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Daichi Shirai
The Canon Institute for Global Studies

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A note on hump-shaped output in the RBC model*

Daichi Shirai†

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Abstract

This note shows that a standard real business cycle model with a specific parameter range can weakly generate a hump-shaped output response output to productivity shocks. This result requires only that the technology shocks are nearly random walk.

Keywords: RBC model; Hump-shaped output

JEL Classification numbers: E10, E32

1 Introduction

The hump-shaped responses of macroeconomic variables to exogenous shocks are one of the most important features of observed business cycles and financial crises.[1] Many studies focus on reproducing this feature by introducing some friction, for example, investment adjustment costs, labor adjustment costs, and agency costs. This note shows that a standard real business cycle (RBC) model without additional friction can weakly generate a hump-shaped output response to persistent productivity shocks.

Previously, it was thought that the output response to productivity shocks in the standard RBC model displays no hump shape, except for a case of complete depreciation of

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†The Canon Institute for Global Studies. Email: shirai.daichi@canon-igs.org

[1]Kobayashi and Shirai (2014) summarize these stylized facts.

capital. However, we find that even in a case of partial depreciation of capital, a standard RBC model can weakly reproduce hump-shaped responses. To reproduce this feature, technology shocks have to be sufficiently persistent, that is, almost a random walk. King and Rebelo (1999) point out that the standard RBC model requires a persistent exogenous shock to replicate the main statistical features of business cycles. In addition, a persistent productivity shock can generate hump-shaped output.

Standard RBC models have two kinds of propagation mechanisms: capital accumulation and intertemporal substitution. We emphasize that capital accumulation is an important propagation mechanism to reproduce the hump-shaped output. When a change in capital stock is larger than a change in productivity shock, then output displays a hump shape. This situation can be generated from a relatively high investment rate, which propagates capital accumulation and makes the output response countercyclical during some periods.

The remainder of this note is organized as follows. Section 2 presents the standard RBC model used in this note. Section 3 analyzes the condition of the hump-shaped output with complete capital depreciation. Section 4 shows the impulse response function (IRF) for output to a productivity shock with partial depreciation. Section 5 concludes.

2 Model

In this section, we describe a standard RBC model with a representative household and firm. All markets are competitive.

2.1 Settings

The representative household maximizes expected discounted lifetime utility defined over consumption, C, and hours of work, L:

$$U = \sum_{t=0}^{\infty} \beta^t \left[\ln(C_t) + \gamma \ln(1 - L_t) \right],$$

subject to a budget constraint:

$$C_t + I_t = w_t L_t + r_t K_t,$$

where β is the subjective discount rate, such that $0 < \beta < 1$, I_t is investment, w_t is the wage, r_t is the rental rate of capital, and K_t is the capital stock, which obeys the usual law of motion:

$$K_{t+1} = (1-\delta)K_t + I_t,$$

where δ is the depreciation rate.

The representative firm produces output according to a Cobb–Douglas production function:

$$Y_t = z_t K_t^{\alpha} L_t^{1-\alpha}, \quad 0 \leq \alpha \leq 1,$$

where z_t is productivity. In this note, we assume that the economy was initially in a steady state and the technology shock makes the initial value of z_1 lower (greater) than the steady state value, $z_1 = \omega z_*, \quad \omega > 0$, in such a way that productivity at time 1 is lower (greater) than its steady state value by 1%, $\omega = 0.99$ ($\omega = 1.01$). Productivity is assumed to follow the process:

$$z_{t+1} = \rho z_t + (1-\rho)z_*, \quad 0 < \rho < 1, \quad t \geq 1.$$

This assumption implies that the evolution of productivity can be represented as follows:

$$z_t = (1 - \rho^{t-1}(1-\omega))z_*, \quad t \geq 1. \tag{1}$$

3 Mechanism of hump-shaped output

In this section we provide an intuitive explanation of the mechanism of the hump-shaped output using a special case of parameter setting, that is, complete depreciation of capital within a one period, or $\delta = 1$. It is well known that exact analytical solutions can be obtained in RBC models with log utility, Cobb–Doglas production functions, and complete depreciation of capital. [2] In this case, the evolution of capital stock is governed by the following:

$$K_{t+1} = \alpha\beta z_t K_t^{\alpha} L^{1-\alpha}. \tag{2}$$

[2] For example, see McCallum (1989), Ljungqvist and Sargent (2012) and Stokey, Lucas and Prescott (1989).

In addition, all variables are obtained analytically:

$$\begin{aligned} C_t &= (1-\alpha\beta) z_t K_t^{\alpha} L^{1-\alpha}, \\ L &= \frac{X}{1+X}, \end{aligned}$$

where $X \equiv \frac{(1-\alpha)}{\gamma(1-\alpha\beta)}$. In this special case, labor inputs are always constant even when the economy is in the transition toward the steady state because of complete balancing of income and substitution effects. In addition, the investment rate, $s_t = \frac{Y_t - C_t}{Y_t}$, is constant and equal to $\alpha\beta$.

A humped-shape variable responds countercyclically to a shock during some periods. Most studies usually judge graphically whether the response of a variable is hump-shaped. In this note, we define "hump-shaped" explicitly for descriptive purposes.

Definition 1. After revelation of a negative (positive) technology shock z_1 at the end of period 0, the hump-shaped response of a variable in period 2 is the same as or decreases (increases) more than its value in period 1.

The following proposition provides a simple characterization of the hump-shaped response of output to the technology shock.

Proposition 1. *Assume the log utility, Cobb–Douglas production, and complete capital stock depreciation in one period. Then the output displays a hump-shaped response to technology shocks when the following condition is satisfied:*

$$\rho \geq \frac{1-\omega^{1-\alpha}}{1-\omega}. \tag{3}$$

Proof. Because the hump-shaped response is determined by the growth rate in period 2, as per Definition 1, the output is the same or decreases (increases) more than its value in period 1 in response to the negative (positive) technology shock. The output growth rate in period 2 is as follows:

$$\frac{Y_2 - Y_1}{Y_1} = \frac{z_2 K_2^{\alpha} L^{1-\alpha}}{z_1 K_1^{\alpha} L^{1-\alpha}} - 1.$$

Equation (1) and (2) imply that

$$\frac{Y_2 - Y_1}{Y_1} = \frac{1-\rho(1-\omega)}{\omega}\omega^{\alpha} - 1. \tag{4}$$

When output in period 2 equals its value in period 1, the left side of equation (4) equals zero, and finally, we can obtain equation (3). □

Proposition 1 implies that the condition (3) satisfies the case of large values of $(\rho,\ \alpha)$.[3] In other words, when the technology shock displays persistence to some extent or the investment rate is high, standard RBC models are capable of producing hump-shaped output responses to technology shocks. The intuition of Proposition 1 can be understood from equation (4), which shows that output growth rate is decomposed into rate of change of productivity, $\frac{1-\rho(1-\omega)}{\omega}$, and rate of change of capital stock, ω^{α}. On the one hand, productivity converges to the steady state monotonically and the response of capital stock is always countercyclical and hump-shaped during some periods. When the capital growth rate is higher than the productivity growth rate, that is, when ρ and α are large, the output response is also hump-shaped. After the impact period of a productivity shock, investment leads to a large change in the relatively high investment rate and the next period's output changes substantially. This effect creates the propagation mechanism. Therefore, when the propagation mechanism of capital accumulation is larger than the productivity growth rate, then the output response is hump-shaped.

Figure 1 depicts the output response to a 1% technology shock, that is, $\omega = 0.99$. We show five cases corresponding to $\rho = 0.5,\ 0.701,\ 0.8,\ 0.9,\ 0.99$ and other parameters are set standard values: $\alpha = 1/3,\ \beta = 0.98,\ \gamma = 1.8,\ \delta = 1,\ z_* = 1$.[4] The result for $\rho = 0.5$, output converges to the steady state monotonically. The result for $\rho = \frac{1-\omega^{1-\alpha}}{1-\omega} = 0.701$ which strictly satisfies the condition, equation (4), output in period 2 is the same level in period 1 and converges to the steady state after period 3. In case of a large ρ, that is, more than 0.701, output is conspicuously hump-shaped.

In Figure 2, we set the value of ρ equal to 0.9 and show five cases corresponding to $\alpha = 0.25,\ \alpha = 0.3,\ \alpha = 1/3,\ \alpha = 0.35,\ \alpha = 0.4$. This figure shows that the higher the

[3] In addition, McCallum (1989) and Romer (2011) show that this special case of the RBC model can generate hump-shaped responses. However, they do not explain explicitly the condition that can generate hump-shaped responses.

[4] In addition, we can consider a large value for α because we interpret output, including remaining capital stock.

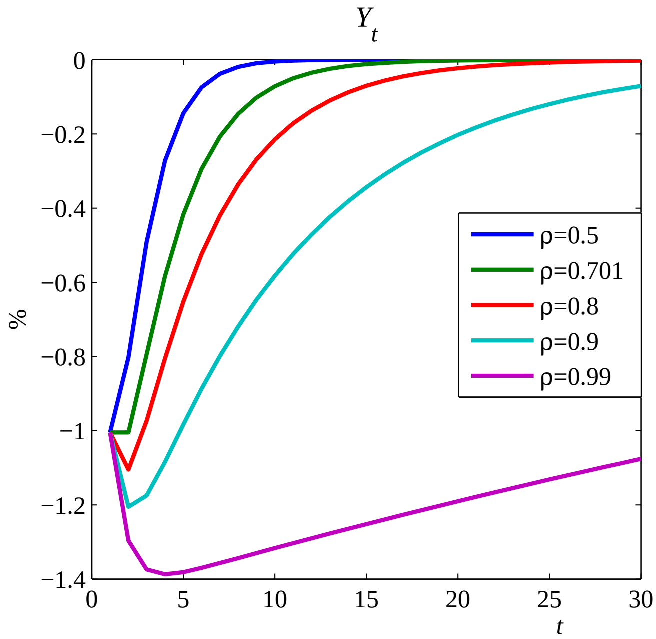

Figure 1: Response to the technology shock (z_t shock) corresponding to each ρ

Note: This figure depicts the nonlinear dynamic response to a 1% technological shock in terms of percentage deviation (log deviation times 100) from the steady state. Parameter settings are $\alpha = 1/3,\ \beta = 0.98,\ \gamma = 1.8,\ \delta = 1$ and $z_* = 1$

value of α, the stronger is the propagation mechanism.

4 Quantitative experiments

The assumption of complete capital stock depreciation in one period allows us to understand the hump-shaped mechanism analytically. In this section, we relax this assumption to introduce capital depreciation partially, $0 < \delta < 1$. In this section, we calculate impulse response functions, and some parameter settings can generate hump-shaped responses.

First, we set the depreciation rate of capital, δ, as 0.025 and compute the impulse responses to the 1% negative technology shock corresponding to $\alpha = 0.25,\ 0.3,\ 1/3,\ 0.35,\ 0.37$ and $\rho = 0.96,\ 0.97,\ 0.98,\ 0.99,\ 0.993$.[5] Table 1 shows the difference between percentage

[5]These impulse responses are calculated with Dynare. In addition, we solve the model using the forward shooting algorithm. This algorithm can solve nonlinear system of equations globally. However, there is no difference between standard impulse response function and forward shooting algorithm. Thus, we do not

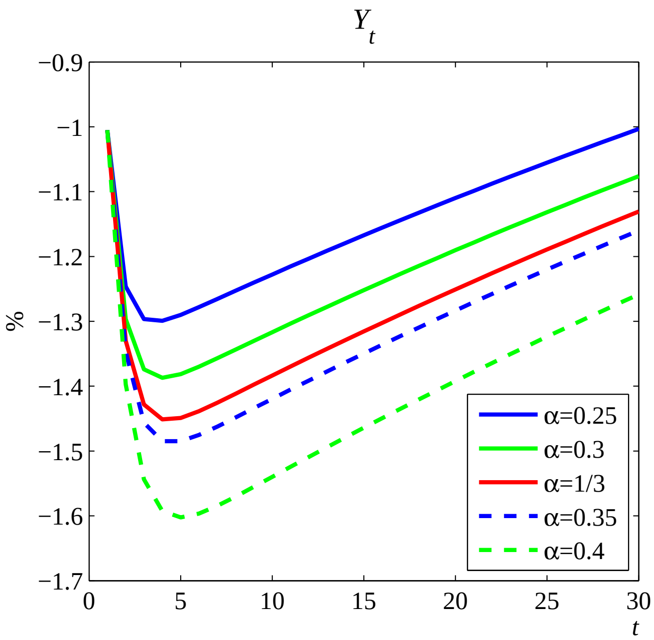

Figure 2: Response to the technology shock (z_t shock) corresponding to each α

Note: This figure depicts the nonlinear dynamic response to a 1% technological shock in terms of percentage deviation (log deviation times 100) from the steady state. Parameter settings are $\beta = 0.98,\ \gamma = 1.8,\ \delta = 1,\ \rho = 0.9,$ and $z_* = 1$

deviation from the steady state in period 2 and period 1 corresponding to each α and ρ. If the response of output to the negative technology shock is hump-shaped, the difference of percentage deviation between periods 2 and 1 is negative by Definition 1. Table 1 reports that a RBC model can weakly generate hump-shaped responses in the case of greater than or equal to $\rho = 0.99$. In addition, Figure 3 depicts the impulse response function: $\alpha = 1/3, \delta = 0.025$, and corresponding to each ρ. When the technology shock is strongly persistent or nearly random walk, the output response is hump-shaped even with a standard parameter setting. Usually, standard RBC models require that the technology shocks are persistent to fit with observed business cycle data and many studies frequently assume that the technology evolves according to a random walk, for example, Chang, Gomes and Schorfheide (2002). This is not a special case, even in empirical studies. Nelson and Plosser (1982) show that productivity contains a unit root. In addition, Table 2 shows that DSGE models estimate that technology shocks have a persistently large value.

Table 1: Difference between percentage deviation from the steady state in periods 2 and 1 corresponding to each α and ρ

		α				
		0.25	0.3	1/3	0.35	0.37
	0.96	0.0516	0.0452	0.0415	0.0399	0.0380
	0.97	0.0358	0.0300	0.0267	0.0252	0.0235
ρ	0.98	0.0207	0.0156	0.0127	0.0113	0.0099
	0.99	0.0069	0.0024	-0.0001	-0.0012	-0.0025
	0.993	0.0030	-0.0011	-0.0035	-0.0046	-0.0058

Notes: These values are based on impulse reponses to the 1 % technology shock under each parameter setting.

However, the RBC model with standard parameter values cannot produce a sufficiently strong hump-shaped response that is not quantitatively similar to observed business cycle data. Therefore, many studies introduce additional mechanisms, such as, the specialized

report the results of the forward shooting algorithm.

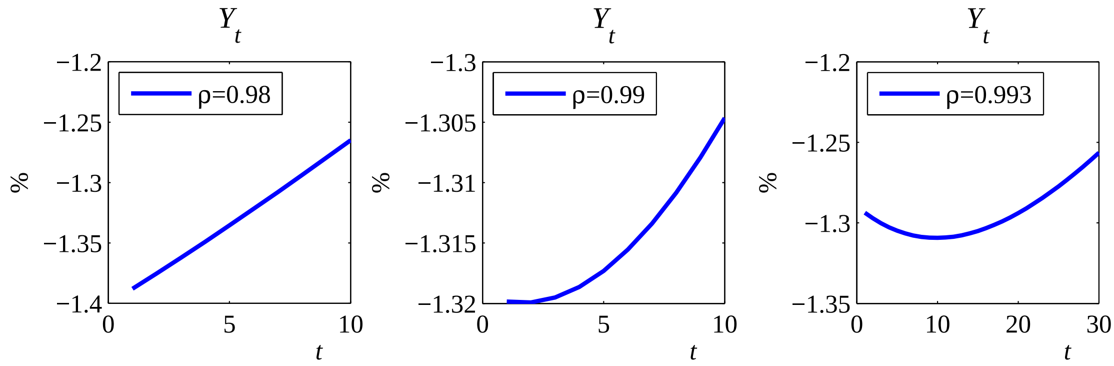

Figure 3: Impulse Responses corresponding to each ρ

Notes: Each panel depicts the impulse response to a 1 standard deviation technology shock in terms of percentage deviation from the steady state. Parameter settings are $\alpha = 1/3$, $\beta = 0.98$, $\delta = 0.025$, $\gamma = 1.8$ and $z_* = 1$

Table 2: Estimation results of persistence of technology shock

	value	data
Smets and Wouters (2003)	0.811	Euro Area
Onatski and Williams (2010)	0.954	Euro Area
Levin, Onatski, Williams and Williams (2006)	0.964	U.S.
Sugo and Ueda (2008)	0.949	Japan
Iiboshi, Nishiyama and Watanabe (2006)	0.818	Japan

adjustment costs of investments, (e.g., Smets and Wouters, 2003; Christiano, Eichenbaum and Evans, 2005), agency costs (e.g., Carlstrom and Fuerst, 1997), learning-by-doing mechanisms, (e.g., Chang et al., 2002; Cooper and Johri, 2002), adjustment costs of labor, (e.g., Cogley and Nason, 1995), and redistribution (e.g., Kobayashi and Shirai, 2014), which amplify propagation mechanisms. These additional mechanisms are partially successful at generating a small hump-shaped impulse response.

This note stresses that the hump-shaped output in the RBC model is due to persistent technology shocks and high investment rates. These factors are important for replicating hump-shaped output and have substantial effects on the dynamics of capital stock. Many studies introduce*investment* adjustment costs; on the other hand, *capital* adjustment costs models are less successful in replication because, as Cogley and Nason (1995) point out, the flow of investment is very small relative to the capital stock in capital adjustment models. Hence, one way to replicate the strong hump-shaped output is to introduce a factor that substantially affects the dynamics of capital stock.

5 Conclusion

In this note, we provided intuitive understanding of the hump-shaped output mechanism. In addition, we showed that a standard RBC model with a specific range of parameters can weakly reproduce a hump-shaped output. This hump shape is very weak, although an additional mechanism is not necessary to reproduce this feature. However, as is well known, RBC models have weak internal propagation mechanisms and do not generate a sufficiently strong hump shape.

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