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Keiichiro Kobayashi
Keio University / The Canon Institute for Global Studies

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General Incorporated Foundation
The Canon Institute for Global Studies
一般財団法人　キヤノングローバル戦略研究所
Phone: +81-3-6213-0550　https://www.canon-igs.org

Growing and collapsing bubbles*

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Abstract

The large fluctuations of asset prices in financial crises are modeled as credit-driven bubbles, where agency problems in the banking sector raise the asset prices to unsustainable levels. The peak of a bubble and the timing of its collapse can be predictable because the bubble collapses when the price hits an endogenous threshold that is determined by structural parameters. Tighter monetary policy can dampen the size of the bubble, whereas tighter prudential regulations that cause credit rationing may exacerbate the bubble. Our theory recommends leaning against the bubbly wind, rather than screening the borrowers, as a stabilization policy.

Key words: Asset prices, bubbles, risk-shifting, lean versus screen.

JEL classification: E30, E44, G12.

1 Introduction

Large and unstable fluctuations of asset prices are observed frequently in the recent episodes of economic crises. The "housing bubbles" in the early 2000s in the United States and the southern European countries are the similar episodes of accelerated growth and sudden collapse of asset prices. These fluctuations in asset prices are explained as a bubble, which cannot be accounted for by economic fundamentals.

Barlevy (2018) argues that, on one hand, the policymakers pay close attention to the asset price bubbles because the bubbles are, they believe, destabilizing and distortionary, whereas, on the other hand, the existing theories of bubbles are not satisfactory, as bubbles in those theories are neither destabilizing nor distortionary. The examples of such models are bubbles due to dynamic inefficiency (Samuelson [1958], Tirole [1985]) or due to borrowing constraint (Farhi and Tirole [2012], Martin and Ventura [2012, 2016], Hirano and Yanagawa [2017]). There are theories of destabilizing and distortionary bubbles such

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as those due to agency problems (Allen and Gorton [1993], Allen and Gale [2000], Barlevy [2014]), while these models are two- or three-period models and therefore they are not compatible with standard DSGE models that are used for study of business cycles. In particular, they have nothing to tell about the "timing" of the collapse of the bubble.

In this paper, we propose a model of asset bubble, which is destabilizing and distortionary, and the timing of the collapse is endogenously determined. Our model is a general equilibrium model of a closed economy with infinite horizon, in which we embed Allen and Gale's (2000) model of credit-driven asset bubbles. We show that the peak of the asset price and the timing of its collapse are endogenously determined, implying that we can reasonably predict the bubbly dynamics of asset prices.

Barlevy(2018) argues that policymakers are also interested in the policy prescription to asset bubbles. After the Global Financial Crisis, the policymakers have agreed upon the necessity of preemptive intervention as a consensus, and the theme of policy debates has become "lean versus screen" as a means of the ex ante intervention, that is, preemptive monetary tightening to dampen the bubble (lean), or macroprudential regulation to decrease the credit to borrowers who invest in the bubbly assets (screen).

In this paper, we theoretically compare preemptive monetary tightening and macroprudential regulation, and find that the monetary tightening can dampen the bubble, whereas the prudential regulation may exacerbate the bubble. This result indicates that monetary tightening is more appropriate than tighter prudential regulation as a means to stabilize bubbly fluctuations of asset prices.

The organization of this paper is as follows. The model is presented and the equilibrium is characterized in the next section. In Section 3, monetary policy and prudential regulation are compared as a means of stabilization of asset prices. Section 4 concludes.

2 Model

We embed Allen and Gale's (2000) two-period model (AG model, hereafter) into the infinite-horizon general equilibrium economy, in which the representative consumer makes bank deposit, and the bank provides loans to the investors, who purchase risky assets and can default on the bank loans when the return on the assets turns out to be low.

2.1 Setup

The model is an infinite-horizon closed economy. Time continues from 0 to infinity, i.e., $t = 0, 1, 2, \cdots$. There exists the representative consumer who maximizes the present discounted value of consumption, which is discounted by the subjective time discount factor $\beta \in (0, 1)$. There is also a unit mass of two-period lived investors, who are born in period t and die in period $t + 1$. The investors work as agents for their owner, who is the representative consumer. There also exists a unit mass of competitive banks which accept deposits from the consumer and make loans to the investors. The investors borrow funds from the banks and invest in the safe and risky assets in order to maximize the dividends to the representative consumer. The representative consumer divides the income into consumption and savings, which is invested in the bank deposit. In this economy, the asset markets are segregated as follows.

Assumption 1. Only the investors can invest in the safe and risky assets, whereas the consumer and the banks cannot invest in those assets. The consumer can hold only bank deposits and the banks can hold only the loans to the investors as their assets.

The segregation of the markets is a reasonable assumption, for both the safe and risky assets are interpreted as corporate capital. The consumer and the banks may be able to invest in the government bond, which do not explicitly appear in our model, whereas they do not have necessary expertize to operate the corporate capital.

Representative consumer: Throughout this paper, we assume that aggregate uncertainties do not exist and only idiosyncratic shocks on investors exist. Thus, there is no uncertainty for the representative consumer and the dynamics of the bubbles turn out to be deterministic. The consumer's utility is given by

$$U = \sum_{t=0}^{\infty} \beta^t C_t, \tag{1}$$

where C_t is the consumption in period t. In every period, the representative consumer provides one unit of labor to the market and receives the wage income w_t. Her income in period t consists of y, the fixed endowment from the nature; π_t, the profit of the investors; w_t, the wage income; and $r_t^d d_{t-1}$, the return on the deposit made in the previous period d_{t-1}, where r_t^d is the rate of return on deposit, which is specified later. The total income is used to purchase C_t and to make bank deposit d_t. Thus the budget constraint is

$$C_t + d_t \leq y + w_t + \pi_t + r_t^d d_{t-1}. \tag{2}$$

The representative consumer maximizes (1) subject to (2). The first order condition (FOC) with respect to d_t in the consumer's problem implies that the following condition is necessary to have the equilibrium amount of deposit being positive, i.e., $d_t > 0$:

$$1 \leq \beta r_{t+1}^d. \tag{3}$$

Safe asset: There are two assets in this economy, i.e., the safe asset and the risky asset. The safe asset equals the consumption goods. x_t units of the safe asset invested in period t generates the return, $r_{t+1} x_t$ units of the consumption goods, in period $t+1$. As in the

AG model, we assume that there are the competitive production firms, that use the safe asset x_t as capital input. The firms use capital x_t and labor n_t, which is provided by the representative consumer, to produce $Ax_t^{\alpha}n_t^{1-\alpha}$ units of the consumption goods in period $t+1$, where A is a productivity parameter. The safe asset x_t completely depreciates to zero after production. Thus the firm solves

$$\max_{x_t, n_t} \; Ax_t^{\alpha}n_t^{1-\alpha} - r_{t+1}x_t - w_{t+1}n_t.$$

As the total supply of labor is one, i.e., $n_t = 1$, in every period, the perfect competition among the firms implies that $\{r_{t+1}, w_{t+1}\}$ are given by

$$r_{t+1} = \alpha A\bar{x}_t^{\alpha-1}, \tag{4}$$

$$w_{t+1} = (1-\alpha)A\bar{x}_t^{\alpha}, \tag{5}$$

where $\bar{x}_t$ is the average level of the safe-capital input in the society: $\bar{x}_t = \int_0^1 x_{jt}dj$, where x_{jt} is the investor j's holdings of safe capital. We denote $f(x) = Ax^{\alpha}$. Then, it is rewritten that $r_{t+1} = f'(\bar{x}_t)$ and $w_{t+1} = (1-\alpha)f(\bar{x}_t)$. Note that $\{r_{t+1}, w_{t+1}\}$ are determined in period t.

Risky asset: In the initial period, the total quantity of the risky asset in the economy is one, i.e., $X_0 = 1$, and the risky asset does not depreciate unless the investor liquidates it.[1] An investor who owns the risky asset can liquidate it *only after he holds it for one full period.* Since the investors live for two periods, this assumption implies that only the old investors can liquidate the risky asset. Suppose that a (young) investor purchases X_t units of the risky asset in period t and enters period $t+1$ with them. Now, the (old) investor pays the dead-weight cost of maintenance, $c(X_t)$, in period $t+1$. Then, the investor can choose to either resell them or liquidate them. If he chooses to resell X_t, then he can just sell them at the market price. If he chooses to liquidate X_t, then the risky asset X_t is transformed into RX_t units of the consumption goods immediately in period $t+1$, where R is a random variable, idiosyncratic to each investor, with the support of $[0, R_{\max}]$, the distribution function $\Phi(R)$, and the density function $\phi(R) = \frac{d}{dR}\Phi(R)$. The investor receives RX_t units of consumption goods, while he pays the additional dead-weight cost Δ in period $t+1$, where Δ is the fixed cost of liquidation. The dead-weight costs, $c(X)$ and Δ, are both consumption goods, with $c(0) = c'(0) = 0$, $c'(X) > 0$, and $c''(X) > 0$.

Investors: In each period a unit mass of investors are born and work for their owner, who is the representative consumer. They live for two periods. An investor has nothing at birth and he borrows fund l_t from a bank. The investor, then, purchases x units of

[1]Suppose that the risky asset is a corporate stock and the dividend is paid out only after the firm is liquidated, by selling all the physical asset in the market.

the safe asset and X units of the risky asset, using the borrowed fund. The assets yield returns and the investor repays the debt in the next period. The bank loan is the debt contract with the loan rate r^l_{t+1} and the investor can default on the loan when the return from the assets turns out to be insufficient to repay the debt.

Lemma 1. *In equilibrium,*

$$r^l_{t+1} = r_{t+1}. \tag{6}$$

Proof. Suppose that $r^l_{t+1} > r_{t+1}$. In this case, no investors invest in the safe asset and x_t becomes zero. Then, $r_{t+1} = f'(x_t) = f'(0) = +\infty$, which is a contradiction. Suppose on the contrary that $r^l_{t+1} < r_{t+1}$. In this case, all investors choose to borrow infinite amount and invest it in the safe asset, i.e., $l_t = x_t = \infty$, Then, $r_{t+1} = f'(x_t) = f'(\infty) = 0$, which is a contradiction. Therefore, $r^l_{t+1} = r_{t+1}$ must hold. □

The budget constraint for the investor born in period t is

$$x_t + P_t X_t \leq l_t, \tag{7}$$

where P_t is the price of the risky asset in terms of the safe asset (or the consumption goods), which is the numeraire in this economy. The young investor chooses (x_t, X_t, l_t) in period t, given that he will choose in period $t+1$ to resell or liquidate X_t to maximize his profit. The old investor in period $t+1$ can default on the loan $r_{t+1}l_t$ when the revenue from the safe and risky assets are smaller than $r_{t+1}l_t$, whereas the cost of maintenance and liquidation is not defaultable.

Assumption 2. The (old) investor in period $t+1$ pays the dead-weight cost of maintenance, $c(X_t)$, if he resells X_t, and the dead-weight cost of liquidation, $c(X_t) + \Delta$, if he liquidates X_t. These is no exemption from these payments, even if he defaults on the bank loan.

First, the problem for the old investor is formulated as

$$\pi^{Old}_{t+1}(X) = \max\{\pi^L_{t+1}(X), \quad \pi^S_{t+1}(X)\},$$

where π^L_{t+1} is the investor's profit when he chooses to liquidate in period $t+1$, and π^S_{t+1} is the profit when he chooses to resell in period $t+1$. π^L_{t+1} and π^S_{t+1} are defined by

$$\begin{aligned}\pi^L_{t+1}(X_t) &= \int_{R^*_t}^{R^{\max}} \{r_{t+1}x_t + RX_t - r_{t+1}(x_t + P_t X_t)\}\phi(R)dR - c(X_t) - \Delta \\ &= \int_{R^*_t}^{R^{\max}} (R - r_{t+1}P_t)X_t\phi(R)dR - c(X_t) - \Delta,\end{aligned}$$

with

$$R^*_t = r_{t+1}P_t,$$

and

$$\pi^S_{t+1}(X_t) = r_{t+1}x_t + P_{t+1}X_t - r_{t+1}(x_t + P_tX_t) - c(X_t) = (P_{t+1} - r_{t+1}P_t)X_t - c(X_t).$$

Next, the problem for the young investor is given as follows. Given the prices $\{P_t, r_{t+1}, P_{t+1}\}$, the value to a young investor in period t is π^{Young}_{t+1}, where

$$\pi^{Young}_{t+1} \equiv \max_{x,X} \pi^{Old}_{t+1}(X) = \max\{\max_{x,X} \pi^L_{t+1}(X), \quad \max_{x,X} \pi^S_{t+1}(X)\}.$$

Banks: In each period a competitive bank is born and lives for two periods. It accepts deposit d_t from the representative consumer and provides loans l_t to the investors, to maximize the profit. Thus, the bank's problem is

$$\max_{d_t,l_t} \mathbb{E}_t[\tilde{r}_{t+1}l_t - r^d_{t+1}d_t],$$
$$\text{s.t. } l_t \leq d_t,$$

where $\mathbb{E}_t$ is the expectations in period t, and $\tilde{r}_t$ is a random return on the loan. The FOC implies that

$$\mathbb{E}_t\tilde{r}_{t+1} = r^d_{t+1}. \tag{8}$$

Given the choice by the investors, $\{x_t, X_t\}$, the value of $\mathbb{E}_t\tilde{r}_{t+1}$ is written as follows. In the case where the investors liquidate the risky asset in period $t + 1$,

$$\mathbb{E}_t\tilde{r}_{t+1} = r_{t+1} \Pr(R \geq R^*_t) + \frac{\int_0^{R^*_t} (r_{t+1}x_t + RX_t)\phi(R)dR}{x_t + P_tX_t}, \tag{9}$$

where $R^*_t \equiv r_{t+1}P_t$. In the case where the investors resell the risky asset in period $t + 1$,

$$\mathbb{E}_t\tilde{r}_{t+1} = r_{t+1} \Pr(P_{t+1} \geq r_{t+1}P_t) + \frac{r_{t+1}x_t + P_{t+1}X_t}{x_t + P_tX_t} \Pr(P_{t+1} < r_{t+1}P_t).$$

2.2 Equilibrium

We focus on the symmetric equilibrium where all investors make the same choice whether to resell or liquidate.

Resource constraint: The consumption goods are used for consumption C_t and investment in the safe asset x_t, whereas they come from the endowment y and the production $f(x_{t-1}) = r_tx_{t-1} + w_t$. Thus, the following equality holds in the symmetric equilibrium.

$$C_t + x_t = y + f(x_{t-1}) - c(1) + \mathbb{L}_{t-1} \times [\bar{R} - \Delta], \tag{10}$$

where $\mathbb{L}_t = 1$ if all risky asset is liquidated in $t + 1$ and $\mathbb{L}_t = 0$ if it is not in $t + 1$, and $\bar{R} \equiv \int_0^{R^{\max}} R\phi(R)dR$ is the mean of R.

The bubble is always inefficient: As the risky asset generates $\bar{R} - \Delta$ only when it is liquidated and nothing otherwise, it is obvious that *liquidating all the risky asset in the initial period is the first best.* This is easily confirmed to see the inter-temporal resource constraint, given that $r_t^d = r_t = \beta^{-1}$ and the risky asset is liquidated at $t = \tau$, i.e.,

$$\sum_{t=0}^{\infty} \beta^t C_t \leq \frac{1}{1-\beta} y + f(x_{-1}) + \sum_{t=0}^{\infty} \beta^t \{\beta f(x_t) - x_t\} - \frac{1-\beta^{\tau+1}}{1-\beta} c(1) + \beta^{\tau}(\bar{R} - \Delta),$$

This constraint implies that the utility $\sum_{t=0}^{\infty} \beta^t C_t$ is maximized when $\tau = 0$. Thus, the immediate liquidation of the risky asset is the necessary condition for attaining the first best. This result implies that *any equilibrium that involves the bubble is inefficient in our model.*

In what follows we characterize the "*bubbly periods*," in which the risky asset is resold in the next periods, and the "*collapsing period*," in which the risky asset is liquidated in the next period, and show that the price of risky asset should grow by $P_{t+1} = \beta^{-1} P_t + c'(1)$ in the bubbly periods and it should be at a particular value $P_T = P^g$ in the collapsing period T. Thus, our characterization implies that the price of risky asset P_t grows for the first several periods until it hits the threshold value P^g, and then, the bubble collapses, i.e., the risky asset is liquidated. Since then, the economy stays at the steady state equilibrium without risky asset.

2.2.1 Bubbly periods

In this subsection, we characterize the "bubbly period," which is the equilibrium for period t, with that all investors resell the risky asset in period $t + 1$. In the bubbly periods, the risky asset is not liquidated and thus there is no uncertainty in the economy. Therefore, there is no default on the bank loan nor the bank deposit. The FOC in the consumer's problem, (3), implies that $r_{t+1}^d = r_{t+1} = \beta^{-1}$. The investor allocates the loan l_t to the safe asset x_t and risky asset X_t to maximize the profit, π_{t+1}^S. Thus, an investor's problem is

$$\max_{x_t \geq 0, X_t \geq 0} \pi_{t+1}^S. \tag{11}$$

The FOC with respect to X_t, together with $X_t = 1$, implies that

$$P_{t+1} = \beta^{-1} P_t + c'(1) \tag{12}$$

Denote by x^* the solution to $f'(x) = \beta^{-1}$. Then, $x_t = x^*$. We assume the following restriction on the parameter values so that the resource constraint (10) is satisfied in the bubbly periods.

$$x^* \leq y + f(x^*) - c(1). \tag{13}$$

Note that $\pi_{t+1}^S = c'(1) - c(1) > 0$ in the bubbly periods.

The condition for the young investors' problem to become maximization of π^S_{t+1} is that, given $\{P_t, r_{t+1}, P_{t+1}\}$, where $r_{t+1} = \beta^{-1}$,

$$\max_X \pi^L_{t+1} = \max_X \int_{\beta^{-1}P_t}^{R^{\max}} (R - \beta^{-1}P_t)\phi(R)dR\, X - c(X) - \Delta \leq c'(1) - c(1) = \pi^S_{t+1}.$$

This condition can be rewritten as

$$P_t \geq P^L, \tag{14}$$

where P^L is defined as the solution to

$$\int_{\beta^{-1}P}^{R^{\max}} (R - \beta^{-1}P)\phi(R)dR = c'(X^L), \tag{15}$$

where X^L is the solution to

$$\pi^L(X) = c'(1) - c(1),$$

where $\pi^L(X)$ is defined as $\pi^L(X) \equiv c'(X)X - c(X) - \Delta$.

The argument in this subsection is summarized as follows.

Proposition 2. *In period t, if $P_t \geq P^L$, $r_{t+1} = \beta^{-1}$, and $P_{t+1} = \beta^{-1}P_t + c'(1)$, then all the investor chooses not to liquidate the risky asset.*

2.2.2 Collapsing period

Here, we characterize the equilibrium for period $t = T$ with that all the risky asset is to be liquidated in the next period. We call period T the "collapsing period." The investor's problem in the collapsing period must be

$$\max_{x_t \geq 0, X_t \geq 0} \pi^L_{t+1}. \tag{16}$$

As $X_t = 1$, the FOC with respect to X_t implies that in equilibrium,

$$R^*_t \equiv r_{t+1}P_t = R^*,$$

where R^* is the solution to the FOC:

$$\int_{R^*}^{R^{\max}} (R - R^*)\phi(R)dR = c'(1), \tag{17}$$

This condition pins down the value of R^* uniquely. We focus on the case where

$$\pi^L_{t+1} = c'(1) - c(1) - \Delta \geq 0,$$

in equilibrium, i.e, we impose the restriction that $\Delta \leq c'(1) - c(1)$. The condition for the young investors' problem to become maximization of π^L_{t+1} is that, given $\{P_t, r_{t+1}, P_{t+1}\}$,

$$\max_X \pi^S_{t+1} = \max_X (P_{t+1} - r_{t+1}P_t)X - c(X) \leq c'(1) - c(1) - \Delta.$$

This condition can be rewritten as

$$P_{t+1} \le r_{t+1}P_t + c'(X^S), \tag{18}$$

where X^S is defined as the solution to

$$c'(X)X - c(X) = c'(1) - c(1) - \Delta. \tag{19}$$

Uniqueness of price in the collapsing period: In what follows we assume that $f(x) = Ax^\alpha$. Given the solution to (17), R^*, and the rate of return on the safe asset, $r_{t+1} = r$, the variables P_t and x_t are given by

$$P_t = P(r) \equiv \frac{R^*}{r},$$
$$x_t = x(r) \equiv \left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}},$$

because $r_{t+1}P_t = R^*$ and $r_{t+1} = f'(x_t)$, respectively. The consumer's FOC (3), together with (8) and (9), implies that for $l_t > 0$, the following condition must hold in equilibrium:

$$1 \le \beta \Pr(R > R^*) r_{t+1} + \beta \frac{\int_0^{R^*} (r_{t+1}x_t + R)\phi(R)dR}{x_t + P_t}.$$

The right-hand side can be written as the function $g(r_{t+1})$, where

$$g(r) \equiv \beta \Pr(R > R^*) r + \beta \frac{\int_0^{R^*} (rx(r) + R)\phi(R)dR}{x(r) + P(r)}.$$

In what follows, we focus on the case where the parameters satisfy the following condition.

$$\omega \equiv \Pr(R > R^*) + \int_0^{R^*} \left(\frac{R}{R^*}\right) \phi(R)dR \ge 2\alpha - 1. \tag{20}$$

We can show the following lemma.

Lemma 3. *Given that the parameters satisfy* (20)*, the equation* $g(r) = 1$ *has a unique solution.*

The proof is given in Appendix A. Denote by r^g the solution to

$$g(r) = 1. \tag{21}$$

Define $P^g \equiv \frac{R^*}{r^g}$. Then we have the following lemma.

Lemma 4. *The price in the collapsing period* T *must satisfy* $P_T = P^g$*.*

The proof is given in Appendix B. The argument in this subsection can be summarized as the following proposition.

Proposition 5. *Only if* $(P_T, r_{T+1}) = (P^g, \frac{R^*}{P^g})$ *and* $P_{T+1} \le R^* + c'(X^S)$*, all the investors in period* T *choose to liquidate the risky asset in period* $T + 1$*.*

Collapse of the bubble in the value of risky asset: The price of the risky asset $P_T = P^g$ is jacked-up by the risk-shifting from the investors to the banks (or the consumer) just like in Allen and Gale (2000). We see P^g is higher than the "fundamental price" of the risky asset, $\bar{P}_T$, which is defined, in the same way as in Allen and Gale (2000), as the price at which the investor who invests his own money is willing to invest in the risky asset, given the risk-free rate r_{T+1}. Thus,

$$\bar{P}_T = \frac{1}{r_{T+1}} \left[\bar{R} - c'(1) \right]. \tag{22}$$

Proposition 1 in Allen and Gale (2000) implies that $P^g > \bar{P}_T$, given $r_{T+1} = r^g$, as (17) can be rearranged to

$$P^g = \frac{1}{r_{T+1}} \left[\frac{\int_{R^*}^{R^{\max}} R\phi(R)dR - c'(1)}{\Pr(R \geq R^*)} \right].$$

Note that (17) implies that $R^* \to R^{\max}$ in the case where the marginal cost of liquidation is negligibly small, i.e., $c'(1) \to 0$. Then,

$$P^g = \frac{R^*}{r_{T+1}} \to \frac{R^{\max}}{r_{T+1}} \quad \text{and} \quad \bar{P}_T \to \frac{\bar{R}}{r_{T+1}}, \quad \text{as} \quad c'(1) \to 0.$$

If $R^{\max} \gg \bar{R}$, P^g can be quite larger than the fundamental price $\bar{P}_T$, when the marginal cost of liquidation $c'(1)$ is sufficiently small. In period $T+1$, the total value of the risky asset turns out to be[2]

$$\bar{R} - c(1) - \Delta,$$

which is much lower than the price in the previous period, $P_T = P^g$. The change in the value of risky asset from $P_T X_T = P^g$ in period T to $\bar{R} - c(1) - \Delta$ in period $T+1$ can be interpreted as the collapse of the bubble.

2.2.3 The deterministic equilibrium with bubble

We have characterized the bubbly periods, when the risky assets are resold from the old to young investors, and the collapsing period, when the price of the risky asset hits the peak value, which is uniquely given as P^g. It is shown as follows that the equilibrium path is deterministic, given that the initial price of the risky asset, P_0, satisfies $P^L < P_0 < P_g$.

Proposition 6. *Suppose that there exist a positive integer T and the sequence of prices $\{P_t\}_{t=0}^{T+1}$, which satisfies $P_0 > P^L$, $P_T = P^g$, $P_{T+1} < R^* + c'(X^C)$, and $P_{t+1} = \beta^{-1} P_t + c'(1)$ for $0 \leq t \leq T-1$. This sequence $\{P_t\}_{t=0}^{T+1}$ is the equilibrium prices, and all old investors choose to resell the risky asset in bubbly periods, i.e., $t = 0, 1, 2, \cdots, T$, and choose to liquidate it in period $T+1$. Thus, the price of risky asset grows for $t = 0, 1, 2, \cdots, T$, and the risky asset is liquidated in period $T+1$.*

[2] Note that R is an idiosyncratic shock to each investor.

Proof. It suffices to prove that $P_t > P^g$ cannot hold in equilibrium. Suppose that in some period t_0, $P_{t_0} > P^g$ and all the risky asset is purchased by the investors who are born in period t_0. Lemma 4 implies that period t_0 cannot be the collapsing period, and then t_0 is the bubbly period, implying that $P_{t_0+1} = \beta^{-1}P_{t_0} + c'(1) > P^g$. Thus, $t_0 + 1$ should be also the bubbly period. By induction, it is shown that all periods $t \geq t_0$ should be the bubbly periods. Thus, $P_{t+1} = \beta^{-1}P_t + c'(1)$, $r_{t+1} = \beta^{-1}$, $x_t = x^*$, $X_t = 1$, and $\pi_t = c'(1) - c(1)$, for all $t \geq t_0$. The budget constraint for the consumer can be rewritten as follows, as $d_t = x_t + P_t$:

$$C_t + x_t + P_t = y + w_t + r_t x_{t-1} + r_t P_{t-1} + c'(1) - c(1).$$

Since $x_t = x^*$, $r_{t+1} = \beta^{-1}$, and $w_t + r_t x_{t-1} = f(x_{t-1})$, this budget constraint can be simplified to

$$C_t + x^* + P_t = y^* + \beta^{-1}P_{t-1},$$

where $y^* = y + f(x^*) + c'(1) - c(1)$. Adding up the period $t_0 + j$ budget multiplied by β^j, we obtain the present value budget:

$$\sum_{j=0}^{J} \beta^j (C_{t_0+j} + x^*) + \beta^J P_{t_0+J} = \sum_{j=0}^{J} \beta^j y^* + \beta^{-1} P_{t_0-1}.$$

The transversality condition for the consumer is thus $\lim_{J\to\infty} \beta^J P_{t_0+J} = 0$, whereas $\beta^J P_{t_0+J} > \beta^{J-1} P_{t_0+J-1} > \cdots > P_{t_0} > 0$, because $P_{t+1} = \beta^{-1}P_t + c'(1)$ for $t \geq t_0$. Thus, the transversality condition is not satisfied. This contradicts the rationality of the consumer. Therefore, P_t cannot be larger than P^g in equilibrium. □

In Appendix C, we show a simple back-of-the-envelope example, in which the bubbly periods continue for the first two periods and the bubble hits its peak in the third period. The risky asset is liquidated in the fourth period.

Uniqueness of the price path: It can be said that the equilibrium path of the price is unique in that the price must hit P^g eventually, i.e., $\exists T$ (≥ 0) such that $P_T = P^g$, and the evolution of the price is deterministic. T, the number of bubbly periods, can vary depending on the initial value P_0, which can take any value on the path as long as $P^L < P_0 < P^g$. Note that there always exists the fundamental equilibrium with $T = 0$, where all the risky asset is liquidated in the initial period, given that $P_0 < P^L$.

3 Policy comparison: *Lean versus screen*

In this section, we evaluate and compare the effects of monetary policy and prudential regulation on the price of risky asset. As we will see below, the most prominent difference between the equilibrium with monetary policy and that with prudential regulation is that $r_{t+1} = r^l_{t+1}$ ($> r^d_{t+1}$) in the collapsing period with binding monetary policy, whereas $r_{t+1} > r^l_{t+1}$ ($> r^d_{t+1}$) in the collapsing period with binding prudential regulation.

3.1 Monetary policy

In our model, where there is no money nor nominal variables, we use a simplified assumption, following Allen and Gale (2000), that the central bank can decide the total amount of bank credit, B_t, which is equalized to the loan demand, l_t, in equilibrium. Thus, (7) implies that

$$B_t = x_t + P_t X_t,$$

in equilibrium. The monetary policy is represented by the amount of B, which is in fact determined as an equilibrium outcome of the central bank's decision of nominal interest rate under the economic environment with nominal rigidities, while these details are not explicitly modeled in this paper.

The noticeable feature of monetary policy in this model is that B is the constraint for the bank's decision-making. When B is decided by monetary policy, it gives the total supply of credit in this economy and B is not a constraint for the investors' decisions, whereas B is the constraint for the individual investors when it is determined by the prudential regulations (see the next subsection).

We distinguish the equilibrium variables in the laissez faire case by putting the superscript LF: $(x_t, r_{t+1}, P_t, P^g) = (x_t^{LF}, r_{t+1}^{LF}, P_t^{LF}, P^{g,LF})$, from those in the case with monetary policy $B = \{B_t\}_{t=0}^{\infty}$: $(x_t, r_{t+1}, P_t, P^g) = (x_t(B), r_{t+1}(B), P_t(B), P^g(B))$.

The following proposition holds for the price in the collapsing period T, $P^g(B)$,

Proposition 7. *$P^g(B)$ is lower than $P^{g,LF}$, when B_T is such that $r_{T+1}(B) > r^{LF}_{T+1}$.*

Proof. As the central bank decides the total amount of credit, B, by setting the nominal interest rate, it does not affect the consumer's nor the investors' problem, and it does affect the banks' problem. Thus, $r_t^d(B) = r_t^{d,LF} = \beta^{-1}$ and $r_t(B) = r_t^l(B)$ for all t, because Lemma 1 holds. The central bank can make $r_t(B) > r_t^d(B) = \beta^{-1}$ by setting B sufficiently small. In this case, $(r_{T+1}(B), P^g(B))$ is the solution to

$$r = f'(B - P),$$

$$P = \frac{R^*}{r}, \qquad \text{where } R^* \text{ is the solution to } \int_{R^*}^{R^{\max}} (R - R^*)\phi(R)dR = c'(1),$$

because $B = x + PX$ and $X = 1$ in equilibrium. As R^* does not depend on B, it is easily confirmed that $r(B)$, the solution to $r = f'(B - \frac{R^*}{r})$, is decreasing in B. Thus, when the central bank sets B sufficiently small such that $r(B) > r^{LF}_{T+1}$, then the peak price of the risky asset becomes lower than that in the laissez faire case, i.e., $P^g(B) = \frac{R^*}{r(B)} < \frac{R^*}{r^{LF}_{T+1}} = P^{g,LF}$. $\square$

This proposition shows that tighter monetary policy can dampen the size of the asset price $P^g(B)$ ($< P^{g,LF}$) and duration of the bubble T, as the asset price evolves by the law of motion $P_{t+1} = r_{t+1}(B)P_t + c'(1)$ during the bubbly periods, and $r_{t+1}(B) > r_{t+1}^{LF} = \beta^{-1}$ for $0 \leq t < T$.

3.2 Prudential regulation

Now we interpret B as the constraint on the credit obtained by the individual investors, rather than the total supply of the credit in the economy. Then, the government decision of B is interpreted as the credit rationing due to the prudential regulation. In this case, each investor faces the credit constraint:

$$x + PX \leq B, \tag{23}$$

and Lemma 1 should be modified to

$$r_{t+1}^l \leq r_{t+1}, \tag{24}$$

where $r_{t+1}^l = r_{t+1}$ if (23) is not binding and $r_{t+1}^l < r_{t+1}$ if it is binding.

The price in the collapsing period, $P^g(B)$ is decided by the maximization of π^L by the investor, which is, in the case where $x + PX \leq B$ is binding,

$$\pi^L = \int_{R^*(X)}^{R^{\max}} \{R - R^*(X)\}X\phi(R)dR - c(X) - \Delta,$$

where $R^*(X) = rP - (r - r^l)\frac{B}{X}$. The FOC with respect to X is $\int_{R^*(X)}^{R^{\max}} [R - R^*(X) + (r - r^l)\frac{B}{X}]\phi(R)dR = c'(X)$. Thus, the FOC and the equilibrium condition $X = 1$ imply that the equilibrium variables $(r_{T+1}, r_{T+1}^l, P^g) = (\hat{r}(B), \hat{r}^l(B), \hat{P}^g(B))$ are decided as the solution to

$$r = f'(B - P), \tag{25}$$

$$\int_{rP-(r-r^l)B}^{R^{\max}} [R - rP + 2(r - r^l)B]\phi(R)dR = c'(1), \tag{26}$$

$$1 = \beta \Pr(R > rP - (r - r^l)B)r^l + \beta \int_0^{rP-(r-r^l)B} \frac{(R - rP + rB)}{B}\phi(R)dR, \tag{27}$$

where (27) is the consumer's FOC, $1 = \beta r^d$. Note that (26) implies that $rP - (r - r^l)B \geq R^*$ for $r \geq r^l$. Then,

$$\hat{R}(B) \equiv rP > R^*, \tag{28}$$

$$\tilde{R}(B) \equiv \hat{R}(B) - \{r(B) - r^l(B)\}B > R^*, \tag{29}$$

for $r > r^l$ and $\hat{R} = \tilde{R} = R^*$ for $r = r^l$. Here we focus on the case that $B < B_{T+1}^{LF}$, where B_{T+1}^{LF} is defined as the amount of credit that each investor obtains in the collapsing period in the case where there is no credit constraint, i.e.,

$$B_{T+1}^{LF} \equiv x_T^{LF} + P^{g,LF}. \tag{30}$$

Therefore,

$$B - \frac{\hat{R}}{r} < B_{T+1}^{LF} - \frac{\hat{R}}{r} \leq B_{T+1}^{LF} - \frac{R^*}{r}.$$

This inequality implies that $f'(B - \frac{\hat{R}}{r}) > f'(B_{T+1}^{LF} - \frac{R^*}{r})$ for all $r > 0$. This inequality implies that

$$\hat{r}(B) > r_{T+1}^{LF}, \tag{31}$$

because $\hat{r}(B)$ is the solution to $r = f'(B - \frac{\hat{R}}{r})$, and r_{T+1}^{LF} is the solution to $r = f'(B_{T+1}^{LF} - \frac{R^*}{r})$. Now, in the following proposition, we can demonstrate a surprising example that credit rationing due to smaller B ($< B^{LF}$) may raise the asset price to a higher level than the laissez faire case, i.e.,

$$\hat{P}^g(B) > P^{g,LF} \qquad \text{for} \quad B < B^{LF}.$$

The intuition behind this result is explained as follows. The tighter prudential regulation lowers B, which raises r in (25), so that r becomes larger than r^l. As the market rate r is raised, while the loan rate r^l is still low, the threshold of default, R^*, is lowered. This is because R^* is determined by $rx + R^*X = r^lB$ and r gets higher, while r^l remains low. As R^* becomes lower, so does the probability of default. The lower probability of default makes the investors' demand for the risky asset increase, leading to an increase in P^g, the price of the risky asset. Another explanation is that, in (26), where the marginal gain of investing in an additional amount of the risky asset is equalized to the marginal cost, the marginal gain (the left-hand side) is larger when $r - r^l > 0$ than when $r = r^l$; thus, the investors' demand for the risky asset increases, leading to an increase in P^g. In this way, the tighter prudential regulation can increase the asset price.

Proposition 8. *Suppose that $f(x) = Ax^\alpha$ with $\alpha = 0.8$ and $R \sim U[0, R^{\max}]$, i.e., $\phi(R) = \frac{1}{R^{\max}}$. There exists the parameter A such that $\frac{d}{dB}\hat{P}^g(B) < 0$ at $B = B^{LF}$, which implies that a tighter prudential regulation, represented by $B < B^{LF}$, raises the asset price, i.e., $\hat{P}^g(B) > P^{g,LF}$.*

The proof is given in Appendix D.

4 Conclusion

A model of credit-driven bubble is presented, where agency problems in the banking sector bid up the asset price to an unsustainable height. In this model, the peak of the bubble and the timing of its collapse (T) can be predicted as the bubble collapses when the price hits an endogenous threshold (P^g). Tighter monetary policy can dampen the size of the bubble, whereas tighter prudential regulations that cause credit rationing may exacerbate the bubble. This model implies that asset bubble is destabilizing and distortionary, and that the bubbly dynamics of the asset price may be reasonably predictable from the structural parameters, as the values (T, P^g) are determined by those parameters.

Although the model in the present paper is made simple for the ease of exposition, it must be easily generalized to a non-linear and stochastic dynamic-general-equilibrium model, where we believe our results should be basically preserved. These applications are left for future research.

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Appendix A: Proof of Lemma 3

The equation $g(r) = 1$ can be rewritten as $F(r) = 0$, where

$$F(r) \equiv (\alpha A)^{\gamma}(\beta r - 1) + R^{*} r^{\gamma-1}(\omega\beta r - 1),$$

where $\gamma = (1-\alpha)^{-1}$. Note that $0 < \omega < 1$. It is obvious that

$$F(r) < 0, \qquad \text{for } 0 < r \leq \beta^{-1}, \tag{32}$$

$$F(r) > 0, \qquad \text{for } r \geq (\omega\beta)^{-1}. \tag{33}$$

Thus, the solution to $F(r) = 0$ should satisfy $\beta^{-1} < r < (\omega\beta)^{-1}$. Now the second derivative of $F(r)$ is calculated as

$$F''(r) = (\gamma - 1)R^{*} r^{\gamma-3}\{\gamma\omega\beta r - (\gamma - 2)\}.$$

Thus, $F''(r) \geq 0$ for $r \geq \hat{r}$, where $\hat{r} \equiv \frac{\gamma-2}{\gamma}(\omega\beta)^{-1}$. The condition (20) implies that $\hat{r} \leq \beta^{-1}$, implying that

$$F''(r) \geq 0, \qquad \text{for } r \geq \beta^{-1}.$$

This concavity and continuity of $F(r)$ imply that the solution to $F(r) = 0$ should be unique if it exists, and conditions (32) and (33) imply that there exists a unique solution in $(\beta^{-1}, (\omega\beta)^{-1})$.

Appendix B: Proof of Lemma 4

Suppose $P_T < P^g$, then $1 < g(\frac{R^*}{P_T})$, which means the return on the bank deposit r^d_{T+1} exceeds β^{-1}. In this case, the consumer chooses $C_T = 0$. Noting that period $T-1$ is the bubbly period, we have $d_T = l_T = y + r_T d_{T-1} + w_T + \pi^S_T = y + f(x_{T-1}) + r_T P_{T-1} + c'(1) - c(1)$. As $l_T = x_T + P_T$ and $P_T = r_T P_{T-1} + c'(1)$, this implies that $x_T = y + f(x_{T-1}) - c(1)$, which exceeds $x(\frac{R^*}{P_T})$, as y is assumed to be sufficiently large. This result contradicts $r_{T+1} = f'(x_T)$. Thus, P_T cannot be strictly smaller than P^g. Suppose $P_T > P^g$, then $1 > g(\frac{R^*}{P_T})$, which means that the return on the bank deposit r^d_{T+1} is below β^{-1}. In this case, consumer chooses $d_T = l_T = 0$ and $C_T = y + f(x_{T-1}) - c(1)$. Then, $x_T = 0$, which contradicts $x_T = x(\frac{R^*}{P_T}) > 0$. Thus, P_T cannot be strictly larger than P^g. Therefore, $P_T = P^g$ in the collapsing period.

Appendix C: A back-of-the-envelope example

Here we describe a simple example in which R follows the uniform distribution in $[0, R^{\max}]$, i.e., $U[0, R^{\max}]$. We set the cost of continuation, $c(X) = \varepsilon X^{1+\gamma}$, and the cost of liquidation, $c(X) + \Delta = \varepsilon X^{1+\gamma} + \Delta$, where $\Delta\,(> 0)$ is a fixed cost. First we derive the solution to (17), R^*. Since (17) is written as

$$\frac{(R^{\max} - R^*)^2}{2R^{\max}} = (1+\gamma)\varepsilon,$$

the solution is

$$R^* = R^{\max}\left\{1 - \sqrt{\frac{2(1+\gamma)\varepsilon}{R^{\max}}}\right\}.$$

We set the values of parameters such that $R^* = 0.8R^{\max}$ and $\gamma = 10$. Then, $c'(1) = (1+\gamma)\varepsilon = 0.02R^{\max}$. Thus, $c(X) = \frac{0.02}{11}R^{\max}X^{11}$.

Next, we derive P^L, where the price must satisfy $P_t \geq P^L$ in the bubbly period t. As X^L solves $c'(X)X - c(X) - \Delta = c'(1) - c(1)$,

$$X^L = \left(1 + \frac{\Delta}{\gamma\varepsilon}\right)^{\frac{1}{1+\gamma}}.$$

Then $R^L = \beta^{-1}P^L$ solves (15), which is written as

$$\frac{(R^{\max} - R^L)^2}{2R^{\max}} = (1+\gamma)\varepsilon\left(1 + \frac{\Delta}{\gamma\varepsilon}\right)^{\frac{\gamma}{1+\gamma}}.$$

Thus,

$$P^L = \beta R^L = \beta R^{\max}\left\{1 - \left(1 + \frac{\Delta}{\gamma\varepsilon}\right)^{\frac{\gamma}{2(1+\gamma)}}\sqrt{\frac{2(1+\gamma)\varepsilon}{R^{\max}}}\right\}.$$

We set Δ such that, in the collapsing period, $\pi^L = c'(1) - c(1) - \Delta = 0$ or $\Delta = \gamma\varepsilon$. In this case, $\left(1 + \frac{\Delta}{\gamma\varepsilon}\right)^{\frac{\gamma}{2(1+\gamma)}} = (2)^{\frac{10}{22}} = 1.37$, and

$$P^L = 0.726\beta R^{\max}. \tag{34}$$

We derive the gross interest rate in the collapsing period, r_{T+1}, by solving (21). Since $\Pr(R \geq R^*) = 0.2$, $g(r)$ in (21) is written as follows.

$$\begin{aligned}
&\beta\Pr(R > R^*)r + \beta\frac{\int_0^{R^*}(rx(r) + R)\frac{dR}{R^{\max}}}{x(r) + P(r)}\\
&= \beta 0.2r + \beta\frac{\{0.8rx(r) + 0.16R^{\max} + \frac{(0.8)^2}{2}R^{\max}\}}{x(r) + P(r)}\\
&= \frac{\beta}{x(r) + P(r)}\{rx(r) + 0.48R^{\max}\}\\
&= \beta\frac{\{x(r)r + 0.48R^{\max}\}}{x(r) + \frac{0.8R^{\max}}{r}} = g(r).
\end{aligned}$$

Now, we consider the case where $A \to \infty$. We guess and verify later that $r = O(1)$ as $A \to \infty$. Then, $x(r) \to \infty$ and (21) can be rewritten as

$$1 \approx \beta r.$$

Therefore, in the case where A is sufficiently large, $r \approx \beta^{-1}$, and thus,

$$P^g = R^*/r \approx 0.8\beta R^{\max}. \tag{35}$$

We set $\beta = 0.99$. In this case $P^g = 0.792R^{\max}$ and $P^L = 0.719R^{\max}$. Suppose that $P_T = P^g = 0.792R^{\max}$. Then, $P_{T-1} = \beta(P^g - c'(1)) = 0.7642R^{\max}$, $P_{T-2} = \beta(P_{T-1} - c'(1)) = 0.7368R^{\max}$, $P_{T-3} = \beta(P_{T-2} - c'(1)) = 0.7097R^{\max} < P^L = 0.719R^{\max}$. Thus, the collapsing period is $T = 2$ and the path of equilibrium prices is $\{P_0 = 0.7368R^{\max}, P_1 = 0.7642R^{\max}, P_2 = P^g = 0.792R^{\max}\}$, with P_3 taking any value satisfying $P_3 < \beta P^g$. In this model, the price of risky asset can grow for 2 periods and collapses in the third period. Note that (22) implies that the fundamental price $\bar{P}_t$ is $\beta[\frac{R^{\max}}{2} - c'(1)] = 0.4752R^{\max}$, as $r_t \approx \beta^{-1}$ for all t.

Appendix D: Proof of Proposition 8

In this case, the FOCs (25)–(27) are written as

$$r = \frac{\alpha A}{(B-P)^{1-\alpha}}, \tag{36}$$

$$\frac{1}{2}\{(R^{\max})^2 - (rP - (r - r^l)B)^2\} + [2(r - r^l)B - rP][R^{\max} - rP + (r - r^l)B] = c'(1)R^{\max}, \tag{37}$$

$$\frac{R^{\max}}{\beta} = \{R^{\max} - rP + (r - r^l)B\}r^l + \frac{1}{2}(r^l)^2 B + r(1 - \frac{P}{B})[rP - (r - r^l)B]. \tag{38}$$

The solution to this system of equations is $(r, r^l, P) = (\hat{r}(B), \hat{r}^l(B), \hat{P}^g(B))$. The differentiation of the above equations with respect to B gives

$$r' = -\frac{(1-\alpha)\alpha A(1 - P')}{(B-P)^{2-\alpha}}, \tag{39}$$

$$\begin{aligned}[rP - (r - r^l)B](r'P + rP' - (r - r^l) - (r' - r^{l'})B] \\ = [2(r - r^l) + 2(r' - r^{l'})B - r'P - rP'][R^{\max} - rP + (r - r^l)B] \\ + [2(r - r^l)B - rP][-r'P - rP' + (r - r^l) + (r' - r^{l'})B],\end{aligned} \tag{40}$$

$$\begin{aligned}0 = (-r'P - rP' + (r - r^l) + (r' - r^{l'})B)r^l + (R^{\max} - rP + (r - r^l)B)r^{l'} + r^l r^{l'} B + \frac{(r^l)^2}{2} \\ + r'(1 - \frac{P}{B})[rP - (r - r^l)B] + r(-\frac{P}{B} + \frac{P}{B^2})[rP - (r - r^l)B] \\ + r(1 - \frac{P}{B})[r'P + rP' - (r' - r^{l'})B - (r - r^l)],\end{aligned} \tag{41}$$

where $r' = \frac{d}{dB}r$, $r^{l'} = \frac{d}{dB}r^l$, and $P' = \frac{d}{dB}P$. Evaluating at $B = B^{LF}$, where $r = r^l$, the second and the third equations imply that at $B = B^{LF}$,

$$r'P + rP' = 2(r' - r^{l'})B, \tag{42}$$

$$0 = (R^{\max} + rB)r^{l'} + \frac{r^2}{2} - 2(r' - r^{l'})rP + \left(\frac{rP}{B}\right)^2. \tag{43}$$

Solving (39), (42), and (43) for P', we obtain

$$\left\{\frac{\frac{(R^{\max}+rB)}{2B}r + \frac{r^2P}{B}}{[R^{\max} + rB(1 - \frac{P}{2B}) - \frac{rP^2}{B}]} - \frac{(1-\alpha)r}{B-P}\right\}P' = \frac{\frac{r^2}{2} + \frac{r^2P^2}{B^2}}{[R^{\max} + rB(1 - \frac{P}{2B}) - \frac{rP^2}{B}]} - \frac{(1-\alpha)r}{B-P}. \tag{44}$$

Now, we will show that $P' = \frac{d}{dB}P < 0$ at $B = B^{LF}$, which implies that the tightening of financial regulation exacerbates the bubble, i.e., $P(B) > P(B^{LF})$ for $B < B^{LF}$. By choosing $c'(1)$ appropriately, we can rewrite (37), evaluated at $B = B^{LF}$, as $P = \frac{p}{r}R^{\max}$, where p is a parameter. We set $p = 0.8$, as we did in Section 4. (Note that $pR^{\max} = R^*$, as P in (37) is P^g in Section 2.) We also set $(\alpha A)^{\frac{1}{1-\alpha}} = bR^{\max}$, where b is a parameter that satisfies $0 \leq b < \infty$. Then, (36), evaluated at $B = B^{LF}$, can be rewritten as $B = \left[p + \frac{b}{r^{\frac{\alpha}{1-\alpha}}}\right]\frac{R^{\max}}{r}$. Then, (38), evaluated at $B = B^{LF}$, can be rewritten as an equation for r,

$$\frac{r - \beta^{-1}}{p + \frac{b}{r^{\frac{\alpha}{1-\alpha}}}} = r\left(\frac{1}{\left(1 + \frac{\frac{b}{p}}{r^{\frac{\alpha}{1-\alpha}}}\right)^2} - \frac{1}{2}\right). \tag{45}$$

Rewriting the left-hand side and the right-hand side of (44), we obtain the (sufficient) condition for $P' < 0$ at $B = B^{LF}$, that is, $r(b)$, the solution to (45) should satisfy both

$$\frac{\frac{1}{2\left(p+\frac{b}{r^{\frac{\alpha}{1-\alpha}}}\right)} + \frac{1}{2} + \frac{p}{p+\frac{b}{r^{\frac{\alpha}{1-\alpha}}}}}{1 + p + \frac{b}{r^{\frac{\alpha}{1-\alpha}}} - \frac{p}{2} - \frac{p^2}{p+\frac{b}{r^{\frac{\alpha}{1-\alpha}}}}} - (1-\alpha)\frac{r^{\frac{\alpha}{1-\alpha}}}{b} > 0, \tag{46}$$

and

$$\frac{\frac{1}{2} + \frac{p}{p+\frac{b}{r^{\frac{\alpha}{1-\alpha}}}}}{1 + p + \frac{b}{r^{\frac{\alpha}{1-\alpha}}} - \frac{p}{2} - \frac{p^2}{p+\frac{b}{r^{\frac{\alpha}{1-\alpha}}}}} - (1-\alpha)\frac{r^{\frac{\alpha}{1-\alpha}}}{b} < 0. \tag{47}$$

By setting $p = 0.8$, $\alpha = 0.8$ and $\beta = 0.99$, we can numerically confirm that, for $b \in (0.15, 0.1725]$, there exists the solution to (45) that satisfies $r(b) \geq \beta^{-1}$ and both (46) and (47).[3] For example, when $b = 0.1725$, $r(b) = \beta^{-1} = 1.01$ and both (46) and (47) are satisfied. Another example is $b = 0.17$ and $r(b) = 1.0183$, which also satisfy (46) and (47). We have shown the examples in which $P' < 0$ at $B = B^{LF}$.

[3]It is easily confirmed that the parameters, $p = 0.8$, $\alpha = 0.8$ and $\beta = 0.99$, and the uniform distribution of R satisfy the condition (20).