) = k^{npl}(s), \quad d^{(1)}(s, D) = d^{npl}(s), \quad V^{(1)}(s, D) = G^{npl}(s),$$

for $D > \hat{D}^{(1)}(s)$. It is also easily confirmed that

$$\hat{D}^{(1)}(s) = \bar{D}^{(1)}(s),$$

where $\bar{D}^{(1)}(s)$ is defined by

$$\begin{aligned} \bar{D}^{(1)}(s_H) &= \max D, \\ \text{s.t. } & D_{+1}(s_H, D) < \bar{D}^{(0)}, \\ \bar{D}^{(1)}(s_L) &= \max D, \\ \text{s.t. } & D_{+1}(s_L, D) < \bar{D}^{(0)}. \end{aligned}$$

Now, we can show the following claim.

Claim 2. $\bar{D}^{(1)}(s_L) \leq \bar{D}^{(1)}(s_H) < \bar{D}^{(0)}$.

(Proof of Claim 2)

We have $\bar{D}^{(1)}(s_L) \leq \bar{D}^{(1)}(s_H)$, and

$$\begin{aligned}\bar{D}^{(1)}(s_H) &= F(s_H, k^{npl}(s_H)) - Rk^{npl}(s_H) - G(s_H, k^{npl}(s_H)) + \beta V_H^* \\ &< F(s_H, k^*(s_H)) - Rk^*(s_H) + \beta V_H^* - G(s_H, k^{npl}(s_H)) \\ &= V_H^* - G(s_H, k^{npl}(s_H)) = \bar{D}^{(0)}.\end{aligned}$$

(End of proof of Claim 2)

Note that $d^{npl}(s) < \bar{D}^{(1)}(s)$ because $V_H^* > G^{npl}(s_H) + d^{npl}(s_H)$ implies that $d^{npl}(s) = b^{npl}(s) + \beta \mathbb{E}d^{npl}(s_{+1}) = F(s, k^{npl}(s)) - Rk^{npl}(s) - G^{npl}(s) + \beta \mathbb{E}[G^{npl}(s_{+1}) + d^{npl}(s_{+1})] < F(s, k^{npl}(s)) - Rk^{npl}(s) - G^{npl}(s) + \beta V_H^* = \bar{D}^{(1)}(s)$.

These explicit solutions directly imply (i)–(vi) of Lemma 12.

G.2 Proof of Lemma 13

Proof of (ii). The assumption (ii') implies that $\mathbb{E}d^{(n)}(s_{+1}, D_{+1}) \leq \mathbb{E}d^{(n-1)}(s_{+1}, D_{+1})$, and the assumption (v') implies that $\Gamma^{(n+1)}(s, D) \subset \Gamma^{(n)}(s, D)$. These facts imply that

$$d^{(n+1)}(s, D) = \max_{b \in \Gamma^{(n+1)}(s, D)} b + \beta \mathbb{E}d^{(n)}(s_{+1}, D_{+1}) \leq \max_{b \in \Gamma^{(n)}(s, D)} b + \beta \mathbb{E}d^{(n-1)}(s_{+1}, D_{+1}) = d^{(n)}(s, D).$$

Since $b^{npl}(s) \in \Gamma^{(n+1)}(s, D)$ and $d^{(n)}(s, D) \geq d^{npl}(s)$ for $D > d^{npl}(s)$,

$$d^{(n+1)}(s, D) = \max_{b \in \Gamma^{(n+1)}(s, D)} b + \beta \mathbb{E}d^{(n)}(s_{+1}, D_{+1}) \geq b^{npl}(s) + \beta \mathbb{E}d^{npl}(s_{+1}) = d^{npl}(s),$$

for $D > d^{npl}(s)$. It is obvious that $d^{(n+1)}(s, D) \geq 0$ for $D \leq d^{npl}(s)$.

Proof of (iii). Assumption (iii') implies that for $D \geq \bar{D}^{(n+1)}(s)$, the values of $(d^{(n+1)}(s, D), V^{(n+1)}(s, D), b^{(n+1)}(s, D), k^{(n+1)}(s, D))$ are given as the solution to the following problem.

$$\begin{aligned}d^{(n+1)}(s, D) &= \max_{b \in \Delta_b(s, D), k \in \Delta_k(s)} b + \beta \mathbb{E}d^{npl}(s), \\ \text{s.t. } &\begin{cases} F(s, k) - Rk - b + \beta \mathbb{E}G^{npl}(s) \geq G(s, k), \\ F(s, k) - Rk \geq b. \end{cases}\end{aligned}$$

Then,

$$V^{(n+1)}(s, D) = F(s, k(s, D)) - Rk(s, D) - b(s, D) + \beta \mathbb{E}G^{npl}(s).$$

It is easily shown that the solution is given by

$$b(s, D) = b^{npl}(s), \quad k(s, D) = k^{npl}(s), \quad d^{(n+1)}(s, D) = d^{npl}(s), \quad V^{(n+1)}(s, D) = G^{npl}(s).$$

Proof of (i). For $D \geq \bar{D}^{(n+1)}(s)$, it is the case that $d^{(n+1)}(s, D + \delta) = d^{npl}(s) \leq d^{(n+1)}(s, D) + \delta$ by the part (iii) above. Next, we consider the case where $D < \bar{D}^{(n+1)}(s)$. We can prove the following claim.

Claim 3. For $D < \bar{D}^{(n+1)}(s)$, $b^{(n+1)}(s, D + \delta)$ is the maximum feasible value, i.e.,

$$b^{(n+1)}(s, D + \delta) = \max_{b \in \Gamma^{(n+1)}(s, D + \delta)} b.$$

(Proof of Claim 3). Suppose that $b^{(n+1)}(s, D + \delta)$ is not the maximum feasible value. Then, $b^{(n+1)}(s, D + \delta) + \beta\delta \in \Gamma^{(n+1)}(s, D + \delta)$. We compare $d^{(n+1)}(s, D + \delta)$ and $X^{(n+1)}(b^{(n+1)}(s, D + \delta) + \beta\delta, s, D + \delta)$, where $X^{(n+1)}(b, s, D) \equiv b + \beta\mathbb{E}d^{(n)}(s_{+1}, \beta^{-1}(D - b))$. Assumption (i') implies that

$$\begin{aligned} & X^{(n+1)}(b^{(n+1)}(s, D + \delta) + \beta\delta, s, D + \delta) \\ &= b^{(n+1)}(s, D + \delta) + \beta\delta + \beta\mathbb{E}d^{(n)}(s_{+1}, \beta^{-1}(D + \delta - b^{(n+1)}(s, D + \delta)) - \delta) \\ &= b^{(n+1)}(s, D + \delta) + \beta\mathbb{E}\{\delta + d^{(n)}(s_{+1}, \beta^{-1}(D + \delta - b^{(n+1)}(s, D + \delta)) - \delta)\} \\ &\geq b^{(n+1)}(s, D + \delta) + \beta\mathbb{E}d^{(n)}(s_{+1}, \beta^{-1}(D + \delta - b^{(n+1)}(s, D + \delta))) \\ &= d^{(n+1)}(s, D + \delta) = \max_b X^{(n+1)}(b, s, D + \delta). \end{aligned}$$

Assumption (iv') implies that

$$\begin{aligned} & V^{(n+1)}(s, D + \delta) = \\ & F(s, k(s, D + \delta)) - Rk(s, D + \delta) - b(s, D + \delta) + \beta\mathbb{E}V^{(n)}(s_{+1}, D_{+1}^{(n+1)}(s, D + \delta)) \leq \\ & F(s, k(s, D + \delta)) - Rk(s, D + \delta) - b(s, D + \delta) + \beta\mathbb{E}(-\delta + V^{(n)}(s_{+1}, D_{+1}^{(n+1)}(s, D + \delta) - \delta)) = \\ & F(s, k(s, D + \delta)) - Rk(s, D + \delta) - b(s, D + \delta) - \beta\delta + \beta\mathbb{E}V^{(n)}(s_{+1}, D_{+1}^{(n+1)}(s, D + \delta) - \delta). \end{aligned}$$

Assumption (iv') applies here as $D + \delta \leq \bar{D}^{(n+1)}(s)$, which implies $D_{+1}^{(n+1)}(s, D + \delta) \leq \bar{D}^{(n)}(s)$. These two inequalities imply that the equilibrium value of b should be $b(s, D + \delta) + \beta\delta$. This contradicts the definition of $b^{(n+1)}(s, D + \delta)$. Therefore, $b^{(n+1)}(s, D + \delta)$ is the maximum feasible value. (End of proof of Claim 3)

This claim implies that it suffices to consider the region $b \geq \delta$, when we evaluate $d^{(n+1)}(s, D + \delta)$. If $b + \delta \in \Gamma^{(n+1)}(s, D + \delta)$ then $b \in \Gamma^{(n+1)}(s, D)$ for $D > F(s, k^*(s)) - Rk^*(s)$.²² Defining \hat{b} by $\hat{b} = b(s, D + \delta) - \delta$, it is easily demonstrated that $\hat{b} \in \Gamma^{(n+1)}(s, D)$. Thus,

$$\begin{aligned} d^{(n+1)}(s, D + \delta) &= b(s, D + \delta) + \beta\mathbb{E}d^{(n)}(s_{+1}, \beta^{-1}(D + \delta - b(s, D + \delta))) \\ &= \delta + \hat{b} + \beta\mathbb{E}d^{(n)}(s_{+1}, \beta^{-1}(D - \hat{b})), \\ &\leq \delta + \max_{b \in \Gamma^{(n+1)}(s, D)} b + \beta\mathbb{E}d^{(n)}(s_{+1}, \beta^{-1}(D - b)) = \delta + d^{(n+1)}(s, D). \end{aligned}$$

²²For $D \leq F(s, k^*(s)) - Rk^*(s)$, $(b, D_{+1}) = (D, 0)$ is feasible. Let $d^{(n+1)}(s, D) = b + \beta\mathbb{E}d^{(n)}(s_{+1}, \beta^{-1}(D - b))$. Assumption (i') implies that, for any $b \geq 0$, $\beta\mathbb{E}d^{(n)}(s_{+1}, \beta^{-1}(D - b)) \leq \beta[\beta^{-1}(D - b)] + \beta\mathbb{E}d^{(n)}(s_{+1}, 0)$. Thus, it must be the case that $d^{(n+1)}(s, D) = D + \beta\mathbb{E}d^{(n)}(s_{+1}, 0)$. Therefore, $d^{(n+1)}(s, D + \delta) = \delta + d^{(n+1)}(s, D)$, for $D \leq F(s, k^*(s)) - Rk^*(s)$.

Proof of (iv). We consider the case where $D + \delta \leq \bar{D}^{(n+1)}(s)$. Define $\tilde{\Delta}_b(s, D) = \{b \in \mathbb{R} \mid b = D - \beta D_{+1}, \text{ where } D_{+1} \in \Delta_{+1}, \text{ and } b \geq 0\} \cup \{b^{npl}(s) - \delta\}$. Define $\tilde{\Gamma}^{(n+1)}(s, D) = \{b \in \tilde{\Delta}_b(s, D) \mid \exists k \in \Delta_k(s), \text{ s.t. } F(s, k) - Rk - b - \delta + \beta \mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - b)) \geq G(s, k), \text{ and } F(s, k) - Rk - b - \delta \geq 0\}$. Let $\tilde{b}(s, D)$ be the maximum value of $\tilde{\Gamma}^{(n+1)}(s, D)$. It is obvious that $\tilde{b}(s, D) \leq b(s, D)$, as $b(s, D)$ is the maximum value of $\Gamma^{(n+1)}(s, D)$. $V^{(n+1)}(s, D + \delta)$ can be written as

$$V^{(n+1)}(s, D + \delta) = -\delta + \tilde{V}^{(n+1)}(s, D), \quad (49)$$

where

$$\begin{aligned} \tilde{V}^{(n+1)}(s, D) &= \max_{k \in \Delta_k(s)} F(s, k) - Rk - \tilde{b}(s, D) + \beta \mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - \tilde{b}(s, D))), \quad (50) \\ \text{s.t. } &F(s, k) - Rk - \tilde{b}(s, D) - \delta + \beta \mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - \tilde{b}(s, D))) \geq G(s, k), \\ &F(s, k) - Rk - \tilde{b}(s, D) - \delta \geq 0. \end{aligned}$$

Let $\tilde{k}(s, D)$ be the solution to (50). The following claim holds:

Claim 4. $\tilde{b}(s, D)$ and $\tilde{k}(s, D)$ satisfy $\tilde{b}(s, D) \leq b(s, D)$ and $\tilde{k}(s, D) \leq k(s, D)$.

(Proof of Claim 4). We know $\tilde{b}(s, D) \leq b(s, D)$ from the above argument. Now, $k(s, D)$ is the maximum k that satisfies

$$\begin{aligned} F(s, k) - Rk - b(s, D) + \beta \mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - b(s, D))) &\geq G(s, k), \\ F(s, k) - Rk - b(s, D) &\geq 0, \end{aligned}$$

while $\tilde{k}(s, D)$ is the maximum k that satisfies

$$\begin{aligned} F(s, k) - Rk - \tilde{b}(s, D) - \delta + \beta \mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - \tilde{b}(s, D))) &\geq G(s, k), \\ F(s, k) - Rk - \tilde{b}(s, D) - \delta &\geq 0. \end{aligned}$$

We will demonstrate that $\tilde{k}(s, D) \leq k(s, D)$ by contradiction. Suppose that $\tilde{k}(s, D) > k(s, D)$. Then, $F(s, \tilde{k}(s, D)) - R\tilde{k}(s, D) - b(s, D) \geq 0$ is satisfied. The condition for $\tilde{b}(s, D)$ implies

$$F(s, \tilde{k}(s, D)) - R\tilde{k}(s, D) - \tilde{b}(s, D) - \delta + \beta \mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - \tilde{b}(s, D))) \geq G(s, \tilde{k}(s, D)). \quad (51)$$

By definition of $\tilde{\Gamma}^{(n+1)}(s, D)$, the fact that $\tilde{b}(s, D) \leq b(s, D)$ implies that there exists an integer $m (\geq 0)$ such that $\tilde{b}(s, D) + m\beta\delta = b(s, D)$. Then,

$$\begin{aligned} -\tilde{b}(s, D) + \beta \mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - \tilde{b}(s, D))) &= -b(s, D) + m\beta\delta + \beta \mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - b(s, D) + m\beta\delta)) \\ &\leq -b(s, D) + \beta \mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - b(s, D))), \end{aligned}$$

where the inequality is due to assumption (iv'). This inequality together with (51) implies that

$$F(s, \tilde{k}(s, D)) - R\tilde{k}(s, D) - b(s, D) - \delta + \beta \mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - b(s, D))) \geq G(s, \tilde{k}(s, D)).$$

This condition and the nonnegativity condition ($F(s, \tilde{k}(s, D)) - R\tilde{k}(s, D) - b(s, D) \geq 0$) imply that $\tilde{k}(s, D) \in \Gamma^{(n+1)}(s, D)$, which implies that $\tilde{k}(s, D) \leq k(s, D)$, a contradiction. Thus, it must be the case that $\tilde{k}(s, D) \leq k(s, D)$. (*End of proof of Claim 4*).

Let $(k, b) = (k(s, D), b(s, D))$ and $(\tilde{k}, \tilde{b}) = (\tilde{k}(s, D), \tilde{b}(s, D))$. Then, Claim 4 implies that there exist a non-negative integer m and a non-negative real number ε such that

$$\begin{aligned} F(s, \tilde{k}) - R\tilde{k} &= F(s, k) - Rk - \varepsilon, \\ \tilde{b} &= b - m\beta\delta. \end{aligned}$$

Thus,

$$\begin{aligned} \tilde{V}^{(n+1)}(s, D) &= F(s, \tilde{k}) - R\tilde{k} - \tilde{b} + \beta \mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - \tilde{b})), \\ &= F(s, k) - Rk - \varepsilon - b + m\beta\delta + \beta \mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - b) + m\delta), \\ &= -\varepsilon + F(s, k) - Rk - b + \beta \mathbb{E}[m\delta + V^{(n)}(s_{+1}, \beta^{-1}(D - b) + m\delta)] \\ &\leq -\varepsilon + F(s, k) - Rk - b + \beta \mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - b)) \\ &= -\varepsilon + V^{(n+1)}(s, D) \leq V^{(n+1)}(s, D), \end{aligned}$$

where the first inequality is from Assumption (iv'). Note that Assumption (iv') applies, since $\beta^{-1}(D - \tilde{b}) < D^{(n)}(s)$ because $D + \delta < D^{(n+1)}(s)$. (49) implies that $V^{(n+1)}(s, D + \delta) = -\delta + \tilde{V}^{(n+1)}(s, D) \leq -\delta + V^{(n+1)}(s, D)$.

Proof of (v). For $D > \bar{D}^{(n+1)}(s)$, it is the case that $V^{(n+1)}(s, D) = G^{mpl}(s)$ as proven at part (iii). Next, we consider the case where $D \leq \bar{D}^{(n+1)}(s)$. For a fixed (s, D) , Assumption (v') implies that $\Gamma^{(n+1)}(s, D) \subset \Gamma^{(n)}(s, D)$ and $\Lambda^{(n+1)}(s, D) \subset \Lambda^{(n)}(s, D)$. The following claim holds.

Claim 5. The variables for $(n + 1)$ -th problem satisfy $b^{(n+1)}(s, D) \leq b^{(n)}(s, D)$ and $k^{(n+1)}(s, D) \leq k^{(n)}(s, D)$.

(*Proof of Claim 5*). Since $\Gamma^{(n+1)}(s, D) \subset \Gamma^{(n)}(s, D)$, Claim 3 implies that $b^{(n+1)}(s, D) \leq b^{(n)}(s, D)$. Next, we prove $k^{(n+1)}(s, D) \leq k^{(n)}(s, D)$. Denote by $(C^{(n)})$ and $(C^{(n+1)})$ the following conditions:

$$\begin{aligned} (C^{(n)}) &\quad \begin{cases} F(s, k) - Rk - b + \beta \mathbb{E}V^{(n-1)}(s_{+1}, \beta^{-1}(D - b)) \geq G(s, k), \\ F(s, k) - Rk - b \geq 0, \end{cases} \\ (C^{(n+1)}) &\quad \begin{cases} F(s, k) - Rk - b + \beta \mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - b)) \geq G(s, k), \\ F(s, k) - Rk - b \geq 0, \end{cases} \end{aligned}$$

- Case 1: Suppose that $b^{(n+1)} = b^{(n)}$

In this case, $k^{(n+1)} \leq k^{(n)}$ should hold because $(C^{(n+1)})$ is (weakly) tighter than $(C^{(n)})$ for $b = b^{(n+1)} = b^{(n)}$.

- Case 2: Suppose that $b^{(n+1)} < b^{(n)}$.

In this case, we first prove that the following condition holds:

$$0 \leq F(s, k^{(n+1)}(s, D)) - Rk^{(n+1)}(s, D) - b^{(n+1)}(s, D) < \delta(s, k^{(n+1)}(s, D)) + \beta\delta, \quad (52)$$

where $\delta(s, k^{(n+1)}(s, D))$ is defined by $\delta(s, k^{(n+1)}(s, D)) \equiv F(s, k^{(n+1)}(s, D)) - Rk^{(n+1)}(s, D) - F(s, k_-^{(n+1)}(s, D)) + Rk_-^{(n+1)}(s, D)$, where $k_-^{(n+1)}(s, D)$ is defined by $f(s, k_-^{(n+1)}(s, D)) - f(s, k^{(n+1)}(s, D)) = \beta\delta$. Thus, $k_-^{(n+1)}(s, D)$ is the value of k , which is smaller than and adjacent to $k^{(n+1)}(s, D)$. The condition (52) is proven by contradiction.²³ Then, as $b^{(n)}(s, D) \geq b^{(n+1)}(s, D) + \beta\delta$, the condition (52) implies that

$$F(s, k_-^{(n+1)}(s, D)) - Rk_-^{(n+1)}(s, D) - b^{(n)}(s, D) < 0,$$

which implies that $k^{(n)}(s, D) > k_-^{(n+1)}(s, D)$, which means $k^{(n)}(s, D) \geq k^{(n+1)}(s, D)$.

(End of proof of Claim 5).

Let $(k, b) = (k^{(n)}(s, D), b^{(n)}(s, D))$ and $(\tilde{k}, \tilde{b}) = (k^{(n+1)}(s, D), b^{(n+1)}(s, D))$. The above claim implies that there exists a non-negative integer m and a non-negative real number ε such that $F(s, \tilde{k}) - R\tilde{k} = F(s, k) - Rk - \varepsilon$ and $\tilde{b} = b - m\beta\delta$. Thus,

$$\begin{aligned} V^{(n+1)}(s, D) &= F(s, \tilde{k}) - R\tilde{k} - \tilde{b} + \beta\mathbb{E}V^{(n)}(s_{+1}, \beta^{-1}(D - \tilde{b})), \\ &\leq F(s, k) - Rk - \varepsilon - b + m\beta\delta + \beta\mathbb{E}V^{(n-1)}(s_{+1}, \beta^{-1}(D - b) + m\delta), \\ &= -\varepsilon + F(s, k) - Rk - b + \beta\mathbb{E}[m\delta + V^{(n-1)}(s_{+1}, \beta^{-1}(D - b) + m\delta)] \\ &\leq -\varepsilon + F(s, k) - Rk - b + \beta\mathbb{E}V^{(n-1)}(s_{+1}, \beta^{-1}(D - b)) \\ &= -\varepsilon + V^{(n)}(s, D) \leq V^{(n)}(s, D), \end{aligned}$$

where the first inequality is from Assumption (v') and the second inequality is from Assumption (iv'). Note that Assumption (iv') applies since $D \leq \bar{D}^{(n+1)}(s)$, which implies that $\beta^{-1}(D - b) \leq \bar{D}^{(n)}(s) \leq \bar{D}^{(n-1)}(s)$. The fact that $k^{(n+1)}(s, D) \geq k^{npl}(s)$ and the enforcement constraint $[V^{(n+1)}(s, D) \geq G(s, k^{(n+1)}(s, D))]$ directly imply that

$$V^{(n+1)}(s, D) \geq G^{mpl}(s).$$

²³Suppose that $F(s, k^{(n+1)}(s, D)) - Rk^{(n+1)}(s, D) - b^{(n+1)}(s, D) \geq \delta(s, k^{(n+1)}(s, D)) + \beta\delta$. Then, $k = k_-^{(n+1)}(s, D)$ and $b = b^{(n+1)}(s, D) + \beta\delta$ satisfies $(C^{(n+1)})$, as follows. First, the limited liability ($F(s, k) - Rk - b \geq 0$) is obviously satisfied. Second, since $F(s, k^{(n+1)}(s, D)) - Rk^{(n+1)}(s, D) - G(s, k^{(n+1)}(s, D)) = F(s, k_-^{(n+1)}(s, D)) - Rk_-^{(n+1)}(s, D) - G(s, k_-^{(n+1)}(s, D)) - \beta\delta$ and $V^{(n)}(s_{+1}, \beta(D - b^{(n+1)}(s, D))) \leq V^{(n)}(s_{+1}, \beta(D - b^{(n+1)}(s, D) - \beta\delta))$, the enforcement constraint is satisfied for $k = k_-^{(n+1)}(s, D)$ and $b = b^{(n+1)}(s, D) + \beta\delta$. Thus, they are in $\Gamma^{(n+1)}(s, D)$. Then, the solution to $(n+1)$ -th problem should be $b^{(n+1)}(s, D) + \beta\delta$, instead of $b^{(n+1)}(s, D)$. This is a contradiction.

Proof of (vi). First, we prove $\bar{D}^{(n+1)}(s) \leq \bar{D}^{(n)}(s)$ by contradiction. Suppose that $\exists s, \bar{D}^{(n+1)}(s) > \bar{D}^{(n)}(s)$. Then, we can pick D such that $\bar{D}^{(n)}(s) < D \leq \bar{D}^{(n+1)}(s)$, which satisfies

$$\begin{aligned} D_{+1}^{(n+1)}(s, D) &= \beta^{-1}[D - b^{(n+1)}(s, D)] < \bar{D}^{(n)}(s_H) \leq \bar{D}^{(n-1)}(s_H), \\ D_{+1}^{(n)}(s, D) &= \beta^{-1}[D - b^{(n)}(s, D)] \geq \bar{D}^{(n-1)}(s_H). \end{aligned}$$

These inequalities imply $b^{(n+1)}(s, D) > b^{(n)}(s, D)$, while $b^{(n+1)}(s, D)$ is feasible in (n) -th problem:

$$b^{(n+1)}(s, D) \in \Gamma^{(n+1)}(s, D) \subset \Gamma^{(n)}(s, D).$$

Therefore, $b^{(n)}(s, D)$ and $b^{(n)}(s, D) + \beta\delta$ are both feasible in (n) -th problem. Assumption (i') implies

$$\begin{aligned} d^{(n)}(s, D) &= b^{(n)}(s, D) + \beta\mathbb{E}d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s, D)) \\ &\leq b^{(n)}(s, D) + \beta\mathbb{E}[\delta + d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s, D) - \delta)] \\ &= b^{(n)}(s, D) + \beta\delta + \beta\mathbb{E}d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s, D) - \delta). \end{aligned}$$

If $d^{(n)}(s, D) < b^{(n)}(s, D) + \beta\delta + \beta\mathbb{E}d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s, D) - \delta)$, then $b^{(n)} + \beta\delta$ should be the solution to the (n) -th problem. This is a contradiction because $b^{(n)}(s, D)$ is the solution. If $d^{(n)}(s, D) = b^{(n)}(s, D) + \beta\delta + \beta\mathbb{E}d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s, D) - \delta)$, then the fact that $d^{(n)}(s, D) = d^{npl}(s)$ and $b^{(n)}(s, D) = b^{npl}(s)$ for $D > \bar{D}^{(n)}(s)$, and $d^{npl}(s) = b^{npl}(s) + \beta\mathbb{E}d^{npl}(s_{+1})$ imply that

$$\mathbb{E}d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s, D) - \delta) < \mathbb{E}d^{npl}(s_{+1}),$$

which, in turn, implies that $\exists s_{+1}, d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s, D) - \delta) < d^{npl}(s_{+1})$. On the other hand, $D > \bar{D}^{(n)}(s) > d^{npl}(s_H)$ implies that $D \geq d^{npl}(s_H) + 2\delta$, which, in turn, implies that $D_{+1}^{(n)}(s, D) - \delta \geq D - \delta > d^{npl}(s_H)$. Then, Assumption (ii') implies that $d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s, D) - \delta) \geq d^{npl}(s_{+1})$. Thus, we have demonstrated that $\exists s_{+1}$, such that $d^{npl}(s_{+1}) \leq d^{(n-1)}(s_{+1}, D_{+1}^{(n)}(s, D) - \delta) < d^{npl}(s_{+1})$, which is a contradiction. Therefore, it cannot be the case that $\exists s, \bar{D}^{(n+1)}(s) > \bar{D}^{(n)}(s)$.

H Proof of Lemma 14

Claim 2 implies that $b(s, D) = \lim_{n \rightarrow \infty} b^{(n)}(s, D)$ satisfies $b(s, D) \geq \delta$ for $D < D_{\max}(s)$. For $D \geq D_{\max}(s)$, Lemmas 12 and 13 imply $b(s, D) = b^{npl}(s) \geq \delta$. Therefore, $b(s, D) \geq \delta$ for all (s, D) .

Lemmas 12 and 13 imply that $V(s, D) = \lim_{n \rightarrow \infty} V^{(n)}(s, D)$ and $D_{\max}(s) = \lim_{n \rightarrow \infty} \bar{D}^{(n)}(s)$ satisfy that $V(s, D + \delta) \leq V(s, D) - \delta$ for $D < D_{\max}(s)$.